Photodisintegration of the Deuteron

by

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SUMMARY

The thesis is divided into three parts. In the first part, a general survey of the two-nucleon problem is given, with particular attention paid to those aspects which impinge directly on the photodisintegration of the deuteron.

In the second part, we consider the conventional theory of deuteron photodisintegration, with the radiative interaction being taken as given on the basis of the gauge invariance of the non-relativistic Hamiltonian for the two-nucleon system. Differential cross-section and polarization formulae are presented, and a discussion given of previous calculations in this field. New calculations are carried out using the Gammel-Thaler type Y.L.A.M. phase parameters obtained in the analysis of Breit et al.(44,45)

The transitions considered are

1. Electric dipole $(^3S_1+^3D_1) \longrightarrow {}^3P_o, {}^3P_1, {}^3P_2+{}^3F_2$
2. Magnetic dipole spin-flip $(^3S_1+^3D_1) \longrightarrow {}^1S_o, {}^1D_2$
3. Electric quadrupole $(^3S_1+^3D_1) \longrightarrow {}^3S_1+{}^3D_1;$ ${}^3D_2, {}^3D_3+{}^3G_3$
4. Magnetic quadrupole spin-flip $(^3S_1+^3D_1) \longrightarrow {}^1P_1, {}^1F_3$

The $^3P_2$ - $^3F_2$ coupling is included, but the $^3S_1$ - $^3D_1$ and $^3D_3$ - $^3G_3$ coupling neglected. Wherever possible,

phenomenological wave-functions are used, and where this is not feasible, they are calculated from a suitable Gammel-Thaler potential. Differential cross-sections and polarizations are obtained for photon laboratory energies up to 130 MeV, the calculations being carried out both for a 4% and 6% deuteron D-state probability. Finally the results obtained are compared and contrasted with those of provious calculations, and both sets compared with experiments.

In the third part of the thesis, the calculation of the matrix element for deuteron photodisintegration by dispersion relations is considered. There are twelve invariant amplitudes. The covariant form of the transition amplitude is related to the non-covariant (Pauli-matrix) form, which is further related to the individual multipole transition amplitudes. The Born terms of the covariant amplitudes are derived, and the dispersion relations written down in energy for a fixed difference in the photon-proton and photon-neutron momentum transfers. It is necessary to use this rather than a fixed momentum transfer, in order to exhibit explicitly all the poles in the dispersion relations.

The dispersion relations contain integrals over both positive and negative energies, the latter arising from the crossed diagrams for which the imaginary part of the amplitude is related to processes such as the radiative absorption of an anti-nucleon by a deuteron, and to the structure of the deuteron through the anomalous singularities of the d-np vertex. These complications are ignored, and we retain only the pole terms and the integrals over positive energies.

The relations are restricted to dipole and quadrupole transitions, and by considering the relations at two different "momentum transfers", equations are obtained explicitly for the individual electric dipole and magnetic dipole spin flip transition amplitudes. The equations are solved in a low energy approximation in which the final state n-p rescattering cut and single pion exchange cut only are considered, for the two cases of the Y.L.A.M. and Signell-Marshak phase-parameters. The results obtained are compared with those obtained in part two of the thesis.

Contents

Part 1. A Survey of the Two-Nucleon Problem.

1. Introduction
2. The Nucleon-Nucleon Interaction
3. The Scattering State
4. Polarization in Nucleon-Nucleon Scattering
5. The Bound State
6. The Deuteron Magnetic Moment and Electric Quadrupole Moment
7. Scattering Length and Effective Range
8. Dispersion Relations and Nucleon-Nucleon Scattering
9. The Phase-Shift Analysis.

Part 2. Photodisintegration of the Deuteron

10. Introduction
11. General Form of the Interaction of the Nucleon Field with the Electromagnetic Field.
12. The Multipole Transitions
13. The Differential Cross-Section for Deuteron Photodisintegration
14. Polarization of the Final State Nucleons
15. Photodisintegration of the Deuteron up to 130 MeV
16. Photodisintegration of the Deuteron above 130 MeV

Part 3. Dispersion Relations for the Photodisintegration of the Deuteron

17. Introduction
18. Kinematical Considerations
19. General Form of the Pauli Transition Matrix for the Photodisintegration of the Deuteron.
20. The Application of Unitarity to Deuteron Photodisintegration
21. The Dispersion Relations
22. Vertex Functions and the Discrete Contributions.
23. Solution of the Dispersion Relations at Low Energies
24. Conclusions

Appendices

1. Mass Values and Other Constants
2. The γ-Matrices
3. Relations between Energies in the Laboratory and Centre-of-Momentum Systems.
4. Phenomenological Deuteron Wave-Functions
5. The Allowed Transitions
6. The Transition Amplitudes

Part I. A Survey of the Two-Nucleon Problem.

1. Introduction.

The two nucleon problem dates from the discovery of the neutron by Chadwick in 1932, the existence of the proton already having been established by Rutherford in 1919. The fundamental problem is to determine the interaction between two nucleons. Qualitatively this interaction is known to be strong and of short range, giving the two-nucleon system a characteristically simple spectrum. Neither the proton-proton system nor the neutron-neutron system has a bound state, the only bound state occurring in the neutron-proton system (the deuteron ground state).

Information about the nature of nucleon forces may be obtained in two ways. The first is by investigation of the direct interaction between two nucleons by means of scattering experiments, the properties of the bound state, and transitions between the bound and continuum states. The second is by the study of the properties of complex nuclei.

The latter cannot give any quantitative information on the nucleon-nucleon interaction, but can give useful qualitative information. The approximately linear dependence of nuclear binding energies on the number of particles in the nucleus indicates that the nuclear forces must be of short range - certainly less than the size of any but the lightest nuclei. The equality between the number of protons and neutrons in light nuclei can be interpreted as showing the existence of a strong attractive force between a neutron and a proton, which conclusion is supported by the stability of the deuteron. For the same reason one can conclude that, neglecting electrostatic repulsion, the proton-proton force must be very nearly equal to the neutron-neutron force. The behaviour of mirror nuclei (isobars with a neutron excess of ± 1) further substantiates this conclusion. It can also be concluded that the neutron-proton force is of the same strength as the neutron-neutron and proton-proton forces from the study of such isobaric triads as Be^{10}, B^{10}, C^{10} and C^{14}, N^{14}, O^{14} which exhibit behaviour analogous to that of mirror nuclei. This apparent equality of the nucleon-nucleon forces leads one to the hypothesis of charge independence of nuclear forces.

For more quantitative information about the nuclear forces it is necessary to investigate the nucleon-nucleon interaction directly. Generally speaking, the scattering of two particles is the simplest way of obtaining data on their interaction, and the nucleon-nucleon system is no exception. Other fruitful sources of information about the two-nucleon system are its electromagnetic interactions, of which the photodisintegration of the deuteron is a most promising phenomenon.

In this first part of the thesis, the general theory of the two-nucleon interaction is developed, and applied to the scattering and bound state problems, particular attention being paid to those aspects which are relevant to the photodisintegration of the deuteron.

2. The Nucleon-Nucleon Interaction.

The general form of this interaction may be determined readily from the general invariance principles of non-relativistic quantum mechanics and the principle of charge independence, which require the potential to be of the form (1)

$$V(\underset{\sim}{r}) = V_1(r) + V_2(r)\,\underset{\sim}{\sigma}^{(1)}\cdot\underset{\sim}{\sigma}^{(2)} + V_3(r)\,S_{12} + V_4(r)\,\underset{\sim}{L}\cdot\underset{\sim}{S}$$
$$+ V_5(r)\left[\underset{\sim}{\sigma}^{(1)}\cdot\underset{\sim}{L}\;\underset{\sim}{\sigma}^{(2)}\cdot\underset{\sim}{L} + \underset{\sim}{\sigma}^{(2)}\cdot\underset{\sim}{L}\;\underset{\sim}{\sigma}^{(1)}\cdot\underset{\sim}{L}\right] \qquad (2.1)$$
$$V_j(r) = V_j^s(r) + V_j^v(r)\,\underset{\sim}{\tau}^{(1)}\cdot\underset{\sim}{\tau}^{(2)}$$

where $\underset{\sim}{\sigma}^{(i)}$ ($\underset{\sim}{\tau}^{(i)}$) is the spin (isotopic spin) operator of the i-th nucleon, $S_{12} = \frac{3\,\underset{\sim}{\sigma}^{(1)}\cdot\underset{\sim}{r}\;\underset{\sim}{\sigma}^{(2)}\cdot\underset{\sim}{r}}{r^2} - \underset{\sim}{\sigma}^{(1)}\cdot\underset{\sim}{\sigma}^{(2)}$ is the tensor operator, $\underset{\sim}{S} = \frac{1}{2}(\underset{\sim}{\sigma}^{(1)}+\underset{\sim}{\sigma}^{(2)})$ and $\underset{\sim}{L} = (\underset{\sim}{r}_1 - \underset{\sim}{r}_2)\times(\underset{\sim}{p}_1 - \underset{\sim}{p}_2)$ is the orbital angular momentum.

The problem is now to determine the form of the functions $V_j(r)$. Historically, two methods of attacking the problem have been available - the meson theoretic treatment and the purely phenomenological approach.

In the former, a one-component pseudo-scalar field variable $\phi(x_\mu)$ is taken to describe the meson field, and is assumed to satisfy the Lorentz invariant equation

$$(\Box - \mu^2)\,\phi(x_\mu) = 0 \qquad (2.2)$$

where

$$\Box = \sum_{\mu=0}^{3} \frac{\partial^2}{\partial x_\mu^2} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \underset{\sim}{x}^2} \qquad (2.3)$$

The field equation may be derived by the usual variational principle from the Lorentz invariant Lagrangian

$$L_0 = -\frac{1}{2}\left\{ \sum_\mu \left(\frac{\partial \phi_0}{\partial x_\mu}\right)^2 + \mu^2 \phi_0^2 \right\} \tag{2.4a}$$

for a real field (corresponding to neutral mesons)

$$L_0 = -\left\{ \sum_\mu \frac{\partial \phi^*}{\partial x_\mu} \cdot \frac{\partial \phi}{\partial x_\mu} + \mu^2 \phi^* \phi \right\} \tag{2.4b}$$

for a complex field (corresponding to charged mesons).

The interaction of the mesons with the nucleons may be obtained by constructing a Lorentz invariant term L' from $\phi(x_\mu)$ and $\psi(x_\mu)$, the spinor field variable, which satisfies a Dirac equation of the form

$$\sum_\mu i \gamma_\mu \frac{\partial \psi}{\partial x_\mu} - m\psi \qquad = \quad 0 \tag{2.5}$$

derivable from the Lagrangian

$$L = \frac{i}{2} \sum_\mu \left\{ \bar{\psi}(x) \gamma_\mu \frac{\partial \psi(x)}{\partial x_\mu} - \frac{\partial \bar{\psi}(x)}{\partial x_\mu} \gamma_\mu \psi(x) \right\} - m \bar{\psi} \psi \tag{2.6}$$

The two simplest forms for L' are the pseudo-scalar-pseudoscalar (ps-ps) coupling

$$L' = \begin{cases} -g\, \bar{\psi}_p \gamma_5 \psi_p \phi_0 - g\, \bar{\psi}_n \gamma_5 \psi_n \phi_0 & \text{real field} \\ -g\sqrt{2}\, \bar{\psi}_p \gamma_5 \psi_n \phi - g\sqrt{2}\, \bar{\psi}_n \gamma_5 \psi_p \phi^* & \text{complex field} \end{cases} \tag{2.7}$$

and the pseudoscalar-pseudovector (ps-pv) coupling

$$L' = \begin{cases} -\frac{f}{\mu}\sum_\mu \left[\bar\psi_p \gamma_5 \gamma_\mu \psi_p \frac{\partial\phi_0}{\partial x_\mu} + \bar\psi_n \gamma_5 \gamma_\mu \psi_n \frac{\partial\phi_0}{\partial x_\mu}\right] & \text{real field} \\ -\frac{f}{\mu}\sqrt{2}\sum_\mu \left[\bar\psi_p \gamma_5 \gamma_\mu \psi_n \frac{\partial\phi}{\partial x_\mu} + \bar\psi_n \gamma_5 \gamma_\mu \psi_p \frac{\partial\phi^*}{\partial x_\mu}\right] & \text{complex field} \end{cases} \qquad (2.8)$$

The (ps-ps) and (ps-pv) coupling constants g and f are real, and have the dimensions of an electric charge.

Equations (2.4), (2.7) and (2.8) may be simplified considerably by introducing isotopic spin and assuming a symmetric meson theory. For the latter we introduce two real fields ϕ_1 and ϕ_2 by

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \quad , \quad \phi^* = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad (2.9)$$

and replace ϕ_0 by another neutral field ϕ_3, interacting with the nucleon field by

$$L' = -g\,\bar\psi_p \gamma_5 \psi_p \phi_3 + g\,\bar\psi_n \gamma_5 \psi_n \phi_3 \qquad (2.10)$$

for the (ps-ps) case, and similarly for the (ps-pv) case.

The symmetrical meson-field Lagrangian is then given by

$$L_0 = -\frac{1}{2}\sum_{i=1}^{3}\left\{\sum_\mu \left(\frac{\partial\phi_i}{\partial x_\mu}\right)^2 + \mu^2 \phi_i^2\right\} \qquad (2.11)$$

and the interaction Lagrangian by

$$L' = -g\sum_{i=1}^{3} \bar\psi \gamma_5 \tau_i \psi \phi_i \qquad (ps\text{-}ps) \qquad (2.12a$$

or

$$L' = -\frac{f}{\mu}\sum_{i=1}^{3}\sum_{\mu}\bar{\psi}\gamma_5\gamma_\mu\tau_i\psi\frac{\partial\phi_i}{\partial x_\mu} \qquad (ps-pv) \qquad (2.12b)$$

Starting from the Lagrangians (2.6), (2.11) and (2.12), and following the usual prescription, the Hamiltonian is given by $H+H'$ where

$$H = \bar{\psi}\left\{\sum_{\mu}\gamma_\mu\frac{\partial}{\partial x_\mu}+m\right\}\psi+\frac{1}{2}\sum_{i}\left\{\pi_i^2+(\mathrm{grad}\,\phi_i)^2+\mu^2\phi_i^2\right\} \qquad (2.13a)$$

$$H' = g\sum_{i}\bar{\psi}\gamma_5\tau_i\psi\phi_i \qquad (ps-ps) \qquad (2.13b)$$

or

$$H' = \frac{f}{\mu}\sum_{i}\sum_{\mu}\bar{\psi}\gamma_5\gamma_\mu\tau_i\psi\frac{\partial\phi_i}{\partial x_\mu}-\frac{f}{\mu}\sum_{i}\bar{\psi}\gamma_5\gamma_0\tau_i\psi\pi_i - \frac{1}{2}\sum_{i}\left(\frac{f}{\mu}\bar{\psi}\gamma_5\gamma_0\tau_i\psi\right)^2 \qquad (2.13c)$$

π_i being the momentum canonically conjugate to ϕ_i

On the basis of this Hamiltonian, several papers were published [2,3,4] on the properties of nuclear systems due to pion exchange between the nucleons, in which use was made of perturbation expansions in the coupling constant, as well as applying the static approximation in which nucleon recoil was neglected. The resulting potential has a strong singularity at the origin, which is aggravated by including higher order terms in the expansion. Since the Schrodinger equation is insoluble for such a potential, the interaction at small distances was replaced by a phenomenological repulsive hard core [2,3] which sufficed to fit most of the low energy scattering data, treating the depths and widths of the repulsive core in the singlet and triplet states as adjustable parameters.

Following on this, the success of the Chew-Low cut-off theory in explaining pion-nucleon scattering and photo-pion production led S. Gartenhaus [5] to derive the corresponding static two-nucleon potential. Using the non-relativistic p-wave extended source Hamiltonian with cut-off,

$$H = (4\pi)^{\frac{1}{2}} \frac{f}{\mu} \sum_{N=1}^{2} \sum_{i} \int d\underline{x}\; \rho(\underline{x} - \underline{x}_N)\, \tau_i^{N}\, \underline{\sigma}^{N} \cdot \underline{\nabla}\, \phi_i(\underline{x}) \qquad (2.14)$$

the second and fourth order terms were calculated using non-relativistic perturbation theory. The resulting potentials give a good fit to all the low energy data, but not to the data at high energies.

In the static limit, any potential obtained is necessarily velocity independent. To improve the high energy results, P.S. Signell and R.E. Marshak [6] added to the Gartenhaus potential a velocity dependent potential of the spin-orbit type

$$V(r) = \left[f_1(r) + P_H \, f_2(r) \right] \underline{L} \cdot \underline{S} \qquad (2.15)$$

where $f_1(r)$, $f_2(r)$ are arbitrary functions, and P_H is the Heisenberg exchange operator. Since the spin-orbit potential vanishes in S-states, the predictions of the Gartenhaus potential are essentially unchanged if the spin-orbit force is chosen to be of sufficiently short range to be masked by the centrifugal barrier for the deuteron D-state. The resulting potential gives a good fit with experiment up to 200 MeV laboratory scattering energy, and qualitative agreement up to 310 MeV.

At the same time J. Gammel and R. Thaler [7] made an extensive computing-machine search for a phenomenological potential, starting from the phase-shift analysis of H.P. Stapp et al.[8]. They looked for Yukawa shaped potentials consisting of central, tensor and spin-orbit terms. The phenomenological potentials so obtained are very similar in form to the Signell-Marshak potentials, differing mainly in the shape and depth of the central cores. The agreement with experimental data is similar to that of Signell and Marshak.

Recently Bryan [9] and Hammada [10] have succeeded in obtaining reasonable agreement with the data up to 310 MeV with models having the one-pion exchange potential tail and a spin-orbit term whose range is compatible with meson theory. This, however, is at the expense of considerable complication in the inner regions of the potential, and Hammada also includes a quadratic $\underset{\sim}{L}.\underset{\sim}{S}$ interaction in the singlet state.

It is thus generally accepted that the nucleon-nucleon interaction is given asymptotically correctly by the one-pion exchange contribution, but the inner regions of the potential, which arise from multi-pion intermediate states and from intermediate states with particles more massive than the pion, must as yet be obtained purely phenomenologically.

3. The Scattering State.

In this section, we follow the argument given initially by J.M. Blatt and L.C. Biedenharn[11].

In the centre of momentum system, the two-nucleon Schrodinger equation reduces to

$$\left\{ -\frac{1}{m} \underset{\sim}{\nabla}^2 + V(\underset{\sim}{r}) \right\} \psi(\underset{\sim}{r}) = E\,\psi(\underset{\sim}{r}) \tag{3.1}$$

In the scattering problem, a solution of (3.1) is required which has the asymptotic form

$$\psi(\underset{\sim}{r}) \xrightarrow[r \to \infty]{} \left\{ e^{ipz} + f(\theta,\phi)\,\frac{e^{ipr}}{r} \right\} \tag{3.2}$$

where $p^2 = mE$

The orbital angular momentum eigenfunctions are the normalized spherical harmonics, which we denote by

$$Y_{L,M}(\theta,\phi) = \sqrt{\frac{(2L+1)(L-M)!}{4\pi\,(L+M)!}} \cdot \frac{(-1)^L}{2^L\,L!} \sin^M\theta \, \frac{d^{L+M}}{(d\cos\theta)^{L+M}} \sin^{2L}\theta \; e^{iM\phi} \tag{3.3}$$

Then denoting the spin eigenfunctions by χ_{S_z}, the eigenfunctions Φ_{JJ_zL} pertaining to eigenvalues J and J_z and quantum number L , are given by

$$\Phi_{JJ_zL} = \sum_{L_z = J_z - 1}^{J_z + 1} C_{JJ_zLL_z}\, Y_{LL_z}(\theta,\phi)\, \chi_{J_z - L_z} \tag{3.4}$$

where $L = J+1$, J or $J-1$ and the numerical coefficients $C_{JJ_zLL_z}$ are the usual Clebsch-Gordon coefficients.

The eigenfunctions $\psi(x)$ belonging to the eigenvalues J, J_z and parity $(-1)^L$ can be written in terms of the Φ_{JJ_zL} as

$$\psi(x) = \sum_{J=0}^{\infty} \sum_{L=J-1}^{J+1} \frac{1}{r} u_{JL}(r) \, \Phi_{JJ_zL}(\theta, \phi, \text{spin}) \tag{3.5}$$

The incident plane-wave e^{ipz} may be similarly expanded as

$$e^{ipz} \chi_{s_z} = \sum_{L=0}^{\infty} \sqrt{4\pi(2L+1)} \; i^L j_L(pr) \, Y_{L0}(\theta,\phi) \; \chi_{s_z} \tag{3.6}$$

where a spin function has been inserted, and $j_L(x)$ is the spherical Bessel function

$$j_L(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} J_{L+\frac{1}{2}}(x) \tag{3.7}$$

The eigenvalue J_z in (3.5) must equal S_z in (3.6) since it is a constant of the motion.

It is convenient to expand (3.6) analogously to (3.5), which may be done by noting that

$$Y_{L0}(\theta,\phi) = \sum_{J=L-1}^{L+1} C_{JS_zL0} \, \Phi_{JS_zL} \tag{3.8}$$

whence

$$e^{ipz}\chi_{sz} = \sum_{J=0}^{\infty}\sum_{L=J-1}^{J+1}\sqrt{4\pi(2L+1)}\; i^L j^L(pr)\, C_{JS_zL0}\,\Phi_{JS_zL}\,\chi_{sz} \quad (3.9)$$

which behaves asymptotically as

$$e^{ipz}\chi_{sz} \xrightarrow[r\to\infty]{} \sum_{J=0}^{\infty}\sum_{L=J-1}^{J+1}\sqrt{4\pi(2L+1)}\; i^L \frac{\sin(pr-\frac{1}{2}L\pi)}{p}\, C_{JS_zL0}\,\Phi_{JS_zL}\,\chi_{sz} \quad (3.10)$$

The case with L = J is the simplest to treat, since this state is itself an eigenstate. (The singlet scattering state is similar to this case). For convenience, denote the three radial wave-functions by $u_J(r)$, $v_J(r)$, $w_J(r)$ for L = J-1, J, J+1 respectively. Asymptotically $v_J(r)$ behaves as a force-free solution, and its most general asymptotic form is given by a linear combination of an incoming and an outgoing wave

$$v_J(r) \xrightarrow[r\to\infty]{} A e^{-i(pr-\frac{1}{2}J\pi)} - B e^{i(pr-\frac{1}{2}J\pi)} \quad (3.11)$$

The relative value of the outgoing amplitude B to the incoming amplitude A is given in terms of the scattering matrix S by

$$B = S\,A \quad (3.12)$$

Since in pure elastic scattering, the flux of the outgoing wave must equal that of the incoming wave, $|S|^2 = 1$

and hence S can be written in the form

$$S = e^{2i\delta_{J,J}} \tag{3.13}$$

where the real quantity $\delta_{J,J}$ is the phase-shift for the partial wave J = L. By substituting (3.13) and (3.12) into (3.10) one gets

$$u_J(r) \xrightarrow[r\to\infty]{} -2i\, A\, e^{i\delta_{J,J}} \sin\left(kr - \tfrac{1}{2}J\pi + \delta_{J,J}\right) \tag{3.14}$$

Because of the tensor force, the two cases L = J-1, L = J+1 are mixed and correspondingly there are two radial wave-functions. Asymptotically, each radial wave-function is a linear superposition of an incoming and an outgoing wave i.e.

$$\begin{aligned} u_J(r) &\longrightarrow A_1\, e^{-i(kr - \frac{1}{2}(J-1)\pi)} - B_1\, e^{i(kr - \frac{1}{2}(J-1)\pi)} \\ w_J(r) &\longrightarrow A_2\, e^{-i(kr - \frac{1}{2}(J+1)\pi)} - B_2\, e^{i(kr - \frac{1}{2}(J+1)\pi)} \end{aligned} \tag{3.15}$$

The scattering matrix is now a (2x2) matrix such that

$$B = SA \tag{3.16a}$$

with

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \quad A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{3.16b}$$

According to general theorems, the S-matrix must be unitary $(S^{\dagger}S = 1)$ and symmetric $(S^{T} = S)$. It can be shown that the most general (2x2) matrix satisfying these conditions contains three independent parameters, and is of the form

$$S = U^{-1} e^{2i\Delta} U \tag{3.17}$$

where U is an orthogonal matrix depending on only one real parameter ε_J

$$U = \begin{pmatrix} \cos\varepsilon_J & \sin\varepsilon_J \\ -\sin\varepsilon_J & \cos\varepsilon_J \end{pmatrix} \tag{3.18}$$

and Δ is a diagonal matrix with real elements

$$\Delta = \begin{pmatrix} \delta_{J,\alpha} & 0 \\ 0 & \delta_{J,\beta} \end{pmatrix} \tag{3.19}$$

From (3.18) and (3.19) the two eigenstates of $A^{(\alpha)}$ and $A^{(\gamma)}$ of S corresponding to eigenvalues $e^{2i\delta_{J,\alpha}}, e^{2i\delta_{J,\gamma}}$ are obtained as

$$\frac{A_2^{(\alpha)}}{A_1^{(\alpha)}} = \tan\varepsilon_J \quad , \quad \frac{A_2^{(\gamma)}}{A_1^{(\gamma)}} = -\cot\varepsilon_J \tag{3.20}$$

and the outgoing amplitudes are given respectively by

$$B^{(\alpha)} = e^{2i\delta_{J,\alpha}} A^{(\alpha)} , \quad B^{(\gamma)} = e^{2i\delta_{J,\gamma}} A^{(\gamma)} \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.15) we get

$$\begin{aligned} u_{J,\alpha}(r) &\xrightarrow[r\to\infty]{} -2i A_\alpha e^{i\delta_{J,\alpha}} \sin\left(kr - \tfrac{1}{2}(J-1)\pi + \delta_{J,\alpha}\right) \\ w_{J,\alpha}(r) &\xrightarrow[r\to-\infty]{} -2i A_\alpha \tan\varepsilon_J\, e^{i\delta_{J,\alpha}} \sin\left(kr - \tfrac{1}{2}(J+1)\pi + \delta_{J,\alpha}\right) \end{aligned} \tag{3.22a}$$

and

$$u_{J,\gamma}(r) \longrightarrow 2i\,A_\gamma \tan\epsilon_J\, e^{i\delta_{J,\gamma}} \sin\left(kr - \tfrac{1}{2}(J-1)\pi + \delta_{J,\gamma}\right) \tag{3.22b}$$
$$w_{J,\gamma}(r) \longrightarrow -2i\,A_\gamma\, e^{i\delta_{J,\gamma}} \sin\left(kr - \tfrac{1}{2}(J+1)\pi + \delta_{J,\gamma}\right)$$

We can thus construct the wave-functions

$$\psi_{J,S_z,\gamma} = \frac{1}{r} u_{J,S_z,\alpha}(r)\, \Phi_{J,S_z,J-1} + \frac{1}{r} w_{J,S_z,\alpha}(r)\, \Phi_{J,S_z,J+1} \tag{3.23}$$
$$\psi_{J,S_z,\gamma} = \frac{1}{r} u_{J,S_z,\gamma}(r)\, \Phi_{J,S_z,J-1} + \frac{1}{r} w_{J,S_z,\gamma}(r)\, \Phi_{J,S_z,J+1}$$

and correspondingly

$$\psi_{J,S_z,\beta} = \frac{1}{r} v_{J,S_z}(r)\, \Phi_{J,S_z,J} \tag{3.24}$$

The phase shifts δ_{LJ} and the mixing parameter ϵ_J are uniquely determined by the requirement that the radial wave-functions vanish with the origin.

In view of (3.23) and (3.24) we can write $\psi(x)$ as

$$\psi(x) = \sum_{J=0}^{\infty} \sum_{\alpha,\beta,\gamma} \psi_{J,S_z,\alpha} \tag{3.25}$$

We must finally remove the ambiguity that there are two eigenstates of the scattering matrix, but so far no prescription has been given for calling one of them an 'α' state and the other a 'β' state. In the limit of the collision energy going to zero, the difference in the centrifugal barrier effects for L = J-1 and L = J+1 is so large that these states become eigenstates i.e. ε_J tends to 0 or $\frac{\pi}{2}$. We define the assignments 'α' and 'β' so that in the limit the α-wave corresponds to the state L = J-1 and the β-wave to the state L = J+1 i.e. we require

$$\lim_{E \to 0} \varepsilon_J = 0 \qquad \text{(all J)} \tag{3.26}$$

Explicit expressions for the amplitudes A may be obtained by requiring the asymptotic form of (3.25) to be of the form of (3.1), by substituting (3.23) and (3.24) for (3.25) in (3.1) and using (3.9) together with the identity

$$e^{i\delta} \sin(kr - \tfrac{1}{2} m\pi + \delta) = \sin(kr - \tfrac{1}{2} m\pi) + (-i)^m e^{i\delta} \sin\delta \, e^{ikr} \tag{3.27}$$

The resulting expressions for $f_{s_z}(\theta, \phi)$ and the differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{1}{3} \sum_{s_z = -1}^{1} \sum_{spin} |f_{s_z}(\theta, \phi)|^2 \tag{3.28}$$

are complicated (12,13,14), and we do not give them here.

4. Polarization in Nucleon-Nucleon Scattering.

The theory of polarization and triple scattering has been given by L. Wolfenstein and J. Ashkin [15] and by L. Wolfenstein [16].

The scattering is described by a matrix M_{ij} in spin space, defined by

$$f_i(\theta, \phi) = \sum_j M_{ij} a_j \tag{4.1}$$

where the a_j are the amplitudes of the various spin-states in the incident plane wave, and the $f_i(\theta, \phi)$ are the scattering amplitudes for these states. The M_{ij} are functions of the phase-shifts and coupling parameters [8]. The polarization and triple scattering data are given in terms of four parameters P, D, R and A, which may be related directly to the M_{ij} and hence to the phase-shifts and coupling parameters [8].

To describe the geometry of a multiple scattering experiment, define for each scattering a unit vector

$$\underset{\sim}{n} = \frac{\underset{\sim}{k} \times \underset{\sim}{k}'}{|\underset{\sim}{k} \times \underset{\sim}{k}'|} \tag{4.2}$$

where $\underset{\sim}{k}$, $\underset{\sim}{k}'$ are unit vectors in the incident and outgoing directions respectively.

In the first scattering, an unpolarized beam hits an unpolarized target. The polarization of the scattered nucleon beam is given by

$$I_o \langle \sigma \rangle_1 = \frac{1}{4} Tr \left[M M^{+} \sigma \right] = P n_1 \qquad (4.3)$$

where I_o is the differential scattering cross-section for an unpolarized beam.

In the second scattering, we are interested in the differential scattering cross-section I for a nucleon beam with polarization $P_1 n_1$ incident on an unpolarized target. In this case it can be shown that

$$I_2 = I_{o_2} + P_1 I_{p_2} \qquad (4.4)$$

where I_{o_2} is the differential cross-section for an unpolarized beam and $P_1 I_{p_2}$ is the contribution to the cross-section of the initial polarization. In general, it can be shown that

$$I_{p_2} = I_{o_2} P_2 n_2 \cdot n_1 \qquad (4.5)$$

whence

$$I_2 = I_{o_2} (1 + P_1 P_2 \cos\phi) \qquad (4.6)$$

If double scattering only is being considered, a left-right asymmetry is measured relative to n_2 . This asymmetry is defined by

$$e = \frac{I_2(+) - I_2(-)}{I_2(+) + I_2(-)} \quad (4.7)$$

where $I_2(\pm)$ refers to scattering such that $\underset{\sim}{n}_2$ is parallel to $\pm \underset{\sim}{n}_1$. From equation (4.6) we have immediately

$$e = P_1 P_2 \quad (4.8)$$

In the third scattering, a left-right asymmetry is measured relative to the direction $\underset{\sim}{n}_3$ in this case, two directions suffice to specify the polarization. It is usual to consider the two cases when $\underset{\sim}{n}_3$ is parallel to $\underset{\sim}{n}_2$ and when $\underset{\sim}{n}_3$ is parallel to $\underset{\sim}{s}$, where

$$\underset{\sim}{s} = (\underset{\sim}{n}_2 \times \underset{\sim}{k}_2') \quad (4.9)$$

Then it can be shown that

$$I_3 \langle \underset{\sim}{\sigma} \rangle_3 \cdot \underset{\sim}{n}_2 = I_0 [P_3 + D \langle \underset{\sim}{\sigma} \rangle_2 \cdot \underset{\sim}{n}_2] \quad (4.10a)$$

$$I_3 \langle \underset{\sim}{\sigma} \rangle_3 \cdot \underset{\sim}{s} = I_0 [A \langle \underset{\sim}{\sigma} \rangle \cdot \underset{\sim}{k}_2 + R \langle \underset{\sim}{\sigma} \rangle_2 \cdot (\underset{\sim}{n}_2 \times \underset{\sim}{k}_2)] \quad (4.10b)$$

Here P, D, A, R are arbitrary functions of $\underset{\sim}{k}_2, \underset{\sim}{k}_2'$ i.e. of the scattering angle θ

Defining the asymmetries in triple scattering by

$$e_3 = \frac{I_3(+) - I_3(-)}{I_3(+) + I_3(-)} \quad (4.11)$$

we have

$$e_{3n} = \frac{P_3 [P_2 + D P_1 \cos\phi]}{1 + P_1 P_2 \cos\phi}, \quad e_{3s} = \frac{P_3 P_1 R \sin\phi}{1 + P_1 P_2 \cos\phi} \quad \begin{matrix}(4.12a)\\(4.12b)\end{matrix}$$

5. The Bound State.

In the centre of momentum system, the Schrodinger equation is

$$\left[-\frac{1}{m}\nabla^2 + V(\underset{\sim}{r})\right]\psi_D(\underset{\sim}{r}) = \varepsilon\,\psi_D(\underset{\sim}{r}) \tag{5.1}$$

where ε is the binding energy of the deuteron. From the considerations of section 2, it is sufficient to consider the potential in the bound state to be of the form

$$V(\underset{\sim}{r}) = V_c(r) + S_{12}\,V_T(\underset{\sim}{r}) \tag{5.2}$$

where $V_c(r)$ and $V_T(r)$ are respectively the central and tensor forces.

Analogously to equation (3.5), the wave-function may be expanded as

$$\psi_D(\underset{\sim}{r}) = \sum \frac{1}{r}\,u_L(r)\,\Phi_{J\,J_z\,L}(\theta,\varphi,\text{spin}) \tag{5.3}$$

where the radial wave-functions $u_L(r)$ satisfy the coupled equations

$$\frac{d^2u_L(r)}{dr^2} - \left[\alpha^2 + \frac{L(L+1)}{r^2} - v_c(r)\right]u_L(r) + v_T(r)\sum_{L'} S_{JLL'}\,u_{L'}(r) = 0 \tag{5.4}$$

where $\alpha^2 = -m\varepsilon$, $v_i(r) = -V_i(r)$ and

$$S_{JLL'} = \int \langle \Phi_{JJ_zL},\, S\, \Phi_{JJ_zL'}\rangle\, d\Omega \tag{5.5}$$

Since the deuteron ground state is known from experimental evidence to be a $({}^3S_1 + {}^3D_1)$ state [12] we can write

$$\psi_D(\underset{\sim}{r}) = \frac{1}{r} u(r)\, \Phi_{1 S_z 0} + \frac{1}{r} w(r)\, \Phi_{1 S_z 2} \tag{5.6}$$

Retaining only the $^3S_1$ and $^3D_1$ states, the coupled equations (5.4) reduce to

$$\begin{aligned} &\frac{d^2u(r)}{dr^2} - \{\alpha^2 - v_c(r)\}u(r) + 2\sqrt{2}\,v_T(r)\,w(r) = 0 \\ &\frac{d^2w(r)}{dr^2} - \left[\alpha^2 + \frac{6}{r^2} - v_c(r) + 2v_T(r)\right]w(r) + 2\sqrt{2}\,v_T(r)\,u(r) = 0 \end{aligned} \tag{5.7}$$

Now note that

$$S_{12}\; \Phi_{1 S_z 0} = \sqrt{8}\; \Phi_{1 S_z 2} \tag{5.8}$$

which may be obtained readily by noting that, since S_{12} conserves parity and total angular momentum, S_{12} operating on $\Phi_{1 S_z 0}$ can lead only to a linear combination of $\Phi_{1 S_z 0}$ and $\Phi_{1 S_z 2}$ i.e.

$$S_{12}\, \Phi_{1 S_z 0} = a\, \Phi_{1 S_z 0} + b\, \Phi_{1 S_z 2} \tag{5.9}$$

Since S_{12} vanishes when averaged over the direction of $\underset{\sim}{r}$, and since in (5.9) S_{12} acts on a function which is independent of $\underset{\sim}{r}$ (L = 0), the resultant cannot be a spherically symmetric state. Hence a = 0. b can now be evaluated by taking a special case, say $S_z = 1$ and $\underset{\sim}{r}$ in the z- direction. The calculation is straightforward and gives (5.8) immediately.

Thus we can write (5.6) as

$$\psi_{D}(x) = \left[\frac{1}{r} u(r) + \frac{1}{\sqrt{8}} S_{12} \frac{1}{r} w(r) \right] \Phi_{1S10} \qquad (5.10)$$

$$= \frac{1}{\sqrt{4\pi}} \left[\frac{u(r)}{r} + \frac{S_{12}}{\sqrt{8}} \frac{w(r)}{r} \right] \times \text{the appropriate spin function.}$$

Equation (5.10) is the most useful form of the deuteron wave-function for practical purposes.

The normalization of $u(r)$ and $w(r)$ is such that

$$\int \left[u^2(r) + w^2(r) \right] dr = 1 \qquad (5.11)$$

From equation (5.7) one can deduce that $u(r)$ and $w(r)$ must have the asymptotic forms

$$u(r) \propto e^{-\alpha r}$$

$$w(r) \propto e^{-\alpha r} \left(1 + \frac{3}{\alpha r} + \frac{3}{(\alpha r)^2} \right) \qquad (5.12)$$

If we introduce the coupling constant ε_g, we may write

$$u(r) = N u_g(r) , \quad w(r) = N w_g(r) \qquad (5.13)$$

where $u_g(r)$ and $w_g(r)$ have the asymptotic forms

$$u_g(r) \xrightarrow[r \to \infty]{} \cos \varepsilon_g \, e^{-\alpha r}$$

$$w_g(r) \xrightarrow[r \to \infty]{} \sin \varepsilon_g \, e^{-\alpha r} \left[1 + \frac{3}{\alpha r} + \frac{3}{(\alpha r)^2} \right] \qquad (5.14)$$

The exact form of the functions $u_g(r)$, $w_g(r)$ is, of course, dependent on the potential chosen to act in the bound state. However since the deuteron is a loosely bound system, reasonable phenomenological deuteron wave-functions may be constructed by assuming suitable functional forms containing several parameters and adjusting these to fit the existing empirical information on the neutron-proton system. This is discussed for a particular functional form in Appendix 4

6. The Deuteron Magnetic Moment and Electric Quadrupole Moment.

Neglecting relativistic corrections, the magnetic dipole moment of the deuteron is the expectation value of the operator

$$\underset{\sim}{\gamma} = \sum_{i=1}^{2} \left\{ \frac{1+\tau_3^{(i)}}{2} \gamma_p \underset{\sim}{\sigma}^{(i)} + \frac{1-\tau_3^{(i)}}{2} \gamma_n \underset{\sim}{\sigma}^{(i)} + \frac{1+\tau_3^{(i)}}{2} \underset{\sim}{r}_i \times \underset{\sim}{p} \right\} \qquad (6.1)$$

where γ_p and γ_n are respectively the proton and neutron magnetic moments in units of nuclear Bohr magnetons. Equation (6.1) may be written as

$$\begin{aligned} \underset{\sim}{\gamma} = & (\gamma_p+\gamma_n) \frac{\sigma^{(1)}+\sigma^{(2)}}{2} + (\gamma_p-\gamma_n) \frac{\tau_3^{(1)}+\tau_3^{(2)}}{2} \cdot \frac{\sigma^{(1)}+\sigma^{(2)}}{2} \\ & + (\gamma_p-\gamma_n) \frac{\tau_3^{(1)}-\tau_3^{(2)}}{2} \cdot \frac{\sigma^{(1)}-\sigma^{(2)}}{2} \\ & + \left\{ \frac{m_n}{m_n+m_p} + \frac{1}{4}(\tau_3^{(1)}+\tau_3^{(2)}) + \frac{1}{4}(1+\tau_3^{(1)}\tau_3^{(2)}) \frac{m_p-m_n}{m_p+m_n} \right\} \underset{\sim}{r} \times \underset{\sim}{p} \end{aligned} \qquad (6.2)$$

where we have introduced the relative co-ordinates $\underset{\sim}{r}, \underset{\sim}{p}$ by

$$\begin{aligned} & \underset{\sim}{r}_1 = \frac{m_2 \underset{\sim}{r}}{m_1+m_2}, \quad \underset{\sim}{r}_2 = -\frac{m_1 \underset{\sim}{r}}{m_1+m_2}, \quad \underset{\sim}{p}_1 = \underset{\sim}{p} = -\underset{\sim}{p}_2 \\ & m_i = \tfrac{1}{2}(m_n+m_p) + \tfrac{1}{2}\tau_3^{(i)}(m_p-m_n) \end{aligned} \qquad (6.3)$$

The terms containing the factors $\frac{1}{2}(\tau_3^{(1)}+\tau_3^{(2)})$ and $\frac{1}{2}(\sigma^{(1)}-\sigma^{(2)})$ vanish identically when the expectation value of (6.2) is taken, since the deuteron ground state is an isotopic spin singlet and a spin triplet. Introducing the orbital angular momentum operator $\underset{\sim}{L} = (\underset{\sim}{r} \times \underset{\sim}{p})$ and the total angular momentum $\underset{\sim}{J} = \underset{\sim}{L} + \frac{1}{2}(\sigma^{(1)}+\sigma^{(2)})$ we can write

$$\underset{\sim}{\gamma} = (\gamma_p+\gamma_n)\underset{\sim}{J} - (\gamma_p+\gamma_n-\tfrac{1}{2})\underset{\sim}{L} \qquad (6.4)$$

where the neutron-proton mass difference has been neglected.

On taking the expectation value of (6.4), in a state of a given I_z the only non-vanishing component of $\underline{\gamma}$ is γ_z. Then the expectation value of γ_z in the substate belonging to the quantum number I_z is

$$\left\{ \gamma_p + \gamma_n - \tfrac{3}{2}(\gamma_p + \gamma_n - \tfrac{1}{2}) \int_0^\infty w^2(r)\, dr \right\} I_z \qquad (6.5)$$

The coefficient of I_z in (6.5) is usually called the deuteron magnetic moment. The integral occurring in (6.5) is simply the D-state probability, which we denote by P_D Thus

$$\gamma_D = (\gamma_p + \gamma_n) - \tfrac{3}{2}(\gamma_p + \gamma_n - \tfrac{1}{2}) P_D \qquad (6.6)$$

At first sight, it would appear from equation (6.6) that the measurement of γ_D would allow the unique determination of P_D. This, however, is not so, since γ_D differs only very slightly from $\gamma_p + \gamma_n$ and one can only conclude that P_D lies in the range $0.039 < P_D < 0.07$ [12].

The electric quadrupole moment of the deuteron is given by the expectation value of the operator

$$Q_{ik} = \int (3x_i x_k - \delta_{ik} r^2)\, \rho(\underline{r})\, d\underline{r} \qquad (6.7)$$

where $\rho(\underline{r})$ is the charge density of the deuteron. Putting $\rho(\underline{r}) = e\, \psi_D^* \psi_D$ the expectation value of (6.7) is

$$\langle II_z | Q_{ik} | II_z' \rangle = e \int \psi^*_{II_z} (3x_i x_k - \delta_{ik} r^2)\, \psi_{II_z'}\, d\underline{r} \qquad (6.8)$$

where ψ_{JJ_z} is an eigenfunction belonging to the quantum numbers

By a group theoretical argument (17) it can be shown that

$$\frac{1}{e}\langle JJ_z|Q_{ik}|JJ_z'\rangle = C\langle JJ_z|3\frac{J_iJ_k+J_kJ_i}{2} - \delta_{ik}\underline{J}^2|JJ_z'\rangle \qquad (6.9)$$

where C is a constant, which may be determined by taking a special matrix element between two top states e.g.

$$\frac{1}{e}\langle JJ|Q_{33}|JJ\rangle = C\langle JJ|3J_z^2 - \underline{J}^2|JJ\rangle = CJ(2J-1) \qquad (6.10)$$

Thus

$$\langle JJ_z|Q_{ik}|JJ_z'\rangle = \frac{eQ}{J(2J-1)}\langle JJ_z|3\frac{J_iJ_k+J_kJ_i}{2} - \delta_{ik}\underline{J}^2|JJ_z'\rangle \qquad (6.11)$$

where we have defined

$$Q = \frac{1}{e}\langle JJ|Q_{33}|JJ\rangle = \int \psi_{JJ}^*(3z^2-r^2)\psi_{JJ}\,d\underline{r} \qquad (6.12)$$

Q is conventionally called the deuteron quadrupole moment. Substituting equation (5.10) for ψ_D in equation (6.12) it is easily shown that

$$Q = \frac{\sqrt{2}}{10}\int_0^\infty r^2\left(uw - \frac{1}{2\sqrt{2}}w^2\right)dr \qquad (6.13)$$

7. Scattering Length and Effective Range.

At very low scattering energies, all the mixing parameters and phase-shifts may be neglected except since this is the only one which is related to an S-state. Then the total triplet cross-section is given by

$$\sigma_T = \frac{4\pi}{p^2} \sin^2 \delta_{1,\alpha} \qquad (7.1)$$

and the differential cross-section by

$$\frac{d\sigma}{d\Omega} = \frac{1}{p^2} \sin^2 \delta_{1,\alpha} \left[1 - \sin^2 \varepsilon_1 \left(2\cos\varepsilon_1 + \frac{1}{\sqrt{2}} \sin\varepsilon_1 \right)^2 P_2 \cos\theta \right] \qquad (7.2)$$

Experimentally, the differential cross-section is found to be very nearly isotropic, which shows ε_1 to be very small.

The usual way to analyse the low energy data is to employ the approximation

$$p \cot \delta = -\frac{1}{a} + \frac{1}{2} r_0 p^2 \qquad (7.3)$$

where the two constants a and r_0 are called the scattering length and effective range respectively. The theory was first developed by J. Schwinger and later by J.M. Blatt and J.D. Jackson [18], by H.A. Bethe [19] and by L.C. Biedenharn and J.M. Blatt [20].

Denote the 'α' wave-functions at energies E_1 and E_2 by $(u_{\alpha_1}, w_{\alpha_1})$ and $(u_{\alpha_2}, w_{\alpha_2})$ respectively. Then it can be shown as a direct consequence of the Schrodinger equations satisfied by these wave-functions that

$$\frac{d}{dr}\left[u_{\alpha_2}\frac{du_{\alpha_1}}{dr} + w_{\alpha_2}\frac{dw_{\alpha_1}}{dr} - u_{\alpha_1}\frac{du_{\alpha_2}}{dr} - w_{\alpha_1}\frac{dw_{\alpha_2}}{dr}\right] = (k_2^2 - k_1^2)\left[u_{\alpha_1}u_{\alpha_2} + w_{\alpha_1}w_{\alpha_2}\right] \quad (7.4)$$

Now introduce the force-free solutions u_α^0, w_α^0 with normalization

$$\begin{aligned} u_\alpha \xrightarrow[r\to\infty]{} u_\alpha^0 &\longrightarrow \cos\varepsilon \sin(kr+\delta_\alpha)/\sin\delta_\alpha \\ w_\alpha \xrightarrow[r\to\infty]{} w_\alpha^0 &\longrightarrow \sin\varepsilon \sin(kr+\delta_\alpha-\pi)/\sin\delta_\alpha \end{aligned} \quad (7.5)$$

where δ_α and ε are the α-wave phase-shift and the mixing parameter belonging to J = 1, respectively. The complete expressions for u_α^0 and w_α^0 are

$$\begin{aligned} u_\alpha^0 &= \cos\varepsilon \; kr\left[\cot\delta_\alpha \, j_0(kr) - n_0(kr)\right] \\ w_\alpha^0 &= \sin\varepsilon \; kr\left[\cot\delta_\alpha \, j_2(kr) - n_2(kr)\right] \end{aligned} \quad (7.6)$$

where $j_\ell(x)$ is the spherical Bessel function

$$j_\ell(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} J_{\ell+\frac{1}{2}}(x) \quad (7.7a)$$

and is regular at x = 0, and $n_\ell(x)$ is the spherical Neumann function

$$n_\ell(x) = (-1)^{\ell+1}\left(\frac{\pi}{2x}\right)^{\frac{1}{2}} J_{-\ell-\frac{1}{2}}(x) \quad (7.7b)$$

and is irregular at x = 0.

Because of the irregularity of $n_2(x)$ at the origin, w_α^0 as defined by (7.6) diverges at $r=0$, and the free solutions are thus inconvenient for the present purpose. Therefore we define modified asymptotic functions $\bar{u}_\alpha, \bar{w}_\alpha$ which are finite at $r=0$ and approach u_α^0, w_α^0 asymptotically.

$$\begin{aligned} \bar{u}_\alpha &= u_\alpha^0 \\ \bar{w}_\alpha &= w_\alpha^0 - \frac{3\sin\varepsilon}{(pr)^2} \end{aligned} \qquad (7.8)$$

which can be shown to satisfy the differential equations

$$\begin{aligned} \left[\frac{d^2}{dr^2} + p^2\right] \bar{u}_\alpha &= 0 \\ \left[\frac{d^2}{dr^2} - \frac{6}{r^2} + p^2\right] \bar{w}_\alpha &= -3\sin\varepsilon\,\frac{1}{r^2} \end{aligned} \qquad (7.9)$$

Then utilizing (7.9) we can show that

$$\frac{d}{dr}\left[\bar{u}_{\alpha 2}\frac{d\bar{u}_{\alpha 1}}{dr} + \bar{w}_{\alpha 2}\frac{d\bar{w}_{\alpha 1}}{dr} - \bar{u}_{\alpha 1}\frac{d\bar{u}_{\alpha 2}}{dr} - \bar{w}_{\alpha 1}\frac{d\bar{w}_{\alpha 2}}{dr}\right]$$
$$= (p_2^2 - p_1^2)\left[\bar{u}_{\alpha 1}\bar{u}_{\alpha 2} + \bar{w}_{\alpha 1}\bar{w}_{\alpha 2}\right] - \frac{3}{r^2}\left[\sin\varepsilon_1\,\bar{w}_{\alpha 2} - \sin\varepsilon_2\,\bar{w}_{\alpha 1}\right] \qquad (7.10)$$

where ε_1 and ε_2 are the mixing parameters at energies E_1 and E_2 respectively.

Integrating (7.4) and (7.11) and subtracting, we obtain after some trivial algebra

$$\cos(\varepsilon_2-\varepsilon_1)\left[p_1\cot\delta_{\alpha 1} - p_2\cot\delta_{\alpha 2}\right] = (p_1^2-p_2^2)\int_0^\infty\left[\bar{u}_{\alpha 1}\bar{u}_{\alpha 2} + \bar{w}_{\alpha 1}\bar{w}_{\alpha 2} - u_{\alpha 1}u_{\alpha 2} - w_{\alpha 1}w_{\alpha 2}\right]dr \qquad (7.11)$$

Remembering that $\varepsilon \longrightarrow 0$ as the energy tends to zero, and defining the triplet scattering length a_t by

$$\lim_{p^2 \to 0} [p \cot \delta_2] = -\frac{1}{a_t} \tag{7.12}$$

we get from (7.11) on letting $\varepsilon_2 \longrightarrow 0$

$$p \cot \delta_2 = -\frac{1}{a_t} + \frac{p^2}{\cos \varepsilon} \int_0^\infty [\bar{u}_2 \bar{u}_{20} + \bar{w}_2 \bar{w}_{20} - u_2 u_{20} - w_2 w_{20}] \tag{7.13}$$

where the subscript '1' has been dropped and '0' inserted to denote the zero-energy wave-functions.

Now note that $\bar{u}, \bar{w}$ differ from u, w only inside the nuclear force range. Thus the main contribution to the integral in (7.13) comes from the inside region where, for low energies, the potential energy is numerically much larger than the kinetic energy. Thus we can assume the integral to be energy independent and obtain the approximation

$$p \cot \delta_2 = -\frac{1}{a_t} + \frac{1}{2} r_{0t} p^2 \tag{7.14}$$

$$r_{0t} = 2 \int_0^\infty [\bar{u}_{20}^2 - u_{20}^2 - w_{20}^2] \tag{7.15}$$

noting that, by definition, $\bar{w}_{20} = 0$

In order to derive a relationship between the scattering length and effective range as defined above, and the deuteron parameters, we start with the coupled equations (5.7) satisfied by the deuteron wave-functions (5.14). Analogously to (7.8) the asymptotic functions $\bar{u}_g, \bar{w}_g$ are defined by

$$\begin{aligned} \bar{u}_g &= \cos \varepsilon_g \, e^{-\alpha r} \\ \bar{w}_g &= \sin \varepsilon_g \left[e^{-\alpha r}\left(1 + \frac{3}{\alpha r} + \frac{3}{(\alpha r)^2}\right) - \frac{3}{(\alpha r)^2} \right] \end{aligned} \tag{7.16}$$

We may then proceed exactly as in the scattering case to obtain

$$\frac{1}{a_t} = \alpha - \frac{1}{2}\alpha^2 r_D \tag{7.17}$$

with

$$\begin{aligned} r_D &= 2\int_0^\infty \left[\bar{u}_g^2 + \bar{w}_g^2 - u_g^2 - w_g^2 \right] dr \\ &= 2\left[\frac{1}{2\alpha} - \int_0^\infty (u_g^2 + w_g^2)\, dr \right] \end{aligned} \tag{7.18}$$

r_D is called the deuteron effective range.

8. Dispersion Relations for Nucleon-Nucleon Scattering.

In view of the success of dispersion relations in describing the pion-nucleon interaction [21], it was natural to apply dispersion relations to nucleon-nucleon scattering. This was done in the first instance by M.L. Goldberger et al.[22] and independently by S. Matsuyama [23], for fixed momentum transfer and in particular for forward scattering.

Matsuyama starts with the relativistic relation with one subtraction, but neglects entirely the non-physical spectrum (relating to the nucleon-anti-nucleon system) except for the single-pion pole, and also neglects the high-energy part of the physical spectrum. An attempt is made to determine the pion-nucleon coupling constant from triplet scattering, but the result obtained is about twice the generally accepted value. This is not really surprising, however, since the neglected two-pion contributions in the non-physical spectrum are of considerable importance in S-wave scattering.

On the other hand, Goldberger et al. retain the two pion term which is evaluated in perturbation theory. The pion-nucleon coupling constant found is in satisfactory agreement with $f^2 = 0.08$.

The dispersion relations obtained in the non-relativistic limit are similar to those for non-relativistic potential scattering [24,25,26], and in the low energy S-wave region considered, it is found that the usual effective range formula (equation (7.3)) is consistent with the dispersion relations.

Both these treatments suffer from the fact that even for forward scattering, there is a large unphysical region of the nucleon-nucleon scattering cut in which the angular momentum eigenstate expansion is not necessarily convergent.

Following on the general representation for the scattering amplitude proposed by S. Mandelstam [27], M. Cini et al.[28] developed dispersion relations for nucleon-nucleon scattering in which the scattering angle is kept constant. These relations have the advantage as opposed to the earlier ones of not involving any unphysical region of nucleon-nucleon scattering, the whole non-physical contribution coming from nucleon-anti-nucleon scattering. On the basis of these dispersion relations, an extrapolation procedure is developed which is in close analogy to the effective range approximation, and which leads to a determination of the pion-nucleon coupling constant. The result obtained ($f^2 = 0.11 \pm 0.02$) is in reasonable agreement with the generally accepted value.

An essentially equivalent treatment of the effective range approximation was made by H.P. Noyes and D.Y. Wong[29], employing the N/D method proposed in a different context by G.F. Chow and S. Mandelstam [30].

In a series of papers, D. Amati et al.[31] developed the theory for partial wave amplitudes based on the Cini-Fubini [32] method of solution of the Mandelstam representation. Essentially all the singularities of the amplitudes for values of the variables lying near their physical region, are treated taking full advantage of the symmetry of the Mandelstam representation. The spectral functions are calculated using unitarity, in both the nucleon and anti-nucleon channels. In the latter case, the two-pion contribution is retained, but three-pion and higher neglected. Integral equations for the partial wave amplitudes are obtained, and the method of solution described.

H.P. Noyes [33] has obtained integral equations of the same form, starting from the analytic structure of partial waves predicted by the Mandelstam representation using the N/D method. Relativistic formulae are derived for the energy dependence of the phase-shifts for nucleon-nucleon scattering, neglecting inelastic processes. The contribution of the one-pion exchange to the absorptive part of the amplitude is exhibited explicitly and the method of inclusion of the two-pion exchange indicated. The formulae may be generalised to include phenomenological constants to represent the contributions from multi-pion and other particle exchanges. The dependence of the phase-shifts on these parameters is sufficiently simple for it to be applied to fitting experimental data. This programme is at present being carried out by H.P. Stapp et al. [34]

The complete discussion of low energy nucleon-nucleon scattering from the standpoint of double dispersion relations is given by M.L. Goldberger et al. [35] The analytic structure of the partial wave amplitudes is completely analysed, and a set of dynamical equations generated by use of the unitarity condition is obtained. Only one- and two-pion exchanges are considered, but it is felt that this should be sufficient for energies up to 170 MeV. Methods of solution are given, but no explicit calculation is carried out.

We can conclude that although, as yet, dispersion relations have produced no information not derivable from a semi-phenomenological potential, they are on a much more secure fundamental basis, and appear capable of giving a complete and unique description of the nucleon-nucleon scattering system.

9. The Phase-Shift Analysis.

In view of the difficulties and ambiguities encountered in explaining the nucleon-nucleon interaction in the standard field-theoretical and potential-model calculations, it is necessary to obtain as much information as possible from a direct analysis of the experimental data.

The standard method of extracting information from the results of scattering experiments is to find sets of phase-shifts which reproduce the experimental data. In the nucleon-nucleon scattering problem, the data used consists of the differential cross-section and the polarization and triple scattering parameters P, D, R and A.

The first direct determination of the nucleon-nucleon scattering matrix using the above data was carried out by H.P. Stapp et al.[8] at a scattering energy of 310 MeV. This was only partially successful in that eight distinct phase-shift solutions were found, although theory indicated three of these to be incompatible with the final state interaction in the process $\pi^+ + d \rightarrow p + p$. More recently M.J. Moravcsik [36] showed that the one-pion exchange contribution (which is exactly calculable) can be expected to dominate the scattering in the higher angular momentum states. This allowed the ambiguity to be further reduced to two physically distinct phase-shift solutions [37,38].

Very similar solutions are found at 210 MeV [39], at which energy large angle measurements of A indicate that one of the two remaining solutions is spurious [40], leaving as the most probable solution that of Stapp No.1. The close similarity between the phase-shift sets at 210 MeV and 310 MeV make it reasonable to assume that at 310 MeV the Stapp No.1 solution is also the most probable.

At lower energies the position is not nearly so clear. There is certainly more than one way to fit the existing data, although the fact that no potential model so far proposed is compatible with a negative $^1D_2$ phase shift at 98 MeV indicates a unique solution at that energy [41,42], which is reasonable when compared to those at 210 and 310 MeV. At 68 MeV, to obtain a unique solution, it is necessary to impose the restriction that ε_2 be negative. These unique solutions are given in Table 2.

At energies below 68 MeV the situation is even more confused, due largely to the lack of triple scattering data. Polarization is small, and the double scattering experiments have been used only to show the necessity of including the $^3P_2$-$^3F_2$ coupling at 40 MeV [43]. Higher partial waves are small, so the inclusion of the one pion exchange contribution is of no assistance.

In an attempt to obtain a unique solution at all energies up to 380 MeV, H.P. Stapp et al.[34] have initiated an analysis based on expressing $p \cot \delta$ as an analytic function of energy. Their object is to use functional forms incorporating the recently proved analytic properties of partial wave amplitudes, and to use theoretical and experimental information regarding the residues of poles and discontinuities across cuts, together with phenomenological parameters to represent the remaining singularities. The theory for this has been developed by H.P. Noyes [33].

A similar analysis has been made by G. Breit et al.[44,45] who have conducted a gradient search of both proton-proton and neutron-proton data, assuming in the latter case the applicability of strict charge independence. Searches were carried out starting with the phase-shifts from the extended source + spin orbit potential below 150 MeV extrapolated to the Stapp phase-shifts, and with the phase parameters corresponding to the Gammel-Thaler potential. The better fits of both families are found to be essentially the same. The best fit (the so-called YLAM set) is given in Table 1, together with the Gammel-Thaler and Signell-Marshak phase-shifts for comparison.

We thus see that although it is reasonable to assume that a unique (or nearly unique) scattering matrix has been found for energies above 100 MeV, there is still much ambiguity at the lower energies where double and triple scattering experiments are difficult to perform. As yet the theory is not sufficiently advanced to remove these ambiguities at low energies, and for further information it is necessary to turn to some other process. The most convenient to study is the photodisintegration of the deuteron, and we consider this in Part 2.

Table 1. **Nucleon-Nucleon Phase Parameters**

A. Y.L.A.M. set of Breit et al.

E_s	^1S_o	$^1P_1$	$^1D_2$	$^1F_3$	^3P_o	$^3P_1$	$^3P_2$	$^3F_2$	ϵ_2	$^3S_1$	$^3D_1$	ϵ_1	$^3D_2$	$^3D_3$	$^3G_3$	ϵ_3
25	0.855	-0.042	0.019	-0.008	0.182	-0.107	0.061	-0.006	-0.336	1.262	-0.081	-0.022	0.098	0.004	-0.006	0.643
50	0.652	-0.082	0.037	-0.020	0.222	-0.161	0.118	-0.008	-0.318	1.030	-0.144	-0.028	0.197	0.017	-0.023	0.785
75	0.503	-0.123	0.055	-0.032	0.213	-0.203	0.165	-0.006	-0.283	0.845	-0.196	-0.027	0.281	0.035	-0.046	0.760
100	0.384	-0.165	0.071	-0.042	0.167	-0.233	0.200	-0.002	-0.253	0.695	-0.241	-0.025	0.348	0.050	-0.066	0.730
125	0.295	-0.206	0.086	-0.049	0.116	-0.272	0.233	0.000	-0.217	0.585	-0.274	-0.020	0.392	0.060	-0.080	0.709
150	0.221	-0.247	0.102	-0.055	0.066	-0.305	0.257	0.002	-0.190	0.485	-0.301	-0.013	0.424	0.063	-0.091	0.682
175	0.156	-0.293	0.116	-0.060	0.021	-0.336	0.280	0.003	-0.167	0.395	-0.326	0.001	0.441	0.071	-0.093	0.637
200	0.096	-0.341	0.130	-0.063	-0.023	-0.365	0.295	0.004	-0.152	0.316	-0.348	0.028	0.446	0.077	-0.100	0.585
225	0.041	-0.392	0.143	-0.064	-0.069	-0.384	0.301	0.004	-0.143	0.242	-0.366	0.072	0.433	0.082	-0.115	0.511
250	-0.012	-0.445	0.156	-0.063	-0.103	-0.393	0.303	0.004	-0.135	0.177	-0.386	0.127	0.435	0.090	-0.132	0.441

E_s is in MeV, and the phase-parameters are the Blatt-Biedenharn phase parameters in radians.

B. Signell-Marshak Phase Parameters

E_s	^1S_o	$^1P_1$	$^1D_2$	$^1F_3$	^3P_o	$^3P_1$	$^3P_2$	$^3F_2$	ε_2	$^3S_1$	$^3D_1$	ε_1	$^3D_2$	$^3D_3$	$^3G_3$	ε_3
18	0.893	-	0.003	-	0.054	-0.043	0.030	0.000	-0.058	1.290	-0.023	0.078	0.015	0.002	0.000	0.399
40	0.749	-	0.015	-	0.136	-0.111	0.083	-0.002	-0.124	1.002	-0.093	0.109	0.092	0.012	-0.004	0.503
75	0.581	-	0.039	-	0.206	-0.206	0.156	-0.009	-0.208	0.733	-0.212	0.101	0.222	0.056	-0.003	0.522
100	0.490	-	0.067	-	0.213	-0.261	0.193	-0.019	-0.250	0.602	-0.293	0.059	0.337	0.107	-0.056	0.538
150	0.335	-	0.099	-	0.170	-0.341	0.228	-0.035	-0.304	0.394	-0.403	0.068	0.443	0.164	-0.089	0.464

C. Gammel-Thaler Phase Parameters

E_s	^1S_o	$^1P_1$	$^1D_2$	$^1F_3$	^3P_o	$^3P_1$	$^3P_2$	$^3F_2$	ε_2	$^3S_1$	$^3D_1$	ε_1	$^3D_2$	$^3D_3$	$^3G_3$	ε_3
20	0.858	-0.089	0.007	0.000	0.177	-0.087	0.046	-0.003	-0.339	1.498	-0.058	-0.005	0.065	0.018	-0.006	0.609
40	0.695	-0.159	0.024	-0.009	0.236	-0.149	0.099	-0.004	-0.352	1.186	-0.125	-0.021	0.145	0.049	-0.021	0.726
80	0.466	-0.246	0.073	-0.026	0.213	-0.231	0.183	-0.003	-0.303	0.843	-0.213	-0.038	0.285	0.083	-0.052	0.878
100	0.388	-0.277	0.096	-0.035	0.182	-0.263	0.216	0.007	-0.272	0.723	-0.242	-0.040	0.344	0.095	-0.066	0.938
140	0.235	-0.326	0.141	-0.050	0.112	-0.318	0.265	0.014	-0.219	0.530	-0.277	-0.036	0.451	0.109	-0.097	1.049
180	0.120	-0.366	0.181	-0.065	0.041	-0.367	0.297	0.016	-0.177	0.377	-0.293	-0.023	0.544	0.116	-0.132	1.142
220	0.019	-0.398	0.213	-0.076	-0.025	-0.409	0.316	0.016	-0.142	0.248	-0.299	0.001	0.623	0.120	-0.169	1.217

E_s is in MeV, and the phase-parameters are the Blatt-Biedenharn phase parameters in radians

Table 2. Unique Proton-Proton Phase-Shift Solutions.

Scattering Energy	$^1S_0$	$^1D_2$	$^3P_0$	$^3P_1$	$^3P_2$	ε_2	$^3F_2$	$^3F_3$	$^3F_4$
68*	30.45°	2.62°	18.59°	-10.49°	6.96°	-2.38°			
95*	22.17°	3.87°	14.24°	-11.98°	11.17°	-2.78°			
210	6.27°	7.36°	-0.43°	-20.85°	16.92°	-2.09°	-0.15°	-2.24°	0.60°
310	-8.92°	11.87°	-11.27°	-27.49°	16.65°	-1.55°	1.21°	-3.53°	3.54°

Scattering energy in MeV. *Unique only if it is assumed that $^1D_2 > 0$ and that the F phase shifts are given by O.P.E.F. The phase shifts listed above are the "nuclear bar" phase-shifts, related to the Blatt-Biedenham phase-shifts by

$$\bar{\delta}_{J-1} + \bar{\delta}_{J+1} = \delta_\alpha + \delta_\beta$$

$$\tan(\bar{\delta}_{J-1} - \bar{\delta}_{J+1}) = \cos 2\varepsilon_J \tan(\delta_\alpha - \delta_\beta)$$

$$\sin 2\bar{\varepsilon}_J = \sin 2\varepsilon_J \sin(\delta_\alpha - \delta_\beta)$$

For states with no mixing, the two sets are identical.

Part 2. Photodisintegration of the Deuteron.

10. Introduction.

In principle, the investigation of the photodisintegration of the deuteron can give information either on the radiative interaction if the initial and final state wave-functions are known, or on the other hand if the radiative interaction is known useful information can be obtained on the neutron-proton interaction. It is generally with this latter point in mind that deuteron photodisintegration is investigated, since the radiative interaction can be assumed to be well known, at least up to photon laboratory energies of 130 MeV.

In this energy range, it has been shown [46] that explicit inclusion of the mesonic field is unnecessary, and so the interaction with the electromagnetic field may be taken as being given on the basis of the gauge-invariance of the non-relativistic Hamiltonian for the two-nucleon system. The photodisintegration proceeds mainly through electric and magnetic dipole transitions, with the electric dipole transition dominant. Electric and magnetic quadrupole transitions cause a marked interference in the angular distribution, but their contribution to the total cross-section is small. Higher multipoles may be ignored.

For unpolarized radiation we can write the angular distribution as

$$\frac{d\sigma}{d\Omega} = a(1 \pm \beta_1 \cos\theta) + b\sin^2\theta(1 \pm \beta_2 \cos\theta) \tag{10.1}$$

where the plus sign is for the protons and the minus sign for the neutrons. The total cross-section is then given by

$$\sigma_T = 4\pi a + \frac{8\pi}{3} b \tag{10.2}$$

a and b arise mainly from the electric and magnetic dipole transitions, while $a\beta_1$ and $b\beta_2$ come directly from interference between the quadrupole and dipole radiations.

Generally speaking, experimental angular distributions are fitted to the simpler formula

$$\frac{d\sigma}{d\Omega} = (a + b\sin^2\theta)(1 \pm \beta \cos\theta) \tag{10.3}$$

with the total cross-section still given by (10.2).

The experimental values for σ_T, the isotropy factor a/b and β are given in Figures 1, 2 and 3 respectively.

On the basis of the above interaction several authors (47-59) have calculated the angular distributions in the medium energy range and have reached more or less satisfactory agreement with the experimental data. To explain this data it is found necessary to take into account transitions from the deuteron D-state, to describe the final state by phase-shifts which correspond to a repulsive long-range tensor potential in the triplet odd states (such as the Signell-Marshak or Gammel-Thaler potential) and to include the transitions to the final $^3F_2$ state. It is found that the angular distribution parameters are sensitive to the D-state probability.

At energies above 130 MeV meson effects must be included explicitly. Experimentally (Figure 1) the total cross-section is found to have a maximum in the region of 320 MeV which is caused by the resonance occurring in the photoproduction of virtual pions on one nucleon and absorption by the other. Early attempts to account for this behaviour were not very successful (60-64), but recently L.D. Pearlstein and A. Klein (46) have given an explicit prescription for including meson effects with considerable success.

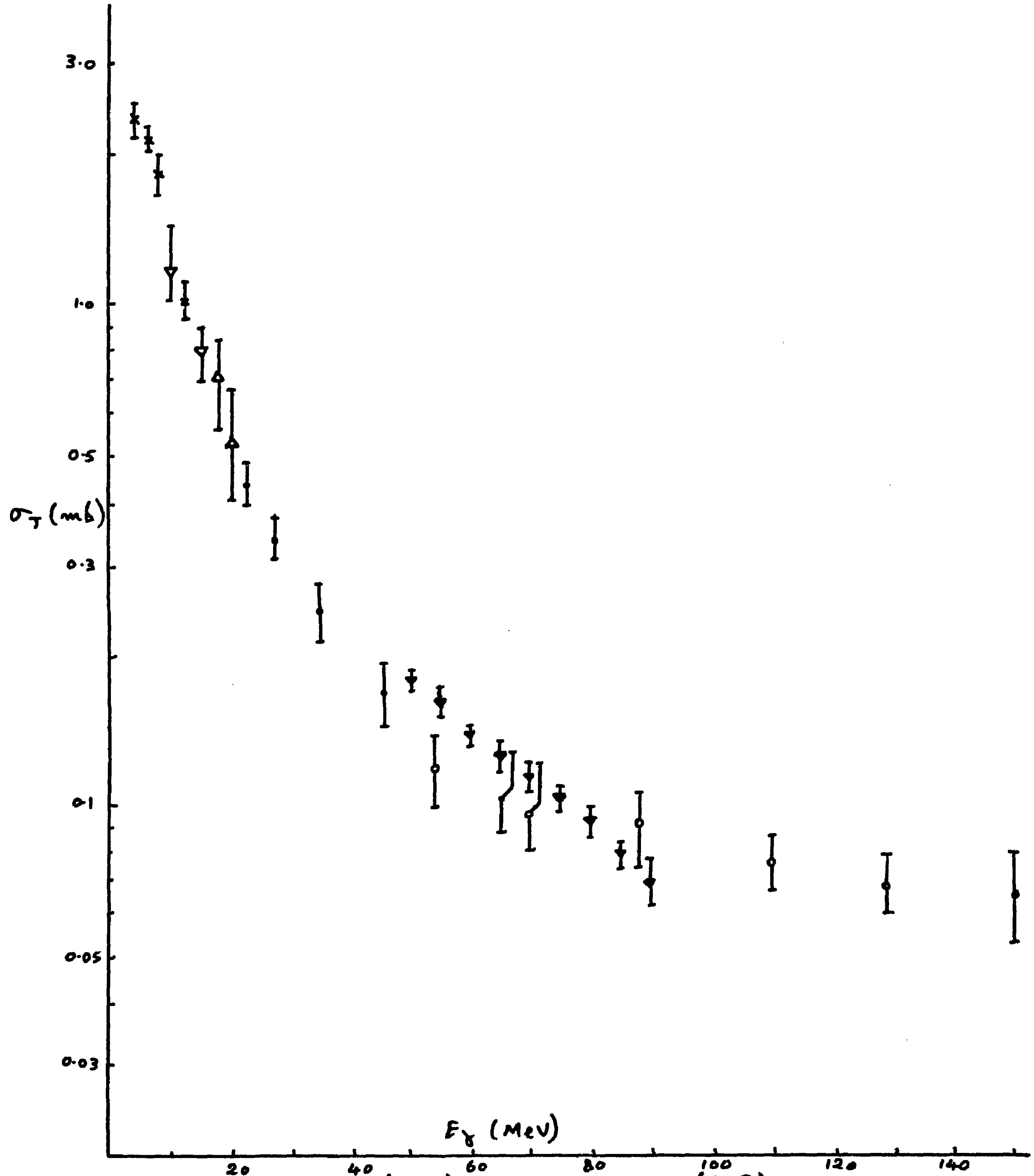
As in the scattering problem, polarization of the outgoing nucleons should provide a sensitive test of the theory. Theoretically it can be shown (65-67) that the polarization of the outgoing nucleons in the direction $\underset{\sim}{n} = (\underset{\sim}{k} \times \underset{\sim}{k})/|\underset{\sim}{k} \times \underset{\sim}{k}|$ is given by

$$\left(\frac{d\sigma}{d\Omega}\right) P = \sin\theta \left[\gamma_0 + \gamma_1 \cos\theta + \gamma_2 \cos^2\theta \right] \qquad (10.4)$$

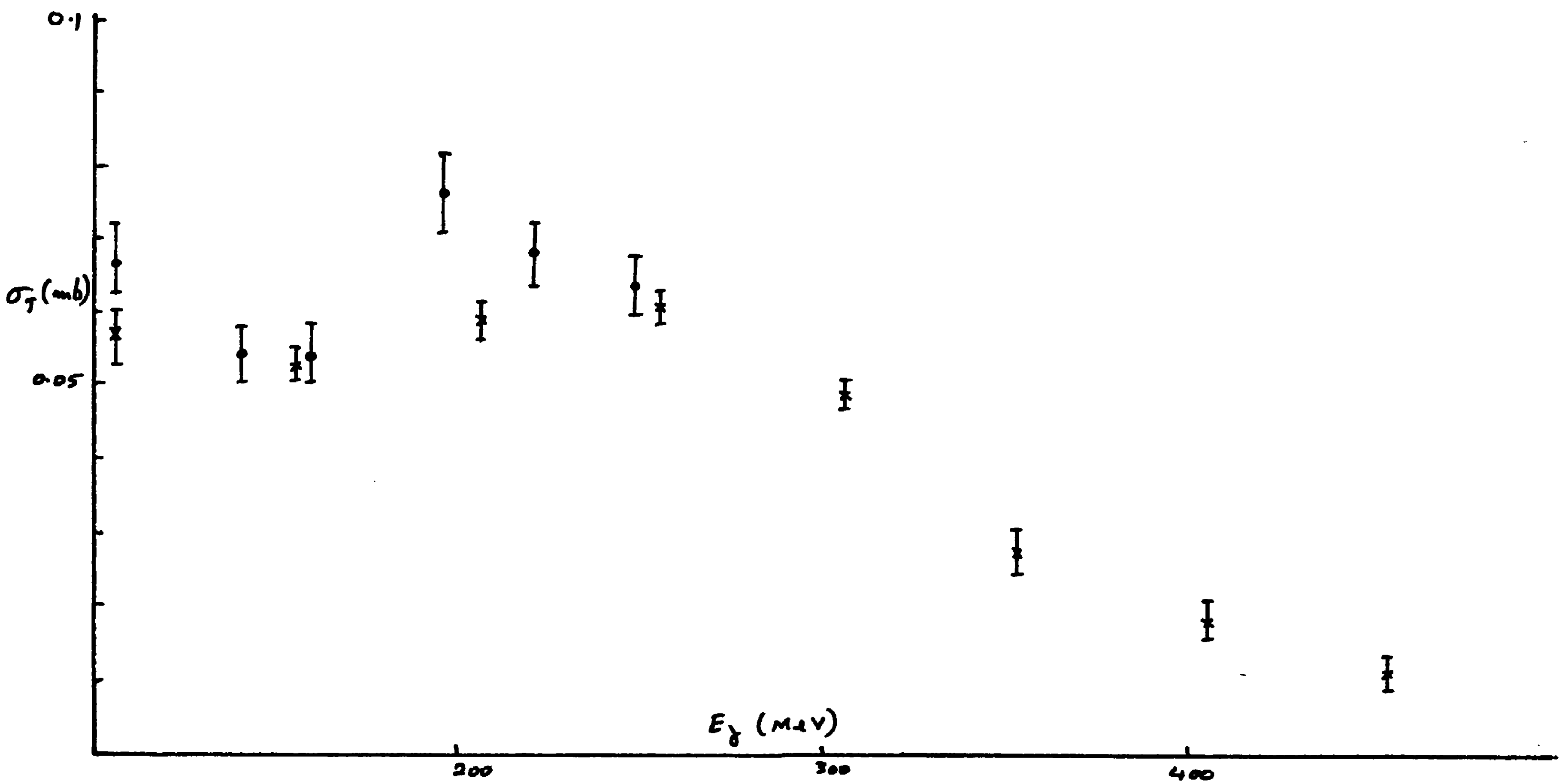
Because there is a difference between the cases in which the proton or the neutron polarization is measured, we must distinguish between $\gamma_{(p)}$ and $\gamma_{(n)}$

Unfortunately, at the moment experimental evidence on the final state polarization is non-existent.

Figure 1. Experimental Total Cross-Section

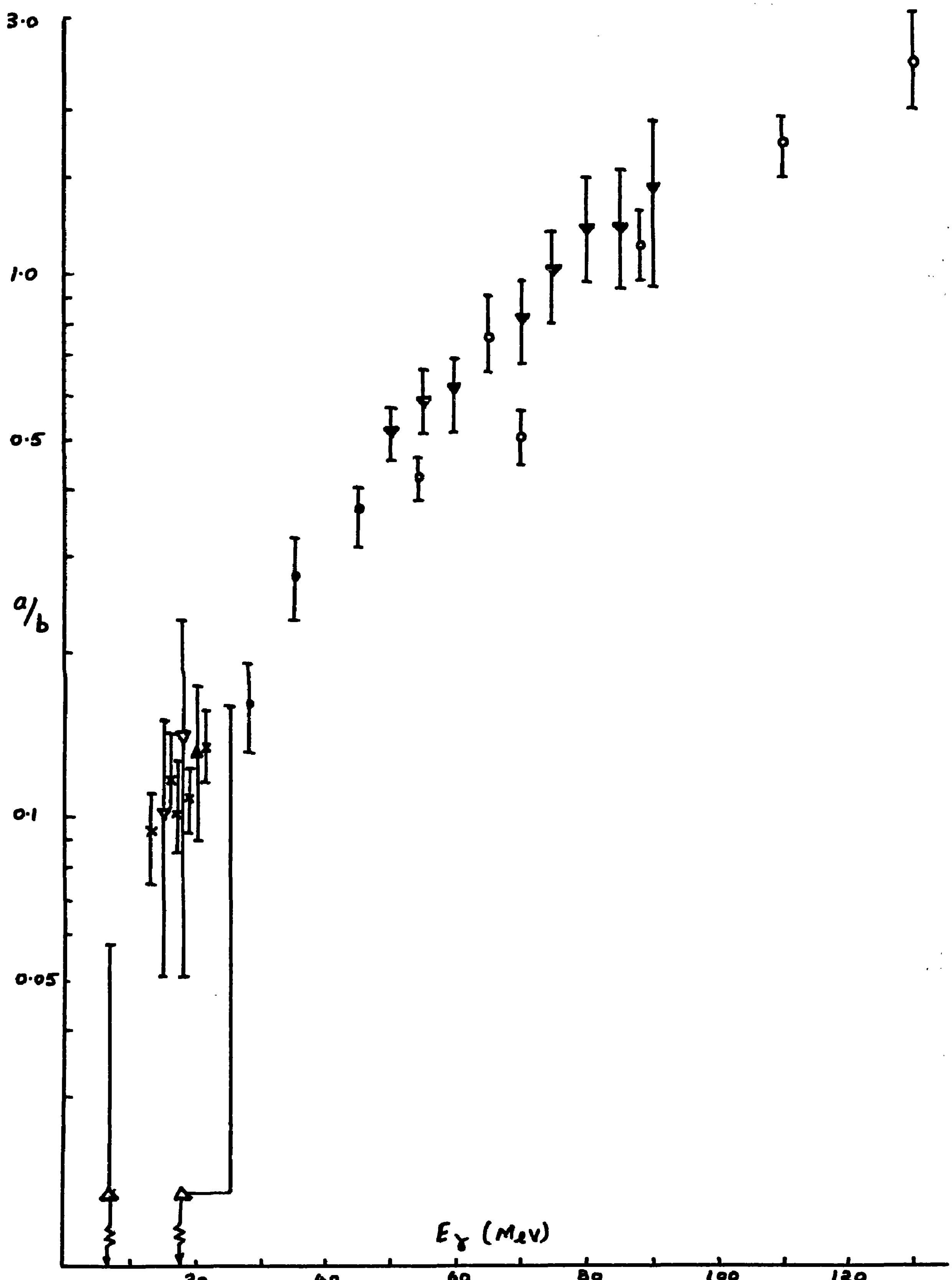


Aleksandrov et al(112)	o	Halpern(108)	▲
Allen(109) (107)	•	Hough(104)	△
		Galey	▼
Carver et al	x	Whetstone et al(113)	▽



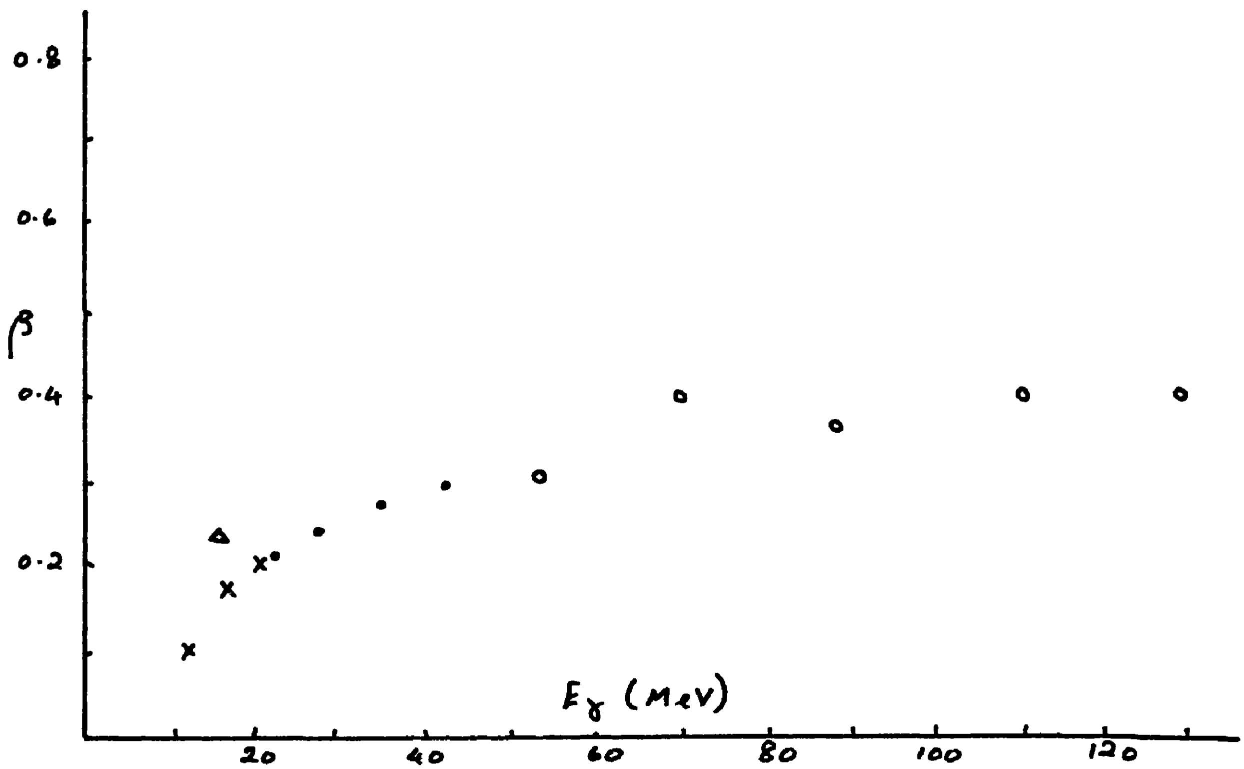
Keck et al[111] ×   Whalin et al[110] •

Figure 2. Experimental Isotropy Ratio



Aleksandrov et al[112] ○
Allen[109] •
Halpern[108] ▲
Hough[104] △
Galey[114] ▼
Waffler et al[106] ▽
Whetstone et al[113] ✱

Figure 3. β



Aleksandrov et al[112] o

Allen[109] •

Waffler et al[106] △

Whetstone et al[113] x

11. General Form of the Interaction of the Nucleon Field with the Electromagnetic Field.

The interaction Hamiltonian for the coupled nucleon and electro-magnetic fields is

$$H' = e \sum_{\mu} \bar{\psi} \frac{1+\tau_3}{2} \gamma_\mu \psi A_\mu - \frac{1}{2} \frac{e}{2m} \sum_{\mu,\nu} \bar{\psi} \Big[(\gamma_p - 1)\frac{1+\tau_3}{2} + \gamma_n \frac{1-\tau_3}{2}\Big] \gamma_\mu \gamma_\nu \psi \left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) \tag{11.1}$$

where the first term gives the usual interaction of the electromagnetic field with a spinor field, and the second term (the Pauli term) is included to account for the anomalous magnetic moment of the nucleon.

Choosing the gauge $A_0 = 0$, H' may be written as

$$H' = -e \psi^* \frac{1+\tau_3}{2} \gamma_0 \underset{\sim}{\gamma} \psi \underset{\sim}{A} - \frac{e}{2m} \psi^* \Big[(\gamma_p - 1)\frac{1+\tau_3}{2} + \gamma_n \frac{1-\tau_3}{2}\Big] \gamma_0 \sigma . \mathrm{curl} \underset{\sim}{A} \tag{11.2}$$

Since the above Hamiltonian can be regarded as a small perturbation, the transition probability of the radiative process is proportional to the matrix element

$$\langle f | \int H' d\underset{\sim}{x} | i \rangle \Big/ \sqrt{\langle f | f \rangle \langle i | i \rangle} \tag{11.3}$$

where $|i\rangle$ and $|f\rangle$ are respectively the initial and final states of the two-nucleon system.

The standard expansion of the nucleon field into plane wave states is

$$\psi_\alpha(x) = \left\{\frac{m}{(2\pi)^3 p_0}\right\}^{\frac{1}{2}} \int dp \sum_{\nu=1,2} \Big\{ e^{ipx} a_\nu^{(+)}(p)\, u_\alpha^{\nu(+)}(p) + e^{-ipx} a_\nu^{(-)}(p)\, u_\alpha^{\nu(-)}(p) \Big\} \tag{11.4a}$$

and of the conjugate field $\bar{\psi}_\alpha(x) = \psi_\alpha^*(x)\gamma_0$

$$\bar{\psi}_\alpha(x) = \left\{\frac{m}{(2\pi)^3 p_0}\right\}^{\frac{1}{2}} \int dp \sum_{\nu=1,2} \left\{ e^{ipx} a_\nu^{*(+)}(p)\, \bar{u}_\alpha^{\nu(+)}(p) + e^{-ipx} a_\nu^{*(-)}(p)\, \bar{u}_\alpha^{\nu(-)}(p) \right\} \quad (11.4b)$$

where $+, -, \nu$ are the quantum numbers defining the particle, anti-particle and isotopic spin states. $a_\nu^{*(+)}(p)$ $a_\nu^{(-)}(p)$ are respectively the creation and annihilation operators for the particle, and $a_\nu^{(+)}(p)$, $a_\nu^{(-)*}(p)$ those of the anti-particle. Thus the two-nucleon state vector is given by

$$|p_1 p_2\rangle = a_\nu^{*(+)}(p_1)\; a_\nu^{*(+)}(p_2)|0\rangle \quad (11.5)$$

where $|0\rangle$ is the vacuum state.

The operators 'a' satisfy the anti-commutation relations

$$\begin{aligned} \left[a_\nu^{*(-)}(p),\; a_{\nu'}^{(+)}(p')\right\} &= \delta_{\nu\nu'}\,\delta(p-p') \\ \left[a_\nu^{(-)}(p),\; a_{\nu'}^{*}(p')\right\} &= \delta_{\nu\nu'}\cdot\delta(p-p') \end{aligned} \quad (11.6)$$

The spinor amplitudes $u, \bar{u}$ satisfy the Dirac equations

$$\begin{aligned} (\gamma\cdot p \pm m)_{\alpha\beta}\, u_\beta^{\pm}(p) &= 0 \\ \bar{u}_\alpha^{(\pm)}(p)\,(\gamma\cdot p \pm m)_{\alpha\beta} &= 0 \end{aligned} \quad (11.7)$$

and have normalization

$$\begin{aligned} \bar{u}^{\nu(\pm)}(p)\;\; u^{\nu'(\mp)}(p) &= \pm\delta^{\nu,\nu'} \\ \bar{u}^{\nu(\pm)}(p)\,\gamma_0\; u^{\nu(\mp)}(p) &= \frac{p_0}{m}\,\delta^{\nu\nu'} \end{aligned} \quad (11.8)$$

The electromagnetic field $\underset{\sim}{A}$ can similarly be expanded as

$$\underset{\sim}{A}(x) = \left\{\frac{1}{(2\pi)^{3/2}\, 2k_0}\right\}^{\frac{1}{2}} \int d\underset{\sim}{k} \left\{ e^{ikx}\, \underset{\sim}{\varepsilon}^{\nu}\, a_{\nu}^{(+)}(\underset{\sim}{k}) + e^{-ikx}\, \underset{\sim}{\varepsilon}^{\nu}\, a_{\nu}^{(-)}(\underset{\sim}{k}) \right\} \qquad (11.9)$$

where $\underset{\sim}{\varepsilon}$ is the polarization vector of the field, and ν denotes the direction of polarization.

The evaluation of the matrix element (11.3) may now be accomplished using the properties (11.5) and (11.6) and noting that

$$u^*(\underset{\sim}{p})\,\gamma_0\,\underset{\sim}{\gamma}\,u(\underset{\sim}{p}') = \left\{\frac{m+p_0}{2m}\right\}^{\frac{1}{2}}\left\{\frac{m+p_0'}{2m}\right\}^{\frac{1}{2}}\left\{\frac{\underset{\sim}{\sigma}.\underset{\sim}{p}\,\underset{\sim}{\sigma}}{p+m} + \frac{\underset{\sim}{\sigma}\,\underset{\sim}{\sigma}.\underset{\sim}{p}'}{p'+m}\right\} \qquad (11.10a)$$

$$u^*(\underset{\sim}{p})\,\gamma_0\,\underset{\sim}{\sigma}\,u(\underset{\sim}{p}') = \left\{\frac{m+p_0}{2m}\right\}^{\frac{1}{2}}\left\{\frac{m+p_0'}{2m}\right\}^{\frac{1}{2}}\left\{\underset{\sim}{\sigma} - \frac{\underset{\sim}{\sigma}.\underset{\sim}{p}\,\underset{\sim}{\sigma}\,\underset{\sim}{\sigma}.\underset{\sim}{p}'}{(p_0+m)(p_0'+m)}\right\} \qquad (11.10b)$$

where $p_0 = \{m^2 + \underset{\sim}{p}^2\}^{\frac{1}{2}}$

Retaining terms of order no higher than v/c equations (11.10a) and (11.10b) become

$$u^*(\underset{\sim}{p})\,\gamma_0\,\underset{\sim}{\gamma}\,u(\underset{\sim}{p}') = \frac{1}{2m}\left[(\underset{\sim}{p}+\underset{\sim}{p}') + i\,\underset{\sim}{\sigma}\times(\underset{\sim}{p}-\underset{\sim}{p}')\right] \qquad (11.11a)$$

$$u^*(\underset{\sim}{p})\,\gamma_0\,u(\underset{\sim}{p}') = \underset{\sim}{\sigma} \qquad (11.11b)$$

Since we are considering an absorption process, we may replace $\underset{\sim}{A}(x)$ by $\left\{\frac{1}{(2\pi)^3\, 2k_0}\right\}^{\frac{1}{2}} \underset{\sim}{\varepsilon}\, e^{i\underset{\sim}{k}.\underset{\sim}{x}}$ to give finally

$$\langle f \,|\, \int H' d\underline{x} \,|\, i \rangle = \left\{ \frac{1}{4\pi k_0} \right\} \iint d\underline{x}_1\, d\underline{x}_2\, \psi_f^*(\underline{x}_1, \underline{x}_2)\, \underline{\varepsilon}^{\nu}$$

$$\left[e^{i\underline{k}\cdot\underline{x}_1} \left\{ \left(-\frac{e\underline{p}_1}{m} \right) \frac{1+\tau_3^{(1)}}{2} - i\underline{\mu}^{(1)} \times \underline{k} \right\} + e^{i\underline{k}\cdot\underline{x}_2} \left\{ \left(-\frac{e\underline{p}_2}{m} \right) \frac{1+\tau_3^{(2)}}{2} - i\underline{\mu}^{(2)} \times \underline{k} \right\} \right] \times \psi_D(\underline{x}_1, \underline{x}_2) \quad (11.12)$$

with

$$\underline{\mu}^{(i)} = \frac{e}{2m} \left[\frac{1+\tau_3^{(i)}}{2} \gamma_p + \frac{1-\tau_3^{(i)}}{2} \gamma_n \right] \underline{\sigma}^{(i)}, \quad i = 1, 2 \quad (11.13)$$

Equation (11.12) is the fundamental equation for the investigation of deuteron photodisintegration at energies for which we may ignore relativistic corrections and the explicit interaction of the meson field. In practice this corresponds to photon energies up to 130 MeV in the laboratory system.

12. The Multipole Transitions.

To utilize equation (11.12), it is convenient to introduce the centre-of-momentum co-ordinate $\underset{\sim}{X}$ and the relative co-ordinate $\underset{\sim}{x}$ of the two nucleons by $\underset{\sim}{x}_1 = \underset{\sim}{X} + \frac{1}{2}\underset{\sim}{x}$ $\underset{\sim}{x}_2 = \underset{\sim}{X} - \frac{1}{2}\underset{\sim}{x}$. For the absorption process, we may assume the centre of momentum (of the two nucleons) to be at rest in the initial state, while in the final state it is moving with momentum $\underset{\sim}{P}$. Then the two wave-functions are given by

$$\psi_D(\underset{\sim}{x}_1, \underset{\sim}{x}_2) = \frac{1}{(2\pi)^{3/2}} \psi_D(\underset{\sim}{x}) \;, \; \psi_f(\underset{\sim}{x}_1, \underset{\sim}{x}_2) = \frac{1}{(2\pi)^{3/2}} e^{i\underset{\sim}{P}\cdot\underset{\sim}{x}} \psi_f(\underset{\sim}{x}) \quad (12.1)$$

and denoting the relative momentum by $\underset{\sim}{k}$ we have

$$\begin{aligned} \langle f | \int H' d\underset{\sim}{x} | i \rangle = \delta(\underset{\sim}{P} - \underset{\sim}{k}) \left\{ \frac{e^2}{4\pi k_0} \right\}^{\frac{1}{2}} \Big[\int \psi_f^*(\underset{\sim}{x}) \; \underset{\sim}{\varepsilon}^{\nu} \left(- \frac{\underset{\sim}{k}}{m} \right) \\ \times \left(e^{i\frac{\underset{\sim}{k}\cdot\underset{\sim}{x}}{2}} \frac{1+\tau_3^{(1)}}{2} - e^{-i\frac{\underset{\sim}{k}\cdot\underset{\sim}{x}}{2}} \frac{1+\tau_3^{(2)}}{2} \right) \psi_D(\underset{\sim}{x}) \, d\underset{\sim}{x} \qquad (12.2) \\ - i k_0 \int \psi_f^*(\underset{\sim}{x}) (\underset{\sim}{k} \times \underset{\sim}{\varepsilon}^{\nu}) \left(e^{i\frac{\underset{\sim}{k}\cdot\underset{\sim}{x}}{2}} \underset{\sim}{\mu}^{(1)} + e^{-i\frac{\underset{\sim}{k}\cdot\underset{\sim}{x}}{2}} \underset{\sim}{\mu}^{(2)} \right) \psi_D(\underset{\sim}{x}) \, d\underset{\sim}{x} \end{aligned}$$

The δ-function merely states the law of conservation of total momentum, and will be omitted in what follows.

Since we have considered the electromagnetic interaction only to first order, equation (12.2) may be divided into two parts, one leading to the isotopic spin singlet final state and the other to the isotopic spin triplet final state. This is effected by making the re-arrangements

$$\left[e^{i\frac{\underline{k}.\underline{x}}{2}}\frac{1+\tau_3^{(1)}}{2} - e^{-i\frac{\underline{k}.\underline{x}}{2}}\frac{1+\tau_3^{(2)}}{2}\right] = \frac{1}{2}\left\{\left(e^{i\frac{\underline{k}.\underline{x}}{2}} - e^{-i\frac{\underline{k}.\underline{x}}{2}}\right) + \left(e^{i\frac{\underline{k}.\underline{x}}{2}} + e^{-i\frac{\underline{k}.\underline{x}}{2}}\right)\frac{\tau_3^{(1)}-\tau_3^{(2)}}{2}\right\} \tag{12.3a}$$

$$\left[e^{i\frac{\underline{k}.\underline{x}}{2}}\underline{\mu}^{(1)} + e^{-i\frac{\underline{k}.\underline{x}}{2}}\underline{\mu}^{(2)}\right] = \frac{1}{2}\frac{e}{2m}\Big[\left(e^{i\frac{\underline{k}.\underline{x}}{2}} + e^{-i\frac{\underline{k}.\underline{x}}{2}}\right)\left\{(\gamma_p+\gamma_n)\frac{\underline{\sigma}^{(1)}+\underline{\sigma}^{(2)}}{2} + (\gamma_p-\gamma_n)\frac{\tau_3^{(1)}-\tau_3^{(2)}}{2}\,\frac{\underline{\sigma}^{(1)}-\underline{\sigma}^{(2)}}{2}\right\} + \left\{e^{i\frac{\underline{k}.\underline{x}}{2}} - e^{-i\frac{\underline{k}.\underline{x}}{2}}\right\}\left\{(\gamma_p+\gamma_n)\frac{\underline{\sigma}^{(1)}-\underline{\sigma}^{(2)}}{2} + (\gamma_p-\gamma_n)\frac{\tau_3^{(1)}-\tau_3^{(2)}}{2}\,\frac{\underline{\sigma}^{(1)}+\underline{\sigma}^{(2)}}{2}\right\}\Big] \tag{12.3b}$$

In the above expressions, terms in $\frac{1}{2}(\tau_3^{(1)}+\tau_3^{(2)})$ have been dropped, since this factor is identically zero when operating on the deuteron ground state.

Since the wave-length λ of the incident photon is related to the photon energy k_0 by

$$\lambda = 1.2396 \times 10^{-10}\,\text{cm}/k_0 \tag{12.4}$$

k_0 being given in MeV, λ is always large compared with the deuteron radius $\frac{1}{\alpha} = 4.3157 \times 10^{-13}$cm., even at energies of 100 MeV. Thus it is legitimate to expand the exponential factors occurring in (12.3a), (12.3b) with respect to $\underline{k}.\underline{x}$ and retain only the first few terms. It is this expansion which leads to the various multipole transitions.

<u>The Electric Dipole Transitions</u>. These arise from the lowest order term in the expansion of (12.3a), which gives

$$\langle f|\int H'd\underline{x}|i\rangle_{E1} = \left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}}\int \psi_f^*(\underline{x})\,\underline{\varepsilon}^{\nu}\left(-\frac{\underline{k}}{m}\right)\psi_D(\underline{x})\,\frac{1}{2}(\tau_3^{(1)}-\tau_3^{(2)}) \tag{12.5}$$

Since the operator occurring in the integrand is odd, it causes a transition from the ground state $({}^3S_1 + {}^3D_1)$ to an odd parity state. The spin configuration of $\psi_D(x)$ is unchanged, hence the final state must be a $({}^3P + {}^3F)$ state, which is an isotopic spin triplet. This implies that the relevant matrix element of $\frac{1}{2}(\tau_3^{(1)} - \tau_3^{(2)})$ is just one.

By means of the Schrodinger equations satisfied by $\psi_D(x)$ and $\psi_f(x)$ equation (12.5) may be re-written as

$$\langle f | \int H' dx | i \rangle = -i \left\{ \frac{e^2}{4\pi k_0} \right\}^{\frac{1}{2}} \int \psi_f^*(x) \frac{k_0 \, x \cdot \varepsilon}{2} \psi_D(x) \, dx \qquad (12.6)$$

<u>The Electric Quadrupole Transitions</u>. The integrand arising from the linear term of the expansion of the retardation factor in (12.3a) is

$$-\frac{ie}{2m}(\varepsilon \cdot p)(k \cdot x) = -\frac{ie}{4m}\left\{(k \times \varepsilon) \cdot (x \times p) + (\varepsilon \cdot p)(k \cdot x) + (\varepsilon \cdot x)(k \cdot p)\right\} \qquad (12.7)$$

As for (12.5), we can show that

$$\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{i}{2} \int \psi_f^*(x) \left[\left(\varepsilon \cdot \frac{x}{2}\right)\left(-\frac{p}{m} \cdot k\right) + \left(k \cdot \frac{x}{2}\right)\left(-\frac{p}{m} \cdot \varepsilon\right) \right] \psi_D(x) \, dx$$
$$= \left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} k_0^2 \int \psi_f^*(x) \frac{x \cdot \varepsilon \; x \cdot k}{8} \psi_D(x) \, dx \qquad (12.8)$$

which gives the electric quadrupole transition. Since the operator in the integrand is even, and does not affect the spin configuration of $\psi_D(x)$, the final state must be a $({}^3S + {}^3D + {}^3G)$ state.

<u>The Magnetic Dipole Transitions</u>. Taking the lowest term in the expansion of equation (12.3b) the integrand of the second term of (12.2) becomes

$$\frac{e}{2m}(\underline{k}\times\underline{\epsilon})\left\{(\gamma_p+\gamma_n)\frac{\underline{\sigma}^{(1)}+\underline{\sigma}^{(2)}}{2}+(\gamma_p-\gamma_n)\frac{\tau_3^{(1)}-\tau_3^{(2)}}{2}\cdot\frac{\underline{\sigma}^{(1)}-\underline{\sigma}^{(2)}}{2}\right\} \quad (12.9)$$

Since this term contains no orbital operators, it cannot change the orbital configuration. If the spin configuration is not changed either, then the relevant matrix element must vanish due to the orthogonality, since it is calculated between states of the same spin and orbital configuration, but different energies. Thus the spin-state must change which requires in turn the change of the isotopic spin state, allowing us to drop the first term of (12.9) and in the second replace the isotopic spin factor by one. The corresponding matrix element is finally given by

$$\langle f|\int H'd\underline{x}\,|i\rangle_{M_1} = i k_0\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}}\int\psi_f^*(x)\,\frac{\gamma_p-\gamma_n}{2}\,i(\underline{\sigma}^{(1)}-\underline{\sigma}^{(2)})\cdot(\underline{k}\times\underline{\epsilon})\psi_D(x) \quad (12.10)$$

which is the magnetic dipole transition due to the magnetic moments of the nucleons, leading to a $({}^1S + {}^1D)$ final state.

The first term on the right-hand-side of equation (12.7) gives the orbital magnetic dipole transition,

$$\langle f | \int H' d\underline{x} | i \rangle = -i \frac{k_0}{m} \left\{ \frac{e^2}{4\pi k_0} \right\}^{\frac{1}{2}} \int \psi_f^*(\underline{x}) (\underline{k} \times \underline{\epsilon}) (\tfrac{\underline{x}}{2} \times \underline{k}) \psi_D(\underline{x}) \, d\underline{x} \qquad (12.11)$$

which leads (as for the electric quadrupole transition) to a $({}^3S + {}^3D + {}^3G)$ final state.

The Magnetic Quadrupole Transitions. This is obtained directly from the linear term in the expansion of (12.3b), to give the spin magnetic quadrupole matrix element,

$$\langle f | \int H' d\underline{x} | i \rangle = - k_0^2 \left\{ \frac{e^2}{4\pi k_0} \right\}^{\frac{1}{2}} \int \psi_f^*(\underline{x}) \frac{\underline{k}\cdot\underline{x}}{2} \left\{ \frac{\gamma_p + \gamma_n}{2} \cdot \frac{i(\underline{\sigma}^{(1)} - \underline{\sigma}^{(2)})}{2} + \frac{(\gamma_p - \gamma_n)}{2m} \cdot \frac{\tau_3^{(1)} - \tau_3^{(2)}}{2} \cdot \frac{\underline{\sigma}^{(1)} + \underline{\sigma}^{(2)}}{2} \right\} \psi_D(\underline{x}) \, d\underline{x} \qquad (12.12)$$

which transition leads to a $({}^3P + {}^3F + {}^1P + {}^1F)$ final state.

By continuing this process, all the multipole transition matrix elements may be obtained. In practice, however, it is sufficient to retain only the dipole and quadrupole matrix elements, and of those only the ones leading to S, P, D or F-wave final states need be considered.

In view of the above discussion, we can give the following table of allowed transitions.

Multipole	Allowed Final States
Electric Dipole	^3P_o, $^3P_1$, $^3P_2$ + $^3F_2$
Electric Quadrupole	$^3S_1$ + $^3D_1$, $^3D_2$, $^3D_3$ + $^3G_3$
Magnetic Dipole	^1S_o, $^1D_2$ (spin transition)
	$^3S_1$ + $^3D_1$, $^3D_2$, $^3D_3$ + $^3G_3$ (orbital transition)
Magnetic Quadrupole	$^3P_1$, $^3P_2$ + $^3F_2$; $^1P_1$, $^1F_3$

The above table is a special case of the general table of allowed transitions given in Appendix 5

13. The Differential Cross-Section for Deuteron Photodisintegration.

We take the deuteron ground-state wave-function to be of the form given by equations (5.10) and (5.13), namely

$$\psi_D(\underline{r}) = \frac{N}{\sqrt{4\pi}} \left\{ \frac{u_g(r)}{r} + \frac{S_{12}}{\sqrt{8}} \frac{w_g(r)}{r} \right\} \chi_m \qquad (13.1)$$

where $u_g(r)$ and $w_g(r)$ have the normalization (5.14). m denotes the initial spin quantum number.

If we now expand the final state wave-function as in equations (3.23), (3.24) and (3.25), and retain only the transitions

$$\begin{array}{lll} \text{E.D.} \longrightarrow & {}^3P_0,\ {}^3P_1,\ {}^3P_2 + {}^3F_2 & \\ \text{M.D.} \longrightarrow & {}^1S_0,\ {}^1D_2 & \\ \text{E.Q.} \longrightarrow & {}^3S_1,\ {}^3D_1,\ {}^3D_2,\ {}^3D_3 & \qquad (13.2) \\ \text{M.Q.} \longrightarrow & {}^1P_1,\ {}^1F_3 & \end{array}$$

then it can be shown (see Appendix 6) that

$$\langle f | \int H' d\underline{r} | i \rangle_{E1} = -\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{\pi k_0}{3 p\alpha} \cdot \frac{\chi_1^{m'\dagger}}{(2\pi)^{3/2}} \cdot \frac{N}{(4\pi)^{1/2}}$$

$$\times \Big\{ \hat{p}\cdot\underline{\epsilon} \Big[E_{10} e^{-i\delta_{10}} + \tfrac{3}{2} E_{11} e^{-i\delta_{11}} + \tfrac{1}{10}(\xi_2 + 4\eta_2) E_{12} e^{-i\delta_{12}} - \tfrac{1}{10}(6\xi_2 - \eta_2) E_{32} e^{-i\delta_{32}} \Big]$$

$$+ i(\underline{\sigma}^{(1)} + \underline{\sigma}^{(2)}).(\hat{k}\times\underline{\epsilon})\ \hat{p}.\hat{k} \Big[\tfrac{1}{2} E_{10} e^{-i\delta_{10}} + \tfrac{3}{4} E_{11} e^{-i\delta_{11}} + \tfrac{1}{20}(4\eta_2 - 29\xi_2) E_{12} e^{-i\delta_{12}}$$

$$- \tfrac{1}{20}(6\xi_2 - 31\eta_2) E_{32} e^{-i\delta_{32}} \Big]$$

$$+ i(\underline{\sigma}^{(1)} + \underline{\sigma}^{(2)}).\hat{k}\ \hat{p}.(\hat{k}\times\underline{\epsilon}) \Big[-\tfrac{1}{2} E_{10} e^{-i\delta_{10}} - \tfrac{3}{4} E_{11} e^{-i\delta_{11}} + \tfrac{1}{20}(29\xi_2 - 4\eta_2) E_{12} e^{-i\delta_{12}}$$

$$- \frac{1}{20}(6\xi_2 - 31\eta_2) E_{32} e^{-i\delta_{32}} \Big]$$

$$+ (\underline{\sigma}^{(1)} \cdot \hat{p}\, \underline{\sigma}^{(2)} \cdot \underline{\varepsilon} + \underline{\sigma}^{(2)} \cdot \hat{p}\, \underline{\sigma}^{(1)} \cdot \underline{\varepsilon}) \Big[-\frac{1}{2} E_{10} e^{-i\delta_{10}} + \frac{3}{4} E_{11} e^{-i\delta_{11}} - \frac{1}{20}(\xi_2 + 4\eta_2) E_{12} e^{-i\delta_{12}}$$

$$+ \frac{1}{20}(6\xi_2 - \eta_2) E_{32} e^{-i\delta_{32}} \Big] \Big\} \chi_1^m$$

$$\langle f | \int H' d\underline{x} | i \rangle_{M1} = i \left\{ \frac{e^2}{4\pi k_0} \right\}^{\frac{1}{2}} \frac{4\pi k_0}{p} \cdot \frac{\gamma_p - \gamma_n}{2m} \cdot \frac{\chi_0^{0\dagger}}{(2\pi)^{3/2}} \cdot \frac{N}{(4\pi)^{1/2}}$$

$$\times \Big\{ i (\underline{\sigma}^{(1)} - \underline{\sigma}^{(2)}) \cdot (\hat{k} \times \underline{\varepsilon}) \Big[M_0 e^{-i\Delta_0} + \frac{1}{2\sqrt{2}} M_2 e^{-i\Delta_2} \Big]$$

$$+ i (\underline{\sigma}^{(1)} - \underline{\sigma}^{(2)}) \cdot \hat{p}\, \hat{p} \cdot (\hat{k} \times \underline{\varepsilon}) \frac{3}{2\sqrt{2}} M_2 e^{-i\Delta_2} - (\underline{\sigma}^{(1)} \times \underline{\sigma}^{(2)}) \cdot (\hat{k} \times \underline{\varepsilon}) \frac{3}{2\sqrt{2}} M_2 e^{-i\Delta_2}$$

$$+ (\underline{\sigma}^{(1)} \times \underline{\sigma}^{(2)}) \cdot \hat{p}\, \hat{p} \cdot (\hat{k} \times \underline{\varepsilon}) \frac{3}{2\sqrt{2}} M_2 e^{-i\Delta_2} \Big\} \chi_1^m$$

$$\langle f | \int H' d\underline{x} | i \rangle_{E2} = \left\{ \frac{e^2}{4\pi k_0} \right\}^{\frac{1}{2}} \frac{4\pi k_0^2}{p\alpha^2} \cdot \frac{N}{(4\pi)^{1/2}} \cdot \frac{\chi_1^{m\dagger}}{(2\pi)^{3/2}} \cdot \frac{1}{16 \cdot 15}\, \hat{p} \cdot \hat{k}$$

$$\times \Big\{ \hat{p} \cdot \hat{\varepsilon} \Big[-18 E_{21} e^{-i\delta_{21}} + 10 E_{22} e^{-i\delta_{22}} - 22 E_{23} e^{-i\delta_{23}} \Big]$$

$$+ i (\underline{\sigma}^{(1)} + \underline{\sigma}^{(2)}) \cdot (\hat{k} \times \underline{\varepsilon})\, \hat{p} \cdot \hat{k} \Big[-10 E_{22} e^{-i\delta_{22}} - 10 E_{23} e^{-i\delta_{23}} \Big]$$

$$+ i (\underline{\sigma}^{(1)} + \underline{\sigma}^{(2)}) \cdot \hat{k}\, \hat{p} \cdot (\hat{k} \times \underline{\varepsilon}) \Big[10 E_{21} e^{-i\delta_{21}} + 20 E_{23} e^{-i\delta_{23}} \Big]$$

$$+ i (\underline{\sigma}^{(1)} + \underline{\sigma}^{(2)}) \cdot \hat{p}\, \hat{p} \cdot (\hat{k} \times \underline{\varepsilon}) \Big[-5 E_{22} e^{-i\delta_{22}} - 10 E_{23} e^{-i\delta_{23}} \Big]$$

$$+ (\underline{\sigma}^{(1)} \cdot \hat{k}\, \underline{\sigma}^{(2)} \cdot \underline{\varepsilon} + \underline{\sigma}^{(2)} \cdot \hat{k}\, \underline{\sigma}^{(1)} \cdot \underline{\varepsilon}) \quad E_{01} e^{-i\delta_{01}}$$

$$+ (\underline{\sigma}^{(1)} \cdot \hat{p}\, \underline{\sigma}^{(2)} \cdot \underline{\varepsilon} + \underline{\sigma}^{(2)} \cdot \hat{p}\, \underline{\sigma}^{(1)} \cdot \underline{\varepsilon}) \Big[9 E_{21} e^{-i\delta_{21}} - 9 E_{22} e^{-i\delta_{22}} + 6 E_{23} e^{-i\delta_{23}} \Big]$$

$$+ (\underline{\sigma}^{(1)} \cdot \hat{p}\, \underline{\sigma}^{(2)} \cdot \hat{k} + \underline{\sigma}^{(2)} \cdot \hat{p}\, \underline{\sigma}^{(1)} \cdot \hat{k})\, \hat{p} \cdot \underline{\varepsilon} \Big[9 E_{21} e^{-i\delta_{21}} - 15 E_{22} e^{-i\delta_{22}} + 6 E_{23} e^{-i\delta_{23}} \Big] \Big\} \hat{\chi}$$

$$\langle f | \int H' d\underline{x} | i \rangle_{M2} = i\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{\pi\, k_0^2}{p\alpha} \cdot \frac{N}{(4\pi)^{1/2}} \cdot \frac{\gamma_p + \gamma_n}{2m} \frac{\chi_0^{0\,+}}{(2\pi)^{3/2}} \hat{p}\hat{p}$$

$$\times\left\{ [i(\underline{\sigma}^{(1)} - \underline{\sigma}^{(2)}) + (\underline{\sigma}^{(1)} \times \underline{\sigma}^{(2)})\} \cdot (\hat{k} \times \underline{\varepsilon}) \left[\tfrac{1}{2} M_1 e^{-i\Delta_1} - \tfrac{3}{2} M_3 e^{-i\Delta_3} \right] \right.$$

$$\left. + \{ i(\underline{\sigma}^{(1)} - \underline{\sigma}^{(2)}) + (\underline{\sigma}^{(1)} \times \underline{\sigma}^{(2)})\} \cdot \hat{p}\hat{p} \cdot (\hat{k} \times \underline{\varepsilon}) \tfrac{5}{2} M_3 e^{-i\Delta_3} \right\} \chi_1^m$$

where the amplitudes E_{LJ}, M_{JJ} are given by

$$E_{10} = \int_0^\infty dr\; \bar{v}_{10}(pr) \left[u_D(r) - \sqrt{2}\, w_D(r) \right] dr \qquad (13.7)$$

$$E_{11} = \int_0^\infty dr\; \bar{v}_{11}(pr) \left[u_D(r) + \frac{1}{\sqrt{2}} w_D(r) \right] dr \qquad (13.8)$$

$$E_{12} = \cos\varepsilon_2 \int_0^\infty dr\; \bar{v}_{12}^{\,1}(pr) \left[u_D(r) - \frac{1}{5\sqrt{2}} w_D(r) \right] dr \qquad (13.9)$$

$$+ \sqrt{\tfrac{3}{2}} \sin\varepsilon_2 \tfrac{3}{5} \sqrt{2} \int_0^\infty dr\; \bar{v}_{32}^{\,1}(pr)\, w_D(r)\, dr$$

$$E_{32} = \cos\varepsilon_2 \tfrac{3}{5} \sqrt{2} \int_0^\infty dr\; \bar{v}_{32}^{\,3}(pr)\, w_D(r) \qquad (13.10)$$

$$- \sqrt{\tfrac{2}{3}} \sin\varepsilon_2 \int_0^\infty dr\; \bar{v}_{12}^{\,3}(pr) \left[u_D(r) - \frac{1}{5\sqrt{2}} w_D(r) \right]$$

$$E_{01} = \frac{1}{\sqrt{2}} \int_0^\infty \bar{v}_{10}(pr)\, w_D(r)\, (\alpha r)^2\, dr \qquad (13.11)$$

$$E_{21} = \int_0^\infty (\alpha r)^2\, \bar{v}_{21}(pr) \left[u_D(r) - \frac{1}{\sqrt{2}} w_D(r) \right] dr \qquad (13.12)$$

$$E_{22} = \int_0^\infty (\alpha r)^2\, \bar{v}_{22}(pr) \left[u_D(r) + \frac{1}{\sqrt{2}} w_D(r) \right] dr \qquad (13.13)$$

$$E_{23} = \int_0^\infty (\alpha r)^2\, \bar{v}_{23}(pr) \left[u_D(r) - \frac{\sqrt{2}}{7} w_D(r) \right] dr \qquad (13.14)$$

$$M_0 = \int_0^\infty v_0(pr)\, u_D(r)\, dr \qquad (13.15)$$

$$M_1 = \int_0^\infty \bar{v}_1(kr)\left[u_D(r) - \frac{\sqrt{2}}{10} w_D(r)\right] \alpha r \, dr \qquad (13.16)$$

$$M_2 = \int_0^\infty v_2(kr)\, w_D(r)\, dr \qquad (13.17)$$

$$M_3 = \frac{3}{5}\sqrt{2}\int_0^\infty \bar{v}_3(kr)\, w_D(r)\,(\alpha r)\, dr \qquad (13.18)$$

The differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\,\frac{1}{6} \sum |\langle H' \rangle|^2 \rho(E) \frac{1}{flux} \qquad (13.19)$$

spin final state

spin initial state

polarization

where the density of final states, $\rho(E)$, is given by

$$\rho(E) = \frac{km}{2} \qquad (13.20)$$

and the incident flux is equal to $(2\pi)^{-3}$.

The evaluation of (13.19) has been given by several authors (49, 52, 53, 67). The result is

$$\frac{d\sigma}{d\Omega} = a + b\sin^2\theta + c\cos\theta + d\sin^2\theta\cos\theta \qquad (13.21)$$

where

$$a = \frac{1}{36} B(k)\Big\{ 4E_{10}^2 + 9E_{11}^2 + (9\xi_2^2 + 4\eta_2^2)E_{12}^2 + (9\xi_2^2 + 9\eta_2^2)E_{32}^2$$

$$-18E_{11}E_{12}\xi_2\cos(\delta_{11}-\delta_{12}) - 8E_{10}E_{12}\eta_2\cos(\delta_{10}-\delta_{12})$$

$$+18E_{11}E_{32}\eta_2\cos(\delta_{11}-\delta_{32}) - 12E_{10}E_{32}\xi_2\cos(\delta_{10}-\delta_{32})$$

$$-6\,E_{12}E_{32}\,\eta_2\,\xi_2\cos(\delta_{12}-\delta_{32}) \tag{13.22}$$

$$+36\,(\gamma_p-\gamma_m)^2\left(\frac{\alpha}{m}\right)^2\left[M_0^2+\tfrac{1}{2}M_2^2-\sqrt{2}\,M_0M_2\cos(\Delta_0-\Delta_2)\right]\Big\}$$

$$b=\frac{1}{24}\,B(k)\Big\{3E_{11}^2+(3\xi_2^2+4\eta_2^2)E_{12}^2+(9\xi_2^2+3\eta_2^2)E_{32}^2$$

$$+8E_{10}E_{11}\,\eta_2\cos(\delta_{10}-\delta_{12})+18\,E_{11}E_{12}\,\xi_2\cos(\delta_{11}-\delta_{12})$$

$$+12E_{10}E_{32}\,\xi_2\cos(\delta_{10}-\delta_{32})+6E_{12}E_{32}\,\eta_2\,\xi_2\cos(\delta_{12}-\delta_{32}) \tag{13.23}$$

$$-18\,E_{11}E_{32}\,\eta_2\cos(\delta_{11}-\delta_{32})\,\xi_2$$

$$+24\,(\gamma_p-\gamma_m)^2\left(\frac{\alpha}{m}\right)^2\left[\frac{3}{4}M_2^2+\frac{3}{\sqrt{2}}M_0M_2\cos(\Delta_0-\Delta_2)\right]\Big\}$$

$$c=\frac{1}{120}\,B(k)\,\frac{k_0}{\alpha}\Big\{E_{10}\big[4E_{01}\cos(\delta_{10}-\delta_{01})+4E_{21}\cos(\delta_{10}-\delta_{21})$$

$$-4E_{23}\cos(\delta_{10}-\delta_{23})\big]+E_{11}\big[-6E_{01}\cos(\delta_{11}-\delta_{01})$$

$$+3E_{21}\cos(\delta_{11}-\delta_{21})+5E_{22}\cos(\delta_{11}-\delta_{22})-8E_{23}\cos(\delta_{11}-\delta_{23})\big]$$

$$+E_{12}\big[2E_{01}\cos(\delta_{12}-\delta_{01})-4E_{21}\cos(\delta_{12}-\delta_{21})-5E_{22}\cos(\delta_{12}-\delta_{22})$$

$$+12E_{23}\cos(\delta_{12}-\delta_{23})\big]+E_{32}\big[-12E_{01}\cos(\delta_{32}-\delta_{01}) \tag{13.24}$$

$$-3E_{21}\cos(\delta_{32}-\delta_{01})+5E_{22}\cos(\delta_{32}-\delta_{22})-2E_{23}\cos(\delta_{32}-\delta_{23})\big]$$

$$+120\,(\gamma_p^2-\gamma_m^2)\left(\frac{\alpha}{m}\right)^2\Big[M_0M_1\cos(\Delta_0-\Delta_1)-\frac{1}{\sqrt{2}}M_1M_2\cos(\Delta_1-\Delta_2)$$

$$-M_0M_2\cos(\Delta_0-\Delta_2)+\frac{1}{\sqrt{2}}M_2M_3\cos(\Delta_2-\Delta_3)\Big]\Big\}$$

$$d=\frac{1}{60}\,B(k)\,\frac{k_0}{\alpha}\Big\{E_{10}\big[5E_{23}\cos(\delta_{10}-\delta_{23})\big]+E_{11}\big[5E_{22}\cos(\delta_{11}-\delta_{22})$$

$$+ 10 E_{23} \cos(\delta_{11} - \delta_{23}) \Big\} + E_{12} \Big[9 E_{21} \cos(\delta_{12} - \delta_{21}) + 10 E_{22} \cos(\delta_{12} - \delta_{22})$$

$$+ 6 E_{23} \cos(\delta_{12} - \delta_{23}) \Big\} + E_{32} \Big[15 E_{01} \cos(\delta_{32} - \delta_{01}) + 6 E_{21} \cos(\delta_{32} - \delta_{21})$$

$$- 10 E_{22} \cos(\delta_{32} - \delta_{22}) + 4 E_{23} \cos(\delta_{32} - \delta_{23}) \Big] \tag{13.25}$$

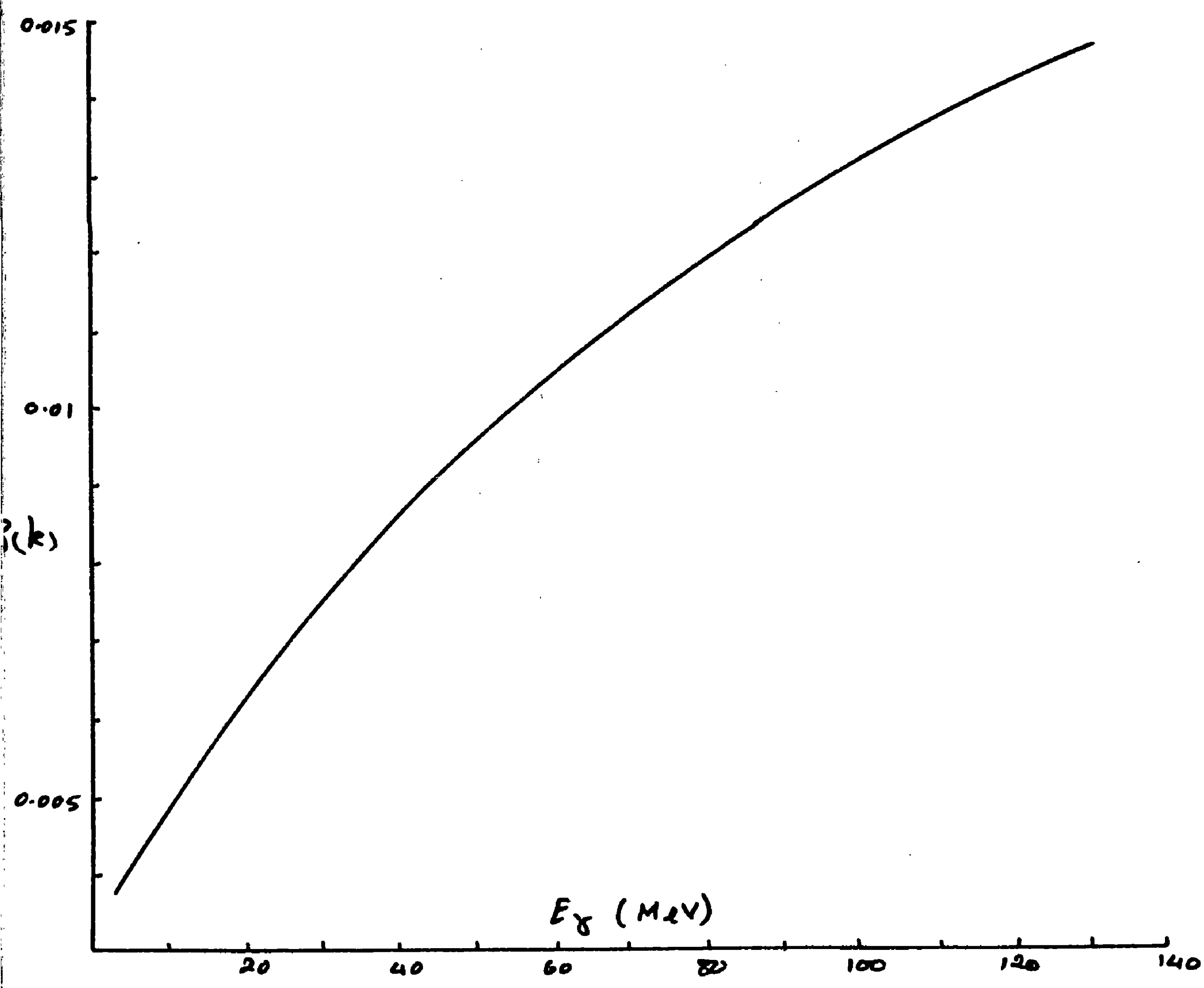
$$+ 60(\gamma_p^2 - \gamma_n^2)\left(\frac{\alpha}{m}\right)^2 \Big\{ \frac{3}{\sqrt{2}} M_1 M_2 \cos(\Delta_1 - \Delta_2) + \frac{5}{2} M_0 M_3 \cos(\Delta_0 - \Delta_3) + \sqrt{2} M_2 M_3 \cos(\Delta_2 - \Delta_3) \Big\}$$

The dimensionless constant B(k) is given by

$$B(k) = \frac{e^2}{12} \cdot \frac{m k_0}{p} \cdot \frac{N^2}{\alpha^2} \tag{13.26}$$

and is shown in Figure 4.

Figure 4. **B(k)**



14. Polarization of the Final State Nucleons.

The theory of the polarization of the final state nucleons in deuteron photodisintegration has been given by W. Czyz and J. Sawicki (66) and by J.J. de Swart (67).

To calculate the polarization, one has to determine the reaction amplitude

$$f\chi = g\chi_0^0 + \sum_{m_s=-1}^{1} f_{m_s}\chi_1^{m_s} \qquad (14.1)$$

and use the equation

$$P^{(i)} = \frac{\{\langle f|\,\sigma^{(i)}\,|i\rangle\}_{Av}}{\{\langle f|\,i\rangle\}_{Av}} \qquad (14.2)$$

where $\{\ \}_{Av}$ indicates averaging over the initial spin and photon polarization.

Confining ourselves to the electric and magnetic dipole, and the electric quadrupole transitions, the radiative interaction operator is proportional to

$$-\,\xi\cdot v + \frac{\gamma_p-\gamma_n}{2}\, i\,(\sigma^{(1)}-\sigma^{(2)})\cdot(k\times\xi) - i\,\frac{k_0}{4}\,\xi\cdot v\; k\cdot r \qquad (14.3)$$

(see equations (12.6), (12.8) and (12.10)).

Choosing the quantization axis along

$$n = \frac{p\times k}{|p\times k|} \qquad (14.4)$$

and defining ϕ by

$$\cos\phi = \underset{\sim}{\xi} \cdot \underset{\sim}{n} \tag{14.5}$$

(14.3) may be written as

$$\begin{aligned}&- \left[z + i \frac{k_0}{4} zy - \frac{\gamma_p - \gamma_n}{2} (\sigma_z^{(1)} - \sigma_x^{(2)}) \right\} \cos\phi \\ &- \left[x + i \frac{k_0}{4} xy + \frac{\gamma_p - \gamma_n}{2} (\sigma_z^{(1)} - \sigma_z^{(2)} \right\} \sin\phi\end{aligned} \tag{14.6}$$

f_{m_s} and g may be expressed in terms of $\cos\phi$ and $\sin\phi$ namely

$$f_{m_s} = f_{m_s}^z \cos\phi + f_{m_s}^x \sin\phi, \quad g = g^z \cos\phi + g^x \sin\phi \tag{14.7}$$

In the frame of reference chosen, P_x and P_y vanish. On substitution of (14.1), (14.6) and (14.7) into (14.2) we get

$$P = P_z = \frac{\sum_{m_s} \left[Re(g^{x*} f_0^*) + 2 Re(g^{z*} f_0^x) + |f_1^x|^2 - |f_{-1}^x|^2 \right]}{\sum_{m_s} \left(|f_{m_s}^x|^2 - |f_{m_s}^z|^2 \right) + |g_x|^2 + |g_z|^2} \tag{14.8}$$

Evaluating this expression gives finally

$$\left(\frac{d\sigma}{d\Omega}\right) P(\phi) = \sin\theta \left[\gamma_0(\phi) + \gamma_1(\phi)\cos\theta + \gamma_2(\phi)\cos^2\theta \right] \tag{14.9}$$

where

$$\begin{aligned}\gamma_0(\phi) = \frac{1}{16} B(k) \Big\{ &\frac{\alpha}{m} (\gamma_p - \gamma_n) \left[4 M_0 E_{11} \sin(\delta_{11} - \Delta_0) \right. \\ &+ \sqrt{2} M_2 E_{11} \sin(\delta_{11} - \Delta_2) - 3\sqrt{2}\, \xi_2 M_2 E_{12} \sin(\delta_{12} - \Delta_2) \Big\}\end{aligned}$$

$$+\frac{1}{90}\frac{k_0}{\alpha}\Big[2E_{10}\Big\{3\sqrt{2}\,E_{01}\sin(\delta_{01}-\delta_{10})-3E_{21}\sin(\delta_{21}-\delta_{10})$$

$$-5E_{22}\sin(\delta_{22}-\delta_{10})+8E_{23}\sin(\delta_{23}-\delta_{10})\Big\}$$

$$+9E_{11}\Big\{-\sqrt{2}\,E_{01}\sin(\delta_{01}-\delta_{10})-2E_{21}\sin(\delta_{21}-\delta_{11})+2E_{23}\sin(\delta_{23}-\delta_{11})\Big\}$$

$$+E_{12}\Big\{-3\sqrt{2}(3\xi_2+4\eta_2)E_{01}\sin(\delta_{01}-\delta_{12})-20\eta_2\,E_{22}\sin(\delta_{22}-\delta_{12})$$

$$-6(3\xi_2+2\eta_2)\sin(\delta_{21}-\delta_{12})+2(9\xi_2+16\eta_2)E_{23}\sin(\delta_{23}-\delta_{12})\Big\}$$

$$+3E_{23}\Big\{-3\sqrt{2}(2\xi_2-\eta_2)E_{01}\sin(\delta_{01}-\delta_{32})+6(\xi_2+\eta_2)E_{21}\sin(\delta_{21}-\delta_{32})$$

$$+10\xi_2\,E_{22}\sin(\delta_{22}-\delta_{32})-2(8\xi_2-3\eta_2)E_{23}\sin(\delta_{23}-\delta_{32})\Big\}\Big]\Big\}$$

$$\gamma_1(p)=\frac{1}{12}B(k)\Big\{\Big[6\eta_2\,E_{11}E_{12}\sin(\delta_{12}-\delta_{11})-4\xi_2\,E_{10}E_{12}\sin(\delta_{10}-\delta_{12})$$

$$+9\xi_2\,E_{11}E_{32}\sin(\delta_{32}-\delta_{11})+(2\eta_2^2+3\xi_2^2)E_{12}E_{32}\sin(\delta_{12}-\delta_{32})$$

$$+4\eta_2\,E_{10}E_{32}\sin(\delta_{10}-\delta_{32})\Big]$$

$$+\frac{1}{30}\frac{k_0}{\alpha}\Big[2E_{10}\Big\{-3\sqrt{2}\sin(\delta_{01}-\delta_{10})+3E_{21}\sin(\delta_{21}-\delta_{10})$$

$$-5E_{32}\sin(\delta_{22}-\delta_{10})+2E_{23}\sin(\delta_{23}-\delta_{10})\Big\}$$

$$+2E_{12}\Big\{3\sqrt{2}\,\eta_2\,E_{01}\sin(\delta_{01}-\delta_{12})-3\eta_2\,E_{21}\sin(\delta_{21}-\delta_{12})$$

$$+5\eta_2\,E_{22}\sin(\delta_{22}-\delta_{12})-2\eta_2\,E_{23}\sin(\delta_{23}-\delta_{12})\Big\}$$

$$+3E_{32}\Big\{3\sqrt{2}\,\xi_2\,E_{01}\sin(\delta_{01}-\delta_{32})-3\xi_2\,E_{21}\sin(\delta_{21}-\delta_{32})$$

$$+5\xi_2\,E_{22}\sin(\delta_{22}-\delta_{32})-2\xi_2\,E_{23}\sin(\delta_{23}-\delta_{32})\Big\}\Big]\Big\}$$

$$\gamma_2(p) = \frac{1}{420} B(k) \frac{k_0}{\alpha} \Big\{ E_{10} \big[20 E_{22} \sin(\delta_{22}-\delta_{10})$$

$$+ 40 E_{23} \sin(\delta_{23}-\delta_{10}) \big]$$

$$+ E_{11} \big[9\sqrt{2} E_{01} \sin(\delta_{01}-\delta_{11}) + 18 E_{21} \sin(\delta_{21}-\delta_{11})$$

$$+ 42 E_{23} \sin(\delta_{23}-\delta_{11}) \big]$$

$$+ E_{12} \big[-3\sqrt{2}(9\xi_2 + 2\eta_2) E_{01} \sin(\delta_{01}-\delta_{12})$$

$$-6(9\xi_2 + 3\eta_2) E_{21} \sin(\delta_{21}-\delta_{12}) - 50\eta_2 E_{22} \sin(\delta_{22}-\delta_{12})$$

$$+ 8(9\xi_2 + \eta_2) E_{23} \sin(\delta_{23}-\delta_{12}) \big]$$

$$+ E_{32} \big[3\sqrt{2}(\xi_2 + 2\eta_2) E_{01} \sin(\delta_{01}-\delta_{32})$$

$$+ 24\xi_2 E_{21} \sin(\delta_{21}-\delta_{32}) + 4\xi_2 E_{22} \sin(\delta_{22}-\delta_{32})$$

$$+ 2(19\xi_2 - 9\eta_2) E_{23} \sin(\delta_{32}-\delta_{23}) \big] \Big\}$$

15. Photodisintegration of the Deuteron up to 130 MeV.

Before embarking on a discussion of the present calculations, we give a brief summary of the results obtained previously.

The total cross-section and angular distribution for the photodisintegration of the deuteron have been known for some time to show reasonably satisfactory agreement with theoretical calculation up to photon laboratory energies of 10 MeV or so [12, 68-71], on applying the Siegert [72] theorem for the electric dipole transitions. Until recently, however, theoretical work on the differential cross-section for photon laboratory energies between 20 and 150 MeV failed to account for the observed angular distribution, particularly the large isotropic component [63,64,73,74,75]. With purely central forces acting between the neutron and proton, the electric dipole term is a pure $\sin^2\theta$, and the magnetic dipole contribution to the isotropic component was found to be small compared to the experimental value. It was not until a more sophisticated potential with tensor and spin-orbit forces was considered (which allows an electric dipole contribution to the isotropic term) that reasonable agreement with experiment was obtained. The calculation of J.J. de Swart and R.E.

Marshak [48] showed clearly the importance of the deuteron D-state and the final $^3F_2$ state. Following on this, several papers were published [49-59] in which coupling [50,54,55,57,58,59], higher radiation multipoles [52,53,54,55,57,58,59] and retardation [50,57,58] were considered. Up to energies of 130 MeV, it is generally accepted that retardation effects are of little importance [50,58,59], but recently M. Matsumoto [76] has reported that they are essential at energies above 80 MeV. The angular distribution parameters appear to be sensitive to the percentage of deuteron D-state chosen. In the calculations of references 48, 49, 50 and 55, a D-state percentage of 6.7 was required to enable theory to fit the experimental data, although references 49, 52 and 58 obtain reasonably good agreement with experiment employing a deuteron with a 4% D-state. However, in the most recent calculations, in which good agreement with experiment is obtained up to photon energies of 150 MeV, M.L. Rustgi et al.[59] employ a modified Signell-Marshak potential, which gives a deuteron D-state percentage of 6.9.

Polarization calculations have been carried out in references 51, 55, 59 and 66. As yet there are no experimental data with which to compare the theoretical predictions.

With the exception of A.F. Nicholson and G.E. Brown [50] and of G. Kramer [56], calculations have all been carried out using Signell-Marshak phase-shifts. Nicholson and Brown use Gammel-Thaler phase-shifts for an electric dipole calculation at 130 MeV and Kramer considers electric dipole transitions at four energies in order to compare different sets of phase shifts, including Signell-Marshak and Gammel-Thaler phase shifts. Both show that the Gammel-Thaler solution is capable of reproducing the folded angular distribution, but no detailed analysis is made. In view of the recent phase-parameter analyses by G. Breit et al. [44,45] it is of interest to carry out a more detailed analysis of deuteron photodisintegration using their best solution, which is a phase-shift set of the Gammel-Thaler type, and to compare and contrast the results with the Signell-Marshak solutions.

This is done for two different D-state percentages - namely 4 and 6. The deuteron wave-functions used are purely phenomenological, of the Hulthin-Sugawara type (see Appendix 3) i.e.

$$\psi(x) = \frac{N}{\sqrt{4\pi}} \left\{ \frac{u_g(r)}{r} + \frac{S_{12}}{\sqrt{8}} \frac{w_g(r)}{r} \right\} \chi_m \qquad (15.1a)$$

where

$$u_g(r) = \cos\epsilon_g \left[1-e^{-\beta(x-x_c)}\right] e^{-x} \qquad (15.1b)$$

$$w_g(r) = \sin\epsilon_g \left[1-e^{-\gamma(x-x_c)}\right]^2 e^{-x} \left[1+\frac{3(1-e^{-\gamma x})}{x}+\frac{3(1-e^{-\gamma x})^2}{x^2}\right] \qquad (15.1c)$$

and $x = \alpha r$, $x_c = \alpha r_c$ where $\alpha = 0.2316 \times 10^{13}\,\mathrm{cm}^{-1}$ and the hard-core radius is taken to be $r_c = 0.4316 \times 10^{-13}\,\mathrm{cm}$.

For the two D-state probabilities chosen, $N^2 = 7.6579 \times 10^{-12}\,\mathrm{cm}^{-1}$ and

$\beta = 7.961$ $\gamma = 3.798$ $\sin\epsilon_g = 0.02666$ for 4% D-state (15.2a)

$\beta = 7.451$ $\gamma = 4.799$ $\sin\epsilon_g = 0.02486$ for 6% D-state (15.2b)

Wherever possible, phenomenological two-nucleon continuum wave-functions are used in the final state. Otherwise they are calculated from the appropriate Gammel-Thaler potential.

The Electric Dipole Transitions. These transitions

$$^3S_1 + {}^3D_1 \longrightarrow {}^3P_0,\ {}^3P_1,\ {}^3P_2 + {}^3F_2$$

are the most important transitions for the photodisintegration at medium energies, and lead to an angular distribution in the centre of momentum system of the form

$$\left(\frac{d\sigma}{d\Omega}\right)_{E1} = a_{E1} + b_{E1}\sin^2\theta \qquad (15.3)$$

The transition emplitudes E_{LS} appropriate to this case are

$$E_{10} = \int_0^\infty dr\ \bar{v}_{10}(kr)\ [u_D(r) - \sqrt{2}\ w_D(r)]\ dr \tag{15.4a}$$

$$E_{11} = \int_0^\infty dr\ \bar{v}_{11}(kr)\ [u_D(r) + \tfrac{1}{\sqrt{2}}\ w_D(r)]\ dr \tag{15.4b}$$

$$E_{12} = \cos\varepsilon_2 \int_0^\infty dr\ \bar{v}'_{12}(kr)\ [u_D(r) - \tfrac{1}{5\sqrt{2}}\ w_D(r)]\ dr + \sqrt{\tfrac{3}{2}}\ \sin\varepsilon_2\ \tfrac{3}{5}\sqrt{2} \int_0^\infty dr\ \bar{v}'_{32}(kr)\ w_D(r)\ dr \tag{15.4c}$$

$$E_{32} = \cos\varepsilon_2\ \tfrac{3}{5}\sqrt{2} \int_0^\infty dr\ \bar{v}^3_{32}(kr)\ w_D(r)\ dr - \sqrt{\tfrac{2}{3}}\ \sin\varepsilon_2 \int_0^\infty dr\ \bar{v}^3_{12}(kr)\ [u_D(r) - \tfrac{1}{5\sqrt{2}}\ w_D(r)]\ dr \tag{15.4d}$$

where the v^λ_{LS} are the final-state radial wave-functions, δ_{LS} the phase-shifts and ε_2 the $^3P_2 - {}^3F_2$ coupling parameter. To evaluate the amplitudes (15.4) exactly it is necessary to know the radial wave-functions, which in turn requires a knowledge of the potential acting in the triplet odd-parity states. However it has been shown (48) that a very good approximation to the triplet odd parity wave-functions is

$$v^\lambda_{LS} = kr \begin{cases} \cos\delta_{\lambda S}\ j_L(kr) - \sin\delta_{\lambda S}\ n_L(kr) & , r \geq R \\ \cos\delta_{\lambda S}\ j_L(kr) \end{cases} \tag{15.5}$$

where $j_L(x)$ and $n_L(x)$ are respectively the spherical Bessel and Neumann functions. R is taken to be 1.4129 x 10^{-13}cm. and the hard-core radius r_c to be 0.4316 x 10^{-13}cm.

<u>The Magnetic Dipole Transitions</u>. The magnetic dipole spin-flip transitions

$$^3S_1 + {}^3D_1 \longrightarrow {}^1S_0,\ {}^1D_2$$

lead to the angular distribution

$$\left(\frac{d\sigma}{d\Omega}\right)_{M1} = a_{M1} + b_{M1} \sin^2\theta \tag{15.6}$$

The appropriate transition amplitudes M_J are

$$M_0 = \int_0^\infty v_0(kr)\ u_D(r)\, dr \tag{15.7a}$$

$$M_2 = \int_0^\infty \bar{v}_2(kr)\ w_D(r)\, dr \tag{15.7b}$$

In this case, the potential acting in the final state is too strong for the radial wave-functions $v_J(kr)$ to be approximated by (15.5). Accordingly the wave-functions have been obtained by solving the Schrodinger equation using the Gammel-Thaler potential

$$V(r) = \begin{cases} +\infty & , r < r_c \\ -V_0 \dfrac{e^{-\mu r}}{\mu r} & r > r_c \end{cases} \tag{15.8}$$

with V_0 = 425.5 MeV, $\mu = 1.45 \times 10^{13} cm^{-1}$ and $r_c = 0.4 \times 10^{-13}$ cm.

This potential gives a good fit to the Breit et al. $^1S_0$ phase-shifts, but the fit to the $^1D_2$ phase-shifts is rather poorer.

The magnetic dipole triplet transitions

$$^3S_1 + {}^3D_1 \longrightarrow {}^3S_1 + {}^3D_1,\ {}^3D_2$$

have been shown [58] to give an entirely negligible contribution in the considered energy range, and are consequently neglected.

<u>The Electric Quadrupole Transitions</u>. The electric quadrupole transitions

$$^3S_1 + {}^3D_1 \longrightarrow {}^3S_1 + {}^3D_1,\ {}^3D_2,\ {}^3D_3 + {}^3G_3$$

are most important through their interference with the electric dipole transitions, which causes a forward asymmetry in the angular distribution. However they also contribute to both the isotropic term and to the term proportional to $\sin^2\theta$, the complete contribution being given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{E2} = a_{E2} + b_{E2}\sin^2\theta + c_{E2}\cos\theta + d_{E2}\cos\theta\sin^2\theta + e_{E2}\cos^2\theta\sin^2\theta \qquad (15.9)$$

Since the electric quadrupole transitions are second order effects, the $^3S_1$ and $^3D_1$ final states are taken as uncoupled, and the $^3G_3$ state neglected. In this approximation the relevant transition amplitudes are

$$E_{01} = -\frac{1}{\sqrt{2}}\int_0^\infty (\alpha r)^2\ \bar{v}_{01}(kr)\ w_D(r)\ dr \qquad (15.10a)$$

$$E_{21} = \int_0^\infty \sigma_2(kr) \left[u_D(r) - \frac{1}{\sqrt{2}} w_D(r) \right] (kr)^2 \, dr \qquad (15.10b)$$

$$E_{22} = \int_0^\infty \sigma_{22}(kr) \left[u_D(r) + \frac{1}{\sqrt{2}} w_D(r) \right] (kr)^2 \, dr \qquad (15.10c)$$

$$E_{23} = \int_0^\infty \sigma_{23}(kr) \left[u_D(r) - \frac{\sqrt{2}}{4} w_D(r) \right] (kr)^2 \, dr \qquad (15.10d)$$

If the approximation (15.5) is a good one for the electric dipole amplitudes, it should be an even better one for the electric quadrupole amplitudes. Accordingly approximation (15.5) is made in evaluating equations (15.10).

The Magnetic Quadrupole Transitions. We retain only the magnetic quadrupole singlet (spin-flip) transitions

$$^3S_1 + {}^3D_1 \longrightarrow {}^1P_1, \; {}^1F_3$$

which interfere with the magnetic dipole transitions, contributing to the parameters c and d, the effect on parameters a, b and ℓ being negligible. The appropriate transition amplitudes

$$M_1 = \int_0^\infty (kr) \, \sigma_1(kr) \left[u_D(r) - \frac{\sqrt{2}}{10} w_D(r) \right] dr \qquad (15.11a)$$

$$M_3 = \frac{3}{5}\sqrt{2} \int_0^\infty kr \; \sigma_3(kr) \, w_D(r) \, dr \qquad (15.11b)$$

are evaluated using the approximation (15.5) for σ_1 and σ_3

This can be justified in this case by the (αr) term in the integrand, which enhances considerably the contribution of the "outside" region of the final state wave-function, at the expense of the "inside".

The magnetic quadrupole triplet transitions

$$^3S_1 + {}^3D_1 \longrightarrow {}^3P_1, \; {}^3P_2 + {}^3F_2$$

are neglected.

Amplitudes (15.4), (15.7), (15.10) and (15.11) have been evaluated using the continuum wave-functions discussed above, the deuteron wave-functions (15.1) and the Y.L.A.M. phase-shifts of Breit et al.[44,45], which have been given in Table 1. The results of this calculation are given in Table 3 and are compared with the corresponding Signell-Marshak solutions in Table 4. As is to be expected, the general behaviour of both sets of amplitudes is similar. The most noticeable difference between the two is the very strong enhancement of E_{32} in the Gammel-Thaler solution at low energies. This is due to the large negative value of the coupling parameter ε_2 at these energies. Conversely, at higher energies the Gammel-Thaler coupling parameter becomes numerically smaller than the Signell-Marshak coupling parameter, with the consequence that E_{32} in the

Gammel-Thaler solution becomes smaller than that of the Signell-Marshak solution. As we shall see, this has important repercussions on the angular distribution parameters. The other important difference is that M_2 in the Gammel-Thaler solution is smaller than M_2 in the Signell-Marshak solution, which again has an important bearing on the angular distribution parameters, particularly at higher energies.

The Angular Distribution. When we have unpolarized radiation, then in the approximations made, we can write the angular distribution as

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = a(1 \pm \beta_1 \cos\theta) + b\sin^2\theta(1 \pm \beta_2 \cos\theta) \tag{15.12}$$

$$a + b\sin^2\theta + c\cos\theta + d\cos\theta\sin^2\theta$$

where a, b, c and d are given by equations (13.22) to (13.25) inclusive.

We can write $a = a_e + a_m$, $b = b_e + b_m$, $c = c_e + c_m$, $d = d_e + d_m$. Here a_e and b_e are the contributions from the E1 transitions, a_m and b_m are the contributions from the M1 spin-flip transitions, c_e and d_e arise from the E1-E2 interference, and c_m and d_m arise from the M1-M2 interference.

a, b, β_1(= c/a) and β_2(= d/b) have been calculated using the Y.L.A.M. set of phase parameters of Breit et al., and the transition amplitudes of Table 3. The results, together with the isotropy factor α = a/b, and the total cross-section σ_T are given in Table 5, and comparison with the results using Signell-Marshak phase shifts made in Table 6. The total cross-sections are given in Figure 5, the ratio a/b in Figure 6 and the quantities β_1 and β_2 in Figure 7. In this latter case, the experimental points are plotted as β obtained from a best fit of the experimental data to the formula

$$\left(\frac{d\sigma}{d\Omega}\right)_b = (a + b\sin^2\theta)(1 \pm \beta\cos\theta) \qquad (15.13)$$

The considerable increase of a_e at low energies in the present calculations, compared to the value obtained using Signell-Marshak phase shifts, can be attributed directly to the enhancement of E_{32} discussed above. It so happens that a_e depends almost entirely on the terms involving E_{32}, the other terms almost cancelling. On the other hand, as the energy increases, the reverse situation holds i.e. a_e calculated from Signell-Marshak phase-shifts becomes greater than that obtained in the present calculations.

also differs considerably in the two cases at higher energies. This is a direct consequence of the smaller value for M_2 obtained in the present calculations compared to that obtained with Signell-Marshak phase-shifts. The values of and b_m in all cases are very similar. As a result of this the ratio a/b found in the present calculations is greater than that of the Signell-Marshak results at the lower energies, but less at the higher.

At energies below 70 MeV, the best fit to the isotropy ratio is obtained by the present calculations with a 4%

D-state. This, however, does not give a reasonable fit at all above this energy. The best fit above this energy is given by the Signell-Marshak results, but they, in turn, give too small a value at energies below 50 MeV. The best fit over the whole range is given by the present calculations using a 6% D-state, which lies intermediate to the other two results. The total cross-section obtained with the present calculations with a 4% D-state is too small above 80 MeV but the other solutions fit reasonably well up to 130 MeV.

The parameters β_1, β_2 differ in the cases considered, but experimental accuracy is not nearly sufficient for any conclusions to be drawn.

Table 5. Differential Cross-Section for Unpolarised Photons.

A. 4% D-state.

E_γ	5	10	15	20	30	40	50	60	70	90	110	130
a_e	0.869	3.548	4.753	5.336	5.907	5.928	5.592	4.976	4.318	3.468	2.907	2.423
a_m	6.517	2.369	0.812	0.325	0.141	0.072	0.036	0.018	0.007	0.008	0.009	0.010
a	7.386	5.917	5.565	5.661	6.048	6.000	5.628	4.994	4.325	3.476	2.916	2.433
b_e	206.7	149.2	104.0	57.3	29.0	17.2	10.93	7.26	4.86	2.638	1.542	0.953
b_m	0.253	0.386	0.468	0.536	0.599	0.611	0.598	0.567	0.533	0.493	0.428	0.341
b	208.0	149.6	104.5	57.8	29.6	17.8	11.53	7.83	5.39	3.131	1.970	1.294
a/b	0.036	0.040	0.053	0.098	0.204	0.336	0.488	0.638	0.802	1.11	1.48	1.88
β_1	0.02	0.03	0.05	0.06	0.08	0.09	0.11	0.14	0.16	0.20	0.21	0.22
β_2	0.17	0.18	0.20	0.24	0.35	0.42	0.50	0.58	0.65	0.70	0.73	0.75

E_γ in MeV; a, a_e, a_m; b, be, b_m in μb/steradian.

D. 6% D-state.

E_γ	5	10	15	20	30	40	50	60	70	90	110	130
a_e	0.019	1.817	3.227	4.193	5.401	5.597	5.403	4.877	4.326	3.774	3.498	3.380
a_m	7.362	2.835	0.740	0.390	0.091	0.044	0.019	0.020	0.031	0.050	0.057	0.061
a	7.381	4.652	3.967	4.583	5.492	5.641	5.422	4.897	4.357	3.782	3.555	3.441
b_e	206.5	148.7	101.9	53.0	29.66	17.39	10.51	6.51	4.131	2.230	1.277	0.845
b_m	0.243	0.376	0.490	0.586	0.684	0.711	0.718	0.710	0.689	0.635	0.604	0.582
b	206.7	149.1	102.4	53.9	30.34	18.08	11.25	7.22	4.820	2.865	1.881	1.427
a/b	0.030	0.031	0.039	0.085	0.181	0.312	0.483	0.678	0.904	1.32	1.89	2.41
β_1	0.03	0.05	0.06	0.07	0.09	0.10	0.12	0.14	0.16	0.19	0.19	0.20
β_2	0.17	0.18	0.19	0.24	0.33	0.41	0.49	0.57	0.64	0.68	0.71	0.72

E_γ in MeV. a, a_e, a_m; b, b_e, b_m in μb/steradian.

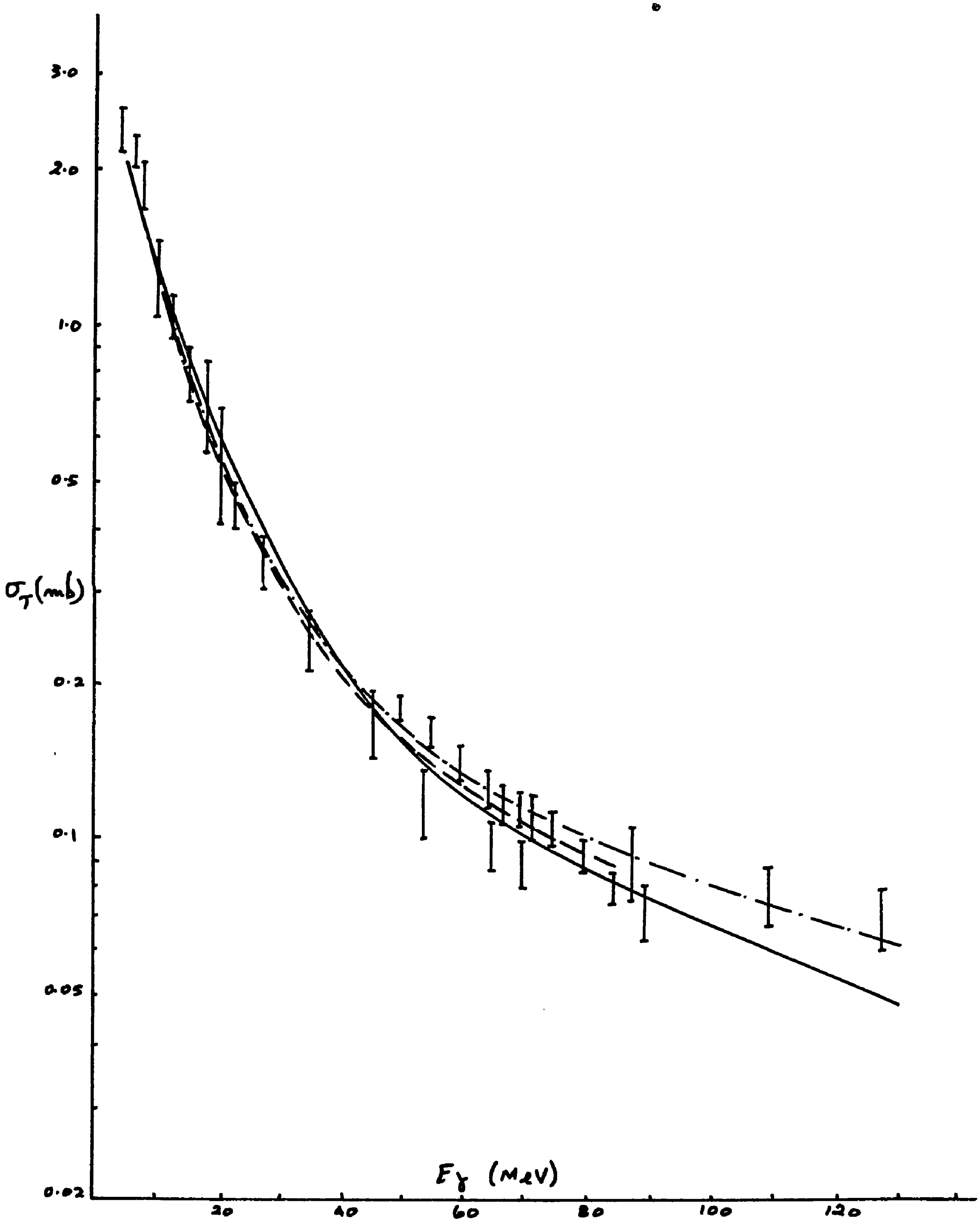
Table 6. Comparison of Gammel-Thaler and Signell-Marshak Angular Distributions.

E_γ	11.23					22.24					39.76				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
a	5.91	-	4.22	-	-	5.62	4.51	4.46	-	5.01	6.00	5.64	4.98	-	5.68
b	140	-	135.7	-	-	57.8	53.9	49.4	-	52.4	17.8	13.08	16.5	-	19.8
a/b	0.04	-	0.031	-	-	0.098	0.095	0.09	-	0.09	0.34	0.31	0.30	-	0.29
β_1	0.03	-	0.05	-	-	0.06	0.07	0.10	-	0.34	0.09	0.10	0.13	-	0.40
β_2	0.17	-	0.20	-	-	0.24	0.24	0.32	-	0.31	0.42	0.41	0.47	-	0.45

E_γ	52.3					77.3				
	1	2	3	4	5	1	2	3	4	5
a	5.36	5.15	5.05	5.49	5.75	4.04	4.16	4.72	4.97	5.21
b	9.40	8.84	9.20	10.6	11.2	4.68	4.11	4.25	3.57	4.1
a/b	0.57	0.58	0.55	0.52	0.51	0.86	1.01	1.12	1.39	1.27
β_1	0.12	0.13	0.12	0.07	0.42	0.17	0.17	0.14	0.06	0.43
β_2	0.54	0.56	0.58	0.54	0.53	0.70	0.75	0.71	0.82	0.65

1. Present calculation 4% D-state.
2. Present calculation 6% D-state.
3. de Swart and Marshak. 6.7% D-state.
4. Kramer and Werntz 4% D-state.
5. Rustgi et al. 6.1% D-state.

Figure 5. Theoretical Total Cross-Section



Y.L.A.M. 4% D-state ———— Signell-Marshak - - - -

Y.L.A.M. 6% D-state —·—

Figure 6. Theoretical Isotropy Ratio

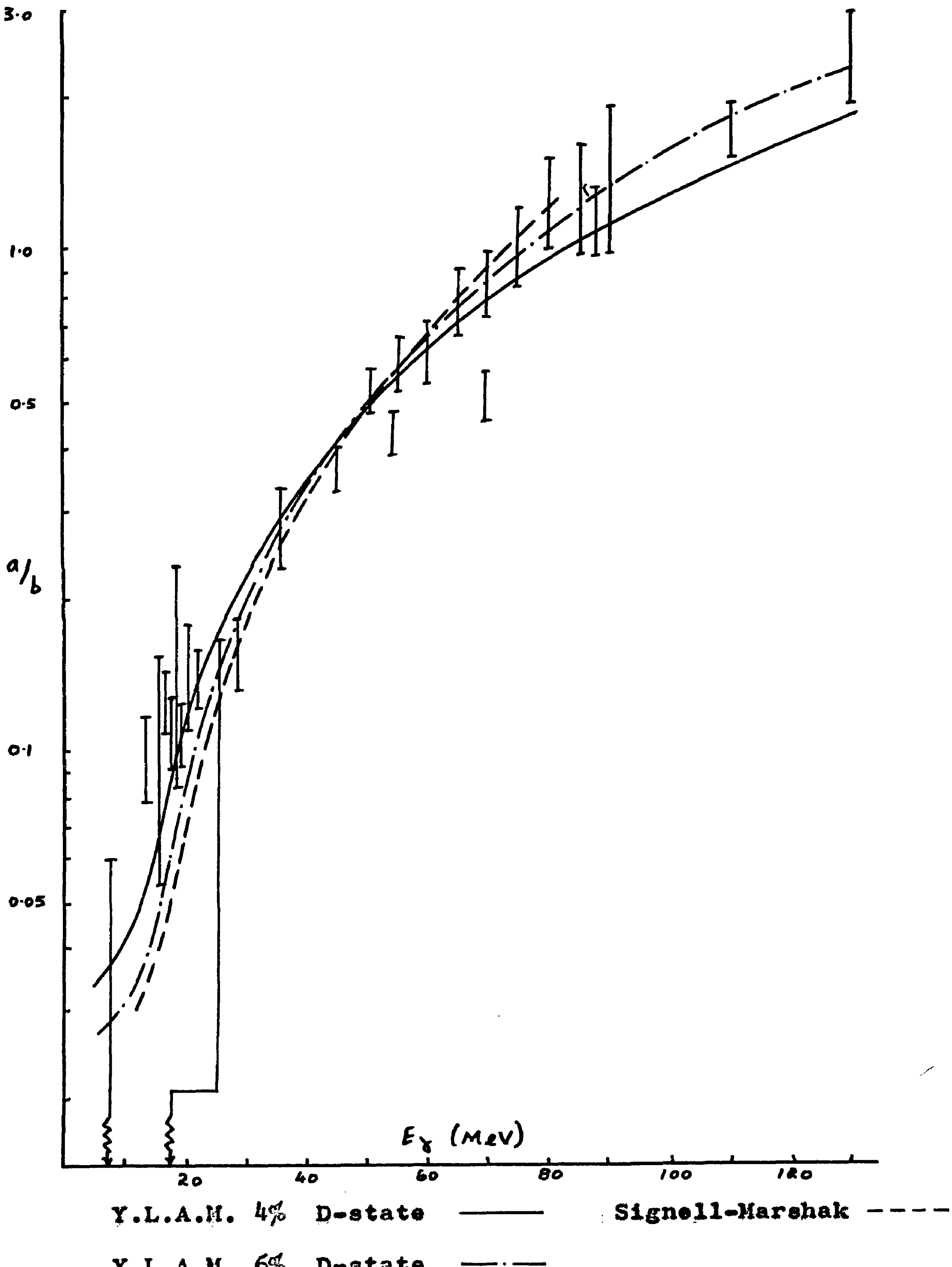
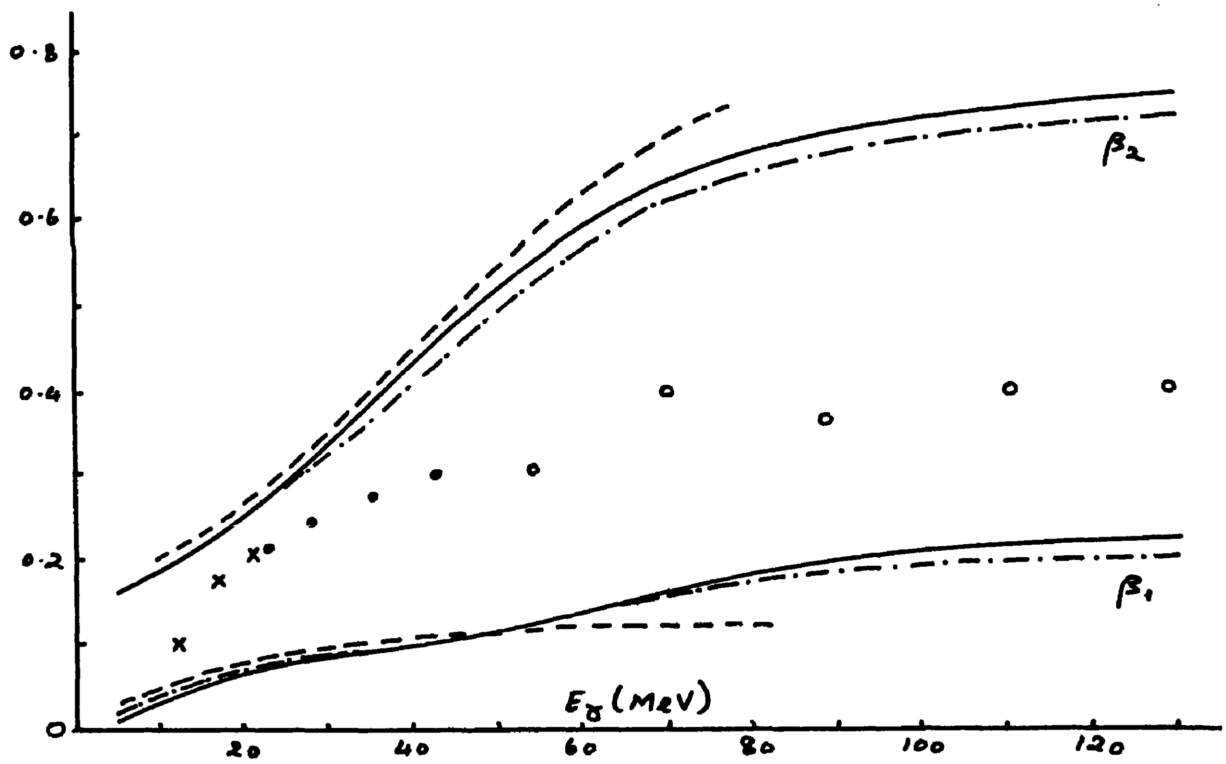


Figure 7. Theoretical β_1 and β_2



Y.L.A.M. 4% D-state ————

Y.L.A.M. 6% D-state —·—

Signell-Marshak - - - -

Linearly Polarized Radiation. When we have partially or totally linearly polarized radiation, the cross-section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(1 + \Sigma_L \Sigma(\theta) \cos 2\chi\right) \tag{15.14}$$

where Σ_L is the degree of linear polarization and χ is the angle between the plane of linear polarization and the azimuthal angle of observation. The function $\Sigma(\theta)$ is

$$\left(\frac{d\sigma}{d\Omega}\right)_0 \Sigma(\theta) = r \sin^2\theta \, (1 \pm \rho \cos\theta) \tag{15.15}$$

with

$$r = b_e - b_m, \quad r\rho = b_{\beta_2} = d \tag{15.16}$$

To compare theory with experiment, the most convenient quantity is

$$s = \frac{b_m}{b_e} = \frac{b - r}{b + r} \tag{15.17}$$

The values of r, ρ and s obtained in the present calculations are given in Table 7, and compared in Table 8 with the corresponding values obtained from the results using Signell-Marshak phase parameters.

Table 7. **Parameters for Polarized Photons**

A 4% D-state

E_γ	5	10	15	20	30	40	50	60	70	90	110	130
σ	206.4	148.8	103.5	56.8	28.4	16.6	10.63	6.69	4.33	2.165	1.114	0.612
ρ	0.17	0.18	0.20	0.24	0.36	0.45	0.54	0.68	0.81	1.01	1.29	1.58
s	0.0012	0.0026	0.0045	0.0093	0.021	0.036	0.055	0.078	0.109	0.187	0.278	0.358

E_γ in MeV; σ in μb/steradian

B 6% D-state

E_γ	5	10	15	20	30	40	50	60	70	90	110	130
σ	206.3	148.3	101.4	52.4	29.0	16.7	9.79	5.80	3.44	1.595	0.673	0.263
ρ	0.17	0.18	0.19	0.25	0.35	0.44	0.56	0.71	0.89	1.62	1.98	3.91
s	0.0012	0.0025	0.0048	0.011	0.023	0.041	0.068	0.109	0.167	0.285	0.472	0.689

E_γ in MeV; σ in μb/steradian

Table 8. Comparison of Gammel-Thaler and Signell-Marshak Parameters for Polarized Photons.

E_γ	11.23			22.24			39.76		
	1	2	3	1	2	3	1	2	3
σ	136.4	135.2	134.7	47.6	45.3	48.0	16.6	16.7	15.1
ρ	0.18	0.18	0.20	0.27	0.28	0.33	0.45	0.44	0.51
s	0.0031	0.0032	0.0038	0.012	0.014	0.0135	0.036	0.041	0.047

E_γ in MeV; σ in μb/steradian.

E_γ	52.3			77.3		
	1	2	3	1	2	3
σ	9.58	8.56	7.76	3.626	2.77	3.07
ρ	0.58	0.60	0.68	0.89	1.19	0.98
s	0.061	0.081	0.085	0.126	0.193	0.161

E_γ in MeV; σ in μb/steradian.

1. Present calculation 4% D-state
2. Present calculation 6% D-state
3. de Swart and Marshak(55) 6.7% D-state

Polarization of the Nucleons.

The polarization of the outgoing nucleons for unpolarized radiation in the direction $\underset{\sim}{n} = \frac{(\underset{\sim}{k} \times \underset{\sim}{k})}{|\underset{\sim}{k} \times \underset{\sim}{k}|}$ is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \sin\theta \left[\gamma_0 + \gamma_1 \cos\theta + \gamma_2 \cos^2\theta \right] \qquad (15.18)$$

where, for the proton polarization, γ_0, γ_1 and γ_2 are given by equations (14.10), (14.11) and (14.12). These coefficients are calculated for the different energies, and are given in Table 9. Comparison with earlier results is given in Table 10.

The differences between the present calculations and previous calculations are much more marked in the polarization parameters than in the cross-section parameters. Unfortunately at the moment there exists no satisfactory experimental results with which to compare the theoretical values.

Table 9. Parameters for Polarization of the Protons.

A. 4% D-state.

E_γ	10	20	30	40	50	60	70	80
γ_0	-15.56	-5.96	-3.42	-2.43	-1.78	-1.37	-1.14	-1.01
γ_1	2.77	4.31	4.25	3.64	3.11	2.59	2.21	1.99
γ_2	0.26	0.37	0.43	0.44	0.44	0.45	0.45	0.44

E_γ in MeV; γ's in μb/steradian.

B. 6% D-state.

E_γ	10	20	30	40	50	60	70	80
γ_0	-14.97	-5.94	-3.49	-2.50	-1.84	-1.41	-1.18	-1.04
γ_1	3.41	4.67	4.56	3.88	3.02	2.19	1.72	1.51
γ_2	0.30	0.44	0.44	0.42	0.38	0.34	0.30	0.26

E_γ in MeV; γ's in μb/steradian.

Table 10. Comparison of Gammel-Thaler & Signell-Marshak Proton Polarization Parameters.

E_γ	11.23			22.24			39.76			52.3			77.3		
	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
γ_0	-12.12	-11.87	-11.64	-5.36	-5.35	-4.47	-2.42	-2.49	-2.22	-1.61	-1.68	-1.68	-1.06	-1.06	-1.10
γ_1	3.02	3.71	2.59	4.52	4.74	4.03	3.89	3.03	3.65	2.71	2.92	3.14	2.09	1.58	2.33
γ_2	0.28	0.33	0.40	0.39	0.45	0.69	0.44	0.42	0.84	0.45	0.37	0.89	0.45	0.40	0.91

E_γ in MeV, γ's in μb/steradian.

1. Present calculation 4% D-state
2. Present calculation 6% D-state
3. de Swart and Marshak 6.7% D-state.

Conclusions. It can be concluded that the Gammel-Thaler type phase-parameters are as suitable for a detailed analysis of deuteron photodisintegration as the Signell-Marshak phase-parameters. The best fit to the angular distribution parameters at low energies is given with a low D-state probability, namely 4%. However to obtain a reasonable fit at photon energies greater than 70 MeV it is necessary to increase this figure to at least 6%. The resulting fit at energies below 70 MeV is not so good as that obtained with the lower D-state probability, but is still fairly satisfactory.

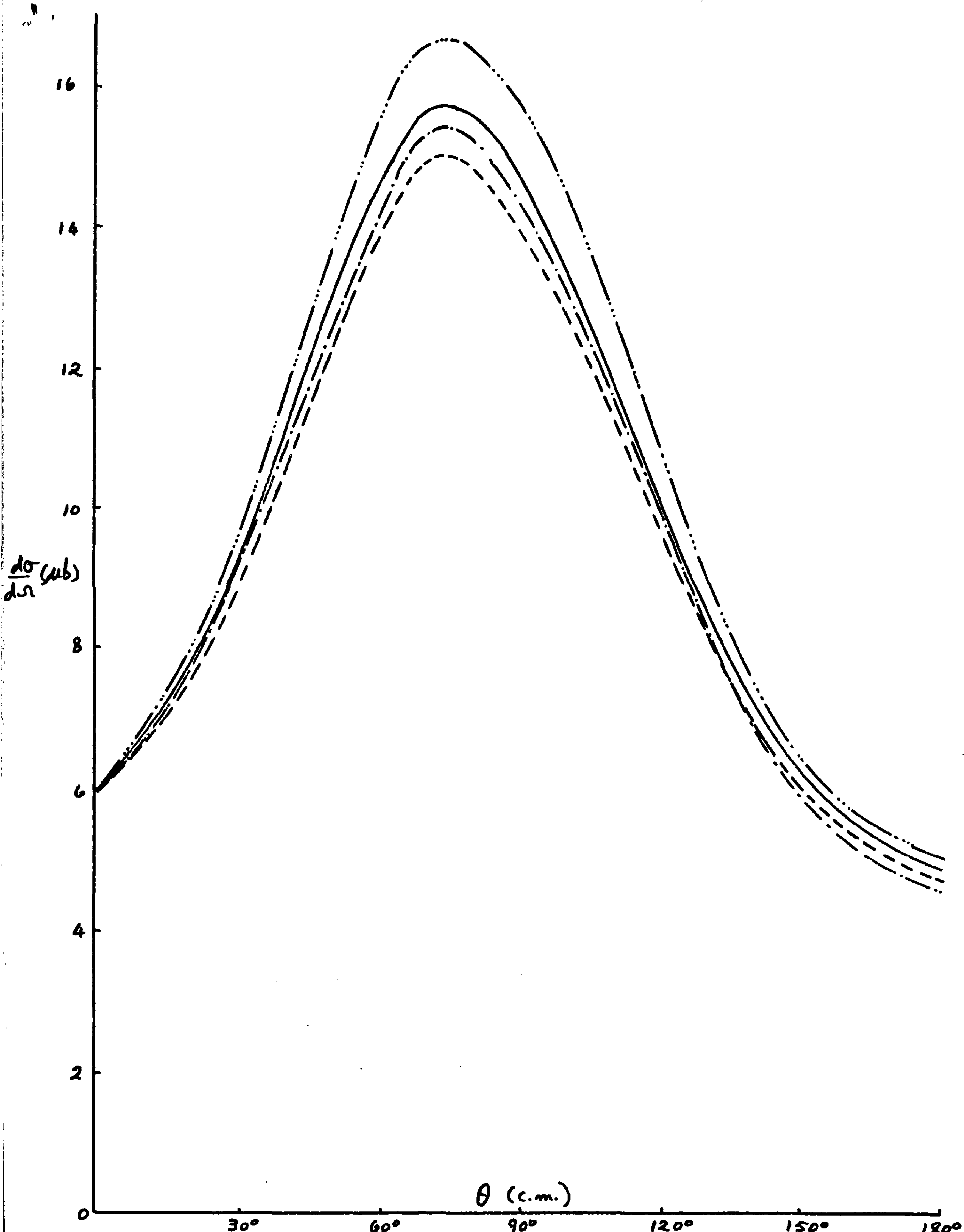
A low D-state probability is to be preferred on other grounds, in that the magnetic moment of the deuteron (equation (6.6)) can then be explained without complicated inclusion of large mesonic and relativistic corrections [12]. It may well be that retardation is of much more significance than has hitherto been supposed [50, 58, 59], as has been argued recently by M. Matsumoto [76], and that a proper inclusion of relativistic corrections would allow the angular distribution to be fitted up to 130 MeV with a low D-state probability.

The question of the correct deuteron D-state probability could well be clarified by accurate experimental angular distributions in the energy range 15-50 MeV, for there the theoretical parameters differ by up to 30% depending on the D-state probability chosen.

In common with other treatments, the M1 spin-flip transition is found to be small at low energies, which one would not expect intuitively. Accurate measurements of a/b and σ_T (and if possible γ_0) at energies up to 15 MeV would help to clarify this situation.

In theory, a complete set of measurements (angular distribution, polarization of the outgoing nucleons, angular distribution with linearly polarized photons) at one energy should suffice to settle many of the outstanding questions. To simplify the analysis, the energy chosen should be one which corresponds to one of the scattering energies at which a "unique" scattering matrix has been determined i.e. scattering energies of 68, 98, 150, 210 or 310 MeV. These correspond to photon laboratory energies of 37, 53, 78, 107 and 158 MeV respectively. 158 MeV is too high, for at this energy relativistic corrections and the inclusion of mesonic effects are necessary. 107 MeV is probably too high also, at least until the question of retardation is settled. At the other end of the scale, 37 MeV is too low for reliable polarization measurements

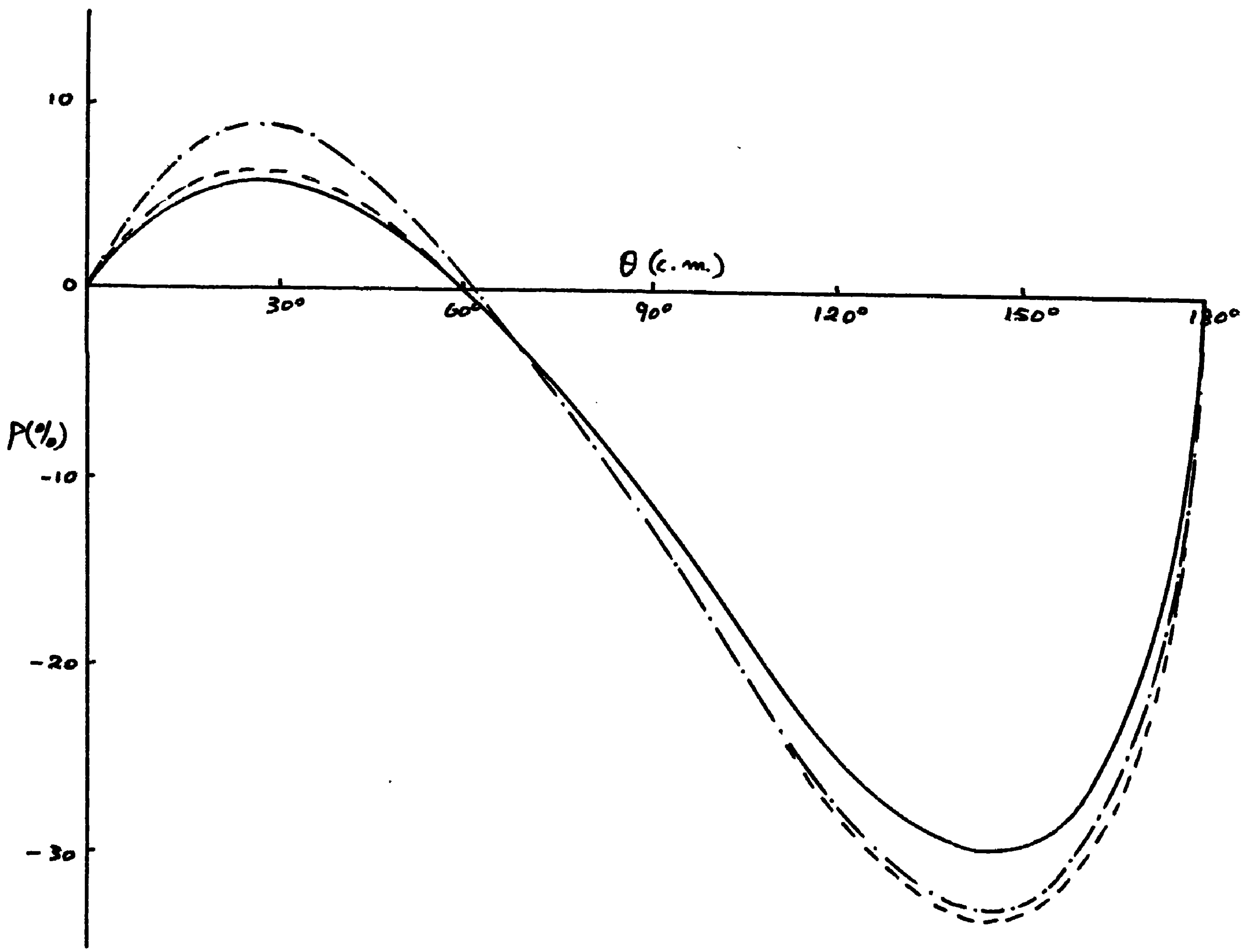
Figure 8. Theoretical Angular Distribution at 52.3 MeV



Y.L.A.M. 4% D-state ——— Signell-Marshak 6.7% D-state —·—

Y.L.A.M. 6% D-state – – – Signell-Marshak 4% D-state —··—

Figure 9. Proton Polarization at 52.3 MeV

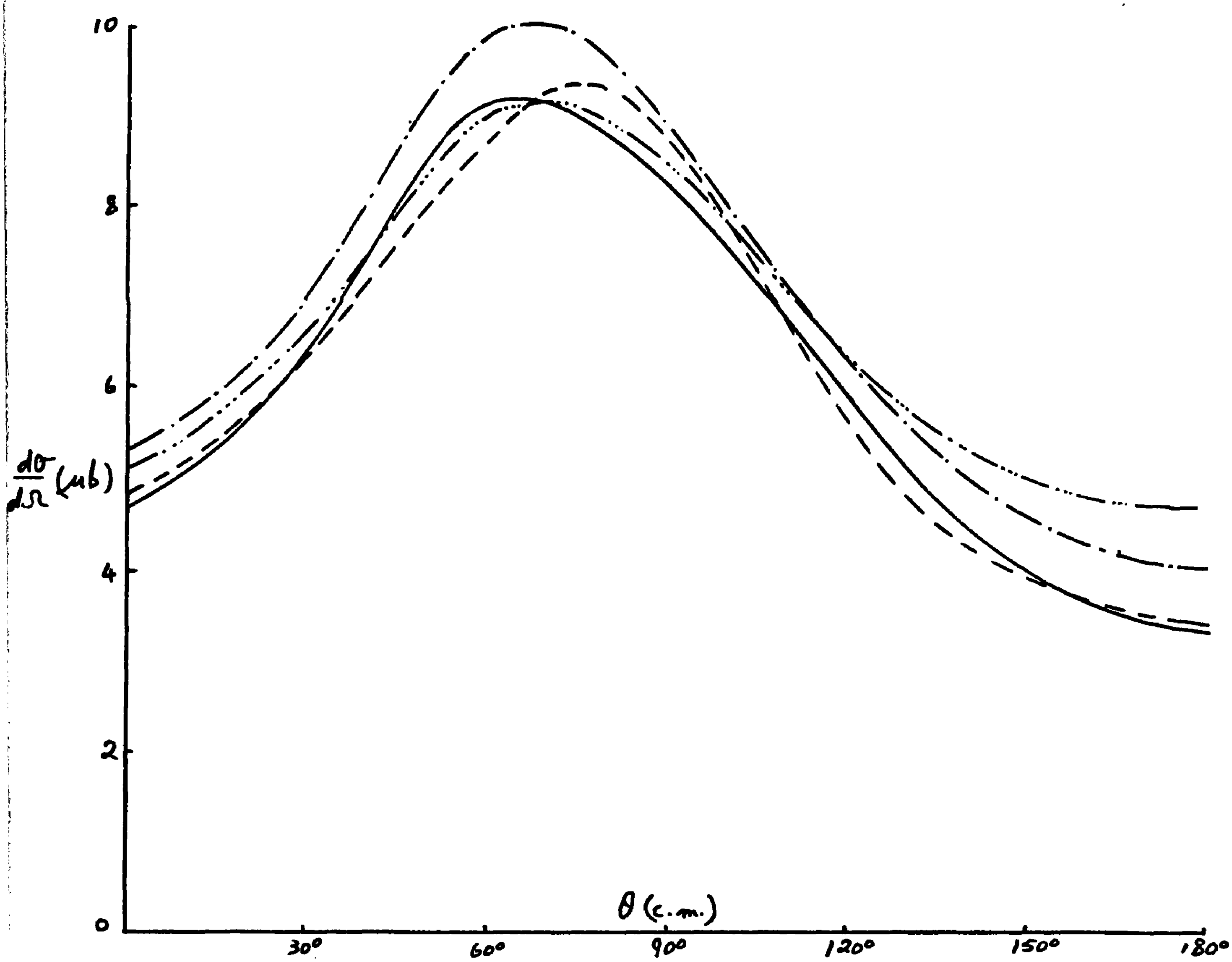


Y.L.A.M. 4% D-state ——————

Y.L.A.M. 6% D-state - - - -

Signell-Marshak 6.7% D-state —·—

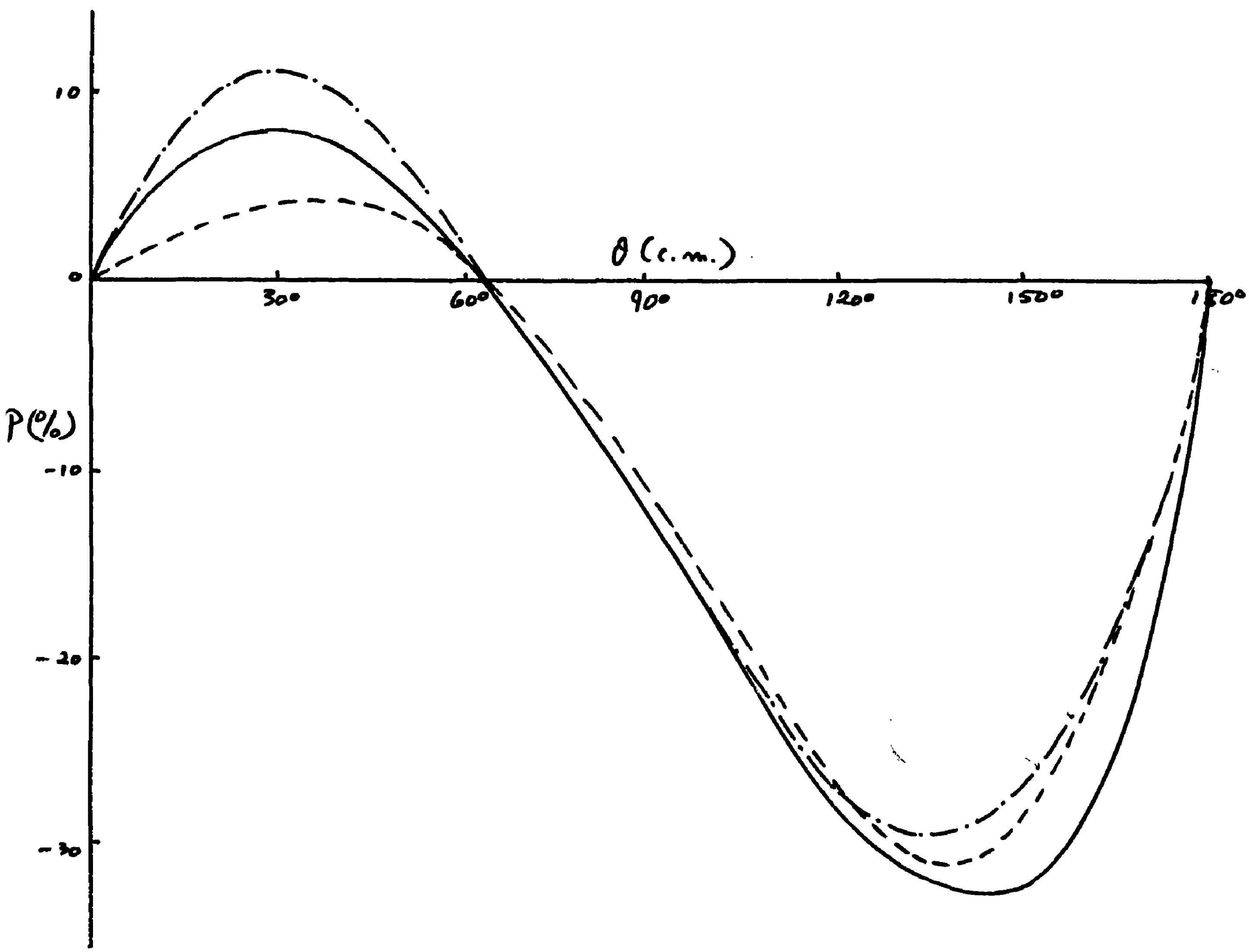
Figure 10. Theoretical Angular Distribution at 77.3 MeV



Y.L.A.M. 4% D-state ———— Signell-Marshak 6.7% D-state —·—

Y.L.A.M. 6% D-state – – – Signell-Marshak 4% D-state —···—

Figure 11. Proton Polarization at 77.3 MeV



Y.L.A.M. 4% D-state ————

Y.L.A.M. 6% D-state - - - -

Signell-Marshak 6.7% D-state —·—

16. Photodisintegration of the Deuteron above 130 MeV.

At energies above 130 MeV it is necessary to take account explicitly of virtual meson effects (4,6,62,63,77). The most successful of the several attempts made to include these effects has been given recently by L.D. Pearlstein and A. Klein (46).

A formal solution for the S-matrix of the photo-disintegration of the deuteron defined by

$$S_{\alpha\beta} = {}_{out}\langle p_1 s_1 ; p_2 s_2 | d, s ; k, \nu \rangle_{in} \qquad (16.1)$$

is exhibited, using the formalism for bound state problems proposed by A. Klein and C. Zemach (78), in which all quantities of interest are developed with the aid of the renormalized many-body Green's functions. To carry out an explicit evaluation it is found necessary to resort to a phenomenological procedure which relates the formally exact expression for $S_{\alpha\beta}$ to parameters available from the more fundamental phenomena of pion-nucleon scattering, photopion production and nucleon-nucleon interactions. By expanding the result in the number of mesons exchanged, and by making a series of non-relativistic approximations, neglecting the pion-pion interaction and assuming that P-wave pions are dominant both in scattering and photopion production, the expression is reduced to one in which the corrections to the conventional matrix element depend only on the amplitude for photopion production, the renormalized pion-nucleon coupling constant and the appropriate two-nucleon wave-functions. Retaining only one-meson effects, it is found that the S-matrix for photodisintegration can be written as

$$S_{\alpha\beta} = S_0(E) + S_0(M) + S_1^{Bo}(E) + S_1^{Bo}(M) + S_1^{(1)}(E) + S_1^{(1)}(M) \qquad (16.2)$$

$S_0(E)$ and $S_0(M)$ turn out to be the conventional electric and magnetic transition amplitudes. $S_1^{Bo}(E)$ and $S_1^{Bo}(M)$ arise from the Born terms of photopion production, and $S_1^{(1)}(E)$ and $S_1^{(1)}(M)$ arise from the complete amplitude for photopion production, excluding the Born terms. It is shown that $S_1^{Bo}(E)$ and $S_1^{Bo}(M)$ are negligible, and considering the magnetic dipole P-wave pion production to be dominant, (16.2) can be simplified to

$$S_{\alpha\beta} = S_0(E) + S_0(M) + S_1^{(1)}(M1) \qquad (16.3)$$

Here $S_1^{(1)}(M1)$ is given by

$$\begin{aligned} i\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{N k_0}{2(2\pi)^2}\chi_0^{0+}\Big\{ & \frac{1}{3}\frac{f^2}{\mu^2}\,(\underline{\sigma}^{(1)}\times\underline{\sigma}^{(2)})\cdot(\hat{k}\times\underline{\epsilon})\,K_{01}\,e^{-i\Delta_0} \\ & - \frac{\gamma_p-\gamma_n}{3Gm}F(k_0)\,i(\underline{\sigma}^{(1)}-\underline{\sigma}^{(2)})\cdot(\hat{k}\times\underline{\epsilon})\,K_{02}\,e^{-i\Delta_2} \\ & - \frac{f^2}{\mu^2}e^{-i\Delta_2}\Big[(\underline{\sigma}^{(1)}\times\underline{\sigma}^{(2)})\cdot\hat{p}\,\hat{p}\cdot(\hat{k}\times\underline{\epsilon}) - \tfrac{1}{3}(\underline{\sigma}^{(1)}\times\underline{\sigma}^{(2)})\cdot(\hat{k}\times\underline{\epsilon})\Big]\Big\} K_{21} \\ & - \frac{f^2}{\mu^2}e^{-i\Delta_2}\Big[3\,\underline{\sigma}^{(1)}\cdot\hat{p}\,\sigma^{(2)}\cdot\hat{p} - 1\Big\}(\underline{\sigma}^{(1)}\times\underline{\sigma}^{(2)})\cdot(\hat{k}\times\underline{\epsilon})\,K_{22} \end{aligned} \qquad (16.4)$$

$$- \frac{f^2}{\mu^2} 4 \{ (\sigma^{(1)} \times \sigma^{(2)}) . \hat{p} \hat{p} . (\hat{k} \times \epsilon) - \tfrac{1}{3} (\sigma^{(1)} \times \sigma^{(2)}) . (\hat{k} \times \epsilon) \} K_{23} e^{-i\Delta_2}$$

$$+ \frac{\gamma_p - \gamma_n}{12 m} F(k_0) \, i (\sigma^{(1)} - \sigma^{(2)}) . (\hat{k} \times \epsilon) [3 \sigma^{(1)} \hat{p} \sigma^{(2)} \hat{p} - 1] p^2 K_{24} e^{i\Delta_2} \} \chi_1^m$$

with

$$F(k_0) = 8\pi \sin \delta_{33} e^{i\delta_{33}} (k_0^2 - \mu^2)^{-3/2} - 3 k_0 \frac{f^2}{\mu^2} \qquad (16.5)$$

and the transition amplitudes K defined by

$$K_{01} = \int_0^\infty \frac{\bar{v}_0(pr)}{pr} e^{-\kappa r} \{ (1 - 2\kappa r) u_D(r) - (1 + \kappa r) \sqrt{2}\, w_D(r) \} dr \qquad (16.6)$$

$$K_{02} = \int_0^\infty \frac{\bar{v}_2(pr)}{(pr)^3} \{ 3 + 3\kappa r + \kappa^2 r^2 \} e^{-\kappa r} \sqrt{8}\, w_D(r) \qquad (16.7)$$

$$K_{21} = \int_0^\infty \frac{\bar{v}_2(pr)}{pr} \{ 1 + \kappa r \} e^{-\kappa r} u_D(r) \, dr \qquad (16.8)$$

$$K_{22} = \int_0^\infty \frac{\bar{v}_0(pr)}{pr} \kappa r \, e^{-\kappa r} \frac{1}{\sqrt{8}} w_D(r) \, dr \qquad (16.9)$$

$$K_{23} = \int_0^\infty \frac{\bar{v}_2(pr)}{pr} \{ 1 + \kappa r \} e^{-\kappa r} \frac{1}{\sqrt{8}} w_D(r) \, dr \qquad (16.10)$$

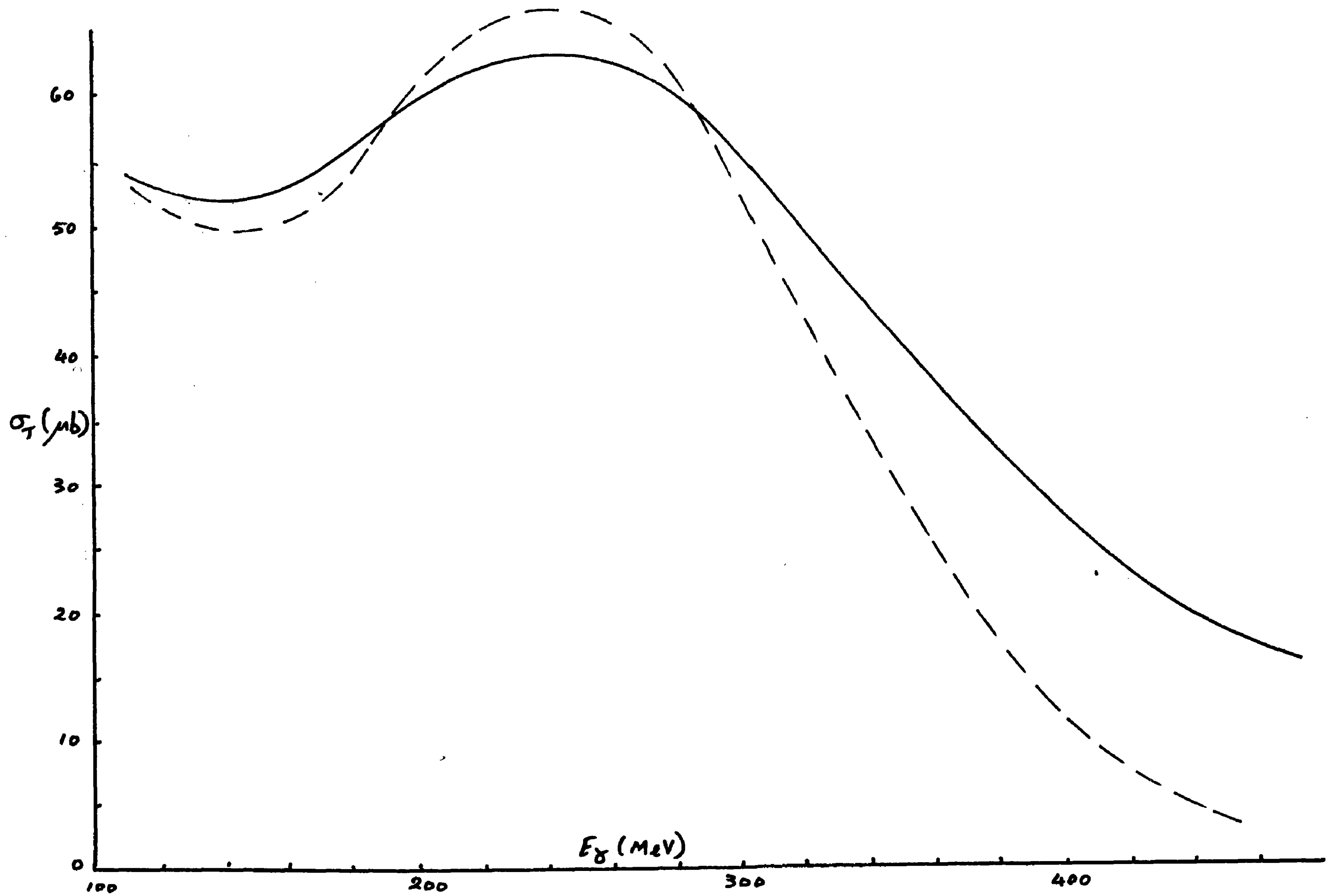
$$K_{24} = \int_0^\infty \frac{\bar{v}_2(pr)}{(pr)^3} \{ 3 + 3\kappa r + \kappa^2 r^2 \} e^{-\kappa r} \{ u_D(r) - \frac{1}{\sqrt{8}} w_D(r) \} dr \qquad (16.11)$$

with

$$\kappa = (\mu^2 - \tfrac{1}{4} k_0^2)^{\frac{1}{2}}, \ \tfrac{1}{2} k_0 < \mu \ ; \ \kappa = - i |\kappa| \ \ \tfrac{1}{2} k_0 > \mu$$

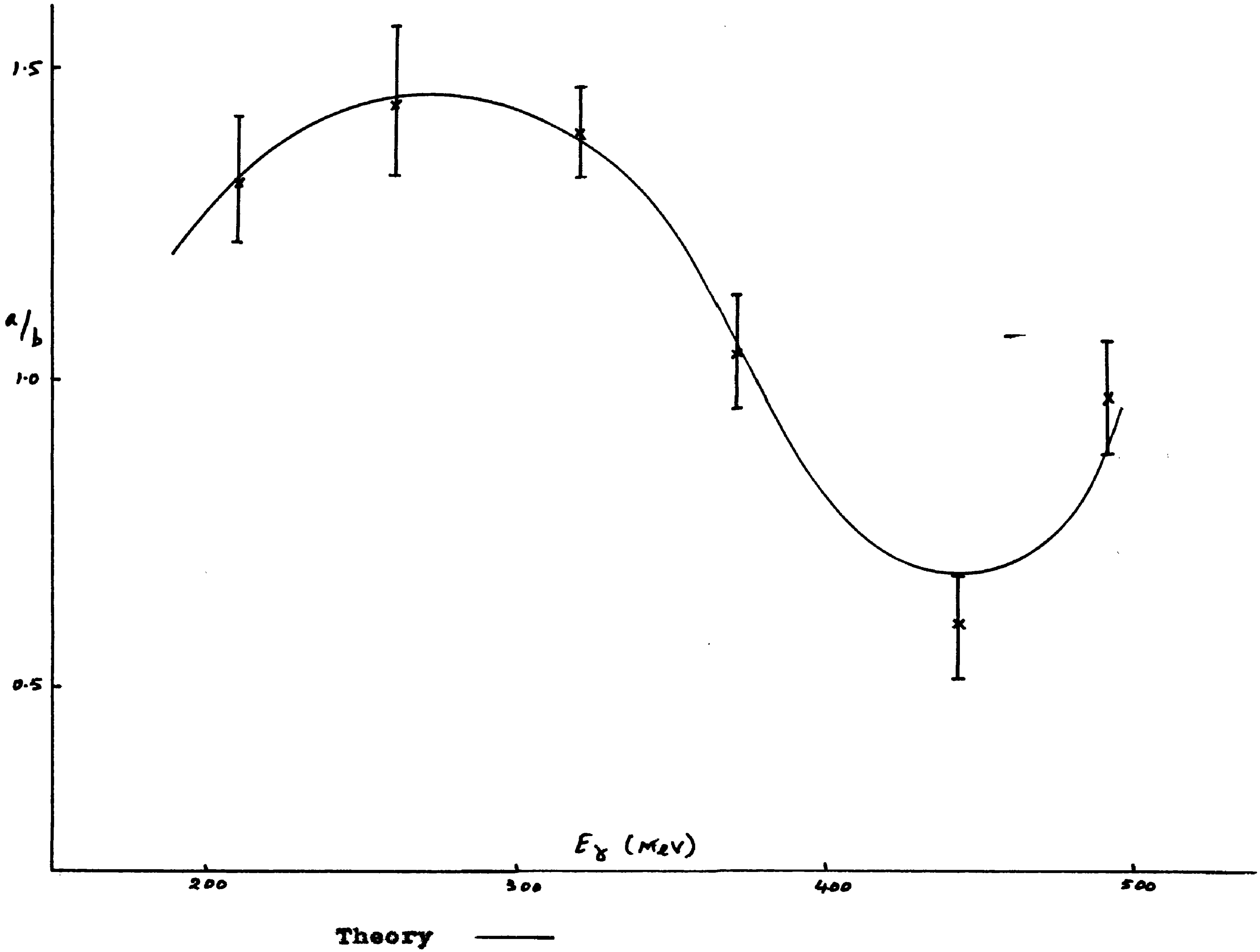
Pearlstein and Klein find that to get reasonable agreement with experiment it is necessary to include a hard-core both in the initial and final states, and to

Figure 12. High Energy Total Cross-Section



Experiment ——— **Theory** - - - -

Figure 13. High Energy Isotropy Ratio



Part III. Dispersion Relations for the Photodisintegration of the Deuteron.

17. Introduction.

In view of the ambiguities and difficulties of principle inherent in the potential approach to deuteron photodisintegration, it is of considerable interest to examine the problem using the techniques of dispersion theory, since this should give, in principle at least, a description of the process independent of any assumptions as regards the form of the interaction involved.

Apart from this, the application of dispersion relation techniques to deuteron photodisintegration is of more fundamental interest. In the last few years such techniques have been applied extensively to processes involving elementary particles. Pion-nucleon scattering [(21)], photopion production on nucleons [(79,80)], K-meson nucleon scattering [(81.82)], nucleon-nucleon scattering [(22,23)], and the structure of the nucleon [(83-86)] have been investigated with comparative success using single dispersion relations, and following on the general representation for the scattering amplitude proposed by S. Mandelstam [(27)], considerable advances have been made in describing processes involving elementary particles e.g. see references (28-35, 87-91) among many others.

The extension of these techniques to problems involving bound states should involve nothing new in principle [92-94], but so far such processes have received little attention. The simplest of such examples is when the bound state remains bound throughout, and may then be treated as an "elementary" particle. Pion-deuteron scattering has been investigated by F. Kaschluhn [95] and elastic neutron-deuteron scattering by R. Blankenbecler et al. [96]. The more complicated situation of the disintegration of a bound state has been considered by R. Blankenbecler and L.F. Cook [97], who consider the deuteron$\longrightarrow$ neutron + proton vertex.

The approach made to the deuteron photodisintegration is similar to the application of dispersion relations to processes involving elementary particles, but differs in that they are used in energy at a fixed difference in two momentum transfers, rather than at fixed momentum transfer, in order to exhibit explicitly all the poles in the dispersion relations. This is necessary since the momentum of, say, the exchanged proton in the proton-pole term is just the momentum transfer between the photon and the final proton, and consequently if the latter were held fixed this pole would not appear explicitly. For the dispersion relations to be equally as valid as fixed momentum-transfer dispersion relations, it is necessary that the amplitude be analytic in both energy and momentum transfer i.e. that the Mandelstam representation is valid for this process. This appears to be true in perturbation theory (up to one-meson exchange diagrams) and is non-relativistic dispersion theory [98-100] for simple potentials.

The dispersion relations contain integrals over both positive and negative energies, the latter arising from the crossed diagrams, for which the imaginary part of the amplitude is related to processes such as the radiative absorption of an anti-nucleon by the deuteron, and to the structure of the deuteron through the anomalous singularities of the d-np vertex. We ignore these complications, and retain only the pole terms and the integrals over positive energies. It is felt that this should be a good approximation for low energies.

18. Kinematical Considerations.

In this section we discuss the kinematics necessary to write the transition amplitude for the photodisintegration of the deuteron in a form suitable for the application of covariant dispersion relations.

Let d,k be the four-momenta of the deuteron and photon respectively, and let p_1, p_2 be the four-momenta of the outgoing nucleons. Let ν characterize the photon polarization, and let s, s_1, s_2 be the spin labels of the deuteron and the two nucleons respectively. Then the S-matrix element describing the process is

$$S_{\beta\alpha} = {}_{out}\langle p_1 s_1;\ p_2 s_2 \,|\, d, s;\ k, \nu \rangle_{in} \tag{18.1}$$

where α and β denote respectively all quantum numbers describing the initial and final states. The transition matrix $T_{\beta\alpha}$ is then defined by

$$S_{\beta\alpha} = \delta_{\beta\alpha} + i\, T_{\beta\alpha} \tag{18.2}$$

It has been shown [20] that, on contracting the deuteron,

$${}_{out}\langle p_1 s_1;\ p_2 s_2 \,|\, d, s;\ k, \nu \rangle_{in} = i^2 \int d^4x_1\, d^4x_2$$

$$\times\, {}_{out}\langle p_1 s_1; p_2 s_2 \,|\, T\{\bar{\psi}(x_1)\bar{\psi}(x_2)\}\,|\, k\nu \rangle\; G^{-1}(x_1 x_2)\quad \chi_d(x_1 x_2) \tag{18.3}$$

where $G^{-1}(x_1 x_2)$ is the inverse two-nucleon Green's function, satisfying an equation of the symbolic form

$$G^{-1}(x_1 x_2) = G^{-1}(x_1)\, G^{-1}(x_2) - I(x_1 x_2) \tag{18.4}$$

where $G^{-1}(x_i)$ is the inverse free-nucleon Green's function, and $I(x_1 x_2)$ the generalized interaction between two nucleons.

$$\chi_d(x_1 x_2) = \langle 0 | T\{ \psi(x_1)\, \psi(x_2) \} | d \rangle \tag{18.5}$$

is the Bethe-Salpeter amplitude for the deuteron.

Writing

$$\chi_d(x_1 x_2) = \frac{1}{\{(2\pi)^3\ 2d_0\}^{\frac{1}{2}}}\, \psi_d(y)\ e^{i d.x} \tag{18.6}$$

where

$$x = \tfrac{1}{2}(x_1 + x_2), \quad y = x_1 - x_2 \tag{18.7}$$

we have

$$\begin{aligned} {}_{out}\langle p_1 s_1;\, p_2 s_2 | d, s; k, \nu \rangle_{in} &= \frac{i^2}{\{(2\pi)^3\, 2d_0\}^{\frac{1}{2}}} \int d^4x\ d^4y \\ &\times {}_{out}\langle p_1 s_1;\, p_2 s_2 | T\{ \bar{\psi}(x+\tfrac{1}{2}y)\, \bar{\psi}(x-\tfrac{1}{2}y)\} | k, \nu \rangle \left[G^{-1}_{(1)}\, G^{-1}_{(2)} - I_{(12)} \right] \psi_d(y)\, e^{i d.x} \end{aligned} \tag{18.8}$$

$G^{-1}_{(1)}$, $G^{-1}_{(2)}$ operating back on $\bar{\psi}(x+\frac{1}{2}y)$, $\bar{\psi}(x-\frac{1}{2}y)$ give rise to the free-nucleon currents. In a complete theory, $I_{(12)}$ operating back would presumably give rise to a complex of currents of all the particles involved in the two-nucleon interaction. If we then assume $\psi_d(y)$ to be separable into "spin" and "space" parts (as it certainly is in the non-relativistic limit),

$$\psi_d(y) = \phi_d(y) \; S(d) \tag{18.9}$$

where $S(d)$ is a covariant 16 component matrix yielding the correct combination of nucleon spin states to form a triplet state, it is natural to define a "deuteron current" formally by

$$J_D(x) = i\int d^4y \; T\{\bar{\psi}(x+\tfrac{1}{2}y)\,\bar{\psi}(x-\tfrac{1}{2}y)\}[G^{(1)}G^{(2)} - I^{(2)}]\,\phi_d(y) \tag{18.10}$$

Then

$$_{out}\langle p_1 s_1 ; p_2 s_2 | d, s ; k, \nu\rangle_{in}$$

$$= \frac{i}{\{(2\pi)^3 2d_0\}^{\frac{1}{2}}}\int d^4x \; {}_{out}\langle p_1 s_1 ; p_2 s_2 | J_D(x) | k\nu\rangle_{in} \; e^{id\cdot x} \; S(d) \tag{18.11}$$

which is exactly of the form obtained on contracting an "elementary" particle.

Equation (2.11) suggests that we write

$$_{out}\langle p_1 s_1 ; p_2 s_2 | d, s ; k, \nu\rangle_{in} = (2\pi)^4 \, i \, \delta(p_1 + p_2 - d - k)$$

$$\times \left\{\frac{m^2}{(2\pi)^{12} 2k_0 2d_0 E_1 E_2}\right\}^{\frac{1}{2}} \bar{u}_\alpha(p_1)\,\bar{u}_\beta(p_2) \; {}^{\alpha\beta}M_{\gamma\delta}{}^{\mu}(p_1 p_2 ; d, k)\; e_\mu^{\nu}(k)\; S_{\gamma\delta}(d) \tag{18.12}$$

where $\bar{u}_\alpha(p_1)$, $\bar{u}_\beta(p_2)$ are the usual Dirac spinors with normalization

$$\bar{u}^{\lambda}(p)\,\bar{u}^{\lambda'}(p) = \delta^{\lambda\lambda'} \tag{18.13}$$

and ${}^{\alpha\beta}M_{\gamma\delta}{}^{\mu}(p_1 p_2 ; d, k)$ is the general covariant matrix element to which the dispersion relations will be applied.

In general, the transition matrix is proportional to the polarization vectors of the photon and deuteron, which we denote respectively by $e_\mu^\nu(k)$ and $\xi_\mu(d)$. $e_\mu^\nu(k)$ satisfies the usual Lorentz condition, and to ensure that the deuteron be in a triplet state in its rest system, $\xi_\mu(d)$ must also satisfy the Lorentz condition

$$d \cdot \xi = 0 \tag{18.14}$$

In order to utilize equation (2.12) it is necessary to know the matrix S explicitly. However, in the absence of a complete theory of the bound state, it is necessary to make some assumption as to the form of this matrix. It must be covariant, and can depend only on the deuteron variables. We assume the form proposed in ref. (21), namely

$$S = -\frac{i}{2\sqrt{2}} \cdot \frac{M + \gamma \cdot d}{M} \gamma \cdot \xi \; C \tag{18.15}$$

which is formally equivalent to

$$S = \frac{1}{2}\left(1 + \frac{E_d}{M}\right)\left(1 + \frac{i\gamma_5^{(1)} \sigma^{(1)} \cdot d}{E_d + M}\right)\left(1 + \frac{i\gamma_5^{(2)} \sigma^{(2)} \cdot \underline{d}}{E_d + M}\right) \tag{18.16}$$

operating on the correct combinations of Pauli spinors.

Energy momentum conservation

$$p_1 + p_2 = d + k \tag{18.17}$$

implies that only three of the four-vectors are independent. We choose to take p_1, p_2 and k.

The mass-shell restrictions

$$p_1^2 = p_2^2 = m^2, \quad k^2 = 0, \quad d^2 = M^2 \tag{18.18}$$

mean that only two independent scalars can be formed from the three independent vectors. We choose

$$\nu = (p_1 + p_2)^2 \quad , \quad \nu_1 = (p_1 - p_2) \cdot k \tag{18.19}$$

Apart from a numerical factor, the latter is the difference between the momentum transfer between the photon and nucleon '2' $(p_2 - k)^2$ and the momentum transfer between the photon and nucleon '1' $(p_1 - k)^2$

In the centre of momentum system,

$$\nu = W^2 \quad , \quad \nu_1 = 2\,\underline{p} \cdot \underline{k} = \frac{(W^2 - 4m^2)^{\frac{1}{2}}\,(W^2 - M^2)}{2W} \cos\theta \tag{18.20}$$

where W is the c.m. energy, θ the angle of the outgoing nucleons w.r.t. the direction of the incoming photon, and $\underline{p}$, $\underline{k}$ are respectively the momenta of one of the outgoing nucleons and of the incident photon.

The most general transition matrix element (18.12) must be a function of Lorentz invariants, which we take to be formed from the three independent vectors p_1, p_2, k and the basic matrices in the spinor spaces of the two nucleons,

$$1^{(i)}, \; i\gamma_5^{(i)}, \; \gamma_\mu^{(i)}, \; i\gamma_5^{(i)}\gamma_\mu^{(i)}, \; \sigma_{\mu\nu}^{(i)} = \tfrac{1}{2}\left[\gamma_\mu^{(i)}\gamma_\nu^{(i)} - \gamma_\nu^{(i)}\gamma_\mu^{(i)}\right], \; i = 1, 2 \tag{18.21}$$

Substantial restrictions are placed on the form of the matrix element by the requirements of gauge invariance which demands that

$$ {}^{\alpha\beta} M_{\gamma\delta}{}^{\mu} k_{\mu} = 0 \qquad (18.22)$$

and by the standard invariance and symmetry requirements. The transformation properties of the basic matrices and the 4-vectors under space inversion ($\mathcal{P}$), charge conjugation ($\mathcal{C}$) and time reversal ($\mathcal{T}$) are listed below.

Basic Matrix	$\mathcal{P}$	$\mathcal{C}$	$\mathcal{T}$
1	1	1	1
γ_0	γ_0	$-\gamma_0$	γ_0
γ_i	$-\gamma_i$	$-\gamma_i$	$-\gamma_i$
$\frac{1}{2}(\gamma_0\gamma_i-\gamma_i\gamma_0)$	$-\frac{1}{2}(\gamma_0\gamma_i-\gamma_i\gamma_0)$	$-\frac{1}{2}(\gamma_0\gamma_i-\gamma_i\gamma_0)$	$\frac{1}{2}(\gamma_0\gamma_i-\gamma_i\gamma_0)$
$\frac{1}{2}(\gamma_i\gamma_j-\gamma_j\gamma_i)$	$\frac{1}{2}(\gamma_i\gamma_j-\gamma_j\gamma_i)$	$-\frac{1}{2}(\gamma_i\gamma_j-\gamma_j\gamma_i)$	$-\frac{1}{2}(\gamma_i\gamma_j-\gamma_j\gamma_i)$
$i\gamma_5\gamma_0$	$-i\gamma_5\gamma_0$	$i\gamma_5\gamma_0$	$i\gamma_5\gamma_0$
$i\gamma_5\gamma_i$	$-i\gamma_5\gamma_i$	$-i\gamma_5\gamma_0$	$i\gamma_5\gamma_i$
$i\gamma_5$	$-i\gamma_5$	$i\gamma_5$	$-i\gamma_5$

		$\mathscr{P}$	$\mathscr{b}$	$\mathscr{y}$
"Spinor" 4-vector	p_0 $\mathbf{p}$	p_0 $-\mathbf{p}$	$-p_0$ $-\mathbf{p}$	p_0 $-\mathbf{p}$
"Boson" 4-vector	k_0 $\mathbf{k}$	k_0 $-\mathbf{k}$	k_0 $\mathbf{k}$	k_0 $-\mathbf{k}$

In constructing the independent covariant forms subject to these restrictions, the requirement of (18.14) is used in the sense that only triplet states are allowed in the initial state. This reduces the number of allowed independent forms from sixteen to twelve, and our choice is given in Table 9. The various linear combinations are taken purely for convenience in the ensuing algebra.

Denoting the independent covariant forms by $m_i(p, p_1, k; \gamma^{(1)}\gamma^{(2)})$ running from 1 to 12, we can then write

$$M = \sum_{i=1}^{12} m_i \; M^i(\nu, \nu_1) \qquad (18.23)$$

where the M^i are scalar functions of the two variables ν, ν_1

Since the electromagnetic interaction is being taken only to first order, it is possible to split each of the M^i into two parts, one arising from the isotopic scalar part of the electromagnetic interaction, and the other from the isotopic vector part i.e. we can write

$$M^i = M^i_S + M^i_V \tfrac{1}{2}(\tau_3^{(1)} - \tau_3^{(2)}) \qquad (18.24)$$

Our selection of the twelve invariant forms is such that all change sign under the operation $p_1 \rightleftharpoons p_2,\ \gamma^{(1)} \rightleftharpoons \gamma^{(2)}$. Including the isotopic spin dependence, (18.24), we see that of the twenty four invariant forms resulting, twelve change sign under the above operation and twelve remain unaltered. But by the general Pauli Principle, the complete amplitude must be unaltered by the above exchange. Under this exchange $\nu_1 \rightarrow -\nu_1$, and so we can meet the required Principle by demanding that

$$M_i^{s}(\nu,\nu_1) = -M_i^{s}(\nu,-\nu_1); \quad M_i^{V}(\nu,\nu_1) = M_i^{V}(\nu,-\nu_1) \tag{18.25}$$

It should be pointed out that the expansion (18.23) is not unique. Invariant forms compatible with the necessary symmetry and invariance requirements can be chosen other than those given in Table 9. Such a set, m_i' say, is related to our chosen set by a linear relationship

$$m_i' = c_{ij}\ m_j \tag{18.26}$$

which in turn requires a linear relationship between the corresponding amplitudes M_i and M_i viz.
cor

$$M_i'(\nu,\nu_1) = c_{ij}\ M_j(\nu,\nu_1)$$

$$M_i(\nu,\nu_1) = c_{ij}\ M_j(\nu,\nu_1) \tag{18.27}$$

The set of C_{i5}'s will then have the necessary symmetry of anti-symmetry properties w.r.t. ν_1 to satisfy the (perhaps different) behaviour of M^i and $M^{i'}$ necessary to satisfy the general Pauli Principle.

Although the standard invariance and symmetry requirements are now exhausted, as yet no use has been made of the unitarity of the S-matrix. As is well known [(101)], for the photodisintegration of the deuteron, unitarity implies that the phase of the amplitude in a single-partial wave is the scattering phase-shift in the two-nucleon final state. However equation (18.23) is not an angular momentum eigenstate expansion, and consequently in order to apply unitarity it is necessary to relate the amplitudes to the partial wave amplitudes.

The first step is to relate the Dirac-matrix form of the amplitude, (18.23) to the Pauli matrix form. In the centre of momentum system, the amplitude H for the photodisintegration may be defined by

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\,\frac{1}{6} \sum_{\substack{\text{spin initial state} \\ \text{spin final state} \\ \text{polarization}}} |\langle H \rangle|^2 \, \rho(E) \, \frac{1}{\text{flux}} \qquad (18.28)$$

where the density of the final states is given by

$$\rho(E) = \frac{pm}{2} \tag{18.29}$$

and the incident flux is equal to $(2\pi)^{-3}$.

Analogously to equation (18.23) we may expand H as

$$H = \sum_{i=1}^{12} h_i H^i \tag{18.30}$$

where the h_i are scalar products of $\underline{\sigma}^{(1)}, \underline{\sigma}^{(2)}, \underline{p}, \underline{k}$ and $\underline{\varepsilon}$ and are given in Table 10.

The restriction to twelve independent forms is again due to the requirement of an initial triplet state. Without this requirement there would be sixteen, the extra four being

$$\underline{\sigma}^{(1)} \cdot \underline{\sigma}^{(2)} \quad \{(\underline{\sigma}^{(1)} \times \underline{\sigma}^{(2)}) - i(\underline{\sigma}^{(1)} - \underline{\sigma}^{(2)})\} \cdot (\hat{k} \times \underline{\varepsilon})$$

$$\{(\underline{\sigma}^{(1)} \times \underline{\sigma}^{(2)}) - i(\underline{\sigma}^{(1)} - \underline{\sigma}^{(2)})\} \cdot \hat{k}\, \hat{p} \cdot (\hat{k} \times \underline{\varepsilon}), \quad \{(\underline{\sigma}^{(1)} \times \underline{\sigma}^{(2)}) - i(\underline{\sigma}^{(1)} - \underline{\sigma}^{(2)})\} \cdot \hat{p}\, \hat{p} \cdot (\hat{k} \times \underline{\varepsilon}) \tag{18.31}$$

However acting on a triplet state, $\underline{\sigma}^{(1)} \cdot \underline{\sigma}^{(2)}$ gives unity and the other three forms are identically zero. We are thus left with the independent forms of Table 10.

The h_i and the m_i may be related by decomposing the Dirac spinors to Pauli spinors. For this, we work in the centre-of-momentum system with

$$p_1 = \left(\frac{W}{2}, \ \underline{p}\right) \tag{18.32a}$$

$$p_2 = \left(\frac{W}{2}, -\underline{p}\right) \tag{18.32b}$$

$$k = \left(\frac{W^2 - M^2}{2W} , \underset{\sim}{k} \right) \tag{18.32c}$$

$$d = \left(\frac{W^2 + M^2}{2W} , - \underset{\sim}{k} \right) \tag{18.32d}$$

A direct comparison then gives a set of linear equations relating the twelve amplitudes H^i to the twelve amplitudes M^i viz.

$$\left\{ 1 + \frac{(W-2m)(W-M)^2}{(W+2m)(W+M)^2} - \frac{8W^2 \nu_1^2}{(W+2m)^2 (W+M)^4} \right\} M_1 + \frac{16MW}{(W+M)^2 (W+2m)} M_3$$

$$- \frac{8}{W+M} \left\{ 1 - \frac{(W-2m)(W-M)}{(W+2m)(W+M)} + \frac{4W \nu_1^2}{(W+2m)^2 (W+M)^3} \right\} M_4 + \frac{16W}{(W+M)^2 (W+2m)} M_{10}$$

$$+ \frac{4}{W+2m} \left\{ (W-m) + \frac{(W+m)(W-M)^2}{(W+M)^2} \right\} M_{11} - \frac{8}{W+M} \left\{ (W-m) - (W+m) \frac{(W-M)(W-2m)}{(W+M)(W+2m)} \right.$$

$$\left. + \frac{4W(W+m)\nu_1^2}{(W+2m)^2 (W+M)^3} \right\} M_{12} = \frac{32 (2\pi)^6 \, mW}{(W+M)^2 (W+2m)} \cdot \left[\frac{W^2+M^2}{(W^2-M^2)(W^2-4m^2)} \right]^{\frac{1}{2}} H_1$$

$$\frac{(W-M)(W-2m)}{(W+M)^2 (W+2m)} \left\{ 1 - \frac{4W^2 \nu_1^2}{(W^2-4m^2)(W^2-M^2)^2} \right\} M_1 + \frac{8MW\nu_1}{(W+2m)(W+M)^3 (W-M)} M_3$$

$$+ \frac{4(W-M)(W-2m)}{W(W+M)^3 (W+2m)} \nu_1 \left\{ 1 - \frac{4W^2 \nu_1^2}{(W^2-4m^2)(W^2-M^2)^2} \right\} M_4 + \frac{8W\nu_1}{(W+2m)(W+M)^3 (W-M)} M_{10}$$

$$+ \left\{ (W-m) + \frac{(W+m)(W-M)^2}{(W+M)^2} \right\} \frac{2\nu_1}{(W+2m)(W^2-M^2)} M_{11} + \left\{ -(W+m)(W+2m) \right.$$

$$\left. - \frac{(W+m)(W-2m)(3W-M)}{W+M} + \frac{8W^2 (W+m)\nu_1^2}{(W+2m)(W+M)^4} \right\} \frac{2\nu_1}{W(W+2m)(W+M)^2} M_{12}$$

$$= \frac{(2\pi)^6}{mW} \cdot \left[\frac{W^2+M^2}{W^2-M^2} \right]^{\frac{1}{2}} H_2$$

$$\frac{8W^4 \nu_1^2}{(W+2m)^2(W+M)^5(W-M)} M_1 - \frac{4W^2}{(W+2m)(W^2-M^2)} M_3 + \frac{32W^3 \nu_1^2}{(W-M)(W+M)^5(W+2m)^2} M_4$$

$$- \frac{4W^2(W-m)(W-M)}{(W+M)^3(W+2m)} M_7 + \frac{8W^2}{(W+2m)(W+M)^3} M_9 - \frac{8W^2}{(W-M)(W+M)^2(W+2m)} M_{10}$$

$$- \frac{4W^2(W-m)}{(W+2m)(W^2-M^2)} M_{11} - \frac{32W^3(W+m)\nu_1^2}{(W+2m)^2(W+M)^5(W-M)} M_{12}$$

$$= \frac{(2\pi)^6 \, 4W^2}{m(W^2-M^2)} \left[\frac{W^2+M^2}{(W^2-M^2)(W^2-4m^2)}\right]^{\frac{1}{2}} H_3$$

$$- \frac{4W^2(W-M)\nu_1}{(W+2m)^2(W+M)^3} M_1 - \frac{8W(3W-M)\nu_1}{(W+2m)^2(W+M)^3} M_4 + \frac{8W(W+m)\nu_1}{(W+2m)^2(W+M)^2} M_8$$

$$+ \frac{8W(W+m)(3W-M)\nu_1}{(W+2m)^2(W+M)^3} M_{12} = \frac{(2\pi)^6 \, 4W}{m(W^2-4m^2)} \cdot \left[\frac{W^2+M^2}{W^2-M^2}\right]^{\frac{1}{2}} H_4$$

$$- \frac{2(W-M)^3}{(W+2m)^2(W+M)} M_1 - \frac{2(W^2-M^2)}{(W+2m)^2} M_2 - \frac{16(W-M)^2}{(W+2m)^2(W+M)} M_4$$

$$- \frac{2(W+m)^2(W^2-M^2)}{(W+2m)^2} M_5 - \frac{2(W-M)^3(W+M)}{(W+2m)^2} M_6$$

$$+ \frac{16(W-m)(W+m)}{(W+2m)^2} M_8 + \frac{16(W+m)(W-M)^2}{(W+2m)^2(W+M)} M_{12}$$

$$= \frac{(2\pi)^6 \, 4W^2}{m(W^2-M^2)} \cdot \left[\frac{W^2+M^2}{(W^2-M^2)(W^2-4m^2)}\right]^{\frac{1}{2}} H_5$$

$$- \frac{2W^2(W-M)(W-2m)}{(W+M)^3(W+2m)} M_1 + \frac{2W^2(W-M)}{(W+M)^3} M_2 + \frac{8W^2(W-M)}{(W+M)^3(W+2m)} M_3$$

$$- \frac{16W^2(W-2m)}{(W+M)^3(W+2m)} M_4 + \frac{2W^2(W-m)^2(W-M)}{(W+M)^3} M_5 + \frac{2W^2(W-M)}{W+M} M_6$$

$$+ \frac{8W^2(W-m)(W-M)}{(W+M)^3(W+2m)} M_7 - \frac{16W^2(W-m)}{(W+M)^3} M_8 - \frac{16W^2}{(W+2m)(W+M)^3} M_{10}$$

$$- \frac{8W^2(W+m)(W-M)}{(W+2m)(W+M)^3} M_{11} + \frac{16W^2(W+m)(W-2m)}{(W+M)^3(W+2m)} M_{12}$$

$$= \frac{(2\pi)^6\, 4W}{m(W^2-4m^2)} \cdot \left[\frac{W^2+M^2}{W^2-M^2}\right]^{\frac{1}{2}} H_6$$

$$\frac{W-M}{(W+M)(W+2m)} M_3 - \frac{4W v_1^2}{(W+2m)^2(W^2-M^2)^2} M_4 - \frac{(W+m)(W+M)}{(W+2m)(W-M)} M_7$$

$$+ \frac{4W(W+m)v_1^2}{(W+2m)^2(W^2-M^2)^2} M_8 + \frac{2}{(W-M)(W+2m)} M_9 - \frac{2}{(W+M)(W+2m)} M_{10}$$

$$- \frac{(W+m)(W-M)}{(W+2m)(W+M)} M_{11} + \frac{4W(W+m)v_1^2}{(W+2m)^2(W^2-M^2)} M_{12}$$

$$= \frac{(2\pi)^6\, W}{m(W-M)^2} \cdot \left[\frac{W^2+M^2}{W^2-M^2}\right]^{\frac{1}{2}} H_7$$

$$- \frac{2(W-M)v_1}{(W+M)^3(W+2m)} M_3 + \frac{2(W-2m)v_1}{W(W+M)^2(W+2m)} M_4 - \frac{2(W-M)(W-m)}{(W+M)^3(W+2m)} v_1 M_7$$

$$+ \frac{4v_1}{(W+M)^3(W+2m)} M_9 + \frac{4v_1}{(W+M)^3(W+2m)} M_{10} + \frac{2(W+m)(W-M)v_1}{(W+M)^3(W+2m)} M_{11}$$

$$- \frac{2(W+m)(W-2m)}{W(W+M)^2(W+2m)} v_1 M_{12} = \frac{(2\pi)^6}{mW^2} \cdot \left[\frac{(W^2+M^2)(W^2-M^2)}{W^2-4m^2}\right]^{\frac{1}{2}} H_8$$

$$\frac{4W^2(W-M)v_1}{(W+2m)^2(W+M)^3}M_1 + \frac{8W^2(3W-M)v_1}{(W+2m)^2(W+M)^3(W-M)}M_4 - \frac{8W(W+m)v_1}{(W+2m)^2(W+M)^2}M_8$$

$$+ \frac{8W(W+m)v_1}{(W+2m)^2(W+M)^2}M_{11} - \frac{32W^2(W+m)v_1}{(W+2m)^2(W+M)^3}M_{12} = \frac{(2\pi)^6 4W}{m(W^2-4m^2)}\left[\frac{W^2+M^2}{W^2-M^2}\right]^{\frac{1}{2}} H_9$$

$$- \frac{W(W-M)(W-2m)}{4(W+M)}\left\{1-\frac{4W^2v_1^2}{(W^2-4m^2)(W^2-M^2)^2}\right\}M_1 - \frac{W(W-M)(W-2m)}{4(W+M)}\left\{1-\frac{4W^2v_1^2}{(W^2-4m^2)(W^2-M^2)^2}\right\}M_2$$

$$+ \frac{2mW(W-M)}{(W+2m)(W+M)}M_3 - \frac{2MW(W-2m)}{(W+M)^2}M_4 + \frac{W(W-M)(W-2m)(W^2-M^2)}{4(W+M)}\left\{1-\frac{4W^2v_1^2}{(W^2-4m^2)(W^2-M^2)^2}\right\}M_5$$

$$+ \frac{W(W-2m)(W-M)^2}{4}\left\{1-\frac{4W^2v_1^2}{(W^2-4m^2)(W^2-M^2)^2}\right\}M_6 + \frac{W(W-M)(W^2-2m^2)}{(W+M)(W+2m)}M_7$$

$$- \frac{W(W-2m)}{(W+M)^2}\left\{2(W^2+mM) - \frac{4W^2v_1^2}{(W^2-4m^2)(W^2-M^2)}\right\}M_8 - \frac{W}{W+M}\left\{1 + \frac{(W-M)(W-2m)}{(W+M)(W+2m)}\right\}M_9$$

$$+ \frac{W}{W+M}\left\{\frac{W-M}{W+M} + \frac{W-2m}{W+2m}\right\}M_{10} + \frac{W(W-M)}{2(W+M)}\left\{(W-m) + \frac{(W+m)(W-2m)}{W+2m}\right\}M_{11}$$

$$+ \frac{W-2m}{2(W+M)}\left\{-\frac{2W}{W+M}\left[(W+M)(W-m) + (W-M)(W+m)\right] + \frac{4W^2v_1^2}{(W^2-4m^2)(W^2-M^2)^2}\left[(W-m)(W-M)\right.\right.$$

$$\left.\left. + (W+m)(W+M)\right]\right\}M_{12} = \frac{(2\pi)^6}{m(W-M)}\left[\frac{W^2+M^2}{W^2-M^2}\right]^{\frac{1}{2}} H_{10}$$

$$- \frac{2W^3v_1}{(W+2m)(W+M)^3(W-M)}M_1 - \frac{2W^3v_1}{(W+2m)(W+M)^3(W-M)}M_2 + \frac{8W^2v_1}{(W+2m)(W+M)^4}M_4$$

$$+ \frac{2W^3(W^2-m^2)v_1}{(W+M)^3(W-M)(W+2m)}M_5 + \frac{2W^3v_1}{(W+M)^2(W+2m)}M_6 - \frac{8W^2(2W^2-m[W-M])}{(W+2m)(W+M)^4(W-M)}v_1 M_8$$

$$+ \frac{4W^2v_1}{(W-M)(W+2m)(W+M)^2}\left\{(W-m) + (W+m)\frac{3W-M}{W+M}\right\}M_{12}$$

$$= \frac{(2\pi)^6 W^2}{m(W^2-M^2)}\left[\frac{W^2+M^2}{(W^2-M^2)(W^2-4m^2)}\right]^{\frac{1}{2}} H_{11}$$

$$\frac{W(W-M)}{(W+2m)(W+M)} M_1 + \frac{W(W-M)}{(W+2m)(W+M)} M_2 + \frac{8MW}{(W+2m)(W+M)^2} M_4 - \frac{W(W-M)(W^2-m^2)}{(W+2m)(W+M)} M_5$$

$$- \frac{W(W-M)^2}{W+2m} M_6 + \frac{8W(W^2+mM)}{(W+2m)(W+M)^2} M_8 - \frac{4W}{(W+2m)^2(W+M)} M_{10}$$

$$+ \frac{4W}{(W+2m)(W+M)^2} \left\{ (W+m)(W-M) + (W-m)(W+M) \right\} M_{12}$$

$$= \frac{(2\pi)^6 4W}{m(W^2-4m^2)} \left[\frac{W^2+M^2}{W^2-M^2} \right]^{\frac{1}{2}} H_{12}$$

which we may write as

$$H = AM \tag{18.34}$$

where $A = (a_{ij})$ is a 12 x 12 matrix and $H = (H_i)$, $M = (M_i)$ are both 12 x 1 matrices.

The next step is to relate the H_i to the individual multipole amplitudes. This is done in the next section.

Table 9 m_i

i	m_i
1	$(p_1\cdot k\, p_2\cdot\varepsilon - p_2\cdot k\, p_1\cdot\varepsilon)\, 1^{(1)} 1^{(2)}$
2	$(p_1\cdot k\, p_2\cdot\varepsilon - p_2\cdot k\, p_1\cdot\varepsilon)\, i\gamma_5^{(1)}\, i\gamma_5^{(2)}$
3	$\{\gamma^{(1)}\cdot k\,(p_1+p_2)\cdot\varepsilon - (p_1+p_2)\cdot k\,\gamma^{(1)}\cdot\varepsilon\}\, 1^{(2)} - \{\gamma^{(2)}\cdot k\,(p_1+p_2)\cdot\varepsilon - (p_1+p_2)\cdot k\,\gamma^{(2)}\cdot\varepsilon\}\, 1^{(1)}$
4	$\{\gamma^{(1)}\cdot k\,(p_1-p_2)\cdot\varepsilon - (p_1-p_2)\cdot k\,\gamma^{(1)}\cdot\varepsilon\}\, 1^{(2)} + \{\gamma^{(2)}\cdot k\,(p_1-p_2)\cdot\varepsilon - (p_1-p_2)\cdot k\,\gamma^{(2)}\cdot\varepsilon\}\, 1^{(1)}$
5	$(p_1\cdot k\, p_2\cdot\varepsilon - p_2\cdot k\, p_1\cdot\varepsilon)\, i\gamma_5^{(1)}\gamma^{(1)}\cdot p_2\, i\gamma_5^{(2)}\gamma^{(2)}\cdot p_1$
6	$(p_1\cdot k\, p_2\cdot\varepsilon - p_2\cdot k\, p_1\cdot\varepsilon)\, i\gamma_5^{(1)}\gamma^{(1)}\cdot k\, i\gamma_5^{(2)}\gamma^{(2)}\cdot k$ $\times (p_1+p_2)\cdot k\}\, i\gamma_5^{(1)}\gamma^{(1)}\cdot p_2$
7	$i\gamma_5^{(1)}\{\gamma^{(1)}\cdot k\,(p_1+p_2)\cdot\varepsilon - (p_1+p_2)\cdot k\,\gamma^{(1)}\cdot\varepsilon\}\, i\gamma_5^{(2)}\gamma^{(2)}\cdot p_1 - i\gamma_5^{(2)}\{\gamma^{(2)}\cdot k\,(p_1+p_2)\cdot\varepsilon - \gamma^{(2)}\cdot\varepsilon$
8	$i\gamma_5^{(1)}\{\gamma^{(1)}\cdot k\,(p_1-p_2)\cdot\varepsilon - (p_1-p_2)\cdot k\,\gamma^{(1)}\cdot\varepsilon\}\, i\gamma_5^{(2)}\gamma^{(2)}\cdot p_1 + i\gamma_5^{(2)}\{\gamma^{(2)}\cdot k\,(p_1-p_2)\cdot\varepsilon - (p_1-p_2)\cdot k\,\gamma^{(2)}\cdot\varepsilon\}$
9	$i\gamma_5^{(1)}\, i\gamma_5^{(2)}(\gamma^{(1)}\cdot\varepsilon\,\gamma^{(2)}\cdot k - \gamma^{(2)}\cdot\varepsilon\,\gamma^{(1)}\cdot k)$ $\times\, i\gamma_5^{(1)}\gamma^{(1)}\cdot p$
10	$(\gamma^{(1)}\cdot k\,\gamma^{(2)}\cdot\varepsilon - \gamma^{(2)}\cdot k\,\gamma^{(1)}\cdot\varepsilon)$
11	$\{\gamma^{(1)}\cdot k\,(p_1+p_2)\cdot\varepsilon - (p_1+p_2)\cdot k\,\gamma^{(1)}\cdot\varepsilon\}\, p_1\cdot\gamma^{(2)} - \{\gamma^{(2)}\cdot k\,(p_1+p_2)\cdot\varepsilon - (p_1+p_2)\cdot k\,\gamma^{(2)}\cdot\varepsilon\}\, p_2\cdot\gamma^{(1)}$
12	$\{\gamma^{(1)}\cdot k\,(p_1-p_2)\cdot\varepsilon - (p_1-p_2)\cdot k\,\gamma^{(1)}\cdot\varepsilon\}\, p_1\cdot\gamma^{(2)} + \{\gamma^{(2)}\cdot k\,(p_1-p_2)\cdot\varepsilon - (p_1-p_2)\cdot k\,\gamma^{(2)}\cdot\varepsilon\}\, p_2\cdot\gamma^{(1)}$

Table 10 h_i

i	h_i
1	$\hat{p}\cdot\boldsymbol{\varepsilon}$
2	$i\,(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot(\hat{k}\times\boldsymbol{\varepsilon})$
3	$i\,(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot\hat{k}\;\hat{p}\cdot(\hat{k}\times\boldsymbol{\varepsilon})$
4	$i\,(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot\hat{p}\;\hat{p}\cdot(\hat{k}\times\boldsymbol{\varepsilon})$
5	$\boldsymbol{\sigma}^{(1)}\cdot\hat{p}\;\boldsymbol{\sigma}^{(2)}\cdot\hat{p}\;\hat{p}\cdot\boldsymbol{\varepsilon}$
6	$\boldsymbol{\sigma}^{(1)}\cdot\hat{k}\;\boldsymbol{\sigma}^{(2)}\cdot\hat{k}\;\hat{p}\cdot\boldsymbol{\varepsilon}$
7	$\boldsymbol{\sigma}^{(1)}\cdot\hat{p}\;\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{\varepsilon}+\boldsymbol{\sigma}^{(2)}\cdot\hat{p}\;\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\varepsilon}$
8	$\boldsymbol{\sigma}^{(1)}\cdot\hat{k}\;\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{\varepsilon}+\boldsymbol{\sigma}^{(2)}\cdot\hat{k}\;\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\varepsilon}$
9	$(\boldsymbol{\sigma}^{(1)}\cdot\hat{p}\;\boldsymbol{\sigma}^{(2)}\cdot\hat{k}+\boldsymbol{\sigma}^{(2)}\cdot\hat{p}\;\boldsymbol{\sigma}^{(1)}\cdot\hat{k})\,\hat{p}\cdot\boldsymbol{\varepsilon}$
10	$\{i(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})+(\boldsymbol{\sigma}^{(1)}\times\boldsymbol{\sigma}^{(2)})\}\cdot(\hat{k}\times\boldsymbol{\varepsilon})$
11	$\{i(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})+(\boldsymbol{\sigma}^{(1)}\times\boldsymbol{\sigma}^{(2)})\}\cdot\hat{k}\;\hat{p}\cdot(\hat{k}\times\boldsymbol{\varepsilon})$
12	$\{i(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})+(\boldsymbol{\sigma}^{(1)}\times\boldsymbol{\sigma}^{(2)})\}\cdot\hat{p}\;\hat{p}\cdot(\hat{k}\times\boldsymbol{\varepsilon})$

19. General Form of the Pauli Transition Matrix for the Photodisintegration of the Deuteron.

The general form of the Pauli transition matrix for deuteron photodisintegration has been given previously[59,67], but not in a form suitable for the application of dispersion relations. To obtain the matrix in such a form, we follow a method due to L.D. Pearlstein and A. Klein [101] who apply it to the special case of dipole and quadrupole transitions to S, P and D-wave final states.

In the centre-of-momentum system, the amplitude required may be written as

$$\langle \hat{p}\, s'\, m_{s'} | T(w) | \hat{k}\, \lambda\, 1\, m_s \rangle \tag{19.1}$$

where $\hat{k}$ and $\hat{p}$ are unit vectors in the direction of incidence of the photon and in the direction of relative motion of the outgoing nucleons respectively, λ specifies the polarization of the photon, s' the spin of the final nucleons and m_s', m_s the z-components of the final and initial spins respectively.

Expanding in angular momentum states,

$$\begin{aligned}\langle \hat{p}\, s' m_s' | T(w) | \hat{k} \lambda 1\, m_s \rangle = & \sum_{JLL'R_L} b_{s'}(JL';LR_L)\, \lambda^{p} \\ & \times \langle L' s'\; m_s - m_s' + \lambda\; m_s' | L' s'\; J\; m_s + \lambda \rangle \\ & \times \langle L 1 J\; m_s + \lambda | L 1 \lambda\, m_s \rangle\; Y_{L'}^{m_s - m_s' + \lambda}(\hat{p}\cdot\hat{k})\end{aligned} \tag{19.2}$$

where $\langle L L' m m' | L L' J M \rangle = C_{LL'J m m'}$ is the usual Clelsch-Gordon coefficient. In equation (19.2) L' refers to the final-state orbital angular momentum, L to the initial state multipole and p = 0 for electric transitions, p = 1 for magnetic transitions, and $\mathcal{P}_L$ denotes the parity

The operator of interest is the one with respect to the spin-space of the deuteron and the emergent nucleons. Defining the triplet spin operator $\underset{\sim}{S}$ by $\underset{\sim}{S} = \frac{1}{2}(\underset{\sim}{\sigma}^{(1)} + \underset{\sim}{\sigma}^{(2)})$ and

$$S_+ = \tfrac{1}{2}(S_x + i S_y) = \tfrac{1}{2}(\sigma_+^{(1)} + \sigma_+^{(2)})$$

$$S_- = \tfrac{1}{2}(S_x - i S_y) = \tfrac{1}{2}(\sigma_-^{(1)} + \sigma_-^{(2)})$$

then any spin operator 0 can be written in the form

$$\begin{aligned}
0 = {} & \langle 11|0|11\rangle \tfrac{1}{2}(1+S_3)S_3 + \langle 10|0|11\rangle \sqrt{2}\, S_- S_3 \\
& + \langle 1\text{-}1|0|11\rangle\, 2S_-^2 + \langle 11|0|10\rangle \sqrt{2}\, S_3 S_+ \\
& + \langle 10|0|10\rangle \tfrac{1}{2}\left(4S_+S_- - S_3[1+S_3]\right) - \langle 1\text{-}1|0|10\rangle \sqrt{2}\, S_3 S_- \\
& + \langle 11|0|1\text{-}1\rangle\, 2S_+^2 - \langle 10|0|1\text{-}1\rangle \sqrt{2}\, S_+ S_3 \qquad (19.3)\\
& - \langle 1\text{-}1|0|1\text{-}1\rangle \tfrac{1}{2}(1-S_3)S_3 \\
& + \langle 00|0|11\rangle \tfrac{1}{\sqrt{2}}\left(\sigma_-^{(2)} - \sigma_-^{(1)} + \sigma_3^{(1)}\sigma_-^{(2)} - \sigma_3^{(2)}\sigma_-^{(1)}\right) \\
& + \langle 00|0|10\rangle \tfrac{1}{4}\left(\sigma_3^{(1)} - \sigma_3^{(2)} + i\left[\sigma_2^{(1)}\sigma_1^{(2)} - \sigma_1^{(1)}\sigma_2^{(2)}\right]\right) \\
& + \langle 00|0|1\text{-}1\rangle \tfrac{1}{\sqrt{2}}\left(\sigma_+^{(1)} - \sigma_+^{(2)} + \sigma_3^{(1)}\sigma_+^{(2)} - \sigma_3^{(2)}\sigma_+^{(1)}\right)
\end{aligned}$$

Applying equation (19.3) to (19.2) we obtain, for the case $\lambda = 1$,

$$\langle p | T(w) | k \lambda \rangle_{\lambda=1} = \sum_{JLL'R_L} b_1(JL';LR_L) \Big\{ \langle L'111 | L'1J2 \rangle \langle L1J2 | L111 \rangle$$

$$\times \tfrac{1}{2}(1+S_3)S_3 \, Y_{L'}^{1}(p.k) + \langle L'120 | L'1J2 \rangle \langle L1J2 | L111 \rangle \sqrt{2}\, S_- S_3 \, Y_{L'}^{2}(p.k)$$

$$+ \langle L'13-1 | L'1J2 \rangle \langle L1J2 | L111 \rangle \, 2 S_-^2 \, Y_{L'}^{3}(p.k)$$

$$+ \langle L'101 | L'1J1 \rangle \langle L1J1 | L110 \rangle \sqrt{2}\, S_3 S_+ \, Y_{L'}^{0}(p.k)$$

$$+ \langle L'110 | L'1J1 \rangle \langle L1J1 | L110 \rangle \tfrac{1}{2}\left(4 S_+ S_- - S_3(1+S_3)\right) Y_{L'}^{1}(p.k)$$

$$- \langle L'12-1 | L'1J1 \rangle \langle L1J1 | L110 \rangle \sqrt{2}\, S_3 S_- \, Y_{L'}^{2}(p.k)$$

$$+ \langle L'1-11 | L'1J0 \rangle \langle L1J0 | L11-1 \rangle \, 2 S_+^2 \, Y_{L'}^{-1}(p.k)$$

$$- \langle L'100 | L'1J0 \rangle \langle L1J0 | L11-1 \rangle \sqrt{2}\, S_+ S_3 \, Y_{L'}^{0}(p.k)$$

$$- \langle L'11-1 | L'1J0 \rangle \langle L1J0 | L11-1 \rangle \tfrac{1}{2}(1-S_3)S_3 \, Y_{L'}^{1}(p.k) \Big\}$$

$$+ \sum_{JLL'R_L} b_0(JL';LR_L) \Big\{ \langle L'020 | L'0J2 \rangle \langle L1J2 | L111 \rangle$$

$$\times \tfrac{1}{\sqrt{2}} \left(\sigma_-^{(2)} - \sigma_-^{(1)} + \sigma_3^{(1)} \sigma_-^{(2)} - \sigma_3^{(2)} \sigma_-^{(1)} \right) Y_{L'}^{2}(p.k)$$

$$+ \langle L'010 | L'0J1 \rangle \langle L1J1 | L110 \rangle \tfrac{1}{2} \left(\sigma_3^{(1)} - \sigma_3^{(2)} + i \left[\sigma_2^{(1)} \sigma_1^{(2)} - \sigma_1^{(1)} \sigma_2^{(2)} \right] \right) Y_{L'}^{1}(p.k)$$

$$+ \langle L'000 | L'0J0 \rangle \langle L1J0 | L11-1 \rangle \tfrac{1}{\sqrt{2}} \left(\sigma_+^{(1)} - \sigma_+^{(2)} + \sigma_3^{(1)} \sigma_+^{(2)} - \sigma_3^{(2)} \sigma_+^{(1)} \right) Y_{L'}^{0}(p.k) \Big\}$$

and similarly for $\lambda = -1$. Rewriting $Y_{LM}(\theta,\varphi)$ in terms of $P_L(\theta)$ we then obtain the following set of operators and their equivalent invariant forms.

(i) $L_+(1+S_3)S_2\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) = \frac{1}{\sqrt{2}}\{ i\, \boldsymbol{S}.\boldsymbol{k}\, \boldsymbol{p}.(\boldsymbol{k}\wedge\boldsymbol{\varepsilon}) + (\boldsymbol{S}.\boldsymbol{k})^2\, \boldsymbol{p}.\boldsymbol{\varepsilon}\}\, P_{L'}'(\boldsymbol{p}.\boldsymbol{k})$

(ii) $(L_+)^2 S_- S_3\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) = -\frac{1}{4\sqrt{2}}\{ L'(L'+1)\, \boldsymbol{S}.\boldsymbol{\varepsilon}\, \boldsymbol{S}.\boldsymbol{k}\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) - 2\, \boldsymbol{S}.\boldsymbol{\varepsilon}\, \boldsymbol{p}.\boldsymbol{k}\, \boldsymbol{S}.\boldsymbol{k}\, P_{L'}'(\boldsymbol{p}.\boldsymbol{k})$

$-2\{(\boldsymbol{S}.\boldsymbol{k})^2\, \boldsymbol{p}.\boldsymbol{\varepsilon}\, \boldsymbol{p}.\boldsymbol{k} - \boldsymbol{p}.\boldsymbol{\varepsilon}\, (\boldsymbol{S}.\boldsymbol{p})(\boldsymbol{S}.\boldsymbol{k})\}P_{L'}''(\boldsymbol{p}.\boldsymbol{k})\}$

(iii) $(L_+)^3(S_-)^2\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) = \frac{1}{2^4\sqrt{2}}\{ [L'(L'+1)[(2\boldsymbol{p}.\boldsymbol{\varepsilon} - (\boldsymbol{S}.\boldsymbol{k})^2\, \boldsymbol{p}.\boldsymbol{\varepsilon} - (\boldsymbol{S}.\boldsymbol{k})\boldsymbol{p}.\boldsymbol{\varepsilon} + 2\boldsymbol{S}.\boldsymbol{\varepsilon}\, \boldsymbol{S}.\boldsymbol{p}$

$-2\boldsymbol{S}.\boldsymbol{\varepsilon}\, \boldsymbol{S}.\boldsymbol{k}\, \boldsymbol{p}.\boldsymbol{k}] - 2\boldsymbol{p}.\boldsymbol{\varepsilon}[2-(\boldsymbol{S}.\boldsymbol{k})^2] + 2[\boldsymbol{p}.\boldsymbol{k}\, \boldsymbol{S}.\boldsymbol{k} - \boldsymbol{S}.\boldsymbol{p}]\boldsymbol{S}.\boldsymbol{\varepsilon}$

$+2\boldsymbol{S}.\boldsymbol{\varepsilon}\, [\boldsymbol{p}.\boldsymbol{k}\, \boldsymbol{S}.\boldsymbol{k} - \boldsymbol{S}.\boldsymbol{p}\}]\, P_{L'}'(\boldsymbol{p}.\boldsymbol{k}) + 4\,[(\boldsymbol{p}.\boldsymbol{k})^2(\boldsymbol{S}.\boldsymbol{k}\, \boldsymbol{S}.\boldsymbol{\varepsilon} + \boldsymbol{S}.\boldsymbol{\varepsilon}\, \boldsymbol{S}.\boldsymbol{k})$

$-\boldsymbol{p}.\boldsymbol{k}\, (\boldsymbol{S}.\boldsymbol{\varepsilon}\, \boldsymbol{S}.\boldsymbol{p} + \boldsymbol{S}.\boldsymbol{p}\, \boldsymbol{S}.\boldsymbol{\varepsilon}) + \boldsymbol{p}.\boldsymbol{\varepsilon}\, \boldsymbol{p}.\boldsymbol{k}\, ((\boldsymbol{S}.\boldsymbol{k})^2 - S^2)]\, P_{L'}''(\boldsymbol{p}.\boldsymbol{k})$

$+4(\boldsymbol{p}.\boldsymbol{\varepsilon})\{\boldsymbol{p}.\boldsymbol{k}\, \boldsymbol{S}.\boldsymbol{k} - \boldsymbol{S}.\boldsymbol{p}\}^2\, P_{L'}'''(\boldsymbol{p}.\boldsymbol{k})$

(iv) $S_3 S_+\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) = -\frac{1}{\sqrt{2}}\, \boldsymbol{S}.\boldsymbol{k}\, \boldsymbol{S}.\boldsymbol{\varepsilon}\, P_{L'}(\boldsymbol{p}.\boldsymbol{k})$

(v) $L_+(4S_+S_- - S_3[1+S_3])\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) = \frac{2}{\sqrt{2}}[1-(\boldsymbol{S}.\boldsymbol{k})^2\}\, \boldsymbol{p}.\boldsymbol{\varepsilon}\, P_{L'}'(\boldsymbol{p}.\boldsymbol{k})$

(vi) $(L_+)^2 S_3 S_-\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) = \frac{1}{2^2\sqrt{2}}\{ -L'(L'+1)\, \boldsymbol{S}.\boldsymbol{k}\, \boldsymbol{S}.\boldsymbol{\varepsilon}\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) + 2\boldsymbol{S}.\boldsymbol{k}\, \boldsymbol{S}.\boldsymbol{\varepsilon}\, \boldsymbol{p}.\boldsymbol{k}\, P_{L'}'(\boldsymbol{p}.\boldsymbol{k})$

$+2[(\boldsymbol{S}.\boldsymbol{k})^2\, \boldsymbol{p}.\boldsymbol{k}\, \boldsymbol{p}.\boldsymbol{\varepsilon} - \boldsymbol{S}.\boldsymbol{k}\, \boldsymbol{S}.\boldsymbol{p}\, \boldsymbol{p}.\boldsymbol{\varepsilon}]\, P_{L'}''(\boldsymbol{p}.\boldsymbol{k})\}$

(vii) $L_-(S_+)^2\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) = \frac{1}{2^2\sqrt{2}}\{ 2\boldsymbol{p}.\boldsymbol{\varepsilon} - (\boldsymbol{S}.\boldsymbol{k})^2\, \boldsymbol{p}.\boldsymbol{\varepsilon} - \boldsymbol{S}.\boldsymbol{k}\, \boldsymbol{p}.\boldsymbol{\varepsilon} + 2\boldsymbol{p}.\boldsymbol{k}\, \boldsymbol{S}.\boldsymbol{k}\, \boldsymbol{S}.\boldsymbol{\varepsilon}$

$-2\boldsymbol{S}.\boldsymbol{p}\, \boldsymbol{S}.\boldsymbol{\varepsilon}\}\, P_{L'}'(\boldsymbol{p}.\boldsymbol{k})$

(viii) $S_+ S_3\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) = -\frac{1}{\sqrt{2}}\, \boldsymbol{S}.\boldsymbol{\varepsilon}\, \boldsymbol{S}.\boldsymbol{k}\, P_{L'}(\boldsymbol{p}.\boldsymbol{k})$

(ix) $L_+(1-S_3)S_3\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) = \frac{1}{\sqrt{2}}\{\boldsymbol{S}.\boldsymbol{k} - (\boldsymbol{S}.\boldsymbol{k})^2\}\, \boldsymbol{p}.\boldsymbol{\varepsilon}\, P_{L'}'(\boldsymbol{p}.\boldsymbol{k})$

(x) $(L_+)^2(\sigma_-^{(1)} - \sigma_-^{(2)} + \sigma_3^{(1)}\sigma_-^{(2)} - \sigma_3^{(2)}\sigma_-^{(1)})\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) = \frac{1}{2^2\sqrt{2}}[L'(L'+1)[i(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}) + (\boldsymbol{\sigma}^{(1)}\times\boldsymbol{\sigma}^{(2)})]$

$[(\boldsymbol{k}\times\boldsymbol{\varepsilon})\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) - 2\boldsymbol{p}.\boldsymbol{k}\, (\boldsymbol{k}\times\boldsymbol{\varepsilon})\, P_{L'}'(\boldsymbol{p}.\boldsymbol{k}) + 2\{\boldsymbol{p}\, \boldsymbol{p}.(\boldsymbol{k}\times\boldsymbol{\varepsilon}) - \boldsymbol{p}.\boldsymbol{k}\, \boldsymbol{k}\, \boldsymbol{p}.(\boldsymbol{k}\times\boldsymbol{\varepsilon})\}P_{L'}''(\boldsymbol{p}.\boldsymbol{k})]$

(xi) $L_+(\sigma_3^{(1)} - \sigma_3^{(2)} + i\{\sigma_2^{(1)}\sigma_1^{(2)} - \sigma_1^{(1)}\sigma_2^{(2)}\})\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) = \frac{1}{\sqrt{2}}[i(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}) + (\boldsymbol{\sigma}^{(1)}\times\boldsymbol{\sigma}^{(2)})\}\boldsymbol{k}\, \boldsymbol{p}.(\boldsymbol{k}\times\boldsymbol{\varepsilon})\, P_{L'}'$

(xii) $(\sigma_+^{(1)} - \sigma_+^{(2)} + \sigma_3^{(1)}\sigma_+^{(2)} - \sigma_3^{(2)}\sigma_+^{(1)})\, P_{L'}(\boldsymbol{p}.\boldsymbol{k}) = -\frac{1}{\sqrt{2}}[i(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\} + (\boldsymbol{\sigma}^{(1)}\times\boldsymbol{\sigma}^{(2)})\}(\boldsymbol{k}\times\boldsymbol{\varepsilon})\, P_{L'}(\boldsymbol{p}.\boldsymbol{k})$

The equivalent invariant forms can be obtained by continued application of the relation

$$\underset{\sim}{L}\;\underset{\sim}{p}\cdot\underset{\sim}{A} = -i\;(\underset{\sim}{p}\times\underset{\sim}{A}) \tag{19.6}$$

for any vector $\underset{\sim}{A}$.

Substituting (19.5) in (19.4) and evaluating the Clelsch-Gordon coefficients, after much straightforward but tedious algebra, one obtains

$$\langle p \,|\, T(w) \,|\, k \rangle = \sum H^i h_i \tag{19.7}$$

where the h_i are given in Table 10, and the H_i by

$$H_1 = b_1(L+1,L;E_L)\left[\frac{2}{L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{\frac{L^2+2L+1}{2(L+1)}P_L'(\cos\theta) - \frac{2\cos\theta}{L+1}P_L''(\cos\theta) - \frac{(\cos\theta)^2}{L+1}P_L'''(\cos\theta)\right\}$$

$$+\,b_1(L,L;E_L)\left[\frac{2(2L+1)}{L(L+1)}\right]^{\frac{1}{2}}\left\{\frac{1}{2L(L+1)}P_L'(\cos\theta) + \frac{2\cos\theta}{L(L+1)}P_L''(\cos\theta) + \frac{(\cos\theta)^2}{L(L+1)}P_L'''(\cos\theta)\right\}$$

$$+\,b_1(L-1,L;E_L)\left[\frac{2}{L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{-\frac{1}{2}P_L'(\cos\theta) - \frac{2}{L}\cos\theta\,P_L''(\cos\theta) + \frac{(\cos\theta)^2}{L}P_L'''(\cos\theta)\right\}$$

$$+\,b_1(L+1,L;E_L)\left[\frac{2}{(L+1)(L+3)(2L+5)}\right]^{\frac{1}{2}}\left\{(L+1)P_L'(\cos\theta) + 2\cos\theta\,P_L''(\cos\theta) + \frac{(\cos\theta)^2}{L+2}P_L'''(\cos\theta)\right\}$$

$$+\,b_1(L-1,L;E_{L-2})\left[\frac{2}{L(L-2)(2L-3)}\right]^{\frac{1}{2}}\left\{L\,P_L'(\cos\theta) - \frac{2}{L-1}\cos\theta\,P_L''(\cos\theta) - \frac{(\cos\theta)^2}{L-1}P_L'''(\cos\theta)\right\}$$

$$+\,b_1(L+1,L;M_{L+1})\left[\frac{2}{L+1}\right]^{\frac{1}{2}}\left\{-P_L'(\cos\theta) + \frac{2\cos\theta}{(L+1)(L+2)}P_L''(\cos\theta) + \frac{(\cos\theta)^2}{(L+1)(L+2)}P_L'''(\cos\theta)\right\}$$

$$+\,b_1(L,L;M_{L+1})\left[\frac{2(2L+1)}{L(L+1)(2L+1)(L+2)}\right]^{\frac{1}{2}}\left\{\frac{1}{2}(L+2)P_L'(\cos\theta) - \frac{2}{L+1}\cos\theta\,P_L''(\cos\theta) + \frac{(\cos\theta)^2}{L+1}P_L'''\right\}$$

$$+ b_1(L,L;\ M_{L-1})\left[\frac{2(2L+1)}{L(L+1)(L-1)(2L-1)}\right]^{\frac{1}{2}}\left\{\frac{(L-2)(L+1)}{2L}P_L'(\cos\theta) - \frac{2}{L}\cos\theta P_L''(\cos\theta) + \frac{(\cos\theta)^2}{L}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;\ M_{L-1})\left[\frac{2}{L}\right]^{\frac{1}{2}}\left\{\frac{3L-1}{4L}P_L'(\cos\theta) - \frac{2}{L(L-1)}\cos\theta P_L''(\cos\theta) + \frac{(\cos\theta)^2}{L(L-1)}P_L'''(\cos\theta)\right\}$$

$$H_2 = b_1(L+1,L;E_L)\left[\frac{2L+1}{2L(L+1)}\right]^{\frac{1}{2}}\left\{-\frac{2L+3}{L+1}P_L'(\cos\theta)\cos\theta + \frac{(2L+3)(1-\cos^2\theta)}{L+1}P_L''(\cos\theta)\right\}$$

$$+ b_1(L,L;\ E_L)\left[\frac{2L+1}{2L(L+1)}\right]^{\frac{1}{2}}\left\{\frac{2L+1}{L(L+1)}P_L'(\cos\theta)\cos\theta - \frac{2L+1}{L(L+1)}(1-\cos^2\theta)\,P_L''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;\ E_L)\left[\frac{1}{2L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{-\frac{(L-2)(L+1)\cos\theta}{2L}P_L'(\cos\theta) - \frac{(2L-1)}{L}(1-\cos^2\theta)\,P_L''(\cos\theta)\right\}$$

$$+ b_1(L+1,L;E_{L+2})\left[\frac{L+3}{2(L+1)(2L+5)}\right]^{\frac{1}{2}}\left\{-\frac{(L-2)}{2(L+2)}\cos\theta P_L'(\cos\theta) + \frac{2}{L+2}(1-\cos^2\theta)\,P_L''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;\ E_{L-2})\left[\frac{L-2}{2L(2L-3)}\right]^{\frac{1}{2}}\left\{-\frac{L+3}{2(L-1)}\cos\theta P_L'(\cos\theta)\right\}$$

$$+ b_1(L+1,L;\ M_{L+1})\left[\frac{2}{L+1}\right]^{\frac{1}{2}}\left\{\frac{(L+1)(L+3)}{2(L+2)}P_L(\cos\theta) + \frac{\cos\theta}{2(L+2)}P_L'(\cos\theta) - \frac{(1-\cos^2\theta)}{2(L+1)}P_L''\right\}$$

$$+ b_1(L,L;M_{L+1})\left[\frac{2L+1}{2L(L+1)(L+2)(2L+3)}\right]^{\frac{1}{2}}\left\{-L(L+1)\,P_L(\cos\theta) + \cos\theta P_L'(\cos\theta) - \frac{L}{L+1}(1-\cos^2\theta)P_L''(\cos\theta)\right\}$$

$$+ b_1(L,L;M_{L-1})\left[\frac{(2L+1)}{L(L+1)(L-1)(2L-1)}\right]^{\frac{1}{2}}\left\{L(L+1)\,P_L(\cos\theta) - \frac{L+1}{L}\cos\theta P_L'(\cos\theta) + \frac{3}{L}(1-\cos^2\theta)\,P_L''\right\}$$

$$+ b_1(L-1,L;\ M_{L-2})\left[\frac{2}{L}\right]^{\frac{1}{2}}\left\{\frac{L^2-2}{2(L-1)}P_L(\cos\theta) - \frac{1}{2L}\cos\theta P_L'(\cos\theta) + \frac{1}{L}(1-\cos^2\theta)P_L''(\cos\theta)\right\}$$

$$H_3 = b_1(L+1,L;E_L)\left[\frac{1}{2L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{-\frac{L^2-L-4}{2(L+1)}P_L'(\cos\theta) + \frac{(2L+3)}{L+1}\cos\theta\, P_L''(\cos\theta)\right\}$$

$$+ b_1(L,L;E_L)\left[\frac{1}{2L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{\frac{L^2+L+2}{2L(L+1)}P_L'(\cos\theta) - \frac{1}{L(L+1)}\cos\theta\, P_L''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;E_L)\left[\frac{1}{2L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{-\frac{L^2+3L-2}{2L}P_L'(\cos\theta) - \frac{2L-1}{L}\cos\theta\, P_L''(\cos\theta)\right\}$$

$$+ b_1(L+1,L;E_{L+2})\left[\frac{1}{2(L+1)(L+3)(2L+5)}\right]^{\frac{1}{2}}\left\{-\frac{L+1}{2}P_L'(\cos\theta) + 2\,\frac{L+3}{L+2}\cos\theta\, P_L''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;E_{L-2})\left[\frac{1}{2L(L-2)(2L-3)}\right]^{\frac{1}{2}}\left\{-\frac{L}{2}P_L'(\cos\theta)\right\}$$

$$+ b_1(L+1,L;M_{L+1})\left[\frac{1}{2(L+1)}\right]^{\frac{1}{2}}\left\{-\frac{L+3}{2(L+1)}P_L'(\cos\theta) - \frac{1}{L+1}\cos\theta\, P_L''(\cos\theta)\right\}$$

$$+ b_1(L,L;M_{L+1})\left[\frac{2L+1}{2L(L+1)(L+2)(2L+3)}\right]^{\frac{1}{2}}\left\{-\frac{(L+2)(L-1)}{2(L+1)}P_L'(\cos\theta)\right\}$$

$$+ b_1(L,L;M_{L+1})\left[\frac{2L+1}{2L(L+1)(L-1)(2L-1)}\right]^{\frac{1}{2}}\left\{-\frac{(L+2)(L-1)}{L}P_L'(\cos\theta) + \cos\theta\, P_L''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;M_{L-1})\left[\frac{2}{L}\right]^{\frac{1}{2}}\left\{-\frac{L-2}{4L}P_L'(\cos\theta)\right\}$$

$$H_4 = b_1(L+1,L;E_L)\left[\frac{1}{2L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{-\frac{2L+3}{L+1}P_L''(\cos\theta)\right\}$$

$$+ b_1(L,L;E_L)\left[\frac{2L+1}{2L(L+1)}\right]^{\frac{1}{2}}\left\{\frac{1}{L(L+1)}P_L''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;E_L)\left[\frac{1}{2L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{\frac{2L-1}{L}P_L''(\cos\theta)\right\}$$

$$+ b_1(L+1, L; E_{L+2}) \left[\frac{1}{2(L+1)(L+3)(2L+5)}\right]^{\frac{1}{2}} \left\{ -\frac{2(L+3)}{L+2} P_L''(\cos\theta) \right\}$$

$$+ b_1(L+1, L; M_{L+1}) \left[\frac{1}{2(L+1)}\right]^{\frac{1}{2}} \left\{ -\frac{1}{L+1} P_L''(\cos\theta) \right\}$$

$$+ b_1(L, L; M_{L-1}) \left[\frac{2L+1}{2L(L+1)(L-1)(2L-1)}\right]^{\frac{1}{2}} \left\{ - P_L''(\cos\theta) \right\}$$

$$H_5 = b_1(L+1, L; E_L) \left[\frac{2}{L(L+1)(2L+1)}\right]^{\frac{1}{2}} \frac{1}{L+1} P_L'''(\cos\theta)$$

$$- b_1(L, L; E_L) \left[\frac{2(2L+1)}{L(L+1)}\right]^{\frac{1}{2}} \frac{1}{L(L+1)} P_L'''(\cos\theta)$$

$$+ b_1(L-1, L; E_L) \left[\frac{2}{L(L+1)(2L+1)}\right] \frac{1}{L} P_L'''(\cos\theta)$$

$$+ b_1(L+1, L; E_{L+2}) \left[\frac{2}{(L+1)(L+3)(2L+5)}\right]^{\frac{1}{2}} \frac{1}{L+2} P_L'''(\cos\theta)$$

$$+ b_1(L-1, L; E_{L-2}) \left[\frac{2}{L(L-2)(2L-3)}\right]^{\frac{1}{2}} \frac{1}{L-1} P_L'''(\cos\theta)$$

$$+ b_1(L+1, L; M_{L+1}) \left[\frac{2}{L+1}\right]^{\frac{1}{2}} \cdot \frac{1}{(L+1)(L+2)} P_L'''(\cos\theta)$$

$$+ b_1(L, L; M_{L+1}) \left[\frac{2(2L+1)}{L(L+1)(L+2)(2L+3)}\right]^{\frac{1}{2}} \frac{1}{L+1} P_L'''(\cos\theta)$$

$$+ b_1(L, L; M_{L-1}) \left[\frac{2(2L+1)}{L(L+1)(L-1)(2L-1)}\right]^{\frac{1}{2}} \frac{1}{L} P_L'''(\cos\theta)$$

$$H_6 = b_1(L+1,L;E_L)\left[\frac{1}{2L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{-\frac{(L-1)(L+2)}{L+1}P_L'(\cos\theta) + \frac{8\cos\theta}{L+1}P_L''(\cos\theta) + \frac{2(\cos\theta)^2}{L+1}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L,L;E_L)\left[\frac{2(2L+1)}{L(L+1)}\right]^{\frac{1}{2}}\left\{\frac{(L+2)(L-1)}{L(L+1)}P_L'(\cos\theta) - \frac{4\cos\theta}{L(L+1)}P_L''(\cos\theta) - \frac{(\cos\theta)^2}{L(L+1)}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;E_L)\left[\frac{2}{L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{-\frac{(L+2)(L-1)}{L}P_L'(\cos\theta) + \frac{4\cos\theta}{L}P_L''(\cos\theta) + \frac{(\cos\theta)^2}{L}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L+1,L;E_{L+2})\left[\frac{2}{(L+1)(L+3)(2L+5)}\right]^{\frac{1}{2}}\left\{(L+4)P_L'(\cos\theta) + \frac{(2L+4)\cos\theta}{L+2}P_L''(\cos\theta) + \frac{(\cos\theta)^2}{L+2}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;E_{L-2})\left[\frac{2}{L(L-2)(2L-3)}\right]^{\frac{1}{2}}\left\{(L-3)P_L'(\cos\theta) - \frac{(2L-5)\cos\theta}{L-1}P_L''(\cos\theta) + \frac{(\cos\theta)^2}{L-1}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L+1,L;M_{L+1})\left[\frac{2}{L+1}\right]^{\frac{1}{2}}\left\{-\frac{2}{L+1}P_L'(\cos\theta) - \frac{(L+5)\cos\theta}{(L+1)(L+2)}P_L''(\cos\theta) - \frac{(\cos\theta)^2}{(L+1)(L+2)}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L,L;M_{L+1})\left[\frac{2(2L+1)}{L(L+1)(2L+3)(L+2)}\right]^{\frac{1}{2}}\left\{\frac{2(L+2)}{L+1}P_L'(\cos\theta) + \frac{(L+5)\cos\theta}{L+1}P_L''(\cos\theta) + \frac{(\cos\theta)^2}{L+1}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L,L;M_{L-1})\left[\frac{2(2L+1)}{L(L+1)(L-1)(2L-1)}\right]^{\frac{1}{2}}\left\{-\frac{2(L-1)}{L}P_L'(\cos\theta) - \frac{(L-4)\cos\theta}{L}P_L''(\cos\theta) + \frac{(\cos\theta)^2}{L}P_L'''\right\}$$

$$+ b_1(L-1,L;M_{L-1})\left[\frac{2}{L}\right]^{\frac{1}{2}}\left\{\frac{2}{L}P_L'(\cos\theta) + \frac{(L-4)\cos\theta}{L(L-1)}P_L''(\cos\theta) - \frac{(\cos\theta)^2}{L(L-1)}P_L'''(\cos\theta)\right\}$$

$$H_7 = b_1(L+1,L;E_L)\left[\frac{1}{2L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{-\frac{1}{L+1}P_L'(\cos\theta) - \frac{2\cos\theta}{L+1}P_L''(\cos\theta)\right\}$$

$$+ b_1(L,L;E_L)\left[\frac{1}{2L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{\frac{1}{L(L+1)}P_L'(\cos\theta) + \frac{2\cos\theta}{L(L+1)}P_L''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;E_L)\left[\frac{L+1}{2L(2L+1)}\right]^{\frac{1}{2}}\left\{-\frac{1}{L(L+1)}P_L'(\cos\theta) - \frac{2\cos\theta}{L(L+1)}P_L''(\cos\theta)\right\}$$

$$+b_1(L+1,L;\,E_{L+2})\left[\frac{1}{2(L+1)(L+3)(2L+5)}\right]^{\frac{1}{2}}\left\{-\,2P_L'(\cos\theta)-\frac{2\cos\theta}{L+2}P_L''(\cos\theta)\right\}$$

$$+b_1(L-1,L;\,E_{L-2})\left[\frac{1}{2L(L-2)(2L-3)}\right]^{\frac{1}{2}}\left\{2P_L'(\cos\theta)-\frac{2}{L-1}\cos\theta\,P_L''(\cos\theta)\right\}$$

$$+b_1(L+1,L;\,M_{L+1})\left[\frac{1}{2(L+1)}\right]^{\frac{1}{2}}\left\{-\frac{L}{L+1}P_L'(\cos\theta)+\frac{2\cos\theta}{(L+1)(L+2)}P_L''(\cos\theta)\right\}$$

$$+b_1(L,L;\,M_{L+1})\left[\frac{2L+1}{2L(L+1)(L+2)(2L+3)}\right]^{\frac{1}{2}}\left\{\frac{L(L+2)}{L+1}P_L'(\cos\theta)-\frac{2\cos\theta}{L+1}P_L''(\cos\theta)\right\}$$

$$+b_1(L,L;\,M_{L-1})\left[\frac{2L+1}{2L(L+1)(L-1)(2L-1)}\right]^{\frac{1}{2}}\left\{\frac{L^2-1}{L}P_L'(\cos\theta)-\frac{2\cos\theta}{L}P_L''(\cos\theta)\right\}$$

$$+b_1(L-1,L;\,M_{L-1})\left[\frac{1}{2L}\right]^{\frac{1}{2}}\left\{-\frac{L+1}{L}P_L'(\cos\theta)+\frac{2}{L(L-1)}\cos\theta\,P_L''(\cos\theta)\right\}$$

$$H_8=b_1(L+1,L;\,E_L)\left[\frac{2}{L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{-L\,P_L(\cos\theta)+\frac{2\cos\theta}{L+1}P_L'(\cos\theta)+\frac{(\cos\theta)^2}{L+1}P_L''(\cos\theta)\right\}$$

$$+b_1(L,L;\,E_L)\left[\frac{2L+1}{2L(L+1)}\right]^{\frac{1}{2}}\left\{2P_L(\cos\theta)-\frac{4\cos\theta}{L(L+1)}P_L'(\cos\theta)-\frac{2(\cos\theta)^2}{L(L+1)}P_L''(\cos\theta)\right\}$$

$$+b_1(L-1,L;\,E_L)\left[\frac{1}{2L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{-2(L+1)P_L(\cos\theta)+\frac{4\cos\theta}{L}P_L'(\cos\theta)+\frac{2(\cos\theta)^2}{L}P_L''(\cos\theta)\right\}$$

$$+b_1(L+1,L;\,E_{L+2})\left[\frac{2(L+3)}{(L+1)(2L+5)}\right]^{\frac{1}{2}}\left\{\frac{L+1}{L+2}P_L(\cos\theta)+\frac{2\cos\theta}{L+3}P_L'(\cos\theta)+\frac{(\cos\theta)^2}{(L+2)(L+3)}P_L''(\cos\theta)\right\}$$

$$+b_1(L-1,L;\,E_{L-2})\left[\frac{2(L-2)}{L(2L-3)}\right]^{\frac{1}{2}}\left\{\frac{L}{L-1}P_L(\cos\theta)-\frac{(2L-3)\cos\theta}{(L-1)(L-2)}P_L'(\cos\theta)\right.$$

$$\left.+\frac{(\cos\theta)^2}{(L-1)(L-2)}P_L''(\cos\theta)\right\}$$

$$+ b_1(L+1,L;M_{L+1})\left[\frac{2}{L+1}\right]^{\frac{1}{2}}\left\{\frac{L^2+2L-1}{2(L+2)}P_L(\cos\theta)+\frac{L^2+L-4}{2(L+1)(L+2)}\cos\theta\, P_L'(\cos\theta)\right.$$
$$\left.-\frac{(\cos\theta)^2}{(L+1)(L+2)}P_L''(\cos\theta)\right\}$$

$$+ b_1(L,L;M_{L+1})\left[\frac{2L(L+2)(2L+1)}{(L+1)(2L+3)}\right]^{\frac{1}{2}}\left\{\frac{L+2}{2(L+1)}P_L(\cos\theta)-\frac{(L^2+L-4)\cos\theta}{2L(L+1)(L+2)}P_L'(\cos\theta)\right.$$
$$\left.+\frac{(\cos\theta)^2}{L(L+1)(L+2)}P_L''(\cos\theta)\right\}$$

$$+ b_1(L,L;M_{L-1})\left[\frac{2(2L+1)}{L(L+1)(L-1)(2L-1)}\right]^{\frac{1}{2}}\left\{\frac{1}{2}(L+1)(L-2)P_L(\cos\theta)-\frac{L^2+L-4}{2}\cos\theta\, P_L'(\cos\theta)\right.$$
$$\left.+\frac{(\cos\theta)^2}{L}P_L''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;M_{L-1})\left[\frac{2}{L}\right]^{\frac{1}{2}}\left\{-\frac{L(L-2)}{2(L-1)}P_L(\cos\theta)\right.$$
$$\left.+\frac{(L^2+L-4)\cos\theta}{2L(L-1)}P_L'(\cos\theta)-\frac{(\cos\theta)^2}{L(L-1)}P_L''(\cos\theta)\right\}$$

$$H_q = b_1(L+1,L;E_L)\left[\frac{2}{L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{-\frac{3}{2(L+1)}P_L''(\cos\theta)-\frac{\cos\theta}{L+1}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L,L;E_L)\left[\frac{2(2L+1)}{L(L+1)}\right]^{\frac{1}{2}}\left\{\frac{3}{2L(L+1)}P_L''(\cos\theta)+\frac{\cos\theta}{L(L+1)}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;E_L)\left[\frac{2}{L(L+1)(2L+1)}\right]^{\frac{1}{2}}\left\{-\frac{3}{2L}P_L''(\cos\theta)-\frac{\cos\theta}{L}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L+1,L;E_{L+2})\left[\frac{2}{(L+1)(L+3)(2L+5)}\right]^{\frac{1}{2}}\left\{-\frac{\cos\theta}{L+2}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L-1,L;E_{L-2})\left[\frac{2(L-2)}{L(2L-3)}\right]^{\frac{1}{2}}\left\{\frac{1}{L-1}P_L''(\cos\theta)-\frac{\cos\theta}{(L-1)(L-2)}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L+1,L;M_{L+1})\left[\frac{2}{L+1}\right]^{\frac{1}{2}}\left\{\frac{L+4}{2(L+1)(L+2)}P_L''(\cos\theta)+\frac{\cos\theta}{(L+1)(L+2)}P_L'''(\cos\theta)\right\}$$

$$+ b_1(L,L;\, M_{L+1}) \left[\frac{2(2L+1)}{L(L+1)(L+2)(2L+3)}\right]^{\frac{1}{2}} \left\{ -\frac{3}{2(L+1)} P_L''(\cos\theta) - \frac{\cos\theta}{L+1} P_L'''(\cos\theta) \right\}$$

$$+ b_1(L,L;\, M_{L-1}) \left[\frac{2(2L+1)}{L(L+1)(L-1)(2L-1)}\right]^{\frac{1}{2}} \left\{ \frac{L-3}{2L} P_L''(\cos\theta) - \frac{\cos\theta}{L} P_L'''(\cos\theta) \right\}$$

$$+ b_1(L-1,L;\, M_{L-1}) \left[\frac{2}{L}\right]^{\frac{1}{2}} \left\{ -\frac{L-3}{L(L-1)} P_L''(\cos\theta) + \frac{\cos\theta}{L(L-1)} P_L'''(\cos\theta) \right\}$$

$$H_{10} = b_0(L,L;\, E_L) \left[\frac{2L+1}{2}\right]^{\frac{1}{2}} \frac{1}{2L(L+1)} \left\{ \tfrac{1}{2} P_L'(\cos\theta) + \cos\theta\, P_L''(\cos\theta) \right\}$$

$$+ b_0(L,L;\, M_{L+1}) \left[\frac{(2L+1)(L+2)}{2(2L+3)}\right]^{\frac{1}{2}} \cdot \frac{1}{2(L+1)} \left\{ -\tfrac{1}{2} P_L'(\cos\theta) - \frac{\cos\theta}{L+2} P_L''(\cos\theta) \right\}$$

$$+ b_0(L,L;\, M_{L-1}) \left[\frac{(2L+1)(L-1)}{2(2L-1)}\right]^{\frac{1}{2}} \frac{1}{2L} \left\{ -\tfrac{1}{2} P_L'(\cos\theta) - \frac{1}{L-1} \cos\theta\, P_L''(\cos\theta) \right\}$$

$$H_{11} = -\, b_0(L,L;\, E_L) \left[\frac{2L+1}{2}\right]^{\frac{1}{2}} \frac{1}{2L(L+1)} P_L''(\cos\theta)$$

$$+ b_0(L,L;\, M_{L+1}) \left[\frac{2L+1}{2(L+2)(2L+3)}\right]^{\frac{1}{2}} \frac{1}{2(L+1)} P_L''(\cos\theta)$$

$$+ b_0(L,L;\, M_{L-1}) \left[\frac{2L+1}{2(L-1)(2L-1)}\right]^{\frac{1}{2}} \frac{1}{2L} P_L''(\cos\theta)$$

$$H_{12} = b_0(L,L;\ EL)\left[\frac{2L+1}{2}\right]^{\frac{1}{2}}\left\{-\tfrac{1}{2}P_L(\cos\theta) + \frac{\cos\theta}{2L(L+1)}\,P_L'(\cos\theta)\right\}$$

$$+\,b_0(L,L;\ M_{L+1})\left[\frac{2L+1}{2(L+2)(2L+3)}\right]^{\frac{1}{2}}\left\{\tfrac{1}{2}P_L(\cos\theta) - \frac{1}{2(L+1)}\cos\theta\,P_L'(\cos\theta)\right\}$$

$$+\,b_0(L,L;\ M_{L-1})\left[\frac{2L+1}{2(L-1)(2L-1)}\right]^{\frac{1}{2}}\left\{\tfrac{1}{2}P_L(\cos\theta) + \frac{1}{2L}\cos\theta\,P_L'(\cos\theta)\right\}$$

20. The Application of Unitarity to Deuteron Photodisintegration.

The application of unitarity to deuteron photodisintegration is well known [101], and we outline it here for the sake of completeness.

For a fixed angular momentum J, parity R_L, final spin and a given multipole, the possible reaction channels are

$$\begin{aligned} n + p &\longrightarrow n + p \\ \gamma + d &\longrightarrow n + p \\ \gamma + d &\longrightarrow \gamma + d \end{aligned} \qquad (20.1)$$

The submatrix of the transition amplitude referring to the processes (20.1) may be taken to be symmetric. Denoting the initial and final orbital angular momenta by L and L′ respectively, the submatrix T takes the form

$$T_1 = \begin{array}{c} L_1' \\ L_2' \\ \\ \end{array} \left(\begin{array}{ccc} \begin{array}{cc} L_1 & L_2 \end{array} & & \\ \left(n+p \rightarrow n+p \right) & & \begin{array}{c} b_1 \\ b_2 \end{array} \\ b_1 \quad b_2 & & A \end{array} \right) \qquad (20.2)$$

where b_1 and b_2 are the appropriate multipole amplitudes and A is the amplitude for $\gamma + d \longrightarrow \gamma + d$. The unitarity of the S-matrix implies that, on the energy shell,

$$T - T^{\dagger} = -2i\, T^{\dagger} T \qquad (20.3)$$

In the representation in which the n-p scattering is diagonal the complete submatrix has the form

$$T_0 = \begin{pmatrix} -\sin\delta_\alpha e^{i\delta_\alpha} & 0 & b_\alpha \\ 0 & -\sin\delta_\beta e^{i\delta_\beta} & b_\beta \\ b_\alpha & b_\beta & \end{pmatrix} \tag{20.4}$$

which is unitarily connected to T_1 by

$$T_1 = \quad U^{-1}\, T_0\, U \tag{20.5}$$

where

$$U = \begin{pmatrix} \cos\varepsilon & \sin\varepsilon & 0 \\ -\sin\varepsilon & \cos\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{20.6}$$

Applying (20.3) to (20.4), we obtain

$$b_\alpha = \; |M_\alpha|\, e^{\,i\delta_\alpha} \tag{20.7a}$$

$$b_\beta = \; |M_\beta|\, e^{\,i\delta_\beta} \tag{20.7b}$$

Making use of equation (20.5) we see that

$$\begin{aligned} b_1 &= \cos\varepsilon\, b_\alpha - \sin\varepsilon\, b_\beta \\ b_2 &= \sin\varepsilon\, b_\alpha + \cos\varepsilon\, b_\beta \end{aligned} \tag{20.8}$$

In the case of uncoupled states, (20.7) simply becomes

$$b = \; |M|\, e^{\,i\delta} \tag{20.9}$$

From equations (20.7), (20.8) and (20.9) we find that the appropriate expressions relating the multipole amplitudes for the deuteron disintegration to the scattering phase shifts are

$$b_0(J,J;\,E_J) = |E_J|\, e^{i\delta_J}$$

$$b_0(J,J;\,M_{J+1}) = |M_{J+1}|\, e^{i\delta_J}$$

$$b_0(J,J;\,M_{J-1}) = |M_{J-1}|\, e^{i\delta_J}$$

$$b_1(J,J;\,E_J) = |E_{JJ}|\, e^{i\delta_{JJ}}$$

$$b_1(J,J+1;\,M_{J+1}) = |M_{J+1,J}|\, e^{i\delta_{J+1,J}}$$

$$b_1(J,J-1;\,M_{J-1}) = |M_{J-1,J}|\, e^{i\delta_{J-1,J}}$$

$$b_1(J,J-1;\,E_{J-1}) = \cos\varepsilon_J\, |E_{J-1,\alpha}|\, e^{i\delta_{J,\alpha}} - \sin\varepsilon_J\, |E_{J-1,\beta}|\, e^{i\delta_{J,\beta}}$$

$$b_1(J,J-1;\,E_{J+1}) = \cos\varepsilon_J\, |E_{J+1,\alpha}|\, e^{i\delta_{J\alpha}} - \sin\varepsilon_J\, |E_{J+1,\beta}|\, e^{i\delta_{J,\beta}}$$

$$b_1(J,J-1;\,M_J) = \cos\varepsilon_J\, |M_{J\alpha}|\, e^{i\delta_{J\alpha}} - \sin\varepsilon_J\, |M_{J,\beta}|\, e^{i\delta_{J,\beta}}$$

$$b_1(J,J+1;\,E_{J+1}) = \sin\varepsilon_J\, |E_{J+1,\alpha}|\, e^{i\delta_{J\alpha}} + \cos\varepsilon_J\, |E_{J+1,\beta}|\, e^{i\delta_{J,\beta}}$$

$$b_1(J,J+1;\,E_{J-1}) = \sin\varepsilon_J\, |E_{J-1,\alpha}|\, e^{i\delta_{J\alpha}} + \cos\varepsilon_J\, |E_{J-1,\beta}|\, e^{i\delta_{J,\beta}}$$

$$b_1(J,J+1;\,M_J) = \sin\varepsilon_J\, |M_{J,\alpha}|\, e^{i\delta_{J\alpha}} + \cos\varepsilon_J\, |M_{J,\beta}|\, e^{i\delta_{J,\beta}}$$

21. The Dispersion Relations.

Using the standard procedures for contracting the photon and one of the nucleons, we may write the S-matrix as a retarded commutator of the photon and nucleon currents, $i_\mu(y)$ and $f_\alpha(x)$, which are defined by

$$\left(i\gamma_\nu \frac{\partial}{\partial x_\nu} - m\right)_{\alpha\beta} \psi_\beta(x) = f_\alpha(x)$$
$$\Box_y A_\mu(y) = i_\mu(y) \tag{21.1}$$

The result is

$${}_{out}\langle p_1 s_1; p_2 s_2 | d, s; k, \nu \rangle_{in} = (2\pi)^4 i\, \delta(p_1 + p_2 - d - k) \left\{ \frac{m}{(2\pi)^6\, 2k_0 E_1} \right\}^{\frac{1}{2}}$$
$$\times \bar{u}_\alpha(p_1) \int d^4X\, e^{i\frac{(p_1+k)}{2}\cdot X} \langle p_2 s_2 | R[f_\alpha(\tfrac{X}{2}), i_\mu(-\tfrac{X}{2})] | d, s\rangle\, \varepsilon_\mu^\nu \tag{21.2}$$
$$+ P_n$$

where the retarded commutator R is defined by

$$R[A(x), B(y)]_- = -i\theta(x_0 - y_0)[A(x), B(y)]_- \tag{21.3}$$

and P_n is a polynomial of arbitrary degree n, arising from an equal-times singularity in a T-product. However we are going to write unsubtracted dispersion relations (21.10), thus implicitly assuming P_n to be zero, and consequently will treat it as such for the remainder of this section.

Defining the amplitude $F_{\beta\alpha}$ by

$$S_{\beta\alpha} = \delta_{\beta\alpha} + (2\pi)^4 i\, \delta(p_1 + p_2 - d - k) \left\{ \frac{m^2}{(2\pi)^{12} 4 k_0 d_0 E_1 E_2} \right\}^{\frac{1}{2}} F_{\beta\alpha} \tag{21.4}$$

so that

$$F_{\beta\alpha} = \bar{u}_\alpha(p_1)\, \bar{u}_\beta(p_2)\, {}^{\alpha\beta} M_{\gamma\delta}{}^{\mu}(p_1 p_2, d, k)\, \underset{\sim}{\varepsilon}^{\nu}(k)\, S_{\gamma\delta}(d) \tag{21.5}$$

we have

$$F_{\beta\alpha} = \left\{ \frac{(2\pi)^6\, 2 d_0 E_2}{m} \right\}^{\frac{1}{2}} \bar{u}_\alpha(p_1) \int d^4x\, e^{i\frac{(p_1+k)}{2}\cdot x} \langle p_2 s_2 | R[f_\alpha(\tfrac{x}{2}), j_\mu(-\tfrac{x}{2})] | d, s \rangle \underset{\sim}{\varepsilon}^{\nu} \tag{21.6}$$

The absorptive amplitude $A_{\beta\alpha}$ can then be obtained from (21.6) in the usual manner. Inserting a complete set of physical states and performing a space-time integration, we find

$$A_{\beta\alpha} = A^{(1)}_{\beta\alpha} + A^{(2)}_{\beta\alpha} \tag{21.7}$$

where

$$A^{(1)}_{\beta\alpha} = -\frac{1}{2} \left\{ \frac{(2\pi)^6\, 2 d_0 E_2}{m} \right\}^{\frac{1}{2}} \bar{u}_\alpha(p_1) \sum_n \int d\underset{\sim}{\ell}_n\, (2\pi)^4\, \delta(k + d - n) \times \langle p_2 s_2 | f_\alpha(0) | n \rangle \langle n | j_\mu(0) | d, s \rangle\, \underset{\sim}{\varepsilon}^{\nu}(k) \tag{21.8}$$

$$A^{(2)}_{\beta\alpha} = \frac{1}{2} \left\{ \frac{(2\pi)^6\, 2 d_0 E_2}{m} \right\}^{\frac{1}{2}} \bar{u}_\alpha(p_1) \sum_n \int d\underset{\sim}{\ell}_n\, (2\pi)^4\, \delta(d - p_1 - n) \times \langle p_2 s_2 | j_\mu(0) | n \rangle \langle n | f_\alpha(0) | d, s \rangle\, \underset{\sim}{\varepsilon}^{\nu}(k) \tag{21.9}$$

The lowest mass states corresponding to $A^{(1)}$ and $A^{(2)}$ are given in Figures 10 and 11 respectively.

The diagrams of Figure 10 have a discrete pole at $\nu = M^2$ (the deuteron intermediate state) and branch cuts starting at the physical threshold for the reaction, namely $\nu = 4m^2$.

A section of this physical cut is "unphysical" in the sense that for values of ν near $4m^2$, $|\cos\theta| > 1$. The explicit relation is, given ν_1 fixed, then this region extends over all values of ν which satisfy

$$\frac{4\nu\nu_1^2}{(\nu-4m^2)(\nu-M^2)^2} > 1 \tag{21.10}$$

i.e. for all ν such that

$$4m^2 < \nu < \nu_{Max} \tag{21.11}$$

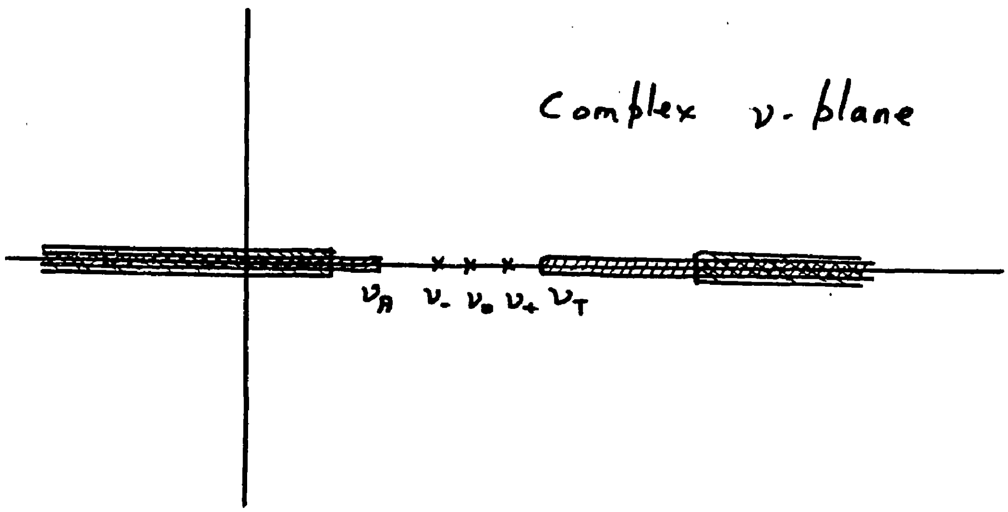
where ν_{Max} is the solution of

$$4\nu\nu_1^2 = (\nu-4m^2)(\nu-M^2)^2 \tag{21.12}$$

In practice this unphysical region can be made arbitrarily small by choosing ν_1 sufficiently small, and vanishes altogether in the limit of $\nu_1 \longrightarrow 0$. This latter limit corresponds to 90° for the outgoing nucleons in the centre of momentum system.

The true unphysical region comes from the crossed cuts of Figure 11. There are two discrete poles at $\nu = M^2 \pm 2\nu_1$ (the one-nucleon intermediate states) plus branch cuts starting at $\nu = M^2 \pm 2\nu_1 - 2\mu(\mu + 2\alpha)$, $\alpha^2 = -m\epsilon$ which arise from the anomalous threshold of the d-np vertex (97). The same anomalous threshold is present also in the next highest mass state, Figure 11b, for which the expected normal thresholds are at $\nu = M^2 \pm 2\nu_1 - 2\mu(\mu + 2m)$

The spectrum of the invariant amplitudes $M_i(\nu, \nu_1)$ is thus



with

$$\nu_A = M^2 + 2|\nu_1| - 2\mu(\mu + 2m) \qquad (21.13a)$$

$$\nu_- = M^2 - 2|\nu_1| \qquad (21.13b)$$

$$\nu_+ = M^2 + 2|\nu_1| \qquad (21.13c)$$

$$\nu_0 = M^2 \qquad (21.13d)$$

$$\nu_T = 4m^2 \qquad (21.13e)$$

For the crossed diagrams, the imaginary part of the amplitude is related to processes such as the radiative absorption of an anti-nucleon by the deuteron, and to the structure of the deuteron through the anomalous singularities of the d-np vertex. Although it is clear in principle how these latter singularities can be included [97], we shall ignore them in calculation, as well as the rest of the unphysical region with the exception of the pole-terms. In this approximation we can write the dispersion relations as

$$M_i(\nu,\nu_1) = \frac{A_i^{(-)}(\nu_-,\nu_1)}{(\nu_- - \nu)} + \frac{A_i^{(+)}(\nu_+,\nu_1)}{(\nu_+ - \nu)} + \frac{A_i^{D}(\nu_D,\nu_1)}{\nu_D - \nu} + \frac{1}{\pi}\int \frac{\mathrm{Im}\, M_i(\nu'\nu_1)\, d\nu'}{\nu' - \nu - i\varepsilon} \qquad (21.14)$$

with the pole terms separated explicitly.

Writing equations (19.8) - (19.19) symbolically as

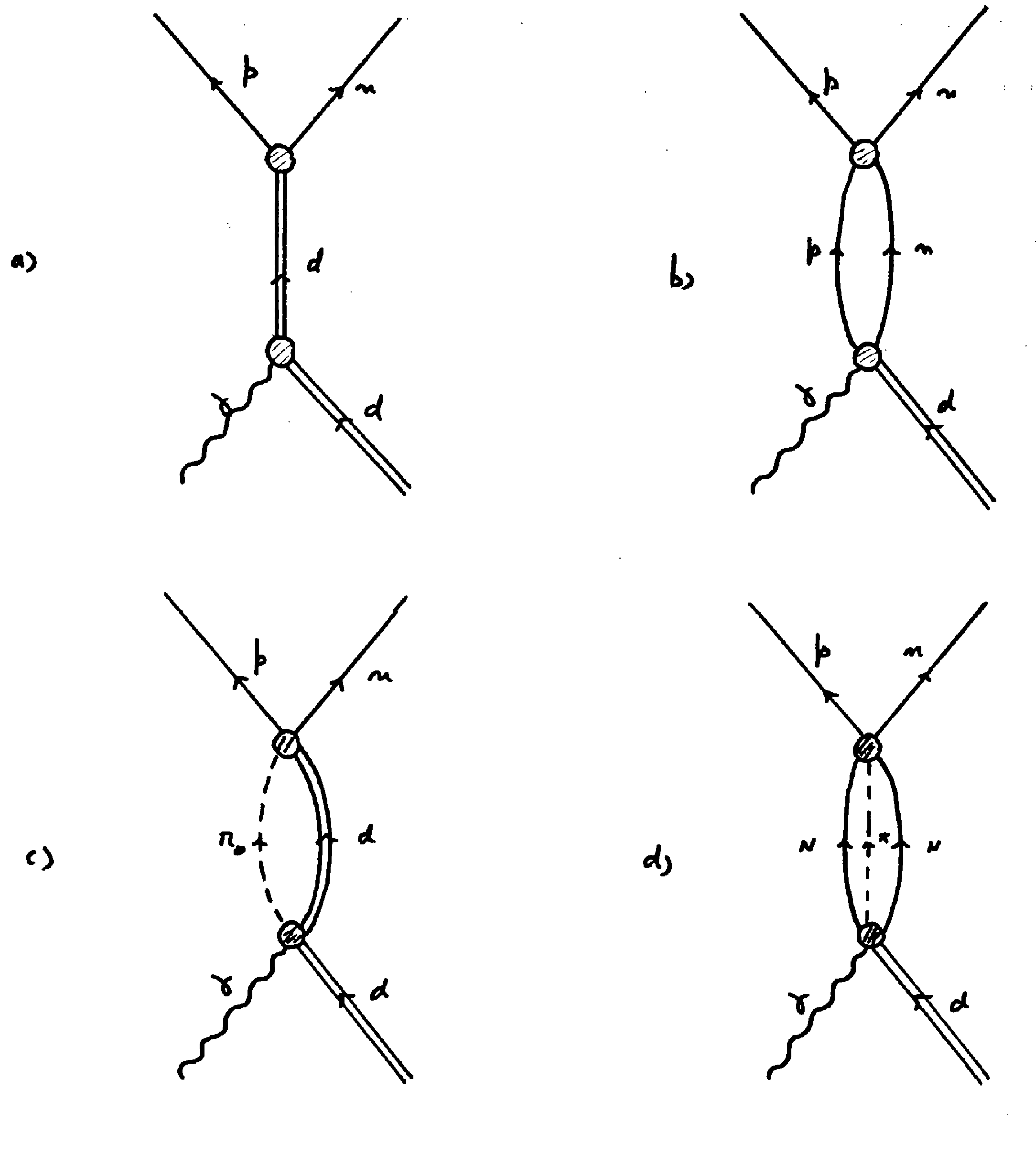
$$H_i = \sum_{J,L,\lambda,s'} \left\{ f_i(J,L,\theta)\, b_{s'}(J,L;\epsilon_\lambda) + g_i(J,L,\theta)\, b_{s'}(J,L;M_\lambda) \right\} \qquad (21.15)$$

we can put

$$M_i = A_{ij}^{-1} H_j = A_{ij}^{-1} \sum \left\{ f_j\, b_s(\epsilon_\lambda) + g_j\, b_{s'}(M_\lambda) \right\} \qquad (21.16)$$

We assume that (21.16) is valid for the whole of the cut on the positive real axis, including the region for which $|\cos\theta| > 1$. Then equations (21.14), (21.16) and (20.10) constitute a complete statement of the problem.

Figure 10.

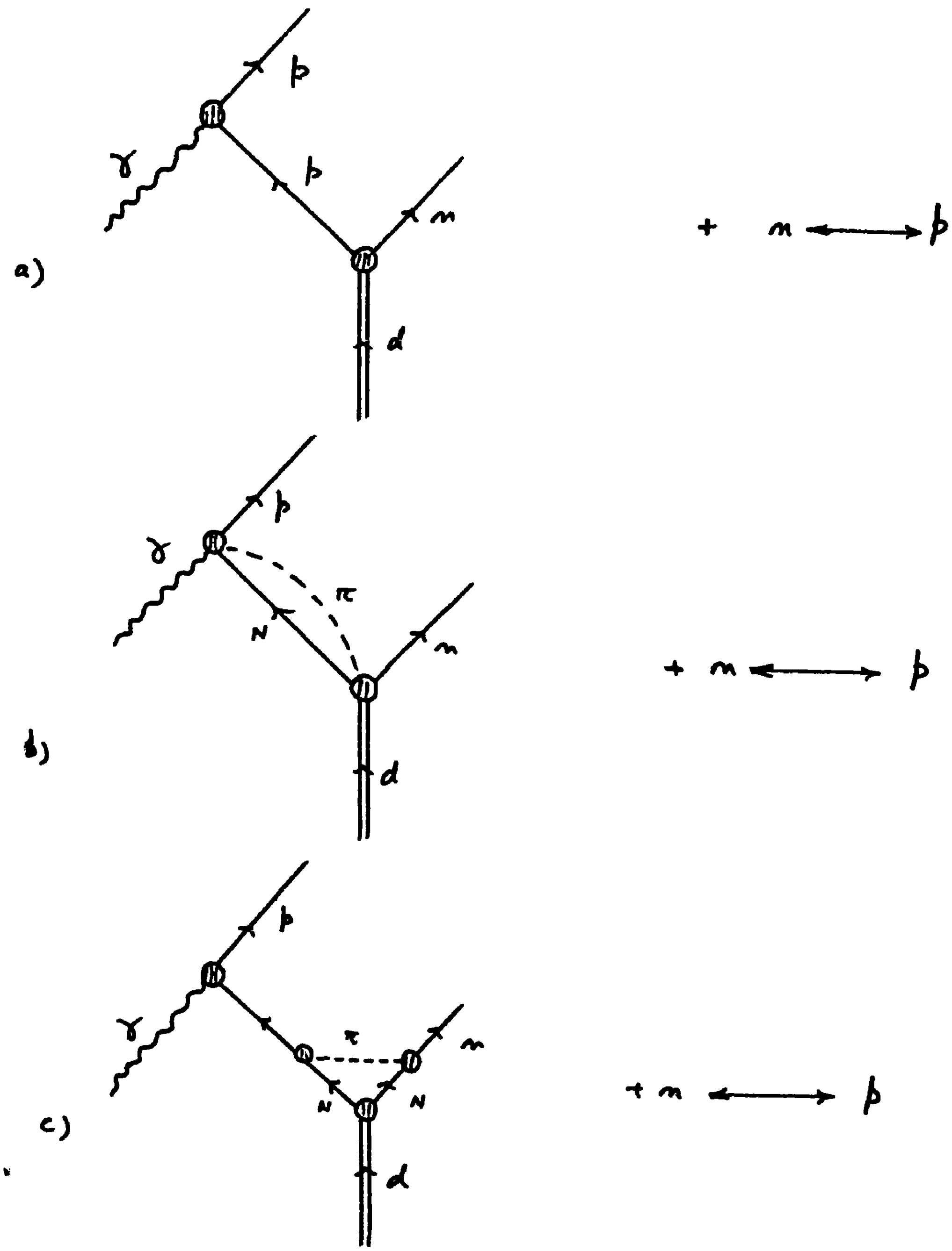


a) deuteron pole

b) elastic n-p rescattering cut

c), d) lowest mass inelastic cuts.

Figure 11



a) proton and neutron crossed poles

b) pion-nucleon crossed cuts

c) part of figure 11(b) exhibiting the nearest anomalous singularity

22. Vertex Functions and the Discrete Contributions.

The contributions of the pole-terms to the dispersion relations may be obtained directly from a knowledge of the γ-p, γ-n, γ-D and D-np vertex functions, on the mass-shell.

1) γ-N vertex.

$$\langle p' | i_\mu(0) | p \rangle = \left\{ \frac{m^2}{(2\pi)^6 E E'} \right\}^{\frac{1}{2}} \bar{u}(p') \left\{ F_1(k^2)\, \gamma_\mu + \tfrac{1}{2} F_2(k^2) \left[(p'-p).\gamma , \gamma_\mu \right]_- \right\} \tag{22.1}$$

where $k^2 = (p'-p)^2$ is the invariant momentum transfer.

$$F_1(0) = \tfrac{1}{2}(1+\tau_3)\, e, \quad F_2(0) = \frac{e}{2m}\left[\tfrac{1}{2}(1+\tau_3)\gamma_p + \tfrac{1}{2}(1-\tau_3)\gamma_n \right] \tag{22.2}$$

where γ_p and γ_n are respectively the proton and neutron anomalous magnetic moments in units of the nucleon Bohr magneton.

2) d-np vertex.

This vertex has been discussed extensively by several authors,[22,96,97]. The argument we give is due to Blankenbecler et al.[96].

Consider the matrix element

$$L = (2\pi)^4\, \delta(p+n-d) \left\{ \frac{(2\pi)^6\, 2 d_0 n_0}{m} \right\}^{\frac{1}{2}} \bar{u}(p) \langle n | f(0) | d \rangle \tag{22.3}$$

Writing the delta-function as an integral, contracting the neutron and using equation (2.6), we find

$$L = \bar{u}(p)\, \bar{u}(n) \int d^4x\, d^4y\; e^{i(P-d).x}\, e^{iQ.y}\, G_p^{-1} G_n^{-1} \psi_d(y) \tag{22.4}$$

where $P = p + n, \quad Q = \frac{1}{2}(p - n)$

The integration over the c.m. co-ordinate may now be performed, and introducing the Fourier transform of $\psi_d(y)$ by

$$\psi_d(Q) = \frac{1}{(2\pi)^2} \int d^4y \; e^{iQ \cdot y} \; \psi_d(y) \qquad (22.5)$$

we find

$$L = (2\pi)^4 \; \delta(p+n-d) \, (2\pi)^2 \, \bar{u}(p) \, \bar{u}(n) \, G_p^{-1} \, G_n^{-1} \; \psi_d(Q)$$

$$= (2\pi)^4 \; \delta(p+n-d) \; (2\pi)^2 \, \bar{u}(p) \, \bar{u}(n) \; i \int d^4Q' \, I(Q', Q) \, \psi_d(Q') \qquad (22.6)$$

where $I(Q', Q)$ is the Fourier transform of the generalised interaction $I(x_1, x_2)$

Equation (22.6) is then a complete covariant description of the d-np vertex. However the functions $I(Q', Q)$ and $\psi_d(Q')$ are not known except in the non-relativistic limit, and to proceed to this limit we follow the prescription given by E.E. Salpeter [102]. In essence, the generalised interaction $I(Q', Q)$ is replaced by a interaction which is quite general except that it depends only on $\underset{\sim}{q} = \underset{\sim}{Q}' - \underset{\sim}{Q}$. This is equivalent to replacing the Lorentz invariant, and hence retarded, interaction by an interaction which is instantaneous. We may then introduce an equal-time amplitude $\varphi_d(\underset{\sim}{Q})$ as the integral over Q_0 of $\psi_d(Q)$. In the non-relativistic limit, $\psi_d(q)$ reduces to the Schrodinger wave-function. If we use the Schrodinger equation in conjunction with the instantaneous potential in equation (22.6), then

$$L = -(2\pi)^4 \delta(p+n-d) \frac{2\pi}{m} \bar{u}(p) \bar{u}(n) (\alpha^2 + q^2) \psi_d(q) \qquad (22.7)$$

where $\alpha^2 = -m\varepsilon$

Since $\delta(p+n-d)$ may be replaced by $\delta(\alpha^2 + q^2)$ (making a slight approximation by adding $\frac{q^2}{2m}$, which is numerically negligible), we require to construct $\psi_d(q)$ in the neighbourhood of $q^2 = -\alpha^2$

It is straightforward to show that we can write

$$\psi_d(q) = \frac{1}{2\pi} \frac{N}{\sqrt{4\pi}} \frac{4\pi}{\alpha^2+q^2} \left[1 + \frac{\rho}{\sqrt{8}} S_n(\hat{q}) \right]$$

$$+ (\alpha^2+q^2)\left[\text{other terms}\right] \qquad (22.8)$$

where ρ is defined by

$$\rho = \lim_{r \to \infty} \frac{w_g(r)}{u_g(r)} \qquad (22.9)$$

N may be expressed in terms of the effective range r_D by

$$N^2 = \frac{2\alpha}{1 - \alpha r_D} \qquad (22.10)$$

to give finally

$$L = (2\pi)^4\, \delta(p+n-d)\, \frac{(4\pi)^{\frac{1}{2}} N}{m}\, \bar{u}(p)\,\bar{u}(n)\left[1 + \frac{\rho}{\sqrt{8}}\, s_m(q)\right\} u(p)\, u(n) \qquad (22.11)$$

with $q = (k - n)$

In general, there are four possible transitions for d→n,p and consequently there should be four invariants associated with the d-n,p vertex. However, on the mass-shell (which is the case in question), this reduces to two(97).

Thus we write

$$\left\{\frac{m}{(2\pi)^3 E_p}\right\}^{\frac{1}{2}} \bar{u}(p)\, \langle n | f(0) | d \rangle$$

$$= \left\{\frac{m^2}{(2\pi)^9\, 2d_0\, E_p E_n}\right\} \bar{u}(p)\,\bar{u}(n) \left\{ A((d-p)^2)\, 1^{(1)} 1^{(2)} + B((d-p)^2)\, i\gamma_5^{(1)}\gamma^{(1)}\cdot p\; i\gamma_5^{(2)}\gamma^{(2)}_\mu \right\} \times S_d \qquad (22.12)$$

which satisfies the usual invariance and symmetry requirements, and reduces to the correct form in the non-relativistic limit. Taking this limit and comparing with (22.11) gives immediately

$$A = \frac{\sqrt{4\pi}\, N}{m}\left(1 - \frac{\rho}{\sqrt{8}}\right)$$

$$B = 12\sqrt{\frac{\pi}{2}}\; \frac{N\rho}{2^2\, m} \qquad (22.13)$$

3) Y-D vertex.

The requirements of Lorentz covariance and gauge invariance lead us to write the following form for the matrix element of the current operator between one-deuteron states

$$\langle d' | j_\mu(0) | d \rangle = \left\{\frac{1}{(2\pi)^6\, 2d_0'\, 2d_0}\right\}^{\frac{1}{2}} S_{d'} \Big\{ (d+d')_\mu A(k^2) + \tfrac{1}{2}\left\{ [\gamma^{(1)}\cdot k, \gamma^{(1)}_\mu]_- + [\gamma^{(2)}\cdot k, \gamma^{(2)}_\mu]_- \right\} B(k^2) + i\gamma_5^{(1)}\gamma^{(1)}\cdot k\; i\gamma_5^{(2)}\gamma^{(2)}\cdot k\, (d+d')_\mu\, C(k^2) \Big\} S_d \qquad (22.14)$$

where $k = d' - d$

To determine the mass-shell values of A, B, C, note that

$$\int d\underset{\sim}{x}\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \langle d' | i_\mu(x) | d \rangle = (2\pi)^3 \delta(d' - k - d) \langle d' | i_\mu(0) | d \rangle \qquad (22.15)$$

In the limit $\underset{\sim}{k} \longrightarrow 0$ the left hand side of equation (9.14) is the expectation value of the total charge operator, and by comparison with (3.13) we can show immediately that

$$A(0) = e \qquad (22.16)$$

the total charge of the deuteron.

Operating with $-i\,\underset{\sim}{\nabla}_k$ on the space component of equation (3.14), we have, in the limit, $\underset{\sim}{k} \longrightarrow 0$

$$\langle d' | \int d\underset{\sim}{x}\; \tfrac{1}{2} (\underset{\sim}{x} \times \underset{\sim}{i}(x)) | d \rangle \qquad (22.17)$$

which is just the magnetic moment operator, and so can be written

$$\frac{e}{2m} \gamma_D \langle d' | \underset{\sim}{S} | d \rangle \quad , \quad \underset{\sim}{S} = \tfrac{1}{2} (\sigma^{(1)} + \sigma^{(2)}) \qquad (22.18)$$

where γ_D is the deuteron magnetic moment in units of nuclear Bohr magnetons.

Comparing this with (4.13) we find

$$B(0) = 2M \frac{e}{2m} \gamma_D = e \frac{M}{m} \gamma_D \qquad (22.19)$$

Finally operate with $\nabla_{\underset{\sim}{k}}^2 - 3\nabla_{k_3}^2$ on the time-component of equation (3.13). Taking the limit of $\underset{\sim}{k} \longrightarrow 0$ we obtain

$$\langle d' | \int d\underset{\sim}{x} \, (3x_3^2 - x^2) \, \rho(x) \, | d \rangle \tag{22.20}$$

which is just the quadrupole-moment operator, and so is equal to

$$- e Q \tag{22.21}$$

where Q is the deuteron quadrupole moment. Comparing this with (3.13) we find

$$C(0) = \frac{1}{2} e Q \tag{22.22}$$

The vertex functions may be combined appropriately according to equations (3.8), (3.9) and the intermediate spin seems trivially performed. After some tedious, but straightforward, algebraic manipulation, we obtain the discrete contributions given in Table 11.

Table 11. The Discrete Contributions

Charge Singlet

i	$A_i^{(+)}$	$A_i^{(-)}$	$A_i^{(D)}$
1	$\frac{2MA\gamma^+e}{m(4M^2-\nu_1)}$	$\frac{2MA\gamma^+e}{m(4M^2+\nu_1)}$	$\left[\frac{4A}{m^2}+4BQ\right]e$
2	$\frac{2MA\gamma^+e}{m\nu_1}$	$-\frac{2MA\gamma^+e}{m\nu_1}$	$\left[\frac{2m^2B}{\nu_1}-\frac{QM^2}{\nu_1}A\right]12e$
3	$\frac{A\gamma^+e}{4m}$	$\frac{A\gamma^+e}{4m}$	
4	$\frac{A\gamma^+e}{4m}$	$\frac{A\gamma^+e}{4m}$	
5	$-\frac{B\gamma^+e}{8mM\nu_1}[2\ell^2+\nu_1]$	$+\frac{B\gamma^+e}{8mM\nu_1}[2\ell^2-\nu_1]$	$\frac{4Be}{\nu_1}$
6	$-\frac{B\gamma^+e}{8mM\nu_1}[2\ell^2+\nu_1]$	$+\frac{B\gamma^+e}{8mM\nu_1}[2\ell^2-\nu_1]$	$\frac{2QAe}{\nu_1}$
7	$-\frac{B\gamma^+e}{4mM}[2\ell^2+\nu_1]$	$-\frac{B\gamma^+e}{4mM}[2\ell^2-\nu_1]$	
8	$-\frac{m^2Be}{4\nu_1}[1+\gamma^+][1+\frac{\nu_1}{4m^2}]$	$+\frac{B\gamma^+e}{4\nu_1}[2\ell^2-\nu_1]$	$\frac{4M^2(m+M)}{m\nu_1}B\gamma_D e$
9	$-\frac{Ae}{4}[1+\gamma^+][1+\frac{\nu_1}{4m^2}]$	$-\frac{m^2Be}{4}[1-\gamma^+][1-\frac{\nu_1}{4m^2}]$	
10	$-\frac{A[\text{illegible}]e}{4[\text{illegible}]}[1+\gamma^+][1+\frac{\nu_1}{4m^2}]$	$-\frac{Ae}{4}[1+\gamma^+][1-\frac{\nu_1}{4m^2}]$	$\frac{2A\nu_1\gamma_D e}{m(M+2m)}$
11	$\frac{A\gamma^+e}{4m^2}$	$\frac{A\gamma^+e}{4m^2}$	
12	$\frac{A\gamma^+e}{4m^2}$	$\frac{A\gamma^+e}{4m^2}$	

The charge triplet can be obtained by replacing Υ^+ by Υ^- and putting all $A_i^{(D)}$ to zero.

23. Solution of the Dispersion Relations at Low Energies

For the restricted calculations considered here, that is the calculation of the dipole transition amplitudes at low energies, it is convenient to express the dispersion relations in terms of the amplitudes $E_{\ell J}$ and M_J of Section 13 and the amplitudes K of Section 16 rather than the general amplitudes $b_{\nu}(J,L;E\lambda)$ and $b_{\nu'}(J,L;M\lambda)$ of Section 19. This can be done directly from equations (13.3) to (13.6) and (16.4) to give

$$H_1 = \left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \cdot \frac{\pi k_0}{60\, p\alpha} \cdot \frac{N}{(2\pi)^{3/2}(4\pi)^{1/2}} \cdot \Big\{ - \Big[20 E_{10} e^{-i\delta_{10}} + 30 E_{11} e^{-i\delta_{11}} + 14(\xi_2 + 4\eta_2) E_{12} e^{-i\delta_{12}} - 14(6\xi_2 - \eta_2) E_{32} e^{-i\delta_{32}} \Big] - \frac{4 W \nu_1}{(W^2-M^2)(W^2-4m^2)^{1/2}} \cdot \frac{k_0}{\alpha} \Big[-9 E_{21} e^{-i\delta_{21}} + 5 E_{22} e^{-i\delta_{22}} - 11 E_{23} e^{-i\delta_{23}} \Big] \Big\}$$

$$H_2 = \left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \cdot \frac{\pi k_0}{6\, p\alpha} \cdot \frac{N}{(2\pi)^{3/2}(4\pi)^{1/2}} \cdot \frac{2 W \nu_1}{(W^2-M^2)(W^2-4m^2)^{1/2}} \Big\{ - \Big[E_{10} e^{-i\delta_{10}} + \tfrac{3}{2} E_{11} e^{-i\delta_{11}} + \tfrac{1}{10}(4\eta_2 - 29\xi_2) E_{12} e^{-i\delta_{12}} - \tfrac{1}{10}(6\xi_2 - 31\eta_2) E_{32} e^{-i\delta_{32}} \Big] - \frac{2 W \nu_1}{(W^2-M^2)(W^2-4m^2)^{1/2}} \frac{k_0}{\alpha} \Big[E_{22} e^{-i\delta_{22}} + E_{23} e^{-i\delta_{23}} \Big] \Big\}$$

$$H_3 = \left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{\pi k_0}{6\, p\alpha} \cdot \frac{N}{(2\pi)^{3/2}(4\pi)^{1/2}} \Big\{ \Big[E_{10} e^{-i\delta_{10}} + \tfrac{3}{2} E_{11} e^{-i\delta_{11}} + \tfrac{1}{10}(29\xi_2 - 4\eta_2) E_{12} e^{-i\delta_{12}} - \tfrac{1}{10}(6\xi_2 - 31\eta_2) E_{32} e^{-i\delta_{32}} \Big] + \frac{2 W \nu_1}{(W^2-M^2)(W^2-4m^2)^{1/2}} \Big[E_{22} e^{-i\delta_{22}} + E_{23} e^{-i\delta_{23}} \Big] \Big\}$$

$$H_4 = -\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{\pi k_0^2}{12 p \alpha^2} \cdot \frac{N}{(4\pi)^{1/2}(2\pi)^{3/2}} \cdot \frac{2 W v_1}{(W^2-4m^2)^{1/2}(W^2-M^2)} \left[E_{22}\, e^{-i\delta_{22}} + 2E_{23}\, e^{-i\delta_{23}} \right]$$

$$H_5 = 0$$

$$H_6 = 0$$

$$H_7 = \left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{\pi k_0}{60 p \alpha} \frac{N}{(4\pi)^{1/2}(2\pi)^{3/2}} \cdot \Big\{ \Big[10 E_{10}\, e^{-i\delta_{10}} - 15 E_{11}\, e^{-i\delta_{11}}$$

$$+ (\xi_2 + 4\eta_2) E_{12}\, e^{-i\delta_{12}} - (6\xi_2 - \eta_2) E_{32}\, e^{-i\delta_{32}} \Big]$$

$$+ \frac{6 W v_1}{(W^2-4m^2)^{1/2}(W^2-M^2)} \cdot \frac{k_0}{\alpha} \left[3 E_{21}\, e^{-i\delta_{21}} - 3 E_{22}\, e^{-i\delta_{22}} + 2 E_{23}\, e^{-i\delta_{23}} \right] \Big\}$$

$$H_8 = \left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{\pi k_0^2}{60 p \alpha^2} \cdot \frac{N}{(4\pi)^{1/2}(2\pi)^{3/2}} \cdot \frac{2 W v_1}{(W^2-4m^2)^{1/2}(W^2-M^2)} \quad E_{01}\, e^{-i\delta_{01}}$$

$$H_9 = \left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{\pi k_0^2}{20 p \alpha^2} \cdot \frac{N}{(4\pi)^{1/2}(2\pi)^{3/2}} \cdot \frac{2 W v_1}{(W^2-4m^2)^{1/2}(W^2-M^2)} \left[3 E_{21}\, e^{-i\delta_{21}} - 5 E_{22}\, e^{-i\delta_{22}} + 2 E_{23}\, e^{-i\delta_{23}} \right]$$

$$H_{10} = i\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{N k_0}{(2\pi)^{3/2}(4\pi)^{1/2}} \Big\{ \frac{2\pi}{p} \cdot \frac{\gamma_p - \gamma_m}{2m} \left[M_0\, e^{-i\Delta_0} - \sqrt{2}\, M_2\, e^{-i\Delta_2} \right]$$

$$+ \frac{2 W v_1}{(W^2-4m^2)^{1/2}(W^2-M^2)} \cdot \frac{k_0}{\alpha} \cdot \frac{\gamma_p + \gamma_m}{2m} \left[\tfrac{1}{2} M_1\, e^{-i\Delta_1} - \tfrac{3}{2} M_3\, e^{-i\Delta_3} \right]$$

$$+ \frac{\gamma_p - \gamma_m}{2m} F(k_0) \left[-\tfrac{1}{12} p^2 K_{02}\, e^{-i\Delta_0} + \tfrac{1}{2} p^2 K_{24}\, e^{-i\Delta_2} \right]$$

$$+ \frac{f^2}{\mu^2} \left[\tfrac{1}{2} K_{01}\, e^{-i\Delta_0} + \left(6K_{22} + \tfrac{1}{2}K_{21} + 2K_{23}\right) e^{-i\Delta_2} \right] \Big\}$$

$$H_{11} = 0$$

$$H_{12} = i\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{N k_0}{(2\pi)^{3/2}(4\pi)^{1/2}} \Big\{ \frac{6\pi}{p} \cdot \frac{\gamma_p - \gamma_m}{2m} \cdot \frac{1}{\sqrt{2}} M_2\, e^{-i\Delta_2}$$

$$+ \frac{2 W v_1}{(W^2-4m^2)^{1/2}(W^2-M^2)} \cdot \frac{k_0}{\alpha} \cdot \frac{\gamma_p + \gamma_m}{2m} \cdot \frac{5}{2} M_3\, e^{-i\Delta_3}$$

$$- \frac{\gamma_p - \gamma_m}{2m} F(k_0)\, 4 p^2 K_{24}\, e^{-i\Delta_2} - \frac{f^2}{\mu^2} \left(6K_{21} + 24K_{23}\right) e^{-i\Delta_2} \Big\}$$

We define the amplitudes K to be zero for

By considering equations (23.1) for two different values of the "momentum transfer" ν_I, say ν_1 and ν_2 we may extract the dipole amplitudes alone, to give

$$-\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{\pi k_0}{60 p_2} \cdot \frac{N}{(2\pi)^{3/2}(4\pi)^{1/2}} \cdot \Big[20 E_{10} e^{-i\delta_{10}} + 30 E_{11} e^{-i\delta_{11}} + 14(\xi_2 + 4\eta_2)\, e^{-i\delta_{12}} E_{12} - 14(6\xi_2 - \eta_2) E_{32} e^{-i\delta_{32}}\Big] = \frac{\nu_2 H_1(\nu_1) - \nu_1 H_1(\nu_2)}{\nu_2 - \nu_1}$$

$$-\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{\pi k_0}{6 p_2} \frac{N}{(2\pi)^{3/2}(4\pi)^{1/2}} \Big[E_{10} e^{-i\delta_{10}} + \tfrac{3}{2} E_{11} e^{-i\delta_{11}} + \tfrac{1}{10}(4\eta_2 - 29\xi_2) E_{12} e^{-i\delta_{12}} - \tfrac{1}{10}(6\xi_2 - 31\eta_2) E_{32} e^{-i\delta_{32}}\Big] = \frac{\nu_2 H_2'(\nu_1) - \nu_1 H_2'(\nu_2)}{\nu_2 - \nu_1}$$

$$\text{with} \quad H_2'(\nu_i) = \frac{2 W \nu_i}{(W^2 - M^2)(W^2 - 4m^2)^{1/2}} \, H_2(\nu_i)$$

$$\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \cdot \frac{\pi k_0}{6 p_2} \cdot \frac{N}{(2\pi)^{3/2}(4\pi)^{1/2}} \Big[E_{10} e^{-i\delta_{10}} + \tfrac{3}{2} E_{11} e^{-i\delta_{11}} + \tfrac{1}{10}(29\xi_2 - 4\eta_2) E_{12} e^{-i\delta_{12}} - \tfrac{1}{10}(6\xi_2 - 31\eta_2) E_{32} e^{-i\delta_{32}}\Big] = \frac{\nu_2 H_3(\nu_1) - \nu_1 H_3(\nu_2)}{\nu_2 - \nu_1}$$

$$\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{\pi k_0}{60 p_2} \cdot \frac{N}{(2\pi)^{3/2}(4\pi)^{1/2}} \Big[10 E_{10} e^{-i\delta_{10}} - 15 E_{11} e^{-i\delta_{11}} + (\xi_2 + 4\eta_2) E_{12} e^{-i\delta_{12}} - (6\xi_2 - \eta_2) E_{32} e^{-i\delta_{32}}\Big] = \frac{\nu_2 H_4(\nu_1) - \nu_1 H_4(\nu_2)}{\nu_2 - \nu_1}$$

$$i\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{N k_0}{(2\pi)^{3/2} \cdot (4\pi)^{1/2}} \cdot \frac{2\pi}{p} \frac{\gamma_p - \gamma_n}{2m} \Big[M_0 e^{-i\Delta_0} - \sqrt{2} M_2 e^{-i\Delta_2}\Big] = \frac{\nu_2 H_{10}'(\nu_1) - \nu_1 H_{10}'(\nu_2)}{\nu_2 - \nu_1}$$

$$i\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{N k_0}{(2\pi)^{3/2}(4\pi)^{1/2}} \cdot \frac{3\pi}{p} \cdot \frac{\gamma_p - \gamma_n}{2m} \sqrt{2} M_2 e^{-i\Delta_2} = \frac{\nu_2 H_{12}(\nu_1) - \nu_1 H_{12}(\nu_2)}{\nu_2 - \nu_1}$$

If we now define the matrices $(\ell_{rj}) \equiv (\ell_{(LJ)j})$ and $(m_{rj}) \equiv (m_{(J)j})$, where $(10) \equiv 1$, $(11) \equiv 2$, $(12) \equiv 3$, $(32) \equiv 4$ and $(0) \equiv 1$, $(2) \equiv 2$, by

$$(\ell_{(LJ)j}) = \begin{pmatrix} -20 & -30 & -14(\xi_2+4\eta_2) & 14(6\xi_2-\eta_2) \\ -10 & -15 & (29\xi_2-4\eta_2) & (6\xi_2-31\eta_2) \\ 10 & 15 & (29\xi_2-4\eta_2) & (31\eta_2-6\xi_2) \\ 10 & -15 & (\xi_2+4\eta_2) & (\eta_2-6\xi_2) \end{pmatrix} \left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{\pi k_0}{60pm} \cdot \frac{N}{(2\pi)^{3/2}(4\pi)^{1/2}} \tag{23.3}$$

and

$$(m_{(J)j}) = \begin{pmatrix} 2 & -2\sqrt{2} \\ 0 & 3\sqrt{2} \end{pmatrix} \left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \frac{\pi k_0}{p} \cdot \frac{N}{(4\pi)^{1/2}(2\pi)^{3/2}} \cdot \frac{\gamma_p-\gamma_n}{2m} \tag{23.4}$$

and denote their inverses by $(\ell^{-1}_{(LJ)j})$ and $(m^{-1}_{(J)j})$ respectively, then

$$\begin{aligned} E_{LJ}(\nu)\, e^{i\delta_{LJ}} &= \sum_j \ell^{-1}_{(LJ)j} \left[\frac{\nu_2 H_j(\nu_1) - \nu_1 H_j(\nu_2)}{\nu_2 - \nu_1}\right] \\ &= \sum_{j,k} \ell^{-1}_{(LJ)j} \frac{1}{\nu_2-\nu_1} \Big[\nu_2\, a_{jk}(\nu,\nu_1)\, M_k(\nu,\nu_1) \\ &\qquad - \nu_1\, a_{jk}(\nu,\nu_2)\, M_k(\nu,\nu_2)\Big] \end{aligned} \tag{23.5}$$

$$\begin{aligned} M_J(\nu)\, e^{i\delta_J} &= \sum_j m^{-1}_{(J)j} \left[\frac{\nu_2 H_j(\nu_1) - \nu_1 H_j(\nu_2)}{\nu_2 - \nu_1}\right] \\ &= \sum_{j,k} m^{-1}_{(J)j} \frac{1}{\nu_2-\nu_1} \Big[\nu_2\, a_{jk}(\nu,\nu_1)\, M_k(\nu,\nu_1) \\ &\qquad - \nu_1\, a_{jk}(\nu,\nu_2)\, M_k(\nu,\nu_2)\Big] \end{aligned} \tag{23.6}$$

Applying the dispersion relations (21.14) to the $M_\ell(\nu,\nu_i)$ occurring in (23.5) and (23.6), then

$$E_{l\mathfrak{J}}(\nu)\cos\delta_{l\mathfrak{J}}(\nu) = \sum_{j,k,\ell}\frac{e^{-1}_{(l\mathfrak{J})j}}{\nu_2-\nu_1}\Big[\nu_2\, a_{jk}(\nu,\nu_1)\Big\{\frac{A_k^{(-)}(\nu,\nu_1)}{\nu_-^{(1)}-\nu}+\frac{A_k^{D}(\nu,\nu_1)}{\nu_D-\nu}+\frac{A_k^{(+)}(\nu,\nu_1)}{\nu_+^{(1)}-\nu}+\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{a^{-1}_{k\ell}(\nu',\nu_1)\,\mathrm{Im}\,H_\ell(\nu',\nu_1)}{\nu'-\nu}d\nu'\Big\} - \nu_1\, a_{jk}(\nu,\nu_2)$$
$$\times\Big\{\frac{A_k^{(-)}(\nu,\nu_2)}{\nu_-^{(2)}-\nu}+\frac{A_k^{D}(\nu,\nu_2)}{\nu_D-\nu}+\frac{A_k^{+}(\nu,\nu_2)}{\nu_+^{(2)}-\nu}+\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{a^{-1}_{k\ell}(\nu',\nu_2)\,\mathrm{Im}\,H_\ell(\nu',\nu_2)}{\nu'-\nu}\Big\} \qquad (23.7)$$

and

$$M_{\mathfrak{J}}(\nu)\sin\Delta_{\mathfrak{J}}(\nu) = \sum_{jk\ell}\frac{m^{-1}_{\mathfrak{J}j}}{\nu_2-\nu_1}\Big[\nu_2\, a_{jk}(\nu,\nu_1)\Big\{\frac{A_k^{(-)}(\nu,\nu_1)}{\nu_-^{(1)}-\nu}+\frac{A_k^{D}(\nu,\nu_1)}{\nu_D-\nu}+\frac{A_k^{(+)}(\nu,\nu_1)}{\nu_+^{(1)}-\nu}+\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{a^{-1}_{k\ell}(\nu',\nu_1)\,\mathrm{Im}\,H_\ell(\nu',\nu_1)}{\nu'-\nu}d\nu'\Big\} - \nu_1\, a_{jk}(\nu,\nu_2) \qquad (23.8)$$
$$\times\Big\{\frac{A_k^{(-)}(\nu,\nu_2)}{\nu_-^{(2)}-\nu}+\frac{A_k^{D}(\nu,\nu_2)}{\nu_D-\nu}+\frac{A_k^{(+)}(\nu,\nu_2)}{\nu_+^{(2)}-\nu}+\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{a^{-1}_{k\ell}(\nu',\nu_2)\,\mathrm{Im}\,H_\ell(\nu',\nu_2)}{\nu'-\nu}d\nu'\Big\}$$

The H_k occurring in the integrands in (23.7) and (23.8) may be related back to all the $E_{l\mathfrak{J}}$ and $M_{\mathfrak{J}}$ of Section 13 by equations (23.1). Since we are interested only in the dipole amplitudes, we can extract the terms in (23.7) and (23.8) depending on the quadrupole amplitudes and the "mesonic" amplitudes K, and write (23.7) and (23.8) symbolically as

$$D_\ell(\nu)\cos\delta_\ell = \pi_\ell(\nu) + q_\ell(\nu) + m_\ell(\nu) + \frac{1}{\pi}\int\frac{g_{\ell\ell'}(\nu,\nu')\,D_{\ell'}(\nu')\sin\delta_{\ell'}}{\nu'-\nu}d\nu' \qquad (23.9)$$

In (23.9) $\pi_\ell(\nu)$ is the contribution arising from the pole-terms, $q_\ell(\nu)$ that arising from the integration over the quadrupole amplitudes and $m_\ell(\nu)$ that arising from the integration over the "mesonic" amplitudes. If $q_\ell(\nu)$ and $m_\ell(\nu)$ are considered as known, then equation (23.9) forms a set of integral equations for the dipole amplitudes.

We solve equations (23.9) for the amplitudes, assuming the phase-shifts to be given, the solution being carried out for the two sets of phase-shifts considered in Part 2 - namely the Y.L.A.M. set of Breit et al[44,45] and the Signell-Marshak set[6] $q_\ell(\nu)$ was evaluated in each case using the appropriate quadrupole amplitudes of Part 2. This contribution is found to be very small, except at very low energies, because of the factor $\frac{2W\nu_i}{(W^2-4m^2)^{1/2}\,(W^2-M^2)}$ occurring in equations (23.1). Even at low energies, the contribution of $q_\ell(\nu)$ is still not considerable, because at these energies the pole contribution $\pi_\ell(\nu)$ is dominant. Even at photon energies of 50 MeV, $\pi_\ell(\nu)$ is still contributing to some 70% of the right-hand-side of equation (23.9) $m_\ell(\nu)$ was evaluated using the amplitudes K of Section 16. These latter were evaluated with a 6% D-state phenomenological deuteron wave-function, and hard-core wave-functions of the type (15.5) with the phase-shifts obtained by a suitable extrapolation of the ${}^1S_0$ and ${}^1D_2$ Y.L.A.M. phase-shifts. The results were checked with experiment and found to give agreement similar to that obtained by L.D. Pearlstein and A. Klein[46].

The solution for the dipole amplitudes E_{LI} and M_I was carried out for photon energies up to 50 MeV. The results are given for the two cases in Table 14 and the corresponding cross-section parameters in Table 15. The latter are compared with experiment in Figures 12 and 13.

Due to the comparitive crudeness of our approximations and calculations, we cannot justifiably draw any rigid conclusions from these results.

The dispersion relations solution is satisfactory up to photon laboratory energies of about 35 MeV, but thereafter starts diverging rather rapidly from the experimental results. That agreement should be obtained as high as 35 MeV must be considered satisfactory in view of having ignored completely the cuts arising from the crossed diagrams and the cut from the anomalous threshold of the d-np vertex.

The transition amplitudes obtained are sufficiently similar to those obtained in the conventional calculations to make it impossible to decide which of the two sets of phase-shifts is to be preferred, although the transition amplitudes obtained from dispersion relations resemble those obtained from the Y.L.A.M. phase parameters more closely than they do those obtained from Signell-Marshak phase parameters.

Table 14. Transition Amplitudes from Dispersion Relations

A. Assuming Y.L.A.M. Phase-Parameters

E_γ	E_{10}	E_{11}	E_{12}	E_{32}	M_o	M_2
5	1.640	1.938	1.793	0.345	1.770	0.045
10	1.021	1.392	1.214	0.352	1.011	0.126
15	0.707	1.112	0.914	0.349	0.646	0.174
20	0.472	0.891	0.667	0.336	0.438	0.201
30	0.256	0.639	0.418	0.287	0.297	0.234
40	0.159	0.504	0.289	0.229	0.221	0.245
50	0.114	0.416	0.221	0.164	0.159	0.247

B. Assuming Signell-Marshak Phase Parameters

E_γ	E_{10}	E_{11}	E_{12}	E_{32}	M_o	M_2
11.23	0.887	1.268	1.104	0.356	8.847	0.155
22.23	0.412	0.835	0.603	0.325	0.417	0.210
39.76	0.161	0.508	0.286	0.222	0.226	0.247

E_γ in MeV; transition amplitudes in units of 10^{-13} cm.

Table 15. Differential Cross-Section Parameters from Dispersion Relations.

A. Assuming Y.L.A.M. Phase Parameters

E_γ	a_e	b_e	a_m	b_m	a	b	a/b	σ_T
5	1.254	206.6	6.458	0.214	7.712	206.8	0.037	1929
10	4.678	148.3	2.463	0.474	7.141	148.8	0.048	1336
15	5.902	104.8	0.939	0.547	6.841	105.7	0.065	971
20	6.347	69.32	0.384	0.589	6.731	62.91	0.107	612
30	6.382	31.75	0.129	0.641	6.511	32.39	0.201	353
40	5.681	17.34	0.064	0.683	5.725	17.40	0.329	218
50	3.874	10.44	0.059	0.717	4.316	10.50	0.411	142

B. Assuming Signell-Marshak Phase Parameters

E_γ	a_e	b_e	a_m	b_m	a	b	a/b	σ_T
11.23	5.392	99.72	1.634	0.532	7.026	100.25	0.070	928
22.23	6.272	43.13	0.356	0.583	6.628	43.71	0.152	449
39.76	4.526	15.70	0.063	0.669	4.590	16.37	0.280	195

E_γ in MeV; a,b in μb/steradian, σ_T in μb;

Figure 12

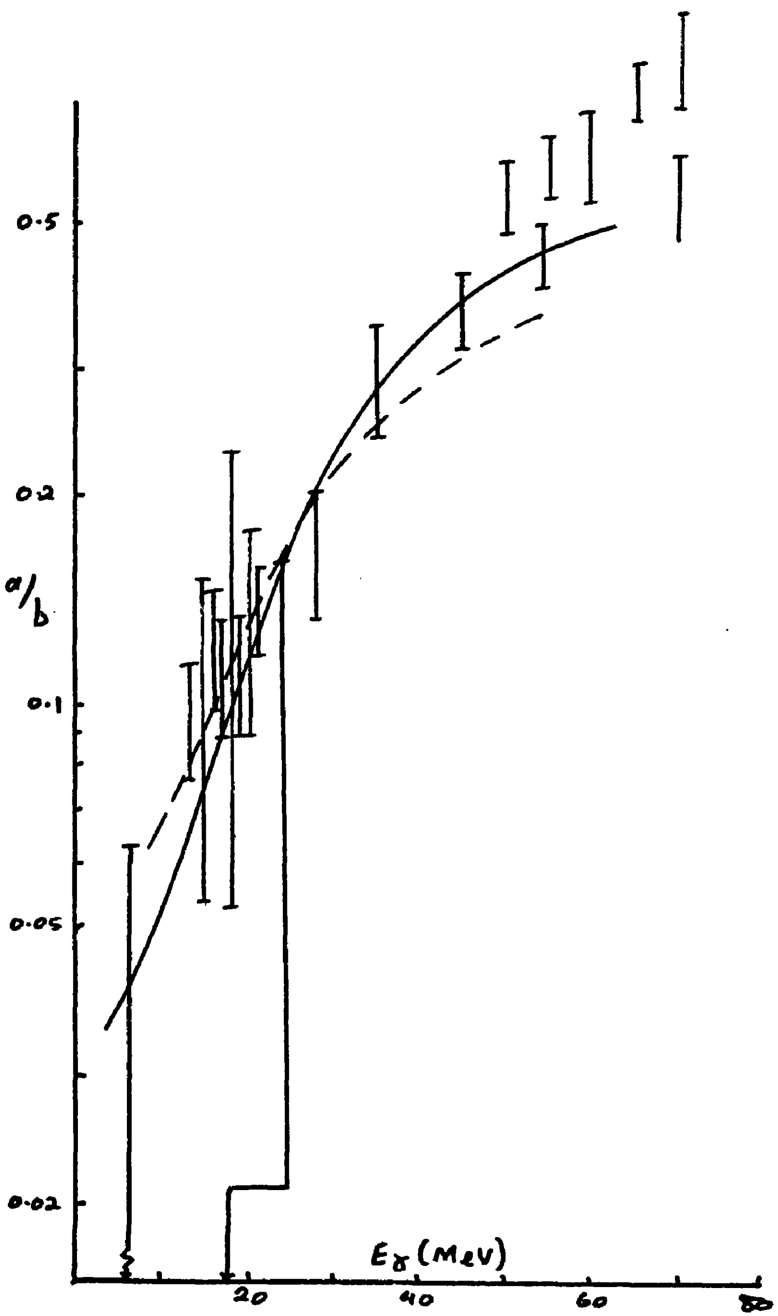
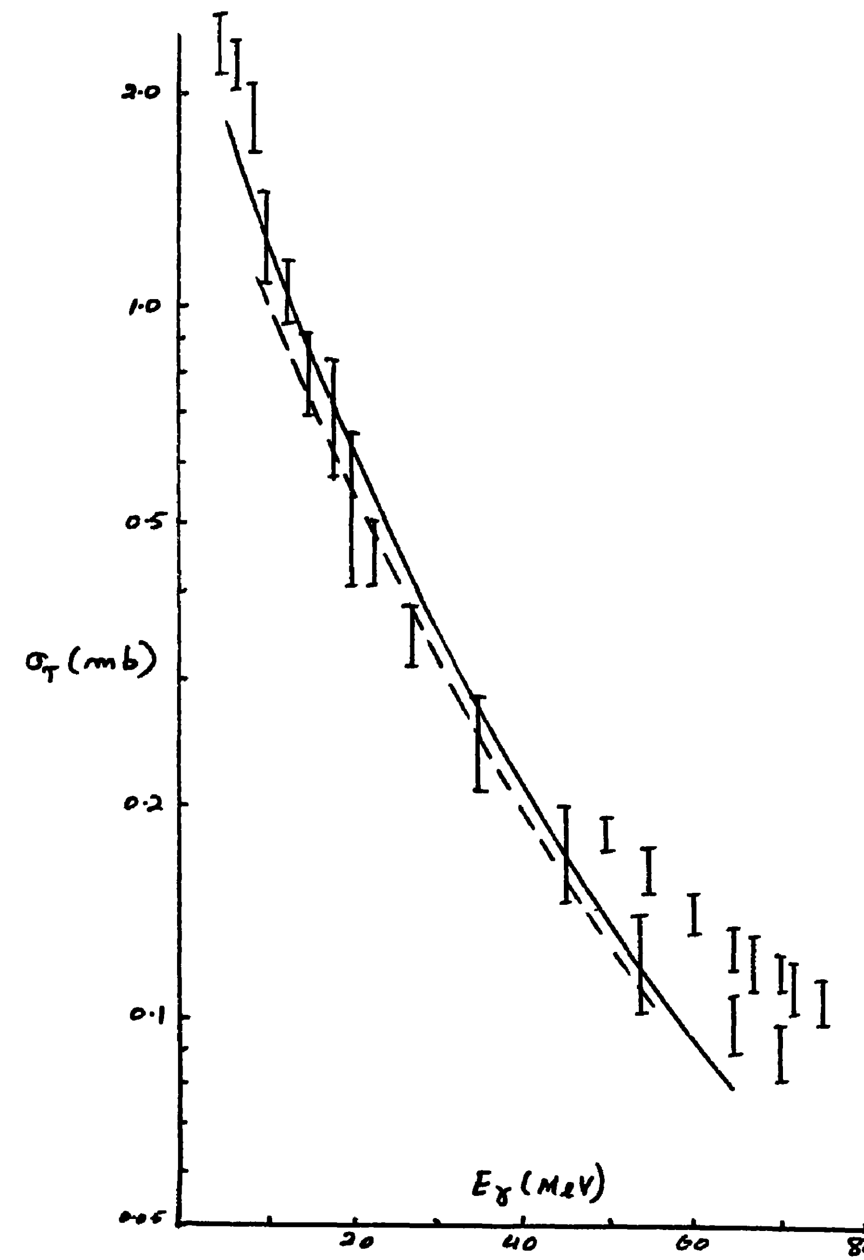


Figure 13



Assuming Y.L.A.M. phase parameters ——— Assuming Signell-Marshak phase parameters - - - -

24. Conclusions

We have seen that the conventional theory of the photodisintegration is sufficient to explain the existing experimental data reasonably well up to photon laboratory energies of 130 MeV. These calculations, however, are not without their ambiguities and uncertainties. By a suitable choice of the deuteron D-state probability, different (albeit similar) sets of phase parameters can reproduce the experimental data within the limits of experimental error. It seems unlikely that any distinction can be made from experiment for some time yet. Some more insight may possibly be obtained by the inclusion of higher multipoles than the dipole and quadrupole (which is at present being carried out by M.L. Rustgi et al[59]) and by a complete assessment of the effects of retardation.

The application of dispersion relations seems the most natural way to proceed, since this avoids the ambiguities and uncertainties of the conventional approach. We have seen that, even in a simple approximation, the dispersion relations can give as good a result as the conventional theory in the energy range at which the approximation is pertinent. To extend the present calculations to higher energies it is necessary to include the cut arising from the anomalous threshold of the d-np vertex. Less important is a more exhaustive treatment of the one pion intermediate state and at least a qualitative assessment of the effect of the two-pion intermediate state.

A discussion of the photodisintegration of the deuteron in the Mandelstam representation should certainly be feasible. A treatment of the photodisintegration analogous to that of H.P. Noyes[33] for nucleon-nucleon scattering, in which the $E_{lJ} \cos \delta_{lJ}$ and $M_J \sin \Delta_J$ say, could be expressed as analytic functions depending on a limited number of parameters would seem the best approach. Such a scheme, taken in conjunction with the recent strides being made in the phase shift analysis, should allow a fairly complete description of deuteron photodisintegration to be made.

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Appendix 1. Mass Values and Other Constants.

We take the currently accepted values for the masses as follows.

Mass of charged pion	$= 139.63 \pm 0.06$ MeV
Mass of neutral pion	$= 135.04 \pm 0.16$ MeV
Mass of proton	$= 938.213 \pm 0.01$ MeV
Mass of neutron	$= 939.506 \pm 0.01$ MeV
Mass of deuteron	$= 1875.5$ MeV

Since we ignore the mass differences between the charged and neutral particles, we take the weighted mean for the pion and the nucleon, to give

$$\mu = 138.10 \text{ MeV}$$

$$\text{and} \quad m = 938.86 \text{ MeV}$$

The binding energy of the deuteron is taken to be

$$\varepsilon = -2.225 \text{ MeV}.$$

The nucleon and deuteron magnetic moments, and the deuteron quadrupole moment are taken to be

$$\Upsilon_p = 2.79276 \pm 0.00006 \text{ Bohr magnetons}$$

$$\Upsilon_n = -1.91304 \pm 0.00010 \text{ Bohr Magnetons}$$

$$\Upsilon_D = 0.857411 \pm 0.000019 \text{ Bohr Magnetons}$$

$$Q = 2.738 \pm 0.014 \times 10^{-27} \text{ cm}^2.$$

The deuteron scattering length is taken to be

$$r_D = 1.704 \times 10^{-13} \text{ cm}.$$

and $\rho = \lim_{r \to \infty} \frac{w_g(r)}{u_g(r)}$ to be

ρ	= 0.02667	4% D-state
	0.02487	6% D-state.

Appendix 2. The γ-Matrices.

Our choice of γ-matrices is

$$\gamma = (\gamma_0, \underset{\sim}{\gamma})$$

where

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \underset{\sim}{\gamma} = \begin{pmatrix} 0 & \underset{\sim}{\sigma} \\ -\underset{\sim}{\sigma} & 0 \end{pmatrix}$$

with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We define

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and the charge conjugate matrix

$$C = \gamma_0\gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}$$

Appendix 3. Relations between Energies in the Laboratory and Centre-of-Momentum Systems.

Denote quantities referring to the laboratory system by primes, and those in the c.m. system unprimed. Let particle '1' be the target (stationary in the laboratory system) and particle '2' the incident particle. Let E_1', E_2' be the energies, including rest energies and $\mathbf{p}_1', \mathbf{p}_2'$ the momenta, with moduli p_1', p_2'. Let P' be the total momentum in the laboratory system and $\mathbf{p}$ $(-\mathbf{p})$ the momenta in the c.m. system, with modulus p, for either particle, and W, W' the total energies in the two systems.

Since the 4-vector scalar product is invariant under Lorentz transformations,

$$P_\mu^2 = P_\mu'^2$$

i.e.

$$\begin{aligned} W^2 &= W'^2 - P'^2 \\ &= (m_1 + E_2')^2 - p_2'^2 \\ &= m_1^2 + m_2^2 + 2m_1 E_2' \end{aligned}$$

Thus for nucleon-nucleon scattering

$$W = \sqrt{2m(m + E_s)}$$

and for deuteron photodisintegration,

$$W = \sqrt{M_D (M_D + 2 E_\gamma)}$$

In the c.m. system, the photon energy k_0 is given by

$$k_0 = \frac{W^2 - M^2}{2W}$$

Photon Laboratory Energy to Centre-of-Momentum Energy.

E_γ	W	E_γ	W
5	1880.43	130	2001.25
10	1885.47	140	2010.47
15	1890.42	150	2019.90
20	1895.40	175	2043.01
25	1900.26	200	2065.08
30	1905.25	225	2088.35
35	1910.24	250	2110.71
40	1914.94	275	2132.83
45	1919.90	300	2154.74
50	1924.84	325	2176.30
60	1934.43	350	2197.90
70	1944.22	375	2219.35
80	1953.71	400	2240.35
90	1963.42	425	2261.15
100	1972.82	450	2281.65
110	1982.42	475	2302.07
120	1991.73	500	2322.49

E_γ and W in MeV.

Nucleon Laboratory Scattering Energy to Centre of Momentum Energy.

E_s	W
10	1882.8
20	1887.6
30	1892.6
40	1897.4
50	1902.6
60	1907.4
70	1912.3
80	1917.3
90	1922.2
100	1927.2
110	1931.8
120	1936.7
130	1941.6
140	1946.5
150	1951.3
160	1956.0

E_s	W
180	1965.7
200	1974.8
220	1984.7
240	1994.0
260	2003.5
280	2012.9
300	2022.1
400	2068.0
500	2113.0
600	2157.0
700	2200.0
800	2242.0
900	2284.0
1000	2325.0

E_s, W in MeV.

Appendix 4. Phenomenological Deuteron Wave-Functions.

Although the exact form of the deuteron wave function can only be obtained by a knowledge of the potential acting in the bound state, reasonable deuteron wave-functions may be constructed by assuming suitable functional forms containing several parameters, which can be varied to fit existing data on the neutron-proton system.

The data which can be used for this purpose are the deuteron binding energy ε (which determines the asymptotic form of the wave-functions by equation (5.14)), the deuteron magnetic moment γ_D (which gives an estimate of the D-state probability through equation (6.6)) the electric quadrupole moment Q (equation (6.13)) and the deuteron effective range, r_D (equation (7.18)).

The functional forms chosen are those suggested by L. Hulthin and M. Sugawara (12) of the form

$$u(x) = N \cos \epsilon_g \left[1 - e^{-\beta(x-x_c)}\right] e^{-x} \qquad x = \alpha r \geq x_c = \alpha r_c$$

$$w(x) = N \sin \epsilon_g \left[1 - e^{-\gamma(x-x_c)}\right]^2 e^{-x}\left[1 + \frac{3(1-e^{-\gamma x})}{x} + \frac{3(1-e^{-\gamma x})^2}{x^2}\right]$$

$u = w = 0, \quad x < x_c$ The normalization factor N is obtained from equation (7.18) as

$$N^2 = \frac{2\alpha}{1 - \alpha r_D}$$

Taking $r_c = 0.4316 \times 10^{-13}$ cm. (i.e. $x_c = 0.1$) and $r_D = 1.704 \times 10^{-13}$ cm., then

$$N^2 = 7.6579 \times 10^{-12}\ cm^{-1}.$$

and the parameters β, γ, $\sin \varepsilon_3$ are given by

P_D	β	γ	$\sin \varepsilon_3$
3%	8.237	3.155	0.02942
4%	7.961	3.798	0.02666
5%	7.699	4.346	0.02514
6%	7.451	4.799	0.02486

The values for deuteron D-state probabilities of 3%, 4% and 5% are given in reference [12]. The last case, $P_D = 6\%$ has been calculated.

Appendix 5. The Allowed Transitions.

Let L be the total angular momentum of the photon, the orbital angular momentum in the final state, and the initial and final spins and J the total angular momentum.

Conservation of angular momentum requires that for triplet-triplet transitions,

$$L = \begin{cases} L'+2 \\ L'+1 \\ L' \\ L'-1 \\ L'-2 \end{cases} \tag{A5.1}$$

and for triplet-singlet transitions,

$$L = \begin{cases} L'+1 \\ L' \\ L'-1 \end{cases} \tag{A5.2}$$

Parity conservation on the other hand, demands that for electric 2^L pole transitions

$$(-1)^{L'} = (-1)^{L}$$

i.e. $L = L' \pm 2n \quad (n = 0, 1, 2, \ldots)$ (A5.3)

and for magnetic 2^L pole transitions,

$$(-1)^{L'} = (-1)^{L\pm 1}$$

i.e. $L = L' - 1 \pm 2n \quad (n = 0, 1, 2, \ldots)$ (A5.4)

Combining the requirements of (A5.1) to (A5.4) we can construct the following table of allowed transitions.

<u>Table of Allowed Transitions.</u>

L'	J	L	Multipole	s	Parity
L'	$L'+1$	L'	Electric 2^L	1	$(-1)^L$
L'	L	L'	"	1	$(-1)^L$
L'	$L'-1$	L'	"	1	$(-1)^L$
L'	$L'+1$	$L'+2$	"	1	$(-1)^L$
L'	$L'-1$	$L'-2$	"	1	$(-1)^L$
L'	$L'+1$	$L'+1$	Magnetic 2^L	1	$(-1)^{L\pm1}$
L'	L'	$L'+1$	"	1	$(-1)^{L\pm1}$
L'	L'	$L'-1$	"	1	$(-1)^{L\pm1}$
L'	$L'-1$	$L'-1$	"	1	$(-1)^{L\pm1}$
L'	L'	L'	Electric 2^L	0	$(-1)^L$
L'	L'	$L'+1$	Magnetic 2^L	0	$(-1)^{L\pm1}$
L'	L'	$L'-1$	"	0	$(-1)^{L\pm1}$

Appendix 6. The Transition Amplitudes.

Note first that the expansion of the final-state wave-function into its component angular momentum states may be written as

$$\psi(\underline{r}) = \frac{1}{(2\pi)^{3/2}} \sum_{\ell, j} i^{\ell}(2\ell+1)\, F_{\ell}(j)\, \frac{v_{\ell j}(kr)}{kr} e^{i\delta_{\ell j}} P_{\ell}(\hat{k}\cdot\hat{r})\, \chi_s^m \qquad (A6.1)$$

for the triplet spin state, and

$$\psi(r) = \frac{1}{(2\pi)^{3/2}} \sum_{j} i^{j}(2j+1) \frac{v_j(kr)}{kr} e^{i\delta_j} P_j(\hat{k}\cdot\hat{r})\, \chi_0^0 \qquad (A6.2)$$

for the singlet case.

$F_{\ell}(j)$ is the projection operator for the state $J = j$, $L = \ell$, $S = 1$ and from a consideration of the values of L, S in the various states of interest, we arrive at

$$F_1(0) = -\frac{1 - (\underline{L}\cdot\underline{S})^2}{3} \qquad (A6.3a)$$

$$F_1(1) = \frac{2 - \underline{L}\cdot\underline{S} - (\underline{L}\cdot\underline{S})^2}{2} \qquad (A6.3b)$$

$$F_1(2) = \frac{2 + 3\underline{L}\cdot\underline{S} + (\underline{L}\cdot\underline{S})^2}{6} \qquad (A6.3c)$$

$$F_3(2) = -\frac{3 + 2\underline{L}\cdot\underline{S} - (\underline{L}\cdot\underline{S})^2}{21} \qquad (A6.3d)$$

$$F_2(1) = -\frac{2 + \underline{L}\cdot\underline{S} - (\underline{L}\cdot\underline{S})^2}{10} \qquad (A6.3e)$$

$$F_2(2) = \frac{6 - \mathbf{L}\cdot\mathbf{S} - (\mathbf{L}\cdot\mathbf{S})^2}{6} \qquad \text{(A6.3f)}$$

$$F_2(3) = \frac{3 + 4\,\mathbf{L}\cdot\mathbf{S} + (\mathbf{L}\cdot\mathbf{S})^2}{15} \qquad \text{(A6.3g)}$$

From (A6.1) we see that we are interested in terms of the form

$$K_\ell(j) = F_\ell(j)\; P_\ell(\hat{p}\cdot\hat{n}) \qquad \text{(A6.4)}$$

which yields for the above cases the following results:

$$K_1(0) = \frac{\hat{p}\cdot\hat{n} - \mathbf{S}\cdot\hat{p}\;\mathbf{S}\cdot\hat{n}}{3} \qquad \text{(A6.5a)}$$

$$K_1(1) = \frac{\mathbf{S}\cdot\hat{n}\;\mathbf{S}\cdot\hat{p}}{2} \qquad \text{(A6.5b)}$$

$$K_1(2) = \frac{4\,\hat{p}\cdot\hat{n} - \mathbf{S}\cdot\hat{p}\;\mathbf{S}\cdot\hat{n} - 3i\,\mathbf{S}\cdot(\hat{n}\times\hat{p})}{6} \qquad \text{(A6.5c)}$$

$$K_3(2) = \frac{5(\hat{p}\cdot\hat{n})^3 + (\hat{p}\cdot\hat{n}) - [\mathbf{S}\cdot\hat{p}\;\mathbf{S}\cdot\hat{n} - 2i\,\mathbf{S}\cdot(\hat{n}\times\hat{p})]\{5(\hat{p}\cdot\hat{n})^2 - 1\} - 10\{\mathbf{S}\cdot(\hat{n}\times\hat{p})\}^2\,\hat{p}\cdot\hat{n}}{14} \qquad \text{(A6.5d)}$$

$$K_2(1) = \frac{3(\hat{p}\cdot\hat{n})^2 + 1 - 3\,\mathbf{S}\cdot\hat{p}\;\mathbf{S}\cdot\hat{n}\;\hat{p}\cdot\hat{n} - 3\,\mathbf{S}\cdot(\hat{n}\times\hat{p})\;\mathbf{S}\cdot(\hat{n}\times\hat{p}) + 3i\,\mathbf{S}\cdot(\hat{n}\times\hat{p})\;\hat{p}\cdot\hat{n}}{10} \qquad \text{(A6.5e)}$$

$$K_2(2) = \frac{(\hat{p}\cdot\hat{n})^2 - 1 + \mathbf{S}\cdot\hat{n}\;\mathbf{S}\cdot\hat{p}\;\hat{p}\cdot\hat{n} + \mathbf{S}\cdot(\hat{n}\times\hat{p})\;\mathbf{S}\cdot(\hat{n}\times\hat{p})}{2} \qquad \text{(A6.5f)}$$

$$K_2(\beta) = \frac{21(\beta\cdot\hat{r})^2 - 3 - 6\,S.\beta\,S.\hat{r}\,\beta.\hat{r} - 6\,S.(\hat{r}\times\beta)\,S.(\hat{r}\times\beta) - 24i\,S.(\hat{r}\times\beta)}{30}\,e^{i\beta\cdot\hat{r}} \qquad (A6.5g)$$

Writing the deuteron wave-function as

$$\psi_D(\underline{r}) = \frac{N}{\sqrt{4\pi}}\left[\frac{u(r)}{r} + \frac{S_{12}}{\sqrt{8}}\,\frac{w_D(r)}{r}\right]\chi_1^m \qquad (A6.6)$$

and incorporating the results of the preceding equations, we obtain after integrating over angle

$$\langle f|H'|i\rangle = \langle f|H'|i\rangle_{E1} + \langle f|H'|i\rangle_{M1} + \langle f|H'|i\rangle_{E2} + \langle f|H'|i\rangle_{M2} \qquad (A6.7)$$

where

$$\langle f|H'|i\rangle_{E1} = -\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}}\cdot\frac{\pi k_0}{3p}\cdot\frac{\chi_1^{m'\dagger}}{(2\pi)^{3/2}}\cdot\frac{N}{(4\pi)^{1/2}}$$

$$\times\Big\{2e^{-i\delta_{10}}\{p.\epsilon - S.\epsilon\,S.p\}\int_0^\infty \bar{v}_{10}(pr)\,[u_D(r) - \sqrt{2}\,w_D(r)]\,r\,dr$$

$$+3e^{-i\delta_{11}}\,S.p\,S.\epsilon\int_0^\infty \bar{v}_{11}(pr)\,[u_D(r) + \tfrac{1}{\sqrt{2}}\,w_D(r)]\,r\,dr$$

$$+\tfrac{1}{5}\,e^{-i\delta_{12}}\Big\{[4p.\epsilon - S.\epsilon\,S.p - 15i\,S.(p\times\epsilon)]\,\xi_2$$

$$+[16p.\epsilon - 4S.\epsilon\,S.p]\,\eta_2\Big\}\Big\{\cos\epsilon_2\int_0^\infty \bar{v}_{12}^{1}(pr)\,[u_D(r)$$

$$-\tfrac{1}{5\sqrt{2}}\,w_D(r)]\,r\,dr + \sqrt{\tfrac{3}{2}}\sin\epsilon_2\,\tfrac{3}{5}\sqrt{2}\int_0^\infty \bar{v}_{32}^{1}(pr)\,w_D(r)\,r\,dr\Big\}$$

$$+\tfrac{1}{5}\,e^{-i\delta_{32}}\Big\{[24p.\epsilon - 6S.\epsilon\,S.p]\,\xi_2 + [-4p.\epsilon + S.\epsilon\,S.p$$

$$-15i\,S.(p\times\epsilon)]\,\eta_2\Big\}\Big\{\cos\epsilon_2\,\tfrac{3}{5}\sqrt{2}\int_0^\infty \bar{v}_{32}^{3}(pr)\,w_D(r)\,r\,dr$$

$$- \sqrt{\tfrac{2}{3}} \sin \varepsilon_2 \int_0^\infty \bar{v}^{\,3}_{12}(pr) \left[u_D(r) - \frac{1}{5\sqrt{2}} w_D(r) \right] r\, dr \Big\} \chi_1^m$$

$$\langle f|H'|i\rangle_{M1} = i \left\{ \frac{e^2}{4\pi k_0} \right\}^{\frac{1}{2}} \frac{4\pi k_0}{p} \cdot \frac{\gamma_p - \gamma_n}{2m} \cdot \frac{\chi_0^{0\dagger}}{(2\pi)^{3/2}} \cdot \frac{N}{(4\pi)^{1/2}}$$

$$\times \Big\{ e^{-i\Delta_0}\, i (\underline{\sigma}^{(1)} - \underline{\sigma}^{(2)}).(\hat{k} \times \underline{\varepsilon}) \int_0^\infty \bar{v}_0(pr)\, u_D(r)\, dr$$

$$+ e^{-i\Delta_2}\, i (\underline{\sigma}^{(1)} - \underline{\sigma}^{(2)}).(\hat{k} \times \underline{\varepsilon}) [1 - 3\underline{\sigma}^{(1)}.\hat{p}\, \underline{\sigma}^{(2)}.\hat{p}] \frac{1}{\sqrt{8}} \int_0^\infty \bar{v}_2(pr)\, w_D(r)\, dr \Big\} \chi_1^m$$

$$\langle f|H'|i\rangle_{E2} = \left\{ \frac{e^2}{4\pi k_0} \right\}^{\frac{1}{2}} \frac{4\pi k_0^2}{p} \cdot \frac{N}{(4\pi)^{1/2}} \cdot \frac{\chi_1^{m\dagger}}{(2\pi)^{3/2}} \cdot \frac{1}{16.15}$$

$$\times \Big\{ \; e^{-i\delta_{01}} [\underline{\sigma}^{(1)}.\underline{\varepsilon}\, \underline{\sigma}^{(2)}.\hat{k} + \underline{\sigma}^{(2)}.\underline{\varepsilon}\, \underline{\sigma}^{(1)}.\hat{k}] \frac{1}{\sqrt{2}} \int_0^\infty \bar{v}_{01}(pr)\, w_D(r)\, r^2\, dr$$

$$- 3 e^{-i\delta_{21}} \Big\{ 2\hat{p}.\hat{k}\, \hat{p}.\underline{\varepsilon} - [\underline{S}.\hat{k}\, \hat{p}.\underline{\varepsilon} + \underline{S}.\underline{\varepsilon}\, \hat{p}.\hat{k}]\, \underline{S}.\hat{p}$$

$$+ i\, [\underline{S}.(\hat{p}\times\underline{\varepsilon})\, \hat{p}.\hat{k} + \underline{S}.(\hat{p}\times\hat{k})\, \hat{p}.\underline{\varepsilon}] - [\underline{S}.(\hat{p}\times\hat{k})\, \underline{S}.(\hat{p}\times\underline{\varepsilon}) + \underline{S}.(\hat{p}\times\underline{\varepsilon})\, \underline{S}.(\hat{p}\times\hat{k})]\Big\}$$

$$\times \int_0^\infty \bar{v}_{21}(pr) \left[u_D(r) - \frac{1}{\sqrt{2}} w_D(r) \right] r^2\, dr \quad - 5 e^{-i\delta_{22}} \Big\{ 2\hat{p}.\hat{k}\, \hat{p}.\underline{\varepsilon}$$

$$+ \underline{S}.\hat{p}\, [\underline{S}.\hat{k}\, \hat{p}.\underline{\varepsilon} + \underline{S}.\underline{\varepsilon}\, \hat{p}.\hat{k}] + [\underline{S}.(\hat{p}\times\underline{\varepsilon})\, \underline{S}.(\hat{p}\times\hat{k})$$

$$+ \underline{S}.(\hat{p}\times\hat{k})\, \underline{S}.(\hat{p}\times\underline{\varepsilon})]\Big\} \int_0^\infty \bar{v}_{22}(pr) \left[u_D(r) + \frac{1}{\sqrt{2}} w_D(r) \right] r^2\, dr$$

$$- e^{-i\delta_{23}} \Big\{ 14\hat{p}.\hat{k}\, \hat{p}.\underline{\varepsilon} - 2[\underline{S}.\hat{k}\, \hat{p}.\underline{\varepsilon} + \underline{S}.\underline{\varepsilon}\, \hat{p}.\hat{k}]\, \underline{S}.\hat{p}$$

$$- 2\, [\underline{S}.(\hat{p}\times\underline{\varepsilon})\, \underline{S}.(\hat{p}\times\hat{k}) + \underline{S}.(\hat{p}\times\hat{k})\, \underline{S}.(\hat{p}\times\underline{\varepsilon})]$$

$$+ 8i\, [\underline{S}.(\hat{p}\times\underline{\varepsilon})\, \hat{p}.\hat{k} + \underline{S}.(\hat{p}\times\hat{k})\, \hat{p}.\underline{\varepsilon}]\Big\} \int_0^\infty \bar{v}_{23}(pr) \left[u_D(r) - \frac{\sqrt{2}}{4} w_D(r) \right] r^2\, dr \Big\} \chi_1^m$$

$$\langle f|H'|i\rangle_{M2} = i\left\{\frac{e^2}{4\pi k_0}\right\}^{\frac{1}{2}} \cdot \frac{\pi\, k_0^2}{p} \cdot \frac{N}{(4\pi)^{1/2}} \cdot \frac{\gamma_p + \gamma_n}{2m} \cdot \frac{\chi_0^{0\dagger}}{(2\pi)^{3/2}}$$

$$\times \Big\{ e^{-i\Delta_1}\, \hat{p}\cdot\hat{k}\;\; i(\sigma^{(1)}-\sigma^{(2)})\cdot(\hat{k}\times\epsilon) \int_0^\infty \mathcal{V}_1(pr)\,[u_D(r) - \tfrac{\sqrt{2}}{10} w_D(r)]\, r\, dr$$

$$+ e^{-i\Delta_3}\; i(\sigma^{(1)}-\sigma^{(2)})\cdot(\hat{k}\times\epsilon)\; [\,\hat{p}\cdot\hat{k} - 5\,\hat{p}\cdot\hat{k}\; \sigma^{(1)}\cdot\hat{p}\; \sigma^{(2)}\cdot\hat{p}$$

$$+ (\sigma^{(1)}\cdot\hat{p}\;\sigma^{(2)}\cdot\hat{k} + \sigma^{(2)}\cdot\hat{p}\;\sigma^{(1)}\cdot\hat{k})\,]\; \frac{3}{5}\sqrt{2} \int_0^\infty \mathcal{V}_3(pr)\; w_D(r)\; r\, dr \Big\}\;\; \chi_1^m$$

with

$$\xi_2 = \cos\epsilon_2 - \sqrt{\tfrac{2}{3}}\,\sin\epsilon_2\,,$$

$$\eta_2 = \cos\epsilon_2 + \sqrt{\tfrac{3}{2}}\,\sin\epsilon_2\,.$$

The above equations may be recast into the form (13.3)-(13.6) by straightforward vector algebra.

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Table 3. The Transition Amplitudes.

A. 4% Deuteron D-state.

E_γ (MeV)	5	10	15	20	30	40	50	60	70	90	110	130
E_{10}	1.622	1.063	0.681	0.461	0.262	0.180	0.136	0.112	0.101	0.091	0.081	0.072
E_{11}	1.930	1.374	1.074	0.852	0.611	0.482	0.397	0.341	0.304	0.239	0.195	0.166
E_{12}	1.791	1.162	0.858	0.624	0.401	0.283	0.207	0.154	0.128	0.085	0.056	0.031
E_{32}	0.380	0.459	0.497	0.462	0.332	0.234	0.178	0.143	0.121	0.086	0.073	0.064
E_{01}	-0.075	-0.074	-0.071	-0.061	-0.036	-0.022	-0.014	-0.008	-0.006	-0.003	-0.002	-0.001
E_{21}	5.591	2.053	1.431	0.952	0.516	0.335	0.241	0.192	0.161	0.124	0.115	0.109
E_{22}	6.951	2.468	1.836	1.192	0.653	0.428	0.292	0.122	0.143	0.082	0.067	0.058
E_{23}	6.123	2.176	1.571	1.059	0.563	0.362	0.271	0.198	0.163	0.096	0.084	0.077
M_o	1.761	0.918	0.600	0.472	0.336	0.242	0.186	0.158	0.124	0.108	0.098	0.091
M_2	0.047	0.114	0.159	0.182	0.211	0.219	0.224	0.221	0.217	0.197	0.172	0.158
M_1	2.139	1.468	1.082	0.867	0.591	0.462	0.379	0.322	0.278	0.221	0.193	0.172
M_3	0.050	0.108	0.137	0.146	0.151	0.149	0.143	0.137	0.131	0.117	0.105	0.096

The amplitudes are in units of 10^{-13} cm.

B. 6% Deuteron D-state.

E_γ (MeV)	5	10	15	20	30	40	50	60	70	90	110	130
E_{10}	1.649	1.107	0.684	0.441	0.238	0.142	0.073	0.064	0.058	0.055	0.053	0.052
E_{11}	1.942	1.354	1.049	0.828	0.619	0.498	0.408	0.352	0.317	0.260	0.209	0.179
E_{12}	1.794	1.164	0.859	0.623	0.399	0.278	0.197	0.144	0.113	0.068	0.040	0.022
E_{32}	0.327	0.339	0.348	0.341	0.306	0.269	0.232	0.198	0.176	0.128	0.104	0.089
E_{01}	-0.055	-0.062	-0.061	-0.053	-0.032	-0.021	-0.014	-0.008	-0.006	-0.003	-0.002	-0.001
E_{21}	5.616	2.108	1.442	0.954	0.539	0.332	0.238	0.178	0.137	0.104	0.090	0.088
E_{22}	6.933	2.457	1.831	1.189	0.658	0.426	0.302	0.201	0.162	0.122	0.091	0.074
E_{23}	6.129	2.177	1.573	1.060	0.564	0.363	0.272	0.199	0.163	0.096	0.084	0.077
M_0	1.745	0.913	0.580	0.461	0.306	0.231	0.182	0.144	0.117	0.088	0.075	0.064
M_2	0.044	0.119	0.171	0.196	0.228	0.243	0.254	0.258	0.257	0.246	0.223	0.197
M_1	2.198	1.481	1.107	0.878	0.587	0.472	0.386	0.321	0.276	0.208	0.182	0.159
M_3	0.047	0.104	0.132	0.142	0.151	0.153	0.151	0.146	0.142	0.132	0.124	0.118

The amplitudes are in units of 10^{-13} cm.

Table 4. **Comparison of Gammel-Thaler and Signell-Marshak Transition Amplitudes**

E_γ (MeV)	11.23			22.24			39.76			52.3			77.3		
	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
E_{10}	0.876	0.879	0.993	0.416	0.392	0.411	0.180	0.412	0.152	0.128	0.079	0.091	0.096	0.055	0.045
E_{11}	1.261	1.239	1.482	0.803	0.782	0.863	0.482	0.498	0.533	0.376	0.391	0.430	0.269	0.280	0.318
E_{12}	1.123	1.124	1.297	0.574	0.575	0.644	0.283	0.278	0.276	0.187	0.174	0.161	0.108	0.087	0.067
E_{32}	0.474	0.346	0.185	0.439	0.327	0.215	0.234	0.269	0.201	0.156	0.212	0.183	0.100	0.153	0.158
E_{01}	-0.073	-0.062	0.050	-0.054	-0.046	-0.019	-0.022	-0.021	-0.028	-0.012	-0.011	-0.030	-0.009	-0.007	-0.021
E_{21}	1.897	1.192	1.97	0.832	0.843	0.76	0.335	0.332	0.32	0.213	0.209	0.21	0.144	0.122	0.12
E_{22}	2.354	2.368	2.42	1.018	1.024	1.06	0.428	0.426	0.45	0.236	0.241	0.28	0.110	0.128	0.13
E_{23}	1.996	2.013	2.08	0.913	0.917	0.89	0.362	0.363	0.43	0.257	0.264	0.31	0.126	0.129	0.18
M_0	0.673	0.671	0.813	0.428	0.426	0.393	0.242	0.231	0.201	0.172	0.158	0.139	0.120	0.104	0.052
M_2	0.132	0.141	0.159	0.195	0.209	0.231	0.219	0.243	0.273	0.224	0.257	0.282	0.210	0.254	0.285

Transition amplitudes in units of 10^{-13} cm.

1. Present calculation 4% D-state
2. Present calculation 6% D-state
3. Taken from de Swart and Marshak[55] 6.7% D-state

SUMMARY

The thesis is divided into three parts. In the first part, a general survey of the two-nucleon problem is given, with particular attention paid to those aspects which impinge directly on the photodisintegration of the deuteron.

In the second part, we consider the conventional theory of deuteron photodisintegration, with the radiative interaction being taken as given on the basis of the gauge invariance of the non-relativistic Hamiltonian for the two-nucleon system. Differential cross-section and polarization formulae are presented, and a discussion given of previous calculations in this field. New calculations are carried out using the Gammel-Thaler type Y.L.A.M. phase parameters obtained in the analysis of Breit et al.[44,45]

The transitions considered are

1. Electric dipole $({}^3S_1+{}^3D_1) \longrightarrow {}^3P_o, {}^3P_1, {}^3P_2+{}^3F_2$
2. Magnetic dipole spin-flip $({}^3S_1+{}^3D_1) \longrightarrow {}^1S_o, {}^1D_2$
3. Electric quadrupole $({}^3S_1+{}^3D_1) \longrightarrow {}^3S_1+{}^3D_1;$ ${}^3D_2, {}^3D_3+{}^3G_3$
4. Magnetic quadrupole spin-flip $({}^3S_1+{}^3D_1) \longrightarrow {}^1P_1, {}^1F_3$

The ${}^3P_2 - {}^3F_2$ coupling is included, but the ${}^3S_1 - {}^3D_1$ and ${}^3D_3 - {}^3G_3$ coupling neglected. Wherever possible,

phenomenological wave-functions are used, and where this is not feasible, they are calculated from a suitable Gammel-Thaler potential. Differential cross-sections and polarizations are obtained for photon laboratory energies up to 130 MeV, the calculations being carried out both for a 4% and 6% deuteron D-state probability. Finally the results obtained are compared and contrasted with those of previous calculations, and both sets compared with experiments.

In the third part of the thesis, the calculation of the matrix element for deuteron photodisintegration by dispersion relations is considered. There are twelve invariant amplitudes. The covariant form of the transitio amplitude is related to the non-covariant (Pauli-matrix) form, which is further related to the individual multipole transition amplitudes. The Born terms of the covariant amplitudes are derived, and the dispersion relations written down in energy for a fixed difference in the photon-proton and photon-neutron momentum transfers. It is necessary to use this rather than a fixed momentum transfer, in order to exhibit explicitly all the poles in the dispersion relations.

The dispersion relations contain integrals over both positive and negative energies, the latter arising from the crossed diagrams for which the imaginary part of the amplitude is related to processes such as the radiative absorption of an anti-nucleon by a deuteron, and to the structure of the deuteron through the anomalous singularities of the d-np vertex. These complications are ignored, and we retain only the pole terms and the integrals over positive energies.

The relations are restricted to dipole and quadrupole transitions, and by considering the relations at two different "momentum transfers", equations are obtained explicitly for the individual electric dipole and magnetic dipole spin flip transition amplitudes. The equations are solved in a low energy approximation in which the final state n-p rescattering cut and single pion exchange cut only are considered, for the two cases of the Y.L.A.M. and Signell-Marshak phase-parameters. The results obtained are compared with those obtained in part two of the thesis.