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An analogue of the Baum-Connes conjecture for Quantum $SL(2, \mathbb{C})$

by

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Abstract

The Baum-Connes conjecture gives a description of the K-theory of the reduced group C^*-algebra of a locally compact second countable group. In the case of a connected Lie group G, Connes reformulated the conjecture in terms of a deformation of G provided by a certain continuous field of C^*-algebras. The conjecture is known to be true for complex semisimple Lie groups, and in 2008 Higson provided a new proof of this result, using Connes reformulation and an observation due to Mackey about the representation theories of a complex semisimple Lie group and an associated group called the Cartan motion group.

In this thesis, we present and prove an analogue of the conjecture for the quantum group quantum $SL(2, \mathbb{C})$ in the spirit of Connes reformulation and Higson's proof. In particular, we define a quantum version of Connes' field, which provides a deformation from quantum $SL(2, \mathbb{C})$ to a quantum analogue of the Cartan motion group. We show that Mackey's observation carries over to the quantum setting, and we then prove an analogue of the conjecture using Higson's method.

We also show there is compatibility between the Baum-Connes conjecture for $SL(2, \mathbb{C})$ and our quantum result, in that we can construct a continuous field which encodes Connes' field and our quantum field, as well as a deformation of $SL(2, \mathbb{C})$ to quantum $SL(2, \mathbb{C})$ and a deformation of the Cartan motion group to the quantum Cartan motion group.

Contents

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Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Introduction

Let G be a locally compact second countable group equipped with left Haar measure. A well-known C^*-algebra associated to G is the **reduced group C^*-algebra** of G, denoted $C_r^*(G)$. This is the closure of $L^1(G)$, acting as bounded convolution operators on $L^2(G)$, with respect to the operator norm.

To each C^*-algebra A there are two abelian groups, $K_0(A)$ and $K_1(A)$, the ***K*-theory groups** of A. These are invariants of A - that is, if B is a C^*-algebra which is isomorphic to A, then $K_*(A) \cong K_*(B)$. Historically, there has been interest in the K-theory of C^*-algebras in the classification programme. For example, a theorem due to Elliott [21, Theorem 7.1] shows that certain separable and nuclear C^*-algebras are classified by K-theoretic information.

Associated to G there is a G-space denoted by $\underline{E}G$. This is constructed from G and the proper actions of G, [2, Definition 1.6]. There are two abelian groups, $K_0^G(\underline{E}G)$ and $K_1^G(\underline{E}G)$, the **equivariant *K*-homology groups** of $\underline{E}G$ (see [2, Section 3]), defined using Kasparov's KK-theory as introduced in [39, Section 2]. One can use index theory (see [2, 3.14]) to define a map

$$\mu : K_*^G(\underline{E}G) \to K_*(C_r^*(G))$$

which is called the **assembly map**. We shall have no need to define the equivariant K-homology groups or the assembly map in general, for often in examples one can describe these in more concrete terms. However, let us state the **Baum-Connes conjecture**, [1], [2, Conjecture 3.15] in generality.

Conjecture. *Let G be a locally compact second countable group. Then the assembly map*

$$\mu : K_*^G(\underline{E}G) \to K_*(C_r^*(G))$$

is an isomorphism.

It is common to refer to the 'left hand side' and the 'right hand side' of the Baum-Connes conjecture, which are the domain and codomain of μ respectively.

If the Baum-Connes conjecture is true, then it provides a means to calculate $K_*(C_r^*(G))$. The Baum-Connes conjecture also has several important corollaries in topology and algebra. In topology, there is the Novikov conjecture, which is related to the classification of high dimensional manifolds (see [62, Section 11, Unsolved Problems] for the original work of Novikov and [2, Theorem 7.11] for the connection to the Baum-Connes conjecture). In algebra, there is the following conjecture, called the **Kadison-Kaplansky conjecture** (see for example [2, Proposition 7.16] for the statement and connection to the Baum-Connes conjecture).

Conjecture. *Let G be a discrete torsion-free group. Then $C_r^*(G)$ contains no non-trivial idempotents.*

The Baum-Connes conjecture is known to hold for a large class of groups, including

1. a-T-menable groups [33, Theorem 1.1], and so in particular amenable groups,
2. Hyperbolic groups [56, Theorem 20], and so in particular free groups,
3. Connected Lie groups [7, Corollary 4.7], and so in particular $SL(n, \mathbb{C})$ for $n \in \mathbb{N}$.

The conjecture was first proved for many special cases within the above classes, in work that spanned many decades. For example, in 1983, Penington and Plymen [64, Theorem 2.1] proved the conjecture held for complex semisimple Lie groups (and in particular $SL(n, \mathbb{C})$ for $n \in \mathbb{N}$). In 2008, Higson provided a different proof [32, 7.1] for this case, based on a reformulation of the conjecture for connected Lie groups due to Connes.

We note that one can generalize the statement of the Baum-Connes conjecture by replacing groups with groupoids, or replacing the reduced group C^*-algebra with a reduced crossed product. However, it is known that such general statements are false, with various counterexamples [34].

Let us now consider the aforementioned reformulation of the Baum-Connes conjecture due to Connes [11, p.g. 145-146]. For concreteness, we will explain this reformulation for $SL(2, \mathbb{C})$, but the reader should bear in mind that one can make appropriate generalizations in the more general setting of an arbitrary connected Lie group. We will let $G = SL(2, \mathbb{C})$ for the rest of the Introduction.

The group $K = SU(2)$ acts on $\mathfrak{k}$, the Lie algebra of K, by the **adjoint action**. This is an action generally defined for Lie groups, but in the case of matrix Lie groups such as K it is simply given by conjugation, see [44, p.g. 44], that is,

$$k \cdot X := kXk^{-1} \tag{1}$$

for $k \in K$ and $X \in \mathfrak{k}$. Therefore K acts on $\mathfrak{k}^*$, the real dual space of $\mathfrak{k}$, given by the formula

$$(k \cdot \phi)(X) := \phi(k^{-1} \cdot X) = \phi(k^{-1} X k) \tag{2}$$

for $k \in K$, $\phi \in \mathfrak{k}^*$ and $X \in \mathfrak{k}$. This action is called the coadjoint action. We can then consider the so-called **Cartan motion group**, given by $G_0 := K \ltimes \mathfrak{k}^*$.

Mackey suggested in [52] there ought to be a correspondence between 'most' of the irreducible unitary representations of G and G_0. This suggests there is a link between G and G_0. In [11, p.g. 146] Connes constructed a 'deformation' of G onto G_0. In particular, Connes constructed a manifold $\mathcal{G}$, which as a set is given by

$$\mathcal{G} = K \ltimes \mathfrak{k}^* \times \{0\} \sqcup G \times (0,1] = G_0 \times \{0\} \sqcup G \times (0,1]$$

but the topology is non-trivial. From the manifold $\mathcal{G}$, Connes constructed a map in K-theory

$$\tilde{\mu} : K_*(C^*(G_0)) \to K_*(C_r^*(G)).$$

This map is induced by a $C([0,1])$-algebra, which we denote by A^C, constructed from $\mathcal{G}$ (here the superscript C stands for *classical*). Roughly speaking, for a locally compact Hausdorff space X, a $C_0(X)$-algebra is a C^*-algebra whose elements can be multiplied by functions in $C_0(X)$. This notion was originally introduced by Kasparov in [39, Definition 1.5]. The $C([0,1])$-algebra A^C has the property that it is trivial away from 0 and it is this property that induces a map in K-theory.

Connes then proved the following theorem relating $\tilde{\mu}$ to the assembly map μ [11, p.g. 145-146, Proposition 8, Proposition 9].

Theorem. *There exists an isomorphism $K_*^G(\underline{E}G) \cong K_*(C^*(G_0))$ such that*

$$\begin{tikzcd} K_*(C^*(G_0)) \arrow[r, "\tilde{\mu}"] & K_*(C_r^*(G)) \\ K_*^G(\underline{E}G) \arrow[u, "\cong"] \arrow[ur, "\mu"'] & \end{tikzcd}$$

commutes.

This theorem reduces the Baum-Connes conjecture for connected Lie groups to studying the $C([0,1])$-algebra A^C and the induced map in K-theory. Higson's proof of the Baum-Connes conjecture for complex semisimple Lie groups [32] using this reformulation is based on Mackey's analogy that there is a correspondence between 'most' of the irreducible unitary representations of G and G_0. Let us briefly describe the key ingredients in the proof given in [32].

The group G has some special unitary representations called the **principal series representations** of G (originally due to Gelfand and Naimark in [26]) which are obtained by inducing characters on a certain subgroup of G to G, see [52, §2]. The principal series representations of G are irreducible, and they allow us to describe $C_r^*(G)$ concretely. There is a bijection between the principal series representations of G and the irreducible unitary representations of G_0, which is a formal description of Mackey's analogy. In particular, there is a bijection between $\mathrm{Spec}(C_r^*(G))$ and $\mathrm{Spec}(C^*(G_0))$, where $\mathrm{Spec}(A)$ denotes the spectrum of a C^*-algebra A. However, this bijection is not a homeomorphism.

Higson analysed certain subquotients of $C_r^*(G)$ and $C^*(G_0)$, and identified the spectra of these subquotients as subspaces of $\mathrm{Spec}(C_r^*(G))$ and $\mathrm{Spec}(C^*(G_0))$ on which the bijection above restricts to a homeomorphism. These subquotients can also be defined for A^C, and these subquotients are also $C([0,1])$-algebras which induce maps in K-theory. Higson then proved that the assembly map is an isomorphism if each of these induced maps is an isomorphism, and used some C^*-algebra theory to prove that the latter is the case.

The objective of this thesis is to provide an analogue of the Baum-Connes theorem for the quantum group **quantum $\boldsymbol{SL(2,\mathbb{C})}$**, as originally introduced by Podleś and Woronowicz in [65]. This result is in the spirit of Connes' reformulation of the classical conjecture, and the proof is in the spirit of the one provided by Higson in the group case.

Let us set the stage by thinking about what we would need to develop a Baum-Connes type result for quantum $SL(2,\mathbb{C})$ in analogy with the work of Connes and Higson.

1. We will need an analogue of the reduced group C^*-algebra for quantum $SL(2,\mathbb{C})$, denoted $C_r^*(G_q)$. It is well known that for a quantum group there are two associated C^*-algebras which are the dual of one another in the sense of quantum groups (see [46, Definition 8.1] for the construction of the dual in the locally compact setting). One can be viewed as an analogue of the function C^*-algebra, and one can be viewed as an analogue of the group C^*-algebra. In [65], the function algebra for $SL_q(2,\mathbb{C})$ is considered. We will therefore consider the dual of this function algebra.

2. The start of Connes' reformulation was the Cartan motion group. We need a quantum version of the Cartan motion group which is a C^*-algebra we denote by $C^*(G_1)$. The reason for this choice of notation will become clear later.

3. We need to construct a $C([0,1])$-algebra A^Q (where Q stands for *quantum*) that induces a map in K-theory

$$K_*(C^*(G_1)) \to K_*(C_r^*(G_q)).$$

   This map would play the role of the assembly map.

4. To show the above map is an isomorphism using Higson's method, we would need to understand the representation theory of the quantum Cartan motion group, and the analogue of the principal series representations for quantum $SL(2,\mathbb{C})$.

These points provide the overall strategy which we take to develop the quantum Baum-Connes theorem. We could also ask if there is any compatibility with the Baum-Connes theorem for $SL(2,\mathbb{C})$.

We conclude the Introduction by describing the structure and content of the thesis in more detail, and give a list of conventions and notation which we will use throughout the thesis.

Thesis Structure and Content

In Chapter 1 we introduce quantum groups. We start with finite quantum groups to set the scene and then move on to discrete and compact quantum groups and the harmonic analysis of compact quantum groups. In this thesis, we will work extensively with quantum $SU(2)$, and so we devote a section to studying the general results in this context. We finish with the construction of the so-called quantum double. This is a particularly important construction for us because the quantum $SL(2,\mathbb{C})$ group can be realized as an example of a quantum double.

We start Chapter 2 by introducing the quantum $SL(2,\mathbb{C})$ group, the corresponding group algebra $C_r^*(G_q)$, and the analogue of the principal series representations in this case. Then, we shall construct a $C([0,1])$-algebra A^Q, which we call the quantum assembly field, which plays the role of Connes' field. This induces a map in K-theory, which is our analogue of the assembly field. In the course of this construction we shall define a quantum analogue of the Cartan motion group, denoted $C^*(G_1)$.

In Chapter 3 we show that the map in K-theory induced by our field A^Q (that is, our quantum assembly map) is an isomorphism. Our proof proceeds in much the same way as Higson's based on the Mackey analogy.

Finally, in Chapter 4, we link our quantum result with the original Baum-Connes theorem for $SL(2,\mathbb{C})$. Specifically, we will prove that there is a commutative square

$$\begin{array}{ccc} K_*(C^*(G_0)) & \longrightarrow & K_*(C_r^*(G)) \\ \downarrow & & \downarrow \\ K_*(C^*(G_1)) & \longrightarrow & K_*(C_r^*(G_q)). \end{array}$$

The reader should note that there is also an appendix which gives a fairly self-contained account of basic $C_0(X)$-algebra theory.

The reader should also note that the author and Voigt have generalized the results of this thesis to cover deformations of complex semisimple Lie groups, of which quantum $SL(2, \mathbb{C})$ is a special case, see [57].

Conventions and Notation

The term *classical* is used to refer to the setting of groups and Lie algebras, as opposed to the *quantum* setting of quantum groups and quantized universal enveloping algebras.

The natural numbers $\mathbb{N}$ is the set $\{1, 2, 3, ...\}$ and $\mathbb{N}_0$ is the set $\{0, 1, 2, 3, ...\}$. The strictly positive real numbers are denoted by $\mathbb{R}_{>0}$.

All vector spaces and algebras are assumed to be over $\mathbb{C}$.

Inner products are conjugate linear in the first variable and linear in the second.

The vector space of bounded linear operators from a normed space V to a normed space W is denoted by $B(V, W)$. The vector space of compact linear operators from V to W is denoted by $K(V, W)$. If $W = V$, then we write $B(V, V) = B(V)$ and $K(V, V) = K(V)$. If $\mathcal{H}$ is a Hilbert space, the group of unitary operators on $\mathcal{H}$ is denoted by $U(\mathcal{H})$. The operator norm is denoted by $\|-\|_{\mathrm{op}}$, and we sometimes omit the subscript when the context is clear.

If A is a C^*-algebra, the multiplier algebra of A is denoted by $M(A)$, and has unit $1_{M(A)}$. The spectrum of A is denoted by $\mathrm{Spec}(A)$ and the primitive ideal space is denoted by $\mathrm{Prim}(A)$. The spectrum of $a \in A$ is denoted $\mathrm{Spec}(a)$. The centre of A is denoted by $Z(A)$.

If A is a C^*-algebra, the $*$-algebra constructed from A by adjoining a unit is denoted by A^+. That is, A^+ is the set $A \times \mathbb{C}$ equipped with the coordinatewise sum and involution, and multiplication given by

$$(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$$

for $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$. The unit in A^+ is given by $(0, 1)$ (even if A is unital). If A is unital with unit 1_A, then the map $A^+ \to A \oplus \mathbb{C}, (a, \lambda) \mapsto (a + \lambda 1_A, \lambda)$ is a $*$-isomorphism. If A is non-unital, then we the map $A^+ \to M(A), (a, \lambda) \mapsto a + \lambda 1_{M(A)}$ is an injective $*$-homomorphism with image $A + \mathbb{C}1_{M(A)} \subseteq M(A)$. Therefore in any case A^+ can be equipped with a norm making A^+ a C^*-algebra. We will denote $(a, \lambda) \in A^+$ by $a + \lambda$.

The set of a positive elements of a C^*-algebra A is denoted by A_+.

We denote the C^*-algebra of $\mathbb{C}$-valued continuous functions on a locally compact Hausdorff space X that vanish at infinity by $C_0(X)$. We denote C^*-algebra of bounded continuous $\mathbb{C}$-valued functions on X by $C_b(X)$. If X is compact, then these C^*-algebras coincide and we instead use the notation $C(X)$.

If G is a locally compact group, then $L^1(G)$ is a Banach $*$-algebra with respect to the convolution product. We denote by $C_c(G)$ the compactly supported $\mathbb{C}$-valued continuous functions on G, and this is a $*$-subalgebra of $L^1(G)$. Even if G is compact, we will use the notation $C_c(G)$ to distinguish this from the C^*-algebra $C(G)$ as defined above.

The symbol $\odot$ denotes the algebraic tensor product, and $\otimes$ denotes the tensor product of Hilbert spaces, or the minimal tensor product of C^*-algebras. In any case, we will use $\otimes$ in elementary tensors.

We will write the full crossed product of a locally compact group G acting on a C^*-algebra A by an action α by $G \ltimes_\alpha A$, and sometimes $G \ltimes A$ when the context is understood. On $C_c(G, A) \subseteq G \ltimes_\alpha A$, the product and involution is given by the formulae

$$(f \star g)(k) = \int_G \alpha_{k^{-1}s}(f(s))g(s^{-1}k)\ \mathrm{d}s$$

and

$$f^*(k) = \Delta_G(k^{-1})\alpha_{k^{-1}}(f(k^{-1})^*)$$

for $f, g \in C_c(G, A)$ and $k \in G$. Here Δ_G is the modular function of G.

The reduced crossed product of G on A is denoted by $G \ltimes_{\alpha,r} A$ or $G \ltimes_r A$.

If a group G acts on a C^*-algebra A, the elements of A fixed by all elements of G are denoted by A^G.

If G is a Lie group, the Lie algebra is denoted by $\mathfrak{g}$, the corresponding letter in the Fraktur typeface.

If G is an abelian group, the Pontryagin dual of G is denoted by $\widehat{G}$.

Chapter 1

Quantum Groups

In this chapter we will present a basic introduction to the theory of quantum groups.

We start with the principle of non-commutative geometry. There is a contravariant functor

C_0 : **locally compact Hausdorff topological spaces** $\longrightarrow$ **commutative** C^***-algebras**.

Here the morphisms on the spaces are proper continuous functions, and the morphisms on the C^*-algebras are non-degenerate $*$-homomorphisms. The functor C_0 is defined by mapping a locally compact Hausdorff space X to the C^*-algebra $C_0(X)$, and a proper continuous function $f : X \to Y$ to

$$C_0(f) : C_0(Y) \to C_0(X), \quad C_0(f)(g) = g \circ f, \quad g \in C_0(Y).$$

In fact, the functor C_0 is an anti-equivalence between the categories. This is a consequence of the work of Gelfand and Naimark, [27, Lemma 4.1]. In particular, we can see that the topological information of a locally compact Hausdorff space X is completely encoded in the algebraic and analytic structure of the C^*-algebra $C_0(X)$. We can therefore take properties of the space X and understand the corresponding properties of $C_0(X)$, as illustrated in the following examples.

1. The space X is compact if and only if $C_0(X)$ is unital.

2. The space X is connected if and only if $C_0(X)$ has no non-trivial projections.

3. If X is a compact measure space with positive measure μ we obtain a positive linear functional
$$C(X) \to \mathbb{C}, \quad f \mapsto \int_X f \; \mathrm{d}\mu.$$
In fact, any positive linear functional on $C(X)$ comes from integration against a

unique positive measure provided that certain regularity properties on the measure are assumed - see [74, Theorem 2.14].

We can drop the commutativity assumption from our C^*-algebras and still make sense of the algebraic properties above. This leads us to the principle of **non-commutative geometry** - that is, we view a C^*-algebra as a function algebra over a 'non-commutative' topological space.

If we are interested in group theory, we might try to restrict C_0 to locally compact groups. That is, if G is a locally compact group, we consider the C^*-algebra $C_0(G)$, and try to understand additional properties of $C_0(G)$ that reflect the fact G is a group. The idea is to then try to define a locally compact **quantum group** in terms of a C^*-algebra equipped with abstract versions of these properties. Strictly speaking, this C^*-algebra is not the quantum group but rather should be viewed as an algebra of functions over the quantum group. Therefore it is common to refer to a quantum group, and an associated C^*-algebra, and we study the latter.

One property of the theory that we would like to have is that when the C^*-algebra associated to a quantum group is commutative, it is isomorphic to $C_0(G)$ for some locally compact group G. Another desirable feature is a notion of duality for quantum groups that extends the usual Pontryagin dual for abelian groups, [66].

There is a definition of a locally compact quantum group due to Kustermans and Vaes [46, Definition 4.1]. However in this chapter we shall instead content ourselves with the general definitions of finite, discrete and compact quantum groups as well as an example of a locally compact quantum group called the **quantum double**.

1.1 Finite, Discrete and Compact Quantum Groups

1.1.1 Finite Quantum Groups

Let us consider finite groups first. Let G be a finite group with unit e. Then we have the following induced maps on $C(G)$.

1. The **comultiplication**, or **coproduct**,

$$\Delta : C(G) \to C(G \times G) = C(G) \odot C(G), \quad \Delta(f)(x, y) = f(xy),$$

for $f \in C(G)$ and $x, y \in G$.

2. The **counit**, $\epsilon : C(G) \to \mathbb{C},\ \epsilon(f) = f(e)$ for $f \in C(G)$.

3. The **antipode**, $S : C(G) \to C(G),\ S(f)(x) = f(x^{-1})$ for $f \in C(G)$ and $x \in G$.

The group axioms mean $C(G)$ is an example of a finite dimensional commutative C^*-algebra which also has the structure of a **Hopf-$*$-algebra**.

Definition 1.1. A **Hopf $*$-algebra** A is a unital $*$-algebra (with multiplication $m : A \odot A \to A$ and unit 1) which is equipped with a unital $*$-homomorphism $\Delta : A \to A \odot A$ and linear maps $\epsilon : A \to \mathbb{C}$ and $S : A \to A$ satisfying

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta, \quad (\epsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}, \quad m(S \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes S)\Delta = \epsilon(-)1.$$

The maps Δ, ϵ and S are called the comultiplication (or coproduct), counit and antipode respectively.

Remark 1.2. The definition of a Hopf algebra originates from the work of Cartier in a seminar in 1955 (see [6, Introduction]). The name is motivated by the work of Hopf in [35], where Hopf introduces a class of spaces (which we today call H-spaces) for which there is a comultiplication in cohomology [6, p.g. 15]. We refer the reader to one of the many contemporary accounts on the background theory of Hopf algebras, for example [78, Chapter 1] for the definition given above, and standard results.

Remark 1.3. Let A be a Hopf-$*$-algebra.

1. It can be shown that ϵ is automatically a $*$-homomorphism and S is anti-multiplicative with $S(S(a^*)^*) = a$ for all $a \in A$, [79, Proposition 2.4]. In particular S is invertible, with $S^{-1}(a) = S(a^*)^*$ for all $a \in A$.

2. The maps ϵ and S are uniquely determined by Δ, [79, Proposition 2.3].

3. The maps ϵ and S are unital. Indeed we see that $(\epsilon \otimes \mathrm{id})\Delta(1) = \epsilon(1)1 = 1$ since Δ is unital, so $\epsilon(1) = 1$. We also have $S(1)1 = \epsilon(1)1 = 1$, so $S(1) = 1$.

4. We will make use of the **Sweedler notation** for coproducts. If $a \in A$ then since $\Delta(a) \in A \odot A$, we can write $\Delta(a) = \sum_{i=1}^{n} a_{i1} \otimes a_{i2}$ for some $n \in \mathbb{N}$ and $a_{i1}, a_{i2} \in A$. In Sweedler notation, we simply write

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$$

or perhaps even $\Delta(a) = a_{(1)} \otimes a_{(2)}$ where in this notation the summation is implied. We write

$$(\mathrm{id} \otimes \Delta)\Delta(a) = (\Delta \otimes \mathrm{id})\Delta(a) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$$

where there is no ambiguity in the notation because of the first axiom in Definition 1.1. Further applications of Δ extend this notation in the natural way.

5. Let A^{op} denote the **opposite** algebra of A. As a vector space $A^{\mathrm{op}} = A$, but if m_A is the multiplication of A, then the multiplication $m_{A^{\mathrm{op}}}$ on A^{op} is given by $m_{A^{\mathrm{op}}}(a \otimes b) = m_A(b \otimes a)$ for $a, b \in A$. The opposite algebra A^{op} is a Hopf-$*$-algebra, with the same coproduct and counit. The antipode is given by S^{-1} because for $a \in A$

$$\begin{aligned} m_{A^{\mathrm{op}}}(S^{-1} \otimes \mathrm{id})\Delta(a) = m_A(a_{(2)} \otimes S^{-1}(a_{(1)})) &= S^{-1}(m_A(a_{(1)} \otimes S(a_{(2)}))) \\ &= S^{-1}(\epsilon(a)1) = \epsilon(a)1 \end{aligned}$$

and similarly $m_{A^{\mathrm{op}}}(\mathrm{id} \otimes S^{-1})\Delta = \epsilon(-)1$.

The axioms in Definition 1.1 reflect the group axioms in the case of $C(G)$. For example $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ is equivalent to the statement that for all $f \in C(G)$ and $x, y, z \in G$ we have

$$f((xy)z) = f(x(yz))$$

and this is equivalent to the statement that $(xy)z = x(yz)$ for all $x, y, z \in G$, which is of course associativity in G. This axiom is sometimes referred to as the **coassociativity** axiom for this reason.

The example of $C(G)$ motivates the following definition.

Definition 1.4. A **finite quantum group** is a finite dimensional C^*-algebra which is also a Hopf-$*$-algebra.

Remark 1.5. The definition we give appears in [53, Definition 2.4]. This has developed over time, for example Van Daele uses this term in [82] to mean a finite dimensional Hopf algebra.

Let us check that this definition satisfies the first of our desired properties for quantum groups.

Proposition 1.6. *Let A be a commutative finite quantum group. Then there exists a finite group G such that $A \cong C(G)$.*

Proof. Since A is a commutative C^*-algebra we see $A \cong C_0(G)$ for some locally compact Hausdorff space G using the equivalence C_0. Since A is finite dimensional, we must have that G is a finite set.

Since $\Delta : C(G) \to C(G) \odot C(G)$ is a $*$-homomorphism, and we can identify $C(G) \odot C(G) \cong C(G \times G)$, then by the equivalence C_0 there exists a map $m : G \times G \to G$ such that

$\Delta(f)(x, y) = f(m(x, y))$ for $f \in C(G)$ and $x, y \in G$, which we will call multiplication. By the coassociativity axiom, we have

$$f(m(m(x, y), z)) = f(m(x, m(y, z)))$$

for all $f \in C(G)$ and $x, y, z \in G$. Therefore we must have $m(m(x, y), z) = m(x, m(y, z))$ for all $x, y, z \in G$, and so the multiplication is associative.

Since $\epsilon : C(G) \to \mathbb{C}$ is a $*$-homomorphism, and we can view $\mathbb{C}$ as the function algebra over a one point space $\{\mathrm{pt}\}$, we obtain a map $\{\mathrm{pt}\} \to G$ using the equivalence C_0. In particular ϵ picks out a single element of G, which we will call e. The axiom

$$(\epsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}$$

means that for all $f \in C(G)$ and $x \in G$,

$$f(m(e, x)) = f(m(x, e)) = f(x).$$

Therefore we must have $m(e, x) = m(x, e) = x$ for all $x \in G$, and so e is an identity for the multiplication.

Finally we use the antipode S to produce multiplicative inverses. First note that $S = S^{-1}$, because $A = A^{\mathrm{op}}$ and the antipode is uniquely determined by the comultiplication (see Remark 1.3). Therefore

$$S(f^*) = S^{-1}(f)^* = S(f)^*$$

for all $f \in C(G)$ and so S is a $*$-homomorphism in this case. We therefore obtain a map $\iota : G \to G$ using the equivalence C_0. The final Hopf axiom

$$m(S \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes S)\Delta = \epsilon(-)1$$

means that

$$f(m(\iota(x), x)) = f(m(x, \iota(x))) = f(e)$$

for all $f \in C(G)$ and $x \in G$. Therefore if $x \in G$, $\iota(x)$ is a multiplicative inverse for x. Hence G is a finite group. □

Remark 1.7. In fact, it is equivalent to say that A is a finite quantum group if A is a finite dimensional Hopf-$*$-algebra such that for $a \in A$, $a^*a = 0$ if and only if $a = 0$, c.f. [45, Definition 2.3]. Indeed, one can use this condition on A (see [82, Proposition 2.1]) to show there exists a positive faithful linear functional on A. One can then use the GNS construction for $*$-algebras (c.f. [59, p.g. 93-94]) to show that A is in fact a C^*-algebra, and hence A is a finite quantum group in the sense of Definition 1.4.

Let us conclude our discussion on finite quantum groups by looking at another example.

Example 1.8. Let G be a finite group. Then $\mathbb{C}G$, the group ring, is an example of a finite quantum group. We will work through the details. We denote the canonical basis for $\mathbb{C}G$ indexed by G by $\{\delta_g\}_{g\in G}$. The $*$-algebra structure of $\mathbb{C}G$ is given by

$$\delta_g\delta_h = \delta_{gh}, \quad (\delta_g)^* = \delta_{g^{-1}}.$$

where $g, h \in G$. As a vector space, we can identify $\mathbb{C}G$ with $C(G)$, where for $g \in G$, the basis element $\delta_g \in \mathbb{C}G$ is identified with the function

$$\delta_g : G \to \mathbb{C}, \quad \delta_g(h) := \begin{cases} 1 & h = g \\ 0 & h \neq g. \end{cases}$$

However this linear isomorphism is not an algebra isomorphism. Note that $\{\delta_g\}_{g\in G} \subseteq L^2(G)$ forms an orthonormal basis. Of course, $L^2(G) = C(G)$ in the finite setting, but we shall keep the distinction in this example so that we can make a generalization shortly.

The group ring is a C^*-algebra, because the left regular representation

$$\lambda : G \to U(L^2(G)), \quad \lambda_g(f)(h) = f(g^{-1}h), \quad f \in L^2(G), \quad g, h \in G$$

extends linearly to a faithful representation of the group ring, which we also denote by λ,

$$\lambda : \mathbb{C}G \to B(L^2(G))$$

which is given by $\lambda(\delta_g) = \lambda_g$ for $g \in G$. This equips $\mathbb{C}G$ with a C^*-norm. Since G is finite, $\mathbb{C}G$ is finite dimensional, and so is complete. Therefore $\mathbb{C}G$ is a finite dimensional C^*-algebra. We can equip $\mathbb{C}G$ with the Hopf structure

$$\Delta(\delta_g) = \delta_g \otimes \delta_g, \quad \epsilon(\delta_g) = 1, \quad S(\delta_g) = \delta_{g^{-1}}$$

(where $g \in G$) on the basis, extending linearly.

If G is abelian, then we note that as C^*-algebras and Hopf-$*$-algebras we have $\mathbb{C}G \cong C(\widehat{G})$, where $\widehat{G}$ denotes the Pontryagin dual of G, i.e. the group of characters of G (introduced by Pontryagin in [66]). The isomorphism is given by

$$\delta_g \mapsto \mathrm{ev}_g$$

for $g \in G$, where if $\omega \in \widehat{G}$, $\mathrm{ev}_g(\omega) = \omega(g)$.

Recall that we also want to extend the notion of Pontryagin duality to quantum groups. Rather than define this in the finite setting, we will return to this later in the compact setting, which subsumes the finite case. However, we note at this stage that Example 1.8 suggests that we should define the dual of a finite quantum group in such a way that the dual of $C(G)$ for a finite group G is $\mathbb{C}G$.

1.1.2 Discrete Quantum Groups

Remark 1.7 tells us that a finite quantum group can be defined by a purely algebraic condition. From the viewpoint of the principle of non-commutative geometry, it should not be too surprising that we don't need to mention C^*-algebras at all because a finite group has the discrete topology. The theory of finite quantum groups could therefore be developed from an algebraic point of view. It is convenient to take this viewpoint in the setting of discrete quantum groups (as originally in [81]), which we will consider now.

Let G be a discrete group. Consider $C_c(G)$, the compactly (and hence finitely) supported functions on G. We can try to define a comultiplication, counit and antipode using the same formulae as in the finite case, but we run into difficulty because $\Delta(f)$ is no longer compactly supported.

To resolve this issue, we use algebraic multiplier algebras and multiplier Hopf-$*$-algebras, as introduced by Van Daele in [80]. We refer the reader to this paper for further details, and for the statements of the definitions.

Definition 1.9. Let A be an algebra.

(a) A **left multiplier** of A is a linear map $L : A \to A$ such that $L(ab) = L(a)b$ for all $a, b \in A$.

(b) A **right multiplier** of A is a linear map $R : A \to A$ such that $R(ab) = aR(b)$ for all $a, b \in A$.

(c) A **multiplier** of A is a pair (L, R) of a left multiplier and right multiplier respectively such that $R(a)b = aL(b)$.

We denote by $M(A)$ the vector space of multipliers of A.

The prototypical example of a multiplier of an algebra A is the pair (L_a, R_a), where $a \in A$, defined by

$$L_a(b) = ab, \quad R_a(b) = ba, \quad b \in A.$$

The space $M(A)$ is an algebra (and a $*$-algebra, if A is a $*$-algebra), with operations

$$(L_1, R_1)(L_2, R_2) := (L_1 \circ L_2, R_2 \circ R_1), \quad (L, R)^* := (R^*, L^*)$$

where, for a linear map $T : A \to A$, we define a linear map $T^* : A \to A$ by $T^*(a) = T(a^*)^*$ for $a \in A$. It is a unital algebra, with identity $1_{M(A)} = (\mathrm{id}, \mathrm{id})$. We call $M(A)$ the (algebraic) multiplier algebra of A. Note that if A is a C^*-algebra, then $M(A)$ is the usual multiplier algebra using the double centralizer definition (see [59, p.g. 38-39]) of the multiplier algebra of a C^*-algebra.

In working with multiplier algebras, often one encounters the term non-degeneracy. Let us recall the notions associated to this now.

Definition 1.10. Let A be an algebra. Then A is said to have **non-degenerate product** if the following conditions are satisfied.

1. If $a \in A$ and $ab = 0$ for all $b \in A$, then $a = 0$.
2. If $a \in A$ and $ba = 0$ for all $b \in A$, then $a = 0$.

If B is an algebra, and $\phi : A \to M(B)$ is a homomorphism, we say ϕ is **non-degenerate** if

$$\mathrm{span}\{\phi(a)b \mid a \in A, b \in B\} = B, \quad \mathrm{span}\{b\phi(a) \mid a \in A, b \in B\} = B.$$

Remark 1.11. In the case where A, B are $*$-algebras and ϕ is a $*$-homomorphism, then we only require one of the conditions in both parts of Definition 1.10.

If A is a C^*-algebra, then A has non-degenerate product, since if $a \in A$ satisfies $ba = 0$ for all $b \in A$, then $a^*a = 0$, and so $\|a^*a\| = \|a\|^2 = 0$, so $a = 0$.

Note that in the case where A and B are C^*-algebras, we weaken the definition of a non-degenerate $*$-homomorphism $\phi : A \to M(B)$, asking for

$$\overline{\mathrm{span}}\{\phi(a)b \mid a \in A, b \in B\} = B.$$

If A has non-degenerate product, then $M(A)$ contains A as a two sided ideal. Indeed,

$$A \to M(A), \quad a \mapsto (L_a, R_a)$$

is a homomorphism, and non-degeneracy implies that the map is injective. One can check that for $(L, R) \in M(A)$, we have

$$(L_a, R_a)(L, R) = (L_{R(a)}, R_{R(a)}), \quad (L, R)(L_a, R_a) = (L_{L(a)}, R_{L(a)})$$

for all $a \in A$, and so $A \subseteq M(A)$ is a two-sided ideal.

Remark 1.12. If A is a commutative algebra with non-degenerate product and $(L, R) \in M(A)$, then $L = R$. This is because for all $a, b \in A$, we have

$$L(a)b = L(ab) = L(ba) = bR(a) = R(a)b.$$

Hence for each $a \in A$, $(L(a) - R(a))b = 0$ for all $b \in B$. Therefore by non-degeneracy $L = R$ as required.

Example 1.13. Let us consider some examples of multiplier algebras.

(a) Let A be a unital $*$-algebra. Then $M(A) \cong A$.

Indeed, A has non-degenerate product, and so $A \subseteq M(A)$ as a two sided ideal. We have that A is unital and $1_{M(A)} = (L_1, R_1)$, the unit in $M(A)$. Therefore $A = M(A)$.

(b) Let X be a discrete space and consider $C_c(X)$, the space of compactly supported $\mathbb{C}$-valued functions on X. In this example, we will consider $C_c(X)$ as an algebra with pointwise multiplication. We will show that $M(C_c(X)) \cong C(X)$.

Note that if $f \in C(X)$, then

$$M_f : C_c(X) \to C_c(X), \quad M_f(g) = fg,$$

for $g \in C_c(X)$ defines a left and right multiplier of $C_c(X)$ and so we obtain an injective homomorphism

$$C(X) \to M(C_c(X)), \quad f \mapsto (M_f, M_f).$$

We need to show that any multiplier of $C_c(X)$ is of the form (M_f, M_f) for some $f \in C(X)$. Since $C_c(X)$ is commutative and has non-degenerate product, then by Remark 1.12, we only need to show that any left multiplier is of the form M_f for some $f \in C_c(X)$.

Let L be a left multiplier of $C_c(X)$. Then for $g \in C_c(X)$ and $x \in X$, we have

$$\begin{aligned} L(g)(x) &= (L(g)\delta_x)(x) \\ &= L(g\delta_x)(x) \\ &= L(g(x)\delta_x)(x) \\ &= L(\delta_x)(x)g(x) \end{aligned}$$

where δ_x is as defined in Example 1.8. Therefore if we define $f \in C(X)$ by

$$f : X \to \mathbb{C}, \quad f(x) := L(\delta_x)(x)$$

for $x \in X$ then $L = M_f$, as required.

(c) Let $(n_i)_{i \in I}$ be a sequence of natural numbers. Let $A = \text{alg} - \bigoplus_{i \in I} M_{n_i}(\mathbb{C})$, the algebraic direct sum of matrix algebras. Then $M(A) \cong \prod_{i \in I} M_{n_i}(\mathbb{C})$.

Note that if $X \in \prod_{i \in I} M_{n_i}(\mathbb{C})$, then

$$L_X : A \to A, \quad L_X(Y) = XY, \quad R_X : A \to A, \quad R_X(Y) = YX,$$

for $Y \in A$ define left and right multipliers of A respectively and so we obtain an injective homomorphism

$$\prod_{i \in I} M_{n_i}(\mathbb{C}) \to M(A), \quad X \mapsto (L_X, R_X).$$

We need to check that every multiplier of A is of the form (L_X, R_X) for some $X \in \prod_{i \in I} M_{n_i}(\mathbb{C})$.

Let $j \in I$, and let $B \in M_{n_j}(\mathbb{C})$. Then B defines an element of A (which we also call B) which is B in the $M_{n_j}(\mathbb{C})$-block and zero in all other blocks. If L is a left multiplier of A, then we claim that $L(B)$ is zero in all blocks other than the $M_{n_j}(\mathbb{C})$-block, and so the L is a direct product of left multipliers in each block. Let I_j be the element of $\prod_{i \in I} M_{n_i}(\mathbb{C})$ which is zero in all blocks other than the $M_{n_j}(\mathbb{C})$-block, where there is an identity matrix. Then

$$L(B)I_j = L(BI_j) = L(B)$$

and so $L(B)$ is zero in all blocks other than the $M_{n_j}(\mathbb{C})$-block. Since each individual block is unital, then the left multipliers in each block are given by a matrix being multiplied on the left. Hence $L = L_X$ for some $X \in \prod_{i \in I} M_{n_i}(\mathbb{C}) \subseteq M(A)$. A similar argument works for the right multiplier R.

We have shown that a multiplier of A is of the form (L_X, R_Y) where $X, Y \in \prod_{i \in I} M_{n_i}(\mathbb{C})$. We need to show that $X = Y$. This follows from the fact that for all $Z, W \in A$, we must have

$$ZYW = R_Y(Z)W = ZL_X(W) = ZXW.$$

Taking $Z = W = I_j$ as defined above, we see X and Y are equal in the $M_{n_j}(\mathbb{C})$-block.

Doing this for each block shows that $X = Y$ as required.

Remark 1.14. The algebraic multiplier algebras satisfy properties we expect from the C^*-algebraic setting. If A and B are $*$-algebras, we have the obvious natural embedding

$$M(A) \odot M(B) \hookrightarrow M(A \odot B)$$

and a non-degenerate $*$-homomorphism $A \to M(B)$ extends to a $*$-homomorphism $M(A) \to M(B)$, see [80, Proposition A.3, A.6].

Using multiplier algebras, we can generalize the definition of a Hopf algebra.

Definition 1.15. Let A be a $*$-algebra with a non-degenerate product. Suppose there exists a $*$-homomorphism $\Delta : A \to M(A \odot A)$ such that the following conditions are satisfied.

1. $\Delta(a)(1 \otimes b) \in A \odot A$ and $(a \otimes 1)\Delta(b) \in A \odot A$ for all $a, b \in A$.
2. Δ satisfies

$$(a \otimes 1 \otimes 1)(\Delta \otimes \mathrm{id})(\Delta(b)(1 \otimes c)) = (\mathrm{id} \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$$

   for all $a, b, c \in A$.
3. The linear maps $T_1, T_2 : A \odot A \to A \odot A$ defined by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b), \quad T_2(a \otimes b) = (a \otimes 1)\Delta(b)$$

   for $a, b \in A$ are bijective.

Then we call A a **multiplier Hopf-$*$-algebra**.

In the case where $A = C_c(G)$, the first condition is simply saying that if $f \in C_c(G)$, and we 'cut down' $\Delta(f)$ in one variable by multiplying by a finitely supported function in this variable then we obtain a finitely supported function. The second condition comes from associativity in G. The third condition is perhaps the most unusual - but this comes from the fact we have multiplicative inverses in G, c.f. [78, Proposition 1.3.19].

It might seem unusual that we do not have a counit or antipode as in Definition 1.1, but these can be reconstructed from this definition, see [80, Definition 3.4 and Theorem 4.6]. In this case, the counit of a multiplier Hopf-$*$-algebra A is $\epsilon : A \to \mathbb{C}$, a $*$-homomorphism such that

$$(\mathrm{id} \otimes \epsilon)((a \otimes 1)\Delta(b)) = ab, \quad (\epsilon \otimes \mathrm{id})(\Delta(a)(1 \otimes b)) = ab$$

for all $a, b \in A$, and the antipode $S : A \to A$ is an anti-multiplicative and linear map such that

$$m(\mathrm{id} \otimes S)((b \otimes 1)\Delta(a)) = b\epsilon(a), \quad m(S \otimes \mathrm{id})(\Delta(a)(1 \otimes b)) = \epsilon(a)b, \quad S(S(a^*)^*) = a$$

for all $a, b \in A$.

We can now give the definition of an (algebraic) discrete quantum group as introduced in [81].

Definition 1.16. A **discrete quantum group** is a multiplier Hopf-$*$-algebra whose underlying $*$-algebra is the algebraic direct sum of full matrix algebras.

Clearly if G is a discrete group then $C_c(G)$ is a direct sum of copies of $\mathbb{C}$ and so satisfies Definition 1.16. In fact, if A is a commutative discrete quantum group, then $A = C_c(G)$ for some discrete group G (see [81, pg. 432]). This definition also includes the finite quantum groups as in Definition 1.4.

The reason for allowing arbitrary matrix algebras in Definition 1.16 is due to Example 1.8. If G is a non-abelian finite group, then $\mathbb{C}G$ is a direct sum of matrix algebras, see [89, Proposition 3.4].

Remark 1.17. Note that since a discrete quantum group has a C^*-norm (because each of the matrix algebras is a C^*-algebra), we can complete to obtain a C^*-algebra. For us, a C^*-algebraic quantum group will be a completion of a discrete quantum group in the sense of Definition 1.16. We should note that on the C^*-algebraic level, Podleś and Woronowicz considered discrete quantum groups to be duals of compact quantum groups, and described the properties of discrete quantum groups without the terminology of multiplier Hopf algebras, [65, Theorem 3.1].

1.1.3 Compact Quantum Groups

Finally we come to the notion of a compact quantum group. In [90] Woronowicz introduced the notion of a compact matrix quantum group, forming the basis for the modern definition of a compact quantum group which can now be found in many sources, such as [61, Definition 1.1.1].

Definition 1.18. A **compact quantum group** is a unital C^*-algebra A with a $*$-homomorphism $\Delta : A \to A \otimes A$ such that

1. $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$, and

2. $\Delta(A)(A \otimes 1)$ and $\Delta(A)(1 \otimes A)$ are dense in $A \otimes A$.

We shall sometimes denote a compact quantum group A by $C(\mathbb{G})$. Then we shall refer to $\mathbb{G}$ as the underlying compact quantum group, and A as the corresponding function algebra. With this terminology, a compact quantum group with commutative function algebra is a compact group, see [61, p.g. 2].

One should notice that a quantum group that is both discrete and compact is a finite quantum group, and that a finite quantum group is an example of a compact quantum group. In analogy to Example 1.8, we have the following.

Example 1.19. Let G be a discrete group. Then we can use the same definitions as in Example 1.8 to equip $\mathbb{C}G$ with a norm. Then the **reduced group C^*-algebra** $C_r^*(G)$ is given by the completion of $\mathbb{C}G$ with respect to this norm.

The C^*-algebra $C_r^*(G)$ is a compact quantum group, with comultiplication defined on $\mathbb{C}G$ by $\Delta(\delta_g) = \delta_g \otimes \delta_g$ for $g \in G$ and then by extending to $C_r^*(G)$. Note that for this one needs to show that Δ is bounded with respect to the above norm, see [45, Example 3.1.3].

One might wonder if one can reconstruct a counit and antipode from the axioms of a compact quantum group. We will see in Section 1.2 that it is possible to define these on a dense subspace, but they need not be bounded there, so do not extend to the whole quantum group. This is illustrated in the following example.

Example 1.20. Let $G = \mathbb{F}_2$ be the free group on two generators. Then by Example 1.19, $A = C_r^*(G)$ is a compact quantum group. The group ring $\mathbb{C}G \subseteq C_r^*(G)$ is a unital $*$-algebra with the given comultiplication. We can define a counit and antipode on $\mathbb{C}G$ as in Example 1.8, and in this way $\mathbb{C}G$ is a Hopf-$*$-algebra. However, the counit here is not bounded. Indeed, suppose it were, and consider the extension, a unital $*$-homomorphism $\epsilon : C_r^*(G) \to \mathbb{C}$. However, $C_r^*(G)$ is simple (see [67]) and so $\mathrm{Ker}(\epsilon) = 0$ or $\mathrm{Ker}(\epsilon) = C_r^*(G)$. Neither are possible.

The main example of a compact quantum group that will concern us is the quantum $SU(2)$ group, as introduced by Woronowicz in [91].

Example 1.21. For $q \in (0,1]$ define $\mathcal{O}(SU_q(2))$ to be the universal unital $*$-algebra generated by α, γ subject to the condition

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ is unitary.}$$

Define $C(SU_q(2))$ to be the universal enveloping C^*-algebra of $\mathcal{O}(SU_q(2))$. That is, we consider unital $*$-representations $\pi : \mathcal{O}(SU_q(2)) \to B(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space, and we define a C^*-norm on $\mathcal{O}(SU_q(2))$ by

$$\|x\| = \sup_{\pi} \|\pi(x)\|_{\mathrm{op}} \tag{1.1}$$

for $x \in \mathcal{O}(SU_q(2))$. There are some details to be checked here - namely, that this makes sense for all $x \in \mathcal{O}(SU_q(2))$ and that the formula above defines a norm. For the former, it is sufficient to check this on α and γ using the triangle inequality. Since the matrix

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is unitary we can apply π to each entry of the matrix to obtain a unitary in $M_2(B(\mathcal{H}))$. If $x \in \mathcal{H}$ with $\|x\|_{\mathcal{H}} \leq 1$, we have

$$\begin{pmatrix} \pi(\alpha) & -q\pi(\gamma^*) \\ \pi(\gamma) & \pi(\alpha^*) \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} \pi(\alpha)x \\ \pi(\gamma)x \end{pmatrix}.$$

Therefore

$$\|\pi(\alpha)x\|_{\mathcal{H}}^2 + \|\pi(\gamma)x\|_{\mathcal{H}}^2 \leq 1$$

and so $\|\pi(\alpha)\|_{\mathrm{op}}, \|\pi(\gamma)\|_{\mathrm{op}} \leq 1$. For the fact (1.1) defines a norm, we refer the reader to [91, (1.8)].

There is a comultiplication map

$$\Delta : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$$

which is given by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma,$$

see [91, Theorem 1.4].

When $q = 1$, $C(SU_q(2))$ is simply the algebra $C(SU(2))$, see [91, p.g. 128, Remark 3]. In this case $\alpha : SU(2) \to \mathbb{C}$ is the coordinate projection to the top left matrix entry, and $\gamma : SU(2) \to \mathbb{C}$ is the coordinate projection to the bottom left matrix entry.

If $q, q' \in (0, 1)$ then the algebras $C(SU_q(2))$ and $C(SU_{q'}(2))$ are isomorphic (see [91, Theorem A2.2]), but this does not mean that they are isomorphic as quantum groups. From the point of view of the principle of non-commutative geometry, the underlying quantum groups are homeomorphic.

1.2 Harmonic Analysis of Compact Quantum Groups

In this section we will review basic representation theory for compact quantum groups, and the construction of the dual of a compact quantum group. This section is mainly theoretical in nature, and we will apply this theory to $SU(2)$ in Section 1.3.

Much of the content here was developed from Woronowicz's original treatment of representations of compact matrix quantum groups in the finite dimensional setting, [90, Section 2]. In what follows however we mainly take definitions and results from [53].

1.2.1 Representation Theory of Compact Quantum Groups

As is the case for groups, the representation theory of quantum groups is of interest. We start by recalling the concept of a (unitary) representation of a compact quantum group and associated notions, as in [53]. In the definition, we use the following leg numbering notation. Let A be a C^*-algebra and let $\sigma : A \otimes A \to A \otimes A$ denote the flip map. If $x \in M(A \otimes K(\mathcal{H}))$ we define $x_{(2,3)}, x_{(1,3)} \in M(A \otimes A \otimes K(\mathcal{H}))$ by

$$x_{(2,3)} = \mathrm{id} \otimes x, \quad x_{(1,3)} = (\sigma \otimes \mathrm{id})(x_{(2,3)}).$$

Definition 1.22. Let $A = C(\mathbb{G})$ be a compact quantum group and $\mathcal{H}$ be a Hilbert space.

(a) A **(unitary) representation** of $\mathbb{G}$ on $\mathcal{H}$ is an invertible (resp. unitary) element $v \in M(A \otimes K(\mathcal{H}))$ such that

$$(\Delta \otimes \mathrm{id})(v) = v_{(1,3)} v_{(2,3)}.$$

(b) A closed subspace $V \subseteq \mathcal{H}$ is said to be **invariant** under a representation v of $\mathbb{G}$ on $\mathcal{H}$ if $(\mathrm{id} \otimes p)v(\mathrm{id} \otimes p) = v(\mathrm{id} \otimes p)$, where p is the orthogonal projection onto V. The representation v is said to be **irreducible** if the only invariant subspaces are 0 and $\mathcal{H}$.

(c) Let $\mathcal{K}$ be a Hilbert space, and suppose $v \in M(A \otimes K(\mathcal{H}))$ and $w \in M(A \otimes K(\mathcal{K}))$ are representations of $\mathbb{G}$ on $\mathcal{H}$ and $\mathcal{K}$ respectively. An intertwiner between v and w is an element $x \in B(\mathcal{H}, \mathcal{K})$ such that $(1 \otimes x)v = w(1 \otimes x)$. If x is invertible (resp.

unitary) then v and w are said to be **equivalent** (resp. **unitarily equivalent**). The set of intertwiners between v and w is denoted by $\mathcal{C}(v,w)$.

(d) If $v \in M(A \otimes K(\mathcal{H}))$ and $w \in M(A \otimes K(\mathcal{K}))$ are representations of $\mathbb{G}$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, then we define the **direct sum** $v \oplus w$, a representation of $\mathbb{G}$ on $\mathcal{H} \oplus \mathcal{K}$, to be the direct sum of v and w viewed as maps $A \otimes K(\mathcal{H}) \to A \otimes K(\mathcal{H})$ and $A \otimes K(\mathcal{K}) \to A \otimes K(\mathcal{K})$ respectively.

Remark 1.23.

1. As expected, if $\mathbb{G} = G$ where G is a compact group, then unitary representations of G in the sense of Definition 1.22 correspond to unitary representations of G in the usual sense, see [53, pg. 20].

2. We will particularly be interested in the case where v is a representation on a finite dimensional Hilbert space $\mathcal{H}$. In this case, if we choose a basis for $\mathcal{H}$ we can write $v = (v_{ij})$, a matrix with entries $v_{ij} \in A$. These entries are often referred to as the **matrix coefficients** of v. The condition on $(\Delta \otimes \mathrm{id})(v)$ then becomes

$$\Delta(v_{ij}) := \sum_k v_{ik} \otimes v_{kj}.$$

Example 1.24. Let $A = C(SU_q(2))$. Consider

$$u^{\frac{1}{2}} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in A \otimes K(\mathbb{C}^2)$$

with respect to the standard orthonormal basis of $\mathbb{C}^2$. By the definition of the comultiplication on $C(SU_q(2))$ this is a unitary representation of $SU_q(2)$. One can also check that this is an irreducible representation.

The unitary representation theory of a compact quantum group is well-behaved like in the classical case of compact groups. For example, we have the following results, see [53, Section 6], which are generalizations from the compact group setting.

Theorem 1.25. *Let $A = C(\mathbb{G})$ be a compact quantum group.*

(a) If V is an invariant subspace for a unitary representation of $\mathbb{G}$, then $V^\perp$ is also invariant.

(b) Any irreducible unitary representation of $\mathbb{G}$ is finite dimensional.

*(c) (**Schur's Lemma**) If v, w are irreducible unitary representations of $\mathbb{G}$, then $\mathcal{C}(u,v) = 0$, or v and w are equivalent and $\mathcal{C}(u,v)$ is one-dimensional.*

(d) *A unitary representation of $\mathbb{G}$ decomposes as a direct sum of irreducible unitary representations of $\mathbb{G}$.*

We can now describe the dense Hopf-$*$-algebra of a compact quantum group, following [53, Section 7]. This is analogous to the space of 'trigonometric polynomials' introduced when studying harmonic analysis on a compact group G, see for example [24, pg. 131].

Theorem 1.26. *Let $A = C(\mathbb{G})$ be a compact quantum group. Let $\mathcal{O}(\mathbb{G})$ be the subspace of $C(\mathbb{G})$ spanned by the matrix coefficients of all finite dimensional unitary representations of $C(\mathbb{G})$.*

(a) *$\mathcal{O}(\mathbb{G})$ is a dense $*$-subalgebra of $C(\mathbb{G})$.*

(b) *Let $\{u^\lambda \mid \lambda \in \Lambda\}$ be a complete set of mutually inequivalent irreducible unitary representations of $\mathbb{G}$. Then the set of all matrix coefficients of this set of representations forms a basis for $\mathcal{O}(\mathbb{G})$.*

(c) *$\mathcal{O}(\mathbb{G})$ is a Hopf-$*$-algebra. With respect to the basis given above, the counit and antipode are given respectively by the formulae*

$$\epsilon(u^\lambda_{ij}) = \delta_{ij}, \quad S(u^\lambda_{ij}) = (u^\lambda_{ji})^*.$$

Example 1.27. Let T be the circle group. We will describe $\mathcal{O}(T)$ as defined Theorem 1.26.

The unitary irreducible representations of T correspond to unitary irreducible representations of T in the sense of Definition 1.22. Each irreducible representation of T is one dimensional, and is given by $z \mapsto z^m$ for some $m \in \mathbb{Z}$ (see [24, Theorem 4.5]).

Then the set of matrix coefficients $\mathcal{O}(T)$ is spanned by the polynomials $z \mapsto z^m$ for $m \in \mathbb{Z}$. Therefore $\mathcal{O}(T) = \mathbb{C}[z, z^{-1}]$.

1.2.2 Haar Integrals and Duals of Compact Quantum Groups

Let G be a locally compact group. Then it is well known (see [30] for the original work due to Haar, or [24, Theorem 2.10] for a contemporary account) that there exists a so-called left **Haar measure** on G. This is a non-zero Radon measure μ on G that is left translation invariant. In terms of integrals, μ satisfies

$$\int_G f(yx)\, \mathrm{d}\mu(x) = \int_G f(x)\, \mathrm{d}\mu(x)$$

for all integrable functions f on G and all $y \in G$. One can also consider right Haar measures, which are right translation invariant. Haar measures on G are unique up to a positive scalar multiple [24, Theorem 2.20], and on a compact group, a left Haar measure is also a right Haar measure and vice-versa (in this case, G is said to be **unimodular**, [24, Section 2.4]). On a compact group G we can normalize our choice of Haar measure by requiring that $\mu(G) = 1$. In particular, on compact groups we talk of 'the' Haar measure without any confusion.

As we saw at the start of the chapter, if X is a compact measure space with positive measure μ we have the positive linear functional on $C(X)$ given by

$$C(X) \to \mathbb{C}, \quad f \mapsto \int_X f(x) \, \mathrm{d}\mu(x).$$

In particular, if G is a compact group with Haar measure μ we obtain the **Haar functional** defined by

$$\phi : C(X) \to \mathbb{C}, \quad \phi(f) = \int_G f(x) \, \mathrm{d}\mu(x)$$

for $f \in C(G)$. The propery $\mu(G) = 1$ is equivalent to $\phi(1) = 1$, where $1 \in C(X)$ is the constant function taking the value $1 \in \mathbb{C}$. The left and right invariance conditions on μ can be written in terms of integrals as

$$(\mathrm{id} \otimes \phi)\Delta(f) = \phi(f) \cdot 1, \quad (\phi \otimes \mathrm{id})\Delta(f) = \phi(f) \cdot 1$$

respectively, for $f \in C(G)$. We can use this to motivate the definition of left and right invariant functionals on a compact quantum group, as seen in [53, Definition 4.1].

Definition 1.28. Let A be a compact quantum group with unit 1. A linear functional ϕ on A is called **left invariant** if

$$(\mathrm{id} \otimes \phi)\Delta(a) = \phi(a) \cdot 1$$

for all $a \in A$ and **right invariant** if

$$(\phi \otimes \mathrm{id})\Delta(a) = \phi(a) \cdot 1$$

for all $a \in A$.

We have the following theorem, [53, Theorem 4.4, Proposition 7.8] which mirrors the classical group case.

Theorem 1.29. *Let $A = C(\mathbb{G})$ be a compact quantum group.*

(a) *There is a unique left invariant state $\phi_{\mathbb{G}}$ on $C(\mathbb{G})$.*

(b) *The unique left invariant state $\phi_{\mathbb{G}}$ is also right invariant.*

(c) *The state $\phi_{\mathbb{G}}$ is faithful on $\mathcal{O}(\mathbb{G})$, i.e. if $a \in \mathcal{O}(\mathbb{G})$ and $\phi_{\mathbb{G}}(a^*a) = 0$, then $a = 0$.*

The state is called the ***Haar functional*** *or* ***Haar state*** *on $C(\mathbb{G})$.*

Let $C(\mathbb{G})$ be a compact quantum group with Haar functional $\phi_{\mathbb{G}}$. Then we set

$$L^2(\mathbb{G}) := \mathrm{GNS}(\phi_{\mathbb{G}}), \tag{1.2}$$

the Hilbert space obtained by carrying out the GNS construction with respect to $\phi_{\mathbb{G}}$, see [59, Section 3.4]. Then $C(\mathbb{G})$ acts by bounded operators on $L^2(\mathbb{G})$. This Hilbert space plays the role of the L^2-space of a classical group. Indeed, if $\mathbb{G}$ is a classical group G, then $L^2(\mathbb{G}) = L^2(G)$.

In the theory of compact groups, the Schur orthogonality relations (see [24, 5.8]) are an important ingredient in the representation theory. Let us recall these relations now.

Theorem 1.30. *Let G be a compact group, and suppose π, ρ are irreducible unitary representations of G on Hilbert spaces $\mathcal{H}_\pi$ and $\mathcal{H}_\rho$ respectively. If $v_\pi, w_\pi \in \mathcal{H}_\pi$ and $v_\rho, w_\rho \in \mathcal{H}_\rho$, then*

$$\int_G \overline{\langle v_\pi, \pi(k) w_\pi\rangle}\langle v_\rho, \rho(k) w_\rho\rangle \, \mathrm{d}\mu(k) = \begin{cases} 0 & \pi \text{ and } \rho \text{ are inequivalent} \\ \frac{1}{\dim(\mathcal{H}_\pi)}\langle v_\rho, v_\pi\rangle\langle w_\pi, w_\rho\rangle & \pi \text{ and } \rho \text{ are u. equivalent.} \end{cases}$$

Note that in the latter case we identify $\mathcal{H}_\pi$ and $\mathcal{H}_\rho$ via an equivariant unitary to calculate the inner products on the right hand side.

Remark 1.31. If π and ρ are unitarily equivalent, then there is an equivariant unitary $U : \mathcal{H}_\pi \to \mathcal{H}_\rho$. Then the right hand side of the formula in Theorem 1.30 is

$$\frac{1}{\dim(\mathcal{H}_\pi)}\langle v_\rho, U v_\pi\rangle\langle U w_\pi, w_\rho\rangle.$$

We remark that the choice of equivariant unitary used here does not matter. Indeed, if $V : \mathcal{H}_\pi \to \mathcal{H}_\rho$ is another unitary, then by Schur's Lemma (Theorem 1.25) there exists $\lambda \in \mathbb{C}$ such that $V = \lambda U$. In fact we must have $|\lambda| = 1$ because V is also unitary. Then

$$\langle v_\rho, V v_\pi\rangle\langle V w_\pi, w_\rho\rangle = |\lambda|\langle v_\rho, U v_\pi\rangle\langle U w_\pi, w_\rho\rangle = \langle v_\rho, U v_\pi\rangle\langle U w_\pi, w_\rho\rangle.$$

Note that the inner products on the left hand side of the formula in Theorem 1.30 are matrix coefficients for π and ρ respectively. Therefore these relations tell us when matrix coefficients are orthogonal in the L^2-norm with respect to the Haar measure.

Let us now discuss the quantum version of these relations. First we need the notion of the contragredient representation, originally developed by Woronowicz for compact matrix quantum groups in [90, Section 3]. We refer the reader to [61, p.g. 11] for the theory for compact quantum groups in general.

Let $A = C(\mathbb{G})$ be a compact quantum group, and let $v \in A \otimes B(\mathcal{H})$ be a finite dimensional representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$. Let $\overline{\mathcal{H}}$ be the conjugate Hilbert space. That is, $\mathcal{H} = \overline{\mathcal{H}}$ as sets, but the scalar multiplication changes. If $h \in \mathcal{H}$, then we write $\overline{h} \in \overline{\mathcal{H}}$ for the corresponding element in $\overline{\mathcal{H}}$. Then for $\lambda \in \mathbb{C}$ and $h \in \mathcal{H}$, we define

$$\lambda \overline{h} = \overline{\overline{\lambda} h}.$$

The inner product of $\overline{\mathcal{H}}$ is given by

$$\langle \overline{h}, \overline{h'} \rangle_{\overline{\mathcal{H}}} = \overline{\langle h, h' \rangle_{\mathcal{H}}}$$

for $\overline{h}, \overline{h'} \in \overline{\mathcal{H}}$.

Recall that $\mathcal{H}^* \cong \overline{\mathcal{H}}$ via the Riesz representation theorem. If $\mathcal{H}$ has basis $\{e_i\}_{i \in I}$, then the corresponding dual basis of $\mathcal{H}^*$ under this identification is given by $\{\overline{e_i}\}_{i \in I}$. Define

$$j : B(\mathcal{H}) \to B(\overline{\mathcal{H}}), \quad j(T)\overline{h} = \overline{T^* h} \tag{1.3}$$

for $T \in B(\mathcal{H})$ and $h \in \mathcal{H}$.

Define the **contragredient representation** v^c of v by

$$v^c := (\mathrm{id} \otimes j)(v^{-1}) \in A \otimes B(\overline{\mathcal{H}})$$

One can show that v^c defines a representation of $\mathbb{G}$, [61, p.g. 11-13], and that in the case where $A = C(G)$ for some compact group G, then we recover the classical contragredient representation. We might ask - if $v = (v_{ij})$ is a unitary representation written in matrix form with respect to a basis in $\mathcal{H}$, how does v^c look with respect to the corresponding dual basis of $\overline{\mathcal{H}}$? What about v^{cc}? Let us answer these questions now.

Proposition 1.32. *Let $A = C(\mathbb{G})$ be a compact quantum group, and let $v = (v_{ij})$ be a finite dimensional unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$, written as a matrix with respect to a chosen orthonormal basis $\{e_i\}_{i \in I}$ of $\mathcal{H}$. Then with respect to the* *corresponding dual basis* $\{\overline{e_i}\}_{i\in I}$ *of* $\overline{\mathcal{H}}$ *we have*

$$(v^c)_{ij} = S(v_{ji}) = v_{ij}^*$$

and with respect to the chosen orthonormal basis of $\mathcal{H}$ *we have*

$$(v^{cc})_{ij} = S^2(v_{ij}).$$

Proof. We can write $v = \sum_{i,j} v_{ij} \otimes e_{ij} \in A \otimes B(\mathcal{H})$, where for $h \in \mathcal{H}$,

$$e_{ij}(h) := \langle e_j, h\rangle_{\mathcal{H}}\ e_i.$$

That is, if we represent e_{ij} by a matrix, it is the usual matrix unit with a 1 in the ij^{th} position and zeroes elsewhere. Since v is unitary, we have that $v^{-1} = v^*$, and so

$$v^{-1} = \sum_{i,j} v_{ij}^* \otimes e_{ij}^* = \sum_{i,j} v_{ij}^* \otimes e_{ji}.$$

Define $\overline{e_{ij}} \in B(\overline{\mathcal{H}})$ by

$$\overline{e_{ij}}(\overline{h}) := \langle \overline{e_j}, \overline{h}\rangle_{\overline{\mathcal{H}}}\ \overline{e_i},$$

for $\overline{h} \in \overline{\mathcal{H}}$. These are the matrix units with respect to the dual basis. One can check that $j(e_{ij}) = \overline{e_{ji}}$, and so

$$v^c = \sum_{i,j} v_{ij}^* \otimes \overline{e_{ij}}.$$

Therefore as a matrix with respect to the basis $\{\overline{e_i}\}_{i\in I}$ of $\overline{\mathcal{H}}$, $(v^c)_{ij} = v_{ij}^* = S(v_{ji})$ as required.

For the double contragredient, we need to find v^{c-1} (since the contragredient representation is not necessarily unitary). We will show that

$$v^{c-1} = \sum_{i,j} S(v_{ij}^*) \otimes \overline{e_{ij}}.$$

We have

$$\left(\sum_{i,j} v_{ij}^* \otimes \overline{e_{ij}}\right)\left(\sum_{k,l} S(v_{kl}^*) \otimes \overline{e_{kl}}\right) = \sum_{i,j,l} v_{ij}^* S(v_{jl}^*) \otimes \overline{e_{il}} = \sum_{i,j,l} (S^{-1}(v_{jl})v_{ij})^* \otimes \overline{e_{il}}.$$

By Remark 1.3, $m_{A^{\mathrm{op}}}(\mathrm{id} \otimes S^{-1})\Delta(v_{il}) = \epsilon(v_{il})1$ and so $\sum_j S^{-1}(v_{jl})v_{ij} = \epsilon(v_{il})1 = \delta_{il}1$.

Therefore

$$\left(\sum_{i,j} v_{ij}^* \otimes \overline{e_{ij}}\right)\left(\sum_{k,l} S(v_{kl}^*) \otimes \overline{e_{kl}}\right) = \sum_i 1 \otimes \overline{e_{ii}} = 1 \in A \otimes B(\overline{\mathcal{H}}).$$

We then finally have that

$$v^{cc} = \sum_{i,j} S(v_{ij}^*) \otimes e_{ji} = \sum_{i,j} S^2(v_{ji}) \otimes e_{ji} = \sum_{i,j} S^2(v_{ij}) \otimes e_{ij}.$$

Therefore as a matrix with respect to the basis $\{e_i\}_{i\in I}$ of $\mathcal{H}$, $(v^{cc})_{ij} = S^2(v_{ij})$ as required.

□

Note that in the classical case $S^2 = \mathrm{id}$ (see the proof of Proposition 1.6) so $v^{cc} = v$. However this is **not** true in the general quantum setting. We will see this with quantum $SU(2)$ in Section 1.3.

The Schur orthogonality relations can now be stated for compact quantum groups, as in [61, Theorem 1.4.3], developing the result by Woronowicz for compact matrix quantum groups in [90, (5.14), (5.15)]. One can show (see [61, pg. 14]) that for an irreducible finite dimensional representation u of a compact quantum group $\mathbb{G}$ on a Hilbert space $\mathcal{H}_u$, the representations u and u^{cc} are equivalent and so by Schur's Lemma (Theorem 1.25) the space $\mathcal{C}(u, u^{cc}) \subseteq B(\mathcal{H}_u)$ is one dimensional. In fact, it is also shown there that $\mathcal{C}(u, u^{cc})$ is spanned by a positive invertible operator $P_u \in B(\mathcal{H}_u)$. We can fix the choice of P_u by requiring $\mathrm{Trace}(P_u) = \mathrm{Trace}(P_u^{-1})$.

Theorem 1.33. *Let $A = C(\mathbb{G})$ be a compact quantum group, and let $v = (v_{ij})$ be a finite dimensional unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$, written as a matrix with respect to a chosen orthonormal basis of $\mathcal{H}$. Then*

$$\phi_{\mathbb{G}}(v_{kl}v_{ij}^*) = \frac{\delta_{ki}(P_v)_{jl}}{\mathrm{Trace}(P_v)}, \quad \phi_{\mathbb{G}}(v_{ij}^* v_{kl}) = \frac{\delta_{jl}(P_v^{-1})_{ki}}{\mathrm{Trace}(P_v)}$$

and if $u = (u_{kl})$ is another finite dimensional unitary representation of $\mathbb{G}$ that is inequivalent to v, then $\phi_{\mathbb{G}}(u_{kl}v_{ij}^*) = \phi_{\mathbb{G}}(u_{ij}^* v_{kl}) = 0$.

Note that if $A = C(G)$, and u is a finite dimensional unitary representation of G, then the operator $P_u = \mathrm{id}$ because $u^{cc} = u$, and then we recover Theorem 1.30 from Theorem 1.33.

Now we turn to duality, and define the dual of a compact quantum group. Since the Pontryagin dual of a compact group is a discrete group [24, Proposition 4.4], we will be constructing a discrete quantum group. The following construction is a special case of that given in [83, Section 4].

Let $A = C(\mathbb{G})$ be a compact quantum group with dense Hopf algebra of matrix coefficients $\mathcal{O}(\mathbb{G})$ and Haar state $\phi_{\mathbb{G}}$. Let $\{u^\lambda \mid \lambda \in \Lambda\}$ be a complete set of mutually inequivalent irreducible unitary representations of $\mathbb{G}$. Let P_λ denote the positive operator intertwining u^λ and $u^{\lambda^{cc}}$ as defined prior to Theorem 1.33. Consider the vector space

$$\mathcal{D}(\mathbb{G}) := \{\omega \in \mathcal{O}(\mathbb{G})^* \mid \omega(f) = \phi_{\mathbb{G}}(fg) \text{ for some } g \in \mathcal{O}(\mathbb{G})\} \subseteq \mathcal{O}(\mathbb{G})^*.$$

This is a convenient subspace of $\mathcal{O}(\mathbb{G})^*$ to work with. We will show that $\mathcal{D}(\mathbb{G})$ contains the dual basis to the basis of matrix coefficients provided by our choice of representations and Theorem 1.26 (b). Set

$$a_{ij}^\lambda := \sum_r \operatorname{Trace}(P_\lambda)(P_\lambda^{-1})_{jr}(u_{ir}^\lambda)^* \in \mathcal{O}(\mathbb{G}) \tag{1.4}$$

where the indices i, j depend on the labels given to a chosen basis for the (finite-dimensional) carrier Hilbert space of the representation u^λ.

Define $\omega_{ij}^\lambda \in \mathcal{O}(\mathbb{G})^*$ by $\omega_{ij}^\lambda(f) = \phi_{\mathbb{G}}(fa_{ij}^\lambda)$ for $f \in \mathcal{O}(\mathbb{G})$. Then $\omega_{ij}^\lambda \in \mathcal{D}(\mathbb{G})$ by construction and if $\mu \in \Lambda$ we have

$$\begin{aligned}
\omega_{ij}^\lambda(u_{kl}^\mu) &= \phi_{\mathbb{G}}(u_{kl}^\mu a_{ij}^\lambda) \\
&= \sum_r \operatorname{Trace}(P_\lambda)(P_\lambda^{-1})_{jr}\phi_{\mathbb{G}}(u_{kl}^\mu(u_{ir}^\lambda)^*) \\
&= \sum_r \operatorname{Trace}(P_\lambda)(P_\lambda^{-1})_{jr}\frac{\delta_{ki}\delta_{\mu\lambda}(P_\lambda)_{rl}}{\operatorname{Trace}(P_\lambda)} \\
&= \delta_{\lambda\mu}\delta_{ik}\delta_{jl}
\end{aligned}$$

where k, l depend on labels given to a chosen basis for the carrier space of u^μ and we have used the orthogonality relations in Theorem 1.33. It follows that the ω_{ij}^λ form a linearly independent set. If $f \in \mathcal{O}(\mathbb{G})$, then we can write $f = \sum_{i,j,\lambda} c_{ij}^\lambda u_{ij}^\lambda$ where $c_{ij}^\lambda \in \mathbb{C}$. If $\omega \in \mathcal{D}(\mathbb{G})$, then there exists $g \in \mathcal{O}(\mathbb{G})$ such that $\omega_{\mathbb{G}}(f) = \phi_{\mathbb{G}}(fg)$. Then

$$\omega(f) = \phi_{\mathbb{G}}(fg) = \sum_{i,j,\lambda} c_{ij}^\lambda \phi_{\mathbb{G}}(u_{ij}^\lambda g) = \sum_{i,j,\lambda} \phi_{\mathbb{G}}(u_{ij}^\lambda g)\omega_{ij}^\lambda(f)$$

so $\mathcal{D}(\mathbb{G})$ is spanned by the ω_{ij}^λ. Therefore $\{\omega_{ij}^\lambda\}$ is a basis for $\mathcal{D}(\mathbb{G})$. We can use the comultiplication on a Hopf algebra to give a multiplication on elements in the dual. We define, for $\omega, \eta \in \mathcal{O}(\mathbb{G})^*$, their product $\omega\eta \in \mathcal{O}(\mathbb{G})^*$ by

$$(\omega\eta)(f) = (\omega \otimes \eta)\Delta(f)$$

for $f \in \mathcal{O}(\mathbb{G})$. We can calculate the product of our basis elements of $\mathcal{D}(\mathbb{G})$, giving

$$(\omega_{ij}^{\lambda}\omega_{kl}^{\mu})(u_{pq}^{\gamma}) = (\omega_{ij}^{\lambda} \otimes \omega_{kl}^{\mu})\left(\sum_r u_{pr}^{\gamma} \otimes u_{rq}^{\gamma}\right) = \delta_{\lambda\gamma}\delta_{\mu\gamma}\delta_{ip}\delta_{jk}\delta_{lq} = \delta_{\lambda\mu}\delta_{jk}\omega_{il}^{\lambda}(u_{pq}^{\gamma}).$$

where $\gamma \in \Lambda$ and p, q depend on labels given to a chosen basis for the carrier space of u^{γ}.

It therefore follows that $\mathcal{D}(\mathbb{G})$ is a subalgebra of $\mathcal{O}(\mathbb{G})^*$ and by examining the formulae for the product of basis elements above, we see that $\mathcal{D}(\mathbb{G})$ is an algebraic direct sum of matrix algebras, where the ω_{ij}^{λ} are matrix units. We can also equip $\mathcal{D}(\mathbb{G})$ with a $*$-structure. For $\omega \in \mathcal{D}(\mathbb{G})$ we define $\omega^* \in \mathcal{O}(\mathbb{G})^*$ by

$$\omega^*(f) = \overline{\omega(S(f)^*)}$$

for $f \in \mathcal{O}(\mathbb{G})$. On the matrix units we have

$$(\omega_{ij}^{\lambda})^*(u_{kl}^{\mu}) = \overline{\omega_{ij}^{\lambda}(S(u_{kl}^{\mu})^*)} = \overline{\omega_{ij}^{\lambda}(u_{lk}^{\mu})} = \omega_{ji}^{\lambda}(u_{kl}^{\mu})$$

so we obtain a $*$-structure on $\mathcal{D}(\mathbb{G})$ and it is the usual one on $\mathcal{D}(\mathbb{G})$ when viewed as a direct sum of matrix algebras. We can go further and show that $\mathcal{D}(\mathbb{G})$ is a multiplier Hopf algebra, with comultiplication defined by

$$\Delta(\omega)(f \otimes g) = \omega(gf) \tag{1.5}$$

for $\omega \in \mathcal{D}(\mathbb{G})$, $f, g \in \mathcal{O}(\mathbb{G})$. Note that if $\mathcal{D}(\mathbb{G}) \cong \mathrm{alg} - \bigoplus_{i \in I} M_{n_i}(\mathbb{C})$, then $M(\mathcal{D}(\mathbb{G}) \odot \mathcal{D}(\mathbb{G})) \cong \prod_{i,j \in I} M_{n_i}(\mathbb{C}) \odot M_{n_j}(\mathbb{C})$, c.f. Example 1.13. It is not immediately clear that $\Delta(\omega)$ as defined above is an element of $M(\mathcal{D}(\mathbb{G}) \odot \mathcal{D}(\mathbb{G}))$. We refer the reader to [83, p.g. 347-348] for the technical details.

The above shows that $\mathcal{D}(\mathbb{G})$ is a discrete quantum group in the sense of Definition 1.16. We can then complete $\mathcal{D}(\mathbb{G})$ to obtain a C^*-algebraic discrete quantum group, which we view as the dual of the compact quantum group $C(\mathbb{G})$.

We can also equip $\mathcal{D}(\mathbb{G})$ with the analogue of a Haar integral. Let us first state the general definition we need.

Definition 1.34. Let H be a multiplier Hopf algebra.

(a) Let $\phi : H \to \mathbb{C}$ be a linear functional. If $a \in A$, we define $(\mathrm{id} \otimes \phi)(\Delta(a)) \in M(A)$ by

$$(\mathrm{id} \otimes \phi)(\Delta(a))b := (\mathrm{id} \otimes \phi)(\Delta(a)(b \otimes 1)), \quad b(\mathrm{id} \otimes \phi)(\Delta(a)) = (\mathrm{id} \otimes \phi)((b \otimes 1)\Delta(a))$$

for all $b \in B$ and $(\phi \otimes \mathrm{id})(\Delta(a)) \in M(A)$ by

$$(\phi \otimes \mathrm{id})(\Delta(a))b := (\phi \otimes \mathrm{id})(\Delta(a)(1 \otimes b)), \quad b(\phi \otimes \mathrm{id})(\Delta(a)) := (\phi \otimes \mathrm{id})((1 \otimes b)\Delta(a))$$

for all $b \in B$.

(b) A linear functional $\phi : H \to \mathbb{C}$ is **left invariant** if

$$(\mathrm{id} \otimes \phi)(\Delta(a)) = \phi(a)1_{M(A)}$$

for all $a \in A$ and **right invariant** if

$$(\phi \otimes \mathrm{id})(\Delta(a)) = \phi(a)1_{M(A)}$$

for all $a \in A$.

(c) An **integral** on H is a non-zero left or right invariant functional.

Remark 1.35. If H is a multiplier Hopf-$*$-algebra, then an integral is automatically faithful, and H has a left integral if and only if it has a right integral, see [83, Theorem 3.7].

The following theorem mirrors Theorem 1.29 in the discrete case, see [83, Proposition 4.8]. Note however that one has to adjust the formulae in [83, Proposition 4.8] slightly due to the difference between the formula (1.5) and [83, Definition 4.4].

Theorem 1.36. *Let $A = C(\mathbb{G})$ be a compact quantum group, with counit ϵ on $\mathcal{O}(\mathbb{G})$. A left invariant integral $\phi_{\widehat{\mathbb{G}}}$ for $\mathcal{D}(\mathbb{G})$ is given by*

$$\phi_{\widehat{\mathbb{G}}}(\phi_{\mathbb{G}}(-f)) = \epsilon(f)$$

for $f \in \mathcal{O}(\mathbb{G})$. This integral is positive, i.e. if $\omega \in \mathcal{D}(\mathbb{G})$, then

$$\phi_{\widehat{\mathbb{G}}}(\omega^*\omega) \geq 0.$$

We define, in analogy with (1.2),

$$L^2(\widehat{\mathbb{G}}) := \mathrm{GNS}(\phi_{\widehat{\mathbb{G}}})$$

the Hilbert space obtained by applying the GNS construction to $\phi_{\widehat{\mathbb{G}}}$. Note that since $\phi_{\widehat{\mathbb{G}}}$ is positive by Theorem 1.36, and faithful by Remark 1.35, $L^2(\widehat{\mathbb{G}})$ is the completion of $\mathcal{D}(\mathbb{G})$

with respect to the inner product

$$\langle \omega, \eta \rangle := \phi_{\widehat{\mathbb{G}}}(\omega^* \eta)$$

for $\omega, \eta \in \mathcal{D}(\mathbb{G})$. We note that each $\omega \in \mathcal{D}(\mathbb{G})$ defines a linear operator $L^2(\widehat{\mathbb{G}}) \to L^2(\widehat{\mathbb{G}})$ by left multiplication. Since $\mathcal{D}(\mathbb{G})$ is an algebraic direct sum of matrix algebras, each $\omega \in \mathcal{D}(\mathbb{G})$ is represented by a finite matrix with respect to the basis $\{\omega_{ij}^{\lambda}\}$ of $L^2(\widehat{\mathbb{G}})$ defined above. Hence ω defines a bounded linear operator on $L^2(\widehat{\mathbb{G}})$.

Let us close this section by linking a compact quantum group and its dual via a quantum analogue of the Fourier transform, originally defined in [83, p.g. 346].

Let $\mathbb{G}$ be a compact quantum group. Define the **Fourier transform**

$$\mathcal{F} : \mathcal{O}(\mathbb{G}) \to \mathcal{D}(\mathbb{G}), \quad \mathcal{F}(f)(g) = \phi_{\mathbb{G}}(gf), \quad f, g \in \mathcal{O}(\mathbb{G}). \tag{1.6}$$

The Fourier transform is a surjective linear map by definition. Suppose $\mathcal{F}(f) = 0$ for some $f \in \mathcal{O}(\mathbb{G})$. Then $\mathcal{F}(f)(f^*) = \phi_{\mathbb{G}}(f^* f) = 0$. By Theorem 1.29 it follows that $f = 0$, and so the Fourier transform is a linear isomorphism.

We should explain why this should be viewed as an analogue of the classical Fourier transform.

Example 1.37. Let $\mathbb{G} = T$, the circle group. Then $\mathcal{O}(\mathbb{G})$ is the space of polynomials on T as seen in Example 1.27, and since all the irreducible representations of T are one-dimensional $\mathcal{D}(\mathbb{G}) = C_c(\mathbb{Z})$.

If $m \in \mathbb{Z}$ and $f \in \mathcal{O}(\mathbb{G})$,

$$\mathcal{F}(f)(z^m) = \phi_T(z^m f) = \int_T f(z) z^m \, \mathrm{d}z.$$

In particular, as a function on $\mathbb{Z}$,

$$\mathcal{F}(f)(m) = \int_T f(z) z^m \, \mathrm{d}z$$

which is the Fourier transform for the circle, because the characters of T are of the form $z \mapsto z^m$. Note however that often one conjugates the character by convention in the definition of the classical Fourier transform.

In the classical setting where G is an abelian group, the Fourier transform gives rise to a unitary isomorphism

$$\mathcal{F} : L^2(G) \to L^2(\widehat{G}).$$

Let us prove the quantum analogue of this result (see [83, Proposition 4.9]).

Lemma 1.38. *Let $\mathbb{G}$ be a compact quantum group, and $\omega, \eta \in \mathcal{D}(\mathbb{G})$ with $\eta = \mathcal{F}(g)$ for some $g \in \mathcal{O}(\mathbb{G})$. Then $\phi_{\widehat{\mathbb{G}}}(\omega\eta) = \omega(S^{-1}(g))$.*

Proof. We have, for $f \in \mathcal{O}(\mathbb{G})$,

$$\begin{aligned}
(\omega\eta)(f) &= (\omega \otimes \eta)\Delta(f) \\
&= \omega(f_{(1)})\eta(f_{(2)}) \\
&= \omega(f_{(1)})\mathcal{F}(g)(f_{(2)}) \\
&= \omega(f_{(1)})\phi_{\mathbb{G}}(f_{(2)}g) \\
&= \omega(f_{(1)})\phi_{\mathbb{G}}(f_{(2)}\epsilon(g_{(1)})g_{(2)}) \\
&= \omega(f_{(1)}\epsilon(g_{(1)}) \cdot 1)\phi_{\mathbb{G}}(f_{(2)}g_{(2)}).
\end{aligned}$$

Let $\Delta^{\mathrm{cop}} = \sigma \circ \Delta$, where σ is the flip map. It follows from Remark 1.3 that

$$m(\mathrm{id} \otimes S^{-1})\Delta^{\mathrm{cop}}(-) = \epsilon(-) \cdot 1$$

and so

$$\begin{aligned}
\omega(f_{(1)}\epsilon(g_{(1)}) \cdot 1)\phi_{\mathbb{G}}(f_{(2)}g_{(2)}) &= \omega(f_{(1)}g_{(2)}S^{-1}(g_{(1)}))\phi_{\mathbb{G}}(f_{(2)}g_{(3)}) \\
&= \omega(f_{(1)}g_{(2)}\phi_{\mathbb{G}}(f_{(2)}g_{(3)})S^{-1}(g_{(1)})) \\
&= \omega(\phi_{\mathbb{G}}(fg_{(2)})S^{-1}(g_{(1)})) \\
&= \omega(S^{-1}(g_{(1)}))\phi_{\mathbb{G}}(fg_{(2)}) \\
&= \omega(S^{-1}(g_{(1)}))\mathcal{F}(g_{(2)})(f).
\end{aligned}$$

Now by Theorem 1.36

$$\phi_{\widehat{\mathbb{G}}}(\omega\eta) = \omega(S^{-1}(g_{(1)}))\epsilon(g_{(2)}) = \omega(S^{-1}(g)).$$

as required. □

Proposition 1.39. *Let $\mathbb{G}$ be a compact quantum group. The Fourier transform extends to a unitary isomorphism*

$$\mathcal{F} : L^2(\mathbb{G}) \to L^2(\widehat{\mathbb{G}}).$$

Proof. We want to show that

$$\langle \mathcal{F}(f), \mathcal{F}(g) \rangle_{L^2(\widehat{\mathbb{G}})} = \langle f, g \rangle_{L^2(\mathbb{G})}$$

for $f, g \in \mathcal{O}(\mathbb{G})$. We have, by Lemma 1.38,

$$\begin{aligned}\langle \mathcal{F}(f), \mathcal{F}(g)\rangle_{L^2(\widehat{\mathbb{G}})} = \phi_{\widehat{\mathbb{G}}}(\mathcal{F}(f)^*\mathcal{F}(g)) &= \mathcal{F}(f)^*(S^{-1}(g)) \\ &= \overline{\mathcal{F}(f)(g^*)} \\ &= \overline{\phi_{\mathbb{G}}(g^* f)} \\ &= \phi_{\mathbb{G}}(f^* g) \\ &= \langle f, g\rangle_{L^2(\mathbb{G})}\end{aligned}$$

as required. □

We finish with a description of the action of $\mathcal{D}(\mathbb{G})$ on $L^2(\mathbb{G})$ provided by the Fourier transform.

Proposition 1.40. *Let $\mathbb{G}$ be a compact quantum group and let $\omega \in \mathcal{D}(\mathbb{G})$, viewed as a bounded operator on $L^2(\widehat{\mathbb{G}})$. Then*

$$\omega \cdot := \mathcal{F}^{-1}\omega\mathcal{F} \in B(L^2(\mathbb{G}))$$

is given by the formula

$$\omega \cdot g = \omega(S^{-1}(g_{(1)}))g_{(2)}$$

for $g \in \mathcal{O}(\mathbb{G})$.

Proof. By the proof of Lemma 1.38, we have, for $f \in \mathcal{O}(G)$,

$$(\omega\mathcal{F}(g))(f) = \omega(S^{-1}(g_{(1)}))\mathcal{F}(g_{(2)})(f)$$

Following by the inverse Fourier transform gives the result. □

We will now apply the results in this section to the quantum group $SU_q(2)$ introduced in Example 1.21.

1.3 Quantum $SU(2)$

Throughout this section, $q \in (0, 1]$, unless explicitly stated otherwise.

Let us start by considering Theorem 1.26 in the context of quantum $SU(2)$.

Example 1.41. One can show that for $\mathbb{G} = SU_q(2)$, $\mathcal{O}(\mathbb{G}) = \mathcal{O}(SU_q(2))$ as defined in Example 1.21. Let us demonstrate why this holds true when $q = 1$. The ideas for $q < 1$

are similar and rely on extending results on the representation theory of compact groups to the quantum setting.

The representation given in Example 1.24 is the standard 2 dimensional unitary representation of $SU(2)$, which we will denote by π in this example. We can consider the tensor product representation $\pi^{\otimes n}$ for $n \geq 0$, and the matrix coefficients of these representations. We can see that $\mathcal{O}(SU(2))$ (in the sense of Example 1.21) can then be described as the linear span of matrix coefficients contained in $\pi^{\otimes n}$ for some n.

Let $\mathcal{E}_\rho$ denote the span of the matrix coefficients of a unitary representation ρ of $SU(2)$. Then the Peter-Weyl theorem [24, Theorem 5.12] states

$$L^2(SU(2)) = \bigoplus_\rho \mathcal{E}_\rho$$

where ρ runs over the (equivalence classes of) irreducible unitary representations of $SU(2)$. Note that $\mathcal{O}(SU(2))$ is dense in $C(SU(2))$ (with respect to the supremum norm) which is dense in $L^2(SU(2))$ (of course, with the L^2-norm), so $\mathcal{O}(SU(2))$ is dense in $L^2(SU(2))$ in the L^2-norm.

Suppose that there is an irreducible representation ρ of $SU(2)$ which does not appear in $\pi^{\otimes n}$ for $n \geq 0$ (that is, when we decompose each $\pi^{\otimes n}$ into irreducible representations of $SU(2)$ using Theorem 1.25, ρ does not appear). Then we must have $\mathcal{O}(SU(2)) \subseteq \mathcal{E}_\rho^\perp \subset L^2(SU(2))$. This contradicts density. Therefore $\mathcal{O}(\mathbb{G}) = \mathcal{O}(SU(2))$.

We will determine, up to unitary equivalence, all of the irreducible unitary representations of $SU_q(2)$ for $q \in (0,1]$. These were originally described in [91, §5], but we will take a contemporary approach. For this, we introduce the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$ of the Lie algebra of $SL(2,\mathbb{C})$ and an appropriate 'quantization' of this algebra. One can find this approach in various sources, such as [43, p.g. 61-65].

The Lie algebra of $SL(2,\mathbb{C})$, which we denote by $\mathfrak{sl}_2(\mathbb{C})$, is given by

$$\mathfrak{sl}_2(\mathbb{C}) = \{X \in M_2(\mathbb{C}) \mid \operatorname{Trace}(X) = 0\}$$

and therefore has basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It follows (see [44, Chapter III] for the general definition of a universal enveloping algebra of a Lie algebra) that $U(\mathfrak{sl}_2(\mathbb{C}))$ (with a slight abuse of notation) is the unital associative algebra generated by elements E, F and H such that

$$EF - FE = H, \quad HF - FH = -2F, \quad HE - EH = 2E. \tag{1.7}$$

We can turn $U(\mathfrak{sl}_2(\mathbb{C}))$ into a Hopf-$*$-algebra with the formulae

$$E^* = F, \quad F^* = E, \quad H^* = H,$$

$$\Delta(H) = 1 \otimes H + H \otimes 1, \quad \Delta(E) = 1 \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes 1$$

and

$$S(E) = -E, \quad S(F) = -F, \quad S(H) = -H, \quad \epsilon(E) = \epsilon(F) = \epsilon(H) = 0. \tag{1.8}$$

This algebra can be 'quantized' as seen in the following definition. These were originally defined in generality by Drinfeld [19] and Jimbo [37].

Definition 1.42. Let $q \in (0, 1)$. The **quantized universal enveloping algebra** $U_q(\mathfrak{sl}_2(\mathbb{C}))$ is the unital Hopf $*$-algebra generated by K, K^{-1}, E and F, such that

$$KK^{-1} = 1 = K^{-1}K, \quad KEK^{-1} = qE, \quad KFK^{-1} = q^{-1}F, \quad EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}}$$

with the $*$-structure

$$E^* = K^2F, \quad F^* = EK^{-2}, \quad K^* = K,$$

and Hopf-structure

$$\Delta(K) = K \otimes K, \quad \Delta(E) = 1 \otimes E + E \otimes K^2, \quad \Delta(F) = K^{-2} \otimes F + F \otimes 1,$$

$$S(E) = -EK^{-2}, \quad S(F) = -K^2F, \quad S(K) = K^{-1}, \quad \epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = 1.$$

Note that this definition does not make sense at $q = 1$, and so the connection to the classical enveloping algebra is not immediately clear. An alternative presentation is given in [40, p.g. 125] where one can make sense of the case where $q = 1$, and realize $U(\mathfrak{sl}_2(\mathbb{C}))$ as a quotient of the resulting algebra. Note that there are several conventions on the definition of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ in the literature, including in [40], and so one would need to make several adjustments for this to be compatible with our given presentation. Instead, let us set $U_1(\mathfrak{sl}_2(\mathbb{C})) := U(\mathfrak{sl}_2(\mathbb{C}))$ so that we can work simultaneously with the classical and quantized enveloping algebras in the sequel.

We will work with q-numbers throughout this section. For $q \in (0,1)$ and $n \in \mathbb{Z}$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

We will write $[n]$ when the value of q is clear. Note that as $q \to 1$, $[n]_q \to n$, so $[n]_q$ should be viewed as a 'quantization' of n.

Example 1.43. Define the $*$-representation $\pi_{\frac{1}{2}} : U_q(\mathfrak{sl}_2(\mathbb{C})) \to M_2(\mathbb{C})$ by

$$\begin{cases} \pi_{\frac{1}{2}}(E) := \begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \quad \pi_{\frac{1}{2}}(F) := \begin{pmatrix} 0 & 0 \\ q^{\frac{1}{2}} & 0 \end{pmatrix} \quad \pi_{\frac{1}{2}}(K) := \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix} & q \neq 1 \\ \pi_{\frac{1}{2}}(E) := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \pi_{\frac{1}{2}}(F) := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \pi_{\frac{1}{2}}(H) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & q = 1 \end{cases}$$

with respect to the standard orthonormal basis of $\mathbb{C}^2$. We will label the standard orthonormal basis of $\mathbb{C}^2$ by $e^{\frac{1}{2}}_{\frac{1}{2}}$ and $e^{\frac{1}{2}}_{-\frac{1}{2}}$, i.e. $e^{\frac{1}{2}}_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e^{\frac{1}{2}}_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Since $\pi_{\frac{1}{2}}(U_q(\mathfrak{sl}_2(\mathbb{C})))' = \mathbb{C} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the only projections commuting with $\pi_{\frac{1}{2}}$ are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ whence $\pi_{\frac{1}{2}}$ is irreducible.

We will determine, up to equivalence, the irreducible $*$-representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$. Later we will establish the link between these representations and the irreducible unitary representations of $SU_q(2)$.

Lemma 1.44. *Suppose $\pi : U_q(\mathfrak{sl}_2(\mathbb{C})) \to \mathrm{End}(V)$ is a finite dimensional representation on a vector space V, and $v \in V$ is an eigenvector of the operator*

$$\begin{cases} \pi(K) & q \neq 1 \\ \pi(H) & q = 1 \end{cases}$$

with eigenvalue λ. Then

$$\begin{cases} \pi(K)\pi(E)v = q\lambda\pi(E)v, \quad \pi(K)\pi(F)v = q^{-1}\lambda\pi(F)v & q \neq 1 \\ \pi(H)\pi(E)v = (\lambda+2)\pi(E)v, \quad \pi(H)\pi(F)v = (\lambda-2)\pi(F)v & q = 1 \end{cases}$$

and there exists an $n \in \mathbb{N}$ such that $\pi(E)^n v = 0$. In particular, there exists a vector $w \in V$

such that w is an eigenvector for

$$\begin{cases} \pi(K) & q \neq 1 \\ \pi(H) & q = 1 \end{cases}$$

and $\pi(E)w = 0$.

Proof. The first part follows from straightforward calculations using the relations in Definition 1.42 and (1.7). The second part follows from the fact that V is finite dimensional. $\square$

Definition 1.45. Suppose $\pi : U_q(\mathfrak{sl}_2(\mathbb{C})) \to \mathrm{End}(V)$ is a finite dimensional representation on a vector space V. An eigenvector $w \in V$ for

$$\begin{cases} \pi(K) & q \neq 1 \\ \pi(H) & q = 1 \end{cases}$$

such that $\pi(E)w = 0$ is called a **highest weight vector**, and the corresponding eigenvalue is called the **highest weight**.

Lemma 1.46. *Suppose $\pi : U_q(\mathfrak{sl}_2(\mathbb{C})) \to \mathrm{End}(V)$ is a finite dimensional irreducible representation on a vector space V, and $w \in V$ is a highest weight vector, with highest weight λ. Then*

$$\{f_n := \pi(F)^n w \mid 0 \leq n \leq d\}$$

is a basis for V, where $d = \dim(V) - 1$. Moreover, $\lambda = \begin{cases} \omega q^{\frac{d}{2}} & q \neq 1 \\ d & q = 1 \end{cases}$ where $\omega \in \{1, -1, i, -i\}$.

Proof. Using the relations in $U_q(\mathfrak{sl}_2(\mathbb{C}))$ one can show that $V_0 = \mathrm{span}\{\pi(F)^n w \mid n \geq 0\}$ is invariant under $\pi(U_q(\mathfrak{sl}_2(\mathbb{C})))$, and so $V = V_0$ by irreducibility. Note that $\{\pi(F)^n w \mid n \geq 0\}$ is a set of linearly independent vectors, being eigenvectors of $\pi(K)$ (resp. $\pi(H)$) with distinct eigenvalues by Lemma 1.44, and so forms a basis for V. Since V is finite dimensional, then we must have $V = \mathrm{span}\{\pi(F)^n w \mid 0 \leq n \leq d\}$, where $d = \dim(V) - 1$.

To obtain the formulae for λ, note that $\mathrm{Trace}(\pi(EF - FE)) = 0$. We therefore have

$$0 = \begin{cases} \mathrm{Trace}(\pi(K^2 - K^{-2})) & q \neq 1 \\ \mathrm{Trace}(\pi(H)) & q = 1. \end{cases}$$

Both are equations involving λ, which can be solved to give

$$\lambda = \begin{cases} \omega q^{\frac{d}{2}} & \omega \in \{1, -1, i, -i\},\ q \neq 1 \\ d & q = 1. \end{cases} \qquad \square$$

Let $m \in \frac{1}{2}\mathbb{N}_0$ and suppose V is the carrier space of a finite dimensional irreducible representation of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ of dimension $2m+1$. Set

$$V_j := \begin{cases} \omega q^j \text{ eigenspace of } \pi(K) & q \neq 1 \\ 2j \text{ eigenspace of } \pi(H) & q = 1 \end{cases}$$

for $j \in \{-m, -m+1, ..., m-1, m\}$. By Lemmas 1.44 and 1.46 each eigenspace is one-dimensional, and we have the diagram

$$\begin{tikzcd} V & = & V_m \arrow[rr, bend left, "F"] & \oplus & V_{m-1} \arrow[ll, bend left, "E"] \arrow[rr, bend left, "F"] & \oplus & \cdots \arrow[ll, bend left, "E"] \arrow[rr, bend left, "F"] & \oplus & V_{-m}. \arrow[ll, bend left, "E"] \end{tikzcd}$$

It is convenient in this situation to reindex the basis given in Lemma 1.46. For $0 \leq n \leq 2m$, set $v_{m-n} = f_n$. Then the basis $\{v_j\}$ is indexed by $j \in \{-m, -m+1, ..., m-1, m\}$ and $v_j \in V_j$. Such a basis of V obtained by applying Lemma 1.46 to a highest weight vector $w \in V_m$ is called a **weight basis** for V corresponding to w.

One can use the relations in $U_q(\mathfrak{sl}_2(\mathbb{C}))$ to determine the action of an irreducible representation with respect to a weight basis, c.f. [86, Lemma 2.29].

Theorem 1.47. *Suppose $\pi : U_q(\mathfrak{sl}_2(\mathbb{C})) \to \mathrm{End}(V)$ is a finite dimensional irreducible representation on a vector space V with $\dim(V) = 2m+1$ for some $m \in \frac{1}{2}\mathbb{N}_0$. Let $w \in V_m$ be a highest weight vector, with corresponding weight basis $\{v_j\}_{j=-m}^{m}$ of V. Then*

$$\begin{cases} \pi(E)v_j = \omega^2[m-j][1+m+j]v_{j+1}, \quad \pi(F)v_j = v_{j-1}, \quad \pi(K)v_j = \omega q^j v_j & q \neq 1 \\ \pi(E)v_j = (m-j)(1+m+j)v_{j+1}, \quad \pi(F)v_j = v_{j-1}, \quad \pi(H)v_j = 2jv_j & q = 1 \end{cases}$$

where $\omega \in \{1, -1, i, -i\}$. One sets $v_{-m-1} = v_{m+1} = 0$ so the formulae are consistent.

Comparing the formulae we see that the actions of E and F in the quantum case tend to those of the classical case as $q \to 1$, provided we take $\omega = 1$. From now let us take $\omega = 1$.

Suppose $m \in \frac{1}{2}\mathbb{N}_0$, and let V be a $2m+1$-dimensional vector space. If $\{v_j\}_{j=-m}^{m}$ is a basis, then using the above formulae one can define a representation $U_q(\mathfrak{sl}_2(\mathbb{C})) \to \mathrm{End}(V)$. Therefore for each $m \in \frac{1}{2}\mathbb{N}_0$ one can construct an irreducible representation of dimension

$2m+1$ and then by Lemma 1.46 and Theorem 1.47 these are unique up to equivalence. Therefore, for each $m \in \frac{1}{2}\mathbb{N}_0$, there is a unique irreducible representation of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ of dimension $2m+1$.

We note that if we take $q \neq 1$, $m = \frac{1}{2}$, $V = \mathbb{C}^2$ and $v_j = e_j^{\frac{1}{2}}$ as defined in Example 1.43, then this construction defines a representation that is equivalent to $\pi_{\frac{1}{2}}$ after rescaling the basis, but it is not a $*$-representation with respect to the canonical Hilbert space structure on $\mathbb{C}^2$. Ideally, for each $m \in \frac{1}{2}\mathbb{N}_0$ we would like to fix a Hilbert space $V(m)$ of dimension $2m+1$ onto which both classical and quantum algebras are represented irreducibly in such a way that both are $*$-representations.

We have already constructed $V(\frac{1}{2})$ in Example 1.43. To construct the remaining $V(m)$ we make use of tensor products of representations.

Definition 1.48. Let π_1, π_2 be $*$-representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Define the tensor product representation on $\mathcal{H}_1 \otimes \mathcal{H}_2$ by the formula

$$\pi_1 \otimes \pi_2 : U_q(\mathfrak{sl}_2(\mathbb{C})) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2), \quad (\pi_1 \otimes \pi_2)(X) := (\pi_1 \otimes \pi_2)\Delta(X)$$

for $X \in U_q(\mathfrak{sl}_2(\mathbb{C}))$. This is a $*$-representation.

We construct $V(m)$ inductively. Note that the assumptions of the following theorem are satisfied by $V(\frac{1}{2})$, and so this rather complicated looking theorem is just the inductive step in this process.

Theorem 1.49. *Let $m \in \frac{1}{2}\mathbb{N}_0$ with $m \geq \frac{1}{2}$. Suppose we have constructed a Hilbert space $V(m)$ of dimension $2m+1$ onto which $U_q(\mathfrak{sl}_2(\mathbb{C}))$ acts irreducibly as a $*$-representation for all $q \in (0,1]$. Assume we have constructed an orthonormal weight basis $\{e_j^m\}_{j=-m}^m$ for $V(m)$, where e_m^m is a highest weight vector, and that the actions of E, F and K (resp. H) are given by the formulae*

$$\begin{aligned}
E \cdot e_j^m &= \begin{cases} q^j[m-j]^{\frac{1}{2}}[m+j+1]^{\frac{1}{2}} e_{j+1}^m & q \neq 1 \\ (m-j)^{\frac{1}{2}}(m+j+1)^{\frac{1}{2}} e_{j+1}^m & q = 1 \end{cases}, \\
F \cdot e_j^m &= \begin{cases} q^{-(j-1)}[m-j+1]^{\frac{1}{2}}[m+j]^{\frac{1}{2}} e_{j-1}^m & q \neq 1 \\ (m-j+1)^{\frac{1}{2}}(m+j)^{\frac{1}{2}} e_{j-1}^m & q = 1 \end{cases}, \\
K \cdot e_j^m &= q^j e_j^m, \\
H \cdot e_j^m &= 2j e_j^m.
\end{aligned}$$

(a) There exist irreducible subspaces $V\left(m-\frac{1}{2}, q\right)$ and $V\left(m+\frac{1}{2}, q\right)$ in $V(m) \otimes V\left(\frac{1}{2}\right)$ of

dimension $2(m-\frac{1}{2})+1$ *and* $2(m+\frac{1}{2})+1$ *respectively such that*

$$V(m)\otimes V\left(\frac{1}{2}\right)=V\left(m-\frac{1}{2},q\right)\oplus V\left(m+\frac{1}{2},q\right).$$

(b) The vectors

$$\begin{aligned} e^{m+\frac{1}{2}}_{m+\frac{1}{2}} &:= e^m_m\otimes e^{\frac{1}{2}}_{\frac{1}{2}},\\ e^{m-\frac{1}{2}}_{m-\frac{1}{2}} &:= \frac{1}{\sqrt{1+q^{2m+1}[2m]}}e^m_{m-1}\otimes e^{\frac{1}{2}}_{\frac{1}{2}}-\frac{1}{\sqrt{1+q^{-(2m+1)}[2m]^{-1}}}e^m_m\otimes e^{\frac{1}{2}}_{-\frac{1}{2}} \end{aligned}$$

for $q\neq 1$ *and*

$$\begin{aligned} e^{m+\frac{1}{2}}_{m+\frac{1}{2}} &:= e^m_m\otimes e^{\frac{1}{2}}_{\frac{1}{2}},\\ e^{m-\frac{1}{2}}_{m-\frac{1}{2}} &:= \frac{1}{\sqrt{1+2m}}e^m_{m-1}\otimes e^{\frac{1}{2}}_{\frac{1}{2}}-\frac{1}{\sqrt{1+(2m)^{-1}}}e^m_m\otimes e^{\frac{1}{2}}_{-\frac{1}{2}} \end{aligned}$$

for $q=1$ *are unit highest weight vectors in* $V\left(m-\frac{1}{2},q\right)$ *and* $V\left(m+\frac{1}{2},q\right)$ *respectively.*

(c) There are orthonormal weight bases $\{e^k_j\}^k_{j=-k}$ *of* $V(k,q)$, $k=m\pm\frac{1}{2}$ *and the actions of* E, F *and* K *(resp.* H*) on these bases are given by*

$$\begin{aligned} E\cdot e^k_j &= \begin{cases} q^j[k-j]^{\frac{1}{2}}[k+j+1]^{\frac{1}{2}}e^k_{j+1} & q\neq 1\\ (k-j)^{\frac{1}{2}}(k+j+1)^{\frac{1}{2}}e^k_{j+1} & q=1 \end{cases},\\ F\cdot e^k_j &= \begin{cases} q^{-(j-1)}[k-j+1]^{\frac{1}{2}}[k+j]^{\frac{1}{2}}e^k_{j-1} & q\neq 1\\ (k-j+1)^{\frac{1}{2}}(k+j)^{\frac{1}{2}}e^k_{j-1} & q=1 \end{cases},\\ K\cdot e^k_j &= q^je^k_j,\\ H\cdot e^k_j &= 2je^k_j. \end{aligned}$$

Proof. The tensor product representation $V(m)\otimes V\left(\frac{1}{2}\right)$ is a finite dimensional $*$-representation, and so decomposes as a direct sum of irreducible representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$. The set

$$\left\{e^m_j\otimes e^{\frac{1}{2}}_n \mid j=-m,-m+1,...,m,\ n=\frac{1}{2},-\frac{1}{2}\right\}$$

is an orthonormal basis of $V(m)\otimes V\left(\frac{1}{2}\right)$ consisting of eigenvectors for the action of K (resp. H), with eigenvalues q^{j+n} (resp. $2(j+n)$).

It is easy to check $e^{m+\frac{1}{2}}_{m+\frac{1}{2}} := e^m_m\otimes e^{\frac{1}{2}}_{\frac{1}{2}}$ is a highest weight vector of weight $q^{m+\frac{1}{2}}$ (resp. $2(m+\frac{1}{2})$). By repeatedly applying F, and normalising the resulting vectors, we obtain a sequence $\{e_j^{m+\frac{1}{2}}\}_{j=-(m+\frac{1}{2})}^{m+\frac{1}{2}}$ of orthonormal vectors, which span a $2(m+\frac{1}{2})+1$ dimensional subspace, denoted $V\left(m+\frac{1}{2},q\right)$ on which $U_q(\mathfrak{sl}_2(\mathbb{C}))$ acts irreducibly as a $*$-representation, c.f. the proof of Lemma 1.46.

Let us now show there is a highest weight vector of weight $q^{m-\frac{1}{2}}$ (resp. $2(m-\frac{1}{2})$) in $V(m)\otimes V\left(\frac{1}{2}\right)$. Let us consider $v=\sum_{j,n} c_{j,n} e_j^m \otimes e_n^{\frac{1}{2}}$, the general form of a vector in $V(m)\otimes V\left(\frac{1}{2}\right)$. If we apply K (resp. H), then we can see that if v is an eigenvector of K (resp. H) of the desired weight if and only if v is of the form

$$v = c_{m,-\frac{1}{2}} e_m^m \otimes e_{-\frac{1}{2}}^{\frac{1}{2}} + c_{m-1,\frac{1}{2}} e_{m-1}^m \otimes e_{\frac{1}{2}}^{\frac{1}{2}}$$

If we now apply E, we on the one hand should obtain zero, but on the other hand we can use the formulae for the actions of E in $V(m)$ and $V(\frac{1}{2})$. We obtain

$$\begin{cases} c_{m-1,\frac{1}{2}} q^m [2m]^{\frac{1}{2}} + c_{m,-\frac{1}{2}} q^{-\frac{1}{2}} = 0 & q \neq 1 \\ c_{m-1,\frac{1}{2}} (2m)^{\frac{1}{2}} + c_{m,-\frac{1}{2}} = 0 & q = 1. \end{cases}$$

In particular, we see that v must be in the span of the vector

$$\begin{cases} e_{m-1}^m \otimes e_{\frac{1}{2}}^{\frac{1}{2}} - q^{m+\frac{1}{2}} [2m]^{\frac{1}{2}} e_m^m \otimes e_{-\frac{1}{2}}^{\frac{1}{2}} & q \neq 1 \\ e_{m-1}^m \otimes e_{\frac{1}{2}}^{\frac{1}{2}} - (2m)^{\frac{1}{2}} e_m^m \otimes e_{-\frac{1}{2}}^{\frac{1}{2}} & q = 1. \end{cases}$$

Normalizing produces a unit highest weight vector $e_{m-\frac{1}{2}}^{m-\frac{1}{2}}$ of weight $q^{m-\frac{1}{2}}$ (resp. $2(m-\frac{1}{2})$) given by the formulae in part (b). By repeatedly applying F, and normalizing the resulting vectors, we obtain a sequence $\{e_j^{m-\frac{1}{2}}\}_{j=-(m-\frac{1}{2})}^{m-\frac{1}{2}}$ of orthonormal vectors, which span a $2(m-\frac{1}{2})+1$ dimensional subspace, denoted $V\left(m-\frac{1}{2},q\right)$ on which $U_q(\mathfrak{sl}_2(\mathbb{C}))$ acts irreducibly as a $*$-representation, c.f. the proof of Lemma 1.46. Note that

$$V(m)\otimes V\left(\frac{1}{2}\right) = V\left(m-\frac{1}{2},q\right) \oplus V\left(m+\frac{1}{2},q\right)$$

by comparing dimensions. It remains to check the action of E, F and K (resp. H) on the orthonormal bases we have constructed for $V\left(m-\frac{1}{2},q\right)$ and $V\left(m+\frac{1}{2},q\right)$.

Let $k=m\pm\frac{1}{2}$, and let $v_j^k := F^{k-j}\cdot e_k^k$. Then by the formulae in Theorem 1.47 we have

$$\begin{cases} E\cdot e_j^k = \frac{[k-j][1+k+j]\left\|v_{j+1}^k\right\|}{\left\|v_j^k\right\|} e_{j+1}^k, \quad F\cdot e_j^k = \frac{\left\|v_{j-1}^k\right\|}{\left\|v_j^k\right\|} e_{j-1}^k, \quad K\cdot e_j^k = q^j e_j^k & q\neq 1, \\ \\ E\cdot e_j^k = \frac{(k-j)(1+k+j)\left\|v_{j+1}^k\right\|}{\left\|v_j^k\right\|} e_{j+1}^k, \quad F\cdot e_j^k = \frac{\left\|v_{j-1}^k\right\|}{\left\|v_j^k\right\|} e_{j-1}^k, \quad H\cdot e_j^k = 2j e_j^k & q=1. \end{cases}$$

To compute these norms, note that in the case $q \neq 1$, we have, by Theorem 1.47

$$\begin{aligned}
\left\|v_j^k\right\|^2 = \langle v_j^k, v_j^k\rangle &= \frac{1}{[m-j+1][m+j]}\langle E\cdot v_{j-1}^k, v_j^k\rangle \\
&= \frac{1}{[m-j+1][m+j]}\langle v_{j-1}^k, E^*\cdot v_j^k\rangle \\
&= \frac{1}{[m-j+1][m+j]}\langle v_{j-1}^k, K^2F\cdot v_j^k\rangle \\
&= \frac{q^{2(j-1)}}{[m-j+1][m+j]}\langle v_{j-1}^k, v_{j-1}^k\rangle \\
&= \frac{q^{2(j-1)}}{[m-j+1][m+j]}\left\|v_{j-1}^k\right\|^2
\end{aligned}$$

from which we can obtain the formulae for the actions. We can carry out an entirely similar argument in the case where $q = 1$. □

Theorem 1.49, applied to $V(\frac{1}{2})$, tells us that in particular there are subspaces $V(0,q)$ and $V(1,q)$ of $V(\frac{1}{2})\otimes V(\frac{1}{2})$ of dimension 1 and 3 respectively, on which the action of $\pi_{\frac{1}{2}}\otimes\pi_{\frac{1}{2}}$ gives the 1 and 3 dimensional irreducible representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$. We can consider the Hilbert spaces $V(0)$ and $V(1)$ of dimension 1 and 3 respectively, with orthonormal bases $\{e_0^0\}$ and $\{e_1^1, e_0^1, e_{-1}^1\}$. We can identify $V(0)\cong V(0,q)$ and $V(1)\cong V(1,q)$ by identifying the orthonormal bases we have chosen.

Iteratively taking tensor products allows us to fix a Hilbert space $V(m)$ for any $m\in\frac{1}{2}\mathbb{N}_0$ onto which $U_q(\mathfrak{sl}_2(\mathbb{C}))$ acts as a $*$-representation for all $q\in(0,1]$. We also have a canonical choice of orthonormal basis for each $V(m)$, and formulae for the actions of E, F and K (resp. H). We will from now write $V(m)$ as the (carrier space of the) irreducible $*$-representation of $U_q(\mathfrak{sl}_2(\mathbb{C}))$.

We will use the term **type I** to describe $*$-representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ that decompose as a direct sum of these representations. For $q = 1$, this is always the case, but in the case of $q\neq 1$, we have additional irreducible representations arising from the case $\omega\in\{-1,\pm i\}$ in Theorem 1.47.

Let us now note some additional consequences of Theorem 1.49.

Denote by $C_q^{(m)}$ the change of basis matrix from the tensor basis of $V(m)\otimes V(\frac{1}{2})$ to the orthonormal weight basis of $V(m+\frac{1}{2})$ and $V(m-\frac{1}{2})$. We need a convention for how to label matrix entries of $C_q^{(m)}$. The columns are naturally labelled by the chosen basis of $V(m+\frac{1}{2})$ and $V(m-\frac{1}{2})$ and the rows are labelled by elements of the tensor basis. Let $C_q^{(m)}((k,\ell),(i,j))$ denote the entry corresponding to the row labelled by $e_k^m\otimes e_\ell^{\frac{1}{2}}$ and column labelled by e^i_j. We can write

$$e^i_j = \sum_{k,\ell} C_q^{(m)}((k,\ell),(i,j)) e^m_k \otimes e^{\frac{1}{2}}_\ell$$

for $i \in \{m \pm \frac{1}{2}\}$ and $j \in \{-i, -i+1, ..., i\}$. From the formulae in Theorem 1.49 for the highest weight vectors, and the formulae for the action of E and F on these, we see that for each $m \in \frac{1}{2}\mathbb{N}_0$, the map

$$q \mapsto C_q^{(m)}$$

defines an element of $C((0,1], M_{2(2m+1)}(\mathbb{C}))$.

For $m = \frac{1}{2}$ and $q < 1$, we have

$$e^1_1 = e^{\frac{1}{2}}_{\frac{1}{2}} \otimes e^{\frac{1}{2}}_{\frac{1}{2}}, \quad e^1_0 = \frac{q^{-\frac{1}{2}}}{[2]_q^{\frac{1}{2}}} e^{\frac{1}{2}}_{\frac{1}{2}} \otimes e^{\frac{1}{2}}_{-\frac{1}{2}} + \frac{q^{\frac{1}{2}}}{[2]_q^{\frac{1}{2}}} e^{\frac{1}{2}}_{-\frac{1}{2}} \otimes e^{\frac{1}{2}}_{\frac{1}{2}},$$

$$e^1_{-1} = e^{\frac{1}{2}}_{-\frac{1}{2}} \otimes e^{\frac{1}{2}}_{-\frac{1}{2}}, \quad e^0_0 = \frac{q^{\frac{1}{2}}}{[2]_q^{\frac{1}{2}}} e^{\frac{1}{2}}_{\frac{1}{2}} \otimes e^{\frac{1}{2}}_{-\frac{1}{2}} - \frac{q^{-\frac{1}{2}}}{[2]_q^{\frac{1}{2}}} e^{\frac{1}{2}}_{-\frac{1}{2}} \otimes e^{\frac{1}{2}}_{\frac{1}{2}}$$

and the corresponding formulae for $q = 1$ are given by taking the limits in the above as $q \to 1$.

Consider the change of basis matrix $C_q^{(\frac{1}{2})}$ from $\{e^1_1, e^1_0, e^1_{-1}, e^0_0\}$ to $\{e^{\frac{1}{2}}_{\frac{1}{2}} \otimes e^{\frac{1}{2}}_{\frac{1}{2}}, e^{\frac{1}{2}}_{\frac{1}{2}} \otimes e^{\frac{1}{2}}_{-\frac{1}{2}}, e^{\frac{1}{2}}_{-\frac{1}{2}} \otimes e^{\frac{1}{2}}_{\frac{1}{2}}, e^{\frac{1}{2}}_{-\frac{1}{2}} \otimes e^{\frac{1}{2}}_{-\frac{1}{2}}\}$ of $V\left(\frac{1}{2}\right) \otimes V\left(\frac{1}{2}\right)$. Then the above formulae tell us $C_q^{(\frac{1}{2})}$ and $C_q^{(\frac{1}{2})^{-1}}$ is given by

$$C_q^{(\frac{1}{2})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q^{-\frac{1}{2}}}{[2]_q^{\frac{1}{2}}} & 0 & \frac{q^{\frac{1}{2}}}{[2]_q^{\frac{1}{2}}} \\ 0 & \frac{q^{\frac{1}{2}}}{[2]_q^{\frac{1}{2}}} & 0 & -\frac{q^{-\frac{1}{2}}}{[2]_q^{\frac{1}{2}}} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad C_q^{(\frac{1}{2})^{-1}} = C_q^{(\frac{1}{2})^T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q^{-\frac{1}{2}}}{[2]_q^{\frac{1}{2}}} & \frac{q^{\frac{1}{2}}}{[2]_q^{\frac{1}{2}}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{q^{\frac{1}{2}}}{[2]_q^{\frac{1}{2}}} & -\frac{q^{-\frac{1}{2}}}{[2]_q^{\frac{1}{2}}} & 0 \end{pmatrix} \tag{1.9}$$

with respect to the ordering of the bases given above.

Having determined the irreducible $*$-representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$, we now explain the connection with $SU_q(2)$.

We will consider a certain subalgebra of the dual of $U_q(\mathfrak{sl}_2(\mathbb{C}))$. First, note that the comultiplication of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ induces a multiplication on the dual defined by the formula

$$u, w \in U_q(\mathfrak{sl}_2(\mathbb{C}))^*, \quad X \in U_q(\mathfrak{sl}_2(\mathbb{C})), \quad (uw)(X) = (u \otimes w)\Delta^{\mathrm{cop}}(X). \tag{1.10}$$

Here Δ^{cop} means Δ followed by the flip map, see the proof of Lemma 1.38. The choice of using Δ^{cop} instead of Δ is one of convention. We will see that this choice behaves well with respect to our other conventions shortly.

Remark 1.50. Note that 'cop' stands for coopposite. This name arises from the following context. Given a Hopf-$*$-algebra A, one can define the coopposite Hopf-$*$-algebra A^{cop} as follows. As an algebra, $A^{\text{cop}} = A$, but we equip A^{cop} with the comultiplication Δ^{cop}. If ϵ is the counit of A, then the counit axiom for A tells us that ϵ is a counit for A^{cop}. In Remark 1.3 we saw that S^{-1} is the antipode of A^{op}. We can write the antipide axiom for A^{op} as

$$m_{A^{\text{op}}}(\text{id} \otimes S^{-1})\Delta = m_A(S^{-1} \otimes \text{id})\Delta^{\text{cop}} = \epsilon(-)1$$

and

$$m_{A^{\text{op}}}(S^{-1} \otimes \text{id})\Delta = m_A(\text{id} \otimes S^{-1})\Delta^{\text{cop}} = \epsilon(-)1.$$

Therefore S^{-1} is an antipode for A^{cop}, and so A^{cop} is a Hopf-$*$-algebra.

One can think of the construction of A^{cop} from A as the coalgebra (that is, algebras with a coproduct) analogue of the construction of the algebra A^{op} from A.

By the Hopf axioms in Definition 1.1, ϵ is the multiplicative identity with respect to the multiplication in (1.10). We can also define a $*$-structure on the dual via the formula

$$u^*(X) = \overline{u(S^{-1}(X)^*)}$$

for $u \in U_q(\mathfrak{sl}_2(\mathbb{C}))^*$ and $X \in U_q(\mathfrak{sl}_2(\mathbb{C}))$. We can define a comultiplication only on a subalgebra of $U_q(\mathfrak{sl}_2(\mathbb{C}))^*$. The natural choice is to try to define

$$\Delta(u)(X \otimes Y) = u(XY)$$

for $u \in U_q(\mathfrak{sl}_2(\mathbb{C}))^*$ and $X, Y \in U_q(\mathfrak{sl}_2(\mathbb{C}))$, however it need not be the case that $\Delta(u) \in U_q(\mathfrak{sl}_2(\mathbb{C}))^* \otimes U_q(\mathfrak{sl}_2(\mathbb{C}))^*$. We will shortly restrict attention to a setting where this is the case. In this case the counit and antipode are defined by, for $u \in U_q(\mathfrak{sl}_2(\mathbb{C}))^*$ and $X \in U_q(\mathfrak{sl}_2(\mathbb{C}))$,

$$\epsilon(u) = u(1), \quad S(u)(X) = u(S^{-1}(X)).$$

Define, for $m \in \frac{1}{2}\mathbb{N}_0$ and $i, j \in \{-m, -m+1, ..., m\}$, the linear functionals

$$u_{ij}^m : U_q(\mathfrak{sl}_2(\mathbb{C})) \to \mathbb{C}, \quad X \mapsto \langle e_i^m, X e_j^m \rangle_{V(m)}.$$

These are the matrix coefficients of the representation $V(m)$ with respect to the orthonormal basis we have constructed. Let B_0 be the $*$-subalgebra of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ generated by the set of all such matrix coefficients. Since

$$\Delta(u^m_{ij})(X\otimes Y) = u^m_{ij}(XY) = \sum_k u^m_{ik}(X)u^m_{kj}(Y) = \left(\sum_k u^m_{ik}\otimes u^m_{kj}\right)(X\otimes Y) \tag{1.11}$$

for $X, Y \in U_q(\mathfrak{sl}_2(\mathbb{C}))$, we have $\Delta(u^m_{ij}) \in B_0 \otimes B_0$. Also, one can check u^0_{00} is the counit of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ using the formulae in Theorem 1.49 and the definition of the counit given in Definition 1.42 and (1.8). Therefore u^0_{00} is a unit for B_0 and in particular B_0 is a Hopf-$*$-algebra.

It follows from Theorem 1.49 that the matrix coefficients u^m_{ij} for $m \geq 1$ can be expressed as products and sums of the matrix coefficients for $m = \frac{1}{2}$. In particular we can describe B_0 as the unital $*$-subalgebra of $U_q(\mathfrak{sl}_2(\mathbb{C}))^*$ generated by the matrix coefficients $\{u^{\frac{1}{2}}_{ij}\}^{\frac{1}{2}}_{i,j=-\frac{1}{2}}$.

Lemma 1.51. *We have the following product-to-sum and matrix coefficient-to-product formulae.*

(a) For $m \in \frac{1}{2}\mathbb{N}_0$, $i, j \in \{-m, -m+1, ..., m\}$ and $k, l \in \{-\frac{1}{2}, \frac{1}{2}\}$ we have

$$u^{\frac{1}{2}}_{k\ell}u^m_{ij} = \sum_{a=m-\frac{1}{2},m+\frac{1}{2}} \sum_b \sum_c C^{(m)}_q((i,k),(a,b))\overline{C^{(m)}_q((j,\ell),(a,c))}u^a_{bc}$$

where all sums are over the appropriate indices.

(b) For $m \geq \frac{1}{2}$ and $i, j \in \{-m, -m+1, ..., m\}$ we have

$$u^m_{ij} = \sum_{k,\ell}\sum_{a,b} \overline{C^{(m-\frac{1}{2})}_q((a,b),(m,i))}C^{(m-\frac{1}{2})}_q((k,\ell),(m,j))u^{\frac{1}{2}}_{b\ell}u^{m-\frac{1}{2}}_{ak}$$

where all sums are over the appropriate indices.

Proof.

1. We have the change of basis formula

$$e^m_j \otimes e^{\frac{1}{2}}_\ell = \sum_{a=m-\frac{1}{2},m+\frac{1}{2}} \sum_c C^{(m)^{-1}}_q((a,c),(j,\ell))e^a_c = \sum_{a=m-\frac{1}{2},m+\frac{1}{2}} \sum_c \overline{C^{(m)}_q((j,\ell),(a,c))}e^a_c.$$

Applying $X \in U_q(\mathfrak{sl}_2(\mathbb{C}))$ to both sides, and taking the inner product with $e^m_i \otimes e^{\frac{1}{2}}_k$ gives

$$\left\langle e^m_i \otimes e^{\frac{1}{2}}_k, Xe^m_j \otimes e^{\frac{1}{2}}_\ell\right\rangle = \left\langle e^m_i \otimes e^{\frac{1}{2}}_k, X\left(\sum_{a=m-\frac{1}{2},m+\frac{1}{2}} \sum_c \overline{C^{(m)}_q((j,\ell),(a,c))}e^a_c\right)\right\rangle.$$

Writing

$$e_i^m \otimes e_k^{\frac{1}{2}} = \sum_{d=m-\frac{1}{2},m+\frac{1}{2}} \sum_b \overline{C_q^{(m)}((i,k),(d,b))} e_b^d$$

gives

$$\left\langle e_i^m \otimes e_k^{\frac{1}{2}}, X e_j^m \otimes e_\ell^{\frac{1}{2}} \right\rangle = \sum_{a=m-\frac{1}{2},m+\frac{1}{2}} \sum_b \sum_c C_q^{(m)}((i,k),(a,b)) \overline{C_q^{(m)}((j,\ell),(a,c))} \langle e_b^a, X e_c^a \rangle.$$

Using the definition of the tensor product representation and the definition of multiplication in $U_q(\mathfrak{sl}_2(\mathbb{C}))^*$ we have

$$u_{k\ell}^{\frac{1}{2}} u_{ij}^m(X) = \sum_{a=m-\frac{1}{2},m+\frac{1}{2}} \sum_b \sum_c C_q^{(m)}((i,k),(a,b)) \overline{C_q^{(m)}((j,\ell),(a,c))} u_{bc}^a(X).$$

2. We have the change of basis formula

$$e_j^m = \sum_{k,\ell} C_q^{(m-\frac{1}{2})}((k,\ell),(m,j)) e_k^{m-\frac{1}{2}} \otimes e_\ell^{\frac{1}{2}}.$$

Applying $X \in U_q(\mathfrak{sl}_2(\mathbb{C}))$ to both sides, and taking the inner product with e_i^m gives

$$u_{ij}^m(X) = \left\langle e_i^m, \sum_{k,\ell} C_q^{(m-\frac{1}{2})}((k,\ell),(m,j)) X e_k^{m-\frac{1}{2}} \otimes e_\ell^{\frac{1}{2}} \right\rangle.$$

Writing

$$e_i^m = \sum_{a,b} C_q^{(m-\frac{1}{2})}((a,b),(m,i)) e_a^{m-\frac{1}{2}} \otimes e_b^{\frac{1}{2}}$$

gives

$$u_{ij}^m(X) = \sum_{k,\ell} \sum_{a,b} \overline{C_q^{(m-\frac{1}{2})}((a,b),(m,i))} C_q^{(m-\frac{1}{2})}((k,\ell),(m,j)) \left\langle e_a^{m-\frac{1}{2}} \otimes e_b^{\frac{1}{2}}, X e_k^{m-\frac{1}{2}} \otimes e_\ell^{\frac{1}{2}} \right\rangle.$$

Using the definition of the tensor product representation and the definition of multiplication in $U_q(\mathfrak{sl}_2(\mathbb{C}))^*$ we have

$$u_{ij}^m(X) = \sum_{k,\ell} \sum_{a,b} \overline{C_q^{(m-\frac{1}{2})}((a,b),(m,i))} C_q^{(m-\frac{1}{2})}((k,\ell),(m,j)) u_{b\ell}^{\frac{1}{2}} u_{ak}^{m-\frac{1}{2}}(X). \qquad \square$$

Using Lemma 1.51 together with the matrix $C_q^{(\frac{1}{2})}$ defined in (1.9), one can check that the

matrix
$$\begin{pmatrix} u^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} & u^{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}} \\ u^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}} & u^{\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}} \end{pmatrix}$$
is unitary. In particular we obtain a surjective homomorphism $\mathcal{O}(SU_q(2)) \to B_0$, sending $\alpha \mapsto u^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}$ and $\gamma \mapsto u^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}}$. One can check that this map is a Hopf-$*$-homomorphism (i.e. it preserves the Hopf structure in the natural way). It is also possible to show that this map is an isomorphism (c.f. [43, pg. 121-122]). In fact, there one proves something slightly stronger, which we discuss now. First, we introduce the notion of a skew pairing of Hopf-$*$-algebras.

Definition 1.52. Let A and B be Hopf-$*$-algebras. A **skew pairing** between A and B is a bilinear map
$$\langle -, - \rangle : A \times B \to \mathbb{C}$$
such that
$$\langle aa', b\rangle = \langle a \otimes a', \Delta(b)\rangle, \quad \langle a, bb'\rangle = \langle \Delta^{\mathrm{cop}}(a), b \otimes b'\rangle, \quad \langle a, 1\rangle = \epsilon(a), \quad \langle 1, b\rangle = \epsilon(b)$$
$$\langle S^{-1}(a), b\rangle = \langle a, S(b)\rangle, \quad \langle a^*, b\rangle = \overline{\langle a, S(b)^*\rangle}$$
for all $a, a' \in A$ and $b, b' \in B$.

The skew pairing is said to be **non-degenerate in the first variable** if $\langle a, b\rangle = 0$ for all $b \in B$ implies $a = 0$, and the skew pairing is said to be **non-degenerate in the second variable** if $\langle a, b\rangle = 0$ for all $a \in A$ implies $b = 0$. The skew pairing is said to be **non-degenerate** if it is non-degenerate in both variables.

Remark 1.53. Note that in Definition 1.52 we have used the bilinear map $\langle -, -\rangle$ to define another bilinear map, which we also denote by $\langle -, -\rangle$, given by
$$\langle -, - \rangle : (A \odot A) \times (B \odot B) \to \mathbb{C}, \quad \langle a \otimes a', b \otimes b'\rangle := \langle a, a'\rangle\langle b, b'\rangle$$
for all $a, a' \in A$ and $b, b' \in B$. We have also abused notation in using the same notation for the counits and antipodes of A and B. From context one will be able to tell which is being used.

We can define a skew pairing
$$\langle -, - \rangle : U_q(\mathfrak{sl}_2(\mathbb{C})) \times \mathcal{O}(SU_q(2)) \to \mathbb{C} \tag{1.12}$$
by viewing $\mathcal{O}(SU_q(2)) = B_0 \subseteq U_q(\mathfrak{sl}_2(\mathbb{C}))^*$ and then using the natural pairing given by evaluation of functionals. In [43, pg. 121-122] it is shown this pairing is non-degenerate.

This pairing sets up a correspondence between the representation theories of $SU_q(2)$ and $U_q(\mathfrak{sl}_2(\mathbb{C}))$ as shown in the following proposition.

Proposition 1.54.

(a) *Let v be a unitary representation of $SU_q(2)$ on a finite dimensional Hilbert space $\mathcal{H}$, written in terms of matrix coefficients $v = (v_{ij})$. Then*

$$\pi_v : U_q(\mathfrak{sl}_2(\mathbb{C})) \to B(\mathcal{H}), \quad \pi_v(X)_{ij} := \langle X, v_{ij} \rangle, \quad X \in U_q(\mathfrak{sl}_2(\mathbb{C}))$$

is a type I $*$-representation of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on $\mathcal{H}$.

(b) *Let $\pi : U_q(\mathfrak{sl}_2(\mathbb{C})) \to B(\mathcal{H})$ be a type I representation on a finite dimensional Hilbert space $\mathcal{H}$. Then the matrix coefficients*

$$v_{ij}^{\pi} : U_q(\mathfrak{sl}_2(\mathbb{C})) \to \mathbb{C}, \quad v_{ij}^{\pi}(X) := \pi(X)_{ij}, \quad X \in U_q(\mathfrak{sl}_2(\mathbb{C}))$$

define a unitary representation $v^{\pi} = (v_{ij}^{\pi})$ of $SU_q(2)$ on $\mathcal{H}$.

These constructions are mutually inverse, and preserve equivalence and direct sums. In particular, irreducibility is preserved under these processes.

Proof.

(a) Let us first check that π_v is a $*$-homomorphism. We have, for $X, Y \in U_q(\mathfrak{sl}_2(\mathbb{C}))$,

$$\pi_v(XY) = (\langle XY, v_{ij} \rangle) = \langle X \otimes Y, \Delta(v_{ij}) \rangle = \left(\sum_k \langle X, v_{ik} \rangle \langle Y, v_{kj} \rangle \right) = \pi_v(X)\pi_v(Y)$$

and

$$\pi_v(X^*) = (\langle X^*, v_{ij} \rangle) = (\overline{\langle X, S(v_{ij})^* \rangle}) = (\overline{\langle X, v_{ji} \rangle}) = \pi_v(X)^*.$$

It remains to check that π_v is a type I representation. Note that if $q = 1$, there is nothing to check and so we now assume $q < 1$.

Since π_v is a $*$-representation, it is completely reducible, and so can be written as a direct sum of irreducible representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$. We wish to show that each of these irreducible representations are of the form $V(m)$ for some $m \in \frac{1}{2}\mathbb{N}_0$. This can be determined by considering the eigenvalues of $\pi_v(K)$. Indeed, if the eigenvalues are all positive, we must have that $\omega = 1$ in each irreducible representation by looking at the form of the eigenvalues in Lemma 1.46.

Note that $\mathbb{C}[K, K^{-1}]$ is a sub-Hopf-$*$-algebra of $U_q(\mathfrak{sl}_2(\mathbb{C}))$. The skew pairing between $U_q(\mathfrak{sl}_2(\mathbb{C}))$ and $\mathcal{O}(SU_q(2))$ restricts to a skew pairing between $\mathbb{C}[K, K^{-1}]$ and

$\mathcal{O}(SU_q(2))$. Let us fix $X \in \mathbb{C}[K, K^{-1}]$ and consider the functional

$$\phi_X : \mathcal{O}(SU_q(2)) \to \mathbb{C}, \quad \phi_X(x) = \langle X, x\rangle, \quad x \in \mathcal{O}(SU_q(2)).$$

We define a $*$-homomorphism

$$\pi : \mathcal{O}(SU_q(2)) \to \mathcal{O}(T), \quad \pi(\alpha) = z, \quad \pi(\gamma) = 0 \tag{1.13}$$

where $\mathcal{O}(T)$ as defined in Example 1.27. We then have an induced map

$$\pi^* : \mathcal{O}(T)^* \to \mathcal{O}(SU_q(2))^*, \quad \pi^*(\phi)(x) = \phi(\pi(x)), \quad \phi \in \mathcal{O}(T)^*, \quad x \in \mathcal{O}(SU_q(2)).$$

Define the pairing

$$\langle -, -\rangle_T : \mathbb{C}[K, K^{-1}] \times \mathcal{O}(T) \to \mathbb{C}$$

by setting

$$\langle K, z\rangle_T = q^{\frac{1}{2}}, \quad \langle K^{-1}, z\rangle_T = q^{-\frac{1}{2}}, \quad \langle K, z^{-1}\rangle_T = q^{-\frac{1}{2}}, \quad \langle K^{-1}, z^{-1}\rangle_T = q^{\frac{1}{2}}$$

and then by extending to the remaining elements according to the rules in Definition 1.52.

Since

$$\langle K, \alpha\rangle = q^{\frac{1}{2}}, \quad \langle K^{-1}, \alpha\rangle = q^{-\frac{1}{2}}, \quad \langle K, \alpha^*\rangle = q^{-\frac{1}{2}}, \quad \langle K^{-1}, \alpha^*\rangle = q^{\frac{1}{2}}$$

by Example 1.43 we have, for each $X \in \mathbb{C}[K, K^{-1}]$,

$$\phi_X = \pi^*(\langle X, -\rangle_T).$$

Therefore for each $X \in \mathbb{C}[K, K^{-1}]$ and $x \in \mathcal{O}(SU_q(2))$ we have

$$\langle X, x\rangle = \langle X, \pi(x)\rangle_T. \tag{1.14}$$

Since π is a homomorphism of Hopf-$*$-algebras, $\pi(v)$ is a unitary representation of $C(T)$, which is a unitary representation of T. Then

$$\pi_v(K) = (\langle K, v_{ij}\rangle) = (\langle K, \pi(v_{ij})\rangle_T)$$

and the latter matrix is equivalent to a positive diagonal matrix, as required.

(b) We need to show that

$$\Delta(v^{\pi}_{ij}) = \sum_k v^{\pi}_{ik} \otimes v^{\pi}_{kj} \quad \text{and} \quad v^{\pi} \text{ is a unitary}$$

by Remark 1.23. The first point comes from the calculation of the comultiplication in B_0, see (1.11). For unitarity we have

$$\begin{aligned}
(v^{\pi} v^{\pi *})_{ij} &= \sum_k v^{\pi}_{ik} (v^{\pi *})_{kj} \\
&= \sum_k v^{\pi}_{ik} {v^{\pi}_{jk}}^* \\
&= \sum_k v^{\pi}_{ik} S(v^{\pi}_{kj}) \\
&= m(\mathrm{id} \otimes S)\Delta(v^{\pi}_{ij}) \\
&= \epsilon(v^{\pi}_{ij})1 \\
&= v^{\pi}_{ij}(1)1 \\
&= \delta_{ij}1
\end{aligned}$$

and similarly the other way around.

One can directly check that these constructions are mutually inverse, preserve equivalence (using linearity of the pairing) and preserve direct sums. □

Remark 1.55. Let us understand the pairing (1.12) in the case of $q = 1$, which will give us an alternative interpretation of Proposition 1.54 in this case.

Recall that for a Lie group G with Lie algebra $\mathfrak{g}$ there is an exponential map $\exp : \mathfrak{g} \to G$. For matrix Lie groups this is given by the matrix exponential map.

One can identify $U(\mathfrak{sl}_2(\mathbb{C}))$ with an algebra of differential operators on $C^\infty(SL(2,\mathbb{C}))$, the smooth $\mathbb{C}$-valued functions on $SL(2,\mathbb{C})$, as follows. The unit in $U(\mathfrak{sl}_2(\mathbb{C}))$ corresponds to the identity operator. Let $X = X_1 ... X_n \in U(\mathfrak{sl}_2(\mathbb{C}))$, where $n \in \mathbb{N}$ and each $X_i \in \mathfrak{sl}_2(\mathbb{C})$. Then we define, for $f \in C^\infty(SL(2,\mathbb{C}))$, the function $Xf \in C^\infty(SL(2,\mathbb{C}))$ by the formula

$$(Xf)(g) := \left. \frac{\partial^n}{\partial t_1 ... \partial t_n} \right|_{t_1 = ... = t_n = 0} f(g \exp(t_1 X_1) ... \exp(t_n X_n))$$

for $g \in SL(2,\mathbb{C})$. We can extend to linear combinations of elements of the same form as X above in the obvious way.

Define a bilinear map

$$\langle -, - \rangle : U(\mathfrak{sl}_2(\mathbb{C})) \times \mathcal{O}(SU(2)) \to \mathbb{C}$$

by $\langle X, f\rangle = (Xf)(I)$, where I is the identity matrix in $SL(2,\mathbb{C})$, $X \in U(\mathfrak{sl}_2(\mathbb{C}))$, $f \in \mathcal{O}(SU(2))$ and X acts on f as a differential operator. Note here we have used the fact that the coordinate maps α and γ generating $\mathcal{O}(SU(2))$ extend to maps on $SL(2,\mathbb{C})$ in the obvious way to make this definition.

One can check that $\langle -,-\rangle$ is a skew pairing and then by checking the values of the pairing on the generators E, F, H of $U(\mathfrak{sl}_2(\mathbb{C}))$ and $\alpha, \gamma, \alpha^*, \gamma^*$ of $\mathcal{O}(SU(2))$ that $\langle -,-\rangle$ agrees with the skew pairing (1.12).

Now if $\pi : SU(2) \to U(\mathcal{H})$ is a unitary representation of $SU(2)$ on a finite dimensional Hilbert space $\mathcal{H}$, then applying Proposition 1.54 we obtain a Lie algebra representation $\rho : \mathfrak{sl}_2(\mathbb{C}) \to B(\mathcal{H})$ given by $\rho(X) = \frac{d}{dt}\Big|_{t=0} \pi(\exp(tX))$ for $X \in \mathfrak{su}(2)$. That is, on $\mathfrak{su}(2) \subseteq \mathfrak{sl}_2(\mathbb{C})$ we have $\rho = d\pi$, the differential of π at the identity.

We note that

$$\mathfrak{sl}_2(\mathbb{C}) = \{X \in M_2(\mathbb{C}) \mid \mathrm{Trace}(X) = 0\}$$

and

$$\mathfrak{su}(2) = \{X \in M_2(\mathbb{C}) \mid \mathrm{Trace}(X) = 0,\ X^* = -X\}$$

and so we see that $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}(2) + i\ \mathfrak{su}(2)$ and this sum is direct. In particular, in the above, if $X \in \mathfrak{sl}_2(\mathbb{C})$, we can write $X = X_1 + iX_2$ for some $X_1, X_2 \in \mathfrak{su}(2)$ and we have

$$\rho(X) = \rho(X_1) + i\rho(X_2) = d\pi(X_1) + id\pi(X_2).$$

Properties of the exponential map also tell us that

$$\begin{array}{ccc} \mathfrak{su}(2) & \xrightarrow{d\pi} & B(\mathcal{H}) \\ \big\downarrow{\scriptstyle \exp} & & \big\downarrow{\scriptstyle \exp} \\ SU(2) & \xrightarrow{\pi} & U(\mathcal{H}) \end{array}$$

commutes.

We can use Proposition 1.54 to transfer our knowledge of representation theory of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ to $SU_q(2)$. In particular, we denote by u^m the irreducible representation of $SU_q(2)$ on $V(m)$, with matrix coefficients (by a slight abuse of notation) u^m_{ij} obtained by applying Proposition 1.54 to $V(m)$. Note that under this correspondence, $u^{\frac{1}{2}}$ as given in Example 1.24 corresponds to $\pi_{\frac{1}{2}}$ as given in Example 1.24.

We can now apply Theorem 1.26 (b) to see that $\mathcal{O}(SU_q(2))$ has basis $\{u^m_{ij} \mid m \in \frac{1}{2}\mathbb{N}_0\}$.

Note that by construction we have

$$u^0_{00} = 1, \quad u^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} = \alpha, \quad u^{\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}} = \alpha^*, \quad u^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}} = \gamma, \quad u^{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}} = -q\gamma^*.$$

We have a useful lemma concerning products of matrix coefficients.

Lemma 1.56. *For any $q \in (0,1]$, the product of two matrix coefficients $u^n_{k\ell}u^m_{ij}$ in $\mathcal{O}(SU_q(2))$ can be expressed as a linear combination of matrix coefficients*

$$u^n_{k\ell}u^m_{ij} = \sum_{a,b,c} C(n,k,\ell,m,i,j,a,b,c,q)u^a_{bc},$$

where the coefficient function $q \mapsto C(n,k,\ell,m,i,j,a,b,c,q) \in \mathbb{C}$ is continuous.

Proof. We proceed by strong induction on $n \in \frac{1}{2}\mathbb{N}_0$. The statement is true for all $m \in \frac{1}{2}\mathbb{N}_0$ when $n = \frac{1}{2}$ by Lemma 1.51. Assume that any product of the form $u^s_{k\ell}u^m_{ij}$ for $s \in \frac{1}{2}\mathbb{N}_0$ with $s < n$ can be expressed as a linear combination of matrix coefficients, where each scalar coefficient depends continuously on $q \in (0,1]$. Now by Lemma 1.51 we have

$$u^n_{k\ell}u^m_{ij} = \left(\sum_{p,r}\sum_{a,b} \overline{C^{(n-\frac{1}{2})}_q((a,b),(n,k))} C^{(n-\frac{1}{2})}_q((p,r),(n,\ell)) u^{\frac{1}{2}}_{br} u^{n-\frac{1}{2}}_{ap} \right) u^m_{ij}$$

The product $u^{n-\frac{1}{2}}_{ap}u^m_{ij}$ can be evaluated by the inductive hypothesis. The remaining products are of the form $u^{\frac{1}{2}}_{k\ell}u^s_{ij}$, which is the base case. The final result is a linear combination of matrix coefficients, where each scalar coefficient depends continuously on $q \in (0,1]$. $\square$

Let us now calculate the Schur orthogonality relations (Theorem 1.33) for $A = C(SU_q(2))$ with respect to our chosen basis. For this we need to examine what happens to the contragredient representation under the correspondence provided by Proposition 1.54.

Define, for a $*$-representation $\pi : U_q(\mathfrak{sl}_2(\mathbb{C})) \to B(\mathcal{H})$ on a finite dimensional Hilbert space $\mathcal{H}$ the contragredient representation $\pi^c : U_q(\mathfrak{sl}_2(\mathbb{C})) \to B(\mathcal{H}^*)$ by the formula

$$\pi^c(X)(\omega)(h) = \omega(\pi(S^{-1}(X))h).$$

for $X \in U_q(\mathfrak{sl}_2(\mathbb{C}))$, $\omega \in \mathcal{H}^*$ and $h \in \mathcal{H}$. We will make use of the usual identification $\mathcal{H}^* \cong \overline{\mathcal{H}}$ in what follows.

Proposition 1.57.

(a) Let $\pi : U_q(\mathfrak{sl}_2(\mathbb{C})) \to B(\mathcal{H})$ be a $*$-representation of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on a finite dimensional Hilbert space $\mathcal{H}$ with basis $\{e_i\}_{i\in I}$. With respect to the corresponding dual basis in $\overline{\mathcal{H}}$, we have $\pi^c(X)_{ij} = \pi(S^{-1}(X))_{ji}$ for $X \in U_q(\mathfrak{sl}_2(\mathbb{C}))$.

(b) *Let* $v = (v_{ij})$ *be a finite dimensional unitary representation of* $SU_q(2)$ *on a Hilbert space* $\mathcal{H}$*, written as a matrix with respect to a chosen orthonormal basis of* $\mathcal{H}$*. In the notation of Proposition 1.54 we have that* $\pi_{v^c} = \pi_v^c$*, and* $\pi_{v^{cc}}(X)_{ij} = \pi_v(S^{-2}(X))_{ij}$*.*

Proof. By the Riesz Representation theorem, if $\omega \in \mathcal{H}^*$, there exists a unique $\eta \in \mathcal{H}$ such that $\omega = \langle \eta, - \rangle$. Then for $X \in U_q(\mathfrak{sl}_2(\mathbb{C}))$ we have

$$\pi^c(X)(\langle \eta, - \rangle) = \langle \eta, \pi(S^{-1}(X)) - \rangle = \langle \pi(S^{-1}(X))^* \eta, - \rangle$$

and so viewed as an operator on $\overline{\mathcal{H}}$, we have

$$\pi^c(X)\overline{\eta} = \overline{\pi(S^{-1}(X))^* \eta}.$$

In particular

$$\begin{aligned} \pi^c(X)_{ij} &:= \langle \overline{e_i}, \pi^c(X)\overline{e_j} \rangle \\ &= \langle \overline{e_i}, \overline{\pi(S^{-1}(X))^* e_j} \rangle \\ &:= \langle \pi(S^{-1}(X))^* e_j, e_i \rangle \\ &= \langle e_j, \pi(S^{-1}(X)) e_i \rangle \\ &= \pi(S^{-1}(X))_{ji}. \end{aligned}$$

Now, the ij-component of $\pi_{v^c}(X)$ is given by

$$\begin{aligned} \pi_{v^c}(X)_{ij} &= \langle v_{ij}^c, X \rangle \\ &= \langle v_{ij}^*, X \rangle \\ &= \overline{\langle v_{ij}, S^{-1}(X)^* \rangle} \\ &= \overline{\pi_v(S^{-1}(X))^*)_{ij}} \\ &= \pi_v(S^{-1}(X))_{ji} \\ &= \pi_v^c(X)_{ij} \end{aligned}$$

as required. Finally

$$\pi_{v^{cc}}(X)_{ij} = \pi_{v^c}(S^{-1}(X))_{ji} = \pi_v(S^{-2}(X))_{ij}.$$ ∎

Note that using the relations in Definition 1.42 we have, for $q \neq 1$,

$$S^{-2}(E) = K^{-2}EK^2, \quad S^{-2}(F) = K^{-2}FK^2, \quad S^{-2}(K) = K = K^{-2}KK^2,$$

and so by Proposition 1.57 we have

$$\pi_{v^{cc}}(X) = \pi_v(S^{-2}(X)) = \pi_v(K^{-2}XK^2) = \pi_v(K^2)^{-1}\pi_v(X)\pi_v(K^2).$$

Therefore an intertwiner between π_v and $\pi_{v^{cc}}$ is given by $\pi_v(K^{-2})$. Under the correspondence provided by Proposition 1.54 $\pi_v(K^{-2})$ is also an intertwiner between v and v^{cc}.

If we now let $v = u^n$ (where $n \in \frac{1}{2}\mathbb{N}_0$), then

$$\pi_v(K^{-2}) = \begin{pmatrix} q^{-2n} & & & \\ & q^{-(2n-2)} & & \\ & & \ddots & \\ & & & q^{2n} \end{pmatrix} \tag{1.15}$$

by Theorem 1.49, and one can check that $\mathrm{Trace}(\pi_v(K^{-2})) = \mathrm{Trace}(\pi_v(K^2)) = [2n+1]$.

We can now derive the orthogonality relations for $SU_q(2)$.

Theorem 1.58. *For $q \in (0,1)$ we have the orthogonality relations*

$$\phi_{SU_q(2)}(u^m_{kl}(u^n_{ij})^*) = \frac{\delta_{mn}\delta_{ki}\delta_{lj}q^{-2j}}{[2n+1]}, \quad \phi_{SU_q(2)}((u^n_{ij})^* u^m_{kl}) = \frac{\delta_{mn}\delta_{ki}\delta_{lj}q^{2i}}{[2n+1]}.$$

As $q \to 1$ we recover the classical orthogonality relations for $SU(2)$. We also have $\phi_{SU_q(2)}(u^n_{ij}) = \delta_{0,n}$.

Proof. The relations follow from Theorem 1.33, noting that P_{u^n} is given by $\pi_{u^n}(K^{-2})$, and using (1.15). As $q \to 1$, $[2n+1] \to 2n+1$ and we see the second formulae converges to the Schur relations in the classical case. For the final relation, put $n = i = j = 0$ into the formulae. □

Finally, let us understand the dual of $SU_q(2)$.

Example 1.59. By our calculation of the irreducible unitary representations of $SU_q(2)$,

$$\mathcal{D}(SU_q(2)) \cong \text{alg-}\bigoplus_{m \in \frac{1}{2}\mathbb{N}_0} M_{2m+1}(\mathbb{C})$$

(using our description of the dual as a direct sum of matrix algebras, Section 1.2.2) and the completion, which we denote by $C^*(SU_q(2))$, is given by

$$C^*(SU_q(2)) \cong \bigoplus_{m \in \frac{1}{2}\mathbb{N}_0} M_{2m+1}(\mathbb{C}).$$

Note that if $q = 1$, this is the classical group C^*-algebra of $SU(2)$ (c.f. the construction in Examples 1.8, 1.19 and for example [89, Proposition 3.4]).

The basis of $\mathcal{D}(SU_q(2))$ dual to $\{u^n_{ij}\}$ is denoted by $\{\omega^n_{ij}\}$. By Theorem 1.58, we see that $\phi_{SU_q(2)} = \omega^0_{00}$.

Note that we have an inclusion

$$U_q(\mathfrak{sl}_2(\mathbb{C})) \hookrightarrow \prod_{m\in\frac{1}{2}\mathbb{N}_0} M_{2m+1}(\mathbb{C}) = M(\mathcal{D}(SU_q(2))) \tag{1.16}$$

given by $X \mapsto (\pi_m(X))$, where π_m is the $2m+1$ dimensional irreducible representation of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on $V(m)$ constructed earlier. It is non-trivial to show injectivity - one needs to prove that the irreducible representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ separate points of $U_q(\mathfrak{sl}_2(\mathbb{C}))$, see for example [86, Theorem 2.32].

Let $\Delta_{\mathcal{D}(SU_q(2))}$ be the comultiplication on $\mathcal{D}(SU_q(2))$. By Definition 1.15, $\Delta_{\mathcal{D}(SU_q(2))}$ is non-degenerate, and so extends to a $*$-homomorphism (see Remark 1.14) which we also call $\Delta_{\mathcal{D}(SU_q(2))}$,

$$\Delta_{\mathcal{D}(SU_q(2))} : M(\mathcal{D}(SU_q(2))) \to M(\mathcal{D}(SU_q(2)) \odot \mathcal{D}(SU_q(2))).$$

Since $U_q(\mathfrak{sl}_2(\mathbb{C})) \subseteq M(\mathcal{D}(SU_q(2)))$ and $U_q(\mathfrak{sl}_2(\mathbb{C}))\odot U_q(\mathfrak{sl}_2(\mathbb{C})) \subseteq M(\mathcal{D}(SU_q(2))\odot\mathcal{D}(SU_q(2)))$, we can restrict $\Delta_{\mathcal{D}(SU_q(2))}$ to $U_q(\mathfrak{sl}_2(\mathbb{C}))$, and ask whether this agrees with the comultiplication on $U_q(\mathfrak{sl}_2(\mathbb{C}))$, which we denote by $\Delta_{U_q(\mathfrak{sl}_2(\mathbb{C}))}$ given in Definition 1.42.

Define a linear pairing

$$(-,-) : \mathcal{D}(SU_q(2)) \times \mathcal{O}(SU_q(2)) \to \mathbb{C}, \quad (\omega, f) \mapsto \omega(f).$$

The way we have defined the multiplier Hopf-$*$-algebraic structure on $\mathcal{D}(SU_q(2))$ means this is a skew pairing of multiplier Hopf-$*$-algebras, which is essentially the same as a pairing in the sense of Definition 1.52. The difference is that the coproduct in $\mathcal{D}(SU_q(2))$ may be an infinite sum of tensor products of matrix units. However, $\mathcal{O}(SU_q(2))$ will only be supported on a finite part of this sum, so the pairing still makes sense. For this reason we can also extend the pairing to

$$(-,-) : \prod_{n\in\frac{1}{2}\mathbb{N}_0} M_{2n+1}(\mathbb{C}) \times \mathcal{O}(SU_q(2)) \to \mathbb{C}$$

and then if we restrict the first factor to $U_q(\mathfrak{sl}_2(\mathbb{C}))$ we obtain a pairing (for which we use the same notation)

$$(-,-):\ U_q(\mathfrak{sl}_2(\mathbb{C}))\times \mathcal{O}(SU_q(2))\to\mathbb{C}.$$

We will show that this is the same as the pairing $\langle -,-\rangle$ obtained by viewing $\mathcal{O}(SU_q(2))$ as the Hopf-$*$-algebra of matrix coefficients $B_0\in U_q(\mathfrak{sl}_2(\mathbb{C}))^*$, as defined in (1.12). It is enough to check this on the basis $\{u^m_{ij}\}$ for $\mathcal{O}(SU_q(2))$. We have, for $X\in U_q(\mathfrak{sl}_2(\mathbb{C}))$,

$$\langle X, u^n_{ij}\rangle =\ ij^{\text{th}} \text{ matrix element of } \pi_n(X)=(X,u^n_{ij}).$$

In particular, for $X\in U_q(\mathfrak{sl}_2(\mathbb{C}))$ and $f,g\in\mathcal{O}(SU_q(2))$,

$$\begin{aligned}(\Delta_{\mathcal{D}(SU_q(2))}(X)-\Delta_{U_q(\mathfrak{sl}_2(\mathbb{C}))}(X),f\otimes g) &= (\Delta_{\mathcal{D}(SU_q(2))}(X),f\otimes g)-(\Delta_{U_q(\mathfrak{sl}_2(\mathbb{C}))}(X),f\otimes g)\\ &= (\Delta_{\mathcal{D}(SU_q(2))}(X),f\otimes g)-\langle(\Delta_{U_q(\mathfrak{sl}_2(\mathbb{C}))}(X),f\otimes g\rangle\\ &= (X,fg)-\langle X,fg\rangle\\ &= (X,fg)-(X,fg)=0.\end{aligned}$$

Non-degeneracy of the pairing then implies that $\Delta_{\mathcal{D}(SU_q(2))}\big|_{U_q(\mathfrak{sl}_2(\mathbb{C}))}=\Delta_{U_q(\mathfrak{sl}_2(\mathbb{C}))}$.

To understand the left Haar integral $\phi_{\widehat{SU_q(2)}}$ on $\mathcal{D}(SU_q(2))$, we need a formulae for the Fourier transform (see the formula in Theorem 1.36).

Recall from the general construction of the dual that $\omega^n_{ij}=\phi_{K_q}(-a^n_{ij})$, where

$$a^n_{ij}:=\sum_r[2n+1]\delta_{jr}q^{2r}(u^n_{ir})^*=q^{2j}[2n+1](u^n_{ij})^*,$$

see equation (1.4). Therefore

$$\mathcal{F}^{-1}(\omega^n_{ij})=q^{2j}[2n+1](u^n_{ij})^*. \tag{1.17}$$

Hence by the formula in Theorem 1.36, we have

$$\phi_{\widehat{SU_q(2)}}(\omega^n_{ij})=q^{2j}[2n+1]\epsilon(u^n_{ij})^*=q^{2j}[2n+1]\delta_{ij}. \tag{1.18}$$

We conclude by noting that

$$(u^n_{ij})^*=(-1)^{2n+i+j}q^{j-i}u^n_{-i\ -j}, \tag{1.19}$$

see [84, p.g. 9]. This requires a more careful analysis of the coefficients in Lemma 1.56, c.f. [12, p.g. 733], which we omit.

1.4 The Quantum Double

Let $A = C(\mathbb{G})$ be a compact quantum group, with dense Hopf-$*$-algebra $\mathcal{O}(\mathbb{G})$ and dual $\mathcal{D}(\mathbb{G})$.

Generalizing Example 1.59, the way we have defined the multiplier Hopf-$*$-algebraic structure on $\mathcal{D}(\mathbb{G})$ sets up a skew pairing

$$(-,-) : \mathcal{D}(\mathbb{G}) \times \mathcal{O}(\mathbb{G}) \to \mathbb{C}, \quad (\omega, f) \mapsto \omega(f)$$

of multiplier Hopf-$*$-algebras.

In this section we will construct an example of a quantum group that is not finite, discrete or compact. This construction starts with an **algebraic quantum group**. This notion arose out of Van Daele's work in [83], and is given in [47, Definition 1.2].

Definition 1.60. An **algebraic quantum group** is a multiplier Hopf-$*$-algebra with a positive left integral or a positive right integral.

We now define the quantum double. This is sometimes called the Drinfeld double, and was introduced by Drinfeld in [19, Section 13] for a pair of Hopf algebras. The construction we show here is a slight extension of Drinfeld's original construction, because we will define the double for a pair of algebras, one being a Hopf algebra and the other a multiplier Hopf algebra. This generalization is introduced in [18, Section 3]. The reader should note that there are various conventions in the literature that amount to small changes in the formulae that follow. We will use the same conventions as [86, Section 1.4].

The **quantum double** of $\mathcal{O}(\mathbb{G})$, denoted $\mathcal{Q}(\mathbb{G})$, is defined as

$$\mathcal{Q}(\mathbb{G}) := \mathcal{D}(\mathbb{G}) \odot \mathcal{O}(\mathbb{G})$$

as a vector space. We can equip $\mathcal{Q}(\mathbb{G})$ with a $*$-algebra structure defined by

$$(x \otimes f)(y \otimes g) = x(y_{(1)}, f_{(1)})y_{(2)} \otimes f_{(2)}(S(y_{(3)}), f_{(3)})g, \tag{1.20}$$

$$(x \otimes f)^* := (x^*_{(1)}, f^*_{(1)})x^*_{(2)} \otimes f^*_{(2)}(S(x^*_{(3)}), f^*_{(3)}) \tag{1.21}$$

for $x, y \in \mathcal{D}(\mathbb{G})$ and $f, g \in \mathcal{O}(\mathbb{G})$. Note that we are using the Sweedler notation (see Remark 1.3), even though $\mathcal{D}(\mathbb{G})$ is a multiplier Hopf-$*$-algebra. This is because if we identify $\mathcal{D}(\mathbb{G}) = \bigoplus_{i \in I} M_{n_i}(\mathbb{C})$ for some indexing set I, we can label the legs of elements in the image of Δ, viewed as elements in $\prod_{i,j \in I} M_{n_i}(\mathbb{C}) \otimes M_{n_j}(\mathbb{C})$.

We can turn $\mathcal{Q}(\mathbb{G})$ into a multiplier Hopf-$*$-algebra using the formula

$$\Delta = (\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\Delta_{\mathcal{D}(\mathbb{G})} \otimes \Delta_{\mathcal{O}(\mathbb{G})})$$

where we have used subscripts on each comultiplication to ensure there is no confusion as to where each is defined, and $\sigma : \mathcal{D}(\mathbb{G}) \odot \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{G}) \odot \mathcal{D}(\mathbb{G})$ denotes the flip map. To emphasize that we are using a different multiplication on $\mathcal{D}(\mathbb{G}) \odot \mathcal{O}(\mathbb{G})$, we will write $\mathcal{Q}(\mathbb{G}) = \mathcal{D}(\mathbb{G}) \bowtie \mathcal{O}(\mathbb{G})$ and denote elementary tensors by $x \bowtie f$, where $x \in \mathcal{D}(\mathbb{G})$ and $f \in \mathcal{O}(\mathbb{G})$.

It is easy to check that

$$\phi_{\mathcal{Q}(\mathbb{G})} = \phi_{\widehat{\mathbb{G}}} \otimes \phi_{\mathbb{G}} \tag{1.22}$$

defines a left invariant integral on the quantum double, where $\phi_{\widehat{\mathbb{G}}}$ is the left integral on $\mathcal{D}(\mathbb{G})$ and $\phi_{\mathbb{G}}$ is the Haar state on $C(\mathbb{G})$, provided by Theorems 1.36 and 1.29 respectively. Then $\phi_{\mathcal{Q}(\mathbb{G})}$ is then a faithful integral by Remark 1.35.

In fact, $\phi_{\mathcal{Q}(\mathbb{G})}$ is also a positive integral. This is non-trivial, but is a consequence of the fact that $\mathcal{Q}(\mathbb{G})$ is the dual of an algebraic quantum group with positive integral (see [86, Proposition 3.11]), and under the duality construction for algebraic quantum groups, this positive integral induces a positive integral on the dual [83, Propositions 4.8 and 4.9] which is, in our case precisely the integral $\phi_{\mathcal{Q}(\mathbb{G})}$. Therefore $\mathcal{Q}(\mathbb{G})$ is an example of an algebraic quantum group in the sense of Definition 1.60.

Now let us turn to the notion of a **Yetter-Drinfeld module**. These were first introduced by Yetter in [93, Definition 3.6], and Majid [54, Proposition 2.2] showed that Yetter-Drinfeld modules correspond to representations of the quantum double.

Let V be an inner product space, and suppose we have a $*$-homomorphism $\pi : \mathcal{O}(\mathbb{G}) \to B(V)$ and a non-degenerate $*$-homomorphism $\rho : \mathcal{D}(\mathbb{G}) \to B(V)$. We ask what conditions we need to impose on π and ρ so that $\rho \otimes \pi : \mathcal{D}(\mathbb{G}) \odot \mathcal{O}(\mathbb{G}) \to B(V)$ defines a $*$-representation of $\mathcal{Q}(\mathbb{G})$.

The map $\rho \otimes \pi$ is a homomorphism if and only if

$$(\rho \otimes \pi)\left((x \bowtie f)(y \bowtie g)\right) = (\rho \otimes \pi)(x \bowtie f)(\rho \otimes \pi)(y \bowtie g) = \rho(x)\pi(f)\rho(y)\pi(g)$$

for all $x, y \in \mathcal{D}(\mathbb{G})$ and $f, g \in \mathcal{O}(\mathbb{G})$. Now

$$\begin{aligned}(\rho \otimes \pi)\left((x \bowtie f)(y \bowtie g)\right) &= (\rho \otimes \pi)\left(x(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}(S(y_{(3)}), f_{(3)})g\right) \\ &= \rho(x)(y_{(1)}, f_{(1)})\rho(y_{(2)})(S(y_{(3)}), f_{(3)})\pi(f_{(2)})\pi(g)\end{aligned}$$

and so $\rho \otimes \pi$ is a homomorphism if and only if

$$\pi(f)\rho(y) = (y_{(1)}, f_{(1)})\rho(y_{(2)})(S(y_{(3)}), f_{(3)})\pi(f_{(2)}).$$

In fact, if ρ and π satisfy this condition, then $\rho\otimes\pi$ is a $*$-homomorphism, since if $x \in \mathcal{D}(\mathbb{G})$ and $f \in \mathcal{O}(\mathbb{G})$ then

$$\begin{aligned}(\rho \otimes \pi)((x \bowtie f)^*) &= (\rho \otimes \pi)((x^*_{(1)}, f^*_{(1)})x^*_{(2)} \bowtie f^*_{(2)}(S(x^*_{(3)}), f^*_{(3)})) \\ &= (x^*_{(1)}, f^*_{(1)})\rho(x^*_{(2)})\pi(f^*_{(2)})(S(x^*_{(3)}), f^*_{(3)}) \\ &= \pi(f^*)\rho(x^*) = \pi(f)^*\rho(x)^* = (\rho(x)\pi(f))^* = (\rho \otimes \pi)(x \bowtie f)^*.\end{aligned}$$

In this case will write $\rho \otimes \pi$ as $\rho \bowtie \pi$ to emphasize that we are viewing this linear map as a $*$-homomorphism on the double.

Definition 1.61. Let $A = C(\mathbb{G})$ be a compact quantum group. A unitary **Yetter-Drinfeld module** for $\mathcal{Q}(\mathbb{G})$ is an inner product space V together with a unital $*$-homomorphism $\pi : \mathcal{O}(\mathbb{G}) \to B(V)$ and a non-degenerate $*$-homomorphism $\rho : \mathcal{D}(\mathbb{G}) \to B(V)$ satisfying the Yetter-Drinfeld compatibility condition

$$\pi(f)\rho(y) = (y_{(1)}, f_{(1)})\rho(y_{(2)})(S(y_{(3)}), f_{(3)})\pi(f_{(2)}) \tag{1.23}$$

for all $f \in \mathcal{O}(\mathbb{G})$, $y \in \mathcal{D}(\mathbb{G})$.

We have seen that a unitary Yetter-Drinfeld module V with $*$-homomorphisms $\pi : \mathcal{O}(\mathbb{G}) \to B(V)$ and $\rho : \mathcal{D}(\mathbb{G}) \to B(V)$ determines a representation $\rho \bowtie \pi : \mathcal{D}(\mathbb{G}) \to B(V)$. Conversely, we have the following.

Proposition 1.62. *Let $\nu : \mathcal{Q}(\mathbb{G}) \to B(V)$ be a non-degenerate $*$-homomorphism, where V is an inner product space. Then V is a unitary Yetter-Drinfeld module equipped with $*$-homomorphisms $\pi : \mathcal{O}(\mathbb{G}) \to B(V)$ and $\rho : \mathcal{D}(\mathbb{G}) \to B(V)$ such that $\nu = \rho \bowtie \pi$.*

Proof. Since ν is non-degenerate, we can extend ν to a $*$-homomorphism which we also denote by ν,

$$\nu : M(\mathcal{Q}(\mathbb{G})) \to B(V).$$

If $f \in \mathcal{O}(\mathbb{G})$, we can define an element $1 \bowtie f \in M(\mathcal{Q}(\mathbb{G}))$ by extending the comultiplication on $\mathcal{D}(\mathbb{G})$ to $M(\mathcal{D}(\mathbb{G}))$ and then using the formulae in the definition of $\mathcal{Q}(\mathbb{G})$ to define left and right multiplication by $1 \bowtie f$, viewing 1 as the unit in $M(\mathcal{D}(\mathbb{G}))$. By construction, $\mathcal{O}(\mathbb{G}) \to M(\mathcal{Q}(\mathbb{G}))$, $f \mapsto 1 \bowtie f$ is a $*$-homomorphism. We also have, for $x \in \mathcal{D}(\mathbb{G})$ and $f \in \mathcal{O}(\mathbb{G})$,

$$x \bowtie f = (x \bowtie 1)(1 \bowtie f) \tag{1.24}$$

and so $\nu(x \bowtie f) = \nu(x \bowtie 1)\nu(1 \bowtie f)$. Then we define

$$\pi : \mathcal{O}(\mathbb{G}) \to B(V), \quad f \mapsto \nu(1 \bowtie f)$$

and

$$\rho : \mathcal{D}(\mathbb{G}) \to B(V), \quad x \mapsto \nu(x \bowtie 1).$$

This pair necessarily makes V into a Yetter-Drinfeld module, and we have $\nu = \rho \bowtie \pi$. $\square$

By Proposition 1.62 we have a correspondence between representations of the double and unitary Yetter-Drinfeld modules for the double.

Let us finish this section by describing how we can construct C^*-algebras from the quantum double. We can construct two C^*-algebras - a full (universal) C^*-algebra, and a reduced C^*-algebra using the techniques of [47, Section 2].

We can apply the GNS construction to the positive functional $\phi_{\mathcal{Q}(\mathbb{G})}$ defined in (1.22) to obtain a Hilbert space $\mathcal{H}_{\mathcal{Q}(\mathbb{G})}$. The quantum double $\mathcal{Q}(\mathbb{G})$ acts on the dense subspace $\mathcal{Q}(\mathbb{G}) \subseteq \mathcal{H}_{\mathcal{Q}(\mathbb{G})}$ by left multiplication. It is a non-trivial fact that the left multiplication is bounded (see [47, Lemmas 2.3 and 2.4]). Therefore we obtain a faithful non-degenerate $*$-representation $\lambda : \mathcal{Q}(\mathbb{G}) \to B(\mathcal{H}_{\mathcal{Q}(\mathbb{G})})$.

For the reduced C^*-algebra, we define $C^*_r(\mathcal{Q}(\mathbb{G})) = \overline{\lambda(\mathcal{Q}(\mathbb{G}))}^{\|-\|_{\mathrm{op}}}$. We will denote the norm on $C^*_r(\mathcal{Q}(\mathbb{G}))$ by $\|-\|_r$.

The full C^*-algebra, denoted $C^*(\mathcal{Q}(\mathbb{G}))$, is the universal enveloping C^*-algebra of $\mathcal{Q}(\mathbb{G})$ (c.f. Example 1.21). That is, we consider all non-degenerate $*$-represention $\nu : \mathcal{Q}(\mathbb{G}) \to B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Equip $\mathcal{Q}(\mathbb{G})$ with the seminorm

$$\|x\|_f := \sup_{\nu} \|\nu(x)\| \tag{1.25}$$

for $x \in \mathcal{Q}(\mathbb{G})$. First, note that (1.25) is finite. It suffices, by the triangle inequality, to check that there exists $R > 0$ such that $\left\|\nu(\omega^\lambda_{ij} \bowtie u^\mu_{kl})\right\| \leq R$, where u^μ_{kl} is an element of the basis of $\mathcal{O}(\mathbb{G})$ provided by Theorem 1.26, and ω^λ_{ij} is an element of the corresponding basis of $\mathcal{D}(\mathbb{G})$ constructed in Section 1.2.2.

By Proposition 1.62, there exist $*$-homomorphisms $\pi : \mathcal{O}(\mathbb{G}) \to B(\mathcal{H})$ and $\rho : \mathcal{D}(\mathbb{G}) \to B(\mathcal{H})$ such that $\nu = \rho \bowtie \pi$. We have

$$\left\|\nu(\omega^\lambda_{ij} \bowtie u^\mu_{kl})\right\| \leq \left\|\rho(\omega^\lambda_{ij})\right\| \left\|\pi(u^\mu_{kl})\right\|,$$

and

$$\left\|\rho(\omega^{\lambda}_{ij})\right\|^2 = \left\|\rho((\omega^{\lambda}_{ij})^*\omega^{\lambda}_{ij})\right\| = \left\|\rho(\omega^{\lambda}_{jj})\right\| \leq 1$$

because $\rho(\omega^{\lambda}_{jj}) \in B(\mathcal{H})$ is a projection. One can show $\|\pi(u^{\mu}_{kl})\| \leq 1$ by an entirely similar argument to that given in Example 1.21. Hence we have

$$\left\|\nu(\omega^{\lambda}_{ij} \bowtie u^{\mu}_{kl})\right\| \leq 1. \tag{1.26}$$

The seminorm (1.25) is in fact a norm on $\mathcal{Q}(\mathbb{G})$ because by definition $\|-\|_r \leq \|-\|_f$. We can then define $C^*(\mathcal{Q}(\mathbb{G}))$ to be the completion of $\mathcal{Q}(\mathbb{G})$ with respect to (1.25).

We will sometimes refer to the norm $\|-\|_r$ as the reduced norm, and the norm $\|-\|_f$ as the full norm.

Chapter 2

The Quantum Assembly Field

In this chapter, we will define a quantum analogue of Connes' assembly field, as described in the Introduction. This will be termed the **quantum assembly field**, and we shall see that it induces a map in K-theory. In Chapter 3 we shall show this map is an isomorphism, so proving a quantum analogue of the Baum-Connes conjecture.

We start by defining quantum $SL(2,\mathbb{C})$ by using the double construction from Chapter 1, Section 1.4. We will introduce the **principal series representations** of quantum $SL(2,\mathbb{C})$ and give a result which uses these representations to describe the reduced C^*-algebra of quantum $SL(2,\mathbb{C})$. We then move on to the construction of the quantum assembly field. The main technical difficulty is in checking the field we construct has the desired fibres.

2.1 Quantum $SL(2,\mathbb{C})$ and Principal Series Representations

2.1.1 Quantum $SL(2,\mathbb{C})$

We will work with a specific example of the quantum double construction given in Section 1.4, where we take $\mathbb{G} = SU_q(2)$ for $q \in (0,1]$. In this case, we will use the notation

$$\mathcal{D}(SL_q(2,\mathbb{C})) := \mathcal{Q}(SU_q(2)) := \mathcal{D}(SU_q(2)) \bowtie \mathcal{O}(SU_q(2)).$$

Remark 2.1. It is not clear from the definition why this quantum double is related to $SL(2,\mathbb{C})$, as indicated by our choice of notation. It is easier to illustrate the reasoning behind this definition by first trying to construct an appropriate quantized function algebra for $SL(2,\mathbb{C})$, namely $C_0(SL_q(2,\mathbb{C}))$.

There is a decomposition

$$SL(2,\mathbb{C}) = K \times A \times N \tag{2.1}$$

as a topological product, where K and A and N are the subgroups

$$K := SU(2), \quad A = \left\{ \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} \mid x \in \mathbb{R} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}$$

of $SL(2,\mathbb{C})$. This is called the **Iwasawa decomposition**, for it originally appears in the work of Iwasawa, [36, Lemma 3.12]. For the case of $SL(n,\mathbb{C})$ in particular, see [44, Chapter VI, First Example in Section 4].

Therefore

$$C_0(SL(2,\mathbb{C})) \cong C(K) \otimes C_0(AN).$$

We have an appropriate quantization of $C(K)$, namely $C(SU_q(2))$. It is not immediately clear how one should quantize the AN-part.

In [65, Equations (1.9)-(1.25)], the authors construct a quantized algebra of matrix coefficients of $SL(2,\mathbb{C})$, which can be seen as analogous to how one obtains $\mathcal{O}(SU_q(2))$ from the definition of $SU(2)$. However matrix coefficients of elements in $SL(2,\mathbb{C})$ are not bounded, and in particular do not vanish at infinity. Therefore we would not expect to be able to directly obtain a sensible definition of $C_0(SL_q(2,\mathbb{C}))$ from this quantized algebra. Instead, one can use of Woronowicz's theory of affiliated elements, as later introduced in [92], and see that the quantized algebra can be realized as an algebra of elements affiliated to the C^*-algebra

$$C_0(SL_q(2,\mathbb{C})) := C(SU_q(2)) \otimes C^*(SU_q(2)), \tag{2.2}$$

see [65, Theorem 5.4, Part 4]. Note that we see in this quantization that $C_0(AN)$ has been replaced with $C^*(SU_q(2))$, the dual of $C(SU_q(2))$ as a quantum group.

Inside $C_0(SL_q(2,\mathbb{C}))$ there is the dense $*$-subalgebra

$$\mathcal{O}(SL_q(2,\mathbb{C})) := \mathcal{O}(SU_q(2)) \odot \mathcal{D}(SU_q(2))$$

which is an example of an algebraic quantum group, by the results in [65, p.g. 407-408]. The comultiplication on $\mathcal{O}(SL_q(2,\mathbb{C}))$ is complicated by virtue of the fact that the Iwasawa decomposition is not a direct or semi-direct product of groups.

It is possible to generalize the duality construction in Section 1.2.2 to algebraic quantum groups, see [83, Section 4]. If one applies this construction to $\mathcal{O}(SU_q(2))$ one obtains

$$\mathcal{D}(SL_q(2,\mathbb{C})) = \mathcal{D}(SU_q(2)) \bowtie \mathcal{O}(SU_q(2)),$$

see [86, Proposition 3.11]. Intuitively, we have taken the dual of $\mathcal{O}(SU_q(2))$ and obtained $\mathcal{D}(SU_q(2))$ and vice-versa. The multiplication on $\mathcal{D}(SL_q(2,\mathbb{C}))$ arises from the comultiplication on $\mathcal{O}(SL_q(2,\mathbb{C}))$.

To ease notation in what follows we set, for $q \in (0,1]$, $G_q := SL_q(2,\mathbb{C})$ and $K_q := SU_q(2)$. We set $K_1 = K$, but we will **not** do the same for G_1 and $SL(2,\mathbb{C})$, for reasons that will become clear shortly.

Recall from (1.22) that we have a left integral $\phi_{G_q} = \phi_{\widehat{K_q}} \otimes \phi_{K_q}$ on $\mathcal{D}(G_q)$, where $\phi_{\widehat{K_q}}$ is the left integral on $\mathcal{D}(K_q)$ and ϕ_{K_q} is the Haar state on $C(K_q)$. We denote the GNS space corresponding to ϕ_{G_q} by $L^2(\widehat{G_q})$. The 'hat' in the notation here is to recognise the fact we are working with a quantum object that is the 'dual' of G_q. Then we obtain, as explained in Section 1.4, a faithful and non-degenerate $*$-homomorphism $\lambda : \mathcal{D}(G_q) \to B(L^2(\widehat{G_q}))$, and so we obtain a C^*-algebra which we term the reduced group C^*-algebra of G_q given by

$$C^*_r(G_q) := \overline{\lambda(\mathcal{D}(G_q))}^{\|-\|_{\mathrm{op}}}.$$

We saw we could also complete $\mathcal{D}(G_q)$ in the full norm and obtain the full group C^*-algebra $C^*(G_q)$.

For $q = 1$, we do **not** recover the classical group C^*-algebras of $SL(2,\mathbb{C})$. Rather, we have the following result, which is folklore. Note that in the case of a Lie group H acting on itself by conjugation, i.e.

$$H \times H \to H, \quad (h,m) \mapsto h \cdot m := hmh^{-1}, \tag{2.3}$$

then the action is called the adjoint action (c.f. (1)).

Proposition 2.2. *We have*

$$C^*_r(G_1) \cong K \ltimes_{\mathrm{adj},r} C(K), \quad C^*(G_1) \cong K \ltimes_{\mathrm{adj}} C(K), \tag{2.4}$$

where K acts on itself by the adjoint action (see equation (2.3)), so inducing an action on the C^*-algebra $C(K)$ given by

$$\mathrm{adj} : K \to \mathrm{Aut}(C(K)), \quad \mathrm{adj}_k(f)(s) = f(k^{-1} \cdot s)$$

for $k, s \in K$ and $f \in C(K)$. In fact $C^*_r(G_1) \cong C^*(G_1)$ because the reduced and full crossed *products in (2.4) are the same.*

Proof. Let us start with the notation we will use in this proof. Consider the convolution algebra $C_c(K, C(K))$. Recall that our convention is that we still retain the subscript c in our notation, to differentiate it from the C^*-algebra $C(K, C(K))$. If $f, g \in C(K)$, then the elementary tensor $f \otimes g \in C_c(K, C(K))$ is the function

$$(f \otimes g)(k) = f(k)g \in C(K)$$

where $k \in K$. Let us now recall the definition of the regular representation in this context. Define

$$\rho : C(K) \to B(L^2(K, L^2(K))), \quad (\rho(f)\xi)(k)(s) = f(ksk^{-1})\xi(k)$$

and

$$U : K \to U(L^2(K, L^2(K))), \quad (U_s\xi)(k) = \xi(s^{-1}k)$$

for $s, k \in K$, $f \in C(K)$ and $\xi \in L^2(K, L^2(K))$. This is a covariant pair (see [89, Definition 2.10]), and then the regular representation is

$$U \ltimes \rho : C_c(K, C(K)) \to B(L^2(K, L^2(K))), \quad (U \ltimes \rho)(f) = \int_K U_k\rho(f(k))\ \mathrm{d}k$$

for $f \in C_c(K, C(K))$. This is faithful, see [89, Lemma 2.26], and

$$K \ltimes_{\mathrm{adj},r} C(K) = \overline{(U \ltimes \rho)(C_c(K, C(K)))}^{\|-\|_{\mathrm{op}}}.$$

We refer the reader to [89] for an account of crossed products.

Consider the map

$$\mathcal{F}^{-1} \otimes \mathrm{id} : \mathcal{D}(SL_q(2, \mathbb{C})) \to C_c(K, C(K)), \quad x \bowtie f \mapsto \mathcal{F}^{-1}(x) \otimes f. \tag{2.5}$$

Here $\mathcal{F} : \mathcal{O}(K) \to \mathcal{D}(K)$ is the Fourier transform, equation (1.6). It is clear that (2.5) is an injective linear map, because the Fourier transform is injective. The image of (2.5) is the vector space $\mathcal{O}(K) \odot \mathcal{O}(K) \subseteq C_c(K, C(K))$ which is dense in $C_c(K, C(K))$ in the L^1-norm, and in particular in the full and reduced norms, see [89, Lemma 1.87]. We will show that (2.5) is a $*$-homomorphism.

We let α denote the adjoint action to ease notation. We have, using the definition of multiplication in $C_c(K, C(K))$, for $x, y \in \mathcal{D}(K)$, $f, g \in \mathcal{O}(K)$ and $k \in K$,

$$(\mathcal{F}^{-1}(x) \otimes f)(\mathcal{F}^{-1}(y) \otimes g)(k) = \int_K \alpha_{k^{-1}s}((\mathcal{F}^{-1}(x) \otimes f)(s))(\mathcal{F}^{-1}(y) \otimes g)(s^{-1}k)\ \mathrm{d}s$$

$$= \int_K \mathcal{F}^{-1}(x)(s)\mathcal{F}^{-1}(y)(s^{-1}k)\alpha_{k^{-1}s}(f)g \text{ d}s.$$

The image of $(x \bowtie f)(y \bowtie g)$ under (2.5) is given by

$$\mathcal{F}^{-1}(xy_{(2)}) \otimes (y_{(1)}, f_{(1)})f_{(2)}(S(y_{(3)}), f_{(3)})g,$$

using (1.20). To understand $\mathcal{F}^{-1}(xy_{(2)})$ we consider, for $f, g, h \in \mathcal{O}(K)$,

$$\begin{aligned}
(y_{(1)} \otimes xy_{(2)} \otimes y_{(3)})(f \otimes g \otimes h) &= y_{(1)}(f)(xy_{(2)})(g)y_{(3)}(h) \\
&= y_{(1)}(f)x(g_{(1)})y_{(2)}(g_{(2)})y_{(3)}(h) \\
&= x(g_{(1)})y(hg_{(2)}f) \\
&= \int_K \int_K \mathcal{F}^{-1}(x)(s)g_{(1)}(s)\mathcal{F}^{-1}(y)(k)h(k)g_{(2)}(k)f(k) \text{ d}k \text{ d}s \\
&= \int_K \int_K \mathcal{F}^{-1}(x)(s)g(sk)\mathcal{F}^{-1}(y)(k)h(k)f(k) \text{ d}k \text{ d}s \\
&= \int_K \int_K \mathcal{F}^{-1}(x)(s)g(u)\mathcal{F}^{-1}(y)(s^{-1}u)h(s^{-1}u)f(s^{-1}u) \text{ d}u \text{ d}s \\
&= \int_K (\mathcal{F}^{-1}(x) * \mathcal{F}^{-1}(y)hf)(u)\ g(u) \text{ d}u.
\end{aligned}$$

Therefore

$$\begin{aligned}
(\mathcal{F}^{-1}(xy_{(2)}) \otimes (y_{(1)}, f_{(1)})f_{(2)}(S(y_{(3)}), f_{(3)})g)(k) &= \int_K \mathcal{F}^{-1}(x)(s)\mathcal{F}^{-1}(y)(s^{-1}k)\alpha_{k^{-1}s}(f)g \text{ d}s \\
&= (\mathcal{F}^{-1}(x) \otimes f)(\mathcal{F}^{-1}(y) \otimes g)(k).
\end{aligned}$$

We now check that (2.5) preserves $*$. If $x \in \mathcal{D}(K)$, $f \in \mathcal{O}(K)$ and $k \in K$, we have

$$(\mathcal{F}^{-1}(x) \otimes f)^*(k) = \alpha_{k^{-1}}((\mathcal{F}^{-1}(x) \otimes f)(k^{-1})^*) = \overline{\mathcal{F}^{-1}(x)(k^{-1})}\alpha_{k^{-1}}(f^*)$$

using the definition of the involution in $C_c(K, C(K))$, and

$$(\mathcal{F}^{-1} \otimes \text{id})((x \bowtie f)^*)(k) = \mathcal{F}^{-1}(x^*_{(2)})(k)(x^*_{(1)}, f^*_{(1)})f^*_{(2)}(S(x^*_{(3)}), f^*_{(3)}).$$

using (1.21). Note that if $g \in \mathcal{O}(K)$ and $y \in \mathcal{D}(K)$ with $\mathcal{F}(g) = y$, then for $h \in \mathcal{O}(K)$ we have

$$\begin{aligned}
\mathcal{F}(g)^*(h) &= (\mathcal{F}(g)^*, h) \\
&= \overline{(\mathcal{F}(g), S(h^*))} \qquad \text{(by Definition 1.52)} \\
&= \overline{\int_K g(k)S(h^*)(k) \text{ d}k}
\end{aligned}$$

$$
\begin{aligned}
&= \int_K \overline{g(k)} h(k^{-1}) \, \mathrm{d}k \\
&= \int_K \overline{g(k^{-1})} h(k) \, \mathrm{d}k \\
&= \int_K S(g^*)(k) h(k) \, \mathrm{d}k \\
&= \mathcal{F}(S(g^*))(h).
\end{aligned}
$$

It follows that $\mathcal{F}^{-1}(y^*) = S(\mathcal{F}^{-1}(y)^*)$. Therefore

$$
\begin{aligned}
(\mathcal{F}^{-1} \otimes \mathrm{id})((x \bowtie f)^*)(k) &= \mathcal{F}^{-1}(x^*)(k) \alpha_{k^{-1}}(f^*) \\
&= \overline{\mathcal{F}^{-1}(x)(k^{-1})} \alpha_{k^{-1}}(f^*) \\
&= ((\mathcal{F}^{-1} \otimes \mathrm{id})(x \bowtie f))^*(k)
\end{aligned}
$$

as required.

Let us now prove that $C_r^*(G_1) \cong K \ltimes_{\mathrm{adj},r} C(K)$. The $*$-algebra $\mathcal{D}(G_1)$ acts on $L^2(\widehat{G_1})$ by left multiplication and on $L^2(K, L^2(K))$ by the regular representation. We need to identify $L^2(\widehat{G_1})$ with $L^2(K, L^2(K))$ in such a way that these actions are identified. This would then show the $*$-homorphism defined in (2.5) above preserves the reduced norms. In particular, we would obtain the desired reduced isomorphism.

We should recall from Remark 2.1 that $\mathcal{D}(G_1)$ is the dual of an algebraic quantum group, which as a $*$-algebra is given by

$$
\mathcal{O}(G_1) := \mathcal{O}(K) \odot \mathcal{D}(K).
$$

It can be shown (see [86, Proposition 3.11]) that the left integral on $\mathcal{O}(G_1)$ is given by $\phi_K \otimes \phi_{\widehat{K}}$, where ϕ_K is the Haar state on $C(K)$ and $\phi_{\widehat{K}}$ is the left integral on $\mathcal{D}(K)$. Therefore the GNS space corresponding to $\phi_K \otimes \phi_{\widehat{K}}$, denoted by $L^2(G_1)$, is given by

$$
L^2(G_1) \cong L^2(K) \otimes L^2(\widehat{K}) \cong L^2(K, L^2(K))
$$

where the latter isomorphism is given by $\mathrm{id} \otimes \mathcal{F}^{-1}$.

In the same way as for the duality between discrete and compact quantum groups, we have a unitary Fourier transform

$$
\mathcal{F}_{G_1} : L^2(G_1) \to L^2(\widehat{G_1})
$$

given by the formula $\mathcal{F}_{G_1} = \mathcal{F} \otimes \mathcal{F}^{-1}$ (see [86, Theorem 1.13]). Then for $x \bowtie f \in \mathcal{D}(G_1)$

viewed as bounded operators on $L^2(\widehat{G}_1)$, we need to check that the diagram

$$\begin{tikzcd} L^2(\widehat{G_1}) \arrow[r, "\mathcal{F}^{-1}\otimes\mathcal{F}"] \arrow[d, "x\bowtie f"'] & L^2(G_1) \arrow[r, "\mathrm{id}\otimes\mathcal{F}^{-1}"] & L^2(K)\otimes L^2(K) \arrow[d, "(U\ltimes\rho)((\mathcal{F}^{-1}\otimes\mathrm{id})(x\bowtie f))"] \\ L^2(\widehat{G_1}) \arrow[r, "\mathcal{F}^{-1}\otimes\mathcal{F}"] & L^2(G_1) \arrow[r, "\mathrm{id}\otimes\mathcal{F}^{-1}"] & L^2(K)\otimes L^2(K) \end{tikzcd}$$

commutes. It suffices to check that for $y \bowtie g \in L^2(\widehat{G_1})$, where $y \in \mathcal{D}(K_q)$ and $g \in \mathcal{O}(K_q)$, we have

$$(\mathcal{F}^{-1} \otimes \mathrm{id})((x \bowtie f)(y \bowtie g)) = (U \ltimes \rho)((\mathcal{F}^{-1} \otimes \mathrm{id})(x \bowtie f))(\mathcal{F}^{-1} \otimes \mathrm{id})(y \bowtie g).$$

For $k \in K$,

$$\begin{aligned} &((U \ltimes \rho)((\mathcal{F}^{-1} \otimes \mathrm{id})(x \bowtie f))(\mathcal{F}^{-1} \otimes \mathrm{id})(y \bowtie g))(k) \\ &= \int_K (U_s\rho((\mathcal{F}^{-1}(x) \otimes f)(s))(\mathcal{F}^{-1}(y) \otimes g))(k)\ \mathrm{d}s \\ &= \int_K \rho((\mathcal{F}^{-1}(x) \otimes f)(s))(\mathcal{F}^{-1}(y) \otimes g)(s^{-1}k)\ \mathrm{d}s \\ &= \int_K \mathcal{F}^{-1}(x)(s)\mathcal{F}^{-1}(y)(s^{-1}k)\alpha_{k^{-1}s}(f)g\ \mathrm{d}s \\ &= (\mathcal{F}^{-1}(x) \otimes f)(\mathcal{F}^{-1}(y) \otimes g)(k) \\ &= (\mathcal{F}^{-1} \otimes \mathrm{id})((x \bowtie f)(y \bowtie g))(k) \end{aligned}$$

as required.

Now let us turn to the isomorphism $C^*(G_1) \cong K \ltimes_{\mathrm{adj}} C(K)$. Note that the full crossed product is the completion of $C_c(K, C(K))$ in the norm

$$\|f\| := \sup\{\|(\rho \ltimes \pi)(f)\| \mid (\pi, \rho) \text{ is a covariant pair of representations}\}$$

for $f \in C_c(K, C(K))$, see [89, Lemma 2.27]. Note that here $\pi : C(K) \to B(\mathcal{H})$ and $\rho : K \to U(\mathcal{H})$ are representations satisfying the covariance condition of [89, Definition 2.10].

We therefore need to show that if $\nu : \mathcal{D}(G_1) \to B(\mathcal{H})$ is a non-degenerate $*$-representation on a Hilbert space $\mathcal{H}$, then ν extends to a representation on $C_c(K, C(K))$ of the form $\rho \ltimes \pi$ as above. Then $\|-\| = \|-\|_f$ where the latter is as defined in (1.25).

By Proposition 1.62, there exist representations $\pi : \mathcal{O}(K) \to B(\mathcal{H})$ and $\rho : \mathcal{D}(K) \to B(\mathcal{H})$ such that $\nu = \rho \bowtie \pi$. We can extend π to $C(K)$ by universality (see Example 1.21), and ρ to $M(\mathcal{D}(K))$ and hence $C^*(K)$ by Remark 1.14. Then one can check (π, ρ) is a covariant pair using the Yetter-Drinfeld compatibility condition (1.23), and that $\rho \ltimes \pi = \rho \bowtie \pi$ on $\mathcal{D}(G_1)$.

Finally, since K is compact (and hence amenable), the full and reduced norm on the convolution algebra $C_c(K, C(K))$ are equal, see [76, Proposition 2.2]. □

2.1.2 The Principal Series Representations

Throughout this section, for $q \in (0,1]$, we write $G_q = SL_q(2,\mathbb{C})$ and $K_q = SU_q(2)$, with $K_1 = K$.

Let T be the circle group, and consider the associated compact quantum group $C(T)$. Recall from Example 1.27 that $\mathcal{O}(T) = \mathbb{C}[z, z^{-1}] \subseteq C(T)$ and from the proof of Proposition 1.54 that there is a $*$-homomorphism

$$\pi : \mathcal{O}(SU_q(2)) \to \mathcal{O}(T), \quad \pi(\alpha) = z, \quad \pi(\gamma) = 0.$$

Recall that we constructed a basis $\{u^n_{ij}\}$ for $\mathcal{O}(SU_q(2))$ in Section 1.3. For calculations it is convenient to understand π on this basis, which is the content of the next proposition, another folklore result.

Proposition 2.3. *If $\pi : \mathcal{O}(SU_q(2)) \to \mathcal{O}(T)$ is the $*$-homomorphism*

$$\pi : \mathcal{O}(SU_q(2)) \to \mathcal{O}(T), \quad \pi(\alpha) = z, \quad \pi(\gamma) = 0,$$

then we have $\pi(u^n_{ij}) = \delta_{ij} z^{2j}$, where $\{u^n_{ij}\}$ is the basis for $\mathcal{O}(SU_q(2))$ constructed in Section 1.3.

Proof. We split the proof into two cases - for $q < 1$ and for $q = 1$.

For $q < 1$, we have the pairing

$$\langle -, - \rangle_T : \mathbb{C}[K, K^{-1}] \times \mathcal{O}(T) \to \mathbb{C}$$

defined in the proof of Proposition 1.54 by setting

$$\langle K, z \rangle_T = q^{\frac{1}{2}}, \quad \langle K^{-1}, z \rangle_T = q^{-\frac{1}{2}}, \quad \langle K, z^{-1} \rangle_T = q^{-\frac{1}{2}}, \quad \langle K^{-1}, z^{-1} \rangle_T = q^{\frac{1}{2}}$$

and then by extending to the remaining elements according to the rules in Definition 1.52. This pairing is non-degenerate in the second variable, i.e. if $f \in \mathcal{O}(T)$ and we have $\langle X, f \rangle_T = 0$ for all $X \in \mathbb{C}[K, K^{-1}]$, then $f = 0$. This is because if we write

$f = \sum_{m\in\mathbb{Z}} c_m z^m \in \mathcal{O}(T)$, where only finitely many of the $c_m \in \mathbb{C}$ are non-zero and we have, for all $l \in \mathbb{Z}$,

$$\left\langle K^l, \sum_{m\in\mathbb{Z}} c_m z^m \right\rangle = \sum_{m\in\mathbb{Z}} c_m q^{\frac{lm}{2}} = 0,$$

then $c_m = 0$ for all m. Now for $l \in \mathbb{Z}$,

$$\langle K^l, \pi(u^n_{ij}) - \delta_{ij} z^{2j} \rangle_T = \langle K^l, \pi(u^n_{ij})\rangle_T - \delta_{ij}\langle K^l, z^{2j}\rangle_T = \langle K^l, \pi(u^n_{ij})\rangle_T - \delta_{ij} q^{li}.$$

Finally we note that $\langle K^l, \pi(u^n_{ij})\rangle_T = \langle K^l, u^n_{ij}\rangle = \delta_{ij} q^{li}$, by equation (1.14) and the formulae in Theorem 1.49.

For $q = 1$, we can see that π is the restriction of functions in $\mathcal{O}(K)$ to $T \subseteq K$. Let $z = e^{ix}$ for some $x \in \mathbb{R}$. We have

$$\pi(u^n_{ij})\left(\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}\right) = \left\langle e_i, \pi_n\left(\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}\right) e_j \right\rangle_{V(n)},$$

where $\pi_n : K \to U(V(n))$ is the standard $2n+1$-dimensional irreducible representation of K obtained by applying Proposition 1.54 to the standard $2n+1$-dimensional irreducible representation of $U(\mathfrak{sl}_2(\mathbb{C}))$ constructed in Section 1.3, which we also denote by π_n. By Remark 1.55 the following diagram commutes,

$$\begin{array}{ccccc} \mathfrak{t} & \hookrightarrow & \mathfrak{k} & \xrightarrow{\pi_n} & B(V(n)) \\ \downarrow{\scriptstyle\exp} & & \downarrow{\scriptstyle\exp} & & \downarrow{\scriptstyle\exp} \\ T & \hookrightarrow & K & \xrightarrow{\pi_n} & U(V(n)) \end{array}$$

where exp is the matrix exponential. Since, for $x \in \mathbb{R}$,

$$\pi_n(xH) = x\begin{pmatrix} -2n & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 2n \end{pmatrix}$$

by the formulae in Theorem 1.49, we have

$$\pi_n\left(\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}\right) = \begin{pmatrix} e^{-2nix} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{2nix} \end{pmatrix} = \begin{pmatrix} z^{-2n} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & z^{2n} \end{pmatrix}.$$

Then $\pi(u^n_{ij})\left(\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}\right) = \delta_{ij} z^{2j}$ as required. □

The Haar state ϕ_{K_q} on $C(K_q)$ provided by Theorem 1.29 defines a semi-definite form

$$\langle f, g\rangle_{K_q} := \phi_{K_q}(f^* g), \quad f, g \in C(K_q).$$

In fact the Haar state is faithful on $C(K_q)$ (see [60, Theorem 1.1]), and so this form defines an inner product on $C(K_q)$, and therefore $L^2(K_q)$ as defined in (1.2) is the completion of $C(K_q)$ with respect to this inner product.

By Theorem 1.58, $\{u^n_{ij}\}$ forms an orthogonal basis for $L^2(K_q)$, with

$$\left\|u^n_{ij}\right\|^2_{L^2(K_q)} = \phi_{K_q}((u^n_{ij})^* u^n_{ij}) = \frac{q^{2i}}{[2n+1]}. \tag{2.6}$$

We can then normalise this basis to obtain an orthonormal basis of $L^2(K_q)$, which we denote by $\{e^n_{ij}\}$, where

$$e^n_{ij} := q^{-i}\sqrt{[2n+1]}u^n_{ij}. \tag{2.7}$$

For $m \in \frac{1}{2}\mathbb{Z}$, define

$$\mathcal{O}(\mathcal{E}^q_m) := \left\{f \in \mathcal{O}(K_q) \mid (\mathrm{id} \otimes \pi)\Delta(f) = f \otimes z^{-2m}\right\}, \tag{2.8}$$

a subspace of the Hilbert space $L^2(K_q)$. Equip $\mathcal{O}(\mathcal{E}^q_m)$ with the inner product restricted from $C(K_q)$. We denote the closure of $\mathcal{O}(\mathcal{E}^q_m)$ in $L^2(K_q)$ by $\mathcal{H}^q_m$.

Proposition 2.4. *An orthogonal basis for $\mathcal{H}^q_m$ is given by $\{u^n_{i\,-m}\}$, where $n \geq |m|$, $n+m \in \mathbb{Z}$ and $i \in \{-n, -n+1, ..., n\}$.*

Proof. Let $f = \sum_{n,i,j} c^n_{ij} u^n_{ij} \in \mathcal{O}(K_q)$, where $c^n_{ij} \in \mathbb{C}$. The condition for $f \in \mathcal{O}(\mathcal{E}^q_m)$ means that

$$\sum_{n,i,j} c^n_{ij}(u^n_{ij} \otimes z^{2j}) = \left(\sum_{n,i,j} c^n_{ij} u^n_{ij}\right) \otimes z^{-2m}$$

by Proposition 2.3, which is equivalent to

$$\sum_{n,i,j} u^n_{ij} \otimes c^n_{ij}(z^{2j} - z^{-2m}) = 0.$$

Since the u^n_{ij} are linearly independent, we must have $c^n_{ij}(z^{2j} - z^{-2m}) = 0$. Therefore either $c^n_{ij} = 0$ or $z^{2j} = z^{-2m}$ for all z. Suppose for some n', i', j' that $c^{n'}_{i'j'} \neq 0$. Then $z^{2j'} = z^{-2m}$ for all $z \in \mathbb{C}$ and so $j' = -m$. If $j \neq -m$, then $z^{-2m} \neq z^{2j}$ and so $c^n_{ij} = 0$. In particular, $f = \sum_{n,i} c^n_{i\,-m} u^n_{i\,-m}$. Note that for $u^n_{i\,-m}$ to be defined, we require that $-n \leq -m \leq n$ (or equivalently $n \geq |m|$) and that there exists $k \in \mathbb{N}$ such that $n - k = -m$ (or equivalently $n + m \in \mathbb{Z}$).

Finally note that a direct calculation using Proposition 2.3 shows that an element of $\mathcal{O}(K_q)$ of the form $\sum_{n,i} c^n_{i\ -m} u^n_{i\ -m}$ with $n \geq |m|$ and $n + m \in \mathbb{Z}$ is contained in $\mathcal{O}(\mathcal{E}^q_m)$. □

Remark 2.5. By Proposition 2.4, $\mathcal{H}^q_m \cong \ell^2(\{e^n_{i\ -m}\}) := \mathcal{H}_m$, independently of q.

In the case of $SL(2,\mathbb{C})$, there are the so-called principal series representations of $SL(2,\mathbb{C})$, originally due to Gelfand and Naimark in [26]. These are obtained by inducing characters of the upper-triangular subgroup of $SL(2,\mathbb{C})$ to $SL(2,\mathbb{C})$, see [52, §2]. The principal series representations of $SL(2,\mathbb{C})$ allow one to describe $C^*_r(SL(2,\mathbb{C}))$ concretely, because the regular representation may be decomposed, in an appropriate sense, into principal series representations using Harish-Chandra's Plancherel formula, [31].

Remark 2.6. The aforementioned description of $C^*_r(SL(2,\mathbb{C}))$ is given in [64, Proposition 4.1]. We will return, in more detail, to the classical case in Chapter 4. For now the reader should note that an analogue of the principal series for the quantum setting would be desirable, so that we can describe $C^*_r(SL_q(2,\mathbb{C}))$.

A complete classification of the irreducible unitary representations of $SL_q(2,\mathbb{C})$ was given by Pusz in [68], including the quantum principal series representations (see also the remarks in [69, p.g. 1-2]). However these representations are introduced in terms of the function algebra $C_0(SL_q(2,\mathbb{C}))$ (see (2.2)). We will instead proceed by introducing the principal series representations in terms of Yetter-Drinfeld modules, following the approach of Voigt and Yuncken in [86, Section 5.3].

Recall from Section 1.4 that (non-degenerate) representations of $\mathcal{D}(G_q)$ correspond to unitary Yetter-Drinfeld modules as in Definition 1.61. Therefore a quantum analogue of the principal series representations can be described as a family of unitary Yetter-Drinfeld modules, as follows.

Proposition 2.7. *Let $q < 1$. Define, for $m \in \frac{1}{2}\mathbb{Z}$ and $\lambda \in i\mathbb{R}$,*

$$\mathcal{O}(K_q) \to B(\mathcal{O}(\mathcal{E}^q_m)), \quad f \mapsto f\cdot \tag{2.9}$$

where for $f \in \mathcal{O}(K_q)$ and $\xi \in \mathcal{O}(\mathcal{E}^q_m)$ we define

$$f \cdot \xi = f_{(1)} \xi S(f_{(3)}) \left(K^{2+2\lambda}, f_{(2)}\right).$$

Here K is as in Definition 1.42. For $m \in \frac{1}{2}\mathbb{Z}$, we define

$$\mathcal{D}(K_q) \to B(\mathcal{O}(\mathcal{E}^q_m)), \quad \omega \mapsto \omega\cdot \tag{2.10}$$

where for $\omega \in \mathcal{D}(K_q)$ and $\xi \in \mathcal{O}(\mathcal{E}^q_m)$ we define

$$\omega \cdot \xi = \big(S(\omega), \xi_{(1)}\big)\, \xi_{(2)}.$$

Then $\mathcal{O}(\mathcal{E}^q_m)$ equipped with (2.9) and (2.10) is a unitary Yetter-Drinfeld module for $\mathcal{D}(G_q)$.

Proof. This is a direct calculation - one checks that the condition (1.23) is satisfied. $\square$

Note that in the first formula in Proposition 2.7, we use the fact that $U_q(\mathfrak{sl}_2(\mathbb{C})) \hookrightarrow \prod_{m\in\frac{1}{2}\mathbb{N}_0} M_{2m+1}(\mathbb{C})$ (see (1.16)) to be able to calculate $K^{2+2\lambda}$, and we extend the pairing

$$(-,-) : \mathcal{D}(SU_q(2)) \times \mathcal{O}(SU_q(2)) \to \mathbb{C}, \quad (\omega, f) \mapsto \omega(f)$$

to

$$(-,-) : \prod_{n\in\frac{1}{2}\mathbb{N}_0} M_{2n+1}(\mathbb{C}) \times \mathcal{O}(SU_q(2)) \to \mathbb{C}.$$

We refer the reader back to Example 1.59 for the full details.

Recall from Example 1.59 that we have a basis $\{\omega^n_{ij}\}$ of $\mathcal{D}(K_q)$, dual to the basis $\{u^n_{ij}\}$ of $\mathcal{O}(K_q)$. By Proposition 1.40, each $\omega \in \mathcal{D}(K_q)$ gives rise to a bounded operator on $L^2(K_q)$, and we see from the formula there that the action of ω on $\mathcal{O}(\mathcal{E}^q_m)$ given in Proposition 2.7 is just the restriction of the former operator to $\mathcal{O}(\mathcal{E}^q_m)$. We wish to understand this restricted operator.

Let us start by calculating a formula for the action of the basis elements of $\mathcal{D}(K_q)$ on the basis elements of $\mathcal{O}(K_q) \subseteq L^2(K_q)$.

Lemma 2.8. *The action of $\mathcal{D}(K_q)$ as bounded linear operators on $L^2(K_q)$ (as defined in Proposition 1.40) is given by the formula*

$$\omega^m_{kl} \cdot u^n_{ij} = \delta_{nm}\delta_{l\ -i}(-1)^{k+i} q^{-k-i} u^m_{-k\ j}$$

for $m, n \in \frac{1}{2}\mathbb{N}_0$ and $k, l \in \{-m, -m+1, ..., m\}$ and $i, j \in \{-n, -n+1, ..., n\}$.

Proof. From Example 1.59, we have

$$\mathcal{F}^{-1}(\omega^n_{ij}) = q^{2j}[2n+1](u^n_{ij})^* \tag{2.11}$$

and so

$$\mathcal{F}((u^n_{ij})^*) = q^{-2j}[2n+1]^{-1}\omega^n_{ij}.$$

Since

$$(u^n_{ij})^* = (-1)^{2n+i+j} q^{j-i} u^n_{-i\ -j}, \tag{2.12}$$

(see (1.19)), we have

$$\mathcal{F}(u^n_{-i\ -j}) = (-1)^{-2n-i-j} q^{-j+i} (q^{-2j}[2n+1]^{-1}\omega^n_{ij}).$$

Replacing $-i$ with i and $-j$ with j, we see that

$$\mathcal{F}(u^n_{ij}) = (-1)^{-2n+i+j} q^{j-i} (q^{2j}[2n+1]^{-1}\omega^n_{-i\ -j}).$$

Then

$$\begin{aligned}
\omega^m_{kl}\mathcal{F}(u^n_{ij}) &= \omega^m_{kl}((-1)^{-2n+i+j} q^{j-i}(q^{2j}[2n+1]^{-1}\omega^n_{-i\ -j})) \\
&= \delta_{nm}\delta_{l,-i}(-1)^{-2n+i+j} q^{j-i}(q^{2j}[2n+1]^{-1}\omega^m_{k\ -j}) \\
&= \delta_{nm}\delta_{l,-i}(-1)^{-2m+i+j} q^{j-i}(q^{2j}[2n+1]^{-1}\omega^m_{k\ -j}).
\end{aligned}$$

Applying $\mathcal{F}^{-1}$, and then using (2.11) and (2.12) we have

$$(\mathcal{F}^{-1}\omega^m_{kl}\mathcal{F})(u^n_{ij}) = \delta_{nm}\delta_{l,-i}(-1)^{-2m+i+j} q^{j-i}(u^m_{k\ -j})^* = \delta_{nm}\delta_{l,-i}(-1)^{k+i} q^{-k-i} u^m_{-k\ j}. \qquad \square$$

Proposition 2.9. *Let $m \in \frac{1}{2}\mathbb{Z}$. Then, for $l \in \frac{1}{2}\mathbb{N}_0$, $k, j \in \{-l, -l+1, ..., l\}$, ω^l_{kj} acts as a finite rank operator on $\mathcal{H}_m$, with image independent of q.*

Proof. By Lemma 2.8, we have, for $n \in \frac{1}{2}\mathbb{N}_0, n \geq |m|, n+m \in \mathbb{Z}$ and $i \in \{-n, -n+1, ..., n\}$,

$$\omega^l_{kj} \cdot u^n_{i\ -m} = \delta_{nl}\delta_{j,-i}(-1)^{k+i} q^{-k-i} u^l_{-k\ -m}$$

so we obtain using (2.7)

$$\omega^l_{kj} \cdot e^n_{i\ -m} = \delta_{nl}\delta_{j,-i}(-1)^{k+i} q^{-2k-2i} e^l_{-k\ -m}$$

In particular, if $X = \sum_{n,i} c^n_{i\ -m} e^n_{i\ -m} \in \mathcal{O}(\mathcal{E}^q_m)$, then

$$\begin{aligned}
\omega^l_{kj} \cdot X &= \sum_{n,i} c^n_{i\ -m}\omega^l_{kj} \cdot e^n_{i\ -m} \\
&= \sum_{n,i} c^n_{im}\delta_{nl}\delta_{j\ -i}(-1)^{k+i} q^{-2k-2i} e^l_{-k\ -m} \\
&= c^l_{-jm}(-1)^{k-j} q^{-2k+2j} e^l_{-k\ -m}
\end{aligned}$$

and so $\mathrm{Im}(\omega^l_{kj}\cdot) = \mathrm{span}(e^l_{-k\ -m})$. $\square$

By Proposition 2.7 and our remarks in Section 1.4 we obtain, for each $m \in \frac{1}{2}\mathbb{Z}$ and $\lambda \in i\mathbb{R}$, a non-degenerate $*$-homomorphism

$$\pi^q_{(m,\lambda)} : \mathcal{D}(G_q) \to B(\mathcal{H}^q_m).$$

By Proposition 2.9, the image of $\pi^q_{(m,\lambda)}$ is contained in $K(\mathcal{H}^q_m)$. The universal enveloping algebra of $\mathcal{D}(G_q)$ is $C^*(G_q)$, and so we obtain a non-degenerate $*$-homomorphism

$$\pi^q_{(m,\lambda)} : C^*(G_q) \to K(\mathcal{H}^q_m). \tag{2.13}$$

For a classical group G, a non-degenerate representation of $C^*(G)$ on a Hilbert space $\mathcal{H}$ corresponds to a unitary representation of the group G on $\mathcal{H}$, see [24, pg. 73]. Therefore, we may view $\pi^q_{(m,\lambda)}$ as a representation of the underlying quantum group G_q, and we call these the (unitary) **principal series representations** of G_q. We will sometimes denote $\mathcal{H}^q_m$ by $\mathcal{H}^q_{(m,\lambda)}$ in this case to emphasise the dependence of the actions on λ.

The dependence on λ of the actions in Proposition 2.7 is periodic. Indeed, the only action that depends on λ in Proposition 2.7 is the action of $\mathcal{O}(K_q)$. There the dependence on λ comes from the pairing $\left(K^{2+2\lambda}, -\right)$, where $K \in U_q(\mathfrak{sl}_2(\mathbb{C}))$. For each $n \in \frac{1}{2}\mathbb{N}_0$, recall (c.f. (1.15)) that K is represented on $V(n)$ by the matrix

$$\begin{pmatrix} q^n & & & \\ & q^{n-1} & & \\ & & \ddots & \\ & & & q^{-n} \end{pmatrix}$$

with respect to the standard orthonormal weight basis for $V(n)$ constructed in Theorem 1.49. Since $q \in (0,1]$, there exists $h \in (-\infty, 0]$ such that $q = e^h$. Then if $\lambda = \mu + \frac{2\pi i}{h}$,

$$q^{n(2+2\lambda)} = q^{n(2+2\mu+\frac{4\pi i}{h})} = q^{n(2+2\mu)} q^{\frac{4n\pi i}{h}} = q^{n(2+2\mu)} e^{4n\pi i}$$

for all $n \in \frac{1}{2}\mathbb{N}_0$. Since $n \in \frac{1}{2}\mathbb{N}_0$, $4n \in 2\mathbb{Z}$ and so $e^{4n\pi i} = 1$. Therefore $\pi^q_{(m,\lambda)} = \pi^q_{(m,\lambda+\frac{2\pi i}{h})}$. In particular the principal series representations of G_q are indexed by points in

$$M_q := \frac{1}{2}\mathbb{Z} \times {}^{i\mathbb{R}}\!\big/\!{}_{2\pi i h^{-1}\mathbb{Z}}. \tag{2.14}$$

In what follows, we will write $(m, \lambda) \in M_q$, even when $\lambda \in i\mathbb{R}$. Strictly speaking we are picking a representative of λ in ${}^{i\mathbb{R}}\!\big/\!{}_{2\pi i h^{-1}\mathbb{Z}}$, but the above periodicity in our irreducible representations means the choice of representative is not important.

We have the following useful, but non-trivial facts about principal series representations. These were known to Pusz [68], but we refer the reader to [86, Theorem 5.49] for the statement in the same framework which we have set up here.

Theorem 2.10. *Let $m \in \mathbb{Z}$ and $\lambda \in i\mathbb{R}$.*

(a) $\pi^q_{(m,\lambda)}$ is an irreducible representation of G_q on $\mathcal{H}^q_m$.

(b) $\pi^q_{(m,\lambda)}$ and $\pi^q_{(m',\lambda')}$ are equivalent if and only if

$$(m,\lambda) = (m',\lambda') \quad \text{or} \quad (m,\lambda) = (-m',-\lambda').$$

Let $W = \mathbb{Z}_2 = \{1,-1\}$. This is called the **Weyl group** (we will see the reasoning for this terminology when we come back to the classical setting). We define an action of W on M_q by $w(m,\lambda) = (wm, w\lambda)$. By Theorem 2.10, two representations $\pi^q_{(m,\lambda)}$ and $\pi^q_{(n,\mu)}$ are equivalent if and only if (m,λ) and (n,μ) are in the same W-orbit.

The quotient ${M_q}/{W}$ can be viewed as a subset of M_q obtained by identifying points $(m,\lambda) \sim (-m,-\lambda)$. Therefore in ${M_q}/{W}$, no two distinct points represent equivalent irreducible representations.

We now turn to describing the reduced group C^*-algebra of G_q. For this, we require several constructions.

For each point in $(m,\lambda) \in M_q$, we have an associated Hilbert space $\mathcal{H}^q_{(m,\lambda)}$, the carrier space of the principal series representation with parameter (m,λ). We can define a bundle $\mathcal{H}^q$ over M_q with surjection

$$p^q : \bigsqcup_{m \in \frac{1}{2}\mathbb{Z}} \mathcal{H}^q_m \times {i\mathbb{R}}/{2\pi i h^{-1}\mathbb{Z}} \to M_q \tag{2.15}$$

sending $(v_m,\lambda) \in \mathcal{H}^q_m \times {i\mathbb{R}}/{2\pi i h^{-1}\mathbb{Z}}$ to $(m,\lambda) \in M_q$. The fibre over $(m,\lambda) \in M_q$ is $\mathcal{H}^q_{(m,\lambda)} \cong \mathcal{H}^q_m$. Therefore each fibre is a Hilbert space. We in fact have a Hilbert bundle in the sense of Fell, [22, Section 1.1]. We refer the reader to [23, Chapter 2, Section 13] for an account of the background theory.

Note that when we fix $m \in \frac{1}{2}\mathbb{Z}$, the restriction of the bundle $\mathcal{H}^q$ to $\{m\} \times {i\mathbb{R}}/{2\pi i h^{-1}\mathbb{Z}} \subseteq M_q$ is the trivial Hilbert bundle

$$\mathcal{H}^q_m \times {i\mathbb{R}}/{2\pi i h^{-1}\mathbb{Z}} \to {i\mathbb{R}}/{2\pi i h^{-1}\mathbb{Z}}$$

therefore $\mathcal{H}^q$ is a locally trivial bundle.

The continuous sections of the bundle vanishing at infinity is a Hilbert $C_0(M_q)$-module. Taking the compact operators on this Hilbert module gives a continuous field of C^*-

algebras over M_q, with fibres $K(\mathcal{H}^q_{(m,\lambda)})$, see [17, 10.7]. We denote this field by $C_0(M_q, K(\mathcal{H}^q))$. The local triviality of the bundle means that this field is a trivial field when restricted to $\{m\} \times {}^{i\mathbb{R}}\!/\!_{2\pi i h^{-1}\mathbb{Z}}$, and so

$$C_0(M_q, K(\mathcal{H}^q)) = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} C\left({}^{i\mathbb{R}}\!/\!_{2\pi i h^{-1}\mathbb{Z}}, K(\mathcal{H}^q_m)\right). \tag{2.16}$$

Note that Theorem 2.10 (b) provides unitaries $U^q_{(m,\lambda)} : \mathcal{H}^q_{(m,\lambda)} \to \mathcal{H}^q_{(-m,-\lambda)}$ which intertwine $\pi^q_{(m,\lambda)}$ and $\pi^q_{(-m,-\lambda)}$. These unitaries are known to be unique up to a scalar multiple of modulus 1, [86, Theorem 5.42]. In fact, Voigt and Yuncken provide explicit formulae there for these intertwiners. We fix the aforementioned scalar so that the formulae described there hold.

We obtain an action of W on $C_0(M_q, K(\mathcal{H}^q))$ given by

$$(-1 \cdot f)(m, \lambda) = (U^q_{(m,\lambda)})^{-1} f(-m, -\lambda) U^q_{(m,\lambda)}. \tag{2.17}$$

for $f \in C_0(M_q, K(\mathcal{H}^q))$, $(m, \lambda) \in M_q$. The fact this formula defines an element of $C_0(M_q, K(\mathcal{H}^q))$ follows from the aforementioned formula in [86, Theorem 5.42] - these formulae are continuous in λ.

We have the following result which determines the reduced group C^*-algebra. The proof is due to appear in [85], and we provide the proof that will be given there. Recall that if a group G acts on a C^*-algebra A, the elements of A fixed by all elements of G are denoted by A^G.

Theorem 2.11. *Let $q \in (0, 1)$ and let $\mathcal{H}^q = (\mathcal{H}^q_{(m,\lambda)})$ be the Hilbert space bundle of unitary principal series representations of G_q over M_q introduced in (2.15). Then there is an isomorphism*

$$C^*_r(G_q) \cong C_0(M_q, K(\mathcal{H}^q))^W.$$

Proof. First, we consider a unitary principal series representation with parameter $(m, \lambda) \in M_q$,

$$\pi^q_{(m,\lambda)} : C^*(G_q) \to K(\mathcal{H}^q_m).$$

In fact, this $*$-homomorphism factors through the reduced group C^*-algebra. This is a consequence of the Plancherel theorem for G_q, which will appear in greater generality in [85], but is already known for G_q, see [5, Theorem 12]. Therefore we obtain a $*$-homomorphism

$$\pi^q_{(m,\lambda)} : C^*_r(G_q) \to K(\mathcal{H}^q_m).$$

We can consider the $*$-homomorphism

$$C_r^*(G_q) \to C_0(M_q, K(\mathcal{H}^q)), \quad f \mapsto ((m,\lambda) \mapsto \pi^q_{(m,\lambda)}(f)). \tag{2.18}$$

Note that the image of (2.18) is contained in $C_0(M_q, K(\mathcal{H}^q))^W$ by construction. Another application of the Plancherel theorem tells us that (2.18) is injective. It remains to check surjectivity.

The irreducible representations of $C_0(M_q, K(\mathcal{H}^q))$ are described as follows. If $(m,\lambda) \in M_q$, then we define

$$\tilde{\pi}_{(m,\lambda)} : C_0(M_q, K(\mathcal{H}^q)) \to K(\mathcal{H}^q_m), \quad \tilde{\pi}_{(m,\lambda)}(f) = f(m,\lambda)$$

for $C_0(M_q, K(\mathcal{H}^q))$. This is an irreducible representation of $C_0(M_q, K(\mathcal{H}^q))$, and each irreducible representation of $C_0(M_q, K(\mathcal{H}^q))$ is of this form - indeed, see the description of $C_0(M_q, K(\mathcal{H}^q))$ given in (2.16). Note that if we consider $C_r^*(G_q) \subseteq C_0(M_q, K(\mathcal{H}^q))$ by (2.18) then for $(m,\lambda) \in M_q$, we have $\tilde{\pi}_{(m,\lambda)}\big|_{C_r^*(G_q)} = \pi^q_{(m,\lambda)}$. The result then follows from standard results about subalgebras of postliminal C^*-algebras, see [17, 11.1.1 and 11.1.6]. $\square$

One should compare Theorem 2.11 to the classical case originally given in [64, Proposition 4.1]. We will return to this in Chapter 4.

2.2 The Quantum Assembly Field

Throughout this section, for $q \in (0,1]$, we write $G_q = SL_q(2,\mathbb{C})$ and $K_q = SU_q(2)$, with $K_1 = K$.

The objective of the rest of this chapter is to construct, for a fixed $0 < q_0 < 1$, a continuous $C([q_0,1])$-algebra A^Q with fibres $A^Q_q = C_r^*(G_q)$.

2.2.1 Constructing the Field

Consider

$$B := \prod_{q \in [q_0,1]} B(L^2(\widehat{G_q})),$$

the C^*-algebra of norm bounded sections with values in the family $B(L^2(\widehat{G_q}))$. The norm is given by

$$\|b\| := \sup_{q\in[q_0,1]} \|b(q)\|, \quad b \in B.$$

For each n and m in $\frac{1}{2}\mathbb{N}_0$, and $i, j \in \{-n, -n+1, ..., n\}$ and $k, l \in \{-m, -m+1, ..., m\}$, and for each $q \in (0, 1]$, we have the element

$$\omega^n_{ij} \bowtie u^m_{kl} \in \mathcal{D}(G_q). \tag{2.19}$$

Since $\mathcal{D}(G_q) = \mathcal{D}(K_q) \odot \mathcal{O}(K_q)$ as a vector space, these elements form a basis for $\mathcal{D}(G_q)$.

Let us denote the elements in (2.19) by $(\omega^n_{ij} \bowtie u^m_{kl})(q)$ to signify the fact that this element is in $\mathcal{D}(G_q)$. We have constructed a section

$$q \mapsto (\omega^n_{ij} \bowtie u^m_{kl})(q) \tag{2.20}$$

of the algebraic direct product alg $- \prod_{q\in[q_0,1]} \mathcal{D}(G_q)$. We now show that the sections (2.20) are norm bounded, and so define elements of B. That is, we need to show that $\sup_{q\in[q_0,1]} \left\|(\omega^n_{ij} \bowtie u^m_{kl})(q)\right\|_r < \infty$, where $\|-\|_r$ denotes the norm of $C^*_r(G_q)$.

Note that it suffices to show $\sup_{q\in[q_0,1]} \left\|(\omega^n_{ij} \bowtie u^m_{kl})(q)\right\|_f < \infty$, where $\|-\|_f$ is the norm of $C^*(SL_q(2,\mathbb{C}))$, because the reduced norm is bounded above by the full norm. By equations (1.25) and (1.26),

$$\left\|(\omega^n_{ij} \bowtie u^m_{kl})(q)\right\|_f \leq 1$$

and so

$$\sup_{q\in[q_0,1]} \left\|(\omega^n_{ij} \bowtie u^m_{kl})(q)\right\|_f \leq 1$$

as required.

Let A denote the $*$-subalgebra of B generated by the sections $q \mapsto (\omega^n_{ij} \bowtie u^m_{kl})(q)$. Note that $C([q_0, 1]) \subseteq B$ as a C^*-subalgebra as the sections

$$q \mapsto f(q) \cdot 1_{L^2(\widehat{G_q})}.$$

Let $\mathcal{D}(\mathcal{G}) = C([q_0, 1])A$, again a $*$-subalgebra of B. Let A^Q be the C^*-subalgebra of B generated by $\mathcal{D}(\mathcal{G})$. We will show that A^Q is a $C([q_0, 1])$-algebra, see Definition A.1.

If $\gamma \in \mathcal{D}(\mathcal{G})$, then we may write $\gamma = \sum_{i\in I} f_i\gamma_i$, where $f_i \in C([q_0, 1])$, $\gamma_i \in A$, and I is a finite indexing set. Therefore if $f \in C([q_0, 1])$, $f\gamma = \sum_{i\in I} f(f_i\gamma_i) = \sum_{i\in I}(ff_i)\gamma_i \in \mathcal{D}(\mathcal{G})$. Therefore if $f \in C([q_0, 1])$, we can define the multiplication operator

$$M_f : \mathcal{D}(\mathcal{G}) \to \mathcal{D}(\mathcal{G}) \subseteq A^Q, \quad M_f(\gamma) = f\gamma, \quad \gamma \in \mathcal{D}(\mathcal{G}).$$

Note that M_f is bounded on $\mathcal{D}(\mathcal{G})$ because

$$\|M_f\gamma\|_{A^Q} = \|f\gamma\|_{A^Q} = \|f\gamma\|_B \leq \|f\|_B \, \|\gamma\|_B = \|f\|_\infty \cdot \|\gamma\|_{A^Q}$$

and so M_f extends to a bounded linear operator $M_f : A^Q \to A^Q$. Consider A^Q as a Hilbert-A^Q module, with A^Q-valued inner product

$$\langle \gamma, \mu \rangle_{A^Q} = \gamma^* \mu, \quad \gamma, \mu \in A^Q.$$

Let us show that M_f is an adjointable operator. For $\gamma, \mu \in \mathcal{D}(\mathcal{G})$ we have

$$\langle M_f\gamma, \mu \rangle = (M_f\gamma)^*\mu = (f\gamma)^*\mu = \gamma^*(f^*\mu) = \gamma^*(M_{f^*}\mu) = \langle \gamma, M_{f^*}\mu \rangle.$$

From the continuity of the inner product, M_f and M_{f^*}, we have that M_f is adjointable with adjoint M_{f^*}. In particular, M_f defines a multiplier of A^Q. Since

$$(M_f\gamma)\mu = \gamma(M_f\mu)$$

for all $\gamma, \mu \in \mathcal{D}(\mathcal{G})$, we have that $M_f \in ZM(A^Q)$.

The map

$$\phi : C([q_0, 1]) \to ZM(A^Q), \quad f \mapsto M_f$$

is clearly a unital $*$-homomorphism. In particular, A^Q is a $C([q_0, 1])$-algebra. In the following section we identify the fibres of A^Q.

2.2.2 Identifying the Fibres of the Assembly Field

For each $q \in [q_0, 1]$ we have a canonical evaluation map $\mathrm{ev}_q : B \to B(L^2(\widehat{G_q}))$, which we can restrict to $A^Q \subseteq B$. If $\gamma \in \mathcal{D}(\mathcal{G})$, we can write

$$\gamma = \sum_{n,m,i,j,k,l} f^{n,m}_{i,j,k,l} \, \omega^n_{ij} \bowtie u^m_{kl}$$

where the indices run over a finite set, and each $f^{n,m}_{i,j,k,l} \in C([q_0, 1])$. Then

$$\mathrm{ev}_q(\gamma) = \sum_{n,m,i,j,k,l} f^{n,m}_{i,j,k,l}(q)(\omega^n_{ij} \bowtie u^m_{kl})(q) \in \mathcal{D}(G_q).$$

In fact, as the elements

$$\{(\omega^n_{ij} \bowtie u^m_{kl})(q)\}_{n,m,i,j,k,l} \tag{2.21}$$

form a basis for $\mathcal{D}(G_q)$, we have that $\mathrm{ev}_q(\mathcal{D}(\mathcal{G})) = \mathcal{D}(G_q)$. Therefore $\mathrm{ev}_q(A^Q) = C_r^*(G_q)$.

It is not clear that this canonical evaluation map ev_q is the quotient map to the fibre of A^Q at q. However, we will show that this is indeed the case.

By the definition of the fibre of a $C_0(X)$-algebra (A.1), we need to show

$$A_q^Q := {A^Q}\big/{J_q A^Q} \cong C_r^*(G_q)$$

where J_q is the ideal of functions in $C([q_0, 1])$ vanishing at q. Note that $J_q A^Q \subseteq \mathrm{Ker}(\mathrm{ev}_q)$ and so there is a canonical surjective $*$-homomorphism

$$\nu_q : {A^Q}\big/{J_q A^Q} \to {A^Q}\big/{\mathrm{Ker}(\mathrm{ev}_q)} \cong C_r^*(G_q). \tag{2.22}$$

We will show that this map is an isomorphism. In particular, this will identify the quotient map to the fibre at q with ev_q.

Let us prove an analogue of Lemma 1.56 for quantum $SL(2, \mathbb{C})$.

Lemma 2.12. *For any $q \in [q_0, 1]$, the product of two basis elements (2.21) in $\mathcal{D}(G_q)$ can be expressed as a linear combination of basis elements (2.21),*

$$(\omega_{ij}^n \bowtie u_{kl}^m)(q)(\omega_{bc}^a \bowtie u_{ef}^d)(q) = \sum_{\mu,\eta,p,r,v,w} C(\mu, \eta, p, r, v, w, q)(\omega_{pr}^\mu \bowtie u_{vw}^\eta)(q),$$

where the coefficient function $q \mapsto C(\mu, \eta, p, r, v, w, q) \in \mathbb{C}$ is continuous.

Proof. For the proof we will suppress the dependence on q to avoid excessive notation. We have

$$\begin{aligned}
&(\omega_{ij}^n \bowtie u_{kl}^m)(\omega_{bc}^a \bowtie u_{ef}^d) \\
&= (\omega_{ij}^n \bowtie u_{kl}^m)(\omega_{bc}^a \bowtie 1)(1 \bowtie u_{ef}^d) \\
&= \left(\sum_{s,t} ((\omega_{bc}^a)_{(1)}, u_{ks}^m)\omega_{ij}^n(\omega_{bc}^a)_{(2)}((\omega_{bc}^a)_{(3)}, S^{-1}(u_{tl}^m)) \bowtie u_{st}^m \right)(1 \bowtie u_{ef}^d).
\end{aligned} \tag{2.23}$$

Let us fix s and t and consider

$$((\omega_{bc}^a)_{(1)}, u_{ks}^m)\omega_{ij}^n(\omega_{bc}^a)_{(2)}((\omega_{bc}^a)_{(3)}, S^{-1}(u_{tl}^m)) \bowtie u_{st}^m.$$

The first leg is an element of $\mathcal{D}(K_q)$. We can evaluate this functional at $g \in \mathcal{O}(K_q)$ to obtain

$$((\omega_{bc}^a)_{(1)}, u_{ks}^m)\omega_{ij}^n(\omega_{bc}^a)_{(2)}((\omega_{bc}^a)_{(3)}, S^{-1}(u_{tl}^m))(g) = \omega_{bc}^a(u_{ks}^m g_{(2)} S^{-1}(u_{tl}^m))\omega_{ij}^n(g_{(1)}).$$

Since this leg is an element of $\mathcal{D}(K_q)$, it may be expressed as a linear combination of elements of the form $\omega^{\alpha}_{\beta\gamma}$. To understand the coefficient of such an element, we evaluate the above expression at $g = u^{\alpha}_{\beta\gamma}$. This is given by

$$((\omega^a_{bc})_{(1)}, u^m_{ks})\omega^n_{ij}(\omega^a_{bc})_{(2)}((\omega^a_{bc})_{(3)}, S^{-1}(u^m_{tl}))(u^{\alpha}_{\beta\gamma}) = \sum_{\delta} \omega^a_{bc}(u^m_{ks}u^{\alpha}_{\delta\gamma}S^{-1}(u^m_{tl}))\omega^n_{ij}(u^{\alpha}_{\beta\delta}).$$

By Lemma 1.56 and the formulae in Example 1.59, we see that this scalar depends continuously on $q \in [q_0, 1]$. Going back to 2.23, we have

$$\begin{aligned}(\omega^n_{ij} \bowtie u^m_{kl})(\omega^a_{bc} \bowtie u^d_{ef}) &= \left(\sum_{s,t}((\omega^a_{bc})_{(1)}, u^m_{ks})\omega^n_{ij}(\omega^a_{bc})_{(2)}((\omega^a_{bc})_{(3)}, S^{-1}(u^m_{tl})) \bowtie u^m_{st}\right)(1 \bowtie u^d_{ef}) \\ &= \sum_{s,t}((\omega^a_{bc})_{(1)}, u^m_{ks})\omega^n_{ij}(\omega^a_{bc})_{(2)}((\omega^a_{bc})_{(3)}, S^{-1}(u^m_{tl})) \bowtie u^m_{st}u^d_{ef}\end{aligned}$$

we see that again by Lemma 1.56, the linear combination of matrix elements obtained from $u^m_{st}u^d_{ef}$, where the coefficient functions are continuous. $\square$

Let us return to identifying the fibres of A^Q. We first consider the case where $q = 1$. We will see this is somewhat of a special case.

Proposition 2.13. $A^Q_1 = C^*_r(G_1)$.

Proof. Let

$$\pi_1 : A^Q \to {A^Q}/{J_1 A^Q} \tag{2.24}$$

denote the quotient map.

Recall that $\mathcal{D}(\mathcal{G})$ is generated as a $*$-algebra by the sections

$$q \mapsto f(q)\omega^n_{ij} \bowtie u^m_{kl}(q)$$

where $f \in C([q_0, 1])$. Since π_1 is surjective, the quotient ${A^Q}/{J_1 A^Q}$ has a dense $*$-algebra generated by the elements

$$\pi_1(q \mapsto f(q)\omega^n_{ij} \bowtie u^m_{kl}(q)).$$

Define a linear map

$$\rho_1 : \mathcal{D}(G_1) \to {A^Q}/{J_1 A^Q}$$

by first defining the map on the basis elements $(\omega^n_{ij} \bowtie u^m_{kl})(1)$ by

$$(\omega^n_{ij} \bowtie u^m_{kl})(1) \mapsto \pi_1(q \mapsto \omega^n_{ij} \bowtie u^m_{kl}(q))$$

and then extending linearly.

Let us show that ρ_1 is a $*$-homomorphism. The product $(\omega^n_{ij} \bowtie u^m_{kl})(1)(\omega^a_{bc} \bowtie u^d_{ef})(1)$ can be written, by Lemma 2.12,

$$(\omega^n_{ij} \bowtie u^m_{kl})(1)(\omega^a_{bc} \bowtie u^d_{ef})(1) = \sum_{p,s,t,\alpha,\beta,\gamma} C(p,s,t,\alpha,\beta,\gamma,1)(\omega^p_{st} \bowtie u^\alpha_{\beta\gamma})(1)$$

where $C(p,s,t,\alpha,\beta,\gamma,-) : [q_0,1] \to \mathbb{C}$ is a continuous function. Then

$$\rho_1((\omega^n_{ij} \bowtie u^m_{kl})(1)(\omega^a_{bc} \bowtie u^d_{ef})(1)) = \sum_{p,s,t,\alpha,\beta,\gamma} C(p,s,t,\alpha,\beta,\gamma,1)\pi_1(q \mapsto (\omega^p_{st} \bowtie u^a_{bc})(q)).$$

By Proposition A.4,

$$\begin{aligned} &\sum_{p,s,t,\alpha,\beta,\gamma} C(p,s,t,\alpha,\beta,\gamma,1)\pi_1(q \mapsto (\omega^p_{st} \bowtie u^a_{bc})(q)) \\ &= \pi_1\left(q \mapsto \sum_{p,s,t,\alpha,\beta,\gamma} C(p,s,t,\alpha,\beta,\gamma,q)\ (\omega^p_{st} \bowtie u^a_{bc})(q)\right). \end{aligned}$$

Now we note that

$$\begin{aligned} &\rho_1((\omega^n_{ij} \bowtie u^m_{kl})(1))\rho_1((\omega^a_{bc} \bowtie u^d_{ef})(1)) \\ &= \pi_1(q \mapsto (\omega^n_{ij} \bowtie u^m_{kl})(q))\pi_1(q \mapsto (\omega^a_{bc} \bowtie u^d_{ef})(q)) \\ &= \pi_1(q \mapsto (\omega^n_{ij} \bowtie u^m_{kl})(q)(\omega^a_{bc} \bowtie u^d_{ef})(q)) \\ &= \pi_1\left(q \mapsto \sum_{p,s,t,\alpha,\beta,\gamma} C(p,s,t,\alpha,\beta,\gamma,q)\ (\omega^p_{st} \bowtie u^a_{bc})(q)\right) \end{aligned}$$

so ρ_1 is multiplicative. One can similarly show that the map preserves the involution.

We can therefore extend the $*$-homomorphism

$$\rho_1 : \mathcal{D}(G_1) \to A^Q \big/ J_1 A^Q$$

to a $*$-homomorphism

$$\rho_1 : C^*_r(G_1) \to A^Q \big/ J_1 A^Q.$$

using the fact $C^*_r(G_1) = C^*(G_1)$, see Proposition 2.2.

The composition

$$\nu_1 \circ \rho_1 : C^*_r(G_1) \to A^Q \big/ J_1 A^Q \to C^*_r(G_1)$$

(where ν_1 is defined in (2.22)) is seen to be the identity map on the dense $*$-subalgebra

$\mathcal{D}(G_1)$ of $C_r^*(G_1)$ and so is the identity on $C_r^*(G_1)$. The composition

$$\rho_1 \circ \nu_1 : {A^Q}\big/{J_1 A^Q} \to C_r^*(G_1) \to {A^Q}\big/{J_1 A^Q}$$

is seen to be the identity map on the dense $*$-subalgebra $\pi_1(\mathcal{D}(\mathcal{G}))$ of ${A^Q}\big/{J_1 A^Q}$. Therefore we have $A_1^Q \cong C_r^*(G_1)$, and the quotient map is given by ev_1. $\square$

Notice that the case where $q = 1$ is special. We were able to extend a map on $\mathcal{D}(G_1)$ to a map on $C_r^*(G_1)$ in the proof of Proposition 2.13 using the fact $C_r^*(G_1) = C^*(G_1)$. This is not possible for $q < 1$, so we must find another approach for the remaining fibres.

Recall that by Theorem 2.11, for each $q \in (0,1)$, we can view $C_r^*(G_q) \subseteq C_0(M_q, K(\mathcal{H}_q))$. We will identify a fixed C^*-algebra which contains $C_r^*(G_q)$ as a C^*-subalgebra for all $q \in (0,1)$.

Define

$$M := \frac{1}{2}\mathbb{Z} \times {i\mathbb{R}}\big/{2\pi i\mathbb{Z}}.$$

For $q \in (0,1)$, we write $q = e^h$ for some $h \in (-\infty, 0)$. Then M_q (recall (2.14)) is homeomorphic to M by the formula

$$M \to M_q, \quad (m,\mu) \mapsto (m, h^{-1}\mu). \tag{2.25}$$

Recall also that $\mathcal{H}_m^q$ can be identified for each $q \in (0,1)$, with the fixed Hilbert space $\mathcal{H}_m$, see Remark 2.5. Let

$$V_m^q : \mathcal{H}_m^q \to \mathcal{H}_m$$

denote this isomorphism, which is given on the orthogonal basis of Proposition 2.4 by the formula

$$u^n_{i\ -m} \mapsto \frac{q^i}{\sqrt{[2n+1]}} e^n_{i\ -m}. \tag{2.26}$$

by the formula (2.6).

Consider the bundle of Hilbert spaces

$$p : \bigsqcup_{m \in \frac{1}{2}\mathbb{Z}} \mathcal{H}_m \times {i\mathbb{R}}\big/{2\pi i\mathbb{Z}} \to M$$

sending $(v_m, \lambda) \in \mathcal{H}_m \times {i\mathbb{R}}\big/{2\pi i\mathbb{Z}}$ to $(m,\lambda) \in M$. As in the case of $\mathcal{H}^q$, we obtain a locally trivial bundle $\mathcal{H}$ of Hilbert spaces, and a continuous field of C^*-algebras over M, given by the $C_0(M)$-algebra

$$C_0(M, K(\mathcal{H})) = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} C\left({}^{i\mathbb{R}}\!/\!_{2\pi i\mathbb{Z}}, K(\mathcal{H}_m) \right).$$

We will show that for each $q \in (0,1)$, $C_0(M_q, K(\mathcal{H}_q))$ and $C_0(M, K(\mathcal{H}))$ are isomorphic as $C_0(M)$-algebras. Note here we use the homeomorphism between M and M_q defined in (2.25) to view $C_0(M_q, K(\mathcal{H}_q))$ as a $C_0(M)$-algebra.

Proposition 2.14. *For each $q \in (0,1)$, $C_0(M_q, K(\mathcal{H}_q))$ and $C_0(M, K(\mathcal{H}))$ are isomorphic as $C_0(M)$-algebras.*

Proof. Let $f \in C_0(M_q, K(\mathcal{H}_q))$. Define an element $F \in \prod_{(m,\lambda)\in M} K(\mathcal{H}_{(m,\lambda)})$ by the formula

$$F(m, \lambda) = V_m^q f(m, h^{-1}\lambda)(V_m^q)^*, \quad (m, \lambda) \in M. \tag{2.27}$$

We will show $F \in C_0(M, K(\mathcal{H}))$. To do so we fix m and check continuity in the second variable. However continuity in the second variable is immediate as f is continuous in this variable.

It is easy to see that (2.27) defines a $*$-homomorphism $C_0(M_q, K(\mathcal{H}_q)) \to C_0(M, K(\mathcal{H}))$. This homomorphism is also a $C_0(M)$-module homomorphism. One can construct, in an entirely similar way, an inverse $*$-homomorphism and $C_0(M)$-module homomorphism $C_0(M, K(\mathcal{H})) \to C_0(M_q, K(\mathcal{H}_q))$. □

Proposition 2.14 together with Theorem 2.11 tells us that for each $q \in (0,1)$ we have an inclusion

$$\iota_q : C_r^*(G_q) \hookrightarrow C_0(M, K(\mathcal{H})).$$

Lemma 2.15. *For each $y \in \mathcal{D}(\mathcal{G})$, the map*

$$[q_0, 1) \to C_0(M, K(\mathcal{H})), \quad q \mapsto \iota_q(\mathrm{ev}_q(y)) \tag{2.28}$$

is continuous.

Proof. We wish to show that (2.28) is an element of $C([q_0,1), C_0(M, K(\mathcal{H})))$. This is equivalent to the statement that for all δ such that $q_0 < \delta < 1$, we have that the formula in (2.28) defines an element of

$$C([q_0, 1-\delta], C_0(M, K(\mathcal{H}))).$$

Since

$$C_0(M, K(\mathcal{H})) = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} C\left({}^{i\mathbb{R}}\!/_{2\pi i\mathbb{Z}}, K(\mathcal{H}_m)\right)$$

we have

$$\begin{aligned} C([q_0, 1-\delta], C_0(M, K(\mathcal{H}))) &= C\left([q_0, 1-\delta], \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} C\left({}^{i\mathbb{R}}\!/_{2\pi i\mathbb{Z}}, K(\mathcal{H}_m)\right)\right) \\ &\cong \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} C\left([q_0, 1-\delta], C\left({}^{i\mathbb{R}}\!/_{2\pi i\mathbb{Z}}, K(\mathcal{H}_m)\right)\right) \\ &\cong \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} C\left([q_0, 1-\delta] \times {}^{i\mathbb{R}}\!/_{2\pi i\mathbb{Z}}, K(\mathcal{H}_m)\right). \end{aligned}$$

It therefore suffices to fix $m \in \frac{1}{2}\mathbb{Z}$ and show that

$$[q_0, 1-\delta] \times {}^{i\mathbb{R}}\!/_{2\pi i\mathbb{Z}} \to K(\mathcal{H}_m), \quad (q, \lambda) \mapsto \iota_q(\mathrm{ev}_q(y))(m, \lambda)$$

is an element of $C_0\left([q_0, 1-\delta] \times {}^{i\mathbb{R}}\!/_{2\pi i\mathbb{Z}}, K(\mathcal{H}_m)\right)$.

If $y \in \mathcal{D}(\mathcal{G})$, we can write

$$y = \sum_{n,m,i,j,k,l} f^{n,m}_{i,j,k,l}\ \omega^n_{ij} \bowtie u^m_{kl}$$

where the indices run over a finite set, and each $f^{n,m}_{i,j,k,l} \in C([q_0, 1])$ by Lemma 2.12. Then since sums preserve continuity, it suffices to assume $y = \omega^n_{ij} \bowtie u^m_{kl}$. We need to show that if $(q_\nu, \lambda_\nu) \in [q_0, 1-\delta] \times {}^{i\mathbb{R}}\!/_{2\pi i\mathbb{Z}}$ is a sequence converging to $(q, \lambda) \in [q_0, 1-\delta] \times {}^{i\mathbb{R}}\!/_{2\pi i\mathbb{Z}}$, then

$$\|\iota_{q_\nu}(\mathrm{ev}_{q_\nu}(y))(m, \lambda_\nu) - \iota_q(\mathrm{ev}_q(y))(m, \lambda)\|_{K(\mathcal{H}_m)} \to 0.$$

Recall that the strong-$*$-topology on bounded subsets of $B(\mathcal{H}_m)$ coincides with the strict topology, viewing $M(K(\mathcal{H}_m)) = B(\mathcal{H}_m)$ (see [70, Lemma C.6]). Since for all q and λ, $\|\iota_q(\mathrm{ev}_q(y))(m, \lambda)\|_{\mathrm{op}} \le \|y\|_{A^Q}$, we are considering operators in a bounded set. Therefore if we can show

$$\|\iota_{q_\nu}(\mathrm{ev}_{q_\nu}(y))(m, \lambda_\nu)\xi - \iota_q(\mathrm{ev}_q(y))(m, \lambda)\xi\|_{\mathcal{H}_m} \to 0$$

and

$$\|\iota_{q_\nu}(\mathrm{ev}_{q_\nu}(y))^*(m, \lambda_\nu)\xi - \iota_q(\mathrm{ev}_q(y))^*(m, \lambda)\xi\|_{\mathcal{H}_m} \to 0$$

for any $\xi \in \mathcal{H}_m$, then for any compact operator $S \in K(\mathcal{H}_m)$, we have

$$\|S\iota_{q_\nu}(\mathrm{ev}_{q_\nu}(y))(m, \lambda_\nu) - S\iota_q(\mathrm{ev}_q(y))(m, \lambda)\|_{\mathrm{op}} \to 0.$$

Note that the operators $\iota_{q_\nu}(\mathrm{ev}_{q_\nu}(y))(m, \lambda_\nu)$, $\iota_q(\mathrm{ev}_q(y))(m, \lambda)$ have finite rank by Proposition 2.9. The image only depends on the choice of y, and not on q or λ. Then we can choose S to be the orthogonal projection onto this fixed subspace and by construction we will have

$$\|\iota_{q_\nu}(\mathrm{ev}_{q_\nu}(y))(m,\lambda_\nu) - \iota_q(\mathrm{ev}_q(y))(m,\lambda)\|_{\mathrm{op}} \to 0.$$

Therefore it suffices to show that for all $y \in \mathcal{D}(\mathcal{G})$ and for $\xi \in \mathcal{H}_m$ that

$$\|\iota_{q_\nu}(\mathrm{ev}_{q_\nu}(y))(m,\lambda_\nu)\xi - \iota_q(\mathrm{ev}_q(y))(m,\lambda)\xi\|_{\mathcal{H}_m} \to 0.$$

The action of the first leg of $y \in \mathcal{D}(\mathcal{G})$ on $\mathcal{H}_m$ depends continuously on q (see the formulae in Proposition 2.8) and is independent of the circular parameter. Therefore it suffices to understand the action of the second leg on $\mathcal{H}_m$. It also suffices to check this for $\xi = e^n_{i\ -m} \in \mathcal{H}_m$ by Proposition 2.4, where $n \in \frac{1}{2}\mathbb{N}_0$, $i \in \{-n, -n+1, ..., n\}$. If we consider $u^a_{bc}(q) \in C(K_q)$ with the formula for V^q_m, (2.26), we have

$$\begin{aligned} V^q_m u^a_{bc}(q)(V^q_m)^* e^n_{i\ -m} &= q^{-i}\sqrt{[2n+1]_q} V^q_m\ u^a_{bc}(q)\cdot u^n_{i\ -m}(q) \\ &= q^{-i}\sqrt{[2n+1]_q} V^q_m \sum_{r,s} u^a_{br}(q) u^n_{i\ -m}(q) u^a_{sc}(q)(K^{2+2\lambda}, u^a_{rs}(q)). \end{aligned}$$

The product $u^a_{br}(q)u^n_{i\ -m}(q)u^a_{sc}(q)$ can be expressed as a linear combination of the standard basis elements, with continuous coefficients in q, by Lemma 1.56. The unitary V^q_m will then rescale these basis elements by a continuous function in q, see the formula (2.26). Overall, the result is a linear combination of basis elements of $\mathcal{H}_m$, where the coefficients depend continuously on q and λ. It then follows that

$$\|\iota_{q_\nu}(\mathrm{ev}_{q_\nu}(y))(m,\lambda_\nu)\xi - \iota_q(\mathrm{ev}_q(y))(m,\lambda)\xi\|_{\mathcal{H}_m} \to 0.$$

as required. □

Corollary 2.16. *There is a $*$-homomorphism $\iota : A^Q \to C_b([q_0,1), C_0(M, K(\mathcal{H})))$, such that*

$$\iota(y)(q) = \iota_q(\mathrm{ev}_q(y))$$

for all $y \in A^Q$ and $q \in [q_0, 1)$. The map is injective on $A^Q\big|_{[q_0,1)} = C_0([q_0,1))A^Q$. In particular, for $q < 1$, the map $q \mapsto \|\mathrm{ev}_q(y)\|$ is continuous for each $y \in A^Q$.

Proof. Define $\iota : \mathcal{D}(\mathcal{G}) \to C([q_0,1), C_0(M, K(\mathcal{H})))$ by

$$\iota(y)(q) = \iota_q(\mathrm{ev}_q(y)), \quad y \in \mathcal{D}(\mathcal{G}), \quad q \in [q_0, 1).$$

By Lemma 2.15 this is a well defined $*$-homomorphism. The image is contained in

$C_b([q_0,1), C_0(M, K(\mathcal{H})))$ because for $y \in \mathcal{D}(\mathcal{G})$,

$$\|\iota(y)\| = \sup_{q\in[q_0,1)} \|\iota(y)(q)\| = \sup_{q\in[q_0,1)} \|\iota_q(\mathrm{ev}_q(y))\| = \sup_{q\in[q_0,1)} \|\mathrm{ev}_q(y)\| \leq \|y\| < \infty. \qquad (2.29)$$

The above calculation also shows that ι is a bounded $*$-homomorphism on $\mathcal{D}(\mathcal{G})$ and so ι extends to A^Q. If $y \in A^Q\big|_{[q_0,1)}$, then $\mathrm{ev}_1(y) = 0$, and so $\|y\| = \sup_{q\in[q_0,1)} \|\mathrm{ev}_q(y)\|$. We therefore have equality in the last step of (2.29) and therefore ι is isometric, and hence injective. $\square$

Recall that we have an action of the Weyl group $W = \{\pm 1\}$ on $C_0(M_q, K(\mathcal{H}^q))$, see (2.17). Therefore after identifying $C_0(M, K(\mathcal{H})) \cong C_0(M_q, K(\mathcal{H}^q))$ by Proposition 2.14, we have a W-action on $C_0(M, K(\mathcal{H}))$ for each q. Since the formulae for the intertwiners provided by Theorem 2.10 depend continuously on q, see [86, Theorem 5.42], we have an action of W on $C([q_0,1), C_0(M, K(\mathcal{H})))$ and $\iota(A^Q) \subseteq C([q_0,1), C_0(M, K(\mathcal{H})))^W$. If $f \in C([q_0,1), C_0(M, K(\mathcal{H})))^W$ then

$$f(q) \in C_0(M, K(\mathcal{H}))^W$$

where the action of W here depends on q. We have that

$$M/W \cong \{0\} \times [0,\pi] \sqcup \bigsqcup_{m\in\frac{1}{2}\mathbb{N}} \{m\} \times i\mathbb{R}/2\pi i\mathbb{Z}.$$

Note here the distinction at $m = 0$. This is because we identify $(0,\lambda) \sim (0,-\lambda)$, so we quotient $i\mathbb{R}/2\pi i\mathbb{Z}$ further. We identify $i\mathbb{R}/2\pi i\mathbb{Z}$ with $\mathbb{R}/2\pi\mathbb{Z}$, and then the latter with S^1. If $z \in S^1$, the relation means we identify $z \sim z^{-1} = \overline{z}$. Therefore at $m = 0$, we are left with the upper half semicircle, which is homeomorphic to $[0,\pi]$.

For the remaining $m \geq \frac{1}{2}$, we have identified the upper half semicircle of $\{m\} \times i\mathbb{R}/2\pi i\mathbb{Z}$ with the lower half semicircle of $\{-m\} \times i\mathbb{R}/2\pi i\mathbb{Z}$, and the upper half semicircle of $\{-m\} \times i\mathbb{R}/2\pi i\mathbb{Z}$ with the lower half semicircle of $\{m\} \times i\mathbb{R}/2\pi i\mathbb{Z}$. This leaves us with a single circle $\{m\} \times i\mathbb{R}/2\pi i\mathbb{Z}$.

Therefore we can identify

$$\begin{aligned} C_0(M, K(\mathcal{H}))^W &\cong \left(\bigoplus_{m\in\frac{1}{2}\mathbb{Z}} C_0\left(i\mathbb{R}/2\pi i\mathbb{Z}, K(\mathcal{H}_m)\right)\right)^W \\ &\cong C_0\left([0,\pi], K(\mathcal{H}_0)\right) \oplus \bigoplus_{m\in\frac{1}{2}\mathbb{Z}_{\geq 1}} C_0\left(i\mathbb{R}/2\pi i\mathbb{Z}, K(\mathcal{H}_m)\right) =: D. \qquad (2.30) \end{aligned}$$

for each q. The dependence of q here is only in identifying $\mathcal{H}_m \cong \mathcal{H}^q_m \cong \mathcal{H}^q_{-m} \cong \mathcal{H}_{-m}$, but this dependence is continuous in q because of the formula (2.26) and the formulae for the intertwiners $U^q_{(m,\lambda)}$ given in [86, Theorem 5.42]. Then if $f \in C([q_0,1), C_0(M, K(\mathcal{H})))^W$, we obtain, using these identifications, a map $q \mapsto f(q) \in D$ that defines an element of $C([q_0,1), D)$, where we have identified $f(q)$ with its image in D via the above isomorphism (2.30).

Theorem 2.17. *A^Q is a continuous $C([q_0,1])$-algebra with fibres $A^Q_q = C^*_r(G_q)$, and the quotient map to the fibre at q is ev_q. The restriction $A^Q\big|_{[q_0,1)}$ is a trivial field.*

Proof. The result about the fibres for $q < 1$ follows from Corollary 2.16 and Proposition A.10. We proved that $A^Q_1 = C^*_r(G_1)$ in Proposition 2.13. The quotient map to the fibre at $q \in [q_0,1]$ is ev_q.

Next we check continuity. We only need to consider continuity at the fibre at 1, because we have continuity at $q < 1$ by Corollary 2.16. We only need to show lower-semicontinuity by Proposition A.7. For this we will use Proposition A.47. Recall that we have a faithful left integral $\phi_{\widehat{G_q}} = \phi_{\widehat{K_q}} \otimes \phi_{K_q}$ on $\mathcal{D}(G_q)$, where $\phi_{\widehat{K_q}}$ is the left integral on $\mathcal{D}(K_q)$ and ϕ_{K_q} is the Haar functional on $C(K_q)$. One can extend $\phi_{\widehat{G_q}}$ to a faithful weight on each fibre as in [47, Definition 6.1]. We now check the conditions of propositon A.7 hold for these weights.

Since each weight is faithful, the corresponding GNS representations are injective by Theorem A.45. The family of sections $\mathcal{D}(\mathcal{G})$ is dense in A^Q by definition, and $\mathcal{D}(\mathcal{G})$ consists of integrable sections, so the third condition holds. The fourth condition holds because $\mathcal{D}(G_q)$ is dense in the GNS space for $\phi_{\widehat{G_q}}$. It remains to check the continuity condition. It is enough to see that $\phi_{\widehat{K_q}}(\omega^n_{ij})$ and $\phi_{K_q}(u^n_{ij})$ are continuous in q and the fact that as $q \to 1$ these formulae behave as expected. This follows from Theorem 1.58 and Example 1.59, (1.18).

Therefore A^Q is a continuous $C([q_0,1])$-algebra as required. Finally, we note that $A^Q\big|_{[q_0,1)} \subseteq C_0([q_0,1), D)$, where D is as defined in (2.30) - we need to show that $A^Q\big|_{[q_0,1)} = C_0([q_0,1), D)$. This follows from the Dixmier-Stone-Weierstrass theorem for continuous fields, Theorem A.23. $\square$

Chapter 3

The Quantum Assembly Map

By Theorem 2.17 and Proposition A.29, the continuous field A^Q induces a map in K-theory

$$\mu : K_*(C^*(G_1)) \to K_*(C^*_r(G_{q_0})) \tag{3.1}$$

for a fixed $q_0 \in (0, 1)$. We shall refer to this map as the **quantum assembly map**. In this chapter we will show that the quantum assembly map is an isomorphism.

To do this, we will adapt the method used by Higson in [32] to prove that the classical assembly map is an isomorphism for all complex semisimple Lie groups. The inspiration for this method is due to Mackey. In [52] Mackey suggested that there ought to be a correspondence between 'most' of the irreducible unitary representations of a connected semisimple Lie group G and a semi-direct product group G_0, which we call the **Cartan motion group** of G. In the case of $SL(2, \mathbb{C})$,

$$G_0 = K \ltimes \mathfrak{k}^*$$

where K acts on $\mathfrak{k}^*$ by the coadjoint action defined in (2) in the Introduction. Later Connes [11, p.g. 145-146, Proposition 8, Proposition 9] proved that there is a continuous field over $[0, 1]$, which we denote by A^C, with fibres

$$A_t := \begin{cases} C^*(G_0) & t = 0 \\ C^*_r(G) & t > 0 \end{cases}$$

such that $A|_{(0,1]}$ is trivial, and that the induced map in K-theory

$$K_*(C^*(G_0)) \to K_*(C^*_r(G))$$

provided by Proposition A.29 is the assembly map in the Baum-Connes conjecture, after identifying $K_*(C^*(G_0))$ with the left hand side of the Baum-Connes conjecture.

In [32], Higson analysed this field and showed that the induced map is an isomorphism. His method relies on the fact that the spectra of $C^*(G_0)$ and $C^*_r(G)$ are the same as sets, which formally describes the correspondence Mackey envisaged. However the topologies on the spectra are different. These differences motivated Higson to study certain subquotients of these group C^*-algebras, which are constructed in such a way so as to remove this difference in the topologies. A careful analysis of the corresponding subquotients of the field allows one to prove that the assembly map is an isomorphism.

This method carries over in essentially the same way to our assembly field. First, we will understand the representation theory of $C^*(G_1)$ and $C^*_r(G_{q_0})$ and the topology of the spectra. This will motivate the subquotients that we will study, and then finally we will conclude by using these subquotients to show that μ is an isomorphism.

3.1 The Mackey Analogy

Recall (see, for example, [89, Definition 2.6]) that a **dynamical system** is a triple (A, G, α), where A is a C^*-algebra, G is a locally compact group and $\alpha : G \to \mathrm{Aut}(A)$ is a continuous action.

If a locally compact group G acts on a locally compact Hausdorff space X then G acts on $C_0(X)$ by

$$\alpha : G \to \mathrm{Aut}(C_0(X)), \quad \alpha_g(f)(x) = f(g^{-1} \cdot x), \quad g \in G, \quad f \in C_0(X), \quad x \in X.$$

Then $(C_0(X), G, \alpha)$ is a dynamical system and we can form the crossed product $G \ltimes_\alpha C_0(X)$. In this section we will seek to understand such crossed products in the case where G is compact. Recall from propostion 2.2 that

$$C^*(G_1) \cong K \ltimes_{\mathrm{adj}} C(K)$$

and so we will be able to apply general results in this section to understand the representation theory of $C^*(G_1)$.

3.1.1 Crossed Products as $C_0(X)$-algebras

In this section, we will state a result that will allow us to see that certain crossed products are $C_0(X)$-algebras where X is some locally compact Hausdorff space X.

Let A be a C^*-algebra. We refer the reader to the literature for generalities concerning what follows in the next three paragraphs, for example [17, Chapter 3]. Recall that $\mathrm{Prim}(A)$ is the set of **primitive ideals**, that is, kernels of the irreducible representations, with the **Jacobson topology**. This is defined by specifying the closure operation on subsets $S \subseteq \mathrm{Prim}(A)$ by

$$\overline{S} = \left\{ I \in \mathrm{Prim}(A) \mid \bigcap_{J \in S} J \subseteq I \right\}. \tag{3.2}$$

The **spectrum** $\mathrm{Spec}(A)$ is the set of equivalence classes of irreducible representations of A. We clearly have a surjection $\mathrm{Spec}(A) \to \mathrm{Prim}(A)$, $\pi \mapsto \mathrm{Ker}(\pi)$. We can use this map to define a topology on $\mathrm{Spec}(A)$ by the prescription

$$S \subseteq \mathrm{Spec}(A) \text{ is open} :\iff \{\mathrm{Ker}(\pi) \mid \pi \in S\} \text{ is open.}$$

This topology ensures that the canonical surjection $\mathrm{Spec}(A) \to \mathrm{Prim}(A)$ is continuous and open. The spectrum $\mathrm{Spec}(A)$ is locally compact (see [17, 3.38]), and so $\mathrm{Prim}(A)$ is also locally compact (being the continuous and open image of a locally compact space, see [41, p.g. 147]).

If (A, G, α) is a dynamical system then we have a continuous action of G on $\mathrm{Prim}(A)$ defined by

$$G \times \mathrm{Prim}(A) \to \mathrm{Prim}(A), \quad (s, I) \mapsto s \cdot I := \alpha_s(I),$$

see [70, Lemma 5.44]. Let

$$\sigma : \mathrm{Prim}(A) \to G\backslash \mathrm{Prim}(A)$$

be the quotient map, which is continuous and open.

Since the continuous and open image of a locally compact space is locally compact (indeed, the image of a compact neighbourhood of a point is a compact neighbourhood of the image of the point), $G\backslash \mathrm{Prim}(A)$ is locally compact.

Assume now that $G\backslash \mathrm{Prim}(A)$ is also Hausdorff. Lee's theorem (see [50, Theorem 4]) tells us that A is a continuous $C_0(G\backslash \mathrm{Prim}(A))$-algebra, with $C_0(G\backslash \mathrm{Prim}(A))$-action

$$\mu_A : C_0(G\backslash \mathrm{Prim}(A)) \to ZM(A) \cong C_b(\mathrm{Prim}(A)), \quad f \mapsto f \circ \sigma. \tag{3.3}$$

Note here that we identify $ZM(A)$ with $C_b(\mathrm{Prim}(A))$ using the Dauns-Hofmann Theorem, originally in [13, Corollary 8.13 and Corollary 8.16].

The action α on A is a field of actions of G on A, as in Definition A.35. Indeed, if $G \cdot I \in G\backslash \mathrm{Prim}(A)$ (where $I \in \mathrm{Prim}(A)$), then $J_{G\cdot I}A$ is a closed, two sided ideal that is invariant under α, see [89, Lemma 8.1]. Then as explained in Remark A.36 we obtain an action $\alpha^{G\cdot I} : G \to \mathrm{Aut}(A_{G\cdot I})$ of G on the fibre $A_{G\cdot I}$ given by the formula

$$\alpha_s^{G\cdot I}(a + J_{G\cdot I}A) = \alpha_s(a) + J_{G\cdot I}A$$

for $a \in A$ and $s \in G$. Note that the quotient map $A \to A_{G\cdot I}$ is α-$\alpha^{G\cdot I}$ equivariant, so we obtain a $*$-homomorphism

$$\pi_{G\cdot I} : G \ltimes_\alpha A \to G \ltimes_{\alpha^{G\cdot I}} A_{G\cdot I}. \tag{3.4}$$

We have the following useful result that allows one to determine the irreducible representations of certain crossed products, which is a special case of [89, Proposition 8.7].

Proposition 3.1. *Let (A, G, α) be a dynamical system and suppose G is amenable and $G\backslash \mathrm{Prim}(A)$ is Hausdorff. Then $G \ltimes_\alpha A$ is a continuous $C_0(G\backslash \mathrm{Prim}(A))$-algebra with fibres*

$$(G \ltimes_\alpha A)_{G\cdot I} = G \ltimes_{\alpha^{G\cdot I}} A_{G\cdot I}, \quad G\cdot I \in G\backslash \mathrm{Prim}(A)$$

with the evaluation map to the fibre $(G \ltimes_\alpha A)_{G\cdot I}$ given by $\pi_{G\cdot I}$, the $*$-homomorphism (3.4) induced by the equivariant quotient map $A \to A_{G\cdot I}$.

Being able to equip a C^*-algebra A with a $C_0(X)$-structure allows one to use Theorem A.21 to understand the irreducible representations of the crossed product. In the following section we will see how to apply Proposition 3.1 and induction of group representations to understand the representation theory of crossed products arising from dynamical systems $(K, C_0(X), \alpha)$, where K is a compact group and X is a locally compact Hausdorff space.

3.1.2 Induction of Representations

In this section we discuss induction of group representations. The notion of induction was first introduced for finite groups by Frobenius in [25], and generalized by Mackey to locally compact groups in [51]. We refer the reader to [24, Chapter 6] for a contemporary account of induced representations. Note that here we restrict attention to compact groups which reduces the technical difficulties.

Let K be a compact group and let H be a closed subgroup of K. There is a process called **induction** which takes a unitary representation of H and produces a unitary representation of K. Let $\pi : H \to U(V)$ be a unitary representation of H on the Hilbert space V. Then define

$$\mathrm{Ind}_H^K(V) := \{\xi \in L^2(K,V) \mid \xi(kh) = \pi(h^{-1})\xi(k) \text{ for all } k \in K,\ h \in H\} \tag{3.5}$$

equipped with the inner product given by

$$\langle f,g\rangle_{L^2(K,V)} := \int_K \langle f(s),g(s)\rangle_V \,\mathrm{d}s, \quad f,g \in L^2(K,V)$$

on $L^2(K,V)$. This is a Hilbert space. We then have a unitary representation $\tilde{\pi} : K \to U(\mathrm{Ind}_H^K(V))$ given by

$$(\tilde{\pi}(k)\xi)(s) = \xi(k^{-1}s) \tag{3.6}$$

for $k,s \in K$ and $\xi \in \mathrm{Ind}_H^K(V)$.

Let us consider some basic examples of induction.

Example 3.2. Let K be a compact group with identity e, and $\pi : K \to U(V)$ be a unitary representation. Then

$$\mathrm{Ind}_K^K(V) \to V, \quad \xi \mapsto \xi(e)$$

is an equivariant unitary. That is, induction from K to itself does not change the equivalence class of the representation being induced.

Example 3.3. Let K be a compact group with identity e and let H be a closed (and hence compact) subgroup. We can induce the left regular representation on H,

$$\lambda : H \to U(L^2(H))$$

to K. Then if $\xi \in \mathrm{Ind}_H^K(L^2(H))$, $k \in K$ and $h \in H$ we have

$$\xi(k)(h) = \xi(k)(he) = (\lambda_{h^{-1}}\xi(k))(e) = \xi(kh)(e).$$

It follows that

$$\mathrm{Ind}_H^K(L^2(H)) \to L^2(K), \quad \xi \mapsto (k \mapsto \xi(k)(e))$$

is an equivariant unitary between the induced representation and the left regular representation of K. That is, induction of the left regular representation from a closed subgroup of K is the left regular representation (up to unitary equivalence) on K.

We have the following result about induction and tensor products that will be useful to us later.

Proposition 3.4. *Let K be a compact group, H be a closed subgroup of K and W be a Hilbert space. Let π be a unitary representation of H on a Hilbert space V and let 1 be the trivial representation of H on W, so that*

$$\pi \otimes 1 : H \to U(V \otimes W), \quad (\pi \otimes 1)(h) = \pi(h) \otimes 1_W$$

is a unitary representation of H on the Hilbert space $V \otimes W$. Then

$$\mathrm{Ind}_H^K(V) \otimes W = \mathrm{Ind}_H^K(V \otimes W)$$

where we view $\mathrm{Ind}_H^K(V) \otimes W \subseteq \mathrm{Ind}_H^K(V \otimes W)$ by

$$\xi \otimes w \mapsto (k \mapsto \xi(k) \otimes w) \tag{3.7}$$

for $\xi \in \mathrm{Ind}_H^K(V)$, $w \in W$ and $k \in K$.

Proof. We start by noting that we can easily check that on the dense subspace $\mathrm{Ind}_H^K(V) \odot W$ of $\mathrm{Ind}_H^K(V) \otimes W$ that the map (3.7) in the statement of the proposition is isometric, and so the map (3.7) is indeed injective. We will now show that $\mathrm{Ind}_H^K(V) \otimes W$ is dense in $\mathrm{Ind}_H^K(V \otimes W)$.

Since the subspace $C_c(K, V \otimes W) \subseteq L^2(K, V \otimes W)$ is dense in L^2-norm, then by an averaging argument,

$$\{f \in C_c(K, V \otimes W) \mid f(kh) = (\pi(h^{-1}) \otimes 1_W) f(k) \text{ for all } k \in K,\ h \in H\} \tag{3.8}$$

is dense in $\mathrm{Ind}_H^K(V \otimes W)$.

We will now find a dense subspace of (3.8) contained in $\mathrm{Ind}_H^K(V) \otimes W$, from which the result will follow. First note that $C_c(K, V \otimes W) = C(K, V \otimes W)$ now equipped with the supremum norm,

$$\|f\|_\infty := \sup_{k \in K} \|f(k)\|_{V \otimes W}, \quad f \in C(K, V \otimes W),$$

is a Hilbert $C(K)$-module, and is isomorphic to $C(K, V) \otimes W$ where the tensor product is the exterior tensor product of Hilbert modules (see [48, p.g. 34-35]). The subspace $C(K, V) \odot W$ is dense in $C(K, V \otimes W)$ with respect to the supremum norm. Note that if

$f \in C_c(K, V \otimes W) \subseteq L^2(K, V \otimes W)$, we have

$$\|f\|^2_{L^2(K,V\otimes W)} = \int_K \|f(k)\|^2_{V\otimes W} \, \mathrm{d}k \leq \sup_{k\in K} \|f(k)\|^2_{V\otimes W} = \|f\|^2_{C(K,V\otimes W)}.$$

Therefore $C(K, V) \odot W \subseteq C_c(K, V \otimes W)$ densely in the L^2-norm, and so in particular $C(K, V) \odot W$ is dense in $L^2(K, V \otimes W)$ in the L^2-norm. We then have that

$$\{f \in C(K, V) \odot W \mid (\mathrm{ev}_{kh} \otimes 1_W)(f) = (\pi(h^{-1}) \circ \mathrm{ev}_k \otimes 1_W)(f) \text{ for all } k \in K,\, h \in H\}$$

is dense in $\mathrm{Ind}_H^K(V \otimes W)$ and contained in $\mathrm{Ind}_H^K(V) \otimes W$. Therefore $\mathrm{Ind}_H^K(V) \otimes W = \mathrm{Ind}_H^K(V \otimes W)$. $\square$

Let K be a compact group, H a closed subgroup of K and V a unitary representation of H on a Hilbert space V. We have a dynamical system $(C(K/H), K, \mathrm{lt})$, where lt denotes the action by left translation, i.e. $(k \cdot f)(sH) = f(k^{-1}sH)$ for $k, s \in K$ and $f \in C(K/H)$.

We will now describe two constructions of a representation of $K \ltimes_{\mathrm{lt}} C(K/H)$ on $\mathrm{Ind}_H^K(V)$.

Construction 3.5. Note that $\mathrm{Ind}_H^K(V)$ is a left $C(K/H)$-module with the action given by pointwise multiplication, because if $f \in C(K/H)$, $\xi \in \mathrm{Ind}_H^K(V)$, $k \in K$ and $h \in H$ we have

$$(f\xi)(kh) = f(khH)\xi(kh) = f(kH)\pi(h^{-1})\xi(k) = \pi(h^{-1})(f\xi)(k).$$

Therefore we obtain a $*$-homomorphism $\rho : C(K/H) \to B(\mathrm{Ind}_H^K(V))$. The representations $\tilde{\pi}$ (recall the definition from (3.6)) and ρ form a covariant pair for the dynamical system $(C(K/H), K, \mathrm{lt})$. This is because

$$\begin{aligned}
(\tilde{\pi}(k)\rho(f)\tilde{\pi}(k)^*\xi)(s) &= \tilde{\pi}(k)(\rho(f)\tilde{\pi}(k)^*\xi)(s) \\
&= f(k^{-1}sH)(\tilde{\pi}(k)^*\xi)(k^{-1}s) \\
&= (k \cdot f)(sH)\xi(s) \\
&= (\rho(k \cdot f)\xi)(s)
\end{aligned}$$

for all $k \in K$, $f \in C(K/H)$, $\xi \in \mathrm{Ind}_H^K(V)$ and $s \in K$. Therefore we have a $*$-representation $\tilde{\pi} \ltimes \rho$ of the crossed product $K \ltimes C(K/H)$ on $\mathrm{Ind}_H^K(V)$.

Construction 3.6. There is a **Morita equivalence** between the C^*-algebras $C^*(H)$ and $K \ltimes C(K/H)$ (which is originally constructed in [29, Section 2]). We avoid any generalities about Morita equivalences, but this is a notion originally defined for rings by Morita in [58], and by Rieffel for C^*-algebras in [71].

Two Morita equivalent C^*-algebras have the 'same' representation theories. More precisely, if A and B are C^*-algebras, we can consider the categories of non-degenerate $*$-

representations Rep(A) and Rep(B) of A and B respectively. The objects in these categories are the $*$-representations of each algebra, and the morphisms are the intertwiners between the representations. If A and B are Morita equivalent algebras, then Rep(A) and Rep(B) are equivalent categories, [70, 3.30]. This equivalence induces a homeomorphism between Spec(A) and Spec(B), which follows from [70, Corollary 3.33].

In particular, by the equivalence a unitary representation π of H corresponds to a $*$-representation of $K \ltimes C(K/H)$. This is of the form $L = \tilde{\pi} \ltimes \rho$, where $\tilde{\pi}$ is a unitary representation of K and ρ is a representation of $C(K/H)$ on the same Hilbert space, see [70, Corollary C.31].

In Constructions 3.5 and 3.6, we have described two ways in which we can take a unitary representation of H and produce a $*$-representation of $K \ltimes C(K/H)$. It can be shown (see [20, Corollary 7.8]) that these processes give rise to equivalent representations. We denote (the equivalence class of) such representations by $\mathrm{Ind}_H^K(V)$ as well.

In particular, if we take an irreducible representation of H, we can use Construction 3.5 to construct a representation of $K \ltimes C(K/H)$, and by the theory of Construction 3.6, this is an irreducible representation of $K \ltimes C(K/H)$, and all the irreducible representations of $K \ltimes C(K/H)$ arise in this way.

Using these facts we can describe the representation theory of crossed products arising from actions of compact groups on locally compact spaces. We note that this is a special case of the Mackey-Rieffel-Green theorem concerning represenions of certain crossed products, see for example [20, Theorem 7.29]. We instead follow a more direct approach, based on a suggestion by Voigt, which seems to be folklore.

Proposition 3.7. *Let K be a compact group acting on a locally compact Hausdorff space X. Then*

(a) The space $K\backslash X$ is Hausdorff and $K \ltimes C_0(X)$ is a continuous $C_0(K\backslash X)$-algebra with fibres

$$(K \ltimes C_0(X))_{K\cdot x} = K \ltimes C(K \cdot x)$$

for $K \cdot x \in K\backslash X$, where the action of K on $K \cdot x$ is the restriction of the action of K to $K \cdot x$. The evaluation maps

$$\pi_{K\cdot x} : K \ltimes C_0(X) \to K \ltimes C(K \cdot x)$$

are induced by the restriction maps $C_0(X) \to C(K \cdot x)$.

(b) For each $x \in X$, there is a homeomorphism $K \cdot x \cong K/K_x$, where K_x is the stabilizer of x under the action of K. This homeomorphism is equivariant for the action of K

on $K \cdot x$ and the action of K on K/K_x induced by the action of K on itself by left translation.

(c) *All irreducible $*$-representations of $K \ltimes C_0(X)$ are of the form*

$$\mathrm{Ind}_{K_x}^{K}(V) \circ \pi_{K \cdot x}$$

where $x \in X$, V is an irreducible unitary representation of the stabilizer group K_x of x, $\pi_{K\cdot x}$ is the evaluation map to $(K \ltimes C_0(X))_{K\cdot x} = K \ltimes C(K \cdot x)$, which we identify with $K \ltimes C(K/K_x)$ using (b), and $\mathrm{Ind}_{K_x}^{K}(V)$ is the irreducible representation of $K \ltimes C(K/K_x)$ defined by Constructions 3.5 and 3.6.

(d) *Let $x \in X$, and V be an irreducible unitary representation of the stabilizer group K_x of x. Define*

$$\pi_{(V,x)} : K \to U(\mathrm{Ind}_{K_x}^{K}(V)), \quad (\pi_{(V,x)}(k)\xi)(s) = \xi(k^{-1}s),$$

$$\rho_{(V,x)} : C_0(X) \to B(\mathrm{Ind}_{K_x}^{K}(V)), \quad (\rho_{(V,x)}(f)\xi)(s) = f(s \cdot x)\xi(s).$$

for $s, k \in K$, $f \in C_0(X)$ and $\xi \in \mathrm{Ind}_{K_x}^{K}(V)$. This is a covariant pair for the action of K on $C_0(X)$. Moreover,

$$\pi_{(V,x)} \ltimes \rho_{(V,x)} = \mathrm{Ind}_{K_x}^{K}(V) \circ \pi_{K\cdot x}.$$

(e) *If $x, y \in X$ and V, W are irreducible unitary representations of K_x and K_y respectively, then $\mathrm{Ind}_{K_x}^{K}(V) \circ \pi_{K\cdot x}$ and $\mathrm{Ind}_{K_y}^{K}(W) \circ \pi_{K\cdot y}$ are equivalent if and only if $y = t \cdot x$ for some $t \in K$ and W corresponds to V under the identification of K_x and K_y induced by conjugation by t.*

Proof. We have each of the following facts.

1. The space $K\backslash X$ is Hausdorff, because K acts on the locally compact Hausdorff space X continuously and properly, see [49, Proposition 12.24].

2. For each $x \in X$, the orbit $K \cdot x \subseteq X$ is closed, being the continuous image of a compact space.

3. For each $x \in X$, K_x is compact, being a closed subset of the compact group K.

4. The map $K/K_x \to K \cdot x$, $kK_x \mapsto k \cdot x$ is a homeomorphism, because it is a continuous bijection between a compact space and a Hausdorff space.

The first point here allows us to apply Proposition 3.1. We therefore need to understand

$C_0(X)$ as a $C_0(K\backslash X)$-algebra to complete the proof of (a).

The $C_0(K\backslash X)$-structure on $C_0(X)$ is given by (3.3) applied to this situation. In particular, we are in the setting of Example A.12 where the map in this Example is given by the quotient map $q : X \to K\backslash X$. Therefore the fibre of $C_0(X)$ at the orbit $K \cdot x \in K\backslash X$ of $x \in X$ is $C_0(q^{-1}(\{K \cdot x\})) = C(K \cdot x)$, with evaluation map to the fibre at the point $K \cdot x \in K\backslash X$ given by restricton of functions on X to the (compact) orbit $K \cdot x \subseteq X$.

It is clear by the definition of the action on $C_0(X)_{K\cdot x}$ as defined in Section 3.1.1 that this is simply the action of K on X restricted to $K \cdot x \subseteq X$. This completes the proof of (a). Part (b) follows from the fourth point above, which can easily be seen to be an equivariant map.

For (c), we note that each irreducible representation of a $C_0(X)$-algebra factors through an evaluation map (see Theorem A.21). Therefore, we only need to understand the irreducible representations of the crossed product $K \ltimes C(K/K_x)$ for each $x \in X$. However, by the Rieffel correspondence, these arise from the irreducible representations of K_x via the process described above this proposition.

It is easy to check that the formulae in (d) defines a covariant pair, and so we have a $*$-homomorphism, for each $x \in X$,

$$\pi_{(V,x)} \ltimes \rho_{(V,x)} : K \ltimes C_0(X) \to B(\mathrm{Ind}_{K_x}^K(V)). \tag{3.9}$$

One can easily check that on $C_c(K, C_0(X)) \subseteq K \ltimes C_0(X)$ that (3.9) is the same as $\mathrm{Ind}_{K_x}^K(V) \circ \pi_{K\cdot x}$, giving the result. Note here that we are using the concrete definition of $\mathrm{Ind}_{K_x}^K(V)$ given in Construction 3.5.

For part (e), suppose $y = t \cdot x$ for some $t \in K$ and W corresponds to V under the identification of K_x and K_y induced by conjugation with t. Let us therefore assume that $V = W$ and that the representations of K_x and K_y on V are denoted by π_{K_x} and π_{K_y} respectively. Then as

$$K_x \to K_y, \quad k \mapsto tkt^{-1}$$

is the isomorphism induced by conjugation, we have $\pi_{K_x}(k) = \pi_{K_y}(tkt^{-1})$ for $k \in K_x$. Define

$$T : \mathrm{Ind}_{K_x}^K(V) \to \mathrm{Ind}_{K_y}^K(W), \quad (T\xi)(k) = \xi(kt), \quad \xi \in \mathrm{Ind}_{K_x}^K(V),\ k \in K.$$

This is well defined because if $k \in K$, $s \in K_y$ and $\xi \in \mathrm{Ind}_{K_x}^K(V)$, then

$$(T\xi)(ks) = \xi(kst) = \xi(ktt^{-1}st) = \pi_{K_x}(t^{-1}st)^{-1}\xi(kt) = \pi_{K_y}(s)^{-1}(T\xi)(k).$$

Clearly T is a unitary isomorphism. It intertwines $\pi_{(V,x)}$ and $\pi_{(W,x)}$ because left translation is not affected by right translation. It also intertwines $\rho_{(V,x)}$ and $\rho_{(W,y)}$ because

$$\begin{aligned}(T\rho_{(V,x)}(f)T^{-1}\xi)(s) &= (\rho_{(V,x)}(f)T^{-1}\xi)(st)\\ &= f(st\cdot x)(T^{-1}\xi)(st)\\ &= f(s\cdot(t\cdot x))\xi(s)\\ &= f(s\cdot y)\xi(s)\\ &= (\rho_{(W,y)}(f)\xi)(s)\end{aligned}$$

for $f \in C_0(X)$, $\xi \in \operatorname{Ind}_{K_y}^K(W)$ and $s \in K$. Therefore T provides an equivalence between $\operatorname{Ind}_{K_x}^K(V) \circ \pi_{K\cdot x}$ and $\operatorname{Ind}_{K_y}^K(W) \circ \pi_{K\cdot y}$ by part (d).

Conversely, suppose $\operatorname{Ind}_{K_x}^K(V) \circ \pi_{K\cdot x}$ and $\operatorname{Ind}_{K_y}^K(W) \circ \pi_{K\cdot y}$ are equivalent. For this to happen, they must factor through the same fibre of $K \ltimes C_0(X)$, i.e. $K \cdot x = K \cdot y$. That is, there exists $t \in K$ with $y = t \cdot x$. Using conjugation, we can view V and W as irreducible representations of K_x. For these to induce to the same representation on the crossed product, we must have that V and W are equivalent because the Morita equivalence (as explained in Construction 3.6) establishes a one-to-one correspondence between the representation theories of K_x and $K \ltimes C(K/K_x)$. $\square$

3.1.3 The Mackey Analogy in the Quantum Setting

Let us now apply Proposition 3.7 to our situation. Recall that if $q \in (0,1]$ and $G_q := SL_q(2,\mathbb{C})$, we have

$$C^*(G_1) \cong K \ltimes_{\mathrm{adj}} C(K)$$

where $K = SU(2)$ and K acts on itself by the adjoint action (2.3). Proposition 3.7 tells us that for each $x \in K$, and irreducible representation V of K_x, the stabilizer of x in K, we obtain an irreducible representation of G_1 and all irreducibles are obtained in this way. Note that K_x is the centralizer of x in K in this case.

Proposition 3.7 also tells us we can choose any representative in the conjugacy class of x for this process. Recall that we can view the circle group $T \subseteq K$ as the matrices of the form

$$\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}$$

where $z \in T \subseteq \mathbb{C}$. For any $x \in K$ we can choose a representative for the conjugacy class of x that lies in T. Indeed, all elements of K are normal and so are unitarily diagonalizable, i.e. there exists $u \in U(\mathbb{C}^2)$ such that uxu^* is diagonal. We can rescale our unitary u so that $\det(u) = 1$, and so we can assume $u \in K$. Then $uxu^* \in T \subseteq K$.

Lemma 3.8. *Let $x \in T \subseteq K$. Then K_x, the centralizer of x in K is*

$$K_x = \begin{cases} K & x = \pm I \\ T & x \neq \pm I \end{cases}$$

where I is the identity matrix. Two distinct elements $x, y \in T$ are conjugate in K if and only if $x = y^{-1}$, and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = x^{-1}.$$

Proof. For a given $x \in T$, we can write

$$x = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \in T$$

for some $t \in \mathbb{R}$. Direct calculation tells us that if $\begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in K_x$, then $\alpha^*\gamma = 0$ or $e^{it} = e^{-it}$. Therefore $\alpha = 0$, $\gamma = 0$ or $t = 0, \pi$.

If $t = 0,\ \pi$, then $x = I, -I$ respectively, and clearly $K_x = K$.

If $\alpha = 0$ and $\gamma \neq 0$, $t \neq 0, \pi$, then

$$\begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

which cannot happen. The case where $\gamma = 0$ can occur, and we then see $K_x = T$ for $x \neq I, -I$.

Note that this calculation also tells us that two distinct elements $x, y \in T$ are conjugate in K if and only if $x = y^{-1}$, and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = x^{-1}.$$

□

The irreducible representations of T are usually indexed by integers. For notational convenience in what follows, we will use half integers instead. That is, each irreducible representation of T is of the form

$$\tau_m : T \to U(\mathbb{C}), \quad \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} \mapsto z^{2m} \tag{3.10}$$

for $m \in \frac{1}{2}\mathbb{Z}$ (see for example [24, Theorem 4.5]). Therefore for each $x \in T \setminus \{I, -I\}$ and $m \in \frac{1}{2}\mathbb{Z}$ we obtain an irreducible representation (using the notation as defined in Proposition 3.7 (d))

$$\pi_{(m,x)} := \pi_{(\tau_m,x)} \ltimes \rho_{(\tau_m,x)} : K \ltimes_{\mathrm{adj}} C(K) \to B(\mathcal{H}_{(m,x)}) \tag{3.11}$$

where

$$\mathcal{H}_{(m,x)} := \mathrm{Ind}_T^K(\tau_m) \cong \{\xi \in L^2(K) \mid \xi(kt) = \tau_m(t)^{-1}\xi(k) \text{ for all } k \in K,\, t \in T\}. \tag{3.12}$$

Note that it is because τ_m is one dimensional that we can view $\mathcal{H}_{(m,x)} = \mathrm{Ind}_T^K(\tau_m) \subseteq L^2(K)$.

Remark 3.9. The inclusion $T \subseteq K$ induces the surjective $*$-homomorphism

$$C(K) \to C(T), \quad f \mapsto f|_T\,.$$

Note that this agrees with the $*$-homomorphism from Proposition 2.3 on $\mathcal{O}(SU(2))$, and so for $t \in T$ we have, by Proposition 2.3,

$$u^n_{ij}(t) = \delta_{ij}\tau_j(t). \tag{3.13}$$

Then for $k \in K$ and $t \in T$

$$u^n_{ij}(kt) = \sum_l u^n_{il}(k)u^n_{lj}(t) = \tau_j(t)u^n_{ij}(k).$$

Therefore $u^n_{ij} \in \mathcal{H}_{(m,x)}$ if and only if $j = -m$. Setting $q = 1$ in Proposition 2.4 we see that $\{u^n_{i\ -m}\}$ forms an orthogonal basis for $\mathcal{H}_{(m,x)}$, where $n \geq |m|$, $n + m \in \mathbb{Z}$ and $i \in \{-n, -n+1, ..., n\}$. Therefore

$$\mathcal{H}_{(m,x)} \cong \ell^2(\{e^n_{i\ -m}\}) := \mathcal{H}_m, \tag{3.14}$$

c.f. Remark 2.5.

Recall from Section 1.3 that the irreducible representations of K are indexed by elements of $\frac{1}{2}\mathbb{N}_0$. That is, if $m \in \frac{1}{2}\mathbb{N}_0$, there is a $2m+1$ dimensional representation $V(m)$ of K. Therefore for each $m \in \frac{1}{2}\mathbb{N}_0$ we obtain a pair of irreducible representations,

$$\pi_{(m,\pm I)} := \pi_{(V(m),\pm I)} \ltimes \rho_{(V(m),\pm I)} : K \ltimes_{\mathrm{adj}} C(K) \to B(\mathcal{H}_{(m,\pm I)}) \tag{3.15}$$

where $\pi_{(V(m),\pm I)}$, $\rho_{(V(m),\pm I)}$ are as defined in Proposition 3.7 (d) and

$$\mathcal{H}_{(m,\pm I)} := \mathrm{Ind}_K^K(V(m)) \cong V(m)$$

by Example 3.2. Under this identification $\pi_{(V(m),\pm I)} : K \to U(\mathcal{H}_{(m,\pm I)})$ is simply the representation $V(m)$, and

$$\rho_{(V(m),\pm I)} : C(K) \to B(\mathcal{H}_{(m,\pm I)}), \quad \rho_{(V(m),\pm I)}(f) = f(\pm I)1_{V(m)}.$$

Since $\{I\}$ and $\{-I\}$ are singleton conjugacy classes in K, then by Proposition 3.7 (e), $\pi_{(m,\pm I)}$ represent two distinct equivalence classes of representations in $\mathrm{Spec}(C_r^*(G_1))$, and both of these classes cannot contain any of the representations $\pi_{(m,x)}$ for $m \in \frac{1}{2}\mathbb{Z}$ and $x \in T \setminus \{\pm I\}$.

Let us now consider equivalences among the representations defined in (3.11).

Proposition 3.10. *Let $x, y \in T \setminus \{I, -I\}$ and $m, n \in \frac{1}{2}\mathbb{Z}$. Then $\pi_{(m,x)}$ is unitarily equivalent to $\pi_{(n,y)}$ if and only if*

$$(m, x) = (n, y), \quad \text{or} \quad (m, x) = (-n, y^{-1})$$

and in the latter case the unitary intertwiner is given by

$$\mathcal{H}_{(m,x)} \to \mathcal{H}_{(-m,x^{-1})}, \quad \xi \mapsto (k \mapsto \xi(kw))$$

where $\xi \in \mathcal{H}_{(m,x)}$ and $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Proof. From Lemma 3.8 two distinct elements $x, y \in T$ are conjugate if and only if $x = y^{-1}$, and

$$wxw^{-1} = x^{-1}$$

Under conjugation of T by w the representations τ_m and τ_{-m} as defined in (3.10) are identified. The result then follows from Proposition 3.7 (e). □

Consider the set

$$M := \frac{1}{2}\mathbb{Z} \times T. \tag{3.16}$$

There is an action of the Weyl group $W = \mathbb{Z}_2 = \{\pm I\}$ on M by

$$-1 \cdot (m, x) = (-m, x^{-1}).$$

We can now describe (as a set) the spectrum of $K \ltimes_{\mathrm{adj}} C(K)$.

Proposition 3.11. *There is a bijection*

$$M/W \to \mathrm{Spec}(K \ltimes_{\mathrm{adj}} C(K)).$$

Proof. Consider the class of $(m, \pm I)$ in M/W. If $m < 0$, we can replace m by $-m$ and the class is unchanged, so we can assume that $m > 0$. Define the map on these classes by $[(m, \pm I)] \mapsto \pi_{(m,\pm I)}$.

If $x \in T \setminus \{\pm I\}$ and $m \in \frac{1}{2}\mathbb{Z}$, the class of (m, x) has two representatives, namely

$$(m, x), (-m, x^{-1}).$$

By Proposition 3.10, we can define the map on these classes by $[(m, x)] \mapsto \pi_{(m,x)}$, because different representatives for the same class give rise to equivalent representations.

The map we have defined is clearly a bijection. □

We now compare this to the situation for $C_r^*(G_q)$ for $q \in (0, 1)$. From (2.30) we can see that we can identify

$$\mathrm{Spec}(C_r^*(G_q)) \cong M/W \tag{3.17}$$

as topological spaces, where we view M/W as a subspace of M, which in turn is equipped with the product topology. Therefore, as sets, $\mathrm{Spec}(C^*(G_1))$ and $\mathrm{Spec}(C_r^*(G_q))$ are the same. This is the quantum version of Mackey's analogy.

However, this identification as sets is **not** a homeomorphism. In the following section we will understand the topology of $\mathrm{Spec}(C^*(G_1))$ using an alternative description of the crossed product.

3.1.4 The Spectrum of the Quantum Cartan Motion Group

We start with a preparatory lemma.

Lemma 3.12. *Let A be a C^*-algebra, and K be a compact group acting on A via two actions*

$$\alpha : K \to \mathrm{Aut}(A), \quad \beta : K \to \mathrm{Aut}(A),$$

that commute, i.e. $\alpha_k \beta_s = \beta_s \alpha_k$ for all $s, k \in K$. Then β induces an action of K on $K \ltimes_\alpha A$ fixing $K \subseteq M(K \ltimes_\alpha A)$ (which we also denote by β), α restricts to an action on A^β, and

$$(K \ltimes_\alpha A)^\beta \cong K \ltimes_\alpha A^\beta.$$

Proof. For each $s \in K$, the map

$$\beta_s : A \to A$$

is α-equivariant because α and β commute. Therefore the $*$-homomorphism

$$\mathrm{id} \ltimes \beta_s : C_c(K, A) \to C_c(K, A), \quad (\mathrm{id} \ltimes \beta_s)(f)(k) = \beta_s(f(k)), \quad f \in C_c(K, A), \quad k \in K$$

extends to a map $K \ltimes_\alpha A \to K \ltimes_\alpha A$ (see [89, Corollary 2.48]) which we also call $\mathrm{id} \ltimes \beta_s$. Clearly on $C_c(K, A)$ have $(\mathrm{id} \ltimes \beta_s)(\mathrm{id} \ltimes \beta_k) = \mathrm{id} \ltimes \beta_{sk}$ for all $s, k \in K$, and so the extensions define an action of K on $K \ltimes_\alpha A$, which we also call β.

The copy of K inside $M(K \ltimes_\alpha A)$ is defined by

$$K \to M(K \ltimes_\alpha A), \quad k \mapsto U_k, \quad (U_k f)(s) := \alpha_s(f(k^{-1}s))$$

for $f \in C_c(K, A)$, $s \in K$, see [89, Proposition 2.34]. One can easily check that for each $k \in K$, $\beta_s(U_k) = U_k$ for all $s \in K$.

If $a \in A^\beta$, then $\alpha_s(a)$ is invariant under β because α and β commute. Therefore α restricts to an action on A^β.

Consider the inclusion homomorphism

$$\iota : A^\beta \to A.$$

This is clearly α-equivariant and so as above we obtain a $*$-homomorphism

$$\mathrm{id} \ltimes \iota : K \ltimes_\alpha A^\beta \to K \ltimes_\alpha A. \tag{3.18}$$

Since K is compact, reduced crossed products are equal to full crossed products. It is well known that in the reduced case, faithful homomorphisms on C^*-algebras induce faithful homomorphisms between the reduced crossed products, see [20, pg. 12-13]. Therefore $\mathrm{id} \ltimes \iota$ is injective.

We have $(\mathrm{id} \ltimes \iota)(K \ltimes_\alpha A^\beta) \subseteq (K \ltimes_\alpha A)^\beta$ because for $f \in C_c(K, A^\beta)$ and $k, s \in K$ we have

$$(\mathrm{id} \ltimes \beta_s)(\mathrm{id} \ltimes \iota)(f)(k) = \beta_s((\mathrm{id} \ltimes \iota)(f)(k)) = \beta_s(\iota(f(k))) = \iota(f(k)) = (\mathrm{id} \ltimes \iota)(f)(k).$$

We will use an averaging argument to show that $(\mathrm{id} \ltimes \iota)(K \ltimes_\alpha A^\beta) = (K \ltimes_\alpha A)^\beta$.

If $f \in K \ltimes_\alpha A$, then the map $K \to K \ltimes_\alpha A$ defined by $s \mapsto (\mathrm{id} \ltimes \beta_s)(f)$ is continuous and

$$\int_K \|(\mathrm{id} \ltimes \beta_s)(f)\|_{K \ltimes_\alpha A} \, \mathrm{d}s = \|f\|_{K \ltimes_\alpha A}. \tag{3.19}$$

Since $K \to K \ltimes_\alpha A, s \mapsto (\mathrm{id} \ltimes \beta_s)(f)$ for $f \in K \ltimes_\alpha A$ is continuous we can define

$$\mathrm{av}(f) = \int_K (\mathrm{id} \ltimes \beta_s)(f)\ \mathrm{d}s \in K \ltimes_\alpha A.$$

By construction $(\mathrm{id} \ltimes \beta_s)(\mathrm{av}(f)) = \mathrm{av}(f)$, so $\mathrm{av}(f) \in (K \ltimes_\alpha A)^\beta$. Note that if $f \in (K \ltimes_\alpha A)^\beta$, then $\mathrm{av}(f) = f$.

If $f \in (K \ltimes_\alpha A)^\beta$ and $\epsilon > 0$ there exists $g \in C_c(K, A)$ such that $\|f - g\|_{K \ltimes_\alpha A} < \epsilon$. Then

$$\mathrm{av}(g)(k) = \int_K \beta_s(g(k))\ \mathrm{d}s \in A$$

and in particular $\mathrm{av}(g) \in C_c(K, A^\beta)$. Then

$$\begin{aligned}
\|f - \mathrm{av}(g)\|_{K \ltimes_\alpha A} &= \|\mathrm{av}(f) - \mathrm{av}(g)\|_{K \ltimes_\alpha A} \\
&= \left\| \int_K (\mathrm{id} \ltimes \beta_s)(f - g)\ \mathrm{d}s \right\|_{K \ltimes_\alpha A} \\
&\leq \int_K \|(\mathrm{id} \ltimes \beta_s)(f - g)\|_{K \ltimes_\alpha A}\ \mathrm{d}s \\
&= \|f - g\|_{K \ltimes_\alpha A} < \epsilon
\end{aligned}$$

where the final equality follows from (3.19), and so $(\mathrm{id} \ltimes \iota)(K \ltimes_\alpha A^\beta)$ is dense in $(K \ltimes_\alpha A)^\beta$, as required. □

We now have a useful identification of certain crossed products, which originally appears in [72, Proposition 4.3], with a proof given for finite groups.

Theorem 3.13. *Let K be a compact group acting on a locally compact Hausdorff space X. Then*

$$K \ltimes C_0(X) \cong C_0(X, K(L^2(K)))^K$$

where K acts on $C_0(X)$ via the action induced from that of K on X, and K acts on $K(L^2(K))$ with the conjugation action with respect to the right regular representation on $L^2(K)$. If $f \in C_c(K, C_0(X)) \subseteq K \ltimes C_0(X)$, then

$$X \to K(L^2(K)), \quad x \mapsto T, \quad (T\xi)(r) := \int_K f(s)(s^{-1}r \cdot x)\xi(s^{-1}r)\ \mathrm{d}s, \quad \xi \in L^2(K), \quad r \in K$$

is the corresponding element of $C_0(X, K(L^2(K)))^K$ under this isomorphism.

In addition, $K \ltimes C_0(X)$ is a postliminal C^*-algebra.

Proof. For this proof, we will use many K-actions. Let us start by introducing them.

1. Let $\alpha : K \to \mathrm{Aut}(C_0(X))$ be the action of K on $C_0(X)$ induced by the action of K on X.
2. Let $\mathrm{lt} : K \to \mathrm{Aut}(C(K))$ be the action of K on $C(K)$ by left translation, i.e. $\mathrm{lt}(g)(f)(s) = f(g^{-1}s)$ for $g, s \in K$ and $f \in C(K)$.
3. Let $\mathrm{rt} : K \to \mathrm{Aut}(C(K))$ be the action of K on $C(K)$ by right translation, i.e. $\mathrm{rt}(g)(f)(s) = f(sg)$ for $g, s \in K$ and $f \in C(K)$.
4. Let $\lambda : K \to U(L^2(K))$ be the left regular representation.
5. Let $\rho : K \to U(L^2(K))$ be the right regular representation.
6. Let γ be the conjugation action of K on $K(L^2(K))$ by the right regular representation, i.e. $\gamma_k(T) = \rho_k T \rho_{k^{-1}}$ for $T \in K(L^2(K))$.
7. Let $\iota : K \to \mathrm{Aut}(C_0(X))$ be the trivial action of K on $C_0(X)$.

By the Stone-von Neumann theorem (see for instance [89, Theorem 4.24]),

$$K \ltimes_{\mathrm{lt}} C(K) \cong K(L^2(K)).$$

The isomorphism is implemented by considering the multiplication representation

$$M : C(K) \to B(L^2(K)), \quad M_f(g) = fg, \quad f \in C(K), \quad g \in L^2(K)$$

and showing $\lambda \ltimes M : K \ltimes_{\mathrm{lt}} C(K) \to B(L^2(K))$ is faithful, and with image equal to the compact operators (indeed, $(\lambda \ltimes M)(C_c(K, C(K)))$ are Hilbert-Schmidt operators, and we can also obtain all rank one operators in this way).

Under the isomorphism provided by the Stone-von Neumann theorem, the action γ of K on $K(L^2(K))$ corresponds to the action rt of K on $K \ltimes_{\mathrm{lt}} C(K)$ (see the proof of Lemma 3.12). This is because for $f \in C_c(K, C(K))$ and $k \in K$,

$$\begin{aligned}\gamma_k((\lambda \ltimes M)(f)) &= \gamma_k\left(\int_K \lambda_s M(f(s))\,\mathrm{d}s\right)\\ &= \int_K \rho_k \lambda_s M(f(s))\rho_k^*\,\mathrm{d}s\\ &= \int_K \lambda_s \rho_k M(f(s))\rho_k^*\,\mathrm{d}s\end{aligned}$$

and one can easily check $\rho_k M(f(s))\rho_k^* = M(\mathrm{rt}_k(f(s)))$ for all $s \in K$.

The actions $\alpha \otimes \mathrm{rt}$ and $\iota \otimes \mathrm{lt}$ on $C_0(X) \otimes C(K)$ commute and so $\alpha \otimes \mathrm{rt}$ defines a $\iota \otimes \mathrm{lt}$-equivariant automorphism (and consequently a group action) on $K \ltimes_{\iota\otimes\mathrm{lt}} (C_0(X) \otimes C(K))$, which we also call $\alpha \otimes \mathrm{rt}$.

Now

$$K \ltimes_{\iota\otimes\mathrm{lt}} (C_0(X) \otimes C(K)) \cong C_0(X) \otimes (K \ltimes_{\mathrm{lt}} C(K))$$

(c.f. [89, Theorem 2.75]), and a direct calculation tells us that the action $\alpha \otimes \mathrm{rt}$ on the left hand side corresponds to $\alpha \otimes \mathrm{rt}$ on the right hand side. Let us now apply all these facts. We have

$$\begin{aligned} C_0(X, K(L^2(K)))^K = (C_0(X) \otimes K(L^2(K)))^{\alpha\otimes\gamma} &\cong (C_0(X) \otimes (K \ltimes_{\mathrm{lt}} C(K)))^{\alpha\otimes\mathrm{rt}} \\ &\cong (K \ltimes_{\iota\otimes\mathrm{lt}} (C_0(X) \otimes C(K)))^{\alpha\otimes\mathrm{rt}} \\ &\cong K \ltimes_{\iota\otimes\mathrm{lt}} (C_0(X) \otimes C(K))^{\alpha\otimes\mathrm{rt}} \end{aligned}$$

where in the final step we have used Lemma 3.12. Identifying $C_0(X) \otimes C(K)$ with $C(K, C_0(X))$, we see that a function $f \in C(K, C_0(X))$ is invariant under $\alpha \otimes \mathrm{rt}$ if and only if

$$\alpha_k(f(sk)) = f(s)$$

for all $s, k \in K$, or equivalently if $e \in K$ is the identity element,

$$f(s) = \alpha_{s^{-1}}(f(e))$$

for all $s \in K$. In particular, evaluation at e produces an $(\iota\otimes\mathrm{lt})$-$\alpha$ equivariant isomorphism $C(K, C_0(X)) \to C_0(X)$, and so we can finally identify

$$K \ltimes_{\iota\otimes\mathrm{lt}} (C_0(X) \otimes C(K))^{\alpha\otimes\mathrm{rt}} \cong K \ltimes_\alpha C_0(X)$$

as required.

One can take an element of $C_c(K, C_0(X)) \subseteq K \ltimes_\alpha C_0(X)$ and apply the isomorphisms above to obtain the stated formula.

Finally, we note $C_0(X, K(L^2(K)))$ is postliminal - indeed the irreducible representations factor through the fibre algebras of this $C_0(X)$-algebra by Theorem A.21, and each fibre is $K(L^2(K))$, which has a unique class of irreducible unitary representations represented by the identity. Any subalgebra of a postliminal algebra is again postliminal and the result follows. □

If $f \in C_0(X, K(L^2(K)))^K$ as defined in Theorem 3.13 and if $x \in X$, then

$$f(x) \in K(L^2(K))^{K_x}. \tag{3.20}$$

Indeed, if we use the notation from the proof of Theorem 3.13 we have

$$(\alpha_k \otimes \gamma_k)(f)(x) = \gamma_k(f(k^{-1} \cdot x)) = \rho_k f(k^{-1} \cdot x)\rho_k^{-1} = f(x)$$

for all $k \in K$. In particular, for $k \in K_x$ we have $\rho_k f(x)\rho_k^{-1} = f(x)$, so $f(x) \in K(L^2(K))^{K_x}$. Let us try to understand the fixed point algebra $K(L^2(K))^{K_x}$. First we need a general result about compact operators between Hilbert spaces.

Proposition 3.14. *Let $\mathcal{H}, \mathcal{K}, \mathcal{L}$ and $\mathcal{M}$ be Hilbert spaces. Then*

$$K(\mathcal{H} \otimes \mathcal{K}, \mathcal{L} \otimes \mathcal{M}) = K(\mathcal{H}, \mathcal{L}) \otimes K(\mathcal{K}, \mathcal{M})$$

where

$$\begin{aligned} K(\mathcal{H}, \mathcal{L}) \otimes K(\mathcal{K}, \mathcal{M}) &:= \overline{K(\mathcal{H}, \mathcal{L}) \odot K(\mathcal{K}, \mathcal{M})} \\ &:= \overline{\operatorname{span}}\{T \otimes S \mid T \in K(\mathcal{H}, \mathcal{L}), S \in K(\mathcal{K}, \mathcal{M})\} \subseteq B(\mathcal{H} \otimes \mathcal{K}, \mathcal{L} \otimes \mathcal{M}). \end{aligned}$$

Proof. Let $h \in \mathcal{H}$ and $l \in \mathcal{L}$. We have the standard rank one operators

$$\theta_{h,l} : \mathcal{H} \to \mathcal{L}, \quad \theta_{h,l}(h') = \langle h, h' \rangle l, \quad h' \in \mathcal{H}.$$

We first note that if $T \in K(\mathcal{H}, \mathcal{L})$ and $S \in K(\mathcal{K}, \mathcal{M})$ then $T \otimes S \in B(\mathcal{H} \otimes \mathcal{K}, \mathcal{L} \otimes \mathcal{M})$, and in fact is a compact operator. This is because the tensor product of two finite rank operators is again a finite rank operator because

$$(\theta_{h,l} \otimes \theta_{k,m})(h' \otimes k') = \langle h, h' \rangle \langle k, k' \rangle l \otimes m = \theta_{h \otimes k, l \otimes m}(h' \otimes k')$$

for $h, h' \in \mathcal{H},\ k, k' \in \mathcal{K},\ l \in \mathcal{L}$ and $m \in \mathcal{M}$. This calculation also shows that we can obtain all the finite rank operators $\mathcal{H} \otimes \mathcal{K} \to \mathcal{L} \otimes \mathcal{M}$ in this way, and hence all the compact operators. □

Proposition 3.15. *Let K be a compact group acting on a locally compact Hausdorff space X. Suppose K acts on $K(L^2(K))$ by the conjugation action with the right regular representation on $L^2(K)$. Then*

$$K(L^2(K))^{K_x} \cong \bigoplus_{\pi \in \mathrm{Irr}(K_x)} K(\mathrm{Ind}_{K_x}^K(\pi))$$

where K_x is the stabilizer subgroup of a point $x \in X$ and $\mathrm{Irr}(K_x)$ is the set of unitary equivalence classes of irreducible representations of K_x.

Proof. Let $\pi : K_x \to U(W_\pi)$ be an irreducible unitary representation of K_x on a Hilbert space W_π. As a consequence of the Peter-Weyl theorem (see [24, 5.12], but take care with inner products - there the inner products are linear in the first variable) applied to K_x, we have

$$L^2(K_x) \cong \bigoplus_{\pi \in \mathrm{Irr}(K_x)} W_\pi \otimes W_\pi^*$$

which is equivariant with respect to the action of K on $L^2(K_x)$ by the left regular representation and the action of K on $W_\pi \otimes W_\pi^*$ by π on the first leg and the trivial representation in the second. This isomorphism (from the right hand side to the left) is given by

$$v \otimes \overline{w} \mapsto (k \mapsto \dim(W_\pi)^{\frac{1}{2}} \langle w, \pi(k^{-1})v \rangle).$$

where $v, w \in W_\pi$ and $k \in K_x$. We can induce the left regular representation of K_x on $L^2(K_x)$ to K. By Example 3.3, the result is (up to equivalence) the left regular representation on K. By Proposition 3.4,

$$L^2(K) \cong \bigoplus_{\pi \in \mathrm{Irr}(K_x)} \mathrm{Ind}_{K_x}^K(\pi) \otimes W_\pi^*$$

where the map from the right hand side to the left is given by

$$\xi \otimes \overline{w} \mapsto (k \mapsto \dim(W_\pi)^{\frac{1}{2}} \langle w, \xi(k) \rangle). \tag{3.21}$$

where $\xi \in \mathrm{Ind}_{K_x}^K(\pi)$, $w \in W_\pi$ and $k \in K$. Note that this isomorphism is K_x-equivariant with respect to the action of K_x on W_π^* by the contragredient of π (which we denote by π^*, see Proposition 1.32 and compare this with the definition in [24, p.g. 69]) and the right translation action of K_x on $L^2(K)$. Therefore

$$K(L^2(K))^{K_x} \cong K\left(\bigoplus_{\pi \in \mathrm{Irr}(K_x)} \mathrm{Ind}_{K_x}^K(\pi) \otimes W_\pi^* \right)^{\oplus_\pi (1 \otimes \pi^*(K_x))}.$$

For notational convenience, we set $H_{\pi,x} := \mathrm{Ind}_{K_x}^K(\pi)$. Now we can break a compact operator on a direct sum of Hilbert spaces into blocks of compact operators between Hilbert spaces by viewing operators in $K\left(\bigoplus_{\pi \in \mathrm{Irr}(K_x)} H_{\pi,x} \otimes W_\pi^*\right)$ as infinite matrices of the form

$$\begin{pmatrix} \ddots & \vdots & \vdots & \iddots \\ \cdots & K(H_{\pi,x} \otimes W_\pi^*) & K(H_{\pi',x} \otimes W_{\pi'}^*, H_{\pi,x} \otimes W_\pi^*) & \cdots \\ \cdots & K(H_{\pi,x} \otimes W_\pi^*, H_{\pi',x} \otimes W_{\pi'}^*) & K(H_{\pi',x} \otimes W_{\pi'}^*) & \cdots \\ \iddots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The invariance condition restricted to $K(H_{\pi,x} \otimes W_\pi^*, H_{\pi',x} \otimes W_{\pi'}^*)$ means we consider the operators

$$\{T \in K(H_{\pi,x} \otimes W_\pi^*, H_{\pi',x} \otimes W_{\pi'}^*) \mid (1 \otimes \pi'^*(k))T(1 \otimes \pi^*(k^{-1})) = T\}. \tag{3.22}$$

Inside this operator space we have the subspace

$$U := \operatorname{span}\{T \otimes S \mid T \in K(H_{\pi,x}, H_{\pi',x}), S \in K(W_\pi^*, W_{\pi'}^*),\ T \otimes \pi'^*(k)S\pi^*(k^{-1}) = T \otimes S\}.$$

In fact U is a dense subspace of (3.22). Indeed, given $\epsilon > 0$, and T in the operator space (3.22) we can find a $T' \in K(H_{\pi,x}, H_{\pi',x}) \odot K(W_\pi^*, W_{\pi'}^*)$ such that $\|T - T'\| < \epsilon$ by Proposition 3.14. Define

$$T'' := \int_K (1 \otimes \pi'^*(k))T'(1 \otimes \pi^*(k^{-1}))\, \mathrm{d}k.$$

Then T'' satisfies the invariance condition in the definition of (3.22), $T'' \in U$, and $\|T - T''\| < \epsilon$.

Now, if $T \otimes S$ is an elementary tensor in U, for all $h \in H_{\pi,x}$, $w \in W_\pi^*$ we have:

$$(T \otimes S)(h \otimes w) = Th \otimes Sw = Th \otimes \pi'^*(k)S\pi^*(k^{-1})w$$

In particular, if $Th \neq 0$, we must have $Sw = \pi'^*(k)S\pi^*(k^{-1})w$ for all w (see [87, T.2.8]). This means S is an intertwiner between π'^* and π^*. Schur's lemma (Theorem 1.25) tells us S is non zero if and only if $\pi = \pi'$, and in this case, S is a scalar. Therefore

$$K(L^2(K))^{K_x} \cong \bigoplus_{\pi \in \operatorname{Irr}(K_x)} K(H_{\pi,x}) \otimes \mathbb{C}1_{W_\pi^*} \cong \bigoplus_{\pi \in \operatorname{Irr}(K_x)} K(H_{\pi,x})$$

as required. □

Let us consider Proposition 3.15 in our situation, where $X = K$, and K acts on itself via the adjoint action. By Lemma 3.8, if $x = \pm I$, then $K_x = K$, and so $\operatorname{Ind}_{K_x}^K(V(n)) = V(n)$ by Example 3.2. Therefore

$$K(L^2(K))^K \cong \bigoplus_{n \in \frac{1}{2}\mathbb{N}_0} K(V(n))$$

by Proposition 3.15, where $L^2(K) \cong \bigoplus_{n \in \frac{1}{2}\mathbb{N}_0} V(n) \otimes V(n)^*$ via the left regular representation. If we are given $T = (T_n) \in \bigoplus_{n \in \frac{1}{2}\mathbb{N}_0} K(V(n))$ then T acts on this decomposition by $\oplus_{n \in \frac{1}{2}\mathbb{N}_0} T_n \otimes 1_{V(n)^*}$.

Now suppose $x \in T \setminus \{\pm I\}$. Recall that for $m \in \frac{1}{2}\mathbb{Z}$,

$$\mathcal{H}_{(m,x)} = \mathrm{Ind}_T^K(\tau_m) = \{\xi \in L^2(K) \mid \xi(kt) = \tau_m(t)^{-1}\xi(k) \text{ for all } t \in T,\, k \in K\} \subseteq L^2(K)$$

see (3.12). Then by Lemma 3.8, $K_x = T$, and by Proposition 3.15,

$$K(L^2(K))^T \cong \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} K(\mathcal{H}_{(m,x)})$$

where $L^2(K) \cong \bigoplus_{m\in\frac{1}{2}\mathbb{Z}} \mathcal{H}_{(m,x)} \otimes W_m^*$, and W_m is the carrier space of τ_m. If we are given $T = (T_m) \in \bigoplus_{m\in\frac{1}{2}\mathbb{Z}} K(\mathcal{H}_{(m,x)})$ then T acts on this decomposition by $\oplus_{m\in\frac{1}{2}\mathbb{Z}} T_m \otimes 1_{W_m^*}$.

Let us identify $\mathcal{H}_{(m,x)} \subseteq L^2(K)$ as a subspace of the right hand side of the isomorphism

$$L^2(K) \cong \bigoplus_{n \in \frac{1}{2}\mathbb{N}_0} V(n) \otimes V(n)^*.$$

Applying the right regular representation ρ_t on the left hand side means applying $\oplus_n 1_{V(n)} \otimes \pi_n^*(t)$ on the right hand side, where π_n is the standard $2n+1$ dimensional irreducible representation of K on $V(n)$, constructed using Theorem 1.49 and Proposition 1.54. On $\mathcal{H}_{(m,x)}$ ρ_t must be the same as $\oplus_n 1_{V(n)} \otimes \tau_m(t)^{-1} 1_{V(n)^*}$, see the definition of $\mathcal{H}_{(m,x)}$ above. We therefore have

$$\mathcal{H}_{(m,x)} \cong \bigoplus_{n \in \frac{1}{2}\mathbb{N}_0} V(n) \otimes (V(n))^*_{-m} \tag{3.23}$$

where $(V(n))^*_{-m} := \{\overline{v} \in V(n)^* \mid \pi_n^*(t)\overline{v} = \tau_m(t)^{-1}\overline{v}\}$.

One can check using (3.13) that

$$(V(n))^*_{-m} = \mathbb{C}\overline{e^n_m}, \tag{3.24}$$

where $\{e_i^n\}$ is the orthonormal weight basis of $V(n)$ constructed in Theorem 1.49. Note therefore that $(V(n))^*_{-m}$ is non-zero if and only if $n \geq |m|$, and $n + m \in \mathbb{Z}$.

The description of the fixed point algebras given in Proposition 3.15 allows us to describe the irreducible representations of the crossed product arising from the action of a compact group on a locally compact Hausdorff space under the isomorphism provided by Theorem 3.13.

Proposition 3.16. *Let K be a compact group acting on a locally compact Hausdorff space X, $x \in X$, and V be an irreducible unitary representation of the stabilizer group K_x of x. Then, under the isomorphism*

$$K \ltimes C_0(X) \cong C_0(X, K(L^2(K)))^K$$

provided by Theorem 3.13, the irreducible representation $\pi_{(V,x)} \ltimes \rho_{(V,x)}$ *corresponds to evaluation of functions in* $C_0(X, K(L^2(K)))^K$ *at* x *followed by projection onto the summand* $K(\mathrm{Ind}_{K_x}^K(V))$ *of* $K(L^2(K))^{K_x}$ *(see Proposition 3.15).*

Proof. Recall from (3.21) that the isomorphism

$$L^2(K) \cong \bigoplus_{\pi \in \mathrm{Irr}(K_x)} \mathrm{Ind}_{K_x}^K(\pi) \otimes W_\pi^*$$

is given by the formula (from the right hand side to left hand side)

$$\xi \otimes \overline{w} \mapsto (k \mapsto \dim(W_\pi)^{\frac{1}{2}} \langle w, \xi(k) \rangle) \tag{3.25}$$

where $\xi \in \mathrm{Ind}_{K_x}^K(\pi)$, $w \in W_\pi$ and $k \in K$. Let $f \in C_c(K, C_0(X)) \subseteq K \ltimes C_0(X)$. Then by Theorem 3.13, f corresponds to an element $g \in C_0(X, K(L^2(K)))^K$ with

$$(g(x)\eta)(k) = \int_K f(s)(s^{-1}k \cdot x)\eta(s^{-1}k)\ \mathrm{d}s, \quad \eta \in L^2(K), \quad k \in K.$$

Taking η to be the map $(k \mapsto \dim(W_\pi)^{\frac{1}{2}} \langle w, \xi(k) \rangle)$ as in (3.25), we have

$$\begin{aligned}(g(x)\eta)(k) &= \dim(W_\pi)^{\frac{1}{2}} \left\langle w, \int_K f(s)(s^{-1}k \cdot x)\xi(s^{-1}k)\ \mathrm{d}s \right\rangle \\ &= \dim(W_\pi)^{\frac{1}{2}} \left\langle w, ((\pi_{(V,x)} \ltimes \rho_{(V,x)})(f)\xi)(k) \right\rangle.\end{aligned}$$

The result follows. □

Let us write $A = C^*(G_1)$. We will now determine the topology of $\mathrm{Spec}(A)$. By Theorem 3.13, we have $A \cong C(K, K(L^2(K)))^K$ and so A is postliminal. In particular the canonical surjection $\mathrm{Spec}(A) \to \mathrm{Prim}(A)$ is a homeomorphism (see [17, Theorem 4.3.7]). Therefore to determine the spectrum of A as a topological space it suffices to work with primitive ideals and the Jacobson topology. We will work with the description of A as functions on K taking values in $K(L^2(K))$ in the following.

We will determine all the closed sets of $\mathrm{Prim}(A)$ and hence we will understand the topology of $\mathrm{Prim}(A)$. Let

$$F = \{\mathrm{Ker}(\pi_{(m,x)}) \mid (m, x) \in S \subseteq M\},$$

where

$$M := \frac{1}{2}\mathbb{Z} \times T$$

(see (3.16)) with the usual topology, and S is a subset of M. Any subset of $\mathrm{Prim}(A)$ can be written this way for a suitable choice of S. We view S as a subspace of M. Recall from

(3.2) that

$$\overline{F} = \left\{ P \in \mathrm{Prim}(A) \mid \bigcap_{J \in F} J \subseteq P \right\}.$$

Suppose (m, x) is a limit point of S, with $x \neq \pm I$ (we will deal with the case $x = \pm I$ separately). We claim that $\mathrm{Ker}(\pi_{(m,x)}) \in \overline{F}$.

Since (m, x) is a limit point of S, there exists a sequence $(m_i, x_i) \in S$ such that (m_i, x_i) converges to (m, x). Eventually $m_i = m$ and $x_i \neq \pm I$ and so by reindexing we obtain a sequence $(m, x_i) \in S$ converging to (m, x), with $x_i \neq \pm I$.

Suppose $f \in \bigcap_{J \in F} J$. In particular, $f \in \bigcap_i \mathrm{Ker}(\pi_{(m,x_i)})$. By continuity of f, $f(x_i) \to f(x)$ in $K(L^2(K))$. Let P_m denote the projection to the m^{th} summand of

$$K(L^2(K))^T \cong \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} K(\mathrm{Ind}_T^K(\tau_m)).$$

Then $P_m(f(x_i))$ vanishes for all i. Therefore we must have $P_m(f(x)) = 0$. In particular $f \in \mathrm{Ker}(\pi_{(m,x)})$, and so $\mathrm{Ker}(\pi_{(m,x)}) \in \overline{F}$ as required.

Next we suppose $(m, \pm I)$ is a limit point of S. We claim that $\mathrm{Ker}(\pi_{(m,\pm I)}) \in \overline{F}$.

We can assume there exists a sequence $(m, x_i) \in S$, $x_i \neq \pm I$ with $x_i \to \pm I$. Suppose $f \in \bigcap_{J \in F} J$. In particular, $f \in \bigcap_i \mathrm{Ker}(\pi_{(m,x_i)})$. We know $f(x_i) \to f(\pm I)$ in $K(L^2(K))$. We have

$$L^2(K) \cong \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_{(m,x)} \cong \bigoplus_{m \in \mathbb{Z}} \left(\bigoplus_{n \in \frac{1}{2}\mathbb{N}_0} V(n) \otimes (V(n))^*_{-m} \right).$$

Since $f(\pm I)$ is in the image of the left regular representation, it acts on each $V(n)$ in the decomposition $L^2(K) \cong \bigoplus_{n \in \frac{1}{2}\mathbb{N}_0} V(n) \otimes V(n)^*$ by $\oplus_n \pi_n \otimes 1_{V(n)^*}$. In particular, it acts on $V(n) \otimes (V(n))^*_{-m}$ in the same way, independently of m. Since each $f(x_i)$ vanishes on $V(n) \otimes (V(n))^*_{-m}$, then $f(\pm I)$ must do the same. In particular, $f(\pm I)$ vanishes on those $V(n)$ for which $|m| \leq n$, $n + m \in \mathbb{Z}$. Therefore $f \in \mathrm{Ker}(\pi_{(n,\pm I)})$ $n \geq |m|$, $n + m \in \mathbb{Z}$.

So far we see that

$$\{\mathrm{Ker}(\pi_{(m,x)}) \mid (m, x) \in \overline{S}, x \neq \pm I\} \cup \{\mathrm{Ker}(\pi_{(n,\pm I)}) \mid (m, \pm I) \in \overline{S}, n \geq |m|, n+m \in \mathbb{Z}\} \subseteq \overline{F}.$$

We claim this is the whole of $\overline{F}$.

Let $\pi_{(m,x)}$ be an irreducible representation of A whose kernel is not in the left hand side. One can construct elements f of A such that $\pi_{(m,x)}(f) \neq 0$ but $\pi_{(n,y)}(f) = 0$ for all

$(n, y) \in S$. Therefore

$$\overline{F} = \{\mathrm{Ker}(\pi_{(m,x)}) \mid (m,x) \in \overline{S}, x \neq \pm I\} \cup \{\mathrm{Ker}(\pi_{(n,\pm I)}) \mid (m, \pm I) \in \overline{S}, n \geq |m|, n+m \in \mathbb{Z}\}.$$

In particular we have determined all the closed sets of $\mathrm{Prim}(A)$, and by taking complements we can find all open sets.

It is difficult to write down this topology in a more concrete way, however we can see that away from the points for which $x = \pm I$, we have the usual topology of M, and that any sequence converging to $\pi_{(m,\pm I)}$ (where $m \in \frac{1}{2}\mathbb{N}_0$) converges to $\pi_{(n,\pm I)}$, $m \leq n \in \frac{1}{2}\mathbb{N}_0$, $n + m \in \mathbb{Z}$. In particular the spectrum is not Hausdorff.

The method we employ in the following sections relies on the observation that this unusual behaviour in the topology in the spectrum disappears when we fix $m \in \frac{1}{2}\mathbb{N}_0$ and consider only the subspace $\{\pi_{(m,x)}\}$ where x runs over the appropriate set. To this end, we will consider appropriate subquotients of $C^*(G_1)$, which have these subspaces as spectrum.

To finish this section, we provide a folklore lemma that will be useful to us in understanding the subquotients in the sequel (see [32, Lemma 6.1] for the statement without proof).

Lemma 3.17. *Let A be a C^*-algebra and let $p \in M(A)$ be a projection. Suppose that for every irreducible $*$-representation π of A the operator $\pi(p)$ is a rank-one projection onto a one-dimensional subspace of the carrier Hilbert space $\mathcal{H}_\pi$ of π, spanned by a unit vector $v_\pi \in \mathcal{H}_\pi$. Then*

(a) $ApA = A$, A is a liminal C^*-algebra, and pAp is commutative.

(b) $\mathrm{Spec}(A)$ is a locally compact Hausdorff space.

(c) If $a \in pAp$ and $\pi \in \mathrm{Spec}(A)$, then $\pi(a) = \widehat{a}(\pi)\pi(p)$, where

$$\widehat{a}(\pi) := \langle v_\pi, \pi(a)v_\pi \rangle \in \mathbb{C}$$

This defines a $*$-isomorphism

$$pAp \to C_0(\mathrm{Spec}(A)), \quad a \mapsto \widehat{a}.$$

(d) The inclusion $pAp \hookrightarrow ApA = A$ induces an isomorphism in K-theory $K_*(pAp) \to K_*(A)$.

Proof.

(a) Let π be an irreducible representation of A on a Hilbert space $\mathcal{H}_\pi$. Then $\pi(p) \in K(\mathcal{H}_\pi)$ (being a rank one projection) and so $\pi(ApA) \subseteq K(\mathcal{H}_\pi)$.

If $h \in \mathcal{H}_\pi$ and $a, b \in A$, we have

$$\pi(a)\pi(p)\pi(b)h = \langle v_\pi, \pi(b)h \rangle \pi(a)v_\pi$$

and by appropriate choice of $a, b \in A$ and irreducibility of π, we have that $\pi(ApA)$ contains all rank one operators. Therefore $\pi(ApA) = K(\mathcal{H}_\pi)$.

Since ApA is a closed ideal of A, it is the intersection of the primitive ideals containing it. However, $\pi(ApA) \neq 0$ for any irreducible representation, and so ApA is not contained in any primitive ideal of A. Hence $ApA = A$.

Note that in particular, we must have $\pi(A) = K(\mathcal{H}_\pi)$, and so A is necessarily a liminal C^*-algebra.

To show pAp is commutative, it suffices to show that $\pi(pAp)$ is commutative for all irreducible representations π. Using the notation above,

$$\pi(pap)h = \langle v_\pi, h \rangle \langle v_\pi, \pi(a)v_\pi \rangle v_\pi.$$

Therefore

$$\begin{aligned}\pi(pap)\pi(pbp)h &= \langle v_\pi, h \rangle \langle v_\pi, \pi(b)v_\pi \rangle \pi(pap)v_\pi \\ &= \langle v_\pi, h \rangle \langle v_\pi, \pi(b)v_\pi \rangle \langle v_\pi, \pi(a)v_\pi \rangle v_\pi\end{aligned}$$

Interchanging the roles of a and b we see that $\pi(pap)\pi(pbp) = \pi(pbp)\pi(pap)$, and so pAp is commutative.

(b) It is well known that $\operatorname{Spec}(A)$ is locally compact, see [17, Corollary 3.3.8].

For the Hausdorff condition, we will find, for two irreducible representations $\pi_1, \pi_2 \in \operatorname{Spec}(A)$, a continuous function on $\operatorname{Spec}(A)$ that separates π_1 and π_2.

First, we use ideas from Dixmier [17, Lemma 4.4.2]. Let $y \in M(A)$ be a positive element such that $y \leq p$. We claim that the map $\pi \mapsto \operatorname{Trace}(\pi(y))$ is continuous on $\operatorname{Spec}(A)$. This is generally a lower semicontinuous function (see [17, Proposition 3.5.9]) regardless of the choice of y.

Let $x = p - y \geq 0$. Then

$$\operatorname{Trace}(\pi(p)) - \operatorname{Trace}(\pi(y)) = \operatorname{Trace}(\pi(x))$$

and since $\pi(p)$ is of rank one, we have

$$\mathrm{Trace}(\pi(x)) + \mathrm{Trace}(\pi(y)) = \mathrm{Trace}(\pi(p)) = 1$$

Then $\mathrm{Trace}(\pi(y)) = 1 - \mathrm{Trace}(\pi(x))$ is an upper semicontinuous function, being the difference of a continuous and lower semicontinuous function.

We will now pick y appropriately so that $\mathrm{Trace}(\pi_1(y)) \neq 0$ and $\mathrm{Trace}(\pi_2(y)) = 0$. Since A is liminal, we may find $0 \neq x \in A$ such that $\pi_1(x) = \pi_1(p)$ and $\pi_2(x) = 0$, see [17, Proposition 4.2.5]. We may assume that x is positive, because $\pi_1(x^*x) = \pi_1(p^*p) = \pi_1(p)$ and $\pi_2(x^*x) = 0$.

One can check that $y := \frac{1}{\|x\|} pxp \leq p$ using Cauchy-Schwarz. Therefore the map $\pi \mapsto \mathrm{Trace}(\pi(y))$ is continuous by what we have seen above. Now we note that

$$\mathrm{Trace}(\pi_1(y)) = \frac{1}{\|x\|} \mathrm{Trace}(\pi_1(pxp)) = \frac{1}{\|x\|} \mathrm{Trace}(\pi_1(p^3)) = \frac{1}{\|x\|}$$

and

$$\mathrm{Trace}(\pi_2(y)) = 0.$$

Therefore $\mathrm{Spec}(A)$ is Hausdorff.

(c) Let $a \in pAp$. Then if $\pi \in \mathrm{Spec}(A)$,

$$\pi(a) = \pi(pap) = \langle v_\pi, \pi(a) v_\pi \rangle \pi(p)$$

Note that $\widehat{a}(\pi) = \langle v_\pi, \pi(a) v_\pi \rangle$ is independent of the choice of representative for the class of π, because if π_1 and π_2 are equivalent, we can choose an intertwining unitary that will send v_{π_1} to v_{π_2}.

We now need to check that $\widehat{a}$ is continuous on $\mathrm{Spec}(A)$ and vanishes at ∞. For the fact $\widehat{a}$ vanishes at infinity, note that

$$|\widehat{a}(\pi)| = |\langle v_\pi, \pi(a) v_\pi \rangle| = \|\pi(pap)\| = \|\pi(a)\|.$$

Since the set $\{\pi \in \mathrm{Spec}(A) \mid \|\pi(a)\| \geq k\}$ is compact for each $k > 0$ (see [17, Proposition 3.3.7]), we have that the map $\pi \mapsto \widehat{a}(\pi)$ vanishes at infinity. For continuity, note that because $a = pap$, $a \leq \|a\| p$ using a similar argument to that in part (b). Hence $\pi \mapsto \frac{1}{\|a\|} \mathrm{Trace}(\pi(a))$ is continuous, and so $\pi \mapsto \mathrm{Trace}(\pi(a))$ is continuous. Then we note that $\mathrm{Trace}(\pi(a)) = \langle v_\pi, \pi(a) v_\pi \rangle$, and the result follows.

We therefore have a well-defined map

$$pAp \to C_0(\mathrm{Spec}(A)), \quad a \mapsto \widehat{a} \tag{3.26}$$

and one can easily check this is a $*$-homomorphism. Since $|\widehat{a}(\pi)| = \|\pi(a)\|$ for all $\pi \in \mathrm{Spec}(A)$ the $*$-homomorphism (3.26) is isometric and hence injective.

For surjectivity, if $\pi_1, \pi_2 \in \mathrm{Spec}(A)$ are distinct points, then there exists $a \in A$ such that $\pi_1(a) = \pi_1(p)$ and $\pi_2(a) = 0$. Then $\widehat{a}(\pi_1) = 1$ and $\widehat{a}(\pi_2) = 0$, and so surjectivity follows by Stone-Weierstrass.

(d) This follows from the fact there is a Morita equivalence between pAp and $ApA = A$, see [32, Remark 6.2]. □

3.2 Subquotients of $C_r^*(G_q)$

In this section we will study certain subquotients of $C_r^*(G_q)$ for $q \in (0,1]$. We start with the case of $q = 1$. Recall that $C_r^*(G_1) \cong K \ltimes_{\mathrm{adj}} C(K)$ by Proposition 2.2.

For $n \in \frac{1}{2}\mathbb{N}_0$, set $p_n := \omega^n_{nn} \bowtie 1 \in \mathcal{D}(G_1) \subseteq K \ltimes_{\mathrm{adj}} C(K)$. Then p_n is a projection, because by (1.20) we have

$$\begin{aligned}
(\omega^n_{nn} \bowtie 1)(\omega^n_{nn} \bowtie 1) &= \omega^n_{nn}((\omega^n_{nn})_{(1)}, 1)(\omega^n_{nn})_{(2)} \bowtie 1(S((\omega^n_{nn})_{(3)}), 1)1 \\
&= \omega^n_{nn}\epsilon((\omega^n_{nn})_{(1)})(\omega^n_{nn})_{(2)}\epsilon((\omega^n_{nn})_{(3)}) \bowtie 1 \\
&= \omega^n_{nn}\omega^n_{nn} \bowtie 1 \\
&= \omega^n_{nn} \bowtie 1
\end{aligned}$$

and by (1.21) we have

$$(\omega^n_{nn} \bowtie 1)^* = ((\omega^n_{nn})_{(1)}, 1)(\omega^n_{nn})_{(2)} \otimes 1(S((\omega^n_{nn})_{(3)}), 1) = \omega^n_{nn} \bowtie 1.$$

Note that as an element of the convolution algebra $C_c(K, C(K)) \subseteq K \ltimes_{\mathrm{adj}} C(K)$, $p_n = (2n+1)(u^n_{nn})^* \otimes 1$, see the proof of Proposition 2.2 and the formula for the inverse Fourier transform (1.17) given in Example 1.59.

The fact that p_n is a projection tells us that the image of p_n under any representation of $K \ltimes_{\mathrm{adj}} C(K)$ is an orthogonal projection, and we shall shortly determine the corresponding subspaces in the carrier space of each irreducible representation of $K \ltimes_{\mathrm{adj}} C(K)$.

If $n \in \frac{1}{2}\mathbb{N}_0$ and $i, j \in \{-n, -n+1, ..., n\}$ we define $d^n_{ji} \in C_c(K)$ by

$$d^n_{ji} : K \to \mathbb{C}, \quad d^n_{ji}(s) = (2n+1)\langle \pi_n(s)e^n_i, e^n_j\rangle_{V(n)} = (2n+1)(u^n_{ji})^*(s). \tag{3.27}$$

where e^n_i is an element of the usual orthonormal basis for $V(n)$, the carrier space for the $2n+1$ dimensional irreducible representation π_n of K, constructed using Theorem 1.49 and Proposition 1.54. Also note that in the formula (3.27) the involution $(u^n_{ji})^*$ is taken in $\mathcal{O}(K) \subseteq C(K)$. The index labelling is chosen so the subscripts on the d and u match.

One can directly check (using the Schur orthogonality relations, Theorem 1.25) that

$$d^n_{nn} \star d^n_{nn} = d^n_{nn}, \quad (d^n_{nn})^* = d^n_{nn}$$

where $\star$ is the convolution product and the involution is taken inside $C_c(K)$, so d^n_{nn} is a projection in $C_c(K)$. Note that $p_n = d^n_{nn} \otimes 1 \in C_c(K, C(K))$.

We can also view $d^n_{ji} \in L^2(K)$. Again using Schur orthogonality relations we have $\left\|d^n_{ji}\right\|^2_{L^2(K)} = 2n+1$. Set

$$e^n_{ji} := (2n+1)^{-\frac{1}{2}} d^n_{ji} = (2n+1)^{\frac{1}{2}}(u^n_{ji})^*(s) \tag{3.28}$$

so that $\left\|e^n_{ji}\right\|_{L^2(K)} = 1$. Note that $e^n_{ji} \in \mathcal{H}_{(i,x)}$ for all $x \in T \setminus \{\pm I\}$, and that this is not the same as the definition of the unit vectors given in (2.7).

Lemma 3.18. *Let $n \in \frac{1}{2}\mathbb{N}_0$. Under the isomorphism*

$$L^2(K) \cong \bigoplus_{m \in \frac{1}{2}\mathbb{N}_0} V(m) \otimes V(m)^*$$

the operator $\lambda(d^n_{nn}) \in B(L^2(K))$ corresponds to $p_{\mathbb{C}e^n_n} \otimes 1_{V(n)^*} \in B(V(n) \otimes V(n)^*)$, where λ is the left regular representation of K and $p_{\mathbb{C}e^n_n} : V(n) \to V(n)$ is the rank one projection onto $\mathbb{C}e^n_n$.

Proof. As an operator on $\bigoplus_{m \in \frac{1}{2}\mathbb{N}_0} V(m) \otimes V(m)^*$, $\lambda(d^n_{nn})$ is $\oplus_m \left(\pi_m(d^n_{nn}) \otimes 1_{V(m)^*}\right)$. Let us therefore understand $\pi_m(d^n_{nn})$, which is again a projection. Let $\eta, \xi \in V(m)$. Then

$$\begin{aligned}
\langle \eta, \pi_m(d^n_{nn})\xi\rangle_{V(m)} &= \int_K \langle \eta, d^n_{nn}(s)\pi_m(s)\xi\rangle \, \mathrm{d}s \\
&= \int_K (2n+1)\langle \pi_n(s)e^n_n, e^n_n\rangle_{V(n)}\langle \eta, \pi_m(s)\xi\rangle_{V(m)} \, \mathrm{d}s \\
&= (2n+1)\int_K \langle \eta, \pi_m(s)\xi\rangle_{V(m)}\overline{\langle e^n_n, \pi_n(s)e^n_n\rangle_{V(n)}} \, \mathrm{d}s \\
&= \delta_{nm}\langle \eta, e^n_n\rangle_{V(n)}\langle e^n_n, \xi\rangle_{V(n)}
\end{aligned}$$

where in the last equality we use the Schur orthogonality relations, Theorem 1.30. Therefore $\pi_m(d^n_{nn}) = 0$ unless $m = n$, and $\pi_n(d^n_{nn}) = p_{\mathbb{C}e^n_n}$, the projection onto $\mathbb{C}e^n_n$. $\square$

Lemma 3.19. *Let $n \in \frac{1}{2}\mathbb{N}_0$. Let e^n_n denote the highest weight vector in the representation $V(n)$ of K. Then if $m \in \frac{1}{2}\mathbb{Z}$ and $x \in T \setminus \{\pm I\}$,*

$$\pi_{(m,x)}(p_n) = \begin{cases} \text{orthogonal projection onto } \mathbb{C}e^n_{nm} \subseteq \mathcal{H}_{(m,x)} & |m| \leq n, n+m \in \mathbb{Z} \\ 0 & \text{else} \end{cases}$$

and if $m \in \frac{1}{2}\mathbb{N}_0$

$$\pi_{(m,\pm I)}(p_n) = \begin{cases} \text{orthogonal projection onto } \mathbb{C}e^n_n \subseteq V(m) & n = m \\ 0 & \text{else.} \end{cases}$$

Proof. Consider the case where $x \neq \pm I$ first. Recall from (3.11) that

$$\pi_{(m,x)} := \pi_{(\tau_m,x)} \ltimes \rho_{(\tau_m,x)} : K \ltimes_{\text{adj}} C(K) \to B(\mathcal{H}_{(m,x)})$$

where

$$(\pi_{(\tau_m,x)}(k)\xi)(s) = \xi(k^{-1}s), \quad (\rho_{(\tau_m,x)}(f)\xi)(s) = f(sxs^{-1})\xi(s)$$

for $k, s \in K$, $\xi \in \mathcal{H}_{(m,x)}$ and $f \in C(K)$. Then because $p_n = d^n_{nn} \otimes 1$,

$$\begin{aligned} \pi_{(m,x)}(p_n) &= \int_K \pi_{(\tau_m,x)}(s)\rho_{(\tau_m,x)}(p_n(s)) \, \mathrm{d}s \\ &= \int_K \pi_{(\tau_m,x)}(s) d^n_{nn}(s) \rho_{(\tau_m,x)}(1) \, \mathrm{d}s \\ &= \int_K d^n_{nn}(s)\pi_{(\tau_m,x)}(s) \, \mathrm{d}s. \end{aligned}$$

Therefore

$$\pi_{(m,x)}(p_n) = \pi_{(\tau_m,x)}(d^n_{nn})$$

which is in turn given by the restriction of $\lambda(d^n_{nn})$ to $\mathcal{H}_{(m,x)}$, viewed as the subspace

$$\{\xi \in L^2(K) \mid \xi(kt) = \tau_m(t)^{-1}\xi(k) \text{ for all } t \in T,\ k \in K\} \subseteq L^2(K).$$

Recall from (3.23) that

$$\mathcal{H}_{(m,x)} \cong \bigoplus_{n \in \frac{1}{2}\mathbb{N}_0} V(n) \otimes (V(n))^*_{-m} \subseteq \bigoplus_{n \in \frac{1}{2}\mathbb{N}_0} V(n) \otimes V(n)^*$$

where $(V(n)^*)_{-m} = \mathbb{C}\overline{e^n_m}$. Therefore $\pi_{(\tau_m,x)}(d^n_{nn})$ is the projection onto $\mathbb{C}e^n_n \otimes \mathbb{C}\overline{e^n_m} \subseteq$

$\mathcal{H}_{(m,x)} \subseteq L^2(K)$ by Lemma 3.18. Viewing $e_n^n \otimes \overline{e_m^n} \in L^2(K)$, we have

$$(e_n^n \otimes \overline{e_m^n})(s) = \dim(V(n))^{\frac{1}{2}} \langle \pi_n(s) e_m^n, e_n^n \rangle = (2n+1)^{-\frac{1}{2}} d_{nm}^n(s) = e_{nm}^n(s)$$

and so $\pi_{(\tau_m,x)}(d_{nn}^n)$ is the projection onto $\mathbb{C}e_{nm}^n \subseteq \mathcal{H}_{(m,x)}$. This is non-zero if and only if $|m| \leq n, n+m \in \mathbb{Z}$.

Now we consider the case where $x = \pm I$. Recall from (3.15) that

$$\pi_{(m,\pm I)} := \pi_{(V(m),\pm I)} \ltimes \rho_{(V(m),\pm I)} : K \ltimes_{\text{adj}} C(K) \to B(V(m))$$

where

$$(\pi_{(V(m),\pm I)}(k)\xi)(s) = \xi(k^{-1}s), \quad \rho_{(V(m),\pm I)}(f) = f(\pm I) 1_{V(m)}$$

for $k, s \in K$, $\xi \in \mathcal{H}_{(m,\pm I)} \cong V(m)$ and $f \in C(K)$. Therefore

$$\begin{aligned} \pi_{(m,\pm I)}(p_n) &= \int_K \pi_{(V(m),\pm I)}(s) \rho_{(V(m),\pm I)}(p_n(s)) \, \mathrm{d}s \\ &= \int_K (2n+1)(u_{nn}^n)^*(s) \pi_m(s) \, \mathrm{d}s \\ &= \pi_m(d_{nn}^n) \end{aligned}$$

where $\pi_m : K \to B(V(m))$ is the $2m+1$-dimensional irreducible representation of K. Note that $\pi_m(d_{nn}^n)$ is the projection onto $\mathbb{C}e_n^n \subseteq V(m)$ by Lemma 3.18, which is non-zero if and only if $n = m$. $\square$

For $n \in \frac{1}{2}\mathbb{N}_0$, define

$$J_n^1 := (K \ltimes_{\text{adj}} C(K)) p_n (K \ltimes_{\text{adj}} C(K))$$

This is a closed two sided ideal in $K \ltimes_{\text{adj}} C(K)$. By standard results about the spectrum of ideals of C^*-algebras (see for example [70, Proposition A.27]),

$$\begin{aligned} \operatorname{Spec}(J_n^1) &= \{\pi \in \operatorname{Spec}(K \ltimes_{\text{adj}} C(K)) \mid \pi|_{J_n^1} \neq 0\} \\ &= \{\pi \in \operatorname{Spec}(K \ltimes_{\text{adj}} C(K)) \mid \pi(p_n) \neq 0\}. \end{aligned}$$

As a consequence of Lemma 3.19,

$$\operatorname{Spec}(J_n^1) = \{\pi_{(m,x)} \mid x \in T \setminus \{\pm I\}, |m| \leq n, n+m \in \mathbb{Z}\} \cup \{\pi_{(n,\pm I)}\}.$$

Note that here we regard equivalent representations as equal. Ideally we would like to cut down this spectrum further so that we only have representations of the form $\pi_{(n,x)}$. To that end, define subquotients

$$C_0^1 := J_0^1, \quad C_n^1 := {J_0^1 + J_{\frac{1}{2}}^1 + \dots + J_n^1}\Big/{J_0^1 + J_{\frac{1}{2}}^1 + \dots + J_{n-\frac{1}{2}}^1}, \quad n \in \frac{1}{2}\mathbb{N}.$$

By standard results about the spectrum of quotients of C^*-algebras (see for example [70, Proposition A.27]),

$$\mathrm{Spec}\left(C_n^1\right) \cong \{\pi \in \mathrm{Spec}(K \ltimes_{\mathrm{adj}} C(K)) \mid \pi(p_n) \neq 0,\ \pi(p_i) = 0,\ i < n\}, \quad n \geq \frac{1}{2}.$$

From another application of Lemma 3.19 and our description of the topology on $\mathrm{Spec}(C^*(G_1))$ in Section 3.1.4 we see that

$$\mathrm{Spec}\left(C_0^1\right) = \{\pi_{(0,x)}\}_{x \in W\backslash T} \cong W\backslash T \tag{3.29}$$

and for $n \geq \frac{1}{2}$

$$\mathrm{Spec}\left(C_n^1\right) = \{\pi_{(n,x)}\}_{x \in T} \cong T. \tag{3.30}$$

Proposition 3.20. *There are isomorphisms*

$$p_0 C_0^1 p_0 \cong C\left(T\right)^W, \quad p_n C_n^1 p_n \cong C(T), \quad n \in \frac{1}{2}\mathbb{N}.$$

Proof. Note that p_n is a multiplier of C_n^1 for each n because $J_0^1 + J_{\frac{1}{2}}^1 + \dots + J_n^1$ is an ideal of $C^*(G_1)$ and so p_n acts as a multiplier on this ideal, and this action preserves $J_0^1 + J_{\frac{1}{2}}^1 + \dots + J_{n-\frac{1}{2}}^1 \subseteq J_0^1 + J_{\frac{1}{2}}^1 + \dots + J_n^1$, again because this is an ideal. We also have that p_n is a rank one projection under each irreducible representation of C_n^1 by Lemma 3.19 and our descriptions of the spectra 3.29 and 3.30.

The proposition then follows by a direct application of Lemma 3.17 (c). □

We now consider the corresponding subquotients of $C_r^*(G_q)$ for $q \in (0,1)$, with $q = e^h$ for some $h \in (-\infty, 0)$. As before, for $n \in \frac{1}{2}\mathbb{N}_0$, set $p_n = \omega_{nn}^n \bowtie 1$. Then p_n is a projection in $\mathcal{D}(G_q) \subseteq C_r^*(G_q)$.

For $n \in \frac{1}{2}\mathbb{N}_0$, define $J_n^q := C_r^*(G_q) p_n C_r^*(G_q)$. As in the case of $q = 1$ we have

$$\mathrm{Spec}(J_n^q) = \{\pi \in \mathrm{Spec}(C_r^*(G_q)) \mid \pi|_{J_n^q} \neq 0\} = \{\pi \in \mathrm{Spec}(C_r^*(G_q)) \mid \pi(p_n) \neq 0\}$$

and we define

$$C_0^q := J_0^q, \quad C_n^q := {J_0^q + J_{\frac{1}{2}} + \dots + J_n^q}\Big/{J_0^q + J_{\frac{1}{2}}^q + \dots + J_{n-\frac{1}{2}}^q}, \quad n \in \frac{1}{2}\mathbb{N}.$$

Then

$$\mathrm{Spec}\,(C_n^q) \cong \{\pi \in \mathrm{Spec}(C_r^*(G_q)) \mid \pi(p_n) \neq 0,\ \pi(p_i) = 0,\ i < n\}.$$

Recall from Section 2.1.2, (2.8) that if $(m, \lambda) \in M_q = \frac{1}{2}\mathbb{Z} \times {}^{i\mathbb{R}}\!\big/_{2\pi i h^{-1}\mathbb{Z}}$, we have a Hilbert space $\mathcal{H}^q_{(m,\lambda)}$ that is the closure in the $L^2(K_q)$-norm of

$$\mathcal{O}(\mathcal{E}^q_m) := \left\{ f \in \mathcal{O}(K_q) \mid (\mathrm{id} \otimes \pi)\Delta(f) = f \otimes z^{-2m} \right\} \subseteq L^2(K_q),$$

and the principal series representations (2.13) $\pi^q_{(m,\lambda)} : C^*(G_q) \to K(\mathcal{H}^q_{(m,\lambda)})$. Note that if $n \in \frac{1}{2}\mathbb{N}_0$ and $m \in \{-n, -n+1, ..., n\}$, we have, by Proposition 2.3,

$$\begin{aligned}
(\mathrm{id} \otimes \pi)\Delta((u^n_{nm})^*) &= (\mathrm{id} \otimes \pi)\Delta(u^n_{nm})^* \\
&= (\mathrm{id} \otimes \pi)\left(\sum_k u^n_{nk} \otimes u^n_{km} \right)^* \\
&= (u^n_{nm} \otimes z^{2m})^* \\
&= (u^n_{nm})^* \otimes z^{-2m}
\end{aligned}$$

and so $(u^n_{nm})^* \in \mathcal{O}(\mathcal{E}^q_m)$.

We have

$$\|(u^n_{nm})^*\|^2_{L^2(K_q)} = \phi_{K_q}(u^n_{nm}(u^n_{nm})^*) = \frac{q^{-2m}}{[2n+1]},$$

by Theorem 1.58. Define, for $i, j \in \frac{1}{2}\mathbb{N}_0$ with $i, j \in \{-n, -n+1, ..., n\}$ and $n - j, n - i \in \mathbb{Z}$

$$e^n_{ji} := [2n+1]^{\frac{1}{2}} (u^n_{ji})^* \in \mathcal{O}(SU_q(2)),$$

(c.f. (3.28), and again this is not the same as (2.7)). Then by normalising $(u^n_{nm})^* \in \mathcal{O}(\mathcal{E}^q_m)$, we obtain e^n_{nm}. That is, e^n_{nm} is a unit vector in $\mathcal{H}^q_{(m,\lambda)}$ (again, c.f. (3.28) and the remarks that follow).

Lemma 3.21. *Let $n \in \frac{1}{2}\mathbb{N}_0$ and $\lambda \in {}^{i\mathbb{R}}\!\big/_{2\pi i h^{-1}\mathbb{Z}}$. Then*

$$\pi^q_{(m,\lambda)}(p_n) = \begin{cases} \textit{orthogonal projection onto } \mathbb{C}e^n_{nm} \subseteq \mathcal{H}_{(m,\lambda)} & |m| \leq n,\ n+m \in \mathbb{Z} \\ 0 & \textit{else.} \end{cases}$$

Proof. Note that the collection $\{e^n_{im}\}_{n,i}$ (where $|m| \leq n,\ n+m \in \mathbb{Z}$ and $i \in \{-n, -n+1, ..., n\}$) forms an orthonormal basis for $\mathcal{H}^q_{(m,x)}$. We can calculate, using Proposition 1.40,

$$\begin{aligned}
\pi_{(m,\lambda)}(p_n)e^n_{im} &= \omega^n_{nn} \cdot q^m [2n+1]^{\frac{1}{2}} (u^n_{im})^* \\
&= \sum_l q^m [2n+1]^{\frac{1}{2}} (S(\omega^n_{nn}), (u^n_{il})^*)(u^n_{lm})^*
\end{aligned}$$

$$\begin{aligned}
&= \sum_l q^m [2n+1]^{\frac{1}{2}} (\omega^n_{nn}, S^{-1}((u^n_{il})^*))(u^n_{lm})^* \\
&= \sum_l q^m [2n+1]^{\frac{1}{2}} (\omega^n_{nn}, u^n_{li})(u^n_{lm})^* \\
&= q^m [2n+1]^{\frac{1}{2}} \delta_{ni} (u^n_{nm})^* \\
&= \delta_{ni} e^n_{nm}
\end{aligned}$$

and the result follows. □

Applying Lemma 3.21 and (3.17) we see that

$$\mathrm{Spec}\,(C^q_0) \cong \{\pi_{(0,i\nu)}\}_{\nu \geq 0} \cong W \backslash T$$

and for $n \in \frac{1}{2}\mathbb{N}$

$$\mathrm{Spec}\,(C^q_n) \cong \{\pi_{(n,\lambda)}\}_\lambda \cong \{n\} \times i \left({}^{\mathbb{R}}\!/_{2\pi h^{-1}\mathbb{Z}} \right) \cong T.$$

Proposition 3.22. *There are isomorphisms*

$$p_0 C^q_0 p_0 \cong C\left({}^{\mathbb{R}}\!/_{2\pi h^{-1}\mathbb{Z}} \right)^W, \quad p_n C^q_n p_n \cong C\left({}^{\mathbb{R}}\!/_{2\pi h^{-1}\mathbb{Z}} \right), \quad \in \frac{1}{2}\mathbb{N}.$$

Proof. Lemma 3.17 (c) applies and gives the result immediately. □

3.3 Analysis of the Quantum Assembly Map

If $A := A^Q$ is the quantum assembly field constructed in Chapter 2, then we can define $p_n \in A$ to be the continuous section taking constant values $\omega^n_{nn} \bowtie 1$ in each fibre. Then we can define the following ideals and subquotients of A in analogy with the previous section,

$$J^A_n := A p_n A, \quad n \in \frac{1}{2}\mathbb{N}_0,$$

and

$$C^A_0 = J^A_0, \quad C^A_n = {}^{J^A_0 + J^A_{\frac{1}{2}} + \ldots + J^A_n}\!\Big/_{J^A_0 + J^A_{\frac{1}{2}} + \ldots + J^A_{n-\frac{1}{2}}}, \quad n \in \frac{1}{2}\mathbb{N}.$$

By Propositions A.18 and A.19, these are $C([q_0, 1])$-algebras with fibres

$$(J^A_n)_q := J^q_n, \quad (C^A_n)_q = C^q_n$$

and the obvious evaluation maps.

Proposition 3.23. *For each $q \in [q_0, 1]$,*

$$\overline{\bigcup_{n \in \frac{1}{2}\mathbb{N}_0} J_0^q + \ldots + J_n^q} = C_r^*(G_q)$$

and

$$\overline{\bigcup_{n \in \frac{1}{2}\mathbb{N}_0} J_0^A + \ldots + J_n^A} = A.$$

Proof. Note $I := \overline{\bigcup_n J_0^q + \ldots + J_n^q}$ is a closed 2-sided ideal of $C_r^*(G_q)$. Suppose I is a proper ideal. Then I is contained in some primitive ideal of $C_r^*(G_q)$. In particular, there exists $\pi \in \mathrm{Spec}(C_r^*(G_q))$ such that $\pi|_I = 0$. However we have seen in Lemmas 3.19 and 3.21 that for each $\pi \in \mathrm{Spec}(C_r^*(G_q))$ there exists an $m \in \frac{1}{2}\mathbb{N}_0$ such that $\pi(p_m) \neq 0$. This is a contradiction and so I is not a proper ideal and hence $I = C_r^*(G_q)$.

Now consider $J := \overline{\bigcup_n J_0^A + \ldots + J_n^A}$, a closed 2-sided ideal of A. This is a $C([q_0, 1])$-algebra by Proposition A.18, with fibres $\mathrm{ev}_q(J)$. We will show $\mathrm{ev}_q(J) = I$ as defined above, and then from Proposition A.22 it will follow that $J = A$. Let $a \in J$. Then there exists $b \in \bigcup_n J_0^A + \ldots + J_n^A$ approximating a. Then since $\|a(q) - b(q)\| \leq \|a - b\|$ and $b(q) \in I$, we see $a(q) \in I$. Then $\mathrm{ev}_q(J) = I$. □

The fields (see Proposition A.20) $p_n C_n^A p_n$ have constant fibres by Propositions 3.20 and 3.22. We will show that these fields are trivial.

Proposition 3.24. $p_0 C_0^A p_0 \cong C([q_0, 1], C(T)^W), \quad p_n C_n^A p_n \cong C([q_0, 1], C(T)), \quad n \in \frac{1}{2}\mathbb{N}$

Proof. First, note that if $n \in \frac{1}{2}\mathbb{N}$ and $f \in C_n^A$, then $p_n f p_n \in p_n C_n^A p_n$ and $\mathrm{ev}_q(p_n f p_n) = p_n f(q) p_n \in p_n C_n^q p_n$ by Proposition A.20. The isomorphism $p_n C_n^q p_n$ identifying this corner with continuous functions (see Proposition 3.17) maps $p_n f(q) p_n$ to the following functions in the fibre. In the case $q \neq 1$, with $q = e^h$ for some $h \in (-\infty, 0)$,

$$p_n f(q) p_n \mapsto \tilde{f}(q) \in C\left({}^{i\mathbb{R}}\!/\!_{2\pi i h^{-1}\mathbb{Z}}\right), \quad \tilde{f}(q)(-) = \langle e_{nn}^n, \pi_{(n,-)}^q(f) e_{nn}^n \rangle,$$

by Lemma 3.21. In the case $q = 1$,

$$p_n f(1) p_n \mapsto \left(\tilde{f}(1) : x \mapsto \begin{cases} \langle e_{nn}^n, \pi_{(n,x)}(f) e_{nn}^n \rangle & x \neq \pm I \\ \langle e_n^n, \pi_{(n,\pm I)}(f) e_n^n \rangle & x = \pm I \end{cases} \right) \in C(T).$$

by Lemma 3.19. Note that of course in the case $n = 0$ there are extra conditions on the resulting function in both cases. However as we are only concerned with continuity we will not worry about these extra conditions. Also note that here we have technically chosen a lift of f in A. The functions are independent of the choice of lift.

Let us identify T with ${}^{\mathbb{R}}\!/_{2\pi\mathbb{Z}}$ using the logarithm sending $x = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \mapsto t + 2\pi\mathbb{Z}$, and let us identify ${}^{i\mathbb{R}}\!/_{2\pi i h^{-1}\mathbb{Z}}$ with ${}^{\mathbb{R}}\!/_{2\pi\mathbb{Z}}$ by the obvious rescaling. Then we view each $\tilde{f}(q)$ abvove as a function on ${}^{\mathbb{R}}\!/_{2\pi\mathbb{Z}}$. Define

$$[q_0, 1] \to C\left({}^{\mathbb{R}}\!/_{2\pi\mathbb{Z}}\right), \quad q \mapsto \tilde{f}(q).$$

To show the fields are trivial, we need to show the above is continuous, or equivalently that

$$[q_0, 1] \times {}^{\mathbb{R}}\!/_{2\pi\mathbb{Z}} \to \mathbb{C}, \quad (q, t + 2\pi\mathbb{Z}) \mapsto \tilde{f}(q)(t + 2\pi\mathbb{Z})$$

is continuous. By density of the linear span of the sections $q \mapsto (\omega^a_{bc} \bowtie u^m_{kl})(q)$ in A, it suffices to consider the case $f = \omega^a_{bc} \bowtie u^m_{kl}$. We consider first the case where $q \neq 1$, where we have

$$\begin{aligned}
\tilde{f}(q)(t + 2\pi\mathbb{Z}) &= \langle e^n_{nn}, \pi^q_{(n, ih^{-1}t)}((\omega^a_{bc} \bowtie u^m_{kl})(q)) e^n_{nn} \rangle \\
&= \sum_{r,s} \langle (\omega^a_{bc})^* \cdot e^n_{nn}, u^m_{kr}(q) e^n_{nn} u^m_{sl}(q) (K^{2+2ih^{-1}t}, u^m_{rs}(q)) \rangle \\
&= \sum_{r,s} \langle (\omega^a_{bc})^* \cdot e^n_{nn}, u^m_{kr}(q) e^n_{nn} S(u^m_{sl}(q)) \delta_{rs} q^{r(2+2ih^{-1}t)} \rangle \\
&= \sum_{r} q^{2r} e^{2itr} \langle (\omega^a_{bc})^* \cdot e^n_{nn}, u^m_{kr}(q) e^n_{nn} S(u^m_{sl}(q)) \rangle.
\end{aligned}$$

For $q = 1$, we consider the case where $t \neq 0, \pi$ first. Writing $x = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$, we have

$$\begin{aligned}
&\langle e^n_{nn}, \pi_{(n,x)}((\omega^a_{bc} \bowtie u^m_{kl})(1)) e^n_{nn} \rangle \\
&= \int_K \overline{e^n_{nn}(k)} (\pi_{(n,x)}((\omega^a_{bc} \bowtie u^m_{kl})(1)) e^n_{nn})(k) \; \mathrm{d}k \\
&= \int_K \overline{e^n_{nn}(k)} (\pi_{(\tau_n,x)} \ltimes \rho_{(\tau_n,x)}((\omega^a_{bc} \bowtie u^m_{kl})(1)) e^n_{nn})(k) \; \mathrm{d}k \\
&= \int_K \overline{e^n_{nn}(k)} \int_K \pi_{(\tau_n,x)}(s) \rho_{(\tau_n,x)}((\omega^a_{bc} \bowtie u^m_{kl})(1)(s)) e^n_{nn})(k) \; \mathrm{d}k \; \mathrm{d}s \\
&= \int_K \int_K \overline{e^n_{nn}(k)} \rho_{(\tau_n,x)}((\omega^a_{bc} \bowtie u^m_{kl})(1)(s)) e^n_{nn})(s^{-1}k) \; \mathrm{d}k \; \mathrm{d}s \\
&= \int_K \int_K \overline{e^n_{nn}(k)} \omega^a_{bc}(s) \rho_{(\tau_n,x)}((u^m_{kl})(1)) e^n_{nn})(s^{-1}k) \; \mathrm{d}k \; \mathrm{d}s \\
&= \int_K \int_K \overline{e^n_{nn}(k)} \omega^a_{bc}(s) u^m_{kl}(1)(s^{-1}k \cdot x) e^n_{nn}(s^{-1}k) \; \mathrm{d}k \; \mathrm{d}s.
\end{aligned}$$

Setting $u = s^{-1}k$, we have (after supressing the (1) on the matrix coefficients to ease notation)

$$\begin{aligned}
\langle e^n_{nn}, \pi_{(n,x)}((\omega^a_{bc} \bowtie u^m_{kl}))e^n_{nn}\rangle &= \int_K\int_K \overline{e^n_{nn}(k)}\omega^a_{bc}(s)u^m_{kl}(s^{-1}k\cdot x)e^n_{nn}(s^{-1}k)\ \mathrm{d}k\ \mathrm{d}s \\
&= \int_K\int_K \overline{e^n_{nn}(su)}\omega^a_{bc}(s)u^m_{kl}(u\cdot x)e^n_{nn}(u)\ \mathrm{d}u\ \mathrm{d}s \\
&= \int_K\left(\int_K \overline{e^n_{nn}(su)}\omega^a_{bc}(s)\ \mathrm{d}s\right)u^m_{kl}(u\cdot x)e^n_{nn}(u)\ \mathrm{d}u \\
&= \int_K\left(\int_K \overline{(\omega^a_{bc})^*(s^{-1})e^n_{nn}(su)}\ \mathrm{d}s\right)u^m_{kl}(u\cdot x)e^n_{nn}(u)\ \mathrm{d}u \\
&= \int_K\overline{\left(\int_K (\omega^a_{bc})^*(s^{-1})e^n_{nn}(su)\ \mathrm{d}s\right)}u^m_{kl}(u\cdot x)e^n_{nn}(u)\ \mathrm{d}u \\
&= \int_K \overline{((\omega^a_{bc})^*\cdot e^n_{nn})(u)}u^m_{kl}(u\cdot x)e^n_{nn}(u)\ \mathrm{d}u \\
&= \int_K \overline{((\omega^a_{bc})^*\cdot e^n_{nn})(u)}u^m_{kl}(uxu^{-1})e^n_{nn}(u)\ \mathrm{d}u \\
&= \sum_{r,s}\int_K \overline{((\omega^a_{bc})^*\cdot e^n_{nn})(u)}u^m_{kr}(u)u^m_{rs}(x)u^m_{sl}(u^{-1})e^n_{nn}(u)\ \mathrm{d}u \\
&= \sum_r e^{2itr}\int_K \overline{((\omega^a_{bc})^*\cdot e^n_{nn})(u)}u^m_{kr}(u)e^n_{nn}(u)S(u^m_{sl})(u)\ \mathrm{d}u \\
&= \sum_r e^{2itr}\langle(\omega^a_{bc})^*\cdot e^n_{nn}, u^m_{kr}e^n_{nn}S(u^m_{sl})\rangle.
\end{aligned}$$

Finally for the case where $q = 1$ and $t = 0$ or π,

$$\langle e^n_n, \pi_{(n,\pm I)}((\omega^a_{bc} \bowtie u^m_{kl}))e^n_n\rangle_{V(n)} = \int_K \omega^a_{bc}(s)u^m_{kl}(\pm I)\langle e^n_n, \pi_n(s)e^n_n\rangle_{V(n)}\ \mathrm{d}s$$

Note that we have

$$\begin{aligned}
&\int_K\int_K \overline{e^n_{nn}(k)}\omega^a_{bc}(s)u^m_{kl}(s^{-1}k\cdot\pm I)e^n_{nn}(s^{-1}k)\ \mathrm{d}k\ \mathrm{d}s \\
&= \int_K\int_K \overline{e^n_{nn}(k)}\omega^a_{bc}(s)u^m_{kl}(\pm I)e^n_{nn}(s^{-1}k)\ \mathrm{d}k\ \mathrm{d}s \\
&= u^m_{kl}(\pm I)\int_K \omega^a_{bc}(s)\left(\int_K \overline{e^n_{nn}(k)}e^n_{nn}(s^{-1}k)\ \mathrm{d}k\right)\ \mathrm{d}s.
\end{aligned}$$

Now

$$\begin{aligned}
\int_K \overline{e^n_{nn}(k)}e^n_{nn}(s^{-1}k)\ \mathrm{d}k &= (2n+1)\int_K \overline{\langle\pi_n(k)e^n_n, e^n_n\rangle}\langle\pi_n(s^{-1}k)e^n_n, e^n_n\rangle\ \mathrm{d}k \\
&= (2n+1)\int_K \langle e^n_n, \pi_n(k)e^n_n\rangle_{V(n)}\overline{\langle\pi_n(s)e^n_n, \pi_n(k)e^n_n\rangle}_{V(n)}\ \mathrm{d}k \\
&= (2n+1)\int_K \langle e^n_n, \pi_n(k)e^n_n\rangle_{V(n)}\overline{\langle\pi_n(s)e^n_n, \pi_n(k)e^n_n\rangle}_{V(n)}\ \mathrm{d}k
\end{aligned}$$

$$= \langle e^n_n, \pi_n(s)e^n_n\rangle_{V(n)}.$$

Hence

$$\begin{aligned}\langle e^n_n, \pi_{(n,\pm I)}((\omega^a_{bc} \bowtie u^m_{kl}))e^n_n\rangle_{V(n)} &= \int_K \omega^a_{bc}(s)u^m_{kl}(\pm I)\langle e^n_n, \pi_n(s)e^n_n\rangle_{V(n)}\ \mathrm{d}s \\ &= \int_K\int_K \overline{e^n_{nn}(k)}\omega^a_{bc}(s)u^m_{kl}(s^{-1}k\cdot \pm I)e^n_{nn}(s^{-1}k)\ \mathrm{d}k\ \mathrm{d}s \\ &= \sum_r \langle(\omega^a_{bc})^* \cdot e^n_{nn}, u^m_{kr}e^n_{nn}S(u^m_{sl})\rangle.\end{aligned}$$

It is now clear that we have continuity in both the q and t variables. □

We will now prove that the quantum assembly map is an isomorphism using our understanding of the subquotients we have considered above. This will require some homological algebra, and the notion of direct limits of abelian groups and inductive limits of C^*-algebras. The reader should note we use the same argument to that given in the proof of [32, Conjecture 7.1].

We start by recalling the Five Lemma, see for example [88, p.g. 13]. The proof is given by a standard 'diagram chase' which we omit.

Lemma 3.25. *Let $A, A', B, B', C, C', D, D', E, E'$ be abelian groups and suppose one has the following commutative diagram*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

where all the maps in the diagram are group homomorphisms and the rows of the diagram are exact. If a, b, d and e are isomorphisms, then c is an isomorphism.

We recall the notion of direct limits of direct systems from [77, p.g. 3-8].

Definition 3.26.

(a) A **direct system of abelian groups** (indexed by $\mathbb{N}$) is a sequence of abelian groups $(A_i)_{i\in\mathbb{N}}$ together with a family of homomorphisms $f_{ij} : A_i \to A_j$ for those pairs $(i, j) \in \mathbb{N}\times\mathbb{N}$ with $i \le j$ such that the following properties hold.

1. $f_{ii} = \mathrm{id}$ for all $i \in \mathbb{N}$.
2. If $i, j, k \in \mathbb{N}$ with $i \le j \le k$, then $f_{jk} \circ f_{ij} = f_{ik}$.

We will write (A_i, f_{ij}) to denote the direct system, and refer to the homomorphisms f_{ij} as the connecting homomorphisms.

(b) A **target** for a direct system (A_i, f_{ij}) is an abelian group B together with a collection of homomorphisms $(f_i : A_i \to B)_{i\in\mathbb{N}}$ such that the following compatibility condition is satisfied: for all $i, j \in \mathbb{N}$ with $i \leq j$, the diagram

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, "f_{ij}"] & B \\ A_j \arrow[ur, "f_j"'] & \end{tikzcd} \tag{3.31}$$

commutes.

(c) A **direct limit** for a direct system (A_i, f_{ij}) is a target A (with collection of homomorphisms $(f_i : A_i \to A)_{i\in\mathbb{N}}$ making (3.31) commute) such that the following universal property is satisfied. If B is another target (with collection of homomorphisms $(g_i : A_i \to B)_{i\in\mathbb{N}}$ making (3.31) commute) then there is a unique group homomorphism $f : A \to B$ such that for all $i \in \mathbb{N}$, the diagram

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[dr, "g_i"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commutes.

One can prove that a direct limit always exists for a direct system of abelian groups, [77, Theorem 3.20], and that any two direct limits for a direct system of abelian groups are isomorphic, [77, Theorem 3.16]. Therefore it makes sense to talk of the direct limit for a direct system (A_i, f_{ij}), and we denote this by $\varinjlim A_i$.

We prove the following folklore result.

Lemma 3.27. *Suppose $(A_i, f_{ij})_{i\in\mathbb{N}}$ and $(B_i, g_{ij})_{i\in\mathbb{N}}$ are direct systems of abelian groups with direct limits A and B respectively, and we have a commutative diagram*

$$\begin{tikzcd} A_1 \arrow[r, "f_{12}"] \arrow[d, "\phi_1"] & A_2 \arrow[r, "f_{23}"] \arrow[d, "\phi_2"] & A_3 \arrow[r, "f_{34}"] \arrow[d, "\phi_3"] & \ldots \\ B_1 \arrow[r, "g_{12}"] & B_2 \arrow[r, "g_{23}"] & B_3 \arrow[r, "g_{34}"] & \ldots \end{tikzcd}$$

where for each $i \in \mathbb{N}$, ϕ_i is a group isomorphism. Then $A \cong B$.

Proof. We can see that B is a target for the direct system $(A_i, f_{ij})_{i\in\mathbb{N}}$ by taking $h_i = g_i \circ \phi_i$

for each $i \in \mathbb{N}$. Here $(g_i : B_i \to B)_{i\in\mathbb{N}}$ are the homomorphisms arising from the fact that B is a target for $(B_i, g_{ij})_{i\in\mathbb{N}}$.

Similarly A is a target for the direct system $(B_i, g_{ij})_{i\in\mathbb{N}}$ by taking $k_i = f_i \circ \phi_i^{-1}$ for each $i \in \mathbb{N}$. Here $(f_i : A_i \to A)_{i\in\mathbb{N}}$ are the homomorphisms arising from the fact that A is a target for $(A_i, f_{ij})_{i\in\mathbb{N}}$.

In particular there is a unique group homomorphism $f : A \to B$ making the diagram

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[dr, "h_i"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commute for all $i \in \mathbb{N}$ and there is a unique group homomorphism $g : B \to A$ making the diagram

$$\begin{tikzcd} B_i \arrow[r, "g_i"] \arrow[dr, "k_i"'] & B \arrow[d, "g"] \\ & A \end{tikzcd}$$

commute for all $i \in \mathbb{N}$.

Consider A as a target for the direct system $(A_i, f_{ij})_{i\in\mathbb{N}}$. Because A is the direct limit, there is a unique homomorphism $A \to A$ that makes the diagram

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[dr, "f_i"] & A \arrow[d, dotted] \\ & A \end{tikzcd}$$

commute. Since the identity map from A to itself makes the diagram commute, this must be the homomorphism completing the triangle. We can factorise this diagram as follows.

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, "\phi_i"] \arrow[dr, "h_i"] & A \arrow[d, "f"] \\ B_i \arrow[r, "g_i"] \arrow[d, "\phi_i^{-1}"] \arrow[dr, "k_i"] & B \arrow[d, "g"] \\ A_i \arrow[r, "f_i"] & A \end{tikzcd}$$

and so we must have that $g \circ f$ is the identity map on A. An entirely similar argument, viewing B as a target for $(B_i, g_{ij})_{i\in\mathbb{N}}$, shows that $f \circ g$ is the identity map on B. Therefore $A \cong B$ as required. $\square$

Remark 3.28. We can define direct systems of other objects, such as algebras, and construct the direct limit in this case, by making the obvious replacements of objects and morphisms in the definitions above. When all the objects in a direct system are C^*-algebras, we can also construct the C^*-algebraic inductive limit from the direct limit - see for example [87, Appendix L]. In our case, we will only need the following facts about C^*-algebraic inductive limits. These are stated or follow from statements in [87].

1. Let D be a C^*-algebra, with an increasing sequence of sub-C^*-algebras $(A_i)_{i\in\mathbb{N}}$ of D. We can view this as a direct system of C^*-algebras with the connecting $*$-homomorphisms given by inclusion. Then $A = \overline{\bigcup_{i\in\mathbb{N}} A_i}$ is the C^*-algebraic inductive limit of the sequence.

2. If $(A_i)_{i\in\mathbb{N}}$ is a direct system of C^*-algebras with C^*-algebraic inductive limit A, then $(K_*(A_i))_{i\in\mathbb{N}}$ is a direct system of abelian groups with connecting homomorphisms induced by the connecting $*$-homomorphisms between the C^*-algebras, and

$$K_*(A) \cong \varinjlim K_*(A_i).$$

We can now turn to the proof that the quantum assembly map is an isomorphism. We break this problem down using the following lemma.

Lemma 3.29. *The quantum assembly map is an isomorphism if for each $n \in \frac{1}{2}\mathbb{N}_0$, the evaluation map*

$$\mathrm{ev}_{q_0} : J_0^A + ... + J_n^A \to J_0^{q_0} + ... + J_n^{q_0}$$

induces an isomorphism in K-theory

$$\mathrm{ev}_{q_0} : K_*(J_0^A + ... + J_n^A) \to K_*(J_0^{q_0} + ... + J_n^{q_0}).$$

Proof. The quantum assembly map μ makes the diagram

$$\begin{tikzcd} & K_*(A) \arrow[dl, "\cong"', "ev_1"] \arrow[dr, "ev_{q_0}"] & \\ K_*(C_r^*(G_1)) \arrow[rr, "\mu"] & & K_*(C_r^*(G_{q_0})) \end{tikzcd}$$

commute by the definition of the induced map in K-theory provided by the field A. Therefore to show that μ is an isomorphism it suffices to show that the map induced by evaluation at q_0 from A to $C_r^*(G_{q_0})$ is an isomorphism.

Suppose that for each $n \in \frac{1}{2}\mathbb{N}_0$, the evaluation map at q_0 induces an isomorphism in K-theory, that is,

$$\mathrm{ev}_{q_0} : K_*(J_0^A + ... + J_n^A) \to K_*(J_0^{q_0} + ... + J_n^{q_0})$$

is an isomorphism for each $n \in \frac{1}{2}\mathbb{N}_0$.

Note that because the sequence of C^*-algebras $(J_0^A + ... + J_n^A)_{n \in \frac{1}{2}\mathbb{N}_0}$ is an increasing sequence in A, the inclusion maps give us a direct system of C^*-algebras, and on the level of K-groups, a direct system of abelian groups by Remark 3.28. In the same way, the increasing sequence of C^*-algebras $(J_0^{q_0} + ... + J_n^{q_0})_{n \in \frac{1}{2}\mathbb{N}_0}$ in $C_r^*(G_{q_0})$ give us a direct system of C^*-algebras, and on the level of K-groups, a direct system of abelian groups. We therefore have the following diagram

$$\begin{tikzcd}
K_*(J_0^A) \arrow[r] \arrow[d, "\mathrm{ev}_{q_0}"] & K_*(J_0^A + J_{\frac{1}{2}}^A) \arrow[r] \arrow[d, "\mathrm{ev}_{q_0}"] & K_*(J_0^A + J_{\frac{1}{2}}^A + J_1^A) \arrow[r] \arrow[d, "\mathrm{ev}_{q_0}"] & ... \\
K_*(J_0^{q_0} + J_{\frac{1}{2}}^{q_0}) \arrow[r] & K_*(J_0^{q_0} + J_{\frac{1}{2}}^{q_0}) \arrow[r] & K_*(J_0^{q_0} + J_{\frac{1}{2}}^{q_0} + J_1^{q_0}) \arrow[r] & ...
\end{tikzcd}$$

Since the vertical maps are isomorphisms by assumption, we can apply Lemma 3.27, and we conclude that $K_*(A) \cong K_*(C_r^*(G_{q_0}))$. We now need to show that the map implementing this isomorphism is $\mathrm{ev}_{q_0} : K_*(A) \to K_*(C_r^*(G_{q_0}))$.

From the proof of Lemma 3.27, the isomorphism $K_*(A) \cong K_*(C_r^*(G_{q_0}))$ is implemented by the unique homomorphism making

$$\begin{tikzcd}
K_*(J_0^A + ... + J_n^A) \arrow[r] \arrow[dr] & K_*(A) \arrow[d, dotted] \\
& K_*(C_r^*(G_{q_0}))
\end{tikzcd}$$

commute for all $n \in \frac{1}{2}\mathbb{N}_0$, where the upper arrow is induced by the inclusion map, and the bottom arrow is induced by the evaluation map at q_0, followed by inclusion into $C_r^*(G_{q_0})$. We therefore see that ev_{q_0} completes the diagram, and so must implement the desired isomorphism, as required. $\square$

We can now reduce the problem further using the following lemma.

Lemma 3.30. *If for each $n \in \frac{1}{2}\mathbb{N}_0$, the evaluation map*

$$\mathrm{ev}_{q_0} : C_n^A \to C_n^{q_0}$$

induces an isomorphism in K-theory

$$\mathrm{ev}_{q_0} : K_*(C_n^A) \to K_*(C_n^{q_0})$$

then the homomorphisms

$$\mathrm{ev}_{q_0} : K_*(J_0^A + ... + J_n^A) \to K_*(J_0^{q_0} + ... + J_n^{q_0})$$

are isomorphisms for each $n \in \frac{1}{2}\mathbb{N}_0$.

Proof. We proceed by induction.

We note that $C_0^A = J_0^A$ and $C_0^{q_0} = J_0^{q_0}$ and so

$$\mathrm{ev}_{q_0} : K_*(J_0^A) \to K_*(J_0^{q_0})$$

is an isomorphism by assumption.

Let $k \in \frac{1}{2}\mathbb{N}_0$. Assume that for all $n \in \frac{1}{2}\mathbb{N}_0$ with $n \leq k$ the homomorphisms

$$\mathrm{ev}_{q_0} : K_*(J_0^A + ... + J_n^A) \to K_*(J_0^{q_0} + ... + J_n^{q_0})$$

are isomorphisms.

The short exact sequence

$$0 \longrightarrow J_0^A + ... + J_k^A \longrightarrow J_0^A + ... + J_{k+\frac{1}{2}}^A \longrightarrow C_{k+\frac{1}{2}}^A \longrightarrow 0$$

induces the 6 term exact sequence

$$\begin{tikzcd}
K_0\left(J_0^A + ... + J_k^A\right) \arrow[r] & K_0(J_0^A + ... + J_{k+\frac{1}{2}}^A) \arrow[r] & K_0(C_{k+\frac{1}{2}}^A) \arrow[d] \\
K_1(C_{k+\frac{1}{2}}^A) \arrow[u] & K_1(J_0^A + ... + J_{k+\frac{1}{2}}^A) \arrow[l] & K_1\left(J_0^A + ... + J_k^A\right). \arrow[l]
\end{tikzcd}$$

The short exact sequence

$$0 \longrightarrow J_0^{q_0} + ... + J_k^{q_0} \longrightarrow J_0^{q_0} + ... + J_{k+\frac{1}{2}}^{q_0} \longrightarrow C_{k+\frac{1}{2}}^{q_0} \longrightarrow 0$$

induces the 6 term exact sequence

$$\begin{tikzcd}
K_0\left(J_0^{q_0} + ... + J_k^{q_0}\right) \arrow[r] & K_0(J_0^{q_0} + ... + J_{k+\frac{1}{2}}^{q_0}) \arrow[r] & K_0(C_{k+\frac{1}{2}}^{q_0}) \arrow[d] \\
K_1(C_{k+\frac{1}{2}}^{q_0}) \arrow[u] & K_1(J_0^{q_0} + ... + J_{k+\frac{1}{2}}^{q_0}) \arrow[l] & K_1\left(J_0^{q_0} + ... + J_k^{q_0}\right). \arrow[l]
\end{tikzcd}$$

Let us prove that

$$\mathrm{ev}_{q_0} : K_0(J_0^A + ... + J_{k+\frac{1}{2}}^A) \to K_0(J_0^{q_0} + ... + J_{k+\frac{1}{2}}^{q_0})$$

is an isomorphism. The evaluation map ev_{q_0} induces homomorphisms connecting the two 6 term sequences above, as shown in Figure 3.1.

We can consider the portion of Figure 3.1 shown in Figure 3.2. By the Five Lemma 3.25, we see that

$$\mathrm{ev}_{q_0} : K_0(J_0^A + ... + J_{k+\frac{1}{2}}^A) \to K_0(J_0^{q_0} + ... + J_{k+\frac{1}{2}}^{q_0})$$

is an isomorphism. One can similarly show that

$$\mathrm{ev}_{q_0} : K_1(J_0^A + ... + J_{k+\frac{1}{2}}^A) \to K_1(J_0^{q_0} + ... + J_{k+\frac{1}{2}}^{q_0})$$

is an isomorphism by considering another portion of the diagram in Figure 3.1. Therefore by induction, the homomorphisms

$$\mathrm{ev}_{q_0} : K_*(J_0^A + ... + J_n^A) \to K_*(J_0^{q_0} + ... + J_n^{q_0})$$

are isomorphisms for each $n \in \frac{1}{2}\mathbb{N}_0$. □

We can now prove that the quantum assembly map is an isomorphism.

Theorem 3.31. *The quantum assembly map is an isomorphism.*

Proof. By Lemmas 3.29 and 3.30, it suffices to show that for all $n \in \frac{1}{2}\mathbb{N}_0$ that the homomorphisms

$$\mathrm{ev}_{q_0} : K_*(C_n^A) \to K_*(C_n^{q_0})$$

are isomorphisms.

Consider the diagram

$$\begin{array}{ccc} K_*(C_n^A) & \longrightarrow & K_*(C_n^{q_0}) \\ \uparrow & & \uparrow \\ K_*(p_n C_n^A p_n) & \longrightarrow & K_*(p_n C_n^{q_0} p_n) \end{array}$$

where the vertical maps are induced by the inclusions, and the horizontal maps are induced by the evaluation homomorphisms. The vertical maps are isomorphisms by Lemma 3.17. The bottom arrow here is an isomorphism for each $n \in \frac{1}{2}\mathbb{N}_0$ because each of the fields are trivial by Proposition 3.24, and trivial fields induce isomorphisms in K-theory by Proposition A.30. Therefore the quantum assembly map is an isomorphism. □

$$
\begin{tikzcd}
 & & K_0\left(J_0^A + ... + J_k^A\right) \arrow[rr] \arrow[ddll, dotted, bend right] & & K_0(J_0^A + ... + J_{k+\frac{1}{2}}^A) \arrow[r] \arrow[ddlll, dotted, bend right] & K_0(C_{k+\frac{1}{2}}^A) \arrow[d] \arrow[ddll, dotted, bend right] \\
 & & K_1(C_{k+\frac{1}{2}}^A) \arrow[u] \arrow[ddll, dotted, bend left] & & K_1(J_0^A + ... + J_{k+\frac{1}{2}}^A) \arrow[ll] \arrow[ddlll, dotted, bend left] & K_1\left(J_0^A + ... + J_k^A\right) \arrow[l] \arrow[ddll, dotted, bend left] \\
K_0\left(J_0^{q_0} + ... + J_k^{q_0}\right) \arrow[r] & K_0(J_0^{q_0} + ... + J_{k+\frac{1}{2}}^{q_0}) \arrow[rr] & & K_0(C_{k+\frac{1}{2}}^{q_0}) \arrow[d] & & \\
K_1(C_{k+\frac{1}{2}}^{q_0}) \arrow[u] & K_1(J_0^{q_0} + ... + J_{k+\frac{1}{2}}^{q_0}) \arrow[l] & & K_1\left(J_0^{q_0} + ... + J_k^{q_0}\right) \arrow[ll] & &
\end{tikzcd}
$$

Figure 3.1: The two 6-term exact sequences.

$$
\begin{tikzcd}
K_1(C^A_{k+\frac{1}{2}}) \arrow[r] \arrow[d] & K_0(J^A_0 + ... + J^A_k) \arrow[r] \arrow[d] & K_0(J^A_0 + ... + J^A_{k+\frac{1}{2}}) \arrow[r] \arrow[d] & K_0(C^A_{k+\frac{1}{2}}) \arrow[r] \arrow[d] & K_1(J^A_0 + ... + J^A_k) \arrow[d] \\
K_1(C^{q_0}_{k+\frac{1}{2}}) \arrow[r] & K_0(J^{q_0}_0 + ... + J^{q_0}_k) \arrow[r] & K_0(J^{q_0}_0 + ... + J^{q_0}_{k+\frac{1}{2}}) \arrow[r] & K_0(C^{q_0}_{k+\frac{1}{2}}) \arrow[r] & K_1(J^{q_0}_0 + ... + J^{q_0}_k)
\end{tikzcd}
$$

Figure 3.2: A portion of the 6 term exact sequences.

Chapter 4

Quantization Fields and Deformation Squares

In this chapter we will provide the link between the classical Baum-Connes conjecture for $SL(2,\mathbb{C})$ and Theorem 3.31, thereby providing further evidence that we should view Theorem 3.31 as a quantum analogue of the Baum-Connes conjecture proved for quantum $SL_q(2,\mathbb{C})$.

Recall from the Introduction that if $G = SL(2,\mathbb{C})$, and G_0 is the Cartan motion group, there is a continuous field A^C over $[0,1]$ that induces a map in K-theory

$$K_*(C^*(G_0)) \to K_*(C_r^*(G))$$

which can be identified with the assembly map. In Chapter 2 we saw that the continuous field A^Q induces a map in K-theory

$$K_*(C^*(G_1)) \to K_*(C_r^*(G_q))$$

for a fixed $q \in (0,1)$, which is our quantum assembly map.

In this chapter we will construct two continuous fields A^L and A^R (where the L and R superscripts stand for *left* and *right* respectively, the reason for which will hopefully become clear shortly) which we will call the **left and right quantization fields** respectively. We will see that these continuous fields induce the maps

$$K_*(C^*(G_0)) \to K_*(C^*(G_1))$$

and

$$K_*(C_r^*(G)) \to K_*(C_r^*(G_q))$$

respectively in K-theory.

We will then show that the diagram

$$\begin{array}{ccc} K_*(C^*(G_0)) & \longrightarrow & K_*(C^*_r(G)) \\ \downarrow & & \downarrow \\ K_*(C^*(G_1)) & \longrightarrow & K_*(C^*_r(G_q)) \end{array}$$

commutes, providing a link between the classical and quantum assembly maps. Here the left quantization field provides the map on the left hand edge, and the right quantization field provides the map on the right hand edge. We will prove this by essentially 'gluing' together A^C, A^Q, A^L and A^R to form a certain type of continuous field over $[0,1] \times [0,1]$, which we will call a **deformation square**, and such deformation squares have the property of inducing such commutative squares in K-theory.

4.1 The Classical Assembly Field

We start by recalling some of the details behind the construction of the classical assembly field A^C [11, p.g. 146], as we will need to understand this to construct our deformation square. Let $G = SL(2,\mathbb{C})$ and $K = SU(2)$ throughout.

Recall from (2.1) the Iwasawa decomposition $G = K \times A \times N$ as a topological product (see [44, Theorem 6.46]), where

$$A = \left\{ \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} \mid x \in \mathbb{R} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

If $g \in G$, we can then write $g = kan$ where $k \in K$, $a \in A$ and $n \in N$, and this decomposition is unique.

Note that A normalises N, and so $AN \subseteq G$ is a subgroup. We often use the decomposition in the form $G = K \times AN$. Let $k \in K$, $a \in A$ and $n \in N$, and let p_K, p_{AN} be the projection maps from G to K and AN respectively. We have, for $an \in AN$ and $k \in K$, that

$$ank = p_K(ank)p_{AN}(ank).$$

One can check that that K acts on AN on the right by

$$an \leftharpoondown k := p_{AN}(ank). \tag{4.1}$$

and AN acts on K on the left by

$$an \rightharpoonup k := p_K(ank). \tag{4.2}$$

These actions are called the **right and left dressing actions** respectively, see [8, p.g. 37].

The Iwasawa decomposition on the level of groups induces a decomposition on the level of Lie algebras,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{an}, \tag{4.3}$$

where $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ are the Lie algebras of G, K, A and N respectively (see [44, Theorem 6.46]), where

$$\mathfrak{k} = \{X \in M_2(\mathbb{C}) \mid \operatorname{Trace}(X) = 0,\ X^* = -X\}, \tag{4.4}$$

$$\mathfrak{a} = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\},$$

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \,\middle|\, z \in \mathbb{C} \right\},$$

$$\mathfrak{an} = \mathfrak{a} \oplus \mathfrak{n} = \left\{ \begin{pmatrix} x & z \\ 0 & -x \end{pmatrix} \,\middle|\, x \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

Recall that for a Lie group H, there is an exponential map $\exp : \mathfrak{h} \to H$, where $\mathfrak{h}$ is the Lie algebra of H. For matrix Lie groups, this is given by the matrix exponential map, given by

$$\exp : M_n(\mathbb{C}) \to GL_n(\mathbb{C}), \quad X \mapsto \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

for $n \in \mathbb{N}$, see [44, Proposition 1.86].

We can use the (matrix) exponential map $\exp : \mathfrak{an} \to AN$ to linearize the right dressing action of K on AN to obtain an action of K on $\mathfrak{an}$ by defining

$$X \leftharpoondown k = \left.\frac{d}{dt}\right|_{t=0} p_{AN}(\exp(tX)k)$$

for $X \in \mathfrak{an}$ and $k \in K$. Then

$$X \leftharpoondown k = \left.\frac{d}{dt}\right|_{t=0} p_{AN}(kk^{-1}\exp(tX)k) = \left.\frac{d}{dt}\right|_{t=0} p_{AN}(k^{-1}\exp(tX)k). \tag{4.5}$$

To understand this derivative, we prove the following lemma.

Lemma 4.1. *Let $\mu : \mathbb{R} \to G$ be a differentiable path with $\mu(0) = e$, the identity in G.*

Then

$$\left.\frac{d}{dt}\right|_{t=0} p_{AN}(\mu(t)) = p_{\mathfrak{an}}\left(\mu'(0)\right)$$

where $p_{AN} : G \to AN$, $p_{\mathfrak{an}} : \mathfrak{g} \to \mathfrak{an}$ *are the coordinate projections provided by the Iwasawa decomposition.*

Proof. We can write $\mu(t) = \mu_K(t)\mu_A(t)\mu_N(t)$, where μ_K is a curve in K, μ_A is a curve in A and μ_N is a curve in N. These are differentiable curves passing through e at $t = 0$. By the product rule

$$\mu'(0) = \mu_K'(0) + \mu_A'(0) + \mu_N'(0)$$

and the result follows. □

If $X \in M_n(\mathbb{C})$ then $\left.\frac{d}{dt}\right|_{t=0} \exp(tX) = X$ by differentiating the power series for the matrix exponential term by term. By Lemma 4.1, we then have

$$X \leftharpoonup k = \left.\frac{d}{dt}\right|_{t=0} p_{AN}(k^{-1}\exp(tX)k) = p_{\mathfrak{an}}\left(\left.\frac{d}{dt}\right|_{t=0} k^{-1}\exp(tX)k\right) = p_{\mathfrak{an}}(k^{-1}Xk) \quad (4.6)$$

for $k \in K$ and $X \in \mathfrak{an}$.

We then obtain a left action of K on $\mathfrak{an}$ given by

$$k \rightharpoonup X := X \leftharpoonup k^{-1} = p_{\mathfrak{an}}(kXk^{-1}) \quad (4.7)$$

for $k \in K$ and $X \in \mathfrak{an}$.

We note that

$$\mathfrak{an} \cong \mathfrak{k}^*$$

as vector spaces (and hence groups), induced by the trace pairing

$$\kappa : \mathfrak{k} \times \mathfrak{an} \to \mathbb{R} \quad (X, Y) \mapsto \mathrm{Im}(\mathrm{Trace}(XY)).$$

Let us understand the action of K on $\mathfrak{k}^*$ corresponding to the action (4.7) on $\mathfrak{an}$ under the isomorphism $\mathfrak{an} \cong \mathfrak{k}^*$.

Note that if $Z \in \mathfrak{an}$ and $k \in K$, then $kZk^{-1} = p_{\mathfrak{k}}(kZk^{-1}) + p_{\mathfrak{an}}(kZk^{-1})$, where $p_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$ is the projection provided by the Iwasawa decomposition (4.3). One can check that if $A, B \in \mathfrak{k}$, $\mathrm{Trace}(AB) \in \mathbb{R}$. Therefore if $A \in \mathfrak{k}$, $Z \in \mathfrak{an}$ and $k \in K$, we have

$$\begin{aligned}\kappa(A, k \rightharpoonup Z) = \mathrm{Im}(\mathrm{Trace}(Ap_{\mathfrak{an}}(kZk^{-1}))) &= \mathrm{Im}(\mathrm{Trace}(AkZk^{-1})) \\ &= \mathrm{Im}(\mathrm{Trace}(k^{-1}AkZ))\end{aligned}$$

$$= \kappa(k^{-1} \cdot A, Z),$$

see formula (1) in the Introduction. Hence the corresponding action of K on $\mathfrak{k}^*$ is the coadjoint action, as introduced in the Introduction, formula (2). Hence the Cartan motion group $G_0 = K \ltimes \mathfrak{k}^*$ is isomorphic to $K \ltimes \mathfrak{an}$, where K acts on $\mathfrak{an}$ by (4.7).

Proposition 4.2. *Let $G_0 = K \ltimes \mathfrak{k}^*$ be the Cartan motion group. Then*

$$C^*(G_0) \cong K \ltimes C_0(\mathfrak{k}),$$

where K acts on $\mathfrak{k}$ by the adjoint action.

Proof. We have

$$C^*(G_0) \cong C^*(K \ltimes \mathfrak{k}^*) \cong K \ltimes C^*(\mathfrak{k}^*)$$

where the action of K on $C_c(\mathfrak{k}^*) \subseteq C^*(\mathfrak{k}^*)$ is given by

$$(k \cdot f)(\varphi) = \delta(k) f(k^{-1} \cdot \varphi)$$

for $k \in K$, $f \in C_c(\mathfrak{k}^*)$ and $\varphi \in \mathfrak{k}^*$ and $\delta : K \to \mathbb{R}_{>0}$ is defined by

$$\int_{\mathfrak{k}^*} f(k \cdot \varphi)\, \mathrm{d}\varphi = \delta(k) \int_{\mathfrak{k}^*} f(\varphi)\, \mathrm{d}\varphi, \quad k \in K, \tag{4.8}$$

see [20, Example 3.6]. Note that $\mathfrak{k}^* \cong \widehat{\mathfrak{k}}$, the Pontryagin dual of $\mathfrak{k}$, by the isomorphism

$$\mathfrak{k}^* \to \widehat{\mathfrak{k}}, \quad \varphi \mapsto (X \mapsto \exp(i\varphi(X)))$$

(see [24, Corollary 4.7]) and the action of K on $\mathfrak{k}^*$ corresponds to the action of K on $\widehat{\mathfrak{k}}$ given by

$$(k \cdot \chi)(X) = \chi(k^{-1} X k)$$

for $k \in K$, $\chi \in \widehat{\mathfrak{k}}$ and $X \in \mathfrak{k}$.

Therefore

$$C^*(G_0) \cong K \ltimes C^*(\mathfrak{k}^*) \cong K \ltimes C^*(\widehat{\mathfrak{k}})$$

where the action of K on $C_c(\widehat{\mathfrak{k}}) \subseteq C^*(\widehat{\mathfrak{k}})$ is given by

$$(k \cdot f)(\chi) = \delta(k) f(k^{-1} \cdot \chi)$$

where $k \in K$, $f \in C_c(\widehat{\mathfrak{k}})$ and $\chi \in \widehat{\mathfrak{k}}$ and $\delta : K \to \mathbb{R}_{>0}$ is defined as above in (4.8). Finally, recall that we have the Fourier transform

$$\mathcal{F} : C_c(\widehat{\mathfrak{k}}) \to C_0(\mathfrak{k}), \quad \mathcal{F}(f)(X) = \int_{\widehat{\mathfrak{k}}} f(\chi)\chi(X)\, \mathrm{d}\chi, \quad f \in C_c(\widehat{\mathfrak{k}}), \quad X \in \mathfrak{k},$$

which extends to an isomorphism $C^*(\widehat{\mathfrak{k}}) \cong C_0(\mathfrak{k})$ (see [14, Proposition VII.1.1]). One can directly check that $\mathcal{F}(k \cdot f)(X) = \mathcal{F}(f)(k^{-1}Xk)$ for $f \in C_c(\widehat{\mathfrak{k}})$, $k \in K$ and $X \in \mathfrak{k}$, and so the action of K on $C^*(\widehat{\mathfrak{k}})$ corresponds to the adjoint action of K on $C_0(\mathfrak{k})$. Therefore we finally have

$$C^*(G_0) \cong K \ltimes C_0(\mathfrak{k}). \qquad \square$$

Proposition 4.2 allows us to use the machinery of Section 3.1.2 to determine the representation theory of $C^*(G_0)$. We will come back to this later when we construct the left quantization field. Now let us turn to the construction of A^C.

Define

$$\mathcal{G} = K \ltimes \mathfrak{an} \times \{0\} \sqcup G \times (0,1] = G_0 \times \{0\} \sqcup G \times (0,1]. \tag{4.9}$$

We can view $\mathcal{G}$ as a bundle of groups over $[0,1]$, with fibres G_t. The fibre at 0 is G_0 and for $t > 0$, $G_t = G$. We equip $\mathcal{G}$ with a topology (see [32, Lemma 6.17]). Define

$$K \times \mathfrak{a} \times \mathfrak{n} \times [0,1] \to \mathcal{G}, \quad (k, X, Y, t) \mapsto \begin{cases} (k, X, Y, 0) & t = 0 \\ (k, \exp(tX), \exp(tY), t) & t \neq 0. \end{cases} \tag{4.10}$$

The map (4.10) is a bijection because the (matrix) exponential maps $\mathfrak{a} \to A$ and $\mathfrak{n} \to N$ are homeomorphisms - indeed, in these cases, the matrix exponentials are given by

$$\exp : \mathfrak{a} \to A, \quad \exp\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} \tag{4.11}$$

for $x \in \mathbb{R}$ and

$$\exp : \mathfrak{n} \to N, \quad \exp\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

for $z \in \mathbb{C}$, using the power series definition in each case. We denote the inverse of each of these maps by log.

The domain of (4.10) is a topological space and we equip $\mathcal{G}$ with the topology such that this map is a homeomorphism. The topology on $\mathcal{G}$ restricts to the usual ones on $G_t \times \{t\}$ for each t. Note that every compactly supported continuous function F on $\mathcal{G}$ has the form

$$\begin{cases} F(k,X,Y,0) = f(k,X,Y,0) & k \in K,\ X \in \mathfrak{a},\ Y \in \mathfrak{n} \\ F(k,x,y,t) = f(k,t^{-1}\log(x),t^{-1}\log(y),t) & k \in K,\ x \in A,\ y \in N,\ t \neq 0 \end{cases} \tag{4.12}$$

for some compactly supported continuous function f on $K \times \mathfrak{a} \times \mathfrak{n} \times [0,1]$. If F is a compactly supported continuous function on $\mathcal{G}$ we denote by F_t the restriction of F to G_t. We will now equip each fibre of $\mathcal{G}$ with a Haar measure so that we can construct a field of group algebras from $\mathcal{G}$.

Fix Haar measures on K, $\mathfrak{a}$, and $\mathfrak{n}$, $\mathrm{d}k$, $\mathrm{d}X$ and $\mathrm{d}Y$ respectively. One can show that $G_0 = K \ltimes \mathfrak{an}$ is unimodular, and the Haar measure is given by the product measure $\mathrm{d}k\ \mathrm{d}X\ \mathrm{d}Y$ (see the formulae given in [38, p.g. 9]).

On $SL(2,\mathbb{C})$, a Haar measure is given by

$$\mathrm{d}g = \delta(a)\ \mathrm{d}k\ \mathrm{d}a\ \mathrm{d}n$$

(see [44, Proposition 8.43]) where $\mathrm{d}a$, $\mathrm{d}n$ are Haar measures on A, N respectively, and δ is the group homomorphism defined by

$$\delta : A \to \mathbb{R}_{>0}, \qquad \int_N f(ana^{-1})\ \mathrm{d}n = \delta(a)^{-2} \int_N f(n)\ \mathrm{d}n, \quad a \in A.$$

To fix the Haar measures on G_t for $t > 0$, we define the Haar measures on A and N in G_t (which we denote by A_t and N_t in this case) in terms of scalings of the fixed Haar measures on $\mathfrak{a}$, $\mathfrak{n}$, by the integral formulae

$$\int_{A_t} f(a_t)\ \mathrm{d}a_t := \int_{\mathfrak{a}} f(\exp(tX))\ \mathrm{d}X, \qquad \int_{N_t} f(n_t)\ \mathrm{d}n_t := \int_{\mathfrak{n}} f(\exp(tY))\ \mathrm{d}Y. \tag{4.13}$$

The homomorphism δ is unchanged in the formula for the Haar measure of G_t since both sides of

$$\int_N f(ana^{-1})\ \mathrm{d}n = \delta(a)^{-2} \int_N f(n)\ \mathrm{d}n, \quad a \in A,$$

scale by the same amount when we change the Haar measure on N. Therefore a Haar measure on G_t is given by

$$\mathrm{d}g_t = \delta(a_t)\ \mathrm{d}k\ \mathrm{d}a_t\ \mathrm{d}n_t. \tag{4.14}$$

This choice of rescaling of Haar measure ensures that for each $F \in C_c(\mathcal{G})$,

$$t \mapsto \int_{G_t} F_t\ \mathrm{d}g_t$$

is continuous for $t \in [0,1]$, because, using the notation of (4.12), we have

$$\begin{aligned}\int_{G_t} F_t \, \mathrm{d}g_t &= \int_K \int_{A_t} \int_{N_t} F(k, a_t, n_t, t)\ \delta(a_t)\ \mathrm{d}k\ \mathrm{d}a_t\ \mathrm{d}n_t \\ &= \int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} F(k, \exp(tX), \exp(tY), t)\ \delta(\exp(tX))\ \mathrm{d}k\ \mathrm{d}X\ \mathrm{d}Y \\ &= \int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} f(k, X, Y, t)\ \delta(\exp(tX))\ \mathrm{d}k\ \mathrm{d}X\ \mathrm{d}Y.\end{aligned}$$

Equip $C_c(\mathcal{G})$ with fibrewise convolution (i.e. for $F, G \in C_c(\mathcal{G})$ we have $(F \star G)_t = F_t \star G_t$), and norm

$$\|F\| := \sup_{t\in[0,1]} \|F_t\|, \quad F \in C_c(\mathcal{G}).$$

We can complete $C_c(\mathcal{G})$ in this norm and obtain a C^*-algebra which we denote by A^C.

Pointwise multiplication of elements of $C_c(\mathcal{G})$ by functions in $C([0,1])$ produces elements of $C_c(\mathcal{G})$. This determines a unital $*$-homomorphism

$$\mu_{A^C} : C([0,1]) \to ZM(A^C),$$

and so A^C is a $C([0,1])$-algebra. One can show

$$(A^C)_t := \begin{cases} C^*(G_0) & t = 0 \\ C_r^*(G) & t \neq 0, \end{cases}$$

with canonical fibre maps $F \mapsto F_t$ for $F \in C_c(\mathcal{G})$, and that A^C is a continuous field, see [32, Lemma 6.13]. Also, $A^C\big|_{(0,1]} \cong C_0((0,1], C_r^*(G))$. In particular, there is an induced map in K-theory

$$K_*(C^*(G_0)) \to K^*(C_r^*(G))$$

which, as we discussed in the Introduction, can be identified with the classical assembly map.

Let us conclude this section by giving a description of the principal series representations of G, which can be found in the literature, see for example the description given by Mackey in [52, p.g. 344-345].

The principal series representations are induced from characters of a subgroup of G called the **Borel subgroup**. This subgroup is given by $B = TAN$, where $T \subseteq K$ is the circle group. The characters of B are of the form

$$\chi = \chi_T \times \chi_A \times 1$$

where χ_T and χ_A are characters on T and A, and 1 is the trivial character on N, sending all elements to 1. To see this, note that characters of B factor through the abelianization of B. The commutator subgroup of B is N and the abelianization is $T \times A$.

The characters of T are given by τ_n for $n \in \frac{1}{2}\mathbb{Z}$ (see (3.10)), and the characters of A are given by

$$\tau_y : A \to \mathbb{C}, \quad \tau_y\left(\begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}\right) = e^{2ixy}, \quad x \in \mathbb{R} \tag{4.15}$$

for $y \in \mathbb{R}$, see [24, Theorem 4.5]. Therefore the principal series representations are indexed by parameters $(n, y) \in \frac{1}{2}\mathbb{Z} \times \mathbb{R}$.

Note that in Chapter 3 we only discussed induced representations for compact groups. In this case, we will be inducing representations in the locally compact setting. We will not introduce any general theory - rather we will concretely describe the construction. The reader can consult the literature, for example [24, Chapter 6] for a very general treatment.

Let $q : G \to G/B$ be the canonical quotient map. Consider the vector space

$$\left\{\xi \in C(G) \;\middle|\; q(\mathrm{Supp}(\xi)) \text{ is compact}, \ \xi(gtan) = \tau_n(t)^{-1}\tau_y(a)^{-1}\delta(a)^{-1}\xi(g)\right\} \tag{4.16}$$

(c.f. the definition of $\mathcal{F}_0$ in [24, p.g. 152]). One can equip this vector space with an inner product. The definition of the inner product is a weighted L^2-norm (see the formula in [75, (2.3)]), and we denote the completion of the space (4.16) by $\mathcal{H}_{(n,y)}$. Then we define the principal series representation corresponding to the parameter $(n, y) \in \frac{1}{2}\mathbb{Z} \times \mathbb{R}$ by

$$(\pi_{(n,y)}(g)\xi)(h) = \xi(g^{-1}h) \tag{4.17}$$

where $g, h \in G$ and ξ is an element of the vector space in (4.16).

One can show that the norm on (4.16) is in fact equal to the $L^2(K)$-norm, i.e.

$$\|f\|_{L^2(K)} = \int_K |f(k)|^2 \,\mathrm{d}k, \quad f \in L^2(K)$$

see the argument given in [75, p.g. 188]. Then the restriction of functions in (4.16) defines an isometry

$$\mathrm{res} : \mathcal{H}_{(n,y)} \to \{\xi \in L^2(K) \mid \xi(kt) = \tau_n(t)^{-1}\xi(k) \text{ for all } k \in K \text{ and } t \in T\ \} \cong \mathcal{H}_n,$$

see (3.12) and (3.14). If $\xi \in L^2(K)$ satisfies $\xi(kt) = \tau_n(t)^{-1}\xi(k)$ for all $k \in K$ and $t \in T$, then we can obtain an element of $\mathcal{H}_{(n,y)}$ by setting

$$\xi(kan) := \tau_y(a)^{-1}\delta(a)^{-1}\xi(k) = \tau_{-y}(a)\delta(a)^{-1}\xi(k) \tag{4.18}$$

for $k \in K$, $a \in A$ and $n \in N$. Therefore res induces an isomorphism of Hilbert spaces. In particular $\mathcal{H}_{(n,y)}$ is independent of y, although the action of G is dependent on y. We often refer to this viewpoint of the carrier space as functions on K as the compact picture. In the compact picture, we trade off the simplicity of the definition of $\pi_{(n,y)}$ for a simpler carrier space.

Let us record a non-trivial theorem about principal series representations, originally due to Gelfand and Naimark in [26] (in Russian).

Theorem 4.3. *For each* $(n, y) \in \frac{1}{2}\mathbb{Z} \times \mathbb{R}$,

1. $\pi_{(n,y)}$ *is unitary and irreducible.*
2. $\pi_{(n,y)}$ *is unitarily equivalent to* $\pi_{(-n,-y)}$, *and these are the only equivalences between the principal series representations.*

If we let $\mathcal{H}$ denote the locally trivial bundle of Hilbert spaces over $\frac{1}{2}\mathbb{Z} \times \mathbb{R}$ with fibres $\mathcal{H}_{(m,x)} \cong \mathcal{H}_m$, then as we have done in the quantum setting, we can form the C^*-algebra $C_0(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H}))$. If we consider the Weyl group $W = \mathbb{Z}_2$, then we have a W-action on this C^*-algebra using the intertwiners provided by Theorem 4.3. Then one has the isomorphism

$$C^*_r(SL(2,\mathbb{C})) \cong C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right)^W \tag{4.19}$$

given on $C_c(SL(2,\mathbb{C}))$ by $f \mapsto ((n, y) \mapsto \pi_{(n,y)}(f))$, see [32, Theorem 3.3].

4.2 Lie Algebra to Lie Group Continuous Field

Let us consider an example of a continuous field which is fundamental to the constructions in the rest of this chapter. This is based on a much more general construction, see [55, Kapitel 3].

Example 4.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Define

$$\mathcal{G}_G := \mathfrak{g} \times \{0\} \sqcup (G \times (0, 1]) .$$

This is a groupoid, with partial product

$$\begin{cases} (X,0)(Y,0) = (X+Y,0) & X,Y \in \mathfrak{g},\ t=0 \\ (x,t)(y,t) = (xy,t) & x,y \in G,\ t \neq 0. \end{cases}$$

The inversion is given by

$$\begin{cases} (X,0)^{-1} = (-X,0) & X \in \mathfrak{g},\ t=0 \\ (x,t)^{-1} = (x^{-1},t) & x \in G,\ t \neq 0. \end{cases}$$

We will write $\mathcal{G}$ instead of $\mathcal{G}_G$ in this example to avoid excessive notation.

Let us topologize $\mathcal{G}$. Recall that for a Lie group, there is an exponential map $\exp : \mathfrak{g} \to G$, where $\mathfrak{g}$ is the Lie algebra of G. For matrix Lie groups, this is given by the matrix exponential map. There is a neighbourhood of $0 \in \mathfrak{g}$ on which exp is a diffeomorphism onto its image, and we denote the inverse by log. We refer the reader to the literature for these details, for example [44, p.g. 49 and Proposition 1.86].

Let $V_0 \subseteq \mathfrak{g}$ be a neighbourhood of 0 on which $\exp : \mathfrak{g} \to G$ is a diffeomorphism onto $\exp(V_0)$. Define

$$\phi : \mathfrak{g} \times [0,1] \to \mathcal{G}, \quad (X,t) \mapsto \begin{cases} (X,0) & t=0 \\ (\exp(tX),t) & t \neq 0. \end{cases} \tag{4.20}$$

Let $0 < \epsilon \leq 1$, and $V \subseteq \mathfrak{g}$ be an open subset such that $tV \subseteq V_0$ for all $t < \epsilon$. Define a family $\mathcal{T}$ of subsets of $\mathcal{G}$ of the following two types.

1. $U_{\epsilon,V} = \phi(V \times [0,\epsilon))$.
2. $U \subseteq G \times (0,1]$ open.

Note that on a set $V \times [0,\epsilon)$ of the above form, ϕ is injective.

We check that $\mathcal{T}$ is a basis for a topology on $\mathcal{G}$, using an argument based on that given in [55, p.g. 95-97]. The sets clearly cover $\mathcal{G}$. It remains to check that if $x \in W_1 \cap W_2$, with $W_1, W_2 \in \mathcal{T}$, then there exists $W \in \mathcal{T}$ such that $x \in W \subseteq W_1 \cap W_2$, see [41, Theorem 11, p.g. 47].

This clearly holds if W_1, W_2 are of the second type in our definition of $\mathcal{T}$. Let us consider the case where W_1 and W_2 are of the first type. Here W_i and $\epsilon_i > 0$, for $i = 1,2$, satisfy the appropriate conditions. If $x \in W_1 \cap W_2$, then there exists $(X_1,t_1) \in V_1 \times [0,\epsilon_1)$ and $(X_2,t_2) \in V_2 \times [0,\epsilon_2)$ such that $\phi((X_1,t_1)) = \phi((X_2,t_2)) = x$. By definition of ϕ,

$t_1 = t_2 =: t$. If $t = 0$, then $X_1 = X_2$. Otherwise, $\exp(tX_1) = \exp(tX_2)$ and $t \neq 0$. Since we are working inside V_0, we can invert exp and see that $X_1 = X_2$.

Now set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and $V = V_1 \cap V_2$. Then we have

$$\begin{aligned} x \in U_{\epsilon,V} &= \phi(V \times [0, \epsilon)) \\ &= \phi((V_1 \cap V_2) \times [0, \epsilon)) \\ &= \phi((V_1 \times [0, \epsilon)) \cap (V_2 \times [0, \epsilon))) \\ &\subseteq \phi(V_1 \times [0, \epsilon)) \cap \phi(V_2 \times [0, \epsilon)) \\ &\subseteq W_1 \cap W_2. \end{aligned}$$

Finally, let $U \subseteq G \times (0, 1]$ be open, and $U_{\epsilon,V}$ be of the second type with appropriate $\epsilon > 0$ and $V \subseteq \mathfrak{g}$. The intersection $U \cap U_{\epsilon,V}$ is open in $G \times (0, 1]$ because the intersection of $\phi(V \times [0, \epsilon)) \cap (G \times (0, 1]) = \phi(V \times (0, \epsilon))$, and ϕ is a homeomorphism (and hence open) when restricted to $V \times (0, \epsilon)$ (taking values in $G \times (0, 1]$ - indeed, the formula for ϕ is clearly a homeomorphism in this case). Therefore $\mathcal{T}$ is a basis for a topology on $\mathcal{G}$.

The topology is easily seen to be second countable, locally compact and Hausdorff. Note that by the description of the topology, a sequence $(g_n, t_n) \in G \times (0, 1]$ converges to $(X, 0) \in \mathfrak{g} \times \{0\}$ if and only if

$$t_n \to 0, \quad g_n \in \exp(V_0) \text{ eventually }, \quad t_n^{-1} \log(g_n) \to X. \tag{4.21}$$

This is convenient for checking functions on $\mathcal{G}$ are continuous - we can use the sequential characterization of continuity to check continuity of functions because $\mathcal{G}$ is second (and hence first) countable.

Note that if $V \subseteq \mathfrak{g}$ is open and $\epsilon > 0$ with $tV \subseteq V_0$ for $t < \epsilon$, then $\phi|_{V\times[0,\epsilon)}$ is a homeomorphism onto its image viewed as a subspace of $\mathcal{G}$. Indeed, it suffices to check that $\phi|_{V\times[0,\epsilon)}$ is a continuous open bijection. For openness, we see $\phi(U \times [0, \epsilon))$ and $\phi(U \times (a, b))$ are open sets, where $U \subseteq V$ is open and $0 < a < b \leq \epsilon$, from which the result follows. Continuity is immediate from the above sequential characterisation of the topology.

In particular each point has a neighbourhood that is homeomorphic to an open subset of $G \times (0, 1]$ or $V \times [0, \epsilon)$. It follows that $\mathcal{G}$ is a topological manifold.

Set

$$\mathcal{G}_t := \begin{cases} \mathfrak{g} & t = 0 \\ G & t \neq 0 \end{cases}$$

for $t \in [0,1]$ and $A := C_0(\mathcal{G})$. We can multiply functions in A pointwise by functions in $C([0,1])$ by

$$(g \cdot f)(x,t) = g(t)f(x,t), \quad g \in C([0,1]), \quad f \in C_0(\mathcal{G}), \quad (x,t) \in \mathcal{G}. \tag{4.22}$$

This defines an element of $C_0(\mathcal{G})$ - this follows from the fact the projection map

$$\mathrm{proj} : \mathcal{G} \to [0,1], \quad (x,t) \mapsto t$$

is continuous. Indeed for this the only thing to check is that for $0 < \epsilon \leq 1$, $\mathrm{proj}^{-1}([0,\epsilon))$ is an open set of $\mathcal{G}$. We have

$$\mathrm{proj}^{-1}([0,\epsilon)) = \mathfrak{g} \times \{0\} \sqcup G \times (0,\epsilon).$$

About every point $X \in \mathfrak{g}$, there exists a bounded open neighbourhood V_X, and an $\epsilon_X > 0$ such that for all $t < \epsilon_X$, $tV_X \subseteq V_0$. We can ensure $\epsilon_X < \epsilon$. Then $\mathfrak{g} \times \{0\}$ is covered by the open set $\bigcup_{X \in \mathfrak{g}} U_{\epsilon_X, V_X} \subseteq G \times (0,\epsilon)$. In particular, $\mathrm{proj}^{-1}([0,\epsilon)) = \bigcup_{X \in \mathfrak{g}} U_{\epsilon_X, V_X} \cup G \times (0,\epsilon)$ and therefore is open.

The action of $C([0,1])$ on A gives A the structure of a $C([0,1])$-algebra.

Let us construct some elements of A. We shall refer to these functions as functions of type 1 and type 2 respectively.

1. Let $f \in C_0(G \times (0,1])$ such that there exists $0 < \epsilon \leq 1$ with $f(x,t) = 0$ for all $t < \epsilon$ and $x \in G$. Define

$$F : \mathcal{G} \to \mathbb{C}, \quad (x,t) \mapsto \begin{cases} 0 & t = 0,\ x \in \mathfrak{g} \\ f(x,t) & t \neq 0,\ x \in G. \end{cases}$$

   If $(x_n, t_n) \to (X, 0) \in \mathcal{G}$, then eventually $t_n < \epsilon$ and so eventually $f(x_n,t_n) = 0$. Hence $f(x_n,t_n) \to 0 = F(X,0)$ and so F is continuous. The function F also vanishes at infinity because compact subsets of $G \times (0,1]$ are compact subsets of $\mathcal{G}$.

2. Let $g \in C_c(\mathfrak{g})$. Then there exists a bounded open neighbourhood V of $\mathrm{Supp}(g)$ and then we can find $R > 0$ such that $\frac{1}{R}V \subseteq V_0$. Then for all $t \leq \frac{1}{R}$, $t\,\mathrm{Supp}(g) \subseteq V_0$. Choose a continuous cutoff function $\chi : [0,1] \to \mathbb{R}$ that is identically 1 close to 0 and with support contained in $[0, \frac{1}{R})$. Define

$$G : U_{\frac{1}{R}, V} \to \mathbb{C}, \quad (x,t) \mapsto \begin{cases} g(x) & t = 0,\ x \in \mathfrak{g} \\ \chi(t) g(t^{-1}\log(x)) & t \neq 0,\ x \in G. \end{cases} \tag{4.23}$$

Note that G is continuous on $U_{\frac{1}{R},V} \cap (G \times (0,1])$, being a composition of continuous functions. If $(x_n, t_n) \to (X, 0) \in \mathcal{G}$, then eventually we can define $t_n^{-1}\log(x_n) \in \mathfrak{g}$ and this converges to X. Therefore $G(x_n, t_n) \to G(X, 0)$, and so the function is continuous. Because of the cutoff function, there exists a $t_0 < \frac{1}{R}$ such that $\mathrm{Supp}(G) \subseteq \phi(\mathrm{Supp}(g) \times [0, t_0])$. Since ϕ is a homeomorphism on $V \times [0, \frac{1}{R})$, $\phi(\mathrm{Supp}(g) \times [0, t_0])$ is compact and hence G is compactly supported in $U_{\frac{1}{R},V}$. We can extend G by zero to obtain an element of $C_0(\mathcal{G})$ by Proposition A.24.

Let $\mathcal{D}$ be the $*$-subalgebra of A generated by the functions of type 1 and type 2. It is clear that $\mathcal{D}$ is dense in A by the Stone-Weierstrass theorem.

Let us now determine the fibres of A.

Let $t \in [0,1]$. We can restrict $f \in C_0(\mathcal{G})$ to $\mathcal{G}_t \times \{t\} \subseteq \mathcal{G}$. Note that $\mathcal{G}_t \times \{t\} = \mathrm{proj}^{-1}(\{t\})$ and so is a closed subset of $\mathcal{G}_t$, and by the definition of the topology, is homeomorphic to $\mathcal{G}_t$ with it's usual topology. Then $f|_{\mathcal{G}_t \times \{t\}} \in C_0(\mathcal{G}_t)$ by Proposition A.26 and so we define the restriction homomorphisms

$$\mathrm{res}_{\mathcal{G}_t \times \{t\}} : C_0(\mathcal{G}) \to C_0(\mathcal{G}_t), \quad f \mapsto f(-,t).$$

The restriction homomorphisms are onto by our constructions of functions on $\mathcal{G}$ above. We claim that these are the evaluation maps of the field, from which it would follow that $C_0(\mathcal{G}_t)$ is the fibre of A at t.

We will show that for each $f \in C_0(\mathcal{G})$,

$$[0,1] \to \mathbb{R}, \quad t \mapsto \left\|\mathrm{res}_{\mathcal{G}_t \times \{t\}}(f)\right\|_{C_0(\mathcal{G}_t)} := \sup_{x \in \mathcal{G}_t} |f(x,t)| \tag{4.24}$$

is continuous. Note that if $f \in C_0(\mathcal{G})$,

$$\|f\|_{C_0(\mathcal{G})} = \sup_{(x,t) \in \mathcal{G}} |f(x,t)| = \sup_{t \in [0,1]} \left\|\mathrm{res}_{\mathcal{G}_t \times \{t\}}(f)\right\|_{C_0(\mathcal{G}_t)}.$$

It therefore follows by an $\frac{\epsilon}{3}$-argument that it suffices to show that the map (4.24) is continuous for f in any dense $*$-subalgebra of $C_0(\mathcal{G})$. In particular it suffices to check (4.24) is continuous for elements in $\mathcal{D}$.

Let $f \in \mathcal{D}$. If $t > 0$, then we can choose a closed neighbourhood V of t in $[0,1]$ that is disjoint from 0. Then $f|_{G \times V} \in C_0(G \times V)$ and to check continuity of the norm map (4.24) at t, we can restrict it to V and check continuity of

$$V \to \mathbb{R}, \quad t \mapsto \left\|\mathrm{res}_{\mathcal{G}_t \times \{t\}}(f)\right\|_{C_0(\mathcal{G}_t)} = \sup_{x \in G} |f(x,t)|.$$

This follows from the fact $C_0(G \times V) \cong C_0(V, C_0(G))$. Note that we didn't use the fact that $f \in \mathcal{D}$ here.

It remains to check continuity for $t = 0$. Note that if $f \in \mathcal{D}$, then for V in a sufficiently small closed neighbourhood of 0 in $[0, 1]$, $f|_{\mathrm{proj}^{-1}(V)}$ is the extension of sums and products of elements in $C_c(\mathfrak{g})$ away from 0 as in the construction of type 2 functions above. Since sums and products of elements in $C_c(\mathfrak{g})$ are again elements of $C_c(\mathfrak{g})$, we can assume that $f|_{\mathrm{proj}^{-1}(V)}$ is the extension of a single element of $C_c(\mathfrak{g})$, after possibly shrinking V so that all the cutoff functions involved are identically 1 on V. Since $f|_{\mathrm{proj}^{-1}(V)}$ is then supported inside an open subset of the form $U_{\epsilon,W}$, where W is an open set in $\mathfrak{g}$ and $\epsilon > 0$, with $tW \subseteq V_0$ for $t < \epsilon$, and ϕ is a homeomorphism, we can pull f back to a compactly supported function on $W \times [0, \epsilon)$. Doing this, f is in fact constant with respect to t, and so the norm function is necessarily continuous.

We have shown that the restriction maps satisfy the hypotheses of Proposition A.10 and that for each $f \in C_0(\mathcal{G})$, the map $t \mapsto \|f(-, t)\|_{C_0(\mathcal{G}_t)}$ is continuous. Therefore the evaluation maps to each fibre are given by the above restriction maps, and

$$C_0(\mathcal{G})_t \cong C_0(\mathcal{G}_t).$$

Let us finally consider the restriction of $C_0(\mathcal{G})$ to $(0, 1]$. We have

$$C_0(\mathcal{G})|_{(0,1]} \cong C_0(G \times (0, 1])$$

by Propositions A.24 and A.25. Therefore $C_0(\mathcal{G})$ is trivial away from 0.

Let us illustrate Example 4.4 in the case of the circle, where we can gain geometric intuition.

Example 4.5. Let us consider the case of Example 4.4 where $G = S^1$. We identify $\mathfrak{g}$ with $\mathbb{R}$ and S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. We have the manifold

$$\mathcal{G}_{S^1} := \mathbb{R} \times \{0\} \sqcup S^1 \times (0, 1] = \mathbb{R} \times \{0\} \sqcup \mathbb{R}/2\pi\mathbb{Z} \times (0, 1].$$

Suppose $(x_n + 2\pi\mathbb{Z}, t_n) \in \mathbb{R}/2\pi\mathbb{Z} \times (0, 1]$ is a sequence in $\mathcal{G}_{S^1}$ and $(x_n + 2\pi\mathbb{Z}, t_n) \to (y, 0) \in \mathbb{R} \times \{0\} \in \mathcal{G}_{S^1}$. Then the conditions for convergence of such a sequence in the topology on $\mathcal{G}_{S^1}$ mean in this case that

$$t_n \to 0, \quad \text{eventually } x_n + 2\pi\mathbb{Z} \text{ can be represented by an element in } (-\pi, \pi),$$

and, after possibly changing representatives so that eventually each $x_n \in (-\pi, \pi)$

$$t_n^{-1} x_n \to y \text{ in } \mathbb{R}.$$

The way one should think about this geometrically is that the radii of the circles increase as our field parameter approaches zero. Indeed, this is reflected in our construction as follows.

Let us view the copy of $\mathbb{R}/2\pi\mathbb{Z}$ over a point $t > 0$ in $\mathcal{G}_{S^1}$ as the quotient $\mathbb{R}/2\pi t^{-1}\mathbb{Z}$. This is via the rescaling map

$$\mathbb{R}/2\pi t^{-1}\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}, \quad x + 2\pi t^{-1}\mathbb{Z} \mapsto tx + 2\pi\mathbb{Z}.$$

We can then think of $C_0(\mathcal{G}_{S^1})$ as having fibres

$$C_0(\mathcal{G}_{S^1})_t = \begin{cases} C_0(\mathbb{R}) & t = 0 \\ C\left(\mathbb{R}/2\pi t^{-1}\mathbb{Z}\right) & t \neq 0. \end{cases}$$

Now suppose $(x_n + 2\pi t_n^{-1}\mathbb{Z}, t_n) \in \mathcal{G}_{S^1}$ is a sequence converging to $(y, 0) \in \mathbb{R} \times \{0\} \in \mathcal{G}_{S^1}$. This is the case if and only if

$$t_n \to 0, \quad \text{eventually } x_n + 2\pi t_n^{-1}\mathbb{Z} \text{ can be represented by an element in } (-t_n^{-1}\pi, t_n^{-1}\pi),$$

and, after possibly changing representatives so that eventually each $x_n \in (-t_n^{-1}\pi, t_n^{-1}\pi)$

$$x_n \to y \text{ in } \mathbb{R}.$$

In this viewpoint we can think of the construction of type 2 functions given in Example 4.4 in the following natural way. Take $g \in C_c(\mathbb{R})$. Then for t sufficiently small, t^{-1} is sufficiently large so that the interval $[-t^{-1}\pi, t^{-1}\pi]$ contains $\mathrm{Supp}(g)$. Then by extending g periodically outside of this interval, we can think of g as an element of $C\left(\mathbb{R}/2\pi t^{-1}\mathbb{Z}\right)$ for t small. We then choose a cutoff function to ensure we don't need to worry about when t is large enough that $[-t^{-1}\pi, t^{-1}\pi]$ fails to contain $\mathrm{Supp}(g)$.

4.3 The Left Quantization Field

In this section we will construct a continuous field A^L that induces a map in K-theory

$$K_*(C^*(G_0)) \to K_*(C^*(G_1)).$$

From Proposition 4.2, we have

$$C^*(G_0) \cong K \ltimes C_0(\mathfrak{k})$$

where K acts on $\mathfrak{k}$ via the adjoint action and from Proposition 2.2 we have

$$C^*(G_1) \cong K \ltimes C(K)$$

where K acts on K via the adjoint action. Notice that $C_0(\mathfrak{k})$ and $C(K)$, the C^*-algebras in these crossed products, are the fibres of the field from Example 4.4. In the following example we show that we can build a field with the desired fibres from the field from Example 4.4.

Example 4.6. Let K be a compact Lie group, and let $\mathfrak{k}$ be the Lie algebra of K. The adjoint action of K on both K and $\mathfrak{k}$ is given by conjugation, if we view $K \subseteq M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. This is because a compact Lie group is necessarily a matrix Lie group, see [24, Theorem 5.13].

By Example 4.4, there is a manifold $\mathcal{G}_K$ and $C_0(\mathcal{G}_K)$ is a continuous field over $[0,1]$ with fibres

$$C_0(\mathcal{G}_K)_t = \begin{cases} C_0(\mathfrak{k}) & t = 0 \\ C(K) & t \neq 0. \end{cases}$$

In this example we will show that the adjoint action on each fibre lifts to a field of actions on $C_0(\mathcal{G}_K)$ as defined in Definition A.35. We can assume that the choice of V_0 in the construction of $C_0(\mathcal{G}_K)$ is conjugation invariant, i.e. $kV_0k^{-1} \subseteq V_0$ for all $k \in K$, by choosing an adjoint invariant inner product on $\mathfrak{k}$ and choosing V_0 to be a sufficiently small ball with respect to the resulting norm.

We can define an action of K on $\mathcal{G}_K$ by the formula

$$K \times \mathcal{G}_K \to \mathcal{G}_K, \quad (k, x, t) \mapsto k \cdot (x, t), \quad k \cdot (x, t) = (kxk^{-1}, t).$$

This is a continuous action, because if $(k_n, x_n, t_n) \in \mathcal{G}_K$ converges to $(k, x, t) \in \mathcal{G}_K$, then $k_n \to k$ and $(x_n, t_n) \to (x, t)$. If $t \neq 0$, then eventually $t_n > 0$, and so we can assume that the sequence $(k_n, x_n, t_n) \in K \times K \times (0, 1]$. Clearly we then have

$$k_n \cdot (x_n, t_n) = (k_n \cdot x_n, t_n) \to (k \cdot x, t)$$

because the adjoint action of K on itself is continuous.

If $t = 0$, then $k_n \to k$, $t_n \to 0$ and eventually $x_n \in \exp(V_0)$ with $t_n^{-1} \log(x_n) \to X$ for some $X \in \mathfrak{k}$. We note that exp is adjoint action equivariant - indeed, in the case of a matrix

Lie group this is clear from the formula for the matrix exponential map. Therefore we have $k_n x_n k_n^{-1} \in k_n \exp(V_0) k_n^{-1} = \exp(k_n V_0 k_n^{-1}) \subseteq \exp(V_0)$ eventually, and so we can take logarithms. Then

$$t_n^{-1} \log(k_n x_n k_n^{-1}) = k_n t_n^{-1} \log(x_n) k_n^{-1} \to kXk^{-1}.$$

Hence $k_n \cdot (x_n, t_n) \to (k \cdot X, 0)$ as required.

The action of K on $\mathcal{G}_K$ then produces an action of K on $C_0(\mathcal{G}_K)$ by the formula

$$\alpha : K \to \mathrm{Aut}(C_0(\mathcal{G}_K)), \quad \alpha_k(f)(x,t) = f(k^{-1} \cdot (x,t)) = f(k^{-1}xk, t)$$

for $f \in C_0(\mathcal{G}_K)$ and $(x,t) \in \mathcal{G}_K$. Note that this action preserves the fibres of $C_0(\mathcal{G}_K)$, and so is a field of actions of K on $C_0(\mathcal{G}_K)$. The action on each fibre is clearly the adjoint action.

By Theorem A.38, the crossed product $K \ltimes_\alpha C_0(\mathcal{G}_K)$ is a $C([0,1])$-algebra which has fibres

$$(K \ltimes_\alpha C_0(\mathcal{G}_K))_t = \begin{cases} K \ltimes_{\mathrm{adj}} C_0(\mathfrak{k}) & t = 0 \\ K \ltimes_{\mathrm{adj}} C(K) & t \neq 0 \end{cases}$$

and is trivial away from 0.

Let us take $K = SU(2)$ in Example 4.6. Then the field $K \ltimes_\alpha C_0(\mathcal{G}_K)$ constructed there induces a map in K-theory,

$$\begin{array}{ccc} K_*(K \ltimes_{\mathrm{adj}} C_0(\mathfrak{k})) & \longrightarrow & K_*(K \ltimes_{\mathrm{adj}} C(K)) \\ \| & & \| \\ K_*(C^*(G_0)) & \longrightarrow & K_*(C^*(G_1)). \end{array}$$

This field will be referred to as the **left quantization field**, and the corresponding map the left quantization map.

Let us consider a more concrete viewpoint of the left quantization field that will be useful in the sequel. For the rest of this section, $K = SU(2)$.

By Theorem 3.13,

$$K \ltimes_{\mathrm{adj}} C_0(\mathfrak{k}) \cong C_0(\mathfrak{k}, K(L^2(K)))^K$$

where K acts on $\mathfrak{k}$ via the adjoint action, and on $K(L^2(K))$ by conjugation by the right regular representation. We will provide an alternative description of $C_0(\mathfrak{k}, K(L^2(K)))^K$. First, we need two basic results.

Lemma 4.7. *The restriction mapping*

$$\mathrm{res} : C_0(\mathfrak{k}, K(L^2(K)))^K \to C_0(\mathfrak{t}, K(L^2(K))), \quad f \mapsto f|_{\mathfrak{t}}$$

is injective, where $\mathfrak{t} \subseteq \mathfrak{k}$ is the Lie algebra of $T \subseteq K$.

Proof. If $X \in \mathfrak{k}$, then there exists $k \in K$ such that $k^{-1}Xk \in \mathfrak{t}$, because all elements of $\mathfrak{k}$ are normal (see (4.4)). If $f \in C_0(\mathfrak{k}, K(L^2(K)))^K$ we have

$$f(X) = \gamma_k(f(k^{-1}Xk))$$

where γ_k denotes the conjugation action on $K(L^2(K))$ with respect to the right regular representation, i.e. $\gamma_k(T) = \rho_k T \rho_{k^{-1}}$ for $T \in K(L^2(K))$, where ρ denotes the right regular representation. If $f|_{\mathfrak{t}} = 0$, then $\gamma_k(f(k^{-1}Xk)) = \gamma_k(0) = 0$, and so $f(X) = 0$. Hence $f = 0$. ☐

Note that the elements of $\mathfrak{t} \subseteq \mathfrak{k}$ are given by matrices of the form

$$\begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} \tag{4.25}$$

where $x \in \mathbb{R}$. We now consider an analogue of Lemma 3.8 on the level of Lie algebras.

Lemma 4.8. *Let $X \in \mathfrak{t} \subseteq \mathfrak{k}$. Then K_X, the stabilizer of X under the adjoint action of K on $\mathfrak{k}$ is*

$$K_X = \begin{cases} K & X = 0 \\ T & X \neq 0. \end{cases}$$

Two distinct elements $X, Y \in \mathfrak{t}$ are conjugate under K if and only if $X = -Y$, and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -X.$$

Proof. For a given $X \in \mathfrak{t}$, we can write

$$X = \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix}$$

for some $x \in \mathbb{R}$, see (4.25).

Direct calculation tells us that if $\begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in K_X$, then $\alpha^*\gamma = 0$ or $x = 0$. Therefore $\alpha = 0$, $\gamma = 0$ or $x = 0$.

If $x = 0$ then $X = 0$ and so clearly $K_X = K$.

If $\alpha = 0$ and $\gamma \neq 0$, $x \neq 0$, then

$$\begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} -ix & 0 \\ 0 & ix \end{pmatrix} = \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix}$$

which cannot happen. The case where $\gamma = 0$ can occur, and we then see $K_X = T$ for $X \neq 0$.

Note that this calculation also tells us that two distinct elements $X, Y \in \mathfrak{t}$ are conjugate under K if and only if $X = -Y$, and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -X. \qquad \square$$

Let $f \in C_0(\mathfrak{k}, K(L^2(K)))^K$ and $X \in \mathfrak{t}$. Then by Lemma 4.8 we have $T \subseteq K_X$, and so $f(X) \in K(L^2(K))^{K_X} \subseteq K(L^2(K))^T$, see (3.20). In particular, $\mathrm{res}(f) \in C_0(\mathfrak{t}, K(L^2(K))^T)$.

Recall (see (3.10)) that for each $m \in \frac{1}{2}\mathbb{Z}$, we have the irreducible representation τ_m of T on $W_m = \mathbb{C}$, defined by

$$\tau_m : T \to U(W_m), \quad \tau_m(z) = z^{2m}.$$

We have the Hilbert space

$$\mathcal{H}_m := \mathrm{Ind}_T^K(\tau_m) = \{\xi \in L^2(K) \mid \xi(kt) = \tau_m(t)^{-1}\xi(k) \text{ for all } k \in K,\, t \in T\}$$

as defined in Chapter 3, (3.5). Note that this notation is consistent with the notation in Remark 2.5 and (3.14). By Proposition 3.15,

$$K(L^2(K))^T \cong \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m)$$

where $L^2(K) \cong \bigoplus_{m\in\frac{1}{2}\mathbb{Z}} \mathcal{H}_m \otimes W_m^*$ by inducing the left regular representation of T to K. If we are given $T = (T_m) \in \bigoplus_{m\in\frac{1}{2}\mathbb{Z}} K(\mathcal{H}_{(m,x)})$ then T acts on this decomposition by $\oplus_{m\in\frac{1}{2}\mathbb{Z}} T_m \otimes 1_{W_m^*}$.

Let $\mathcal{H} = (\mathcal{H}_{m,X})$ denote the bundle of Hilbert spaces over $\frac{1}{2}\mathbb{Z} \times \mathfrak{t}$ with locally constant fibres $\mathcal{H}_{m,X} = \mathcal{H}_m$. Then

$$C_0(\mathfrak{t}, K(L^2(K))^T) \cong C_0\left(\mathfrak{t}, \bigoplus_{m\in\frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m)\right) \cong C_0\left(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H})\right)$$

where $C_0(\frac{1}{2}\mathbb{Z}\times\mathfrak{t}, K(\mathcal{H}))$ is the compact operators of the (locally trivial) Hilbert $C_0(\frac{1}{2}\mathbb{Z}\times\mathfrak{t})$-module corresponding to the bundle $\mathcal{H}$, c.f. our construction for Theorem 2.11 and (4.19).

By Lemma 4.7 we can view $C_0(\mathfrak{k}, K(L^2(K)))^K \subseteq C_0(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H}))$. We now need to take into account the invariance condition on $C_0(\mathfrak{k}, K(L^2(K)))^K$ to be able to determine $C_0(\mathfrak{k}, K(L^2(K)))^K$ as a subalgebra of $C_0(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H}))$. For this, we introduce an action of the Weyl group W. First we will introduce the Weyl group in a more abstract sense as opposed to the rather concrete description given previously.

Let $W := {}^{N(T)}\!/_{T}$, where $N(T)$ is the normalizer of T in K. By Lemma 3.8, the normalizer of T consists of the elements

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}, \quad \begin{pmatrix} 0 & -\gamma^* \\ \gamma & 0 \end{pmatrix}, \quad \alpha, \gamma \in S^1 \subseteq \mathbb{C}.$$

Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We note that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^* \end{pmatrix} = \begin{pmatrix} 0 & -\gamma^* \\ \gamma & 0 \end{pmatrix}$$

and so $W = \{T, wT\} \cong \mathbb{Z}_2$. Then W acts on $\mathfrak{t}$ by conjugation, with $w{\cdot}X = -X$ by Lemma 4.8. Strictly speaking, we pick a representative of wT to calculate the conjugation, but the result is independent of the choice of representative.

We will see that W also acts on $C_0(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H}))$. Of course, the class T acts trivially. For each $m \in \frac{1}{2}\mathbb{Z}$, we have the unitary

$$U_w^m : \mathcal{H}_m \to \mathcal{H}_{-m} \tag{4.26}$$

defined by $U_w^m(\xi)(k) = \xi(kw)$ for $\xi in \mathcal{H}_m$ and $k \in K$. Note that strictly speaking we are picking the representative w for the class $wT \in W$. However, the induced isomorphism $K(\mathcal{H}_m) \cong K(\mathcal{H}_{-m})$ induced by conjugation is independent of this choice. Indeed, if $w' \in K$ is another representative for wT, then $w = w't$ for some $t \in T$. Then for $\xi in \mathcal{H}_m$ and $k \in K$

$$U_w^m(\xi)(k) = \xi(kw) = \xi(kw't) = \tau_{-m}(t)^{-1}\xi(kw') = \tau_{-m}(t)^{-1}U_{w'}^m(\xi)(k) \tag{4.27}$$

and so $U_w^m = \tau_{-m}(t)^{-1}U_{w'}^m$. If $T \in K(\mathcal{H}_m)$, we have

$$(U_w^m)^{-1}TU_w^m = (\tau_{-m}(t)^{-1}U_{w'}^m)^{-1}T\tau_{-m}(t)^{-1}U_{w'}^m = (U_{w'}^m)^{-1}TU_{w'}^m. \tag{4.28}$$

Then the formula

$$(w \cdot f)(n, X) = U_w^n f(-n, -X)(U_w^n)^{-1}, \quad (n, X) \in \frac{1}{2}\mathbb{Z} \times \mathfrak{t}$$

defines an action of W on $C_0(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H}))$.

We will check that if $f \in C_0(\mathfrak{k}, K(L^2(K)))^K$, then f is invariant for the W-action on $C_0(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H}))$. Under the isomorphism

$$L^2(K) \cong \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} \mathcal{H}_m \otimes W_m^* \cong \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} \mathcal{H}_m$$

provided by inducing the left regular representation of T to K, the right regular action of w on $L^2(K)$ maps $\mathcal{H}_m$ to $\mathcal{H}_{-m}$.

Now if we view $f \in C_0(\mathfrak{k}, K(L^2(K)))^K \subseteq C_0(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H}))$, we have, for $(n, X) \in \frac{1}{2}\mathbb{Z} \times \mathfrak{t}$,

$$(U_w^n)^{-1} f(-n, -X) U_w^n = (U_w^n)^{-1} p_{-n}(f(-X)) U_w^n = p_n(\rho_{w^{-1}} f(-X) \rho_w) = f(n, X) \tag{4.29}$$

where $p_n : K(L^2(K))^T \to K(\mathcal{H}_n)$ is the projection to the n^{th} summand and ρ is the right regular representation of K. Hence

$$C_0(\mathfrak{k}, K(L^2(K)))^K \subseteq C_0\left(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H})\right)^W. \tag{4.30}$$

For $X \in \mathfrak{t}$ and $f \in C_0(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H}))$ denote by $f_X \in K(L^2(K))$ the element obtained by evaluating f in the second variable at X and viewing the result as an element of

$$\bigoplus_{m \in \frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m) \subseteq K(L^2(K)).$$

With this notation in place we can describe $C_0(\mathfrak{k}, K(L^2(K)))^K \subseteq C_0\left(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H})\right)^W$ completely.

Theorem 4.9. *The C^*-algebra $K \ltimes C_0(\mathfrak{k})$ is isomorphic to the subalgebra*

$$A_0^L := \left\{ f \in C_0\left(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H})\right)^W \mid f_0 \in K(L^2(K))^K \right\}$$

of $C_0(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H}))^W$.

Proof. We already observed that $K \ltimes C_0(\mathfrak{k}) \cong C_0(\mathfrak{k}, K(L^2(K)))^K \subseteq C_0\left(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H})\right)^W$ in (4.30), and if $f \in C_0(\mathfrak{k}, K(L^2(K)))^K$, then $f(0) \in K(L^2(K))^{K_0} = K(L^2(K))^K$. Therefore

$C_0(\mathfrak{k}, K(L^2(K)))^K \subseteq A_0^L$. To show that $K \ltimes C_0(\mathfrak{k}) \cong A_0^L$ we aim to understand the irreducible representations of A_0^L, and compare them to the irreducible representations of $C_0(\mathfrak{k}, K(L^2(K)))^K$ given in Proposition 3.16.

Let $\pi : A_0^L \to B(V)$ be an irreducible representation of A_0^L on a Hilbert space V. By [59, Theorem 5.5.1] there exists an irreducible representation $\overline{\pi} : C_0(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H})) \to B(K)$ on a Hilbert space K such that the restriction of $\overline{\pi}$ to A_0^L is equivalent to π. In particular, we can assume $V \subseteq K$, and the action of π on V is given by $\overline{\pi}$. Since $C_0(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H})) = C_0(\mathfrak{t}, \bigoplus_{\mu\in\frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m)))$, then (after possibly a unitary equivalence) $\overline{\pi}$ must be given by point evaluation at an element $X \in \mathfrak{t}$, followed by projection to a direct summand of $\oplus_{m\in\frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m)$.

In particular, each irreducible representation of A_0^L factors through point evaluation at elements of $\mathfrak{t}$. If $f \in A_0^L$ and $X \in \mathfrak{t}$, then

$$f_X \in K(L^2(K))^{K_X}$$

and by appropriate choice of f, we can obtain all operators in $K(L^2(K))^{K_X}$ in this way. Now by Proposition 3.15

$$K(L^2(K))^{K_0} = K(L^2(K))^K = \bigoplus_{m\in\frac{1}{2}\mathbb{N}_0} K(V(m))$$

and if $X \neq 0$,

$$K(L^2(K))^{K_X} = K(L^2(K))^T = \bigoplus_{m\in\frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m).$$

In particular, the irreducible representations of A_0^L are given by point evaluations at an element $X \in \mathfrak{t}$ followed by projection to one of these direct summands of $K(L^2(K))^{K_X}$. Notice that restricting irreducible representations of A_0^L gives an irreducible representation of $C_0(\mathfrak{k}, K(L^2(K)))^K$, by Proposition 3.16. If we restrict two inequivalent irreducible representations of A_0^L to the image of res, we can also see they remain inequivalent.

The result then follows from standard results about such subalgebras of postliminal C^*-algebras, see [17, 11.1.1 and 11.1.6]. □

We can obtain an analogous result to Theorem 4.9 for $K \ltimes C(K)$. We will only outline the argument in this case. By Theorem 3.13,

$$K \ltimes_{\text{adj}} C(K) \cong C(K, K(L^2(K)))^K$$

where K acts itself via the adjoint action, and on $K(L^2(K))$ by conjugation by the right regular representation. The restriction mapping

$$\mathrm{res} : C(K, K(L^2(K)))^K \to C(T, K(L^2(K))), \quad f \mapsto f|_T$$

is an injection because the conjugacy class of any element of K meets T, see the discussion at the start of Section 3.1.3.

Let $f \in C(K, K(L^2(K)))^K$ and $x \in T$. By Lemma 3.8 the centralizer of x in K is

$$K_x = \begin{cases} K & x = \pm I \\ T & x \neq \pm I. \end{cases}$$

Therefore in any case, $T \subseteq K_x$, we have $f(x) \in K(L^2(K))^{K_x} \subseteq K(L^2(K))^T$, see (3.20). In particular, $\mathrm{res}(f) \in C(T, K(L^2(K))^T)$.

Let $\mathcal{H} = (\mathcal{H}_{m,x})$ denote the bundle of Hilbert spaces over $\frac{1}{2}\mathbb{Z} \times T$ with locally constant fibres $\mathcal{H}_{m,x} = \mathcal{H}_m$. Then

$$C(T, K(L^2(K))^T) \cong C\left(T, \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m)\right) \cong C_0\left(\frac{1}{2}\mathbb{Z} \times T, K(\mathcal{H})\right)$$

where $C_0(\frac{1}{2}\mathbb{Z} \times T, K(\mathcal{H}))$ is the compact operators of the (locally trivial) Hilbert $C_0(\frac{1}{2}\mathbb{Z} \times T)$-module corresponding to the bundle $\mathcal{H}$.

By our observations so far we have $C(K, K(L^2(K)))^K \subseteq C_0(\frac{1}{2}\mathbb{Z} \times T, K(\mathcal{H}))$. We now need to take into account the invariance condition on $C(K, K(L^2(K)))^K$ to be able to determine $C(K, K(L^2(K)))^K$ as a subalgebra of $C_0(\frac{1}{2}\mathbb{Z} \times T, K(\mathcal{H}))$.

By Lemma 3.8, W acts on T by conjugation, as the choice of representative makes no difference to the calculation. Also W acts on $C_0(\frac{1}{2}\mathbb{Z} \times T, K(\mathcal{H}))$ by the formula

$$(w \cdot f)(n, x) = (U_w^n)^{-1} f(-n, x^{-1}) U_w^n, \quad (n, x) \in \frac{1}{2}\mathbb{Z} \times T,$$

where for each $m \in \frac{1}{2}\mathbb{Z}$, we have the same unitaries defined in (4.26), namely

$$U_w^m : \mathcal{H}_m \to \mathcal{H}_{-m}, \quad U_w^m(\xi)(k) = \xi(kw), \quad \xi \in \mathcal{H}_m, \quad k \in K.$$

As in (4.29), if $f \in C(K, K(L^2(K)))^K$, then $f \in C_0(\frac{1}{2}\mathbb{Z} \times T, K(\mathcal{H}))$ is contained in the W-invariant part of $C_0(\frac{1}{2}\mathbb{Z} \times T, K(\mathcal{H}))$.

For $x \in T$ and $f \in C_0(\frac{1}{2}\mathbb{Z} \times T, K(\mathcal{H}))$ denote by $f_x \in K(L^2(K))$ the element obtained by evaluating f in the second variable at x and viewing the result as an element of

$$\bigoplus_{m\in\frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m) \subseteq K(L^2(K)).$$

With this notation in place we can describe $C(K, K(L^2(K)))^K \subseteq C_0\left(\frac{1}{2}\mathbb{Z}\times T, K(\mathcal{H})\right)^W$ completely.

Theorem 4.10. *The C^*-algebra $K \ltimes C(K)$ is isomorphic to the subalgebra*

$$A_1^L := \left\{ f \in C_0\left(\frac{1}{2}\mathbb{Z}\times T, K(\mathcal{H})\right)^W \mid f_{\pm I} \in K(L^2(K))^K \right\}$$

of $C_0(\frac{1}{2}\mathbb{Z}\times T, K(\mathcal{H}))^W$.

Proof. This is essentially the same as the proof of Theorem 4.9. □

Let us consider again the left quantization field $K \ltimes C_0(\mathcal{G}_K)$ in the light of Theorems 4.9 and 4.10. We will define another continuous field of C^*-algebras A^L over $[0,1]$ whose fibres are isomorphic to A_0^L at 0 and A_1^L away from 0. We will then identify this field with the left quantization field introduced above. This will give us a concrete description of the left quantization field.

First, we consider the C^*-algebras

$$C_0\left(\frac{1}{2}\mathbb{Z}\times \mathfrak{t}, K(\mathcal{H})\right) = C_0\left(\mathfrak{t}, \bigoplus_{m\in\frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m)\right) = C_0\left(\mathfrak{t}\right) \otimes \bigoplus_{m\in\frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m) \tag{4.31}$$

$$C_0\left(\frac{1}{2}\mathbb{Z}\times T, K(\mathcal{H})\right) = C\left(T, \bigoplus_{m\in\frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m)\right) = C\left(T\right) \otimes \bigoplus_{m\in\frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m). \tag{4.32}$$

Note here we are abusing notation by using $\mathcal{H}$ for both of our Hilbert bundles. However the context means there should be no confusion.

Combining Example 4.4 for the circle group T and Theorem A.34 with the identifications (4.31) and (4.32) we obtain a field B over $[0,1]$ with fibres

$$B_t = \begin{cases} C_0\left(\frac{1}{2}\mathbb{Z}\times \mathfrak{t}, K(\mathcal{H})\right) & t = 0 \\ C_0\left(\frac{1}{2}\mathbb{Z}\times T, K(\mathcal{H})\right) & t \neq 0. \end{cases}$$

The Weyl group action on each fibre gives a field of actions of W on B in the sense of Definition A.35. Indeed, the action of W on $\mathfrak{t}$ and T give rise to actions on the Lie algebra-

Lie group field, and the action is constant on the compact operators. Then by Proposition A.37, we have obtain a field C over $[0,1]$ with fibres

$$C_t = \begin{cases} C_0\left(\frac{1}{2}\mathbb{Z} \times \mathfrak{t}, K(\mathcal{H})\right)^W & t = 0 \\ C_0\left(\frac{1}{2}\mathbb{Z} \times T, K(\mathcal{H})\right)^W & t \neq 0. \end{cases}$$

The algebras A_0^L and A_1^L defined in Theorems 4.9 and 4.10 are subalgebras of the fibres above. If we take an element $f \in C_c\left(\mathfrak{t}, \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m)\right)$ with the condition that $f_0 \in K(L^2(K))^K$, then $f \in A_0^L$, and we can extend it to a section F of C using the formulae for the type 2 functions defined in Example 4.4, (4.23). By examining these formulae we see that

$$\mathrm{ev}_t(F)(-, I) \in K(L^2(K))^K, \quad \mathrm{ev}_t(F)(-, -I) = 0$$

for all t. In particular, we obtain a subfield of C, which we denote by A^L, with fibres

$$A_t^L \cong \begin{cases} A_0^L & t = 0 \\ A_1^L & t \neq 0. \end{cases}$$

We will now show the left quantization field $K \ltimes C_0(\mathcal{G}_K)$ is isomorphic to A^L.

Theorem 4.11. *The continuous field A^L is isomorphic to the left quantization field.*

Proof. We consider elements of the convolution algebra $C_c(K, C_0(\mathcal{G}_K)) \subseteq K \ltimes C_0(\mathcal{G}_K)$ of the form $g \otimes h$, where $g \in C(K)$ and $h \in C_0(\mathcal{G}_K)$ is of type 1 or type 2, as defined in Example 4.4. The linear span of such functions is dense in $K \ltimes C_0(\mathcal{G}_K)$ by [89, Lemma 1.87]. We will show that such functions define continuous sections of A^L, and then our result will follow from Theorem A.23. In the type 1 case, these functions are elements of $C_c(K, C_0((0,1], C(K)))$ which clearly define continuous sections of A^L.

Now suppose f is of the form $f = g \otimes h$, where $g \in C(K)$ and $h \in C_c(\mathcal{G}_K)$ is a type 2 function obtained by extending a compactly supported function h_0 in $C_c(\mathfrak{k})$ to an element of $C_c(\mathcal{G}_K)$, as explained in Example 4.4. Under the isomorphism provided by Theorems 4.9 and 3.13, we have

$$((g \otimes h_0)(m, X))(\xi)(k) = \int_K g(s) h_0(s^{-1}k \cdot X) \xi(s^{-1}k) \, \mathrm{d}s$$

for $m \in \frac{1}{2}\mathbb{Z}$, $X \in \mathfrak{t}$, $\xi \in \mathcal{H}_m$ and $k \in K$. Similarly, under the isomorphism provided by Theorems 4.10 and 3.13, we have

$$((g \otimes h_t)(m, \exp(X)))(\xi)(k) = \int_K g(s) h_0(t^{-1}s^{-1}k \cdot X) \xi(s^{-1}k) \, \mathrm{d}s$$

for $t > 0$ small. From these formulae and the description of the topology on the field $\mathcal{G}_T$ (see (4.21)) we can see that under these isomorphisms on each fibre, the element f is mapped to a continuous section of A^L, as required. $\square$

4.4 The Right Quantization Field

Let $G = SL(2,\mathbb{C})$ and $K = SU(2)$ throughout this section. For a fixed $q \in (0,1)$ we will construct a continuous field A^R that induces a map in K-theory

$$K_*(C_r^*(G)) \to K_*(C_r^*(G_q))$$

where $G_q = SL_q(2,\mathbb{C})$.

Recall from Section 4.1, (4.19) that

$$C_r^*(G) \cong C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right)^W.$$

Recall from Theorem 2.11 that

$$C_r^*(G_q) \cong C_0\left(\frac{1}{2}\mathbb{Z} \times {\mathbb{R}}/{2\pi h^{-1}\mathbb{Z}}, K(\mathcal{H}^q)\right)^W$$

where $q = e^h$ (and so $h \in (-\infty, 0)$) and $\mathcal{H}_m^q$ is the closure of

$$\mathcal{O}(\mathcal{E}_m^q) := \left\{f \in \mathcal{O}(SU_q(2)) \mid (\mathrm{id} \otimes \pi)\Delta(f) = f \otimes z^{-2m}\right\} \subseteq L^2(K_q)$$

with respect to the L^2-norm on $C(SU_q(2))$, see (2.8). Recall also that $\mathcal{H}_m^q$ can be identified for each $q \in (0,1)$ with a fixed Hilbert space which is isomorphic to $\mathcal{H}_m$, see Remark 2.5. Let

$$V_m^q : \mathcal{H}_m^q \to \mathcal{H}_m$$

denote this unitary, which rescales the standard orthonormal basis elements of $\mathcal{H}_m$ by a scalar which is continuous in q, and converges to 1 as $q \to 1$, see (2.26). Therefore we will make the identification

$$C_r^*(G_q) \cong C_0\left(\frac{1}{2}\mathbb{Z} \times {\mathbb{R}}/{2\pi h^{-1}\mathbb{Z}}, K(\mathcal{H})\right)^W.$$

We see from the above concrete description of $C_r^*(G_q)$ that these algebras are naturally parameterized by h. Since it will be convenient to define A^R to be a $C([0,1])$-algebra, let us fix $q \in (0,1)$ with $q = e^h$, and then use $t \in [0,1]$ to scale h to th, so our quantum parameter is $q^t = e^{th}$, varying between q and 1. We will therefore construct the field A^R to have fibres

$$A_t^R = \begin{cases} C_r^*(G) & t = 0 \\ C_r^*(G_{q^t}) & t \neq 0. \end{cases}$$

By Theorem A.34 and Example 4.5, we have a continuous field B over $[0,1]$ with fibres

$$\begin{aligned} B_0 &:= C_0(\mathbb{R}) \otimes \left(\bigoplus_{m \in \frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m) \right) \\ &= C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H}) \right) \end{aligned} \tag{4.33}$$

and

$$B_t := C_0\left(\mathbb{R}/_{2\pi t^{-1}\mathbb{Z}} \right) \otimes \left(\bigoplus_{m \in \frac{1}{2}\mathbb{Z}} K(\mathcal{H}_m) \right) = C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}/_{2\pi t^{-1}\mathbb{Z}}, K(\mathcal{H}) \right). \tag{4.34}$$

Now we note that for $t > 0$,

$$C_r^*(G_{q^t}) \cong C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}/_{2\pi h^{-1}t^{-1}\mathbb{Z}}, K(\mathcal{H}) \right)^W \subseteq C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}/_{2\pi t^{-1}\mathbb{Z}}, K(\mathcal{H}) \right) = B_t$$

by rescaling the circular parameter using the map

$$\mathbb{R}/_{2\pi h^{-1}t^{-1}\mathbb{Z}} \cong \mathbb{R}/_{2\pi t^{-1}\mathbb{Z}}, \quad x + 2\pi h^{-1}t^{-1}\mathbb{Z} \mapsto hx + 2\pi t^{-1}\mathbb{Z}.$$

Since we have rescaled the quantum case, we need to also scale the classical case, i.e.

$$C_r^*(G) \cong C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H}) \right)^W \subseteq C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H}) \right) = B_0$$

where we rescale the real parameter using the map

$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto hx.$$

We have therefore identified the group C^*-algebras as subalgebras of the fibre algebras of a continuous field B. We now identify a field of actions of W on B so that we may apply Proposition A.37, and obtain a field with the group C^*-algebras themselves as fibres. This is achieved by taking the actions of W on

$$C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H}) \right), \quad C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}/_{2\pi h^{-1}t^{-1}\mathbb{Z}}, K(\mathcal{H}) \right)$$

provided by the equivalences between the principal series representations (Theorems 2.10 and 4.3) and scaling them so that they define actions on B_0 and B_t respectively. If we can then show that the action on each fibre lifts to a field of actions on B, then we can take $A^R = B^W$ (see Proposition A.37) and then A^R will have the desired properties.

We need to understand how the action of the intertwiners in both the classical and quantum cases provided by Theorems 2.10 and 4.3 behave with respect to t.

Let us start with the classical case. For each $m \in \frac{1}{2}\mathbb{Z}$ and $x \in \mathbb{R}$, we have G acting on $\mathcal{H}_m$ by the principal series representation $\pi_{(m,x)} : G \to U(\mathcal{H}_m)$, given in (4.17). Therefore in particular, we can restrict $\pi_{(m,x)}$ to K and decompose $\mathcal{H}_m$ as a direct sum of irreducible representations of K. We saw in Section 3.1.4, (3.23) and (3.24) that

$$\mathcal{H}_m \cong \mathcal{H}_{(m,x)} \cong \bigoplus_{n \geq |m|, n+m \in \mathbb{Z}} V(n).$$

The intertwiner $U_{(m,x)}$ between $\pi_{(m,x)}$ and $\pi_{(-m,-x)}$ is an intertwiner of the underlying K-representations. In particular, by Schur's lemma (Theorem 1.25), $U_{(m,x)}$ is a direct sum of scalar operators

$$U_{(m,x)} : \bigoplus_{n \geq |m|, n+m \in \mathbb{Z}} V(n) \to \bigoplus_{n \geq |m|, n+m \in \mathbb{Z}} V(n), \quad U_{(m,x)} = \bigoplus_{n \geq |m|, n+m \in \mathbb{Z}} \lambda^n_{(m,x)} 1_{V(n)}, \tag{4.35}$$

where the scalars $\lambda^n_{(m,x)} \in S^1 \subseteq \mathbb{C}$ are determined by the fact that $U_{(m,x)}$ is an intertwiner of G-representations.

We should note that we can carry out a similar argument in the quantum case. In this case, for each $m \in \frac{1}{2}\mathbb{Z}$ and $x + 2\pi h^{-1}\mathbb{Z} \in \mathbb{R}/_{2\pi h^{-1}\mathbb{Z}}$, we have that $C^*_r(G_q)$ acts on $\mathcal{H}_m$ by the principal series representation $\pi^q_{(m,ix)} : C^*_r(G_q) \to B(\mathcal{H}_m)$, see (2.13).

We can restrict $\pi^q_{(m,ix)}$ to a $*$-representation of $\mathcal{D}(K_q) \cong \mathcal{D}(K_q) \bowtie 1 \subseteq \mathcal{D}(G_q)$, and then using Remark 1.14 we can extend this to a $*$-representation of $M(\mathcal{D}(K_q))$. Recall that $U_q(\mathfrak{sl}_2(\mathbb{C})) \subseteq M(\mathcal{D}(K_q))$ (see (1.16)), and so $\pi^q_{(m,ix)}$ gives rise to a $*$-representation of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on $\mathcal{H}_m$. By considering the action of K, we can see this is a type 1 representation, and so $\mathcal{H}_m$ decomposes as a direct sum of the irreducible representations constructed in Section 1.3. In fact,

$$\mathcal{H}_m \cong \bigoplus_{n \geq |m|, n+m \in \mathbb{Z}} V(n).$$

The intertwiner between $\pi^q_{(m,ix)}$ and $\pi^q_{(-m,-ix)}$ denoted $U^q_{(m,ix)}$ is a direct sum of scalar operators

$$U^q_{(m,ix)} : \bigoplus_{n \geq |m|, n+m \in \mathbb{Z}} V(n) \to \bigoplus_{n \geq |m|, n+m \in \mathbb{Z}} V(n), \tag{4.36}$$

given by

$$U^q_{(m,ix)} = \bigoplus_{n\geq |m|, n+m\in\mathbb{Z}} \lambda^n_{(m,ix)}(q) 1_{V(n)}. \tag{4.37}$$

We have the following theorem (see [86, Theorem 5.42]) which gives formulae for these scalars.

Theorem 4.12. *Let $m \in \frac{1}{2}\mathbb{Z}$ and $q \in (0,1)$.*

1. *Let $x \in \mathbb{R}$ and $n \in \frac{1}{2}\mathbb{N}_0$, $n+m \in \mathbb{Z}$ with $n \geq |m|$. The scalars $\lambda^n_{(m,x)}$ defining the intertwiners $U_{(m,x)} : \mathcal{H}_m \to \mathcal{H}_{-m}$ in (4.35) are given by*

$$\lambda^m_{(m,x)} = 1, \quad \lambda^n_{(m,x)} = \prod_{k=|m|+1}^{n} \frac{k-ix}{k+ix}.$$

2. *Let $x + 2\pi h^{-1}\mathbb{Z} \in \mathbb{R}/_{2\pi h^{-1}\mathbb{Z}}$ and $n \in \frac{1}{2}\mathbb{N}_0$, $n+m \in \mathbb{Z}$ with $n \geq |m|$. The scalars $\lambda^n_{(m,ix)}(q)$ defining the intertwiners $U^q_{(m,ix)} : \mathcal{H}_m \to \mathcal{H}_{-m}$ in (4.37) are given by*

$$\lambda^m_{(m,ix)}(q) = 1, \quad \lambda^n_{(m,ix)}(q) = \prod_{k=|m|+1}^{n} \frac{[k-ix]_q}{[k+ix]_q}.$$

We should note that the scalars in the classical case in Theorem 4.12 are obtained from the statement in [86, Theorem 5.42] by noting that the coefficients arise from the scalars in Theorem 1.49 which we know are continuous in q and converge to the 'correct' limits as $q \to 1$.

We now define our action on each fibre of B. For $t = 0$, define, for $f \in B_0$ (see (4.33)), the element $w \cdot f \in B_0$

$$(w \cdot f)(m,x) = U^{-1}_{(m,h^{-1}x)} f(-m,-x) U_{(m,h^{-1}x)}$$

where $m \in \frac{1}{2}\mathbb{Z}$ and $x \in \mathbb{R}$. Note this is well defined because the coefficients in Theorem 4.12 depend continuously on $x \in \mathbb{R}$.

For $t > 0$, define for $f \in B_t$ (see (4.34)),

$$(w \cdot f)(m,x) = \left(U^{q^t}_{(m,ih^{-1}x)}\right)^{-1} f(-m,-x) U^{q^t}_{(m,ih^{-1}x)}$$

where $m \in \frac{1}{2}\mathbb{Z}$ and $x \in \mathbb{R}/_{2\pi t^{-1}\mathbb{Z}}$.

Let us now check that $w \cdot f \in B$. For this, consider $(x_n + 2\pi\mathbb{Z}, t_n) \to (y, 0) \in \mathbb{R} \times \{0\} \in \mathcal{G}_{S^1}$

as in Example 4.5, i.e.

$$t_n \to 0, \quad \text{eventually } x_n + 2\pi\mathbb{Z} \text{ can be represented by an element in } (-\pi, \pi),$$

and, after possibly changing representatives so that $x_n \in (-\pi, \pi)$

$$t_n^{-1} x_n \to y \text{ in } \mathbb{R}.$$

We need to show that $U^{q^t}_{(m, ih^{-1} t_n^{-1} x_n)} \to U_{(m, h^{-1} y)}$ strongly. If we consider the scalars in Theorem 4.12, we have that

$$\lambda^n_{(m, ih^{-1} t_n^{-1} x_n)}(q^{t_n}) = \prod_{k=|m|+1}^{n} \frac{[k - ih^{-1} t_n^{-1} x_n]_{q^{t_n}}}{[k + ih^{-1} t_n^{-1} x_n]_{q^{t_n}}} \to \prod_{k=|m|+1}^{n} \frac{k - ih^{-1} y}{k + ih^{-1} y} = \lambda^n_{(m, h^{-1} y)}$$

as required.

Now Proposition A.37 provides a field A^R over $[0,1]$ with fibres

$$A^R_t = \begin{cases} C^*_r(G) & t = 0 \\ C^*_r(G_{q^t}) & t \neq 0 \end{cases}$$

which we call the right quantization field. Note that we have triviality away from 0 because we started with a field with this property, and all the constructions we used from there preserved this property.

Let us now summarize the situation so far. We have constructed a left quantization field A^L that induces a map in K-theory

$$K_*(C^*(G_0)) \to K_*(C^*(G_1))$$

and a right quantization field A^R that induces another map in K-theory,

$$K_*(C^*_r(G)) \to K_*(C^*_r(G_q)).$$

We now ask whether the diagram

$$\begin{tikzcd} K_*(C^*(G_0)) \arrow[r] \arrow[d] & K_*(C^*_r(G)) \arrow[d] \\ K_*(C^*(G_1)) \arrow[r] & K_*(C^*_r(G_q)) \end{tikzcd}$$

commutes, where the upper map is the classical Baum-Connes assembly map, and the lower map is the quantum Baum-Connes assembly map. This would link the classical and quantum Baum-Connes results. We will next give an alternative description of the assembly fields that will allow us to show the above diagram commutes.

4.5 An Alternative Description of the Assembly Fields

Let $G = SL(2,\mathbb{C})$ and $K = SU(2)$ throughout this section.

In this section we will describe a concrete picture of the classical assembly field A^C and quantum assembly field A^Q that we will use to 'glue' together all of our fields.

Recall the Iwasawa decomposition (2.1) $G = K \times A \times N$, where

$$A = \left\{ \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} \mid x \in \mathbb{R} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}$$

and that we have a corresponding Iwasawa decomposition on the level of Lie algebras, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ are the Lie algebras of G, K, A and N respectively, see (4.3).

Recall from (4.11) that for matrices in $\mathfrak{a}$, the exponential map is given explicitly by

$$\exp : \mathfrak{a} \to A, \quad \exp \begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix} = \begin{pmatrix} e^y & 0 \\ 0 & e^{-y} \end{pmatrix}$$

for $y \in \mathbb{R}$, which one can see by using the power series definition of the matrix exponential. The exponential map in this case is bijective, with the inverse given by the logarithm function

$$\log : A \to \mathfrak{a}, \quad \log \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} \log x & 0 \\ 0 & \log x^{-1} \end{pmatrix} = \begin{pmatrix} \log x & 0 \\ 0 & -\log x \end{pmatrix}, \quad x \in \mathbb{R}_{>0}.$$

Note that the logarithm map here is differentiable because the natural logarithm $\log : \mathbb{R}_{>0} \to \mathbb{R}$ is differentiable. We need the following technical result later.

Lemma 4.13. *Let $\mu : \mathbb{R} \to A$ be differentiable path. Then*

$$\mathbb{R} \to \mathfrak{a}, \quad t \mapsto \log \mu(t) \tag{4.38}$$

is differentiable with

$$\frac{d}{dt}\left(\log \mu(t)\right) = \mu(t)^{-1}\mu'(t).$$

Proof. The map (4.38) is the composition of differentiable functions and so is differentiable.

We can write $\mu(t) = \begin{pmatrix} x(t) & 0 \\ 0 & x(t)^{-1} \end{pmatrix}$, where $x : \mathbb{R} \to \mathbb{R}_{>0}$ is a differentiable path. Then

$$\log \mu(t) = \begin{pmatrix} \log x(t) & 0 \\ 0 & \log x(t)^{-1} \end{pmatrix}$$

and so

$$\begin{aligned} \frac{d}{dt}(\log \mu(t)) &= \begin{pmatrix} \frac{d}{dt}(\log x(t)) & 0 \\ 0 & \frac{d}{dt}(\log x(t)^{-1}) \end{pmatrix} \\ &= \begin{pmatrix} x(t)^{-1}x'(t) & 0 \\ 0 & x(t)(x(t)^{-1})' \end{pmatrix} \\ &= \mu(t)^{-1}\mu'(t). \end{aligned}$$

□

Recall from Section 4.1 that the classical assembly field A^C is constructed from the manifold

$$\mathcal{G} = K \ltimes \mathfrak{an} \times \{0\} \sqcup G \times (0,1] = G_0 \times \{0\} \sqcup G \times (0,1].$$

We can view $\mathcal{G}$ as a bundle of groups over $[0,1]$, with fibres G_t. The fibre at 0 is G_0 and for $t > 0$, $G_t = G$. We fixed the Haar measures on K, $\mathfrak{a}$, and $\mathfrak{n}$ to be $\mathrm{d}k$, $\mathrm{d}X$ and $\mathrm{d}Y$ respectively. We fixed the Haar measure on G_t for $t > 0$, namely

$$\mathrm{d}g_t = \delta(a_t)\ \mathrm{d}k\ \mathrm{d}a_t\ \mathrm{d}n_t$$

(see (4.14)), where

$$\int_{A_t} f(a_t)\ \mathrm{d}a_t := \int_{\mathfrak{a}} f(\exp(tX))\ \mathrm{d}X, \quad \int_{N_t} f(n_t)\ \mathrm{d}n_t := \int_{\mathfrak{n}} f(\exp(tY))\ \mathrm{d}Y.$$

Recall from (4.19) that we have the isomorphism

$$C_r^*(G) \cong C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right)^W$$

given on $C_c(SL(2,\mathbb{C}))$ by $f \mapsto ((n,y) \mapsto \pi_{(n,y)}(f))$, where $\pi_{(n,y)}$ is the principal series representation of G with parameter $(n,y) \in \frac{1}{2}\mathbb{Z} \times \mathbb{R}$. We adjust this construction slightly to reflect the fact we are working with G_t. For $t > 0$, define an isomorphism

$$\phi_t : C_r^*(G_t) \cong C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right)^W, \quad \phi_t(f)(n,x) = \pi^t_{(m,t^{-1}x)}(f), \tag{4.39}$$

for $f \in C_c(G_t)$, where

$$(\pi^t_{(n,y)}(f)\xi)(r) = \int_{G_t} f(g_t)\xi(g_t^{-1}r)\ \mathrm{d}g_t \tag{4.40}$$

is the principal series representation of G_t with parameter $(n, y) \in \frac{1}{2}\mathbb{Z} \times \mathbb{R}$, $\xi \in \mathcal{H}_{(n,y)}$ and $r \in K$. Note that this amounts to rescaling the characters of $A_t \subseteq G_t$ from those we described in Section 4.1, (4.15). The W-action on $C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right)$ in this case is given by

$$(w \cdot f)(n, x) = U^{-1}_{(n,t^{-1}x)} f(-n, -x) U_{(n,t^{-1}x)}, \quad (n, x) \in \frac{1}{2}\mathbb{Z} \times \mathbb{R},$$

for $f \in C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right)$.

For the case $t = 0$, we have

$$C^*_r(G_0) \cong \left\{ f \in C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right)^W \mid f(-, 0) \in \bigoplus_{m \in \frac{1}{2}\mathbb{N}_0} K(V(m)) \subseteq K(L^2(K)) \right\}.$$

by Theorem 4.9, after identifying $\mathfrak{t} \cong \mathbb{R}$. Let us denote the above isomorphism by ϕ_0. We will now provide a formula for ϕ_0 by following the construction in Section 4.3.

We can identify $C^*_r(G_0) \cong K \ltimes C^*_r(\mathfrak{an}) \cong K \ltimes C_0(\mathfrak{k})$ via the Fourier transform, given by the formula

$$\mathcal{F} : C_c(\mathfrak{an}) \to C_0(\mathfrak{k}), \quad \mathcal{F}(f)(W) = \int_{\mathfrak{an}} f(Z) e^{i \operatorname{Im}(\operatorname{Trace}(WZ))}\ \mathrm{d}Z, \quad f \in C_c(\mathfrak{an}), \quad W \in \mathfrak{k}.$$

see the proof of Proposition 4.2, where $\mathrm{d}Z = \mathrm{d}X\ \mathrm{d}Y$.

By Theorem 3.13, if we take $f \in C_c(K \times \mathfrak{an})$, then the above maps give an element $F \in C_0(\mathfrak{k}, K(L^2(K)))^K$ defined by

$$\begin{aligned}(F(X)\xi)(r) &= \int_K (\mathrm{id} \otimes \mathcal{F})(f)(s, s^{-1}r \cdot X)\xi(s^{-1}r)\ \mathrm{d}s \\ &= \int_K \int_{\mathfrak{an}} f(s, Z) e^{i \operatorname{Im}(\operatorname{Trace}((s^{-1}r \cdot X)Z))} \xi(s^{-1}r)\ \mathrm{d}s\ \mathrm{d}Z \end{aligned} \tag{4.41}$$

for $X \in \mathfrak{k}$, $\xi \in K(L^2(K))$ and $r \in K$. We can then use the identification of $C_0(\mathfrak{k}, K(L^2(K)))^K$ with A^L_0 to describe the corresponding element of A^L_0 by the formula

$$F(m, x) := p_m(\mathrm{ev}_{ix}(F))$$

for $m \in \frac{1}{2}\mathbb{Z}$, $x \in \mathbb{R}$ and where $p_m : \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} K(\mathcal{H}_n) \to K(\mathcal{H}_m)$ is the coordinate map. Here, and in what follows, we will often identify $i\mathbb{R}$ and $\mathfrak{t}$ (a space of matrices) without changing notation.

Now we have identified each fibre of A^C with a subalgebra of $C_0\left(\frac{1}{2}\mathbb{Z}\times\mathbb{R}, K(\mathcal{H})\right)$, we can consider the $C([0,1])$-algebra

$$B := \left\{ F \in C\left([0,1], C_0\left(\frac{1}{2}\mathbb{Z}\times\mathbb{R}, K(\mathcal{H})\right)\right) \mid F(0)(-,0) \in \bigoplus_{m\in\frac{1}{2}\mathbb{N}_0} K(V(m)) \subseteq K(L^2(K)) \right\}$$

which is a subalgebra of a trivial field and so has the obvious evaluation maps. The fibre B_t contains $(A^C)_t$. We will now show that $A^C \subseteq B$, i.e. if $f \in C_c(\mathcal{G})$, the map

$$(t,n,x) \mapsto \phi_t(f_t)(n,x)$$

defines an element of B, where f_t is the evaluation of f in the fibre at t (i.e. an element of $C_r^*(G_t)$). For this we will have need to consider a convenient dense subspace $\mathcal{D}$ of $C_c(\mathcal{G})$.

Lemma 4.14. *The subspace $\mathcal{O}(K)\odot C_c(\mathfrak{an})\odot C([0,1])$ is dense in $C_c(\mathcal{G})$ in the norm*

$$\|F\| := \sup_{t\in[0,1]} \|F_t\|, \quad F\in C_c(\mathcal{G}).$$

Here we view elements of $\mathcal{O}(K)\odot C_c(\mathfrak{an})\odot C([0,1])$ as functions on $K\times\mathfrak{an}\times[0,1]$ and then as functions on $\mathcal{G}$ via the homeomorphism $K\times\mathfrak{an}\times[0,1]\to\mathcal{G}$.

Proof. Let $f\in C_c(\mathcal{G})$. Viewing f as a function on $K\times\mathfrak{an}\times[0,1]$, there exists compact sets $C_1\subseteq\mathfrak{a}, C_2\subseteq\mathfrak{n}$ such that $\mathrm{Supp}(f)\subseteq K\times C_1\times C_2\times[0,1]$. By inflating C_1, we can assume C_1 is a closed ball around the origin, so that for all $t<1$, if $X\in C_1$, $tX\in C_1$. Let μ denote the product measure on $K\times\mathfrak{an}\times[0,1]$. Set

$$C = \max\{\mu(K\times C_1\times C_2\times[0,1]), \mu(K\times C_1\times C_2\times[0,1])\cdot\sup_{X\in C_1}|\delta(\exp(X))|\}.$$

There exists $g\in\mathcal{O}(K)\odot C_c(C_1\times C_2)\odot C([0,1])\subseteq\mathcal{O}(K)\odot C_c(\mathfrak{an})\odot C([0,1])$ with $\|f-g\|_\infty<\frac{\epsilon}{C}$. We claim that $\|f-g\|<\epsilon$. We have, for $t>0$,

$$\begin{aligned}
&\|f_t-g_t\|_{C_r^*(G_t)}\\
\le\ &\|f_t-g_t\|_{L^1(G_t)}\\
=\ &\int_K\int_{A_t}\int_{N_t}|f_t(ka_tn_t)-g_t(ka_tn_t)|\delta(a_t)\ \mathrm{d}k\ \mathrm{d}a_t\ \mathrm{d}n_t\\
=\ &\int_K\int_{\mathfrak{a}}\int_{\mathfrak{n}}|f_t(k\exp(tX)\exp(tY))-g_t(k\exp(tX)\exp(tY))|\delta(\exp(tX))\ \mathrm{d}k\ \mathrm{d}X\ \mathrm{d}Y\\
=\ &\int_K\int_{\mathfrak{a}}\int_{\mathfrak{n}}|f(k,X,Y,t)-g(k,X,Y,t)|\delta(\exp(tX))\ \mathrm{d}k\ \mathrm{d}X\ \mathrm{d}Y
\end{aligned}$$

$$
\begin{aligned}
&\leq \mu(K \times C_1 \times C_2 \times [0,1]) \sup_{X \in C_1} |\delta(\exp(tX))| \, \|f - g\|_\infty \\
&\leq \mu(K \times C_1 \times C_2 \times [0,1]) \sup_{X \in C_1} |\delta(\exp(X))| \, \|f - g\|_\infty \\
&\leq C \, \|f - g\|_\infty < \epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\|f_0 - g_0\|_{C_r^*(G_0)} &\leq \|f_0 - g_0\|_{L^1(G_0)} \\
&= \int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} |f(k, X, Y, 0) - g(k, X, Y, 0)| \ \mathrm{d}k \ \mathrm{d}X \ \mathrm{d}Y \\
&\leq \mu(K \times C_1 \times C_2 \times [0,1]) \, \|f - g\|_\infty \\
&\leq C \, \|f - g\|_\infty < \epsilon.
\end{aligned}
$$

Therefore $\|f - g\| < \epsilon$, as required. □

We will take $\mathcal{D} := \mathcal{O}(K) \odot C_c(\mathfrak{an}) \odot C([0,1]) \subseteq C_c(\mathcal{G})$. We will see shortly why this is a convenient choice.

Lemma 4.15. *Let $h \in C_c(K \times [0,1])$ with $\sup_{t \in [0,1]} \|h(-,t)\|_{C_r^*(K)} < \infty$. Then h defines a multiplier of A^C with respect to convolution given by*

$$
(h \star f)(k, X, Y, t) := \int_K h(s,t) f(s^{-1}k, X, Y, t) \ \mathrm{d}s, \quad (k, X, Y, t) \in K \times \mathfrak{an} \times [0,1]
$$

for $f \in C_c(\mathcal{G})$.

Proof. Clearly convolution by h defines a linear map $C_c(\mathcal{G}) \to C_c(\mathcal{G})$. The convolution is fibrewise, i.e. we have, for $f \in C_c(\mathcal{G})$ and $t \in [0,1]$,

$$
(h \star f)(-,-,-,t) = h(-,t) \star f(-,-,-,t)
$$

with the latter convolution taking place in $C_c(K \ltimes \mathfrak{an})$. Convolution here defines a bounded linear map, because

$$
\|h(-,t) \star f(-,-,-,t)\| \leq \|h(-,t)\|_{C_r^*(K)} \|f(-,-,-,t)\| \leq \sup_{t \in [0,1]} \|h(-,t)\|_{C_r^*(K)} \|f\| .
$$

Then convolution by h extends to a linear map $A^C \to A^C$. In the same way convolution with h^* extends to a bounded linear map $A^C \to A^C$, which is the adjoint of convolution by h, where we view A^C as a Hilbert A^C-module in the usual way. Note here that

$$
h^*(k,t) := \overline{h(k^{-1},t)}, \quad k \in K, \quad t \in [0,1].
$$

Therefore convolution by h defines a multiplier of A^C. □

Lemma 4.16. *Let $f \otimes g \in \mathcal{O}(K) \odot (C_c(\mathfrak{an}) \odot C([0,1]))$ be an elementary tensor. If $h \in C_c(K)$ (defining an element of $C_c(K \times [0,1])$ that is constant on $[0,1]$) then*

$$h \star (f \otimes g) = (h \star f) \otimes g$$

where the convolution on the right hand side is taken in $C_c(K)$.

Proof. This is a direct calculation. □

Lemma 4.17. *Let $f \in \mathcal{D}$. There exists $h \in \mathcal{O}(K)$ (defining an element of $C_c(K \times [0,1])$ that is constant on $[0,1]$) such that*

$$h \star f = f.$$

Note that h only depends on the $\mathcal{O}(K)$-leg of f.

Proof. We can write $f = \sum_i f_i \otimes g_i \in \mathcal{O}(K) \odot (C_c(\mathfrak{an}) \odot C([0,1]))$, where the summation is finite. Then Lemma 4.16 tells us that for $h \in C_c(K)$, we have $h \star f = \sum_i (h \star f_i) \otimes g_i$. The Fourier transform as defined in Section 1.2.2, (1.6)

$$\mathcal{F} : \mathcal{O}(K) \to \mathcal{D}(K) = \mathrm{alg} - \bigoplus_{n \in \frac{1}{2}\mathbb{N}_0} M_{2n+1}(\mathbb{C})$$

transforms convolution in $\mathcal{O}(K)$ to matrix multiplication in $\mathcal{D}(K)$. Each $\mathcal{F}(f_i)$ is non-zero in only finitely many of the matrix blocks of $\mathcal{D}(K)$ so choose N sufficiently large so that each $\mathcal{F}(f_i)$ can be viewed as an element of $\bigoplus_{n \in \frac{1}{2}\mathbb{N}_0}^N M_{2n+1}(\mathbb{C})$. Since $\mathcal{F}$ is invertible, there exists $h \in \mathcal{O}(K)$ such that $\mathcal{F}(h) = (I_{2n+1}) \in \bigoplus_{n \in \frac{1}{2}\mathbb{N}_0}^N M_{2n+1}(\mathbb{C}) \subseteq \mathcal{D}(K)$, where $I_{2n+1} \in M_{2n+1}(\mathbb{C})$ is the identity matrix. Then $h \star f_i = f_i$ for each i, and the result follows. □

Recall in the following that ϕ_t for $t > 0$ is introduced in (4.39) and ϕ_0 is the isomorphism provided by Theorem 4.9.

Lemma 4.18. *If $h \in C_c(K)$ (defining an element of $C_c(K \times [0,1])$ that is constant on $[0,1]$) then*

$$\phi_t(h(-,t))$$

defines an element of $\prod_{m \in \frac{1}{2}\mathbb{Z}} C_b(\mathbb{R}, K(\mathcal{H}_m))$ that is constant on $\mathbb{R}$ and independent of t.

Proof. For each $t > 0$, and $x \in \mathbb{R}$, direct calculation shows

$$\pi^t_{(n,x)}(h \star f)\xi(r) = \int_K h(k)(f \star \xi)(k^{-1}r)\, \mathrm{d}k$$

for $h \in C_c(K)$, $f \in C_c(G_t)$, $\xi \in \mathcal{H}_m$ and $r \in K$ and so $\pi^t_{(n,x)}(h) = \lambda(h)$ (restricted to $\mathcal{H}_n$) where λ is the left regular representation of K. Therefore $\pi^t_{(n,x)}(h)$ is compact and since $h(-,t) = h(-)$ for all t is independent of t. Therefore $\phi_t(h(-,t))(n,x)$ is independent of $t > 0$ and x.

Now we consider the case of $t = 0$. If $f \in C_c(K \ltimes \mathfrak{an})$, then the element $h \star f \in C_c(K \times \mathfrak{an})$ defines the element H of $C_0(\mathfrak{k}, K(L^2(K)))^K$ given by (4.41), and so we have

$$\begin{aligned}
(H(X)\xi)(r) &= \int_K \int_{\mathfrak{an}} (h \star f)(s, Z) e^{i\,\mathrm{Im}(\mathrm{Trace}((s^{-1}r \cdot X)Z))} \xi(s^{-1}r)\ \mathrm{d}s\ \mathrm{d}Z \\
&= \int_K \int_K \int_{\mathfrak{an}} h(k) f(k^{-1}s, Z) e^{i\,\mathrm{Im}(\mathrm{Trace}((s^{-1}r \cdot X)Z))} \xi(s^{-1}r)\ \mathrm{d}s\ \mathrm{d}Z \\
&= \int_K h(k) \int_K \int_{\mathfrak{an}} f(k^{-1}s, Z) e^{i\,\mathrm{Im}(\mathrm{Trace}((s^{-1}r \cdot X)Z))} \xi(s^{-1}r)\ \mathrm{d}s\ \mathrm{d}Z \\
&= \int_K h(k) \int_K \int_{\mathfrak{an}} f(u, Z) e^{i\,\mathrm{Im}(\mathrm{Trace}((u^{-1}k^{-1}r \cdot X)Z))} \xi(u^{-1}k^{-1}r)\ \mathrm{d}u\ \mathrm{d}Z \\
&= \int_K h(k) (F(X)\xi)(k^{-1}r)\ \mathrm{d}k
\end{aligned}$$

where dZ is the Haar measure on $\mathfrak{an}$, and $X \in \mathfrak{k}$, $\xi \in K(L^2(K))$, $r \in K$ and F is the element of $C_0(\mathfrak{k}, K(L^2(K)))^K$ corresponding to f. Therefore $\phi_0(h(-,0))(n,x) = \lambda(h)$ exactly as above. The result follows. $\square$

We can now prove the desired result.

Theorem 4.19. *For $f \in C_c(\mathcal{G})$, the map*

$$(t, n, x) \mapsto \phi_t(f_t)(n, x)$$

defines an element of B,

$$B := \left\{ F \in C\left([0,1], C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right)\right) \mid F(0)(-,0) \in \bigoplus_{m \in \frac{1}{2}\mathbb{N}_0} K(V(m)) \subseteq K(L^2(K)) \right\}.$$

Proof. It suffices to fix $n \in \frac{1}{2}\mathbb{Z}$ and $f \in C_c(\mathcal{G})$ and show that the map

$$[0,1] \times \mathbb{R} \to K(\mathcal{H}_n), \quad (t, x) \mapsto \phi_t(f_t)(n, x)$$

is continuous. That is, we need to show that if $(t_i, x_i) \in [0,1] \times \mathbb{R}$ is a sequence converging to (t, x), then

$$\|\phi_{t_i}(f_{t_i})(n, x_i) - \phi_t(f_t)(n, x)\|_{\mathrm{op}} \to 0.$$

Recall that the strong-$*$-topology on bounded subsets of $B(\mathcal{H}_n)$ coincides with the strict topology, viewing $B(\mathcal{H}_n) = M(K(\mathcal{H}_n))$. In particular, if we can show that for $f \in \mathcal{D}$ that we have

$$\|\phi_{t_i}(f_{t_i})(n, x_i)\xi - \phi_t(f_t)(n, x)\xi\|_{\mathcal{H}_n} \to 0$$

and

$$\|\phi_{t_i}(f_{t_i})(n, x_i)^*\xi - \phi_t(f_t)(n, x)^*\xi\|_{\mathcal{H}_n} \to 0$$

for any $\xi \in \mathcal{H}_n$, then for any compact operator S, we have

$$\|S\phi_{t_i}(f_{t_i})(n, x_i) - S\phi_t(f_t)(n, x)\|_{\text{op}} \to 0.$$

Since $f \in \mathcal{D}$, we can then find h as in Lemma 4.17. Taking S to be the compact operator provided by Lemma 4.18, we have

$$\|\phi_{t_i}(f_{t_i})(n, x_i) - \phi_t(f_t)(n, x)\|_{\text{op}} \to 0$$

and the proof is complete, because $\mathcal{D}$ is dense in $C_c(\mathcal{G})$ by Lemma 4.14.

It is therefore sufficient to show that for each $f \in C_c(\mathcal{G})$, we have

$$\|\phi_{t_i}(f_{t_i})(n, x_i)\xi - \phi_t(f_t)(n, x)\xi\|_{\mathcal{H}_n} \to 0$$

for any $\xi \in \mathcal{H}_n$. In fact, by density, it suffices to show this for ξ a continuous function in $\mathcal{H}_n$.

We need to examine the formulae for each isomorphism carefully. We have already done this in the case $t = 0$ in (4.41), where we have

$$(\phi_0(f_0)(n, x)\xi)(r) = \int_K \int_{\mathfrak{an}} f(k, X + Y, 0) e^{i\,\text{Trace}((k^{-1}r \cdot x)(X+Y))} \xi(k^{-1}r)\ \mathrm{d}k\ \mathrm{d}X\ \mathrm{d}Y \tag{4.42}$$

where we view $f \in C_c(\mathcal{G})$ as a function on $K \times \mathfrak{an} \times [0, 1]$ using (4.10).

For $t > 0$ we have, using (4.14) and (4.13),

$$\begin{aligned}
&(\phi_t(f_t)(n, x)\xi)(r) \\
&= \int_{G_t} f_t(g_t)(\pi^t_{(n,t^{-1}x)}(g_t)\xi)(r)\ \mathrm{d}g_t \\
&= \int_K \int_{A_t N_t} f(k, a_t n_t, t)\xi(n_t^{-1}a_t^{-1}k^{-1}r)\ \delta(a_t)\ \mathrm{d}k\ \mathrm{d}a_t\ \mathrm{d}n_t \\
&= \int_K \int_{\mathfrak{an}} f(k, \exp(tX)\exp(tY), t)\xi(\exp(-tY)\exp(-tX)k^{-1}r)\delta(\exp(tX))\ \mathrm{d}k\ \mathrm{d}X\ \mathrm{d}Y \\
&= \int_K \int_{\mathfrak{an}} f(k, X + Y, t)\xi(\exp(-tY)\exp(-tX)k^{-1}r)\delta(\exp(tX))\ \mathrm{d}k\ \mathrm{d}X\ \mathrm{d}Y.
\end{aligned}$$

Note that here we have extended ξ from K to G_t using (4.18). This extension satisfies

$$\xi(\gamma(t)k^{-1}r) = \xi(\gamma(t) \rightharpoonup k^{-1}r)\tau_{-t^{-1}x}(p_A(\gamma(t) \leftharpoonup k^{-1}r))\delta(p_A(\gamma(t) \leftharpoonup k^{-1}r))^{-1}$$

where $\gamma : \mathbb{R} \to AN$, $\gamma(t) := \exp(-tY)\exp(-tX)$, $p_A : G \to A$ is the coordinate projection provided by the Iwasawa decomposition, $\tau_{-t^{-1}x}$ is defined by (4.15), and $\leftharpoonup, \rightharpoonup$ are the dressing actions as defined in (4.1) and (4.2). Note that $\gamma'(0) = -X - Y$.

One can directly check that the diagram

$$\begin{tikzcd} A \arrow[r, "\tau_{-t^{-1}x}"] & S^1 \\ \mathfrak{a} \arrow[u, "\exp(t-)"] \arrow[r, "\kappa(-ix,-)"] & \mathbb{R} \arrow[u, "\exp(i-)"'] \end{tikzcd}$$

commutes. Hence

$$\tau_{-t^{-1}x}(p_A(\gamma(t) \leftharpoonup k^{-1}r)) = e^{-i\kappa(ix, t^{-1}\log(p_A(\gamma(t) \leftharpoonup k^{-1}r)))} \tag{4.43}$$

where

$$\kappa : \mathfrak{k} \times \mathfrak{an} \to \mathbb{R} \quad (X, Y) \mapsto \mathrm{Im}(\mathrm{Trace}(XY)).$$

is the trace pairing. Notice that

$$\lim_{t\to 0} t^{-1}\log(p_A(\gamma(t) \leftharpoonup k^{-1}r)) = \left.\frac{d}{dt}\right|_{t=0} \log(p_A(\gamma(t) \leftharpoonup k^{-1}r)).$$

By Lemma 4.13 and the proof of Lemma 4.1, we have

$$\left.\frac{d}{dt}\right|_{t=0} \log(p_A(\gamma(t) \leftharpoonup k^{-1}r)) = \left.\frac{d}{dt}\right|_{t=0} p_A(\gamma(t) \leftharpoonup k^{-1}r) = p_{\mathfrak{a}}\left(\left.\frac{d}{dt}\right|_{t=0} \gamma(t) \leftharpoonup k^{-1}r\right).$$

Finally, we know from (4.6) that linearizing the right dressing action on AN gives

$$\left.\frac{d}{dt}\right|_{t=0} \gamma(t) \leftharpoonup k^{-1}r = p_{\mathfrak{a}}\left(-r^{-1}k(X+Y)k^{-1}r\right) = -p_{\mathfrak{a}}\left(r^{-1}k(X+Y)k^{-1}r\right)$$

and substituting into (4.43) (noting that the trace pairing vanishes on $\mathfrak{n}$) we see that

$$\lim_{t\to 0} \tau_{-t^{-1}x}(p_A(\gamma(t) \leftharpoonup k^{-1}r)) = e^{i\,\mathrm{Trace}((k^{-1}rxr^{-1}k)(X+Y))} = e^{i\,\mathrm{Trace}((k^{-1}r\cdot x)(X+Y))}.$$

If we compare the cases $t = 0$ and $t > 0$, we see that the integrand of the $t > 0$ case converges pointwise to that of the $t = 0$ case as $t \to 0$. We now show that this gives us convergence in the L^2-norm.

Let $I_{(t_i,x_i)}(k, X, Y, r)$ denote the integrand of $(\phi_t(f_t)(n, x)\xi)(r)$. We need to show that

$$\int_K \int_{\mathfrak{an}} I_{(t_i,x_i)}(k, X, Y, r)\ \mathrm{d}k\ \mathrm{d}X\ \mathrm{d}Y \to \int_K \int_{\mathfrak{an}} I_{(t,x)}(k, X, Y, r)\ \mathrm{d}k\ \mathrm{d}X\ \mathrm{d}Y$$

in L^2-norm as $x_i \to x$, $t_i \to t$, i.e. for all $\epsilon > 0$, we can take i sufficiently large so that

$$I_i := \int_K \left| \int_K \int_{\mathfrak{an}} I_{(t_i,x_i)}(k, X, Y, r) - I_{(t,x)}(k, X, Y, r)\ \mathrm{d}k\ \mathrm{d}X\ \mathrm{d}Y \right|^2\ \mathrm{d}r < \epsilon.$$

The measure of $\mathrm{Supp}(f)$ is bounded above by a constant $M > 0$ so that

$$I_i \leq M^2 \sup_{k,X,Y,r} |I_{(t_i,x_i)}(k, X, Y, r) - I_{(t,x)}(k, X, Y, r)|^2$$

Assume that f has support in $K \times C \times [0, 1]$ where C is a compact subset of $\mathfrak{an}$. We can also assume that t_i, t, x_i and x lie in a compact set as we are interested in local behaviour in these variables. We note that $I_{(t,x)}$ depends continuously on k, X, Y, r, t and x. Here we use the fact the dressing action is continuous (being the composition of products and coordinate projection), and δ is a continuous group homomorphism, and the fact that $I_{(t_i,x_i)} \to I_{(0,x)}$ if $t = 0$. Therefore $I_{(-,-)}$ is a continuous function on a compact set, and hence uniformly continuous on this compact set. That is, there exists a $\delta > 0$ such that if $(k, X, Y, r, t, x) \in K \times \mathfrak{an} \times K \times [0, 1] \times \mathbb{R}$ is given, and (k', X', Y', r', t', x') is δ-close to (k, X, Y, r, t, x), then

$$|I_{(t,x)}(k, X, Y, r) - I_{(t',x')}(k', X', Y', r')| < \frac{\sqrt{\epsilon}}{M}.$$

In particular for t_i and x_i sufficiently close to t, x, then

$$I_i \leq M^2 \sup_{k,X,Y,r} |I_{(t_i,x_i)}(k, X, Y, r) - I_{(t,x)}(k, X, Y, r)|^2 < \epsilon$$

and the result follows. □

We then have the following.

Corollary 4.20. *The map $C_c(\mathcal{G}) \to B$ defined by Theorem 4.19 extends to an inclusion $A^C \subseteq B$ of continuous fields.*

Proof. This follows from the fact that $C_c(\mathcal{G})$ is dense in A^C and the map defined by Theorem 4.19 is defined by fibrewise inclusions. □

To complete the identification, we need to define an action of $W = \mathbb{Z}_2$ on B that is a field of actions so that $A^C \cong B^W$. For each $t \in [0, 1]$ we have the action of W on the fibre B_t, where

$$B_0 = \left\{ f \in C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right) \mid f_0 \in \bigoplus_{m \in \frac{1}{2}\mathbb{N}_0} K(V(m)) \subseteq K(L^2(K)) \right\}$$

and

$$B_t = C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right), \quad t > 0.$$

For $t > 0$ this is given by

$$(w \cdot f_t)(n, x) = U^{-1}_{(n,t^{-1}x)} f_t(-n, -x) U_{(n,t^{-1}x)}, \quad (n, x) \in \frac{1}{2}\mathbb{Z} \times \mathbb{R} \tag{4.44}$$

where $f_t \in B_t$ and

$$U_{(n,t^{-1}x)} : \mathcal{H}_n \to \mathcal{H}_{-n}$$

is the intertwiner between the representations $\pi_{(n,t^{-1}x)}$ and $\pi_{(-n,-t^{-1}x)}$ provided by Theorem 4.3. Recall from Theorem 4.12 that this is a direct sum of scalar operators defined by

$$U_{(n,t^{-1}x)} = \bigoplus_{m \geq |n|, n+m \in \mathbb{Z}} \lambda^m_{(n,t^{-1}x)} 1_{V(m)}, \quad \lambda^n_{(n,t^{-1}x)} = 1, \quad \lambda^m_{(n,t^{-1}x)} = \prod_{k=|n|+1}^{m} \frac{k - it^{-1}x}{k + it^{-1}x}.$$

For $t = 0$ this is given by

$$(w \cdot f_0)(n, x) = U_n^{-1} f_0(-n, -x) U_n, \quad (n, x) \in \frac{1}{2}\mathbb{Z} \times \mathbb{R}$$

for $f_0 \in B_0$, where for each $n \in \frac{1}{2}\mathbb{Z}$,

$$U_n : \mathcal{H}_n \to \mathcal{H}_{-n}, \quad U_n(\xi)(k) = \xi(kw), \quad w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

see (4.26), but note the change in notation for convenience here.

Therefore $U_n = \rho_w$, where ρ is the right regular representation of K, restricted to $\mathcal{H}_n$. Decomposing $\mathcal{H}_n$ and $\mathcal{H}_{-n}$ with respect to the left regular representation (see (3.23)) gives

$$\mathcal{H}_n \cong \bigoplus_{m \in \frac{1}{2}\mathbb{N}_0} V(m) \otimes (V(m))^*_{-n}, \quad \mathcal{H}_{-n} \cong \bigoplus_{m \in \frac{1}{2}\mathbb{N}_0} V(m) \otimes (V(m))^*_{n}.$$

In the decompositions above, ρ_w acts by $1_{V(m)} \otimes \pi^*_m(w)$ on $V(m) \otimes (V(m))^*_{-n}$, where π^*_m is the contragredient of the $2m+1$ dimensional irreducible representation π_m of K on $V(m)$.

Since $(V(m))^*_{-n} = \mathbb{C}\overline{e^m_n}$ (see (3.24)), and $\pi^*_m(w)$ must map $(V(m))^*_{-n}$ to $(V(m))^*_n$, we must have that ρ_w decomposes as a sequence of scalar operators $\lambda^m_n 1 : V(m) \otimes (V(m))^*_{-n} \to$

$V(m) \otimes (V(m))_n^*$, and

$$\begin{aligned}
\lambda_n^m &= \langle \overline{e_{-n}^m}, \pi_m^*(w)\overline{e_n^m} \rangle_{V(m)^*} \\
&= \langle \overline{e_{-n}^m}, \overline{\pi_m(w)e_n^m} \rangle_{V(m)^*} \\
&= \overline{\langle e_{-n}^m, \pi_m(w)e_n^m \rangle}_{V(m)} \\
&= \langle \pi_m(w)e_n^m, e_{-n}^m \rangle_{V(m)}. \qquad (4.45)
\end{aligned}$$

Therefore we must understand how w acts on $V(m)$ to determine these scalars. This is done in the following lemma.

Lemma 4.21. *Let $n \in \frac{1}{2}\mathbb{N}_0$ and $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then for $i \in \{-n, -n+1, ..., n\}$, we have $\pi_n(w)e_i^n = (-1)^{n-i}e_{-i}^n$, where π_n is the $2n+1$-dimensional irreducible representation of K on $V(n)$, and e_i^n is one of the basis elements constructed in Theorem 1.49.*

Proof. Recall from Theorem 1.49 that $V(n) \subseteq V\left(n - \frac{1}{2}\right) \otimes V\left(\frac{1}{2}\right)$, and the standard choice of highest weight vector e_n^n under this inclusion is $e_{n-\frac{1}{2}}^{n-\frac{1}{2}} \otimes e_{\frac{1}{2}}^{\frac{1}{2}}$. Note that $\pi_{\frac{1}{2}}(w)e_{\frac{1}{2}}^{\frac{1}{2}} = e_{-\frac{1}{2}}^{\frac{1}{2}}$. We prove that for $n \in \frac{1}{2}\mathbb{N}_0$ $\pi_n(w)e_n^n = e_{-n}^n$ by induction. Assume that for $k \in \frac{1}{2}\mathbb{N}_0$, $k < n$ we have $\pi_k(w)e_k^k = e_{-k}^k$.

The vector e_{-n}^n is a lowest weight vector in $V\left(n - \frac{1}{2}\right) \otimes V\left(\frac{1}{2}\right)$. This just means that e_{-n}^n is an eigenvector for $\pi_n(H)$ and $\pi_n(F)e_{-n}^n = 0$, c.f Definition 1.45. One can check that $e_{-(n-\frac{1}{2})}^{n-\frac{1}{2}} \otimes e_{-\frac{1}{2}}^{\frac{1}{2}}$ is a lowest weight vector, and since our weight spaces are one dimensional $e_{-n}^n = \lambda e_{-(n-\frac{1}{2})}^{n-\frac{1}{2}} \otimes e_{-\frac{1}{2}}^{\frac{1}{2}}$ for some $\lambda \in S^1$. Recall that e_{-n}^n is constructed from e_n^n by repeatedly applying F and normalizing. From the formulas from Theorem 1.49 we see that λ must be a positive scalar, and so $\lambda = 1$.

We then have, using the inductive hypothesis,

$$\begin{aligned}
\pi_n(w)e_n^n &= (\pi_{n-\frac{1}{2}}(w) \otimes \pi_{\frac{1}{2}}(w))(e_{n-\frac{1}{2}}^{n-\frac{1}{2}} \otimes e_{\frac{1}{2}}^{\frac{1}{2}}) \\
&= \pi_{n-\frac{1}{2}}(w)e_{n-\frac{1}{2}}^{n-\frac{1}{2}} \otimes \pi_{\frac{1}{2}}(w)e_{\frac{1}{2}}^{\frac{1}{2}} \\
&= e_{-(n-\frac{1}{2})}^{n-\frac{1}{2}} \otimes e_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= e_{-n}^n.
\end{aligned}$$

Recall from Remark 1.55 that the representation π_n gives rise to a Lie algebra representation of $\mathfrak{sl}_2(\mathbb{C})$ on $V(n)$, which we will denote by ρ_n. The representation ρ_n is given by the formula

$$\rho_n(X) = d\pi_n(X_1) + id\pi_n(X_2)$$

where $X \in \mathfrak{sl}_2(\mathbb{C})$ and we have written $X = X_1 + iX_2$ for some $X_1, X_2 \in \mathfrak{su}(2)$.

The representation ρ_n is the standard irreducible algebra representation of $U(\mathfrak{sl}_2(\mathbb{C}))$ on $V(n)$ as constructed in Section 1.3, by the calculations in Remark 1.55. The actions of E, F and H on $V(n)$ are the same under this correspondence whether viewed as elements of $\mathfrak{sl}_2(\mathbb{C})$ or $U(\mathfrak{sl}_2(\mathbb{C}))$.

If $X \in \mathfrak{sl}_2(\mathbb{C})$ and we write $X = X_1 + iX_2$ for some $X_1, X_2 \in \mathfrak{su}(2)$ then we have

$$\begin{aligned}
\rho_n(wXw^{-1}) &= \frac{d}{dt}\bigg|_{t=0} \pi_n(\exp(twX_1w^{-1})) + i\,\frac{d}{dt}\bigg|_{t=0} \pi_n(\exp(twX_2w^{-1})) \\
&= \frac{d}{dt}\bigg|_{t=0} \pi_n(w\exp(tX_1)w^{-1}) + i\,\frac{d}{dt}\bigg|_{t=0} \pi_n(w\exp(tX_2)w^{-1}) \\
&= \pi_n(w)\left(\frac{d}{dt}\bigg|_{t=0} \pi_n(\exp(tX_1)) + i\,\frac{d}{dt}\bigg|_{t=0} \pi_n(\exp(tX_2))\right)\pi_n(w^{-1}) \\
&= \pi_n(w)\rho_n(X)\pi_n(w^{-1})
\end{aligned}$$

We have that $wFw^{-1} = -E$ in $\mathfrak{sl}_2(\mathbb{C})$, and so

$$\rho_n(-E) = \pi_n(w)\rho_n(F)\pi_n(w^{-1})$$

or $-\rho_n(E)\pi_n(w) = \pi_n(w)\rho_n(F)$.

Let us now consider $\pi_n(w)e^n_{n-k}$ for some $1 \le k \le 2n$. Recall that $e^n_{n-k} = \mu\ \rho_n(F)^k e^n_n$ for some positive scalar μ, provided by the formulae in Theorem 1.49. Then

$$\pi_n(w)e^n_{n-k} = \mu\ \pi_n(w)\rho_n(F)^k e^n_n = (-1)^k\mu\ \rho_n(E)^k\pi_n(w)e^n_n = (-1)^k\mu\ \rho_n(E)^k e^n_{-n}.$$

Now $\rho_n(E)^k e^n_{-n} = \tilde{\mu}e^n_{-n+k} = \tilde{\mu}e^n_{-(n-k)}$ for some positive scalar $\tilde{\mu}$ by Theorem 1.49. Then

$$\pi_n(w)e^n_{n-k} = (-1)^k\mu\tilde{\mu}e^n_{-(n-k)}$$

Since $\pi_n(w)$ is unitary, we must have that $|(-1)^k\mu\tilde{\mu}| = 1$, and so $\mu\tilde{\mu} = 1$. Therefore

$$\pi_n(w)e^n_{n-k} = (-1)^k e^n_{-(n-k)}$$

which gives the formula in the statement of the lemma. □

By Lemma 4.21, we have $\lambda^m_n = \langle \pi_m(w)e^m_n, e^m_{-n}\rangle_{V(m)} = (-1)^{m-n}\langle e^m_{-n}, e^m_{-n}\rangle_{V(m)} = (-1)^{m-n}$. Notice that $\lambda^n_n = (-1)^{2n}$, which is 1 when $n \in \mathbb{Z}$, and -1 if $n \notin \mathbb{Z}$. If $n \notin \mathbb{Z}$, let us redefine U_n by multiplying by -1. The conjugation action is the unchanged, and so we may do this. This choice forces $\lambda^n_n = 1$ in any case.

Let us now show that $\lambda^m_{(n,t^{-1}x)} \to \lambda^m_n$ as $t \to 0$. We have

$$\lambda^m_{(n,t^{-1}x)} = \prod_{k=|n|+1}^{m} \frac{k - it^{-1}x}{k + it^{-1}x} = \prod_{k=|n|+1}^{m} \frac{tk - ix}{tk + ix} \to \prod_{k=|n|+1}^{m} (-1) \quad \text{as } t \to 0.$$

Counting powers of -1 we have that $\lambda^m_{(n,t^{-1}x)} \to \lambda^m_n$ as $t \to 0$. In particular, the action by W on each fibre is continuous in t. Therefore W lifts to a field of actions on B.

We therefore have

$$A^C \cong \left\{ F \in C\left([0,1], C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right)\right)^W \mid F(0)(-,0) \in \bigoplus_{m \in \frac{1}{2}\mathbb{N}_0} K(V(m)) \subseteq K(L^2(K)) \right\}$$

by the analogue of the Stone-Weierstrass theorem, Theorem A.23.

We now provide a similar concrete picture of the quantum assembly field.

Let us fix $q \in (0,1)$ with $q = e^h$ for some $h \in (-\infty, 0)$. Recall that in chapter 3 we constructed the quantum assembly field A^Q over $[q,1]$. This is not an ideal parameterization for us now, because A^C, A^L and A^R are fields over $[0,1]$. The map $[0,1] \to [q,1]$, $t \mapsto q^t$ is a homeomorphism, and we can therefore view A^Q as a field over $[0,1]$ with fibres

$$A^Q_t = C^*_r(G_{q^t}).$$

For $t \neq 0$, we have an isomorphism

$$C^*_r(G_{q^t}) \cong C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/_{2\pi t^{-1}h^{-1}\mathbb{Z}}, K(\mathcal{H})\right)^W.$$

We can identify

$$C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/_{2\pi t^{-1}h^{-1}\mathbb{Z}}, K(\mathcal{H})\right)^W \subseteq C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/_{2\pi h^{-1}\mathbb{Z}}, K(\mathcal{H})\right)$$

by the rescaling the circular parameter using the map

$${}^{\mathbb{R}}\!/_{2\pi t^{-1}h^{-1}\mathbb{Z}} \to {}^{\mathbb{R}}\!/_{2\pi h^{-1}\mathbb{Z}}, \quad x + 2\pi t^{-1}h^{-1}\mathbb{Z} \mapsto tx + 2\pi h^{-1}\mathbb{Z}.$$

Let ϕ_t be the inclusion $C^*_r(G_{q^t}) \subseteq C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/_{2\pi h^{-1}\mathbb{Z}}, K(\mathcal{H})\right)$, and so this is given by, for $f \in C^*_r(G_{q^t})$,

$$\phi_t(f)(n, x + 2\pi h^{-1}\mathbb{Z}) = \pi^{q^t}_{(n,it^{-1}x)}(f)$$

where $\pi^{q^t}_{(n,it^{-1}x)}$ is the quantum principal series representation with parameter $(n, it^{-1}x)$, see (2.13). Compare this formula to that of ϕ_t in the classical case, (4.39).

By Theorem 4.10 we have an isomorphism

$$C^*(G_1) \cong \left\{ f \in C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/\!_{2\pi\mathbb{Z}}, K(\mathcal{H})\right)^W \mid f_0 \in K(L^2(K))^K \right\}. \tag{4.46}$$

We can identify

$$C^*(G_1) \subseteq C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/\!_{2\pi h^{-1}\mathbb{Z}}, K(\mathcal{H})\right)$$

by the rescaling the circular parameter using the map

$${}^{\mathbb{R}}\!/\!_{2\pi\mathbb{Z}} \to {}^{\mathbb{R}}\!/\!_{2\pi h^{-1}\mathbb{Z}}, \quad x + 2\pi\mathbb{Z} \mapsto h^{-1}x + 2\pi h^{-1}\mathbb{Z}.$$

Let ϕ_0 be the inclusion $C_r^*(G_1) \subseteq C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/\!_{2\pi h^{-1}\mathbb{Z}}, K(\mathcal{H})\right)$, and so this is given by, using the concrete description of $C_r^*(G_1)$ in (4.46),

$$\phi_0(f)(n, x + 2\pi h^{-1}\mathbb{Z}) = f(n, hx + 2\pi\mathbb{Z}), \tag{4.47}$$

where $f \in C_r^*(G_1) \subseteq C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/\!_{2\pi\mathbb{Z}}, K(\mathcal{H})\right)$ and $(n, x + 2\pi h^{-1}\mathbb{Z}) \in \frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/\!_{2\pi h^{-1}\mathbb{Z}}$.

Now we have identified each fibre of our quantum assembly field with a subalgebra of

$$C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/\!_{2\pi h^{-1}\mathbb{Z}}, K(\mathcal{H})\right)$$

we can consider the $C([0,1])$-algebra B defined by

$$\left\{ F \in C\left([0,1], C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/\!_{2\pi h^{-1}\mathbb{Z}}, K(\mathcal{H})\right)\right) \mid F(0)(-,0) \in \bigoplus_{m \in \frac{1}{2}\mathbb{N}_0} K(V(m)) \right\},$$

which is a subalgebra of a trivial field and so has the obvious evaluation maps. The fibre B_t contains $C_r^*(G_{q^t})$ using the inclusions above. We will show that $A^Q \subseteq B$, i.e. if we take $f \in A^Q$, the map

$$(t, n, x + 2\pi h^{-1}\mathbb{Z}) \mapsto \phi_t(f_t)(n, x + 2\pi h^{-1}\mathbb{Z})$$

defines an element of B, where f_t is the evaluation of f in A^Q at t (i.e. an element of $C_r^*(G_{q^t})$).

For the classical case, we constructed a special dense $*$-subalgebra on which we checked the above. In the quantum case, we will use the dense $*$-subalgebra $\mathcal{D}(\mathcal{G}) \subseteq A^Q$. Let us recall the definition from Chapter 2, reformulated in this context.

Let A denote the $*$-subalgebra of A^Q generated by the sections $t \mapsto (\omega^n_{ij} \bowtie u^m_{kl})(q^t)$, where $\omega^n_{ij} \in \mathcal{D}(K_{q^t})$ and $u^m_{kl} \in \mathcal{O}(K_{q^t})$ are the usual basis elements. Then $\mathcal{D}(\mathcal{G}) = C([0,1])A$, again a $*$-subalgebra of B.

Theorem 4.22. *For $f \in \mathcal{D}(\mathcal{G})$, the map*

$$(t, n, x + 2\pi h^{-1}\mathbb{Z}) \mapsto \phi_t(f_t)(n, x + 2\pi h^{-1}\mathbb{Z})$$

defines an element of B,

$$\left\{ F \in C\left([0,1], C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!\big/_{2\pi h^{-1}\mathbb{Z}}, K(\mathcal{H})\right)\right) \mid F(0)(-,0) \in \bigoplus_{m \in \frac{1}{2}\mathbb{N}_0} K(V(m)) \right\}.$$

Proof. It is sufficient to prove this in the case $f = \omega^a_{bc} \bowtie u^m_{kl} \in A^Q$, because the $C([0,1])$-linear span of such elements is $\mathcal{D}(\mathcal{G})$.

It is sufficient to fix an $n \in \frac{1}{2}\mathbb{Z}$ and show that the map

$$[0,1] \times {}^{\mathbb{R}}\!\big/_{2\pi h^{-1}\mathbb{Z}} \to K(\mathcal{H}_n), \quad (t, x + 2\pi h^{-1}\mathbb{Z}) \mapsto \phi_t(f_t)(n, x + 2\pi h^{-1}\mathbb{Z})$$

is continuous. By an entirely similar argument to that given at the start of the proof of Theorem 4.19 and Proposition 2.9, it is sufficient to show that for each $\xi \in \mathcal{H}_n$, the map

$$[0,1] \times {}^{\mathbb{R}}\!\big/_{2\pi h^{-1}\mathbb{Z}} \to \mathcal{H}_n, \quad (t, x + 2\pi h^{-1}\mathbb{Z}) \mapsto \phi_t(f_t)(n, x + 2\pi h^{-1}\mathbb{Z})\xi$$

is continuous.

For the ω^a_{bc} part this follows from Lemma 2.8. For the u^m_{kl} part, continuity follows by inspecting the explicit formulae involved. Indeed in the quantum case ($t > 0$), we have the formulae

$$\mathcal{O}(K_{q^t}) \to B(\mathcal{O}(\mathcal{E}^{q^t}_m)), \quad f \cdot \xi = f_{(1)} \xi S(f_{(3)}) \left(K^{2+2it^{-1}x}, f_{(2)} \right),$$

for $f \in \mathcal{O}(K_{q^t})$ and $\xi \in \mathcal{O}(\mathcal{E}^{q^t}_m)$ by Proposition 2.7. Therefore

$$\begin{aligned} u^m_{kl}(q^t) \cdot \xi &= \sum_{r,s} u^m_{kr}(q^t) \xi S(u^m_{sl}(q^t)) \left(K^{2+2it^{-1}x}, u^m_{rs}(q^t) \right) \\ &= \sum_r u^m_{kr}(q^t) \xi S(u^m_{rl}(q^t)) q^{rt(2+2it^{-1}x)} \\ &= \sum_r u^m_{kr}(q^s) \xi S(u^m_{rl}(q^s)) e^{thr(2+2it^{-1}x)} \\ &= \sum_r u^m_{kr}(q^t) \xi S(u^m_{rl}(q^t)) e^{2thr} e^{2irhx}. \end{aligned}$$

For $t = 0$, recall from Proposition 3.7 (d) that the action of $\mathcal{O}(K)$ on $\mathcal{H}_m \subseteq L^2(K)$ is given by

$$(f \cdot \xi)(k) = f\left(k \begin{pmatrix} e^{ihx} & 0 \\ 0 & e^{-ihx} \end{pmatrix} k^{-1}\right) \xi(k)$$

for $f \in \mathcal{O}(K)$, $\xi \in \mathcal{H}_m$ and $k \in K$. Note that the presence of h in this formula is a consequence of the rescaling in (4.47). Therefore

$$\begin{aligned}
(u^m_{kl} \cdot \xi)(k) &= \sum_{r,s} u^m_{kr}(k) u^m_{rs}\left(\begin{pmatrix} e^{ihx} & 0 \\ 0 & e^{-ihx} \end{pmatrix}\right) u^m_{sl}(k^{-1})\xi(k) \\
&= \sum_r u^m_{kr}(k)\xi(k)u^m_{sl}(k^{-1})e^{2irhx} \\
&= \sum_r u^m_{kr}(k)\xi(k)S(u^m_{sl})(k)e^{2irhx}
\end{aligned}$$

and so

$$u^m_{kl} \cdot \xi = \sum_r u^m_{kr} \xi S(u^m_{sl}) e^{2irhx}.$$

The result then follows from the fact that multiplication of matrix elements in $\mathcal{O}(K_{q^t})$ depends continuously on $t \in [0, 1]$ by Lemma 1.56, and a comparison of the formulae above. □

We then have the following.

Corollary 4.23. *The map $\mathcal{D}(\mathcal{G}) \to B$ defined by Theorem 4.22 extends to an inclusion $A^Q \subseteq B$ of continuous fields.*

Proof. This follows from the fact that $\mathcal{D}(\mathcal{G})$ is dense in A^C and the map defined by Theorem 4.19 is defined by fibrewise inclusions. □

To complete the identification, we need to define an action of $W = \mathbb{Z}_2$ on B that is a field of actions so that $A^Q \cong B^W$. For each $t \in [0, 1]$ we have the action of W on the fibre B_t. For $t > 0$ this is given by

$$(w \cdot f_t)(n, x + 2\pi h^{-1}\mathbb{Z}) = \left(U^{q^t}_{(n,it^{-1}x)}\right)^{-1} f_t(-n, -x + 2\pi h^{-1}\mathbb{Z}) U^{q^t}_{(n,it^{-1}x)}$$

where $f_t \in B_t$ and $(n, x + 2\pi h^{-1}\mathbb{Z}) \in \frac{1}{2}\mathbb{Z} \times {\mathbb{R}}/{2\pi h^{-1}\mathbb{Z}}$. For $t = 0$ this is given by

$$(w \cdot f_0)(n, x) = U_n^{-1} f_0(-n, -x) U_n, \quad (n, x) \in \frac{1}{2}\mathbb{Z} \times {\mathbb{R}}/{2\pi h^{-1}\mathbb{Z}}$$

where for each $n \in \frac{1}{2}\mathbb{Z}$,

$$U_n : \mathcal{H}_n \to \mathcal{H}_{-n}, \quad U_n(\xi)(k) = \xi(kw), \quad w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We recall that the intertwining unitaries are direct sums of scalar operators as given in Theorem 4.12. We need to show that these formulae depend continuously on t.

We need to show that (using the same notation as in Theorem 4.12) $\lambda^m_{(n,it^{-1}x)} \to \lambda^m_n$ as $t \to 0$. We have

$$\begin{aligned} \lambda^m_{(n,it^{-1}x)} &= \prod_{k=|n|+1}^{n} \frac{[k - it^{-1}x]_{q^t}}{[k + it^{-1}x]_{q^t}} \\ &= \prod_{k=|n|+1}^{m} \frac{q^{t(k-it^{-1}x)} - q^{-t(k-it^{-1}x)}}{q^{t(k+it^{-1}x)} - q^{-t(k+it^{-1}x)}} \\ &= \prod_{k=|n|+1}^{m} \frac{q^{tk-ix} - q^{-tk+ix}}{q^{tk+ix} - q^{-tk-ix}} \\ &\to \prod_{k=|n|+1}^{m} (-1) \quad \text{as } t \to 0. \end{aligned}$$

Counting powers of -1 we have that $\lambda^m_{(n,it^{-1}x)} \to \lambda^m_n$ as $t \to 0$. In particular, the action by W on each fibre is continuous in t. Therefore W lifts to a field of actions on B.

We therefore have $A^Q \cong B^W$ by the analogue of the Stone-Weierstrass theorem for continuous fields, Theorem A.23.

4.6 Deformation Squares

In this section we will 'glue' together A^C, A^Q, A^L and A^R into one continuous field. The resulting field will be an example of a **deformation square**, a notion that we will define now. Deformation squares are continuous fields which have the special property that they induce commutative diagrams in K-theory, from which our desired result will follow. We note that deformation squares were first introduced by the author and Voigt in [57, Definition 7.1].

Definition 4.24. Let $X := [0,1] \times [0,1]$ be the unit square in $\mathbb{R}^2$. A **deformation square** is a continuous $C(X)$-algebra A such that the following conditions are satisfied.

1. The restriction $A|_{(0,1] \times (0,1]}$ is trivial.

2. The restriction $A|_{\{0\}\times[0,1]}$ is trivial away from $(0,0)$.

3. The restriction $A|_{[0,1]\times\{0\}}$ is trivial away from $(0,0)$.

We can restrict A to the lines $\{1\}\times[0,1]\subseteq X$ and $[0,1]\times\{1\}\subseteq X$. As a result of the first condition in Definition 4.24 and Example A.28, there the field will be trivial away from a single point. Then by Proposition A.29, together with the second and third conditions in Definition 4.24 we have four maps in K-theory, namely

$$\begin{aligned} K_*(A_{(0,0)}) &\to K_*(A_{(0,1)}) \\ K_*(A_{(0,0)}) &\to K_*(A_{(1,0)}) \\ K_*(A_{(0,1)}) &\to K_*(A_{(1,1)}) \\ K_*(A_{(1,0)}) &\to K_*(A_{(1,1)}) \end{aligned}$$

and the following proposition tells us that the maps behave in the way one might hope.

Proposition 4.25. *Let A be a deformation square. Then the diagram*

$$\begin{tikzcd} K_*(A_{(0,1)}) \arrow[r] & K_*(A_{(1,1)}) \\ K_*(A_{(0,0)}) \arrow[r] \arrow[u] & K_*(A_{(1,0)}) \arrow[u] \end{tikzcd}$$

commutes.

Proof. We may assume that A is a unital field. Indeed, if A is not a unital field, we may adjoin a unit to obtain the $C(X)$-algebra $A^+_{C(X)}$ as defined in Section A.1, which is again a deformation square. By Lemma A.33, the maps

$$\begin{aligned} K_*(A_{(0,0)}) &\to K_*(A_{(0,1)}) \\ K_*(A_{(0,0)}) &\to K_*(A_{(1,0)}) \\ K_*(A_{(0,1)}) &\to K_*(A_{(1,1)}) \\ K_*(A_{(1,0)}) &\to K_*(A_{(1,1)}) \end{aligned}$$

are direct summands of the maps

$$\begin{aligned} K_*(A^+_{(0,0)}) &\to K_*(A^+_{(0,1)}) \\ K_*(A^+_{(0,0)}) &\to K_*(A^+_{(1,0)}) \\ K_*(A^+_{(0,1)}) &\to K_*(A^+_{(1,1)}) \\ K_*(A^+_{(1,0)}) &\to K_*(A^+_{(1,1)}) \end{aligned}$$

respectively, where these maps are induced by the fact $A^+_{C(X)}$ is a deformation square. Then if we can show

$$\begin{array}{ccc} K_*(A^+_{(0,1)}) & \longrightarrow & K_*(A^+_{(1,1)}) \\ \uparrow & & \uparrow \\ K_*(A^+_{(0,0)}) & \longrightarrow & K_*(A^+_{(1,0)}) \end{array}$$

commutes, we have that

$$\begin{array}{ccc} K_*(A_{(0,1)}) & \longrightarrow & K_*(A_{(1,1)}) \\ \uparrow & & \uparrow \\ K_*(A_{(0,0)}) & \longrightarrow & K_*(A_{(1,0)}) \end{array}$$

commutes.

Let $p \in M_n(A_{(0,0)})$ be a projection. By the same argument given in the proof of Lemma A.31, we can lift p to a projection $q \in M_n(A)$ such that $q(0,0) = p$. Then the composition

$$K_0(A_{(0,0)}) \to K_0(A_{(0,1)}) \to K_0(A_{(1,1)})$$

maps $[p]_0$ to $[q(1,1)]_0$, and

$$K_0(A_{(0,0)}) \to K_0(A_{(1,0)}) \to K_0(A_{(1,1)})$$

maps $[p]_0$ to $[q(1,1)]_0$, by the description of these maps provided by Lemma A.31. We can also carry out a similar argument for K_1, using Lemma A.32. □

Theorem 4.26. *There is a deformation square A^D such that the following conditions are satisfied.*

1. $A^D\big|_{\{0\}\times[0,1]} \cong A^L$.
2. $A^D\big|_{\{1\}\times[0,1]} \cong A^R$.
3. $A^D\big|_{[0,1]\times\{0\}} \cong A^C$.
4. $A^D\big|_{[0,1]\times\{1\}} \cong A^Q$.

Proof. We build up the deformation square in stages. First, note there is a field B over $[0,1] \times [0,1]$ with fibres

$$\begin{cases} B_{(t,s)} = C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}, K(\mathcal{H})\right) & t \in [0,1], s \in (0,1] \\ B_{(t,0)} = C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right) & t \in [0,1] \end{cases}$$

This can be obtained from Example 4.5 and Theorem A.34 with $D = C([0,1])$. Now we can consider the subfield C with fibres

$$\begin{cases} C_{(t,s)} = C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/\!_{2\pi\mathbb{Z}}, K(\mathcal{H})\right) & t, s \neq 0 \\ C_{(0,s)} = \left\{ f \in C_0\left(\frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/\!_{2\pi\mathbb{Z}}, K(\mathcal{H})\right) \mid f_{\pm I} \in K(L^2(K))^K \right\} & s \neq 0 \\ C_{(t,0)} = C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right) & t \neq 0 \\ C_{(0,0)} = \left\{ f \in C_0\left(\frac{1}{2}\mathbb{Z} \times \mathbb{R}, K(\mathcal{H})\right) \mid f_0 \in K(L^2(K))^K \right\} & \end{cases}$$

Let us now fix $q \in (0,1)$ with $q = e^h$.

Let $s \in (0,1)$. In the construction of the quantum assembly field, one can take the quantum parameter to be $q^s \in (0,1)$. Let us denote this rescaled version of this field by $A^{Q,s}$. Of course we have $A^{Q,1} = A^Q$ with $q_0 = q$ in the construction in chapter 3. From the concrete version of the field seen in Section 4.5, we can define a map

$$A^{Q,s} \to C|_{[0,1]\times\{s\}}, \quad f \mapsto \left((n, x + 2\pi\mathbb{Z}) \mapsto f(n, s^{-1}h^{-1}x + 2\pi s^{-1}h^{-1}\mathbb{Z})\right).$$

This defines an inclusion of $A^{Q,s}$ into $C|_{[0,1]\times\{s\}}$ as fields.

Consider the classical assembly field A^C, for which we have a concrete description provided in Section 4.5. The sections of A^C define sections of $C|_{[0,1]\times\{0\}}$ if we rescale the real parameter by

$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto h^{-1}x.$$

Let $t \in (0,1)$. In the construction of the right quantization field, one can take the quantum parameter to be $q^t \in (0,1)$. Let us denote this rescaled version of this field by $A^{R,t}$. Of course we have $A^{R,1} = A^R$. From the definition of the right quantization field, we can define a map

$$A^{R,t} \to C|_{\{t\}\times[0,1]}, \quad f \mapsto \left((n, x + 2\pi\mathbb{Z}) \mapsto f(n, s^{-1}x + 2\pi s^{-1}\mathbb{Z})\right).$$

This defines an inclusion of $A^{R,t}$ into $C|_{\{t\}\times[0,1]}$ as fields.

Finally, consider the left quantization field A^L, specifically the concrete version provided by Theorem 4.11. The sections of A^L define sections of $C|_{\{0\}\times[0,1]}$ if we rescale the circular parameter by

$${}^{\mathbb{R}}\!/\!_{2\pi s^{-1}\mathbb{Z}} \to {}^{\mathbb{R}}\!/\!_{2\pi s^{-1}h^{-1}\mathbb{Z}}, \quad x + 2\pi s^{-1}\mathbb{Z} \mapsto h^{-1}x + 2\pi s^{-1}h^{-1}\mathbb{Z}$$

and the real parameter by

$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto h^{-1}x.$$

Under these identifications we see that by construction

$$A^L \subseteq C|_{\{0\}\times[0,1]}\,, \quad A^R \subseteq C|_{\{1\}\times[0,1]}\,, \quad A^C \subseteq C|_{[0,1]\times\{0\}}\,, \quad A^Q \subseteq C|_{[0,1]\times\{1\}}\,.$$

Finally we construct an action of $W = \mathbb{Z}_2$ on C using the action of W on each fibre of C coming from the principal series interwiners. Note that from the formulae above, for $(t,s) \in (0,1) \times (0,1)$ the action of W arising from the fact $A_s^{R,t} = A_t^{Q,s} \cong C_r^*(G_{q^{st}})$ is the same, so we have a well-defined action on each fibre. This is given by the formulae

$$(w \cdot f_{(t,s)})(n, x + 2\pi\mathbb{Z}) = \left(U^{q^{st}}_{(n,is^{-1}t^{-1}h^{-1}x)}\right)^{-1} f_{(t,s)}(-n, -x + 2\pi\mathbb{Z})U^{q^{st}}_{(n,is^{-1}t^{-1}h^{-1}x)}$$

for $(t,s) \in (0,1) \times (0,1)$, $f_{(t,s)} \in C_{(t,s)}$, $(n, x + 2\pi\mathbb{Z}) \in \frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/_{2\pi\mathbb{Z}}$, and

$$(w \cdot f_{(0,s)})(n, x + 2\pi\mathbb{Z}) = U_n^{-1} f_{(0,s)}(-n, -x + 2\pi\mathbb{Z})U_n$$

for $s \in [0,1]$, $f_{(0,s)} \in C_{(s,0)}$, $(n, x + 2\pi\mathbb{Z}) \in \frac{1}{2}\mathbb{Z} \times {}^{\mathbb{R}}\!/_{2\pi\mathbb{Z}}$, and

$$(w \cdot f_{(t,0)})(n, x) = U^{-1}_{(n,t^{-1}h^{-1}x)} f_{(t,0)}(-n, -x)U_{(n,t^{-1}h^{-1}x)}$$

for $t \in [0,1]$, $f_{(t,0)} \in C_{(t,0)}$ and $(n,x) \in \frac{1}{2}\mathbb{Z} \times \mathbb{R}$. One can check that if $F \in C$, the action on each fibre depends continuously on all the variables involved by using the formulae in Theorem 4.12 in much the same way that we did for each field individually in this chapter.

We therefore have a field of actions of W on C, and we set $A^D = C^W$. Then by construction we have

$$A^L = C^W\big|_{\{0\}\times[0,1]}\,, \quad A^R = C^W\big|_{\{1\}\times[0,1]}\,, \quad A^C = C^W\big|_{[0,1]\times\{0\}}\,, \quad A^Q = C^W\big|_{[0,1]\times\{1\}}\,.$$

by the analogue of the Stone-Weierstrass theorem for fields, Theorem A.23. □

Corollary 4.27. *We have the commutative square*

$$\begin{array}{ccc} K_*(C^*(G_0)) & \longrightarrow & K_*(C_r^*(G)) \\ \big\downarrow & & \big\downarrow \\ K_*(C^*(G_1)) & \longrightarrow & K_*(C_r^*(G_q)) \end{array}$$

where the top arrow is induced by the classical assembly field A^C, the bottom arrow is induced by the quantum assembly field A^Q, the left arrow is induced by the left quantization field A^L and the right arrow is induced by the right quantization field A^R.

Proof. The deformation square A^D constructed in Theorem 4.26 together with Proposition

4.25 gives the desired commutative diagram in K-theory. □

4.7 Concluding Remarks

Let us conclude the main body of the thesis by discussing extensions to the work carried out in this thesis.

- In [57], the author and Voigt have generalized the results of this thesis to cover deformations of complex semisimple Lie groups, of which quantum $SL(2, \mathbb{C})$ is a special case. In this work, it is shown that the vertical maps in the square appearing in Lemma 4.27 are split injective, which requires some additional KK-theoretic arguments using the Dirac operator.

- Recall that the classical assembly field arises from a groupoid $\mathcal{G}$ (4.9). Therefore one might ask if it is possible to define a notion of a **quantum groupoid**, from which A^Q, and perhaps more generally A^D, arises. One might start by studying the work of Blanchard [4], where fields of quantum groups are introduced.

- On a related note to the above point, the work of de Commer and Floré [16] constructs a field from the C^*-algebras of functions on quantum $GL(n, \mathbb{C})$, with fibre at 1 being $C_0(GL(n, \mathbb{C}))$. This in some sense 'dual' to our right quantization field A^R, and it may be possible to link these two constructions using Blanchard's duality results in [4].

Appendix A

$C_0(X)$-algebras

Throughout this appendix, we let X be a locally compact Hausdorff topological space.

The purpose of this appendix is to give a fairly complete and self-contained study of the basic theory of $C_0(X)$-algebras. Roughly speaking, a $C_0(X)$-algebra is a C^*-algebra in which we can multiply elements by functions in $C_0(X)$. They were originally introduced by Kasparov in [39, Definition 1.5].

A $C_0(X)$-algebra 'fibres' over X. That is, there are C^*-algebras A_x and 'evaluation' maps $\mathrm{ev}_x : A \to A_x$ for each $x \in X$. We can then view A as an algebra of sections of the bundle

$$\coprod_{x \in X} A_x \to X.$$

Later we shall see that generally A should be viewed as a C^*-algebra of upper-semicontinuous sections of this bundle. In certain nice examples, A is in fact an algebra of continuous sections. In this case A is often referred to as a **continuous field** of C^*-algebras.

The fact A fibres over X means that we can often turn problems in studying A to problems in studying each A_x. For example, each irreducible representation of A factors through a fibre A_x for some $x \in X$ [89, Proposition C.5], and A is nuclear if and only if each fibre A_x is nuclear, [4, Proposition 3.23].

First we consider the basic definitions and subalgebras and quotients of $C_0(X)$-algebras. We then study certain $C([0,1])$-algebras that induce maps in K-theory. We also construct new $C_0(X)$-algebras from old using tensor products, group actions and and crossed products. Finally we give a sufficient condition for continuity of a $C_0(X)$-algebra where each fibre is equipped with a weight, extending results from [4].

We will make use of standard references for material on $C_0(X)$-algebras, such as [89,

Appendix C] and [17, Chapter 10].

A.1 Basic Definitions

The starting point is the definition of a $C_0(X)$-algebra, [39, Definition 1.5].

Definition A.1. A $C_0(X)$-algebra is a C^*-algebra A together with a $*$-homomorphism

$$\mu_A : C_0(X) \to ZM(A)$$

that is non-degenerate in the sense that

$$\mu_A(C_0(X))A := \overline{\operatorname{span}}\,\{\mu_A(f)a \mid f \in C_0(X), a \in A\} = A.$$

For brevity, we sometimes write fa for $\mu_A(f)a$. When X is compact, we will say that A is a $C(X)$-algebra. In this case μ_A is non-degenerate if and only if μ_A is unital.

Remark A.2. Note that $\mu_A(C_0(X))$ is a commutative C^*-subalgebra of $ZM(A)$ and so in particular is isomorphic to $C_0(Y)$ for some locally compact Hausdorff space Y. Since μ_A can be viewed as a surjective $*$-homomorphism $C_0(X) \to C_0(Y)$, we have a proper continuous injection $Y \to X$. The inclusion map is closed (because a continuous proper map between locally compact Hausdorff spaces is closed, see [49, Theorem 4.95]), and so we can view $Y \subseteq X$ as a closed subspace via this injective map. From this point of view μ_A is the restriction homomorphism $C_0(X) \to C_0(Y)$.

As a consequence, the action of $C_0(X)$ on A only 'sees' the function restricted to Y, so we can assume (after possibly relabelling X) that $C_0(X) \subseteq ZM(A)$ if necessary. We will take this slightly further in Proposition A.5 and give a sufficient condition for μ_A to be injective, which allows us to see that this is the case in the examples we consider too.

Example A.3. The following are examples of $C_0(X)$-algebras.

(a) Any C^*-algebra A is a $C(\{\text{pt}\})$-algebra. This is because $C(\{\text{pt}\}) \cong \mathbb{C}$ and we have a (unique) unital $*$-homomorphism $\mu_A : \mathbb{C} \to ZM(A)$, $z \mapsto z \cdot 1_{M(A)}$.

(b) Let D be a fixed C^*-algebra. Then $A := C_0(X, D)$ is a $C_0(X)$-algebra. The $*$-homomorphism μ_A is given by

$$\mu_A : C_0(X) \to ZM(A), \quad (\mu_A(f)g)(x) = f(x)g(x), \quad f \in C_0(X), \quad g \in A, \quad x \in X.$$

To show non-degeneracy, it is enough to show $\mu_A(C_0(X))A$ contains $C_0(X) \odot D$ which is dense in A. We note that $\mu_A(C_0(X))(C_0(X) \odot D) = (C_0(X) \cdot C_0(X)) \odot D$ and an approximate identity argument shows that $C_0(X) \cdot C_0(X) = C_0(X)$.

This example is often referred to as the **trivial field** over X with fibre D.

(c) Let $A := \{f \in C([0,1], M_2(\mathbb{C})) \mid f(0) \text{ is diagonal}\}$. This is a $C([0,1])$-algebra with $*$-homomorphism μ_A is given by

$$\mu_A : C([0,1]) \to Z(A), \quad (\mu_A(f)g)(x) = f(x)g(x), \quad f \in C([0,1]), \quad g \in A, \quad x \in X$$

and we can see that μ_A is unital.

Let A be a $C_0(X)$-algebra. For $x \in X$, let $J_x \subseteq C_0(X)$ denote the ideal of functions on X vanishing at x. Set

$$I_x := \mu_A(J_x)A = \overline{\operatorname{span}}\{\mu_A(f)a \mid f \in J_x, a \in A\}.$$

By the Cohen Factorization Theorem [9, Theorem 1], any element $a \in I_x$ admits a factorization $a = fb$, where $f \in J_x$ and $b \in A$. We can 'localize' A at x by considering the quotient space

$$A_x := A/I_x. \tag{A.1}$$

This is called the fibre of A at x.

In Example A.3 (a), we see that $A_{\mathrm{pt}} = A$. We will determine the fibres of the algebras in parts (b) and (c) shortly.

Let $\mathrm{ev}_x : A \twoheadrightarrow A_x$ be the canonical quotient maps to each fibre. We use the notation $\mathrm{ev}_x(a) = a(x)$ for $a \in A$. This notation is intended to be suggestive - indeed, we can think of A as a C^*-algebra of sections of the bundle $\coprod_{x\in X} A_x \to X$.

The evaluation map behaves like we expect on functions, as seen in the following proposition.

Proposition A.4. *Let A be a $C_0(X)$-algebra. If $f \in C_0(X)$ and $a \in A$, then $(fa)(x) = f(x)a(x)$.*

Proof. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit in $C_0(X)$. Then $(fu_\lambda - f(x)u_\lambda)a \in I_x$ for all λ, and converges to $fa - f(x)a$. Since I_x is closed, $fa - f(x)a \in I_x$, so $(fa)(x) = f(x)a(x)$. $\square$

Proposition A.5. *Let A be a $C_0(X)$-algebra.*

(a) If $A_x \neq 0$ for all $x \in X$, then μ_A is injective.

(b) *Identifying $\mu_A(C_0(X)) \cong C_0(Y)$ for some closed subspace $Y \subseteq X$ as in Remark A.2, then $A_x = 0$ for all $x \in X \setminus Y$.*

Proof.

(a) If $\mu_A(f) = 0$ for some $f \in C_0(X)$, then

$$\mu_A(f)a = 0$$

for all $a \in A$. By Proposition A.4, then $f(x)a(x) = 0$ for all $x \in X$ and $a \in A$. If $A_x \neq 0$ for all $x \in X$, then it must be the case that $f(x) = 0$ for all x, so $f = 0$.

(b) We will show that $I_x = \mu_A(J_x)A = A$ for each $x \in X \setminus Y$. To do so, we will show that we can extend a function $f \in C_0(Y)$ to a function $\tilde{f} \in C_0(X)$ that vanishes at x. From this it follows that $\mu_A(J_x) = C_0(Y)$, and $\mu_A(J_x)A = C_0(Y)A = \mu_A(C_0(X))A = A$, and the result follows.

Let $f \in C_0(Y)$ and $x \in X \setminus Y$. Since Y is closed in X, there exists an neighbourhood U of x disjoint from Y. Then by Urysohn's Lemma [74, 2.12] there exists a continuous function $\chi : X \to [0, 1]$ such that $\chi(x) = 1$, and $\mathrm{Supp}(\chi) \subseteq U$.

Because Y is closed in X, the inclusion $Y \subseteq X$ is proper, and so we obtain a surjective $*$-homomorphism $C_0(X) \to C_0(Y)$ which is given by restriction. We can therefore extend f to a C_0 function on X, which we also call f. Then $\tilde{f} := f(1 - \chi) \in C_0(X)$, $\tilde{f}(x) = 0$ and $\tilde{f}(y) = f(y)$ for all $y \in Y$. □

The following proposition records some further basic properties of $C_0(X)$-algebras, as seen in [4, 2.2].

Proposition A.6. *Let A be a $C_0(X)$-algebra.*

(a) *For each $a \in A$, the norm map $N_a : X \to \mathbb{R}$, $x \mapsto \|a(x)\|_{A_x}$ vanishes at infinity.*

(b) *For each $a \in A$, $\|a\|_A = \sup_{x \in X} \|a(x)\|_{A_x} = \sup_{x \in X} N_a(x)$ and the supremum is attained.*

(c) *If A is unital and $a \in A$, $\mathrm{Spec}(a) = \bigcup_{x \in X} \mathrm{Spec}(a(x))$.*

Proof.

(a) By the Cohen Factorization Theorem, any element $a \in A$ admits a factorization $a = fb$ where $f \in C_0(X)$ and $b \in A$. Then $N_a(x) = \|f(x)b(x)\|_{A_x} \leq |f(x)| \, \|b\|_A$ for all $x \in X$. Since f vanishes at infinity, we must have that N_a does too.

(b) From [4, Proposition 2.8]. Since the quotient maps ev_x are necessarily norm-decreasing, we have $\|a(x)\|_{A_x} \leq \|a\|_A$ for all $x \in X$. We need to show that $\|a\|_A$ is attained by some $x \in X$.

Let $a \in A$. Consider the C^*-algebra $B := C^*(C_0(X), a^*a) \subseteq M(A)$. This is a commutative C^*-algebra, so functional calculus tells us there exists a character ϕ on B such that $\phi(a^*a) = \|a\|_A^2$. The restriction of ϕ to $C_0(X)$ is again a character, and so is necessarily an evaluation map at some $x \in X$. Then ϕ descends to a character (which we still denote by ϕ) on $\mathrm{ev}_x(C^*(a^*a)) \subseteq A_x$. In particular

$$\|a\|_A^2 = \phi(a^*a) = \phi(a(x)^*a(x)) \leq \|a(x)\|_{A_x}^2$$

so $\|a\|_A \leq \|a(x)\|_{A_x}$.

(c) From [4, Proposition 2.9]. Clearly if $a - \lambda \cdot 1$ were invertible in A, then $a(x) - \lambda \cdot 1$ would be invertible in A_x for all $x \in X$. This shows that $\bigcup_{x \in X} \mathrm{Spec}(a(x)) \subseteq \mathrm{Spec}(a)$.

In general, if α is a positive element in a C^*-algebra, then α is not invertible if and only if $\|(1+\alpha)^{-1}\| = 1$. To see this fact we can work inside the commutative C^*-algebra $C^*(\alpha)$ which we identify with a function algebra. Then we can see that

$$\begin{aligned} \alpha \text{ is not invertible} &\iff \alpha \geq 0 \text{ with equality attained} \\ &\iff (1+\alpha)^{-1} \leq 1 \text{ with equality attained.} \end{aligned}$$

Also note that an element of a C^*-algebra b is invertible if and only if both the positive elements b^*b and bb^* are invertible. Therefore

$$\begin{aligned} \lambda \in \mathrm{Spec}(a) &\iff b := a - \lambda \cdot 1 \text{ is not invertible} \\ &\iff b^*b,\ bb^* \text{ are not invertible} \\ &\iff \left\|(1+b^*b)^{-1}\right\|_A = \left\|(1+bb^*)^{-1}\right\|_A = 1 \\ &\iff \sup_{x \in X} \left\|(1+b(x)^*b(x))^{-1}\right\|_{A_x} = \sup_{x \in X} \left\|(1+b(x)b(x)^*)^{-1}\right\|_{A_x} = 1. \end{aligned}$$

In particular, by part (b), there exists $y \in X$ such that $\|(1+b(y)^*b(y))^{-1}\|_{A_y} = 1$, which shows $b(y) = a(y) - \lambda \cdot 1$ is not invertible, whence $\lambda \in \mathrm{Spec}(a(y))$. □

One may ask whether the sections of A are continuous in the sense that for each $a \in A$, the norm map N_a is continuous. In generality, one can only say the following.

Proposition A.7. *Let A be a $C_0(X)$-algebra. For each $a \in A$, the norm map $N_a : X \to \mathbb{R}$, $x \mapsto \|a(x)\|_{A_x}$ is upper-semicontinuous.*

Proof. Adapted from [42, Lemma 2.3]. Let $\epsilon > 0$, $a \in A$ and $x \in X$ be given. We need to show there exists a neighbourhood U_x of x such that for all $y \in U_x$, $\|a(y)\|_{A_y} < \|a(x)\|_{A_x} + \epsilon$.

By the definition of the quotient norm, there exists $f \in J_x$ and $b \in A$ such that $\|a - fb\|_A \leq \|a(x)\|_{A_x} + \frac{\epsilon}{2}$. Since $f(x) = 0$, there exists a neighbourhood U_x of x such that $|f(y)| < \frac{\epsilon}{2\|b\|_A}$ for all $y \in U_x$. Then for $y \in U_x$,

$$\begin{aligned}\|a(y)\|_{A_y} &\leq \|a(y) - f(y)b(y)\|_{A_y} + \|f(y)b(y)\|_{A_y} \\ &= \|(a - fb)(y)\|_{A_y} + \|f(y)b(y)\|_{A_y} \\ &< \|a(x)\|_{A_x} + \epsilon.\end{aligned}$$

□

However, there are $C_0(X)$-algebras A and $a \in A$ for which the norm map N_a fails to be lower semicontinuous - see [89, Example C.8] for a basic example. This leads us to the following definition.

Definition A.8. Let A be a $C_0(X)$-algebra. If for each $a \in A$, the norm map $N_a : X \to \mathbb{R}$ is continuous, then we say A is a **continuous $C_0(X)$-algebra**, or a **continuous field of C^*-algebras over X**.

Kirchberg and Wassermann gave a necessary and sufficient condition for lower semicontinuity of the norm functions in [42, Lemma 2.2]. We reformulate the result in our context. First, let us introduce some terminology.

Let A be a C^*-algebra, and let $\{B_i\}_{i \in I}$ a family of C^*-algebras. A family of $*$-homomorphisms $\pi_i : A \to B_i$ is said to be a **faithful** family if for all $a \in A$,

$$\|a\|_A = \sup_{i \in I} \|\pi_i(a)\|_{B_i}.$$

This is equivalent to the condition that if $a \in A$ satisfies $\pi_i(a) = 0$ for all i, then $a = 0$.

For a family $\{B_i\}_{i \in I}$ of C^*-algebras, define

$$p_i : \prod_{i \in I} B_i \to B_i.$$

Lemma A.9. *Let A be a $C_0(X)$-algebra. Then the following are equivalent.*

1. *The norm maps $N_a : X \to \mathbb{R}$, $x \mapsto \|a(x)\|_{A_x}$ are lower semicontinuous for all $a \in A$.*
2. *For any closed subset $X' \subseteq X$ and dense subset Y of X', the morphisms $\mathrm{ev}_Y := \oplus_{x \in Y} \mathrm{ev}_x$ and $\mathrm{ev}_{X'} := \oplus_{x \in X'} \mathrm{ev}_x$ have the same kernel.*

3. *For any closed subset $X' \subseteq X$ and dense subset Y of X', $\{p_x\}_{x \in Y}$ is a faithful family of $*$-homomorphisms on $\mathrm{ev}_{X'}(A)$.*

Proof. We prove that 1 and 2 are equivalent.

Suppose the norm maps are lower semicontinuous and $a \in \mathrm{Ker}(\mathrm{ev}_Y)$. Let $\epsilon > 0$ and $x \in X'$. There exists a neighbourhood U_x of x such that for all $y \in U_x$, $\|a(x)\|_{A_x} - \epsilon < \|a(y)\|_{A_y}$. Since Y is dense in X', then $Y \cap U_x$ is non empty and for all $y \in Y \cap U_x$, $\|a(x)\|_{A_x} < \|a(y)\|_{A_y} + \epsilon = \epsilon$. Therefore for all $x \in X'$, $a(x) = 0$.

Suppose now that N_a is not lower semicontinuous for some $a \in A$ at some $x \in X$. Then for some $\epsilon > 0$, there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ in X such that $x_\lambda \to x$ and $\|a(x_\lambda)\|_{A_{x_\lambda}} \leq \|a(x)\|_{A_x} - \epsilon$ for all λ. The continuous functional calculus tells us that $\||a(x)|\|_{A_x} = \|a(x)\|_{A_x}$ for all x so we can replace a by $|a|$ and still have the same inequality. Therefore we can assume that a is positive.

The inequality $\|a(x_\lambda)\|_{A_{x_\lambda}} \leq \|a(x)\|_{A_x} - \epsilon$ tells us that $\mathrm{Spec}(a(x_\lambda)) \subseteq [0, \|a(x)\|_{A_x} - \epsilon]$, and the spectral radius formula tells us there exists elements of $\mathrm{Spec}(a(x))$ accumulating at $\|a(x)\|_{A_x}$. Choose a continuous function χ on $\mathrm{Spec}(a)$ which vanishes on $[0, \|a(x)\|_{A_x} - \epsilon]$ and is non-zero outside this interval. Then $\chi(a)(x_\lambda) = 0$ and $\chi(a)(x) \neq 0$. Set $Y = \{x_\lambda\}$ and $X' = \overline{Y}$. Then ev_Y and $\mathrm{ev}_{X'}$ do not have the same kernel.

Now we prove that 2 implies 3. Let $c \in \mathrm{ev}_{X'}(A)$ and suppose that $p_x(c) = 0$ for all $x \in Y$. We have that $c = \mathrm{ev}_{X'}(a)$ for some $a \in A$, and so $p_x(c) = 0$ implies $\mathrm{ev}_x(a) = 0$ for all $x \in Y$. Then since $\mathrm{Ker}(\mathrm{ev}_{X'}) = \mathrm{Ker}(\mathrm{ev}_Y)$, and so $\mathrm{ev}_x(a) = 0$ for all $x \in X'$. Therefore $c = 0$.

Finally we prove that 3 implies 2. Let $a \in \mathrm{Ker}(\mathrm{ev}_Y)$. Then $\mathrm{ev}_x(a) = 0$ for all $x \in Y$, and so $p_x(\mathrm{ev}_x(a)) = 0$ for all $x \in Y$. Then $\mathrm{ev}_x(a) = 0$ for all $x \in X'$, and so $a \in \mathrm{Ker}(\mathrm{ev}_{X'})$. □

Sometimes we might find it difficult to identify the fibres of a $C_0(X)$-algebra. However, we may have a faithful family of $*$-epimorphisms which we think should be the evaluation maps. The following proposition helps us with this identification, and it is adapted from [42, Lemma 2.3].

Proposition A.10. *Let A be a $C_0(X)$-algebra, and $\{B_x\}_{x \in X}$ a family of C^*-algebras. Suppose there is a faithful family of $*$-epimorphisms $\pi_x : A \to B_x$ and for each $x \in X$ and all $a \in A$,*

$$\pi_x(\mu_A(f)a) = f(x)\pi_x(a).$$

If the map $X \to \mathbb{R}$, $x \mapsto \|\pi_x(a)\|_{B_x}$ is upper semicontinuous for each $a \in A$, then $\mathrm{Ker}(\pi_x) = \mu_A(J_x)A$. In particular, $A_x = B_x$ and $\mathrm{ev}_x = \pi_x$.

Proof. Clearly $\mu_A(J_x)A \subseteq \mathrm{Ker}(\pi_x)$ by the assumptions about our family of epimorphisms.

If $a \in \mathrm{Ker}(\pi_x)$, then by upper semicontinuity there exists a neighbourhood U_x of x such that for $y \in U_x$, $\|\pi_y(a)\|_{B_y} < \frac{\epsilon}{2}$. By Urysohn's Lemma, there exists $f \in C_0(X)$ satisfying, $0 \leq f \leq 1$, $f(x) = 1$ and $f(y) = 0$ for $y \in X \setminus U_x$. Then $1 - f \in C_b(X)$ vanishes at x. By non-degeneracy, we can write $a = \mu_A(g)b$, where $g \in C_0(X)$ and $b \in A$, and we can extend μ_A to $C_b(X)$. Then $\mu_A(1-f)a = \mu_A((1-f)g)b \in \mu_A(J_x)A$, and

$$\|a - \mu_A(1-f)a\|_A < \epsilon$$

by the assumptions about our family of epimorphisms. Therefore $a \in \mu_A(J_x)A$. $\square$

Example A.11. Let us apply Proposition A.10 to parts (b) and (c) of A.3. In each case, the usual pointwise evaluation maps $\{\mathrm{ev}_x\}$ (a slight abuse of notation which we resolve shortly), where $x \in X$ or $[0,1]$ respectively, is a faithful family of $*$-epimorphisms satisfying the conditions in Proposition A.10. Since the norm maps $x \mapsto \|\mathrm{ev}_x(g)\|$ are continuous (where $x \in X$ or $[0,1]$ respectively) for all $g \in A$, Proposition A.10 applies, and so for part (b), $A_x = D$ for all $x \in X$, and for part (c),

$$A_x = \begin{cases} \mathbb{C}^2 & x = 0 \\ M_2(\mathbb{C}) & x \neq 0 \end{cases}$$

and the evaluation maps are ev_x as expected.

Let us look at another example of a $C_0(X)$-algebra, as seen in [89, Example C.4].

Example A.12. Let Y and Z be locally compact Hausdorff spaces and let $\sigma : Y \to Z$ be a continuous surjective map. Then $C_0(Y)$ is a $C_0(Z)$-algebra, with fibres $C_0(Y)_z \cong C_0(\sigma^{-1}(\{z\}))$ for $z \in Z$, and with evaluation map to the fibre at $z \in Z$ given by restriction of functions from Y to the closed set $\sigma^{-1}(\{z\}) \subseteq Y$.

First, let us describe the $C_0(Z)$-algebra structure on $C_0(Y)$. Recall that $M(C_0(Y))$ can be identified with $C_b(Y)$, the continuous and bounded $\mathbb{C}$-valued functions on Y, see [70, Proposition 2.55]. Using this description of $M(C_0(Y))$, we can define a $*$-homomorphism

$$\mu : C_0(Z) \to M(C_0(Y)), \quad f \mapsto f \circ \sigma.$$

Let us check that μ is non-degenerate. It suffices to see that $\mu(C_0(Z))C_0(Y)$ is an ideal of $C_0(Y)$ which does not vanish at a common point.

Let $y \in Y$. Then we can find a function $f \in C_0(Z)$ such that $f(\sigma(y)) = 1$, and a function $g \in C_0(Y)$ such that $g(y) = 1$, and $(\mu(f)g)(y) = f(\sigma(y))g(y) = 1$. Therefore

$\mu(C_0(Z))C_0(Y)$ is an ideal of $C_0(Y)$ which does not vanish at a common point and so $C_0(Y)$ is a $C_0(Z)$-algebra.

Define for each $z \in Z$ the surjective restriction homomorphisms

$$\mathrm{res}_z : C_0(Y) \to C_0(\sigma^{-1}(\{z\})), \quad f \mapsto f|_{\sigma^{-1}(\{z\})},$$

and the ideals

$$J_z := \{f \in C_0(Z) \mid f(z) = 0\} \subseteq C_0(Z).$$

We will show that $\mathrm{Ker}(\mathrm{res}_z) = \mu(J_z)C_0(Y)$, from which it follows that $C_0(Y)_z \cong C_0(\sigma^{-1}(\{z\}))$ with evaluation map res_z.

We first show that $\mu(J_z)C_0(Y) \subseteq \mathrm{Ker}(\mathrm{res}_z)$. Indeed, it suffices to show that if $f \in J_z$ and $g \in C_0(Y)$, then $\mu(f)g \in \mathrm{Ker}(\mathrm{res}_z)$. We have, for $y \in \sigma^{-1}(\{z\})$,

$$\mathrm{res}_z(\mu(f)g)(y) = (\mu(f)g)(y) = f(\sigma(y))g(y) = f(z)g(y) = 0$$

because $f(z) = 0$.

Now $\mu(J_z)C_0(Y)$ is an ideal of $C_0(Y)$, and so is the intersection of the primitive ideals of $C_0(Y)$ containing it. We need to show that

$$\mu(J_z)C_0(Y) = \bigcap_{y \in \sigma^{-1}(z)} \mathrm{Ker}(\mathrm{ev}_y) = \mathrm{Ker}(\mathrm{res}_z),$$

where for $y \in Y$, ev_y is as in Example A.11. It suffices to show that if $y \notin \sigma^{-1}(z)$, then there exists a function in $\mu(J_z)C_0(Y)$ that does not vanish at y.

Since $y \notin \sigma^{-1}(z)$, then $z' := \sigma(y) \neq z$. Therefore there exists a function $f \in C_0(Y)$ such that $f(z) = 0$ and $f(z') = 1$. In particular $f \in J_z$. We can also choose a function $g \in C_0(Y)$ such that $g(y) = 1$. We then have that $(\mu(f)g)(y) = f(z')g(y) = 1$, and so we must have that $\mathrm{Ker}(\mathrm{res}_z) = \mu(J_z)C_0(Y)$, as required.

To finish this section, we consider adjoining units to $C_0(X)$-algebras. This is in the same spirit as to how one adjoins a unit to a C^*-algebra. This is an adaptation of the definition given in [4, Définition 2.7]. We will restrict to the case where X is compact, as this is all we shall need in the thesis.

Remark A.13. We note that one can generalize what follows to non-compact locally compact spaces X by replacing $C(X)$ with $C_0(X)$, but the result will only be a $C_0(X)$-algebra with unital fibres, rather than a unital $C(X)$-algebra. To obtain a unital algebra, one could directly follow [4, Définition 2.7] where Blanchard constructs, from a $C_0(X)$-algebra

A, a unital $C(X^+)$-algebra $A^+_{C(X^+)}$, where X^+ denotes the one-point compactification of X. Here the fibres are $(A^+_{C(X^+)})_x = A^+_x$ for $x \in X$ and $(A^+_{C(X^+)})_\infty = \mathbb{C}$.

For the compact case, Blanchard's definition gives a slightly larger algebra than the one we will consider, for it includes pathological sections of the form

$$x \mapsto 0 \in A_x \text{ for all } x \in X, \quad \infty \mapsto \lambda \in \mathbb{C}$$

for $\lambda \neq 0$.

For the rest of this section, X is a compact Hausdorff space.

Since A and $C(X)$ are $C(X)$-algebras, the direct sum $A \oplus C(X)$ is a $C(X)$-algebra, with $C(X)$-action

$$\mu : C(X) \to ZM(A \oplus C(X)), \quad \mu(f)(a, g) = (\mu_A(f)a, fg)$$

for $f, g \in C(X)$ and $a \in A$. The fibres are clearly $(A \oplus C(X))_x = A_x \oplus \mathbb{C}$ with the obvious evaluation maps. Note that μ is injective, because for $f \in C(X)$, $\mu(f)(0, 1) = (0, f)$.

Now define $A^+_{C(X)} := A + \mu(C(X)) \subseteq M(A \oplus C(X))$.

Lemma A.14. *Let A be a $C(X)$-algebra. Then*

(a) The algebra $A^+_{C(X)} \subseteq M(A \oplus C(X))$ is a unital C^*-algebra. If A is unital, then $A^+_{C(X)} = A \oplus C(X)$.

(b) There is a split exact sequence of C^*-algebras

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & A^+_{C(X)} \arrow[r] & C(X) \arrow[r] \arrow[l, bend left] & 0. \end{tikzcd}$$

Proof.

(a) We consider the unital and non-unital cases separately.

Suppose that A is unital. Then if $a \in A$ and $f \in C(X)$, we have that $\mu(f) = (f1_A, f) \in A \oplus C(X)$. Therefore

$$a + \mu(f) = (a, 0) + (f1_A, f) = (a + f1_A, f) \in A \oplus C(X)$$

In particular we can obtain any element of $A \oplus C(X)$ in this way, and so $A^+_{C(X)} = A \oplus C(X)$ in this case. This is of course a unital C^*-algebra.

Suppose that A is not unital. We start by noting that $A^+_{C(X)}$ is an algebra, being closed under sums and products. We need to show that $A^+_{C(X)}$ is a C^*-subalgebra of $M(A \oplus C(X))$. We will do this by showing $A^+_{C(X)}$ is complete in norm.

We note that $A \subseteq A^+_{C(X)}$ as a closed subspace. Therefore, the quotient space $A^+_{C(X)}/A$ is a normed space.

Since A is not unital, $A \cap \mu(C(X)) = 0$ in $M(A \oplus C(X))$, and so $A + \mu(C(X)) = A \oplus \mu(C(X))$ as vector spaces. Therefore

$$A^+_{C(X)}/A = A \oplus \mu(C(X))/A \cong \mu(C(X)) \cong C(X)$$

as vector spaces. This linear isomorphism is isometric, because we have in general that for $f \in C(X)$,

$$\|\mu(f) + A\|_{A^+_{C(X)}/A} \leq \|\mu(f)\|_{A^+_{C(X)}} = \|f\|_{C(X)}$$

because the quotient map is decreasing, and for the reverse inequality, we note that

$$\begin{aligned}
\|\mu(f) + A\|_{A^+_{C(X)}/A} &= \inf_{a \in A} \|a + \mu(f)\|_{A^+_{C(X)}} \\
&= \inf_{a \in A} \sup_{\substack{(b,g) \in A \oplus C(X) \\ \|(b,g)\| \leq 1}} \|(a + \mu(f))(b,g)\|_{A \oplus C(X)} \\
&= \inf_{a \in A} \sup_{\substack{(b,g) \in A \oplus C(X) \\ \|(b,g)\| \leq 1}} \|(ab + fb, fg)\|_{A \oplus C(X)} \\
&= \inf_{a \in A} \sup_{\substack{(b,g) \in A \oplus C(X) \\ \|(b,g)\| \leq 1}} \max\left\{\|ab + fb\|_A, \|fg\|_{C(X)}\right\} \\
&= \inf_{a \in A} \max\left\{\sup_{\substack{b \in A \\ \|b\| \leq 1}} \|ab + fb\|_A, \sup_{\substack{g \in C(X) \\ \|g\| \leq 1}} \|fg\|_{C(X)}\right\} \\
&= \inf_{a \in A} \max\left\{\sup_{\substack{b \in A \\ \|b\| \leq 1}} \|ab + fb\|_A, \|f\|_{C(X)}\right\} \geq \|f\|_{C(X)}.
\end{aligned}$$

Therefore $A^+_{C(X)}/A \cong C(X)$ as Banach spaces, and since A, and $C(X)$ are complete, then $A^+_{C(X)}$ is complete. We note that $\mu(1)$ is a unit for $A^+_{C(X)}$.

(b) If A is unital, then clearly we have a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & A^+_{C(X)} \arrow[r] & C(X) \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd}$$

where the map $A^+_{C(X)} = A \oplus C(X) \to C(X)$ is the coordinate projection, and $C(X) \to A^+_{C(X)} = A \oplus C(X)$ is the inclusion into the second factor.

In the non-unital case, the isomorphism of Banach spaces $A^+_{C(X)}/A \cong C(X)$ seen in part (a) is in fact an isomorphism of C^*-algebras, as the maps preserve the algebraic structure too. Therefore we have the split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & A^+_{C(X)} \arrow[r] & C(X) \arrow[r] \arrow[l, bend left, "\mu"] & 0 \end{tikzcd}$$

where the map $A^+_{C(X)} \to C(X)$ is the quotient map. $\square$

Clearly if A is a unital $C(X)$-algebra, then $A^+_{C(X)} = A \oplus C(X)$ is a unital $C(X)$-algebra.

If A is a non-unital $C(X)$-algebra the map μ defines a unital $*$-homomorphism $C(X) \to Z(A^+_{C(X)})$ and so in this case $A^+_{C(X)}$ is also a $C(X)$-algebra. Let us now determine the fibres in this case.

Lemma A.15. *Let A be a non-unital $C(X)$-algebra. Then $(A^+_{C(X)})_x \cong A^+_x$ for $x \in X$, with evaluation maps $a + \mu(f) \mapsto a(x) + f(x)$.*

Proof. Since μ is injective, we can define a linear map, for each $x \in X$,

$$A + \mu(C(X)) = A \oplus \mu(C(X)) \to A^+_x, \quad a + \mu(f) \mapsto a(x) + f(x).$$

By the definition of multiplication in A^+_x, this is a $*$-homomorphism. The kernel is given by

$$\mu_A(J_x)A \oplus J_x$$

as a vector space. Therefore

$$(A^+_{C(X)})_x = {A + \mu(C(X))}/{\mu(J_x)(A + \mu(C(X))} \cong A^+_x$$

as vector spaces, and the isomorphism preserves the multiplication. Therefore this is an isomorphism of C^*-algebras. $\square$

Let us consider a basic example of adjoining a unit to a trivial field.

Example A.16. Let D be a C^*-algebra. Let $A = C(X, D)$. We have that $A^+_{C(X)} \cong C(X, D^+)$. Indeed,

$$A^+_{C(X)} = C(X, D) + C(X) \subseteq M(A \oplus C(X))$$

and we can see that this can be identified with $C(X, D^+)$.

Finally, we note that adjoining a unit to a $C(X)$-algebra does not affect continuity. We follow [4, Proposition 3.2].

Proposition A.17. *Let A be a continuous $C(X)$-algebra. Then $A^+_{C(X)}$ is continuous.*

Proof. This is clear if A is unital, so we suppose A is not unital. Let

$$\mathrm{ev}^+_x : A^+_{C(X)} \to A^+_x, \quad a + \mu(f) \mapsto a(x) + f(x)$$

be the evaluation map to the fibre at $x \in X$. Let $\alpha \in A^+_{C(X)}$. We want to show that

$$X \to \mathbb{R}, \quad x \mapsto \left\|\mathrm{ev}^+_x(\alpha)\right\|_{A^+_x}$$

is continuous. It suffices to show that

$$X \to \mathbb{R}, \quad x \mapsto \left\|\mathrm{ev}^+_x(\alpha)\right\|^2_{A^+_x} = \left\|\mathrm{ev}^+_x(\alpha)^* \, \mathrm{ev}^+_x(\alpha)\right\|_{A^+_x}$$

is continuous, and so we can assume that α is positive. If $\alpha = a + \mu(f)$ for some $a \in A$ and $f \in C(X)$, and $\alpha \geq 0$, then in particular α is self-adjoint and so

$$\left\|\mathrm{ev}^+_x(\alpha)\right\|_{A^+_x} = r(\mathrm{ev}^+_x(\alpha)) = r(a(x) + f(x))$$

(see [59, Theorem 2.1.1]) where r is the spectral radius.

Now $\alpha \geq 0$ in $A^+_{C(X)}$ if and only if there exists $\beta = b + \mu(g) \in A^+_{C(X)}$ (where $b \in A$ and $g \in C(X)$) such that

$$\alpha = \beta^*\beta = b^*b + \mu(g)b^* + \mu(\overline{g})b + \mu(g^*g).$$

Then $a = b^*b + \mu(g)b^* + \mu(\overline{g})b$ is self-adjoint, and $\mu(f) = \mu(g^*g)$ is positive. By the spectral mapping theorem (see [59, Theorem 2.1.14]) we have that

$$\mathrm{Spec}(\mathrm{ev}^+_x(\alpha)) = \mathrm{Spec}(a(x) + f(x)) = \mathrm{Spec}(a(x)) + f(x)$$

and so

$$r(\mathrm{ev}^+_x(\alpha)) = \sup_{z \in \mathrm{Spec}(\mathrm{ev}^+_x(\alpha))} |z| = \sup_{z \in \mathrm{Spec}(\mathrm{ev}^+_x(\alpha))} z = \left(\sup_{z \in \mathrm{Spec}(a(x))} z \right) + f(x).$$

Since a is self-adjoint, then we can write $a = a^+ - a^-$ in terms of its positive and negative parts in A. Then $a(x) = a^+(x) - a^-(x)$ is the corresponding decomposition for $a(x)$. It follows that

$$\left\|\mathrm{ev}_x^+(\alpha)\right\|_{A_x^+} = \left(\sup_{z\in\mathrm{Spec}(a^+(x))} z\right) + f(x) = \left\|a^+(x)\right\|_{A_x^+} + f(x).$$

which is clearly continuous in x, as required. □

A.2 Subalgebras, Ideals and Quotients of $C_0(X)$-algebras

In this section we shall study results concerning subalgebras and quotients of $C_0(X)$-algebras. The Propositions A.18 and A.19 appear to be folklore.

Proposition A.18. *Let A be a $C_0(X)$-algebra, and $I \subseteq A$ be an ideal. Then I is a $C_0(X)$-algebra with fibres $I_x = \mathrm{ev}_x(I) \subseteq A_x$ for each $x \in X$. In particular if A is continuous then I is continuous.*

Proof. If $T \in M(A)$, then T is an adjointable operator from A to itself (viewing A as a Hilbert A-module). We note that $T(I) \subseteq I$, and so T restricts to an element of $M(I)$. Indeed, if (u_λ) is an approximate unit for A and $j \in I$, then $Tu_\lambda j \to Tj$. But $Tu_\lambda j = (Tu_\lambda)j \in I$ since I is an ideal, and since I is closed, we must have that $Tj \in I$.

Therefore we can define a $*$-homomorphism $\mu_I : C_0(X) \to M(I)$, $f \mapsto \mu_A(f)|_I$. Note that the image is contained in the centre of $M(I)$, since it suffices to check that $\mu_I(C_0(X))$ commutes with $I \subseteq M(I)$, and this follows from that fact $\mu_A(C_0(X)) \subseteq ZM(A)$.

Note that if (f_ν) is an approximate unit for $C_0(X)$, then non-degeneracy of μ_A tells us $\mu_A(f_\nu)a \to a$ for all $a \in A$. Then restricting to $a \in I$, we see that $I \subseteq \mu_I(C_0(X)) \cdot I$, and so μ_I is non-degenerate.

Let I_x denote the fibre of I at x. The inclusion $I \hookrightarrow A$ induces a $*$-homomorphism $I_x \to A_x$, $j + J_xI \mapsto j + J_xA$ for $j \in I$. This is in fact an injection, for if $j + J_xA = k + J_xA$ for $j, k \in I$, then $j - k = f \cdot a$ for some $f \in J_x$, $a \in A$. Then if (v_λ) is an approximate unit for I, $(j-k)v_\lambda = f \cdot (av_\lambda) \in J_xI$, converging to $j - k$. Since J_xI is a closed ideal of I, then $j - k \in J_xI$, and so $j + J_xI = k + J_xI$. We can see that

$$\begin{array}{ccc} I & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_x} \\ I_x & \longrightarrow & A_x \end{array}$$

commutes and so the evaluation map on I at $x \in X$ is simply the restriction of the evaluation map on A at $x \in X$ to I, so $I_x = \mathrm{ev}_x(I)$. The continuity statement is then obvious. $\square$

Proposition A.19. *Let A be a $C_0(X)$-algebra, and $I \subseteq A$ be an ideal. Then ${}^{A}\!/\!_{I}$ is a $C_0(X)$-algebra, with fibres $\left({}^{A}\!/\!_{I}\right)_x \cong {}^{A_x}\!/\!_{I_x}$, and evaluation map $ev_x(a+I) = ev_x(a) + I_x$ for each $x \in X$.*

Proof. Define $\mu_{A/I} : C_0(X) \to ZM\left({}^{A}\!/\!_{I}\right)$, $\mu_{A/I}(f)(a+I) = \mu_A(f)a + I$ for $f \in C_0(X)$ and $a \in A$. This is well defined since if $a, b \in A$ with $a - b \in I$ and $f \in C_0(X)$, then $\mu_A(f)(a-b) \in \mu_A(f)I \subseteq I$ by Proposition A.18. The formula defining $\mu_{A/I}$ is clearly a $*$-homomorphism.

Note that if (f_ν) is an approximate unit for $C_0(X)$, then non-degeneracy of μ_A tells us $\mu_A(f_\nu)a \to a$ for all $a \in A$. Then

$$\|(\mu_A(f_\nu)a - a) + I\| \leq \|\mu_A(f_\nu)a - a\| \to 0$$

and so $\mu_{A/I}(f_\nu)(a+I) \to a + I$. Non-degeneracy of $\mu_{A/I}$ follows. Hence ${}^{A}\!/\!_{I}$ is a $C_0(X)$-algebra.

Define, for each $x \in X$, the map

$$ {}^{\left({}^{A}\!/\!_{I}\right)}\!\Big/\!_{J_x\left({}^{A}\!/\!_{I}\right)} \to {}^{\left({}^{A}\!/\!_{J_xA}\right)}\!\Big/\!_{\left({}^{I}\!/\!_{J_xI}\right)}, \quad (a+I) + J_x\left({}^{A}\!/\!_{I}\right) \mapsto (a + J_xA) + {}^{I}\!/\!_{J_xI}. $$

We need to show this map is well defined. Note that here we view ${}^{I}\!/\!_{J_xI} \subseteq {}^{A}\!/\!_{J_xA}$, where the inclusion is induced by the inclusion of I into A (see Proposition A.18).

Suppose $(a+I) + J_x\left({}^{A}\!/\!_{I}\right) = (b+I) + J_x\left({}^{A}\!/\!_{I}\right)$, where $a, b \in A$. We want to show $(a-b) + J_xA \in {}^{I}\!/\!_{J_xI}$, which is the case if and only if there exists an $i \in I$ such that $(a-b) + J_xA = i + J_xA$. This is the case if and only if there exists $f \in J_x$, $c \in A$ such that $a - b - fc = i$.

Since $(a+I) + J_x\left({}^{A}\!/\!_{I}\right) = (b+I) + J_x\left({}^{A}\!/\!_{I}\right)$ then $(a-b) + I \in J_x\left({}^{A}\!/\!_{I}\right)$, which means there exists $f \in J_x$, $c \in A$ such that $(a-b) + I = f(c+I) = fc + I$. Then $a - b - fc \in I$, and we conclude the map is well defined by the above paragraph.

This map is clearly a surjective $*$-homomorphism, so it remains to show injectivity. Suppose $(a+I) + J_x\left({}^{A}\!/\!_{I}\right)$ (where $a \in A$) maps to the zero element under the above map. Then $a + J_xA \in {}^{I}\!/\!_{J_xI}$. Then there exists an $i \in I$ such that $a + J_xA = i + J_xA$, and therefore there also exists an $f \in J_x$, $b \in A$ such that $a - i = fb$.

We want to show that $(a+I) + J_x\left({}^{A}\!/\!_{I}\right) = 0$, i.e. $a + I \in J_x\left({}^{A}\!/\!_{I}\right)$. This is the case if and only if there exists $f \in J_x$, $b \in A$ such that $a + I = f(b+I) = fb + I$, which is the case if and only if there exists $f \in J_x$, $b \in A$ such that $a - fb \in I$. We have exhibited such an f and b in the paragraph above.

To complete the proof, we see that for $x \in X$,the quotient map ${}^{A}\!/_{I} \to \left({}^{A}\!/_{I}\right)_x \cong {}^{A_x}\!/_{I_x}$ is given by $a + I \mapsto \mathrm{ev}_x(a) + I_x$. □

Proposition A.20. *Let A be a $C_0(X)$-algebra, and let $p \in A$ be a projection. Then pAp is a $C_0(X)$-algebra with fibres $p(x)A_xp(x)$. In particular if A is continuous then pAp is continuous.*

Proof. Note that if $T \in ZM(A)$, then $T(pAp) \subseteq pAp$, and so T restricts to an element of $M(pAp)$.

We can then define a $*$-homomorphism $\mu_{pAp} : C_0(X) \to ZM(pAp)$, $f \mapsto \mu_A(f)|_{pAp}$.

For each x, restrict the evaluation ev_x to pAp. This is clearly a surjection onto $p(x)A_xp(x)$. We need to look at the kernel of the evalutation map to identify the fibres.

We have $\mathrm{Ker}(\mathrm{ev}_x|_{pAp}) = pAp \cap \mathrm{Ker}(\mathrm{ev}_x) = pAp \cap J_xA$. We will show $\mathrm{Ker}(\mathrm{ev}_x|_{pAp}) = J_xpAp = pJ_xAp$.

If $d \in pAp \cap J_xA$, then $d = fc = pap$ for some $f \in J_x$, $a \in A$ and $c \in A$. Then $pfcp = ppapp = pap = d$. Therefore $d \in pJ_xAp$.

If $d \in pJ_xAp$, then $d \in pAp$, and $d \in \mathrm{Ker}(\mathrm{ev}_x)$, so $d \in \mathrm{Ker}(\mathrm{ev}_x|_{pAp})$. Therefore we must have $\mathrm{Ker}(\mathrm{ev}_x|_{pAp}) = pJ_xAp$.

The continuity statement is then obvious. □

For the next result concerning subalgebras, we need to understand the irreducible representations of a $C_0(X)$-algebra. We will state the following well-known result without proof (see [89, Proposition C.5] for the statement).

Theorem A.21. *Let A be a $C_0(X)$-algebra. Then every irreducible representation of A is lifted from an irreducible representation of a fibre A_x for some $x \in X$. In particular,*

$$\mathrm{Spec}(A) = \bigsqcup_{x \in X} \mathrm{Spec}(A_x)$$

as sets.

Proposition A.22. *Let A be a $C_0(X)$-algebra, and let $I \subseteq A$ be an ideal. Suppose $I_x = A_x$ for all $x \in X$. Then $I = A$.*

Proof. The spectrum $\mathrm{Spec}(A)$ is as a set the disjoint union of $\mathrm{Spec}(A_x)$ by Theorem A.21. Therefore $\mathrm{Spec}(A) = \mathrm{Spec}(I)$. In particular, no irreducible representations of A vanish on I, and so I is not contained in a primitive ideal of A. Therefore we must have $I = A$. □

Let us conclude this section by considering an extension of Proposition A.22 for subalgebras of continuous fields due to Dixmier, [17, 11.5.3]. It should be viewed as an analogue of the Stone-Weierstrass theorem for continuous fields. It is based on a Stone-Weierstrass result for C^*-algebras due to Glimm, [28, Theorem 1]. We state it without proof.

Theorem A.23. *Let A be a continuous field over X. Suppose $B \subseteq A$ is a C^*-subalgebra, and for $x_1, x_2 \in X$ and $a_1 \in A_{x_1}, a_2 \in A_{x_2}$ (where we require $a_1 = a_2$ when $x_1 = x_2$) there exists $b \in B$ with $b(x_1) = a_1$, $b(x_2) = a_2$. Then $B = A$.*

A.3 K-Theory and Continuous Fields

The results in this section are folklore, and we provide our own proofs.

We can **restrict** $C_0(X)$-algebras to open or closed subspaces of X. That is, if A is a $C_0(X)$-algebra, and V is an open or closed subspace of X, then we will define $A|_V$, the restriction of A to V, which is a $C_0(V)$-algebra. First, we will motivate this definition by considering the most basic $C_0(X)$-algebra, $C_0(X)$. We would expect the restriction of $C_0(X)$ to V in either case to be $C_0(V)$.

Proposition A.24. *Let V be an open subset of X. Then $C_0(V) \subseteq C_0(X)$, by extending C_0-functions on V by zero on $X \setminus V$. Furthermore*

$$C_0(V) \cong \{f \in C_0(X) \mid f(x) = 0 \text{ for all } x \in X \setminus V\}.$$

Proof. Let $f \in C_0(V)$. We check that we can extend f from V to X by setting $f(x) = 0$ for $x \in X \setminus V$. The only potential points of discontinuity are on ∂V. Since V is open $\partial V \cap V = \emptyset$.

Let $\epsilon > 0$ and $x \in \partial V$. There exists a compact set $K \subseteq V$ such that for all $y \in V \setminus K$, $|f(y)| < \epsilon$. Since K is compact in V, K is compact in X, and in particular closed in X, but K does not contain x. Therefore there exists an open neighbourhood U_x of x such that $U_x \cap K = \emptyset$. Then for all $z \in U_x \cap V$,

$$|f(z) - f(x)| = |f(z)| < \epsilon$$

and so the extension to X is continuous.

For the stated isomorphism, we need to show that a function $f \in C_0(X)$ that vanishes for all $x \in X \setminus V$ vanishes at infinity in V. Let $\delta > 0$. Let us find a compact set $K \subseteq V$ such that for all $x \in V \setminus K$, $|f(x)| < \delta$. There exists a compact set $C \subseteq X$ such that for all $x \in X \setminus C$, $|f(x)| < \delta$. It may not be the case that C is contained in V, so we will show we can make C slightly smaller to ensure it is contained in V.

Since V is open, ∂V is disjoint from V, so f vanishes on ∂V. For each point $x \in \partial V$, there is an open neighbourhood U_x of x in X on which $|f(z)| < \delta$ for all $z \in U_x$. By taking the union of all these sets, we obtain an open neighbourhood U of the boundary of V on which $|f(x)| < \delta$. The complement of U, which we denote by W, is closed in X. Note that by construction $W \cap \partial V = \emptyset$.

By the definition of the subspace topology, $C' := C \cap \overline{V} \cap W$ is a closed subset of C because $\overline{V} \cap W$ is closed in X. Hence C' is compact. But C' is contained in V because $V \cup \partial V = \overline{V}$ since V is open, so $C' = C \cap \overline{V} \cap W = C \cap (V \cup \partial V) \cap W = C \cap V \cap W$. Therefore C' is a compact subset of V, and by construction, for $x \in V \setminus C'$, $|f(x)| < \delta$. □

Proposition A.24 suggests we could try to define the restriction of a $C_0(X)$-algebra to an open subset $V \subseteq X$ considering the subalgebra of elements of A that vanish (when viewed as sections) outside of V. The following proposition gives us a concrete way of describing the algebra of such sections.

Proposition A.25. *Let A be a $C_0(X)$-algebra, and suppose $V \subseteq X$ is open. Then*

$$\{a \in A \mid a(x) = 0 \textit{ for all } x \in X \setminus V\} = C_0(V)A \tag{A.2}$$

where $C_0(V) \subseteq C_0(X)$ as functions that vanish outside of V.

Proof. By Proposition A.24, $C_0(V)A$ is contained in the left hand side of (A.2). We will show that $C_0(V)A$ is dense in the left hand side of (A.2), from which the result follows. Note that for elements a in the left hand side of (A.2), the function $x \mapsto \|a(x)\|_{A_x}$ vanishes at infinity in V. This is by essentially the same argument given in the proof of Proposition A.24.

Let $\epsilon > 0$. Then there is a compact set $K \subseteq V$ such that $\|a(y)\|_{A_y} < \frac{\epsilon}{2}$ for all $x \in V \setminus K$. By Urysohn's Lemma, there exists a continuous function $f \in C_c(X)$ such that $0 \leq f \leq 1$, $f \equiv 1$ on K and $\mathrm{Supp}(f) \subseteq V$. In particular, $f \in C_0(V)$ by Proposition A.24 and by construction $\|a - fa\|_A < \epsilon$. The result follows. □

Now we consider the case of closed sets. Extension by zero no longer works. However we can view $C_0(V)$ as a quotient of $C_0(X)$.

Proposition A.26. *Let V be a closed set. The restriction homomorphism*

$$\mathrm{res} : C_0(X) \to C_0(V)$$

is surjective, has kernel

$$\mathrm{Ker}(\mathrm{res}) = C_0(X \setminus V)C_0(X)$$

and

$$C_0(V) \cong {}^{C_0(X)}\!\big/_{J_V C_0(X)}.$$

where $J_V \subseteq C_0(X)$ is the ideal of functions in $C_0(X)$ that vanish on V.

Proof. Let $f \in C_0(X)$. Note that since $V \subseteq X$ is closed, the inclusion map is proper, and so we obtain a surjective $*$-homomorphism $C_0(X) \to C_0(V)$ given by the restriction map. The kernel is

$$\mathrm{Ker}(\mathrm{res}) = \{f \in C_0(X) \mid f(x) = 0 \text{ for all } x \in V\} = C_0(X \setminus V)C_0(X) \cong J_V C_0(X)$$

by Proposition A.25 and Proposition A.24, and the final statement follows from the first isomorphism theorem. □

We are then prompted to make the following definition.

Definition A.27. Let A be a $C_0(X)$-algebra. Let V be a subspace of X.

(a) If V is open, define $A|_V := C_0(V)A$.

(b) If V is closed, define $A|_V := {}^{A}\!\big/_{J_V A}$, where J_V is the ideal of functions in $C_0(X)$ vanishing on V.

We can then see from our general results about quotients and ideals in Propositions A.18 and A.19 that the restriction of a $C_0(X)$-algebra A to an open or closed set is again a $C_0(X)$-algebra. For open subsets, we can easily see that $(A|_V)_x = A_x$, and for closed subsets, we note that

$$(J_V A)_x = 0$$

because if $x \in V$, all elements of J_V evaluate to zero. Therefore $(A|_V)_x = {}^{A_x}\!\big/_{0} \cong A_x$. Since the evaluation maps to each fibre are the usual ones, if A is continuous, then $A|_V$ is continuous.

Example A.28.

(a) Let D be a fixed C^*-algebra and let $A = C_0(X, D)$. Let V be an open or closed subspace of X. Then $A|_V = C_0(V, D)$. Indeed, if V is open, we have

$$A|_V = C_0(V)C_0(X, D) = C_0(V)C_0(X) \otimes D = C_0(V, D)$$

by Propositions A.25 and A.24. If V is closed and J_V denotes the ideal of functions in $C_0(X)$ vanishing on V, then

$$J_V C_0(X, D) = C_0(X \setminus V)C_0(X, D)$$

by Proposition A.24. The surjective restriction map $C_0(X, D) \to C_0(V, D)$ then has kernel $J_V C_0(X, D)$ by Proposition A.25, and so

$$A|_V := {}^{C_0(X, D)}\!\big/_{J_V C_0(X, D)} \cong C_0(V, D).$$

(b) Let $A := \{f \in C([0, 1], M_2(\mathbb{C})) \mid f(0) \text{ is diagonal}\}$. Then

$$A|_{(0,1]} = C_0((0, 1])A = C_0((0, 1], M_2(\mathbb{C})).$$

Part (b) of Example A.28 is special in the sense that we took a non-trivial field and upon restriction obtained a trivial field. In the above example, we say A is **trivial away from 0** in $[0, 1]$. The following proposition allows us to obtain maps in K-theory from such algebras.

Proposition A.29. *Let A be a $C([0, 1])$-algebra. There is a short exact sequence of C^*-algebras*

$$0 \longrightarrow A|_{(0,1]} \longrightarrow A \xrightarrow{\mathrm{ev}_0} A_0 \longrightarrow 0.$$

If $A_{(0,1]}$ is a trivial field, then there is an induced map in K-theory

$$K_*(A_0) \to K_*(A_t)$$

for $t > 0$.

Proof. For exactness at $A|_{(0,1]}$, note that this algebra may be viewed inside A because $(0, 1]$ is open in $[0, 1]$. We have $\mathrm{Ker}(\mathrm{ev}_0) = \{a \in A \mid a(0) = 0\} = A|_{(0,1]}$ by Proposition A.25, which is exactness at A. Surjectivity of ev_0 is immediate and so we have exactness at A_0.

This short exact sequence induces a 6 term exact sequence in K-theory

$$\begin{tikzcd} K_0(A|_{(0,1]}) \arrow[r] & K_0(A) \arrow[r] & K_0(A_0) \arrow[d] \\ K_1(A_0) \arrow[u] & K_1(A) \arrow[l] & K_1(A|_{(0,1]}). \arrow[l] \end{tikzcd}$$

Since $A_{(0,1]}$ is a trivial field, then for $t > 0$, $A_t \cong B$, a fixed C^*-algebra, and $A_{(0,1]} \cong C_0((0,1], B)$. This is homotopy equivalent to the zero C^*-algebra (see [87, Proposition 6.4.7 and Proposition 7.1.6]), which has vanishing K-theory. In the above 6 term sequence we obtain an isomorphism $K_*(A_0) \cong K_*(A)$. Following by the induced map ev_t for $t > 0$ we obtain a map $K_*(A_0) \to K_*(A_t)$. □

The induced map behaves as one might expect on trivial fields over $[0,1]$.

Proposition A.30. *Let D be a fixed C^*-algebra, and let $A = C([0,1], D)$. Then the map induced by A in K-theory, $K_*(D) \to K_*(D)$, is an isomorphism.*

Proof. The induced map in K-theory makes

$$\begin{tikzcd} & K_*(A) \arrow[dl, "\cong"', "ev_0"] \arrow[dr, "ev_1"] & \\ K_*(D) \arrow[rr] & & K_*(D) \end{tikzcd}$$

commute. It therefore suffices to show that the map induced by evaluation at 1 is an isomorphism. This is an isomorphism by exactly the same reason that the left arrow is an isomorphism - we use the six-term exact sequence on

$$0 \longrightarrow C_0([0,1), D) \longrightarrow A \xrightarrow{\mathrm{ev}_1} D \longrightarrow 0$$

and use the fact that $C_0([0,1), D) \cong C_0((0,1], D)$ is contractible. □

Let us conclude this section by giving a concrete description of the map provided by Proposition A.29. We start with the case of A a unital continuous $C([0,1])$-algebra such that $A|_{(0,1]}$ is trivial. Note that in particular in this case each fibre of A is unital.

Lemma A.31. *Let A be a unital continuous $C([0,1])$-algebra such that $A|_{(0,1]}$ is trivial. Let $n \in \mathbb{N}$ and $p \in M_n(A_0)$ be a non-zero projection. Then there exists a projection $q \in M_n(A)$ such that $\mathrm{ev}_0(q) = p$.*

Proof. We can find a positive element $a \in M_n(A)$ such that $\mathrm{ev}_0(a) = p$ because ev_0 is

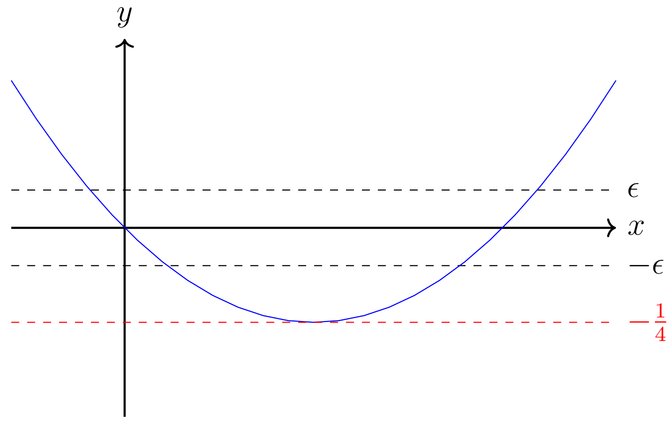

Figure A.1: The graph of $y = x^2 - x$.

surjective. We have that

$$[0,1] \to \mathbb{R}, \quad t \mapsto \left\|\mathrm{ev}_t(a^2 - a)\right\|_{A_t}$$

is continuous, and $\|\mathrm{ev}_0(a^2 - a)\|_{A_0} = \|p^2 - p\|_{A_0} = 0$. Therefore for $\epsilon > 0$, there exists a $\delta > 0$ such that if $t \leq \delta$,

$$\left\|\mathrm{ev}_t(a^2 - a)\right\|_{A_t} < \epsilon.$$

Now for each $t \in [0,1]$, since $\mathrm{ev}_t(a) \in M_n(A_t)$ is positive, it is normal, and so $C^*(\mathrm{ev}_t(a)) \subseteq M_n(A_t)$ is commutative. Therefore there exists a locally compact Hausdorff space X such that $C^*(\mathrm{ev}_t(a)) \cong C_0(X)$. If $\mathrm{ev}_t(a)$ is identified with $f \in C_0(X)$, then $\mathrm{Spec}(\mathrm{ev}_t(a)) = \mathrm{Spec}(f) = \mathrm{Im}(f) \cup \{0\}$. We have

$$\left\|\mathrm{ev}_t(a^2 - a)\right\|_{A_t} < \epsilon \iff \left\|f^2 - f\right\| < \epsilon \iff -\epsilon < f(x)^2 - f(x) < \epsilon \text{ for all } x \in X.$$

In particular, for all $s \in \mathrm{Spec}(\mathrm{ev}_t(a))$, $-\epsilon < s^2 - s < \epsilon$. If we choose ϵ sufficiently small (in fact, $\epsilon < \frac{1}{4}$), then we see that $\mathrm{Spec}(\mathrm{ev}_t(a)) \subseteq [0,1]$ has a gap which includes $s = \frac{1}{2}$ (see Figure A.1).

Therefore there exists $\delta > 0$ such that for all $t \leq \delta$, $\mathrm{Spec}(\mathrm{ev}_t(a))$ does not include $\frac{1}{2}$. For $t > \delta$, we can redefine a so that $\mathrm{ev}_t(a) = \mathrm{ev}_\delta(a)$. This still defines an element of $M_n(A)$ by the triviality of the field away from zero. In particular, we have that a is a lift of p and because $\mathrm{Spec}(a) = \cup_{t\in[0,1]} \mathrm{Spec}(\mathrm{ev}_t(a))$ by Proposition A.6, $\mathrm{Spec}(a)$ does not contain $\frac{1}{2}$. In particular, the function

$$\chi : \mathbb{R} \to \mathbb{C}, \quad t \mapsto \begin{cases} 0 & t < \frac{1}{2} \\ 1 & t > \frac{1}{2} \end{cases}$$

is continuous on $\mathrm{Spec}(a)$ and so we may define $q := \chi(a) \in M_n(A)$ by the continuous functional calculus. The result is a projection, and we have

$$\mathrm{ev}_0(q) = \mathrm{ev}_0(\chi(a)) = \chi(\mathrm{ev}_0(a)) = \chi(p) = p$$

because $\chi = \mathrm{id}$ on $\mathrm{Spec}(p) = \{0, 1\}$. $\square$

From Lemma A.31, the class of a projection $p \in M_n(A_0)$ in $K_0(A_0)$ can be lifted to the class of the projection $q \in M_n(A)$ with $\mathrm{ev}_0(q) = p$, and then under the map given by Proposition A.29, the class $[p]_0$ is mapped to $[\mathrm{ev}_1(q)]_0$. Note that by the definition of the map, the result is independent of the choice of lift q.

We can carry out an entirely similar argument for K_1 as follows.

Lemma A.32. *Let A be a unital continuous $C([0,1])$-algebra such that $A|_{(0,1]}$ is trivial. Let $n \in \mathbb{N}$ and $u \in M_n(A_0)$ be a unitary. Then there exists a unitary $v \in M_n(A)$ such that $\mathrm{ev}_0(v) = u$.*

Proof. The proof proceeds in much the same way as the proof of Lemma A.31. We will only highlight the main ideas.

We can find a lift $b \in M_n(A)$ of u. For $t \in [0, 1]$ small, we have

$$\|\mathrm{ev}_t(b^*b - 1)\|_{A_t} < 1, \quad \|\mathrm{ev}_t(bb^* - 1)\|_{A_t} < 1.$$

In particular, $\mathrm{ev}_t(b^*b)$ and $\mathrm{ev}_t(bb^*)$ are invertible for t small, and so $\mathrm{ev}_t(b)$ is invertible for t small. By redefining b constantly at some point away from 0, we can assume that $\mathrm{ev}_t(b)$ is invertible for all t, and hence we can assume b is invertible by Proposition A.6.

We now note that $v = b|b|^{-1} \in A$ is unitary (the polar decomposition of b is $b = v|b|$), and we have $\mathrm{ev}_0(v) = u$. $\square$

From Lemma A.32, the class of a unitary $u \in M_n(A_0)$ in $K_0(A_0)$ can be lifted to the class of the unitary $v \in M_n(A)$ with $\mathrm{ev}_0(v) = u$, and then under the map given by Proposition A.29, the class $[u]_1$ is mapped to $[\mathrm{ev}_1(v)]_1$. Note that by the definition of the map, the result is independent of the choice of lift v.

We finish by considering the non-unital case. We have the following result.

Lemma A.33. *Let A be a non-unital $C([0,1])$-algebra such that $A|_{(0,1]}$ is trivial. Then $A^+_{C([0,1])}\Big|_{(0,1]}$ is trivial, thus inducing a map in K-theory*

$$K_*(A_0^+) \to K_*(A_1^+).$$

The induced map $K_*(A_0) \to K_*(A_1)$ from the triviality of $A|_{(0,1]}$ is a direct summand of the above map $K_*(A_0^+) \to K_*(A_1^+)$.

Proof. For the first point, we note that

$$A^+_{C([0,1])}\Big|_{(0,1]} = C_0((0,1])(A + C([0,1])) = C_0((0,1])A + C_0((0,1]) \subseteq M(A \oplus C([0,1])).$$

Since $C_0((0,1])A \cong C_0((0,1], A_1)$, we have $A^+_{C([0,1])}\Big|_{(0,1]} \cong C_0((0,1], A_1^+)$, c.f. Example A.16.

For the second point, by Lemma A.14, we have the split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & A^+_{C([0,1])} \arrow[r] & C([0,1]) \arrow[r] \arrow[l, bend left, "\mu"] & 0 \end{tikzcd}$$

and so we obtain an isomorphism $K_*(A^+_{C([0,1])}) \cong K_*(A) \oplus K_*(C([0,1]))$. Now consider the evaluation maps

$$\mathrm{ev}_t^+ : A^+_{C([0,1])} \to A_t^+$$

for $t \in [0,1]$, which restrict to the usual evaluation maps on A and $C([0,1])$. We then have the following diagram

$$\begin{tikzcd} K_*(A_1^+) & \cong & K_*(A_1) & \oplus & K_*(\mathbb{C}) \\ K_*(A^+_{C([0,1])}) \arrow[u, "\mathrm{ev}_1^+"] \arrow[d, "\mathrm{ev}_0^+"'] & \cong & K_*(A) \arrow[u, "\mathrm{ev}_1"] \arrow[d, "\mathrm{ev}_0"'] & \oplus & K_*(C([0,1])) \arrow[u, "\mathrm{ev}_1"] \arrow[d, "\mathrm{ev}_0"'] \\ K_*(A_0^+) & \cong & K_*(A_0) & \oplus & K_*(\mathbb{C}). \end{tikzcd}$$

The first column describes the map $K_*(A_0^+) \to K_*(A_1^+)$, and the second describes the map $K_*(A_0) \to K_*(A_1)$, and the result follows. $\square$

A.4 Tensor Products, Group Actions and Crossed Products

In this section we collect together some general results about constructing new $C_0(X)$-algebras from old, by means of taking tensor products and letting groups act on $C_0(X)$-algebras.

We start with tensor products. This theorem tells us we can take a $C_0(X)$-algebra and take the tensor product with a fixed nuclear C^*-algebra and obtain a new $C_0(X)$-algebra. Parts (a) and (b) are originally in [42, Section 2.2], and parts (c) and (d) are straightforward extensions.

Theorem A.34. *Let A be a $C_0(X)$-algebra and D be a fixed nuclear C^*-algebra. Then*

(a) $A \otimes D$ is a $C_0(X)$-algebra with fibre $A_x \otimes D$ at $x \in X$.

(b) If A is continuous, then $A \otimes D$ is continuous.

(c) If $V \subseteq X$ is open then $(A \otimes D)|_V \cong A|_V \otimes D$.

(d) If $V \subseteq X$ is open and $A|_V$ is a trivial field, then $(A \otimes D)|_V$ is a trivial field.

Proof.

(a) For the proof of part (a), we adapt [42, Lemma 2.4 and Lemma 2.5].

Let $f \in C_0(X)$. Then $\mu_A(f) \in ZM(A)$. Then

$$\mu_A(f) \otimes \mathrm{id} : A \otimes D \to A \otimes D$$

is a central multiplier of $A \otimes D$, and $\mu_{A\otimes D} : C_0(X) \to ZM(A \otimes D)$, $f \mapsto \mu_A(f) \otimes \mathrm{id}$ is a non-degenerate $*$-homomorphism.

Consider, for each $x \in X$, the short exact sequence

$$0 \longrightarrow J_x A \longrightarrow A \longrightarrow A_x \longrightarrow 0.$$

Since D is nuclear, minimal tensor products are the same as maximal ones. Therefore we have an exact sequence

$$0 \longrightarrow (J_x A) \otimes D \longrightarrow A \otimes D \longrightarrow A_x \otimes D \longrightarrow 0,$$

see [3, II.9.6.6]. Note that $\mu_{A\otimes D}(J_x)(A \otimes D) = (J_x A) \otimes D$. Therefore $(A \otimes D)_x = A_x \otimes D$, and the evaluation map is $\mathrm{ev}_x \otimes \mathrm{id}$.

(b) We assume that A is continuous. Upper semicontinuity of $A \otimes D$ is immediate from Proposition A.7. For lower semicontinuity, we use Lemma A.9. Let $X' \subseteq X$ be closed and Y be dense in X'. Then by Lemma A.9 the family of $*$-homomorphisms $\{p_x\}_{x\in Y}$ as defined just before the Lemma is a faithful family on $\mathrm{ev}_{X'}(A)$. Then taking tensor products we see that $\{p_x \otimes \mathrm{id}\}_{x\in Y}$ is a faithful family on $\mathrm{ev}_{X'}(A) \otimes D = (\mathrm{ev}_{X'} \otimes \mathrm{id})(A \otimes D)$. Then applying Lemma A.9 again we see that $A \otimes D$ is continuous.

(c) We have

$$(A \otimes D)|_V = C_0(V)(A \otimes D) = (C_0(V)A) \otimes D = A|_V \otimes D$$

as required.

(d) This follows immediately from part (c). $\square$

We will be interested in constructing $C_0(X)$-algebras when groups act on them. Let us first introduce some terminology which first appears in [73, Definition 3.1].

Definition A.35. Let G be a locally compact group, and A be a $C_0(X)$-algebra. Suppose $\alpha : G \to \mathrm{Aut}(A)$ is an action of G on A, and that

$$\alpha_g(J_x A) \subseteq J_x A$$

for all $g \in G$ and $x \in X$. Then α is said to be a **field of actions of G on A**.

Remark A.36. If A is a $C_0(X)$-algebra and G is a locally compact group that acts on A as a field of actions, then α induces an action of G on each fibre of A, justifying the terminology. Indeed, if $x \in X$, then we define $\alpha^x : G \to \mathrm{Aut}(A_x)$ by

$$\alpha_g^x(a + J_x A) := \alpha_g(a) + J_x A$$

for $a \in A$ and $g \in G$. This is well-defined because if $a + J_x A = b + J_x A$ for $a, b \in A$, then $a - b \in J_x A$ and so $\alpha_g(a - b) \in \alpha_g(J_x A) \subseteq J_x A$. Note that the definition above can be rewritten as

$$\mathrm{ev}_x(\alpha_g(a)) = \alpha_g^x(\mathrm{ev}_x(a)).$$

For finite groups, we have the following general result. This is a folklore result but we note that a generalization to compact groups is present in [15, Lemma 1.7].

Proposition A.37. *Let G be a finite group, and A be a $C_0(X)$-algebra on which G acts by a field of actions α.*

(a) The fixed point algebra A^G is a $C_0(X)$-algebra, with fibre $(A^G)_x = A_x^G$ at $x \in X$.

(b) If A is continuous, then A^G is a continuous field.

(c) If $V \subseteq X$ is open, then $(A|_V)^G \cong (A^G)\big|_V$.

(d) If $V \subseteq X$ is open and $A|_V$ is a trivial field, then $(A^G)\big|_V$ is a trivial field.

Proof.

(a) If $a \in A$, $f \in C_0(X)$, and $g \in G$, we have, for all $x \in X$,

$$\begin{aligned} \mathrm{ev}_x(\alpha_g(fa)) &= \alpha_g^x(\mathrm{ev}_x(fa)) \\ &= \alpha_g^x(f(x)a(x)) \\ &= f(x)\alpha_g^x(a(x)) \\ &= f(x)\,\mathrm{ev}_x(\alpha_g(a)) \\ &= \mathrm{ev}_x(f\alpha_g(a)) \end{aligned}$$

and so $\alpha_g(fa) = f\alpha_g(a)$. Therefore if $a \in A^G$, then $fa \in A^G$. Therefore μ_A restricts to a $C_0(X)$-action on A^G.

Clearly the evaluation map to the fibre at $x \in X$ is given by the restriction of ev_x to A^G. The image is contained in A_x^G by Remark A.36. To see that $\mathrm{ev}_x(A^G) = A_x^G$, we use an averaging argument. Define for $a \in A$ the averaging map

$$\mathrm{av}(a) := \frac{1}{|G|}\sum_{g\in G}\alpha_g(a) \in A^G.$$

If $v \in A_x^G$, then there exists $a \in A$ such that $\mathrm{ev}_x(a) = v$. Then

$$\mathrm{ev}_x(\mathrm{av}(a)) = \frac{1}{|G|}\sum_{g\in G}\mathrm{ev}_x(\alpha_g(a)) = \frac{1}{|G|}\sum_{g\in G}\alpha_g^x(\mathrm{ev}_x(a)) = \frac{1}{|G|}\sum_{g\in G} v = v$$

and so $(A^G)_x = A_x^G$.

(b) This is immediate because $A^G \subseteq A$.

(c) Note that G acts on the restriction $A|_V$ by the calculation in part (a). We have

$$A^G\big|_V = C_0(V)A^G = (C_0(V)A)^G = (A|_V)^G$$

again by the calculation in part (a), as required.

(d) Finally, if $A|_V$ is trivial with constant fibre D, then by part (c), $(A^G)\big|_V = C_0(V, D^G)$, whence $(A^G)\big|_V$ is trivial.

□

Finally we turn to crossed products. Results about crossed products first appear in [73, Section 3]. In [42, p.g. 682], Kirchberg and Wassermann indicate that there is the following result (without proof). One can find a complete statement and proof in [89, Corollary 8.6].

Theorem A.38. *Let A be a $C_0(X)$-algebra and let G be an locally compact group. Suppose*

G acts on A by a field of actions. Then

(a) $G \ltimes_\alpha A$ is a $C_0(X)$-algebra with fibres $G \ltimes_{\alpha^x} A_x$.

(b) If A is continuous and G is amenable, then $G \ltimes_\alpha A$ is continuous.

(b) If $V \subseteq X$ is open, then $(G \ltimes_\alpha A)|_V \cong G \ltimes_\alpha A|_V$.

(c) If $V \subseteq X$ is open and $A|_V$ is a trivial field, then $(G \ltimes_\alpha A)|_V$ is a trivial field.

Proof. For parts (a) and (b) we refer the reader to [89, Corollary 8.6]. However, we should note the $C_0(X)$-action on $G \ltimes_\alpha A$ is given by

$$\mu_{G\ltimes_\alpha A} : C_0(X) \to ZM(G \ltimes_\alpha A), \quad (\mu_{G\ltimes_\alpha A}(f)g)(s) = \mu_A(f)g(s)$$

for $f, g \in C_c(G, A) \subseteq G \ltimes_\alpha A$ and $s \in G$.

(c) We have

$$(G \ltimes_\alpha A)|_V = C_0(V)(G \ltimes_\alpha A) = G \ltimes_\alpha C_0(V)A = G \ltimes_\alpha A|_V$$

as required. Note that in the second equality we use [89, Lemma 3.17].

(d) This follows immediately from part (c). □

A.5 Weights and Continuous Fields

In this final section, we consider $C_0(X)$-algebras with weights on each fibre and give a sufficient condition for continuity of the $C_0(X)$-algebra which is described in terms of the weights. We start with a brief recap of weight theory on C^*-algebras. Weights were originally introduced by Combes in [10]. There are a few sources in the literature that describe the theory of weights - see for example [3] and [63] - and we will state, without proof, results from these sources.

Definition A.39. Let A be a C^*-algebra. A **weight** on a C^*-algebra is a map $\phi : A_+ \to [0, \infty]$ satisfying the following conditions.

1. $\phi(\lambda a) = \lambda\phi(a)$ for all $a \in A_+$ and $\lambda \in \mathbb{R}_{>0}$.

2. $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in A_+$.

Remark A.40. For a weight ϕ on a C^*-algebra A, we have $\phi(0) = 0$. This is because $\phi(0) = \phi(0 + 0) = \phi(0) + \phi(0)$, using the second condition of Definition A.39.

The prototypical example to bear in mind is the case where $A = L^\infty(\mathbb{R})$. Then the Lebesgue integral $\int : L^\infty(\mathbb{R})_+ \to [0,\infty]$ is a weight. To a given weight, there are special associated sets which we now define. The terminology is motivated from this key example.

Definition A.41. For a weight ϕ on a C^*-algebra A, we define the following sets.

(a) $\mathcal{M}_\phi^+ := \{a \in A_+ \mid \phi(a) < \infty\}$, the positive ϕ-integrable elements of A.

(b) $\mathcal{N}_\phi := \{a \in A \mid \phi(a^*a) < \infty\}$, the ϕ-square-integrable elements of A.

(c) $\mathcal{M}_\phi := \operatorname{span} \mathcal{M}_\phi^+$, the ϕ-integrable elements of A.

Proposition A.42. *Let ϕ be a weight on a C^*-algebra A. If $a \in A$, and $b \in \mathcal{N}_\phi$, we have*

$$\phi((ab)^*ab) \leq \|a\|^2 \, \phi(b^*b).$$

It follows from Proposition A.42 that $\mathcal{N}_\phi$ is a left ideal of A.

Proposition A.43. *Let ϕ be a weight on a C^*-algebra A. Then we have the following.*

(a) $\mathcal{M}_\phi$ is a $*$-subalgebra of A and $\mathcal{M}_\phi = \operatorname{span} \mathcal{N}_\phi^* \mathcal{N}_\phi$.

(b) There is a unique, positive, hermitian functional $\mathcal{M}_\phi \to \mathbb{C}$ that extends ϕ on $\mathcal{M}_\phi^+$, and we also call this extension ϕ. This extension satisfies the Cauchy-Schwarz inequality

$$|\phi(a^*b)|^2 \leq \phi(a^*a)\phi(b^*b), \quad a, b \in \mathcal{N}_\phi.$$

Definition A.44. Let A be a C^*-algebra, and let $\phi : A_+ \to [0,\infty]$ be a weight on A. The weight is said to be **faithful** if for $a \in A$, $\phi(a^*a) = 0$ implies $a = 0$.

We also have a GNS construction for weights.

Theorem A.45. *Let A be a C^*-algebra, and let $\phi : A_+ \to [0,\infty]$ be a weight on A. Then there is a $*$-representation π_ϕ of A on a Hilbert space H_ϕ, and a linear map $\Lambda_\phi : \mathcal{N}_\phi \to H_\phi$ such that*

$$\phi(a^*b) = \langle \Lambda_\phi(a), \Lambda_\phi(b) \rangle_{H_\phi}$$

for $a, b \in \mathcal{N}_\phi$ and such that $\pi_\phi(a)\Lambda_\phi(b) = \Lambda_\phi(ab)$ for $a \in A$ and $b \in \mathcal{N}_\phi$. The Hilbert space H_ϕ is the completion of the quotient inner product space

$$\mathcal{H}_\phi := \mathcal{N}_\phi / K_\phi,$$

where $K_\phi := \{a \in \mathcal{N}_\phi \mid \phi(a^*a) = 0\}$. If, in addition, ϕ is faithful, then π_ϕ is a faithful representation.

We will now consider $C_0(X)$-algebras where each fibre has a weight. We will need to restrict attention to elements of these algebras which have desireable properties with respect to the weight. This leads one to the following definitions.

Definition A.46. Let A be a $C_0(X)$-algebra with fibres $\{A_x\}_{x\in X}$. Suppose each fibre carries a weight ϕ_x.

(a) We call a section $a \in A$ **integrable** if for each $x \in X$, $a(x) \in \mathcal{M}_{\phi_x}$. We denote by $A_{\mathcal{M}}$ the set of integrable sections in A.

(b) We call a section $a \in A$ **square integrable** if for each $x \in X$, $a(x) \in \mathcal{N}_{\phi_x}$. We denote by $A_{\mathcal{N}}$ the set of square integrable sections in A.

Note that since $\mathcal{M}_{\phi_x}$ is a $*$-subalgebra of A_x by Proposition A.43, and $\mathcal{N}_{\phi_x}$ is a left ideal of A_x by Proposition A.42, then $A_{\mathcal{M}}$ is a $*$-subalgebra of A and $A_{\mathcal{N}}$ is a left ideal of A.

We now conclude with the result concerning continuity.

Proposition A.47. *Let A be a $C_0(X)$-algebra with fibres $\{A_x\}_{x\in X}$. Suppose each fibre carries a weight ϕ_x, and that the following conditions are satisfied.*

1. *For each $x \in X$, the GNS representation $\pi_{\phi_x} : A \to B(H_{\phi_x})$ associated to the weight ϕ_x (see Theorem A.45) is injective.*
2. *For each $a \in A_{\mathcal{M}}$, the function $X \to \mathbb{C}$, $x \mapsto \phi_x(a(x))$ is continuous.*
3. *$A_{\mathcal{M}}$ is dense in A.*
4. *For each $x \in X$, the space $\{a(x) \mid a \in \mathcal{N}_\phi\}$ is dense in $\mathcal{N}_{\phi_x}$ in the Hilbert space norm.*

Then $X \to \mathbb{R}$, $x \mapsto \|a(x)\|$ is lower semi-continuous (and hence continuous) for each $a \in A$.

Proof. Fix $x \in X$, and let $a \in A_{\mathcal{M}}$. Note that it is enough to prove continuity of the norm map on $A_{\mathcal{M}}$ by the third condition and an $\frac{\epsilon}{3}$-argument. Since π_{ϕ_x} is injective, it is isometric, and therefore

$$\begin{aligned}\|a(x)\|_{A_x} &= \|\pi_{\phi_x}(a(x))\| \\ &= \sup\{|\langle h_x, \pi_{\phi_x}(a(x))k_x\rangle| \ h_x, k_x \in H_{\phi_x}, \|h_x\| \le 1, \|k_x\| \le 1\}.\end{aligned}$$

Let $\epsilon > 0$. Approximating this supremum we find $h_x, k_x \in H_{\phi_x}$ of norm at most 1 such that

$$\|a(x)\|_{A_x} - \epsilon \le |\langle h_x, \pi_{\phi_x}(a(x))k_x\rangle|. \tag{1}$$

By the GNS construction in Theorem A.45, there exists $b_x + K_{\phi_x}, c_x + K_{\phi_x} \in \mathcal{H}_{\phi_x}$ that approximate h_x, k_x respectively. By an appropriate choice of $b_x, c_x \in \mathcal{N}_{\phi_x}$, we can assume

$$\left| \, |\langle h_x, \pi_{\phi_x}(a(x))k_x\rangle| - |\langle b_x + K_{\phi_x}, \pi_{\phi_x}(a(x))(c_x + K_{\phi_x})\rangle| \, \right| < \epsilon$$

by continuity of the inner product. In particular,

$$|\langle h_x, \pi_{\phi_x}(a(x))k_x\rangle| < |\langle b_x + K_{\phi_x}, \pi_{\phi_x}(a(x))(c_x + K_{\phi_x})\rangle| + \epsilon. \tag{2}$$

So by (1) and (2),

$$\|a(x)\|_{A_x} < |\langle b_x + K_{\phi_x}, \pi_{\phi_x}(a(x))(c_x + K_{\phi_x})\rangle| + 2\epsilon = |\phi(b_x^* a(x)c_x)| + 2\epsilon \tag{3}$$

We can also use the fourth assumption to assume that b_x, c_x are evaluations of some square integrable sections - i.e. there are square integrable sections $b, c \in A$ which have evaluations $b(x) = b_x$ and $c(x) = c_x$ at x.

By the second assumption, there exists an open neighbourhood U of x such that for all $y \in U$,

$$\phi_y(b(y)^*b(y)) < 1+\epsilon, \quad \phi_y(c(y)^*c(y)) < 1+\epsilon, \quad |\phi_x(b(x)^*a(x)c(x)) - \phi_y(b(y)^*a(y)c(y))| < \epsilon,$$

where we have used the fact $A_{\mathcal{N}}$ is a left ideal of A to make sense of the third inequality here. The third inequality implies $|\phi_x(b(x)^*a(x)c(x))| < |\phi_y(b(y)^*a(y)c(y))| + \epsilon$. Using this in (3) gives, for $y \in U$,

$$\|a(x)\|_{A_x} < |\phi_y(b(y)^*a(y)c(y))| + 3\epsilon.$$

Now note that, using Cauchy-Schwarz and Proposition A.42,

$$\begin{aligned} |\phi_y(b(y)^*a(y)c(y))|^2 &\leq \phi_y(b(y)^*b(y))\phi_y((a(y)c(y))^*a(y)c(y)) \\ &< \|a(y)\|_{A_y}^2 \phi_y(b(y)^*b(y))\phi_y(c(y)^*c(y)) \\ &< (1+\epsilon)^2 \|a(y)\|_{A_y}^2 . \end{aligned}$$

Putting all this together, and using the fact the quotient map $A \to A_y$ is norm decreasing, we get the estimate

$$\|a(x)\|_{A_x} < \|a(y)\|_{A_y}(1+\epsilon) + 3\epsilon < \|a(y)\|_{A_y} + \epsilon(\|a(y)\|_{A_y} + 3) < \|a(y)\|_{A_y} + \epsilon(\|a\| + 3).$$

for $y \in U$ and lower semicontinuity follows. $\square$

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