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A study of Scottish teachers' beliefs about the interplay of problem solving and problem posing in mathematics education

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BSc PGCE MSc MIMA CMath

Thesis submitted in fulfilment of the degree of Doctor of Philosophy

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June 2017

Abstract

The Scottish Curriculum for Excellence (CfE) advocates that the learning and teaching of mathematical problem solving is no longer compartmentalised but is an overarching feature designed to improve higher order thinking skills at all levels by focusing on conceptual understanding. Comitantly, a growing body of literature acknowledges the interrelated educational benefits of mathematical problem posing within classrooms. Teachers' beliefs are considered powerful indicators of professional practice and can articulate the positionality of teachers with regards to curricula reform. Despite their significance, research into the implementation of mathematical problem solving and mathematical problem posing is, as yet, under-researched particularly in Scotland. The main purpose of this study was to investigate Scottish teachers' beliefs and espoused instructional practices of mathematical problem solving and mathematical problem posing. More prosaically, it explored beliefs regarding the nature of mathematics, the learning of mathematics and the teaching of mathematics.

A mixed methods explanatory design consisting of an online questionnaire followed by semi-structured interviews was selected as the instruments to measure and capture espoused beliefs and reported practices. This study involved a representative sample of 478 participants (229 primary and 249 secondary mathematics practitioners respectively) generated from 21 local education authorities in Scotland. A supplementary feature of the online questionnaire, which harvested 87 volunteered comments, augmented the data collection process. Descriptive and inferential statistics were employed to analyse quantitative data with thematic coding used to organise and interrogate qualitative data.

Factor analysis identified three distinct belief systems consistent with a dominant learner-centred approach (i.e. social constructivist, problem solving and collaborative orientation), mainly learner-centred approach (i.e. social constructivist, problem solving and static transmission orientation) and dominant teacher-centred approach (i.e. static and mechanistic transmission orientation). In other words, teachers' deep-rooted beliefs do not align to one particular group of belief systems but are embedded mutually within a cluster. A mixture of positive, negative and inconsistent beliefs is reported. Significant dissonance exists between the sectors. Characteristics impacting on beliefs include grade and highest level of qualification in the field of education.

This study suggests that the conceptualisation and operationalisation of mathematical problem solving and problem posing may be circumscribed in practice and that primary teachers hold stronger mathematical beliefs than secondary mathematics teachers. Several reasons help to illuminate these findings including a lack of pedagogical content knowledge, ineffective manifestations of mathematical creativity, low mathematics teaching self-efficacy and an over dominant national assessment culture. Implications and recommendations for policy and ITE are discussed.

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Acknowledgement

During the completion of this thesis, I received enormous support and encouragement from several inspirational people. Without their valuable contribution, I would have been unable to complete this work.

Firstly, I will be forever indebted to my main supervisor, Dr. Catherine Fagan for her tireless guidance and help throughout this long-standing process. Similarly, I am equally thankful of Dr. Fiona Patrick for her accompanying role in supporting me. As a partnership, they in turn furnished me with the confidence and motivation to explore various new ideas for suggested reading and to examine critically different methods of thinking. I am extremely appreciative for their friendship, enthusiasm, knowledge, humour, empathy and honesty to listen attentively to all my thoughts and ramblings.

Secondly, to my colleagues at St Leonards School; my sincere thanks to Dawn Pemberton-Hislop, for granting me time off to attend meetings, conferences and training courses. Similarly, I am particularly grateful to Graeme, Jon, Kristina, Louise, David and Ken, who skilfully assisted with the development of the pilot questionnaire and for debating with me over the years, multiple issues associated with mathematics education.

Thirdly, part-time study is an incredibly lonely existence and I would like to extend a special thanks to my friends and fellow students, for encouraging me to stay on course during many challenging periods.

Finally, the most important component in this entire process has been my family. Over the last five years, I have invested many days locked away in my study, missing out on countless gatherings, events and duties. As a consequence, I have underperformed in my role as a father and husband. To my two loving children, Calum and Mhairi; I sincerely thank you so much for your kind understanding and prolonged tolerance of me – Tha gaol agam oirbh. To my bonnie lass Fiona – I will luve thee still, my dear, till a' the seas gang dry. Your everlasting patience and unremitting thoughtfulness throughout this experience has been immeasurable. Without you at my side, this would not have been possible – Bidh gaol agam ort gu bràth.

Authors' declaration

I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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Abbreviations

ACME	Advisory Committee on Mathematics Education
BEd	Batchelor of Education
CfE	Curriculum for Excellence
CPD	Continuous Professional Development
DHT	Deputy Headteacher
HT	Headteacher
GTCS	General Teaching Council for Scotland
IMO	International Mathematical Olympiad
ITE	Initial Teacher Education
LEA	Local Education Authority
LI	Learning Intention
NCTM	National Council of Teachers of Mathematics
P1	First Year of Primary Education
P4	Fourth Year of Primary Education
P7	Seventh Year of Primary Education
PCA	Principal Component Analysis
PGCE	Post Graduate Certificate in Education
PGDE	Professional Graduate Diploma in Education
PISA	Programme for International Student Assessment
PT	Principal Teacher
MProSE	Mathematics Problem Solving for Everyone
OECD	Organization for Economic Co-operation and Development
S2	Second Year of Secondary Education
SC	Success Criteria
SCQF	Scottish Credit and Qualifications Framework
SERA	Scottish Educational Research Association
SPSS	Statistical Package for the Social Sciences
SQA	Scottish Qualifications Authority
SSLN	Scottish Survey of Literacy and Numeracy
TIS	Teacher Induction Scheme
TMBS	Total Mathematics Beliefs Score
TIMMS	Trends in International Mathematics and Science Study
UKMT	United Kingdom Mathematics Trust

CHAPTER ONE

Introduction

In this preliminary chapter, I introduce a background to the study, comprising together my own personal interest in the subject field before moving on to the educational and theoretical justification for the work. This is followed by the presentation of research questions and an overview of the thesis structure.

1.1 A personal journey

The content and arguments submitted in this thesis matured as a result of widespread educational experience gained as a mathematics teacher. Over the years, I have practised within the state and independent sector, obtained promotion to principal teacher, and prepared pupils for a range of Scottish, English and International Baccalaureate qualifications. Much of my time has been consumed thinking about the locus of problem solving and problem posing within the domain of mathematics education. I have conjectured if the policymakers of Curriculum for Excellence have created a paragon of autonomous education that empowers practitioners to flourish at the ultimate didactic level and without undue political interference. My thoughts often deliberate upon the position of Scotland and our teachers from an international perspective. Concomitantly, I have shared classroom experiences and beliefs of problem solving through the production of various publications (e.g. McDonald, 2006, 2013, 2014).

A decision to enter the teaching profession arose after a lengthy period of working in industry within Scotland, other parts of the UK and Europe. Coupled with engineering knowledge acquired from previous employers, I have been able to contextualise mathematical concepts and appreciate the importance of possessing analytical skills, deductive reasoning and logic, which I have applied to enhance learning within the classroom.

Shortly after completing a Master's degree, I was awarded Chartered Mathematician status which acted as a catalyst for my own professional development. Inspired to improve my pedagogical knowledge, I surveyed numerous empirical studies involving teachers' mathematical beliefs of problem solving and problem posing. The impetus for engaging in such enquiry was to address diverse and often enigmatic pedagogical viewpoints expressed by both primary and secondary colleagues during multiple professional collaborations. Curiously, encompassed within the nucleus of teachers, included an eclectic fusion of grades, age and experience from both sectors. In short, it concerned me that dissonance existed with approaches to the learning and teaching of mathematics. Furthermore, I was surprised to discover a paucity of empirical research investigating Scottish teachers compared to other countries. With this in mind and after careful deliberation, I enlisted as a part-time PhD candidate whilst continuing my career as a full-time practitioner, for three reasons. Firstly, in order to fulfil an intellectual challenge; secondly, to critically inform my professional practice; thirdly, to accomplish a long term ambition to become an independent researcher. This has been a decision which has resulted in a doctoral experience awash with feelings of loneliness, isolation, anxiety, frustration and immense satisfaction.

The role of teachers as researchers is nothing new (e.g. Elliot, 1991; McNiff & Whitehead, 2011; Kincheloe, 2012). By connecting to a research learning community, my previously held entrenched views and fragmented philosophies on fundamental constructs have been transformed, empowering me to operationalise a wider range of educational perspectives. I have explored the interface between research and practice, experiencing and augmenting contemporary dimensions to my teaching. In a paper which addresses the synthesis of theory and practice, Beattie (1997) writes:

> It seems a little incongruous to suggest that teachers who have not experienced inquiry in their own lives will be able to create classroom settings which encourage students to question, to pose and solve problems, and to be self-directed learners (p. 114).

Much time and energy has been engaged in reading, thinking, debating, analysing, discussing, arguing, evaluating, writing, assessing and reflecting on learning and teaching of mathematical problem solving and problem posing. Audiences have included pupils, colleagues, probationers, students, parents, academics, strangers, family and friends. My reason for undertaking this study rests on my professional interactions with a range of learners with an assortment of abilities who reported feeling disconnected and ill-equipped in problem posing and problem solving. These feelings of disconnection and lack of skills and knowledge often influenced their choices of employment and higher education.

Monitoring the mathematical progress of my own children, as they shuffled through primary and secondary levels, furnished me with an additional insight into other teachers' espoused views and enacted practices. Significantly, this underpinned my speculative assessment that the implementation of mathematical problem solving and problem posing may be influenced by individual teachers' beliefs.

1.2 Background to the research

Throughout my teaching career, irrespective of curricula requirements, I have exercised the freedom to engage and motivate young people with a repertoire of classic and unfamiliar mathematical problems. Regardless of ability, problems can be differentiated or reformulated to address the mathematical needs of all learners. I have found that mentoring pupils for individual and team competitions has provided me with a powerful mechanism to enrich thinking, inspire creativity and reinforce the mastery of deep conceptual understanding. Various echelons of ingenuity and complexity are packed into problems, demanding increasing levels of mathematical sophistication, and act as a breeding ground for developing flexible, strategic and independent thinking.

It is in sharp contrast to the common emphasis on procedural skills that is encapsulated within memorised driven algorithmic approaches, rooted in traditional mathematics lessons. In my professional experience, the delivery of problem solving and problem posing is not a dichotomy shared by all practitioners. I consider the beliefs of the teacher instrumental to the embodiment of such processes. Hersh (1986, p. 13) argues that "one's conception of what mathematics is affects one's conception of how it should be presented and one's manner of presenting it is an indication of what ones believes to be the most essential in it". I have theorised that the positioning of Scottish teachers appear to be inconsistent with the enactment of policy objectives and research literature available to them. Likewise, probing friends and colleagues to elicit the reasons behind their incongruence to mathematical problem solving and problem posing produced unexpected results. It is on this basis that my enthusiasm regarding the mathematical beliefs of teachers has grown, which energised me to suspect that this notable topic needed further exploration.

1.3 Educational justification for my research

All of this research has been conducted using current teachers within the domain of state schools in Scotland. It is necessary for me to contextualise my research prior to presenting the rational of my study.

The Scottish perspective

Historically, Scotland has always enjoyed an independent educational system and has never been part of what is erroneously but commonly referred to as the 'UK Curriculum'. For example, Boaler (2015a, p. 1), unfittingly used this designate (since it does not exist) when implying that the UK education secretary, a position with a solitary remit for England, has the additional authority to amend the separate education systems which are controlled by the devolved governments of Scotland, Wales and Northern Ireland respectively. In fact, Humes & Bryce (2013, p. 138) point out that "Education has traditionally been identified as one of the three institutions which mark the social and cultural life of Scotland as distinctive, especially when compared to England". Likewise, Menter (2014) asserts:

> We know that Scotland has had a very distinctive education system throughout the history of the Union - that is since 1707. And today we can see much that is strong, innovative and imaginative within Scottish education, especially when contrasted with England (p. 29).

Paradoxically, participants from Scotland, England, Wales and Northern Ireland are integrated in the UK sample for PISA, an international survey that measures young peoples' performance in reading, mathematics and science. Critical proponents of Scottish pupils' mathematical performances (Table 1.1) in such global assessments may wish to accentuate a recent decline of national standards, although this has to be set against limitations of the survey such as issues involving culture, methodology, political influence and interference. Though, the scale of underachievement cannot be overlooked as indicated by inspection reports carried out by Education Scotland. According to OECD (2015, p. 10): "Trends since 2003 in Scotland show a growing proportion of low achievers in maths and a shrinking proportion of high achievers".

Table 1.1 Scottish PISA mathematics performances

Year	2000	2003	2006	2009	2012	2015
Mathematics	533	524	506	500	498	491

Moreover, in a brief analysis of pupils achieving selection to represent the UK team at International Mathematical Olympiads, McDonald (2013) highlights the prolonged conspicuous absenteeism of a Caledonian presence. This prestigious competition demands independent higher order thinking together with formidable problem solving skills. Is it conjectured by me that a decline in the implementation of mathematical problem solving in schools may help to illuminate this phenomenon.

A solitary examination board exists to administer the nation's qualifications, ensuring fairness that all candidates undertake identical external assessments. Unlike other countries, teachers' registration and professional standards are regulated by a teaching council uncontrolled by government. Not only is the GTCS one of the oldest in the world, it is the first independent self-regulating body for teaching (Humes & Bryce, 2013).

1.3.1 Curriculum for Excellence

During the last half century, Scottish teachers have been on the receiving end of an avalanche of curriculum reports and policy reforms but nothing as radical as CfE. Implemented in 2010, this new curriculum has been heralded by its architects, Scottish Government (2008, p. 8), as "one of the most ambitious programmes of education change ever undertaken in Scotland". Cremin & Arthur (2014) claim that it has the potential to engage teachers and to capitalise on the good practice and emotional investment that already exists in schools.

However, it has attracted much criticism for its vagueness in terms of content, design, conceptual clarity and lack of articulated theoretical underpinnings. For example, Priestley (2010, p. 27) maintains that the "curriculum model adapted for CfE is problematic, and symptomatic of a general amnesia in respect of curriculum theory that arguably underpinned earlier developments". Similarly, there is an absence of any coherent reference to what Priestley & Humes (2010, p. 346) describe as the "rich vein of literature in the field of curriculum development" or without due regard to "the insights of research into the curriculum, whether from a philosophical, sociological or psychological standpoint". More recently, Priestley & Sinnema (2014) underline ambiguity with its curricular documentation. Likewise, in their overview of existing Scottish research, Priestley, Minty & Eager (2014) opine that the implementation of the new curriculum depicts an often confused picture, pointing to issues such as teacher anxiety about assessment and a misalignment between teachers' implicit theories about knowledge and learning and the new curriculum.

Nevertheless, CfE has introduced distinctive trajectories underpinning its philosophy, three of which are salient.

Firstly, and possibly the most ambitious design feature is a new expanded curriculum that caters for all young people between three and eighteen. Excluding the early years, the need to support such a coherent mathematics syllabus rests exclusively with primary and secondary mathematics teachers. This accountability has been implemented despite the diversities in entry and training requirements, classroom practice, ethos and values inherent within both professions. It is theorised by me that to provide effective levels of continuity and progression across this continuum, it will be dependent on the outcome of any cross-sector partnership. Despite the laudable existence of primary-secondary transition arrangements, many teachers have limited knowledge of the structures, principles and philosophies of a sector they have not worked in (Corrigan, 2013).

Secondly, it is more strongly predicated than the 5-14 curriculum guidelines on a shift in classroom practices towards more pupil centred approaches to education (Priestley & Minty, 2013). Enshrined within its philosophy is an inherent constructivist view of learning, in contrast with some teachers who perceive knowledge and learning as the transmission of content. MacLellan & Soden (2008) explain that:

> This, in a constructivist perspective, is the basis of all subsequent learning and teachers cannot, therefore, assume that one size fits all. In order to manage the sheer range of learner variability, it is necessary for learners to determine and pursue their own purposes and processes of learning through collaborative work, and accept the constructivist assumption that the locus of intellectual authority resides not in the teacher nor in the resources, but in the discourse facilitated by both teachers and learners (p. 35).

It is regarding the nature of pedagogy that requires a radical change in orientation to practice on the part of many, and in my view, particularly secondary mathematics practitioners, since it calls into question entrenched traditional assumptions of education. In her paper on quality assurance in Scottish schools, Reeves (2008, p. 10) maintains: “The move to privilege constructivist/social constructivist frameworks as opposed to transmissive and behaviourist approaches to teaching alters the role of teachers and hence the assumptions, skills, knowledge and tools that they need to employ in their practice”. In fact, Maclellan & Soden (2008, p. 29) contend that “without understanding of how learners construct knowledge bases through thinking and reasoning, and the teachers' role in facilitating such processes, it is unlikely that the intentions of Curriculum for Excellence can be fully realised".

Thirdly, there is a focus of intent on repositioning teachers as agents of change which according to Priestley, Biesta & Robinson (2013, p. 203) attempt "to reverse a trend of ongoing de-professionalization through regimes of testing and inspection and the establishment of wider cultures of performativity". Teachers are actively encouraged to embark on a crusade of independent professional discourse to support their critical thinking and reflection of everyday curricular practices. During the design phase, the Scottish Executive (2006a, p. 4) emphasised the quintessential role of teachers in shaping classroom practices by aiming "to engage teachers in thinking from first principles about their educational aims and values and their classroom practice" and "through a climate in which reflective practitioners share and develop ideas".

With proposed increased malleability, it is anticipated that schools and teachers can take account of local needs when constructing programmes of education, in order to put learning at the heart of the curriculum. This explicit move away from central prescription towards, as Priestley & Humes (2010, p. 346) express, "a model that relies upon professional capacity to adapt curriculum guidance to meet the needs of local school communities" is a distinctive feature of the new curriculum. Though, in terms of the enactment of experiences and outcomes, Priestly (2010, p. 34) insists that "these should remain as slaves rather than masters of the main purpose of the change". This last point is particularly critical in promoting a focus on a holistic understanding of the subject matter.

1.3.2 Current position

A dilemma faced by some practitioners is their willingness to create classroom environments which are consistent with the universal principles of constructivism and collaborative learning and teaching. However, resistance to change is evident throughout the country. The latest SSLN (Scottish Government, 2016a, p. 25) reported the activities in which the highest percentage of pupils testified that the methods in which they participated 'very often' were to 'listen to the class teacher talk to the class about a topic' (64% in P4, 68% in P7 and 66% in S2 respectively) and to 'work on your own' (59% in P4, 56% in P7 and 59% in S2). These findings are comparable with previous research (Scottish Government, 2014) and continue to suggest a teacher-centred approach which typifies classrooms where mastery of content and basic skills is emphasised and knowledge is transferred by lecture and repetition.

The autonomous vision of the new curriculum policy empowers what is delivered to pupils and how teachers implement it. Within daily practices, a dynamic continuum exists between exercising flexible judgment of teachers' views and rigid adherence to institutional school or local education authority policies. Perhaps accountability for professional learning will further seek to impact on the ongoing tension that exists between the relationship between teachers' beliefs and practice. From my own experience, teachers' mathematical beliefs play a critical role in the selection of pedagogical classroom approaches. A number of researchers endorse this view (e.g. Thompson, 1984, 1985, 1992; Schoenfeld, 1985, 1992, 2011; Ernest, 1989a, 1991; Cross, 2009; Cai & Wang, 2010; Beswick, 2012) while others report on inconsistencies between beliefs and practices (e.g. Cooney, 1985; Raymond, 1997; Skott, 2001).

Regrettably, within the literature I was unable to detect any previous research explicitly involving mathematical problem solving and problem posing beliefs of both Scottish current primary and secondary mathematics teachers. Although, a small quantity of Scottish studies exist that examine student primary teachers' mathematical beliefs (e.g. Macnab & Payne, 2003; Henderson & Hudson, 2011; Henderson, 2012a), newly qualified secondary mathematics teachers beliefs about the learning and teaching of mathematics (e.g. Forrester, 2008), primary teachers' mathematical beliefs (e.g. Hudson, Henderson & Hudson, 2012, 2015), primary and secondary teachers' beliefs about professional development (e.g. Wallace & Priestley, 2011; Biesta, Priestley & Robinson, 2015), primary and secondary teachers' beliefs on the arts (e.g. Wilson et. al, 2008) and secondary science and religious education teachers' collaboration beliefs (e.g. Hall et. al, 2014). Collectively, these miscellaneous studies suggest within Scottish education, the investigation of teachers' beliefs is of interest, all be it on a small scale.

Consequently, in order to afford this research immediate focus, I offer an unconditional definition of teachers' beliefs (I discuss this theme in more detail within chapter four). I refer to Kagan (1992), who expressed:

> Teacher belief is a particular provocative form of personal knowledge that is generally defined as pre- or inservice teachers' implicit assumptions about students, learning, classrooms, and the subject matter to be taught (p. 65-66).

For more than half a century, educational researchers have explored teachers' beliefs in their workplace. Several comprehensive reviews of this literature exist which reflect various approaches to understanding the multiple perspectives and theoretical nature of teachers' specific beliefs and their influences on practice (e.g. Kagan, 1992; Pajares, 1992; Thompson, 1992; Fang, 1996; Richardson, 1996; Phillip, 2007; Fives & Buehl, 2012).

It is important to consider, in the absence of available research, the operationalisation of mathematical problem solving and problem posing within Scottish classrooms. The OECD (2015, p. 11) argues that professional knowledge must be balanced by research or evaluation projects and emphatically state: "There is a clear need to know how CfE is actually being implemented in schools and communities across Scotland". Accordingly, I am postulating that by exploring the nature of Scottish teachers' mathematical beliefs, empirical evidence will be gathered that can stimulate our understanding of current professional practice of CfE within primary and secondary schools. Thus, a worthwhile opportunity exists to ameliorate mathematics educational policy, which may lead to enhanced future classroom experiences for all learners.

1.3.3 Learning theories

In this section, I offer a short review of constructivism and collaborative learning, individually which are driving forces of CfE. In my professional experience, both multidimensional child-centred approaches collectively offer rich learning environments to promote the teaching of mathematical problem solving and problem posing.

However, I begin by justifying the need to engage in such a pedagogical discourse. First, I am charged with a professional obligation to demonstrate a secure knowledge and detailed understanding of learning theories (GTCS, 2012), which has been intensified by political and societal rhetoric to cater for the diversity of learner variability within an ever changing pupil population. Simultaneously interwoven into this requirement is overt curricula awareness resonating with a constructivist view of schooling (Drew & Mackie, 2011). This is in contrast to the entrenched belief held by some teachers that perceived knowledge and learning is the transmission of content (Priestley & Minty, 2012) or at best, restricted to a superficial catchphrase alluded to by Cobb (1994, p. 4) that “students construct their own knowledge”. In rejecting the assumption that ‘one size fits all’, Marton (2007, p. 20) underlines that practitioners must think critically about interpretations of learning by utilising “theoretical tools available for analysing the extent to which the necessary conditions for achieving specific aims for learning are present in certain situations”. Second, the overarching infrastructure of education is underpinned by the presence of theorisations of learning which help to facilitate and shape classroom practice. Moreover, it is logical to posit that explorations of teachers’ beliefs about teaching must concomitantly consider teachers’ beliefs about learning.

Learning theories are conceptual frameworks that describe how individuals learn, often by reference to a particular model of human cognition or development (Illeris, 2009). There are many diverse and in some cases opposing philosophies of how humans acquire knowledge. For example, behaviourism has been intrinsically linked with education for many years and until a few decades ago was the dominant philosophy in multiple classrooms. Based on the model developed by Skinner, it attempts to apply the methods of science to the study of human learning. It is concerned with observable changes in behaviour that results from stimulus-response associations made by the learner. Behaviourism draws on the common practice of reinforcement through reward and punishment such as gold stars for good work and punishment exercises for unsatisfactory behaviour (Lerman, 2014). While practitioners possess a theoretical awareness of various learning styles, this must be reinforced by a pragmatic perspective that encourages all pupils to think metacognitively in order to take responsibility for their own learning. Hiebert & Grouws (2007, p. 373) add a caveat within mathematics education: “Although theories of learning provide some guidance for research on teaching, they do not translate directly into theories of teaching.”

The theory of constructivism is at the core of the revised mathematics curriculum in Scotland. Influenced principally by the seminal contributions of Piaget and Vygotsky, it is an epistemological position which is concerned with how learners construct their own understanding and knowledge of an intellectual world, through experiencing events and reflecting on those experiences. Centred on the learner, Simon (1995, p. 115) asserts that “we construct our knowledge of our world from our perceptions and experiences, which are themselves mediated through our previous knowledge". It differs from behaviourism in that it questions the origin of knowledge where knowledge is seen as created rather than received, mediated by discourse rather than transferred by teacher talk. While there are various forms of constructivism, within education and specifically relevant to this thesis, two branches are most relevant, namely cognitive constructivism and social constructivism.

Cognitive constructivism

Within the conceptualisation of cognitive constructivism, Piaget developed a theory centred on how children pass through stages of intellectual development as a result of biological maturation and environmental experience. The goal of his theory is to explain the mechanisms and processes by which the infant, and then the child, develops into an individual who can reason and think using hypotheses. Piaget (1997) advanced the view that the learner's construction of knowledge is a self-regulating process and that knowledge is not passively received from the world but created as individuals (and groups) adapt to make sense of their experiential worlds. In other words, knowledge is actively built up by a cognising human who needs to adapt to what is fit and viable but that an individual's knowledge is in a constant state of change because humans are subject to an ever fluctuating reality (Von Glasersfeld, 1991, 1995). Maclellan & Soden (2003) articulate Piagetian theory thus:

> Individuals' cognitive schemes allow them to establish an orderliness and predictability in their experiential worlds. When experience does not fit with the individual's schemas, a cognitive disequilibrium results, which triggers the learning process. This disequilibrium leads to adaptation. Reflection on successful adaptive operations (reflection abstraction) leads to new or modified concepts (accommodation), contributing to re-equilibration (p. 111).

The implication for the classroom is that pupils learn through interaction with peers and it is this process of collaboration which results in their existing beliefs and assumptions being challenged, thus initiating change. The role of the teacher is to create stimulating and resourceful learning environments to facilitate appropriate activities to bring about the challenge. Two points are noteworthy here. One is that the classroom culture contributes to learning while the child contributes to the culture of the classroom (Yackel & Cobb, 1996). Second, during disequilibrium a pupil can often feel confused and uncomfortable as they grapple with new concepts for the first time but this can be harnessed positively by the teacher to support learning (Carter, 2008).

An exponent of cognitive constructivism that is promulgated within CfE is active learning. Though there is a lack of conceptual clarity surrounding this construct within national policy documents (Drew & Mackie, 2011), it is delineated on two occasions. Initially, the Scottish Executive (2007, p. 5) define active learning from an early years perspective as “learning which engages and challenges children’s thinking using real-life and imaginary situations”. Later, to acknowledge secondary education, the Scottish Government (2008, p. 30) express that active learning should “build upon children’s enthusiasm, inventiveness and creativity” and “promote the development of logical and creative thinking and encourage a problem-solving approach”. A search of the literature suggests that active learning can be said to encompass four dimensions such as behavioural, cognitive, social and affective (Watkins, Carnell & Lodge, 2007; Drew & Mackie, 2011). Within such topography, Rosenthal (1995) argues that most mathematicians concur the best way to learn mathematics is by actively doing mathematics, which includes discussing it with others and by synthesizing fundamental ideas. Likewise, Ellerton (2013) encourages the active involvement of learners in posing problems during mathematics in order to demonstrate conceptual understanding and to furnish individuals the opportunity to solve, critique and reflect on their own mathematical problems including those of other peers.

Inside a constructivist setting, to allow for the greater participation of pupils, the teacher is someone who promotes and orchestrates classroom discussions. Viewed through the lens of CfE, Reeves & Drew (2013, p. 38) draw attention to the functionality of the teacher “who must relinquish her central position in the classroom; as a source of knowledge, as the most dominant speaker and as the evaluator and assessor of children’s work”. Reeves & Drew (2013) allude that this shift in emphasis has connotations for teacher professional identity and classroom control. Questioning is of paramount importance and can generate fertile learner experiences. Such rich interactions force pupils to communicate their thought processes to represent and reflect on their encounters. The necessity to convey and answer questions verbally forces them to examine and even revise their concepts of reality (Vygotsky, 1978). However, teachers must be openly receptive to the notion that a child might regard a mathematical concept in quite a different way than it is perceived and that this dissonance is not simply reducible to missing pieces or absent techniques or methods (Confrey, 1990). During this communication process, Confrey (1990, p. 109) insists “the teacher must form an adequate model of the students’ ways of viewing an idea and s/he then must assist the student in restructuring those views to be more adequate from the students’ and from the teacher’s perspective". It may be deliberated that pupils' levels of participation in the learning process are inextricably linked to their teachers' levels of participation in the cognitive learning process. Practitioners require a high degree of conceptual expertise and pedagogical knowledge fashioned by a culture of critical reflection.

Within mathematics classrooms, pupils are actively involved in constructing their own mathematical knowledge opposed to memorising algorithmic procedures or endlessly practising a litany of similar examples. Lerman (1993) whilst contending that constructivism offers a justification for mixed ability classes and individualised learning makes a parallel between a powerful metaphor of children developing mathematical structures and the processes involved during mathematical thinking, particularly during problem solving. Another influential contribution on this theme is articulated by Ernest (1991) who makes explicit reference to problem solving and problem posing. He argues that the aim of teaching mathematics is to empower learners to create their own mathematical knowledge and to develop confident and autonomous problem solvers and problem posers.

Towards the end of the last century, Ernest (1998) built his theory of social constructivism by arguing that the learning and teaching of mathematics is indelibly linked to a philosophy of mathematics. However, being furnished with a new set of theoretical or conceptual "lenses" can be empowering for teachers (Prawat, 1992) but may serve to complicate an already overloaded working life. More prosaically, constructivism is open to interpretation. As Cobb (1988, p. 87) cautions: "Although constructivism theory is attractive when the use of learning is considered, deep-rooted problems arise when attempts are made to apply it to instruction". I believe that issues of intersubjectivity can be enhanced by robust pedagogical knowledge. Cobb (1988) highlights two essential criteria for teachers which includes acceptance of the responsibility to facilitate profound cognitive restructuring and conceptual reorganisation along with transference in the belief of what constitutes a successful learning outcome. Regarding the latter aspect, he avows that constructivism does not assume a one-to-one correspondence between pupils' observable behaviours and the underlying conceptual structures. In other words, it is feasible for pupils to use the prescribed methods to solve a particular sets of tasks on which they have received instruction without having developed the desired conceptual structures (Cobb, 1988).

In my professional view, teachers' beliefs are a factor in establishing constructivist environments for mathematical learning. Pirie & Kieren (1992) resonate with this perspective: - drawing upon extensive empirical data, the authors argue that the creation of constructivist environments can produce improvements in the mathematical understanding of pupils. Such claims are supported by other researchers (e.g. Fraser, 2001; Webster & Fisher, 2003). Another critical theme of constructivism that has challenged teachers' beliefs is the important topic of assessment. Reflecting on the contributions of previous researchers, Confrey & Kazak (2006) maintain that providing teachers direct access to artefacts of pupils' work can engage them in examining their own beliefs and in looking more deeply into pupil thinking and reasoning.

Social constructivism

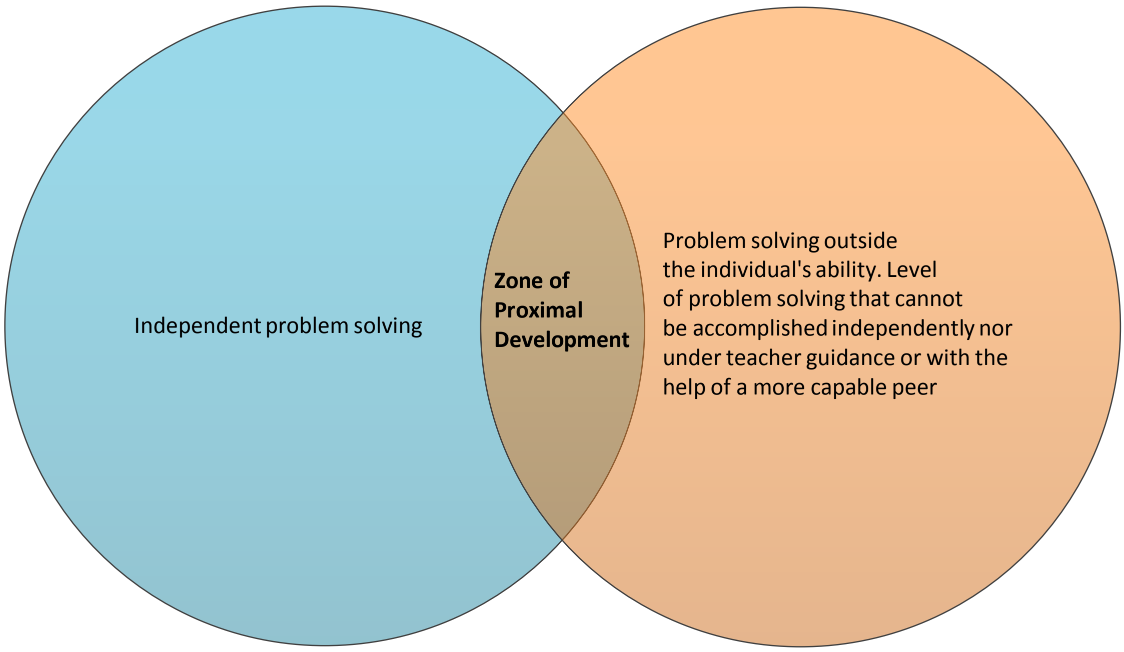
In a similar way that the contribution of Piaget relates to cognitive constructivism, the work of Vygotsky has greatly influenced social constructivism. Grounded on the premise that development cannot be separated from its social and cultural context, social constructivism is a theory of knowledge that examines the knowledge and understandings of the world that are developed in harmonisation with human beings. A major theme of Vygotsky's theoretical framework is that he believed everything is internalised on two levels. Initially, cognitive growth occurs through interaction with others, and then integrated into the individual's mental structure. According to Vygotsky (1978):

> Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological) and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relationships between individuals (p. 57).

From a mathematics education perspective, social constructivism considers knowledge to be driven by human interactions and cultivated by learning communities composed of individual mathematicians. Wood, Cobb & Yackel (1995, p. 402) note: "It is useful to see mathematics as both cognitive activity constrained by social and cultural processes, and as a social and cultural phenomenon that is constituted by a community of actively cognizing individuals". Such a process furnishes teachers with a conceptual framework to understand children's development of knowledge. It requires them to formulate a practice that corresponds with their pupils' method of learning and challenges them to reconstruct what it means to know, do and teach mathematics (Wood, Cobb & Yackel, 1995). Ernest (1991) introduced the notion of mathematical objectivity by linking subjective and objective knowledge in a cycle in which each contributes to the renewal of the other. A practical illustration of this concept can be located during assessment. For instance, approaches which evaluate mathematical creativity, posing problems and reasoning ability are subjective, while traditional methods that focus on computational skills and procedural fluency are objective.

A major doctrine of Vygotsky's (1978) theory is the concept that the potential for cognitive development is limited to a 'Zone of Proximal Development' (ZPD). It is defined by him as "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers" (p. 86). In simple terms, it refers to a range of internal knowledge that may be out of reach for a pupil working independently but is accessible if the pupil has support from a teacher or more capable peer (Figure 1.1). Goos (2004, p. 262) points out that the ZPD "is not a physical space, but a symbolic space created through the interaction of learners with more knowledgeable others and the culture that precedes them". Emblematic interactions infuse semiotic mediation (i.e. words, symbols, graphs, diagrams, etc.) and the culture within and beyond the classroom.

Figure 1.1 Zone of Proximal Development (Vygotsky, 1978)



From a Vygotskian perspective, teachers aid intellectual development in pupils by providing them with information and temporary structural support in carrying out a task, which is gradually reduced as pupil competency increases. This instructional technique is universally known as ‘scaffolding’ and has been characterised as a way of operationalising Vygotsky’s concept of working in the ZPD (Wells, 1999). From my professional experience, many practitioners endorse this position as according to Hammond (2002), it naturally resonates with their own intuitive ideas of what it means to intervene successfully in young people’s learning.

Collaborative learning

The Scottish Government (2008, 2009) emphasises that collaborative learning will encourage children to reason logically and creatively through discussion of mathematical ideas and concepts. Vygotsky’s theories permeate collaborative learning suggesting that group members should have different levels of ability so more advanced peers can help less capable members operate within their ZPD. In its elementary form, Roschelle & Teasley (1995, p. 70) describe this approach as “a coordinated synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem”. It is based on the model that knowledge can be created within a population where members actively interact by sharing experiences and taking on asymmetry roles. During successful collaboration, the agent of inquiry is not the individual, but the knowledge-constructing group; pupils take responsibility for knowing what needs to be known and for ensuring that others know what needs to be known (Hargreaves, 2007).

The majority of studies on collaborative learning adopt constructivism, especially theories from Piaget and Vygotsky, as the theoretical underpinning of peer collaborative learning (e.g. Fawcett & Garton, 2005) because they focus on building meaning through social interactions (John-Steiner & Mahn, 1996). McCrone (2005, p. 111) claims that group interplay can “allow students to test ideas, to hear and incorporate the ideas of others, to consolidate their thinking by putting their ideas into words, and hence, to build a deeper understanding of key concepts”. Discussions invite learners to justify their reasoning which can expose common misconceptions and lead to stronger connections between mathematical topics.

Orchestrating collaborative learning to accommodate multiple perspectives requires teachers to cultivate positive interdependence, be less controlling and pupils to be autonomous individuals who take more responsibility for their own learning (and that of their peers). It is essential for teachers to derogate from a procedural driven agenda and embrace a concept focused culture that promotes critical thinking and reflection. Importantly, teachers must be able to explore meaning and mathematical relationships through non-linear dialogue. In my opinion, collaborative learning is exemplified when presenting opportunities for pupils to tackle challenging mathematical problems prior to offering them guidance and support. This encourages learners to apply pre-existing knowledge that can be used to organise a method or a solution, prior to any intervention. Such an approach is buttressed by extensive empirical research evidence (Swan, 2006).

However, preparing formative and summative assessment information is problematic. In a study of Scottish primary teachers' views of the assessment of collaborative learning, McKechan & Ellis (2014) found that practitioners are challenged to accumulate and collate assessment evidence while simultaneously supporting the learning. The researchers suggest that this tension could be reduced by the use of technology to store data spontaneously.

1.3.4 Mathematical problem solving

Mathematics as a leading subject within Scottish education has been revitalised by the development of a pronounced emphasis and increased cognisance of mathematics within learning. It has been proclaimed by the Scottish Executive (2006b, p. 18) that "to face the challenges of the 21st century, each young person needs to have confidence in using mathematical skills, and Scotland needs both specialist mathematicians and a highly numerate population". On the theme of developing effective contributors, they declare:

> Mathematics offers a host of different contexts to apply skills and understanding creatively and logically to solve problems. Working on suitably challenging problems individually and in groups helps to develop resilience and gives opportunities to communicate solutions. The future prosperity of Scotland within a competitive global economy will depend upon high levels of numeracy across the population and significant numbers of our young people with the mathematical competence to operate in specialist contexts such as research and development environments (p. 19).

The conceptualisation of mathematical problem solving is no longer considered a separate component but has been emphasised as fundamental to effective learning and teaching in all aspects of mathematics, and its assimilation is "addressed within all lines of development rather than appearing as a separate element" (p. 20).

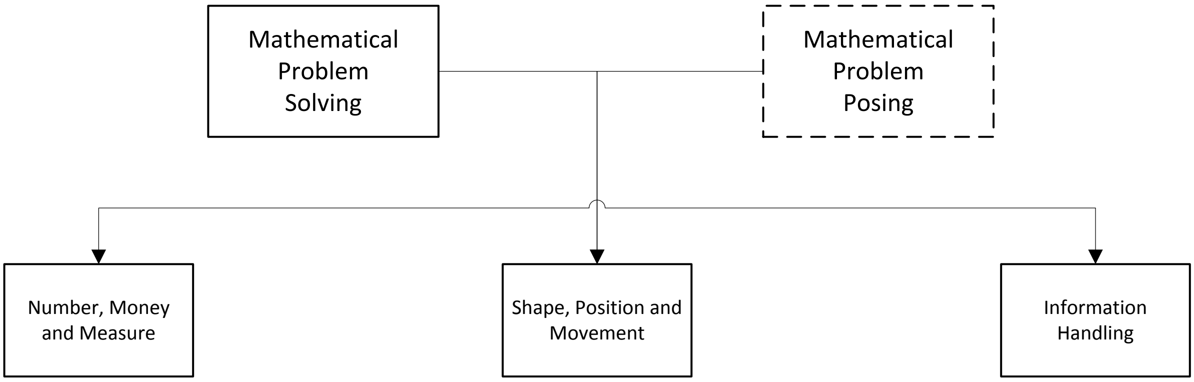
It is recognised that pupils need support in improving their mathematical problem solving competencies. Shortly after the implementation of CfE, evidence provided by Education Scotland (2012, p. 10) acknowledged in the 2011 SSLN that “it is evident from children and young people’s responses, that there is a need to strengthen their capacity to solve problems”. Teachers have a shared responsibility to ensure they address this pedagogical requirement. Halmos (1980, p. 523) warns that “it is the duty of all teachers, and of course teachers of mathematics in particular, to expose their students to problems much more so than facts”. Mathematical problem solving is pivotal to doing, learning and teaching mathematics (Schoenfeld, 1992) and is a central goal of CfE. The Scottish Government (2010c, p. 8) asserts that a “problem solving approach is at the heart of effective learning and teaching of mathematics”. Furthermore, guidelines offered to practitioners (Scottish Government, 2010b, p. 8) instruct teachers to “embody problem solving as an intrinsic element of mathematical approaches”. Halmos (1985, p. 322) highlights that: “A teacher who is not always thinking about solving problems – ones he does not know the answer to – is psychologically simply not prepared to teach problem solving to his students”.

In spite of a wealth of international literature being readily available for the learning and teaching of mathematical problem solving (e.g. Polya, 1957; Schoenfeld, 1985; Silver, 1985; Schroeder & Lester, 1989; Stanic & Kilpatrick, 1989; Lester & Charles, 2003; Schoen & Charles, 2003; Posamentier & Krulik, 2008; Mason, Burton & Stacey, 2010; Lester & Cai, 2016), no manifestation of this theoretical influence has been articulated within any mathematics curricula documentation. For example, no effort has been made to define the construct of a mathematical problem. In his analysis of American research, Lester (1994, p. 661) professed that “problem solving has been the most written about, but possibly the least understood, topic in the mathematics curriculum”. At present, it is unknown to what extent mathematical problem solving is mobilised within Scottish schools.

1.3.5 Mathematical problem posing

On review of CfE mathematical framework, no explicit pedagogical provision is specified for the conceptualisation and operationalisation of mathematical problem posing. Nevertheless, it is widely accepted within the mathematics education research community that problem posing is regarded as a vehicle for promoting conceptual understanding, problem solving and creativity (Cai et al., 2015). I reject the purported notion that problem posing is considered implicit or tacit mathematical knowledge enacted by all teachers but instead argue that it is deemed a field in its own right. Consequently, I posit that mathematical problem posing requires to be embedded within CfE and present a rational for its formal inclusion in this thesis. An illustration of where mathematical problem posing may assimilate within CfE is displayed in Figure 1.2.

Figure 1.2 Proposed location of mathematical problem posing within CfE experiences and outcomes



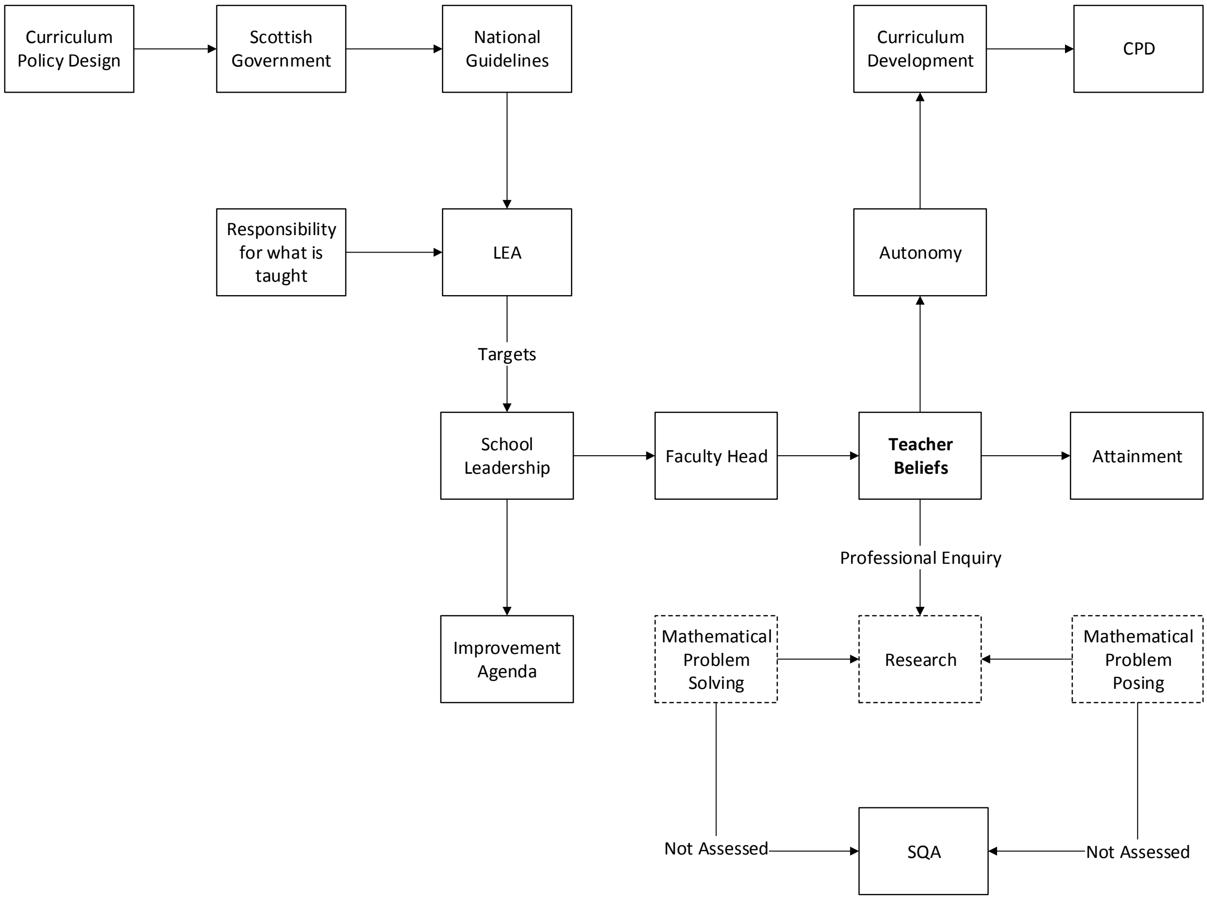
As in the case of mathematical problem solving, it is unknown to what extent mathematical problem posing is being currently mobilised in Scottish schools.

1.4 Aims of the research

Within the structure of CfE, responsibility for the learning and teaching of mathematics is shared reciprocally between primary and secondary establishments. This cross-sector accountability transmits to primary and secondary mathematics teachers respectively, despite the diversities in entry requirements and training required for both interrelated professions. There are two primary aims of this study: Firstly, to critically examine a reconceptualization of the existing mathematics curriculum by addressing a policy gap initiated by the omission of mathematical problem posing. Secondly, to provide empirical evidence of Scottish teachers' beliefs and espoused classroom practices of mathematical problem solving and problem posing. It is suggested that the evidence gathered can help to illuminate current professional practice and shape future policy. Some of the variables that impinge upon the optimisation of integrating mathematical problem solving and problem posing are identified.

In order to attempt to highlight some of the dynamics that may indirectly impact on teachers' mathematical beliefs in Scotland, I have provided a simplistic schematic overview illustrating the conventional position of a teacher within the national framework of CfE (Figure 1.3). It may be observed within this hierarchical structure, both mathematical problem solving and problem posing are not included as national assessment components by the SQA.

Figure 1.3 Overview of teacher within the national framework of CfE



1.5 Research questions

This study sought to answer the following specific research questions:

Question 1

To what extent should mathematical problem posing be embedded within the mathematical framework of Curriculum for Excellence?

Sub questions:

(a) What would be the value for learners of emphasising mathematical problem posing in the curriculum?

(b) What would be the implications for teachers' professional practice of implementing mathematical problem posing in their pedagogy?

Question 2

Are there any differences in the mathematical beliefs of Scottish primary and secondary mathematics teachers?

Sub questions:

(a) What is the importance of learning theory of shaping teachers' mathematical beliefs?

(b) What are teachers' beliefs about mathematical problem solving?

Question 3

What factors impact on the mathematical beliefs of primary and secondary mathematics teachers?

Sub question:

(a) What are the practical characteristics that support or constrain the development of teachers' beliefs?

In order to address these research questions, I initially conducted a literature review of mathematical problem solving, mathematical problem posing and teachers' beliefs. This served to provide several theoretical frameworks for this research.

1.6 Methodology

This study involved two strategies. First, a systematic literature review was undertaken to search for empirical evidence of the educational benefits of mathematical problem posing. Second, a mixed methods explanatory design was employed using questionnaires and semi-structured interviews as the instruments to measure mathematical beliefs and capture levels of current professional practices. Descriptive and inferential statistics (e.g. bar graphs, histograms, factor analysis, independent samples t-tests and ANOVA) were utilised to analyse quantitative data. Thematic analysis was used for the interrogation of qualitative data.

1.7 Outline of the thesis

Prior to providing an outline of the structure of the thesis, it is necessary for me to clarify the nature of the research contained herein. This research is twofold. First, it seeks to address a legitimate gap in knowledge generated by the mathematics policy of CfE which failed to encapsulate changes in contemporary knowledge and emerging research from the mathematics education community. The contribution to knowledge will be a compelling argument for the inclusion of mathematical problem posing within CfE. Second, this research is anchored on the premise that teachers' beliefs are powerful cognitive agents which are intertwined with their professional practice. It seeks to explore an important gap in knowledge in Scottish education by capturing Scottish teachers' mathematical beliefs and espoused professional practices. The contribution to knowledge will be to provide empirical evidence that can serve as importance indicators of current levels of classroom practices of mathematical problem solving and problem posing. Interpretation of findings from the data collection process will be evaluated against existing policy including previous research, from which, it is anticipated that recommendations will emerge that will improve the standard of Scottish education. Following on from this introductory chapter, this thesis is organised thus:

Chapter Two provides a literature review of mathematical problem solving. It addresses the conceptualisation of problem solving making reference to the theoretical work of Polya (1957) and Schoenfeld (1985, 1992), among others. It examines the relationship between heuristics and pupil performance, the use of multiple solutions and factors contributing to successful problem solving. Included is a discussion of the three types of teaching approaches to problem solving described by Stanic & Kilpatrick (1989) and consideration of assessment.

The third chapter focusses on introducing the reader to the conceptualisation of mathematical problem posing. Building on the early work of Kilpatrick (1987) and Brown & Walter (2005), it discusses the importance of problem posing in school mathematics and illustrates multiple theoretical perspectives and frameworks. Incorporated is an examination of the advantages and limitations of problem posing and justification of why it is a rich area for research.

Chapter Four contains a literature review of the construct of teachers' beliefs. An assortment of theoretical perspectives is offered that emphasise the profound influence of this critical dimension. Discussion of the impact of teachers' beliefs and the relationship between practices is presented.

The fifth chapter is concerned with methodology and methods. I have attempted to illuminate the core components of research such as ontology and epistemology that have afforded my research objectivity. Documented is a narrative of my research position and assessment of the interdisciplinarity of this study. Previous empirical studies measuring teachers' beliefs are listed. Justification of the methods selected for each research question including results of a systematic literature review is provided. Information on design, development and piloting are integrated. Ethical considerations are explained in detail.

Chapter Six analyses the data gathered from the questionnaires and addresses the second and third research questions. It presents the statistical results of phase one of the mixed methods explanatory design. Encompassed is validation of the parametric tests and a brief description of volunteered participants' comments.

Chapter Seven analyses the results of interviews undertaken and addresses the second and third research questions. It presents the qualitative results of phase two of the mixed methods explanatory design. Participant information and emergent themes are depicted.

The eighth chapter presents the findings of the research, focusing on each of the three research questions. I discuss the lessons from previous research to make comparisons.

In the final chapter, I draw together the various strands of the thesis in order to tender my conclusions. This consists of limitations of the study as well as implications for policy and practice. Recommendations and suggestions for further research are proposed. The ultimate section is devoted to a synopsis of what I have learned as a doctoral student.

In summary, this chapter has laid the foundations for this thesis. The following chapter provides a literature review of mathematical problem solving.

CHAPTER TWO

Mathematical Problem Solving

During the last fifty years or so, a considerable amount of literature has been published on mathematical problem solving. In this chapter, I attempt to outline the key ideas and theories. In the first section, it is necessary to offer a background in order to contextualise my research.

2.1 Introduction

For more than two thousand years, mathematics has provided a powerful universal language which has acted as an essential ingredient of the anthropological search for knowledge and understanding. It is underpinned by patterns, rigorous proofs and beautiful theorems which postulate truths of mathematical statements, conjectures and by intellectual and logical arguments. During the last century, the accumulation of mathematics and its applications has accelerated resulting in the emergence of contemporary branches of mathematics such as game theory, quantum mechanics, computational mathematics, operational research and graph theory. Intriguingly, mathematics has been portrayed by Carl Frederick Gauss as the queen of sciences, opined as an art (Lockhart, 2009) and paradoxically, by Bertrand Russell, as cited by Garnier & Taylor (2010, p. 52), being "the subject in which we never know what we are talking about, or whether what we are saying is true". Devlin (2003) eloquently captures the essence of what mathematics is all about:

> As the science of abstract patterns, there is scarcely any aspect of our lives that is not affected, to a greater or lesser extent, by mathematics; for abstract patterns are the very essence of thought, of communication, of computation, of society, and of life itself (p. 7).

Mathematical advances have derived both from the attempt to explain the natural world and from the desire to arrive at a form of inescapable truth from careful reasoning. These remain rich and influential motivations for mathematical thinking. Mathematics has been successfully applied to solve numerous complex and profound aspects of the human and societal domain. Archetypal illustrations include biologist's trying to understand the genetic code, development of the internet, predicting population growths, synthesis of new materials, warfare systems design, analysis of traffic patterns, forecasting earthquakes and modelling of social phenomena. Skemp (1987) emphasises the importance of mathematics to society and draws attention to practical applications:

> Mathematics is the most abstract, and so the most powerful, of all theoretical systems. It is therefore potentially the most useful; scientists in particular, but also economists and navigators, businessmen and communications engineers, find it an indispensable ‘tool’ (data-processing system) for their work (p. 17-18).

Today, mathematics remains as a crucial device for shaping the future of mankind, whilst maintaining a prevalent influence on daily life. More prominently, it is perceived to be a major factor which contributes to the wealth of a nation. Napoleon, as cited by Boyer & Merzbach (2011, p. 423) famously proclaimed: “The advancement and perfection of mathematics are intimately connected with the prosperity of the state.” Indigenously, in attempting to raise national awareness of mathematics, The Scottish Government (2016b) warn: “It is essential that our workforce is confident and fluent in maths if Scotland’s economy is to continue to compete internationally” (p. 26). Conversely, the same administration recognises the grave intrinsic challenges that remain by acknowledging our dismal public image of mathematics. They warn that “Scotland has a maths problem. Too many of us are happy to label ourselves as “no good with numbers.” This attitude is deep-rooted and is holding our country back educationally and economically” (p. 3).

The responsibility of mathematics education is designed towards tackling contemporary problems, nurturing creative and critical thinking skills and cultivating productive methods of acquiring and retaining new facets of analytical information within a constantly changing dynamic environment. In schools, pupils need to develop more than tangible mathematical knowledge; transferrable skills such as the ability to reason logically, in order to prepare for a fluctuating competitive market place, particularly as the exponential growth of technology is making larger quantities of information more accessible. On leaving school, young people need to adapt further to unfamiliar or capricious situations (compared to their counterparts in the past) and be equipped with the ability to provide innovative and resourceful solutions to a wide range of challenges. Therefore, schools have an obligation to empower learners to become creative and critical thinkers as well as mathematically literate citizens. Naturally, such a desired outcome is only plausible if pupils are furnished with a myriad of mathematical problem solving skills.

2.2 Problem solving and mathematics education

It is important to consider the position of problem solving within mathematics education. Castaneda, Gibb & McDermit (1982, p. 22) contend that: “A significant proportion of human progress can be attributed to the unique ability of people to solve problems. Not only problem-solving ability critical to human progress and even to survival itself, but is one of the most important aspects of mathematics”. Gagne (1985) in classifying the cognitive process expresses the view that problem solving is the highest form of mathematical learning.

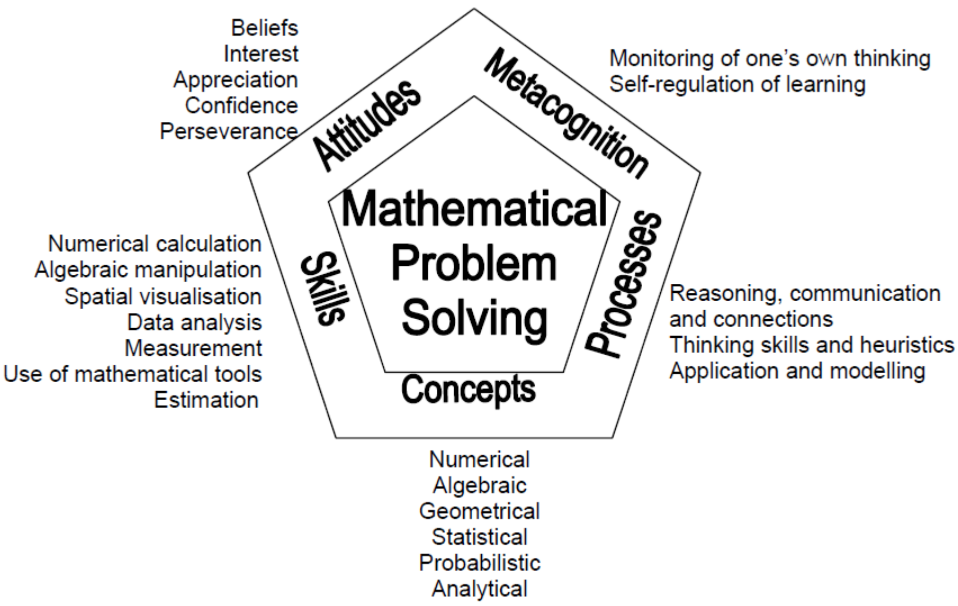
In the same vein, Krulik & Rudlik (1993, p. 9) assert that problem solving is a lifetime activity and that all other activities are subordinate by arguing that the “teaching of problem solving should be continuous. Discussion of problems, proposed solutions, methods of attacking, etc. should be considered at all times”. The authors underline that primary practitioners have the responsibility for initiating this process “and thus laying the foundation for building the child’s capacity to deal successfully with his or her future problem-solving encounters”. Likewise, Lester & Lambdin (2004, p. 192) believe “the primary goals of mathematics learning are understanding and problem solving, and that these goals are inextricably related because learning mathematics with understanding is best supported by engaging in problem solving”. Similarly, prominent mathematician Paul Halmos emphasises the significance of problem solving in schools and argues that it is the responsibility of both sectors to promote. Halmos (1980) maintains:

> The major part of every meaningful life is the solution of problems; a considerable part of the professional life of technicians, engineers, scientists, etc., is the solution of mathematical problems. It is the duty of all teachers and of teachers of mathematics in particular, to expose their students to problems much more than to facts (p. 523).

However, it is essential to distinguish between problem solving as a separate activity and as an approach to mathematics (Schoenfeld, 1992). Consequently, it is of no surprise that for educational systems throughout the world, the conceptualisation and operationalisation of mathematical problem solving is a foremost pedagogical curriculum objective. For example, Xenofontos & Andrews (2012, p. 70) maintain that: “In many European countries, problem-solving and its related skills form key expectations of the intended curriculum for students of all ages”. In England, the Cockcroft Report (1982, p. 71) advocated that “mathematics teaching at all levels should include opportunities for problem solving”. Both Australia (Australian Education Council and Curriculum, 1991) and America (NCTM, 1980, 1989, 2000, 2010, 2014) have strongly recommended that the learning and teaching of school mathematics should be centred on problem solving. The NCTM (2000) postulates "unless students can solve problems, the facts, concepts, and procedures they know are of little use" (p. 182). Significantly, they articulate that problem solving is an "integral part of all mathematics learning, not an isolated part of the mathematics program" and that "Instructional programs should enable all students to build new mathematical knowledge through problem solving" (p. 52).

Similarly, in Singapore, problem solving is the overarching approach to primary and secondary mathematics education and at the heart of learning and teaching (Figure 2.1). As explicitly outlined in the Ministry of Education (MOE, 2007, p. 3): "Problem solving is central to mathematics learning. It involves the acquisition and application of mathematical concepts in a wide range of situations, including non-routine, open-ended and real-world problems." The intimate relationship between mathematics and problem solving was further strengthened in 2011 with the introduction of a research project known as MProSE. Its vision is to integrate problem solving into the everyday teaching of mathematics in all Singaporean schools, regardless of ability or sector.

Figure 2.1 Singapore mathematics curriculum framework (Ministry of Education, 2007)



2.3 Conceptualisation

Whilst the portrayal of mathematics is universally known to all, many researchers and educationalists have described an assortment of different representations of mathematical problem solving without agreeing on a conceptual definition (Lesh & Zawojewski, 2007; English & Sriraman, 2010; Lester, 2013; Schoenfeld, 2013). Critics such as Mamona-Downs & Downs (2005) maintain that the formation of an undisputed description of mathematical problem solving may be elusive whilst others believe that some explanations are outdated (e.g. Lesh, Zawojewski & Carmona, 2003) or have different inferences (e.g. Wilson, Fernandez & Hallaway, 1993). Given innumerable meanings already in use, Grugnetti & Jaquet (2005), as cited by Chamberlin (2008, p. 2), "suggest that a common definition of mathematical problem solving cannot be provided".

The research so far has tended to focus on expanding a collection of illustrations. For example, mathematical problem solving is engaging in a task for which a solution is not known in advance (NCTM, 2000). Schoenfeld (1983, p. 41) declares that a "problem is only a problem (as mathematicians use the word) if you don't know how to go about solving it. A problem that has no 'surprises' in store, and can be solved comfortably by routine or familiar procedures (no matter how difficult!) is an exercise". Interestingly, Schoenfeld (1985, p. 71) provides an alternative vision of problem solving as "a particular relationship between the individual and the task that makes the task a problem for that person". Similarly, Orton & Frobisher (2005, p. 25) proclaim that "a mathematical problem for one learner may be an exercise for another". What is distinctive about the previous two definitions is that they recognise the person experiencing problematicity, which raises an obvious question from a teachers' perspective about how to classify such a task. Polya (1981) conveys problem solving as a feature of human endeavour, by stating that:

> Solving a problem means finding a way out of a difficulty, a way around an obstacle, attaining an aim which was not immediately attainable. Solving problems is the specific achievement of intelligence, and intelligence is the specific gift of mankind: problem solving can be regarded as the most characteristically human activity (p. ix).

Alternatively, Mayer (1985, p. 123) succinctly states that: "A problem occurs when you are confronted with a given situation – let's call it the given state – and you want another situation – let's call that the goal state – but there is no obvious way of accomplishing your goal". According to Cai & Lester (2005, p. 221), problem solving "is an activity requiring the individual to engage in a variety of cognitive actions, each of which requires some knowledge and skill, and some of which are not routine". McLeod (1988, p. 135) outlines mathematical problems as "those tasks where the situation or goal is not immediately attainable and there is no obvious algorithm for the student to use". Likewise, Posamentier & Krulik (2008, p. 1) asserts that “a problem is a situation that confronts a person, that requires resolution, and for which the path to the solution is not immediately known”.

In contrast, Verschaffel, Greer & De Corte (2000, p. ix) highlight a common misconception amongst teachers that word problems are “verbal descriptions of problem situations which one of more questions are raised for which the answer(s) can be obtained by the application of one or more mathematical operations to the numerical data available in the problem statement”. What is troubling about this definition is that it does not refer to cognitive thinking and presumes that a solution is easily achieved by applying a standard procedure. Previously, Smith (1986, p. 16) warns practitioners to exercise better judgment in interpreting “a set of words which are wrapped around some computational exercise”.

Cai & Nie (2007, p. 471) argue that problem solving activities are viewed as a goal to achieve and as an instructional approach supported by experience: “The purpose of teaching problem solving in the classroom is to develop students’ problem solving skills, help them acquire ways of thinking, form habits of persistence, and build their confidence with dealing with unfamiliar situations”. However, Orton (2004, p. 35) takes into account the cognitive processes involved and contends that “successful solutions of problems are dependent on the learner not only having the knowledge and skills required but also being able to tap into the relevant networks and structures in the mind”. Whilst, Lester & Kehle (2003) include reference to earlier experiences, knowledge and cognitive actions:

> Successful problem solving involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an effort to generate new representations and related patterns of inference that resolve the tension or ambiguity (i.e. lack of meaningful representations and supportive inferential moves) that promoted the original problem-solving activity (p. 510).

This definition is inadequate since it only serves to account for effective problem solving. Provocatively, Stanic & Kilpatrick (1989, p. 1) caution that “problem solving has become a slogan, encompassing different views of what education is, of what schooling is, of what mathematics is, and why we should teach mathematics in general and problem solving in particular”. Understandably, with the diversity of operational definitions, Schoenfeld (1992) strongly recommends that clarification is required. He writes:

> The term [problem solving] has served as an umbrella under which radically different types of research have been conducted. At minimum there should be a de facto requirement (now

> the exception rather than the rule) that every study or discussion of problem solving be accompanied by an operational definition of the term and examples of what the author means... Great confusion arises when the same term refers to a multiple of sometimes contradictory and typically underspecified behaviors (p. 363-364).

It has been established that problem solving encompasses more than a special importance in mathematics education and is generally accepted as a means of advancing critical thinking skills (Schoenfeld, 1985). It is the heart and soul of the work of mathematicians, engineers, scientists, economists, computer programmers, including all other professions that require the utilisation of higher-order cognitive processes. Sakshaug, Ollson & Olson (2002) portray the mathematical problem solving process experience as a mission that:

> encompasses the acts of exploring, reasoning, strategising, estimating, conjecturing, testing, explaining and proving. It is a very active process for those involved. Through the problem solving, we are challenged to think beyond the point where we were when we started, we are challenged to think differently. We are challenged to extend our thinking about a situation in a way that is new or different (p. vi).

What is appealing about this definition is that it acknowledges many mathematical functions. In contrast, the following definition offered by Lesh & Zawojewski (2007, p. 782) is intended to embrace creative thinking: “A task, or goal-directed activity, becomes a problem (or problematic) when the “problem solver” (which may be a collaborating group of specialists) needs to develop a more productive way of thinking about a given situation”. While a multiplicity of definitions of mathematical problem solving have been suggested, this thesis will use the definition submitted by PISA who saw it as:

> is an individual’s capacity to engage in cognitive processing to understand and resolve problem situations where a method of solution is not immediately obvious. It includes the willingness to engage with such situations in order to achieve one’s potential as a constructive and reflective citizen (OCED, 2014, p. 30).

This delineation captures a number of important features such as the employment of a strategy, the non-algorithmic nature of the solution and the need for perseverance.

2.3.1 History of mathematical problems

Mathematical problems have existed for thousands of years and have been enriched by various contributions from the likes of Babylonian, Egyptian, Greek and Islamic sources. Euclid’s Data, which is considered the pedagogical strand of Euclid, is a collection of geometrical problems. During the Medieval European mathematics period, Fibonacci posed:

> A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair, which from the second month on becomes productive?

This famous problem directed Fibonacci to the introduction of the Fibonacci sequence, which has been subsequently applied to the curvature of naturally occurring spirals, combinatorics and stock market analysis. One of my favourite mathematical problems involving the transcendental number e was posed during the scientific revolution. Joseph Fourier proved that e was irrational by using contradiction. In modern times, at the beginning of the twentieth century, David Hilbert, outlined twenty three unsolved mathematical problems to the International Congress of Mathematicians in Paris. Several of his problems have since been solved (such as Fermat's last theorem by Andrew Wiles), partially disentangled or remain unanswered (e.g. Riemann hypothesis). Today, many problems remain unsolved in mathematics, mainly attributed to the continuous formation of new problems.

Interestingly, countless historical examples exist where observation and intuition have directed mathematicians to offer logical and accurate solutions to problems. However, there are cases where it has navigated to wrong suppositions or incomplete or erroneous mathematical proofs. For example, Euler once conjectured that the Diophantine equation, $A^4 + B^4 + C^4 = D^4$, has no solution in positive integers. Remarkably, it took more than two hundred years for this statement to be disproved (Elkies, 1988).

It is disingenuous to suggest that inaccurate solutions to mathematical problems are of no intrinsic educational value. On the contrary, they can stimulate rich classroom interactions in order to strengthen deep conceptual understandings. Within my own professional practice, I have regularly posed the fashionable division by zero fallacy: 'Is 1 + 1 = 1 a true statement?' It has generated a positive learning experience whilst anticipating pupils recognising the invalidity of line five, as follows:

Let $\mathrm{a} = \mathrm{b}$

$\Rightarrow \mathrm{a}^2 = \mathrm{b}^2$

$\Rightarrow \mathrm{a}^2 - \mathrm{b}^2 = \mathrm{ab} - \mathrm{b}^2$

$\Rightarrow (\mathrm{a} - \mathrm{b})(\mathrm{a} + \mathrm{b}) = \mathrm{b}(\mathrm{a} - \mathrm{b})$

$$\Rightarrow \frac{(a-b)(a+b)}{(a-b)} = \frac{b(a-b)}{(a-b)}$$

$\Rightarrow a + b = b$

Since $a = b$, we have $2b = b$

Thus, $2 = 1$

It is essential to consider if this previous task can be classified as a legitimate mathematical problem. By undertaking such a deliberation, it focusses attention on the generic structure of mathematical problems. Unequivocally, it is a genuine mathematical problem since it can foster reason, communication, interest and curiosity, while developing a positive disposition towards mathematics.

Furthermore, unsuccessful attempts at solving such a problem or in fact any historical problem is not unproductive. In a study of 25 American undergraduates conducted by Moser et al. (2011), it was found that individuals with a growth mind-set had a greater awareness of errors than individuals with a fixed mind-set and thus were able to rebound better from mistakes. Exhibiting such a tendency during mathematical problem solving can help to promote higher order thinking skills. In describing mathematical thinking processes, Mason (1999) professes that:

> Failure can be more useful than success. One challenging problem teaches you far more than many easy problems. Getting stuck gives you an opportunity to learn - when ideas come too readily, you have no marker to return to, no peg from which to extend your network of cues and triggers (p. ix).

2.4 Fundamental characteristics

It is imperative that a worthwhile mathematical problem should offer a suitable challenge while simultaneously providing an opportunity to learn important mathematics. Such a task should initiate disequilibrium and perplexity (Lambdin, 2003), be intriguing, invite hard work and direct pupils to investigate mathematical ideas and methods of thinking towards the learning goal (Lester & Cai, 2016). In endorsing the mandatory necessity of challenge, Schoenfeld (1985) contends that the task should be an intellectual impasse rather than a computational one.

However, not all mathematical problems are germane for classroom use. Some are contextualised within artificial situations which erroneously promote the employment of algorithmic exercises. Pupils are constrained to implement trivial procedures and manifest the false expectation that a readily available solution exists for every problem. The resulting vulnerability of learners is being ill-equipped in confronting authentic real life mathematical problems, where the solution method is not immediately obvious (Schoenfeld, 1992). Practitioners must consider the needs of all learners before implementing any task. Orton (2004) cautions that sensible attention be afforded when selecting a mathematical task for instruction. He argues that:

> One aspect of problem-solving in mathematics is that often the problems are divorced both from the mainstream subject matter and also from the real world. Such puzzles may contain great interest for some children, but others may not see the point and be demotivated. Such puzzles are unlikely to produce knowledge or rules which are useful or applicable elsewhere (p. 26).

Nonetheless, in order to nurture a community of learners, primary and secondary mathematics teachers should follow the wisdom of Polya (1957), who advocates a great opportunity that awaits:

> If he [teacher] fills his allotted time with drilling his students in routine operations, he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge and helps them to solve their problems with stimulating questions, he may give them a taste for, and some independent means of, independent thinking (p. xxxi).

Selecting an interesting and challenging problem that can stimulate mathematical learning is a fundamental skill for any teacher. Too often, young people are presented with the exclusive drudgery of following 'drill and practice' routines which only serve to augment computational skills whilst instantaneously disengaging learners. Schoenfeld (1994a, p. 60) forewarns all teachers of this danger: "When mathematics is taught as dry, disembodied, knowledge to be received, it is learned (and forgotten or not used) in that way". It has been thought that a good problem can be justified by successfully applying the augmentation and proof strategy of 'convince yourself, convince a friend, convince an enemy' (Mason, Burton & Stacey, 2010).

By directing attention to the type of suitable mathematical problems, focus is drawn to the subjective views of teachers. In short, this places more emphasis on the role of the teacher to select and develop worthwhile mathematical tasks that create opportunities for pupils to develop mathematical understandings, competence, interest and dispositions (NCTM, 1991).

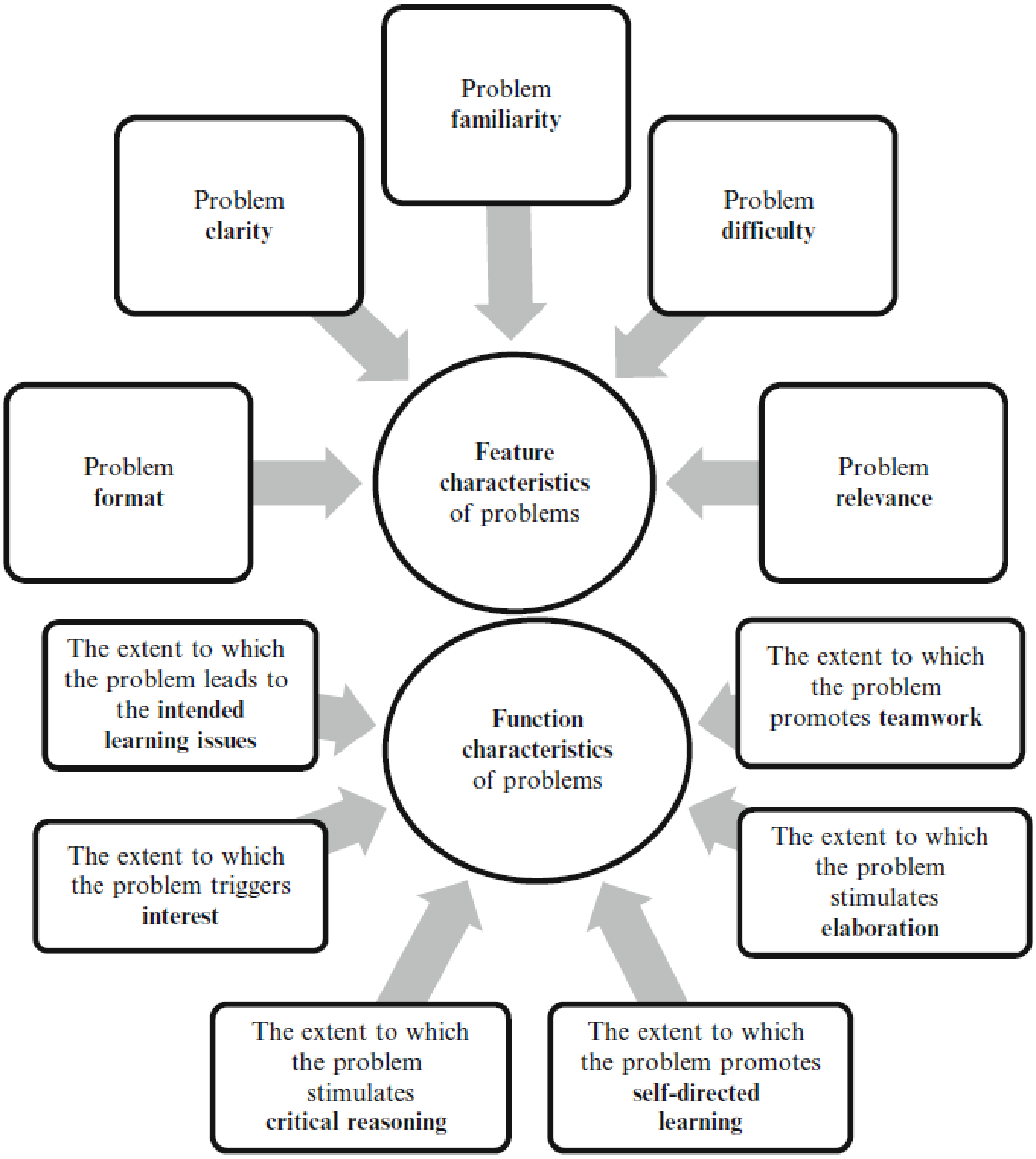
In searching for a definitive set of criteria for defining the characteristics of a worthwhile problem, the NCTM (2010) refer to the work of Lappan & Phillips (1998). The authors offer excellent guidelines in the shape of ten criteria although they emphasise that it is not reasonable to expect that every problem selected satisfies all the criteria but should be dependent on a practitioners instructional goals:

1. The problem has important, useful mathematics embedded in it.
2. The problem requires higher-level thinking and problem solving.
3. The problem contributes to the conceptual development of students.
4. The problem creates an opportunity for the teacher to access what his or her students are learning and where they are experiencing difficulty.
5. The problem can be approached by students in multiple ways using different solution strategies.
6. The problem has various solutions or allows different decisions or positions to be taken and defended.
7. The problem encourages student engagement and discourse.
8. The problem connects to other important mathematical ideas.
9. The problem promotes the skillful use of mathematics.
10. The problem provides an opportunity to practice important skills (p. 1-2).

The NTCM (2010) highlight that the first four criteria should be considered essential in the selection of all mathematical problems. Nevertheless, an important caveat to acknowledge is that an appropriate choice of problem does not guarantee that successful mathematical learning will occur (Lester & Cai, 2016). In my professional experience, this will depend on a number of interrelated dimensions but in particular, the kind of classroom discourse and intervention that normally takes place during mathematics lessons between teacher and pupil. For example, some teachers do not share the belief or have the patience to allow pupils to struggle with challenging mathematical problems, thereby eliminating the requirement to stimulate independent and higher level thinking (Stillman et al., 2010).

In her informative analysis of problem based learning, Sockalingam (2015) provides a valuable insight into the structural elements of a problem. She draws on a previous study of 34 Singaporean biomedical undergraduates which identified eleven characteristics grouped by ‘feature’ and ‘function’ (Figure 2.2). However, her conclusions would have been more persuasive if she had considered studies involving mathematics problems.

Figure 2.2 Function and feature characteristics of problems (Adapted from Sockalingam, 2015)



2.4.1 Classification

One of my satisfying childhood memories was solving the classic puzzle involving a man who has to transport a fox, a chicken and a sack of corn across a river using a rowing boat, which can only carry him and one other object. The circumstances dictated that if the fox and the chicken are left together, the fox will eat the chicken and if the chicken and the corn are left together, the chicken will eat the corn. How does the man do it? While such puzzles are designed to manifest reasoning and thinking processes (Joanssen, 1997), it is debatable if their content neutral and decontextualized nature, evident in many cases, are relevant to the promotion of mathematical problem solving.

Nevertheless, it is important to understand the classification of mathematical problems that can be presented to learners in order to identify the type of solution required (Polya, 1981). Thus far, a number of studies have classified mathematical problem types used in schools. The three most cited studies refer to ‘routine and non-routine’ problems (McLeod, 1994), ‘open and closed’ problems (Boaler, 1998) and ‘well and ill-structured’ problems (Jonassen, 1997). Arguably, all mathematical problems serve a valuable purpose but different categories of problems will accomplish different learning objectives. Specific problems require an element of recall and formulae, some stimulate the need for divergent strategies, others depend on logic and reasoning, selected have multiple solutions or demand decision making and creativity (Jonassen, 1997).

2.4.1.1 Routine and non-routine

Critically, teachers must be able to distinguish between routine mechanical algorithmic tasks accentuated by facts or procedures and unfamiliar activities designed to help pupils construct a deeper understanding of mathematical ideas (e.g. Polya, 1957; Schoenfeld, 1985; Stanic & Kilpatrick, 1989; Krulik & Rudnick, 1993; Mayer, 2003; Orton & Frobisher, 2005; Mason, Burton & Stacey, 2010). Schoenfeld (1988) makes a valuable contribution to this argument. Based on his American case study of secondary pupils, he found that although learning and teaching was successful from a curriculum perspective, learners developed a fragmented conceptual understanding including flawed beliefs about mathematics. For example, participants believed that mathematical problems can be solved within minutes and that it is acceptable to give up quickly. If an activity is reduced to replicating the technique imposed by the educator, it can create an illusion of mathematical competence by simple memorising and reproducing the correct method to manipulate symbols, and may even come to promote the believe that obtaining the correct answer exceeds the need for understanding (Goos, Galbraith & Renshaw, 2004). As Yeo (2007) reminds us:

> If a teacher does not know the differences between the types of mathematical tasks, how is he or she to use them to cultivate different types of skills and thinking? If a teacher refers to standard mathematics textbook tasks as ‘problems’ that the students should ‘solve’, then he or she may not realise that practising this type of task is not mathematical problem solving (p.1).

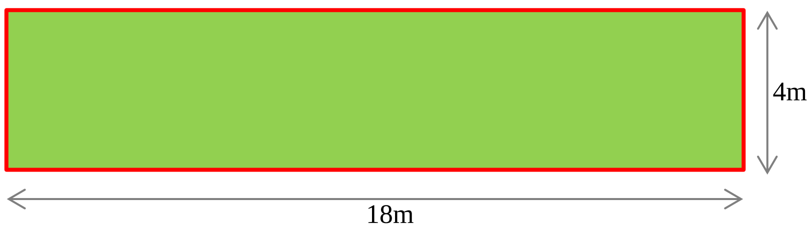
If by poor judgement or otherwise, practitioners restrict pupils to repetitive and computational tasks, many will be unprepared to solve genuine mathematical problems whilst simultaneously extinguishing their motivation and natural curiosity.

Before proceeding to examine other classifications, it is necessary to provide illustrations of archetypal tasks found within mathematics textbooks active within Scottish primary and secondary schools (Figure 2.3). The provision of a real life context is emblematic but surveyed as a solitary feature; its inclusion does not impact on the level of challenge implanted within the parameters aligned to a mathematical task.

Evaluation of the six mathematical tasks illustrated in Figure 2.3, reveal that in spite of the varying levels of complexity, a shared commonality exists with each solution in that a standard procedure can be applied to obtain a correct answer. Such routine tasks resonate with the objective articulated by Lester (1980, p. 31) to "provide students with practice in using standard mathematical procedures, for example, computational algorithms, and use of formulas". The first task is found by multiplying the area of the grass lawn by the unit cost per square meter to obtain £990. The next task requires knowledge of rounding and awareness that the numbers must be whole numbers i.e. 85 and 94. The third task can be answered by the summation of (18 x £0.45) + (6 x £0.30) + £3.99 = £13.89. The fourth task is a recurrence relation which begins with the sequence $u_{n+1} = 0.75u_n + 20$, where $u_0 = 160$ ml. The amount of drug remaining is calculated by finding u_4 (105 ml). The fifth task employs a standard integration formula for volume of solid of revolution to obtain 2570 cm^3. The final task may appear sophisticated since proofs tend to extend mathematical thinking due to an array of theorems, axioms and inferences which are required to construct a rigorous argument. However, in this standard case, $n = 1$ is initially proved and an 'induction rule' applied to establish any arbitrary value. Logically, the degree of challenge within a proof question will determine the nature of classification, although evaluation may be subjective. Proofs that require the creation of new mathematical concepts or derive novel theorems are obvious exemplars of mathematical problems (Powell et al., 2009).

Figure 2.3 Examples of mathematics textbook tasks

Task 1 The plan of a rectangular grass lawn is shown below. Find the total cost of the lawn given that the cost per square metre of grass lawn is £13.75. [Third level]



Task 2 Write down the lower and upper bounds of a crowd of people estimated at 90 to the nearest 10 people. [Second level]

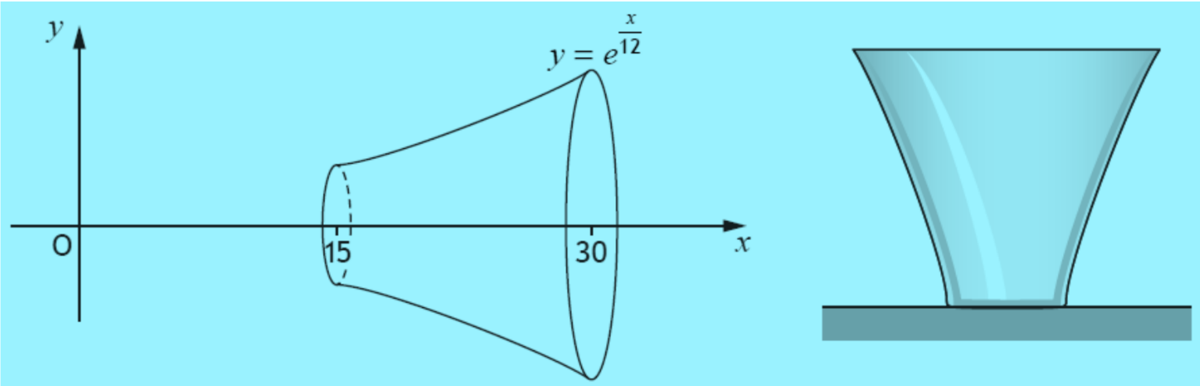
Task 3 An Isle of Lewis photographer charges the following rates:

- 45p per photograph for the first 18 photographs printed
- 30p per photograph for any further photographs printed
- £3·99 for a CD of the photographs

How much will it cost to have 24 photographs printed plus a CD? [Fourth level]

Task 4 A patient is injected with 160 ml of a drug. Every six hours 25% of the drug passes out of her bloodstream. To compensate, a further 20ml dose is given every 6 hours. Calculate the amount of drug remains after 24 hours. [Higher]

Task 5 A plastic bowl is modelled by rotating the curve $y = e^{\frac{x}{12}}$ between $x = 15$ and $x = 30$ through 2π radians about the x-axis as shown in the diagram. Find the volume of the bowl. [Advanced Higher]



Task 6 Prove by mathematical induction,

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2} \quad \forall n \in Z^{+}$$

[Advanced Higher]

Notwithstanding the actuality that any standard mathematical textbook task may be considered ‘sophisticated’ if viewed through the lens of a less experienced individual or someone lacking confidence with performing routine procedures (Schoenfeld, 1985; Orton & Frobisher, 2005), I will proceed with a trajectory to establish additional boundaries between familiar constructs associated with mathematical problem solving. Justification for doing so is further underlined by the misappropriation of terminology such as task, problem, activity and investigation engrained within various CfE narratives (e.g. Scottish Government, 2009, 2010a, 2011a, 2014). For example, in a professional learning resource for practitioners, Education Scotland (2015, p. 2) reiterate their conceptualisation of numeracy by expressing its detachment from mathematics followed by an inference that numerical skills are exclusively associated with solving mathematical problems: “Numeracy is not only a subset of mathematics... We are numerate if we have developed: The confidence and competence in using number which will allow individuals to solve problems". Such contradictory language only serves to confuse teachers and highlights urgency for conceptual clarity. Furthermore, this issue is compounded by the fact that no universally accepted definition exists for each term (Powell et al., 2009).

As explained earlier, word problems are not mathematical problems but are more accurately compartmentalized as routine mathematical tasks. In her study of Australian primary and secondary mathematics teachers, Anderson (2005) found that many participants believed they were implementing mathematical problem solving based on curriculum guidelines. However, it transpired that many of the examples provided to her required lower level thinking. Education Scotland (2015) offers the following strategies for practitioners, which help to promote the use of word problems and thus impede the implementation of problem solving:

> Learners' skills in, and application of, estimating and rounding should be a regular feature of learning. Progression in estimation should involve learners taking account of the impact of real life contexts and using this knowledge when communicating their understanding. For example, when solving word problems, such as those involving division, where the interpretation of the context is required to gain a reasonable answer (p. 8).

2.4.1.2 Open and closed

In her case studies of two English secondary schools, Boaler (1998) draws our attention to alternative pedagogical mathematical methodologies. Phoenix Park implemented open-ended projects such as: 'The volume of a shape is 216, what can it be?' Pupils were encouraged to search out multiple solution methods and answers. This approach accommodates diverse learning styles and can help to promote rich and deep conceptual understanding (Becker & Shimada, 1997; Hiebert et al., 1997). Conversely, in Amber Hill, pupils followed traditional practices which focussed on routine tasks featuring one correct answer, which is characterised as 'closed' (Becker & Shimada, 1997).

However, an interesting dilemma arises when a question contains multiple correct responses. For example: 'Solve the equation $x^2 = 7x$'; By factorising we can obtain $x = 0$ or $x = 7$. Both values of x are required to formulate the correct answer. Similar cases will occur when polynomials, inequalities, trigonometric functions or complex numbers are involved. Yeo

(2017) argues that such cases are considered ‘closed’ since all multiple correct answers can be determined.

2.4.1.3 Well- and ill-structured

Much controversy has surrounded well-structured and ill-structured problems. Simon (1973) found it impossible to construct a formal definition of well-structured problems and opted for a collection of potential characteristics. In the same vein but with less detail, Jonassen (1997) offers similar characteristics, as listed in Table 2.1.

Table 2.1 Characteristics of well-structured problems (Adapted from Jonassen, 1997)

- Present all elements of the problem.
- Are well-defined with a known solution
- Engage the application of a limited number of rules and principles that are organized in a predictive and prescriptive arrangement with well-defined, constrained parameters.
- Involve concepts and rules that appear regular and well-structured in a domain of knowledge that also appears well-structured and predictable.
- Possess correct, convergent answers.
- Possess knowable, comprehensible solutions where the relationship between decision choices and all problem states is known or probabilistic.
- Have a preferred, prescribed solution process.

Other authors point to a clearly specified initial state, goal state and set of operations (e.g. Mayer, 2003). Kilpatrick (1987, p. 134) argues that such problems “can be solved by the application of a known algorithm, and have criteria available for testing the correctness of a solution”. Well-structured problems can be characterised as routine mathematical tasks. Examples include finding the mean number of goals scored in a football competition, length of an unknown side of a triangle, distance travelled by a projected object, surface area of a cuboid, roots of a polynomial function, lines of symmetry of a rhombus and exterior angles of a polygon.

In contrast, ill-structured problems have vaguely defined goals, incomplete or ambiguous information, generate multiple solutions or no solution at all and possess uncertainty about which concepts, rules and principles are necessary for resolution (Simon, 1973; Jonassen, 1997, Mayer, 2003). Typically, they resemble real world situations and in which the solver may not know when they have obtained a final solution (Kilpatrick, 1987). Examples include building a swimming pool using a budget of less than £30,000 and designing a 10 metre long bridge that can hold a 200 kg weight.

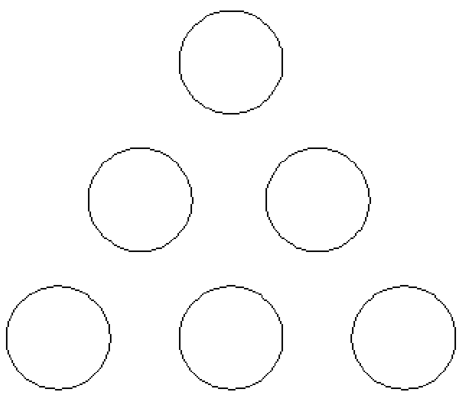
Let us now consider mathematical problems that can be employed to engage pupils in higher-order thinking. I present six examples from my own professional practice (Figure 2.4) that I have used extensively across a continuum of ages and abilities. From a didactical perspective, each problem can be successfully attempted by any pupil as no specific background is required. Such an intrinsic feature is essential if teachers want to promote inclusion, as many practitioners hold the common view that only high achieving pupils can participate in this form of mathematical instruction. The initial problem is well-structured and open-ended since it contains multiple answers. The second and fourth problems are well-structured and closed since they contain a unique answer. In the third problem, the sum of any five odd numbers is odd and therefore cannot be solved; irrespectively, it offers an excellent platform to launch basic number theory, which can be developed to define odd and even numbers in terms of any integer. More critically, as a learning objective, it skilfully alerts pupils to the possibility in mathematics that we can legitimately obtain ‘no solutions’. Such an early growth mind-set is helpful when discussing future linear equations of the form:

$$5x + 10 = 9(x+1) - 4x$$

which produces a false statement (i.e. 10 = 9) or explaining roots of quadratic equations where $b^2 - 4ac < 0$. In sum, this problem is defined as an ill-structured problem. The fifth and sixth problems are well-structured and closed since they contain a unique answer; multiple solutions are possible.

Figure 2.4 Examples of mathematical problems

Problem 1 Is it possible to put the numbers 1, 2, 3, 4, 5, 6 in the circles so that the sums of the three numbers on either side of the triangle are the same?

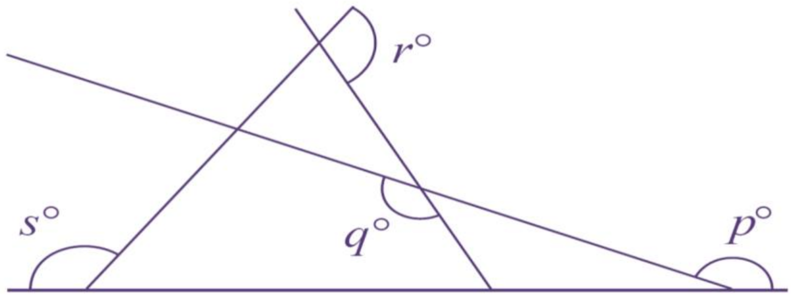


Problem 2 A palindromic number is one which reads the same when its digits are reversed, for example 19391. What is the largest six-digit palindromic number which is exactly divisible by 15?

Problem 3 Find five odd numbers whose sum is 100.

Problem 4 Identify the value of the letters in the equation $\mathrm{CROSS} + \mathrm{ROADS} = \mathrm{DANGER}$

Problem 5 Four straight lines intersect as shown. What is the value of $2p + 2q + 2r + 2s$?



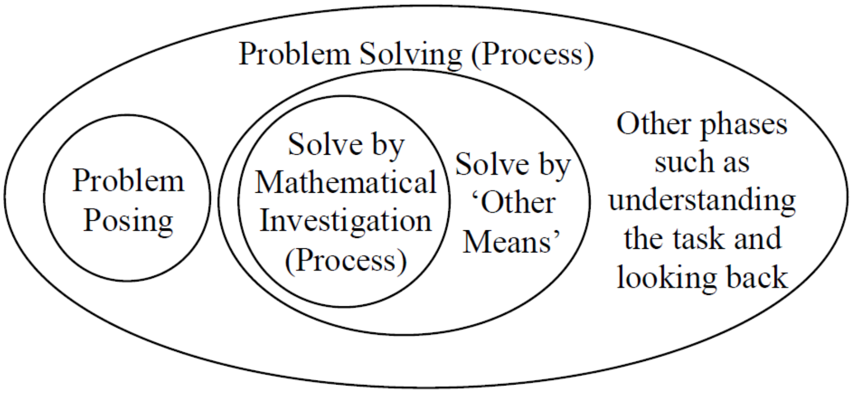
Problem 6 In a classroom with 10 people, everyone shakes hands with everyone else exactly once. Find the total number of handshakes.

2.4.2 Problems and investigations

Whilst the conceptualisation of problems has penetrated deep into mathematics educational discourse, the relationship between problems and investigations highlight inconsistencies in the literature. Orton (2004, p. 85) argues that the "exact distinction between an investigation and a problem has rarely been clarified by advocates of their inclusion in the curriculum, and it still not always clear what is meant when either is being discussed today" and "it is clear that either or both may be developed from the same basic idea or situation". Frobisher (1994) asserts "there is no doubt that a great deal of overlap exists" (p. 152) but that "a distinction should be made between (problem solving) tasks which lead to investigations, and... investigations which have their own existence" (p. 158). Alternatively, some scholars claim that nothing can be gained from establishing any differences. Pirie (1987, p. 2) as cited in Yeo & Yeap (2009) maintains that "no fruitful service will be performed by indulging in the 'investigation' versus 'problem-solving' debate". On the other hand, considerable support exists for associating investigations with having no clear specified goal in the statement. Orton & Frobisher (2005, p. 32) claim that "an open problem is another name for an investigation whilst an open problem is a process problem which gives rise to further problems". The implication of problem posing as an integral component of investigations is reinforced by others (e.g. Cai & Cifarelli, 2005; Yeo & Yeap, 2009; Yeo, 2012). Yeo & Yeap

(2009) illustrates the relationship between mathematical investigation as an activity, mathematical investigation as a process and problem solving as a process (Figure 2.5).

Figure 2.5 Investigative activity for open investigative tasks (Adapted from Yeo & Yeap, 2009)

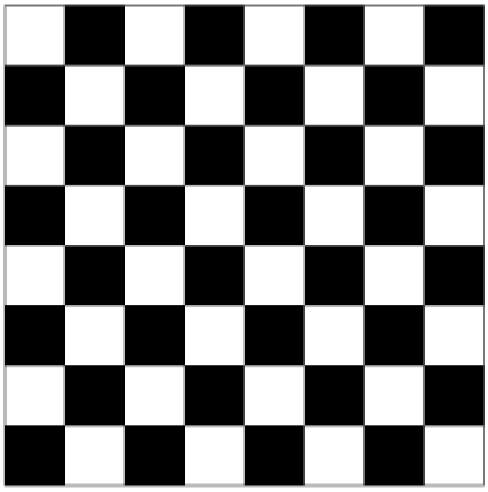


In practice, many primary and secondary mathematics teachers have a propensity to view problems and investigations as one of the same (Orton & Frobisher, 2005). Nationally, this situation is exacerbated by the Scottish Government (2009, p. 3) who do not advocate any dissonance between the two activities by encouraging practitioners to promote learners to "investigate mathematical problems".

However, I firmly believe that problems and investigations are distinct activities due to the unrestricted nature embedded within investigations. Orton & Frobisher (2005, p. 32) opine that: "An investigation provides learners with the freedom to determine the goals they wish to attain. This independence and autonomy is not possible in problems having a precise and unambiguous goal with a known and well-established method of solution". Frobisher (1994) fosters the disparity between problem solving as a convergent activity based on unique solutions and an investigation deemed a divergent activity characterised by multiple solutions and outcomes. He emphasises that both pedagogical approaches to learning mathematics should be welcomed by pupils and "not just something which occurs when the routine of the normal curriculum becomes dreary and tiresome" (p. 169).

Around ten years ago, I successfully introduced a prominent mathematical investigation into my professional practice of which I have shared with many colleagues (McDonald, 2006):

Investigate how many squares are on a chessboard (the answer is not 64!)



The first goal is clearly defined and on obtaining the desired 204 (i.e. $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2$) squares, pupils can extend their mathematical thinking by posing a different problem which involves finding a general rule for any $n \times n$ chessboard [i.e. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6} n\,(n+1)(2n+1), n \geq 1,\ n \in \mathrm{N}$]. Nevertheless, this is not obvious to all pupils since it is not explicitly specified in the statement. It is an important characteristic that demonstrates that an investigation does not depend on whether the activity has a closed or open goal (Yeo & Yeap, 2009).

2.5 Heuristics

As highlighted in chapter one, Curriculum for Excellence advocates that problem solving is no longer compartmentalised as a detached entity but integrated into all levels of learning and teaching of mathematics. However, inextricably absent from this explicit directive are guidelines for practitioners on how to orchestrate this pedagogical approach into practice. In this section, the role of heuristics is introduced followed by a brief discussion of two seminal theoretical frameworks and a brief summary of whether teaching heuristics improves pupil performance.

The conceptualisation of heuristics has been synthesised over the years with many conflicting descriptions available. For example, according to Polya (1957, p. 112): “The aim of heuristics is to study the methods and rules of discovery and invention”. Verschaffel (1999, p. 217) defines heuristic methods as “systematic search strategies for problem analysis and transformation”. De Bono (1984, p. 10) suggests that the idea of heuristics “includes all those aspects of thinking that cannot be fitted into mathematical formulations”. Martinez (1998, p. 606) claims that “Heuristics are crucial because they are the tools by which problems are solved”. Wilson, Fernandez & Hadaway (1993, p. 63) consider heuristics as “kinds of information, available to students in making decisions during problem solving, that are aids to the generation of a solution, plausible in nature rather than perspective, seldom providing infallible guidance, and variable in results”. Whilst a more contemporary perspective is presented by Lesh & Zawojewski (2007) who argue that heuristics involve strategies:

> intended to help problem solvers think about, reflect on, and interpret a problem solving situation more than they are intended to help them decide what to *do* when ‘stuck’ during a solution attempt (p. 768).

2.5.1 Polya (1957)

Mathematician and educationalist, George Polya, is universally famous for his work on mathematical problem solving. The emphasis of Polya's work focussed on the elements of plausible reasoning that lead to the discovery of mathematical assertions to which he referred to this type of reasoning ‘heuristics’, otherwise known as the mental operations typically useful in the process of solving mathematical problems. In his pioneering book, ‘How To Solve It’, (Polya, 1957) proposed four explicit phases to provide a more systematic or planned process approach to mathematical problem solving (Figure 2.6) and which relies on a repertoire of past experiences. A myriad of academics have grounded their research on this book, which Schoenfeld (1987, p. 17) eloquently proclaims as “a charming exposition of the problem-solving introspection”.

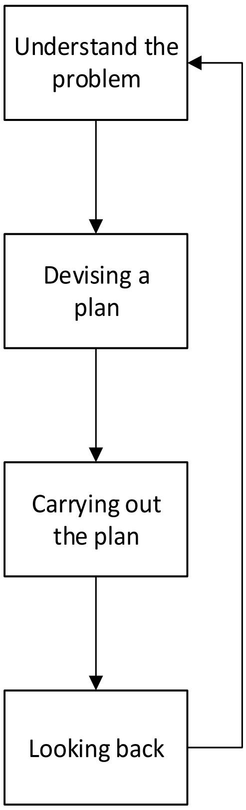
Worldwide, many countries have woven inextricably together the influential works of Polya within their educational systems. For example, in the USA, the NCTM (2000, p. 53) advocate that: “Of the many descriptions of problem-solving strategies, some of the best known can be found in the work of Polya (1957)”. Interestingly, his influence is not confined to mathematics education. For example, within the field of artificial intelligence (amongst other disciplines) his contribution to heuristics is well regarded. Minsky (1961, p. 28) articulates that “everyone should know the work of Polya on how to solve problems”.

I will examine Polya’s (1957) four phase model in more detail:

First Step - Understanding the problem

It may be obvious but this initial operation is crucial to fully understand the given information and the ultimate goal (Lesh & Jawojewski, 2007). Polya (1957, p. 6) maintains that: “It is foolish to answer a question that you do not understand”. Therefore, it is essential to appreciate what a problem is asking. For example, what are the unknown, available data and conditions? This may require becoming more ‘acquainted’ with the problem statement and in some cases, drawing a sketch or a diagram to show connections and relationships, making a table, using a model, working backwards or using a variable. Polya (1957, p. 33) suggests “attention bestowed on the problem may also stimulate your memory and prepare for the recollection of relevant points”. More specifically, he recommends teachers to select problems which are challenging but accessible.

Figure 2.6 Steps to follow when solving a mathematical problem (Adapted from Polya,1957)



Second step - Devising a plan

Many different strategic approaches are available at this stage such as a guess, searching for a pattern or connection between the data and the unknown or recalling a similar solved problem. On the theme of a guess and check strategy, Polya (1957, p. 99) posits that “many a guess has turned out to be wrong but nevertheless useful in leading to a better one”. Conversely, Malloy & Jones (1998, p. 149) argue “if a student guessed but could not explain the solution or did not use the guess to verify the solution, the guess was not considered a strategy". Polya (1957, p. 8) states that a suitable plan is devised "when we know, or know at least in outline, which calculations, computations, or constructions we have to perform in order to obtain the unknown". However, it may be apparent that the selected strategy needs modifying or to be discarded entirely. The application of past experiences may help to relate a similar problem that can be solved (Mayer, 2003). The mechanism to transfer knowledge of related problems requires analogical reasoning. A beautiful illustration of analogical reasoning is presented by Polya (1957) in his solution to solving the centre of gravity of a homogenous tetrahedron by forming a solution using a triangle. Thus, a constant review of the chosen plan is necessary during implementation of this phase.

Third Step - Carrying out the plan

Considered to be the most challenging component of the solution. Pupils must be prepared to validate each stage of the plan and modify any element of the strategy when it is obvious that it will not advance the desired outcome. Polya (1957) states that:

> To devise a plan, to conceive the idea of the solution is not easy. It takes so much to succeed; formerly acquired knowledge, good mental habits, concentration upon the purpose, and one more thing: good luck. To carry out the plan is much easier; what we need is mainly patience (p. 12).

Polya (1957, p. 13) counsels practitioners to insist that the learner should examine each part of the process and in certain cases demonstrate "if they can prove that the step is correct?" Such attention to detail requires perseverance and an awareness that modifications to the plan may result in the abandonment of the original strategy and the creation of a new approach.

Fourth Step - Looking back

This is deemed the most critical stage and extends beyond checking the answer. Detailed examination of the solution will reveal if the argument can be verified, generalised, enhanced, derived differently or applied to another problem. Polya (1957) warns that pupils have a tendency to stop when they have obtained a result and "miss an important and instructive phase of the work" (p. 14). He advises teachers to impress on their students the notion that no problem whatsoever is completely exhausted as there remains always something to do and believes that "we could improve any solution, and, in any case, we can always improve our understanding of the solution" (p. 15). This viewpoint is shared by Watson & Mason (2005, p. xiii-xiv) who advocate that: "No matter how profoundly one thinks one understands it is always possible to probe more deeply and to discover more connections and complexities". In other words, reflection is fundamental to the process in order to solidify mathematical understanding. Lesh & Zawojewski (2007, p. 770) note that: “By describing their own processes, students can use their reflections to develop flexible prototypes of experiences that can be drawn on in future problem solving”. The knowledge gained in looking back nurtures an investment in long time memory that can facilitate forthcoming problem solving encounters (Silver, 1982).

One of the common misunderstandings of Polya’s model is that it is a linear procedure that can be memorised, practiced and habituated. More accurately, it should be designated as dynamic, cyclic and iterative. However, Schoenfeld (1992, p. 353) whilst accepting the validity of Polya’s work maintains that it “did not provide the amount of detail that would enable people who were not already familiar with the strategies to be able to implement them” and suggests they are “descriptive rather than prescriptive”, which according to English & Sriraman (2010, p. 264-265) are mostly “just names for large categories of processes rather than being well-defined processes in themselves”. Lesh & Zawojewski (2007, p. 769) offer a broader perspective on Polya’s heuristics “as not only prompting ways of selecting and carrying out procedures and rules (i.e. “doing” mathematics), but also as a means of developing systems for interpreting and describing situations (i.e. “seeing” mathematically)”.

Although Polya did not include the term ‘metacognitive’ in any of his work, each phase of his four step model of mathematical problem solving are metacognitive in nature. At this point in this thesis, I feel it pertinent to provide a definition of metacognition and refer to Flavell (1976):

> Metacognition refers to one’s knowledge concerning one’s own cognitive processes and products or anything related to them, e.g., the learning relevant properties of information or data... Metacognition refers, among other things, to active monitoring and consequent regulation and orchestration of these [cognitive] processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete goal or objective (p. 232).

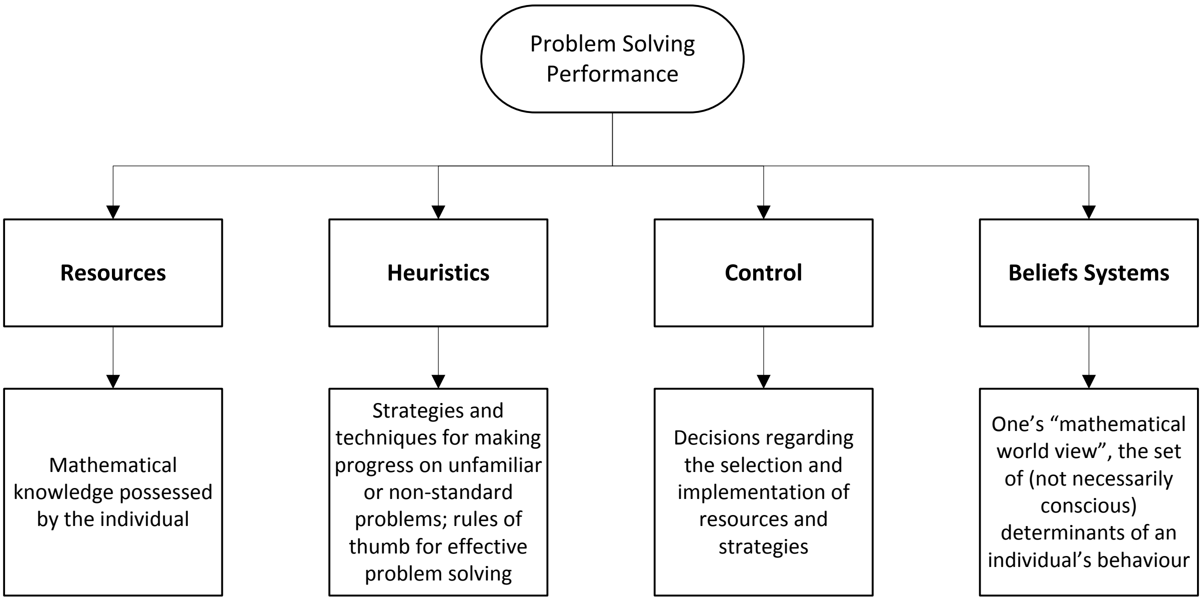
Silver (1982, p. 21) reinforces this viewpoint when he declares that: “If we adopt a metacognitive perspective, we can view many of Polya’s heuristic suggestions as metacognitive prompts”.

2.5.2 Schoenfeld (1985)

In his book, 'Mathematical Problem Solving', Schoenfeld (1985) offers a comprehensive overview of his framework and methodological approach to the exploration of undergraduate mathematical problem solving of which can be respectively applied to school mathematics. On reflection, Schoenfeld (2013, p. 10) describes this publication as "a framework for the analysis of the success of failure problem solving attempts, in mathematics and hypothetically in all problem solving domains".

Schoenfeld (1985) proposes a model grounded on the theory that a learner's ability to solve problems is determined by their individual functioning cognitive and emotional characteristics and identifies four categories of knowledge and behaviour fundamental for mathematical problem solving performance (Figure 2.7). In essence, he argues that it is possible to explain an individual's success or failure in trying to solve a problem on the basis of four categories:

Figure 2.7 Framework for solving mathematical problems (Adapted from Schoenfeld, 1985)



<u>Resources</u>

According to Schoenfeld (1985, p. 17), these refer to "an inventory of all the facts, procedures, and skills – in short, the mathematical knowledge" that the individual is capable of bringing to bear on a particular problem". Examples include intuitions, informal knowledge, algorithmic procedures, non-algorithmic procedures and understandings about the agreed-upon on rules for working in the domain.

Heuristics

Much development work is based on promoting these types of strategic processes. Exemplars include “exploiting analogies, introducing auxiliary elements in a problem or working auxiliary problems, arguing by contradiction, working forward from the data, decomposing and recombining, exploiting related problems, drawing figures, generalizing” (p. 23). Schoenfeld discusses a series of major obstacles to the ascendancy of heuristics strategies such as the lack of prescriptive detail afforded to strategies by claiming “that their definitions are too vague to serve as a guide to their implementation” (p. 95). Moreover, he emphasises the need to have a strong general knowledge base of mathematics as “even a good mastery of heuristics cannot be expected to replace shaky mastery of subject matter” (p. 96).

(Metacognitive) control

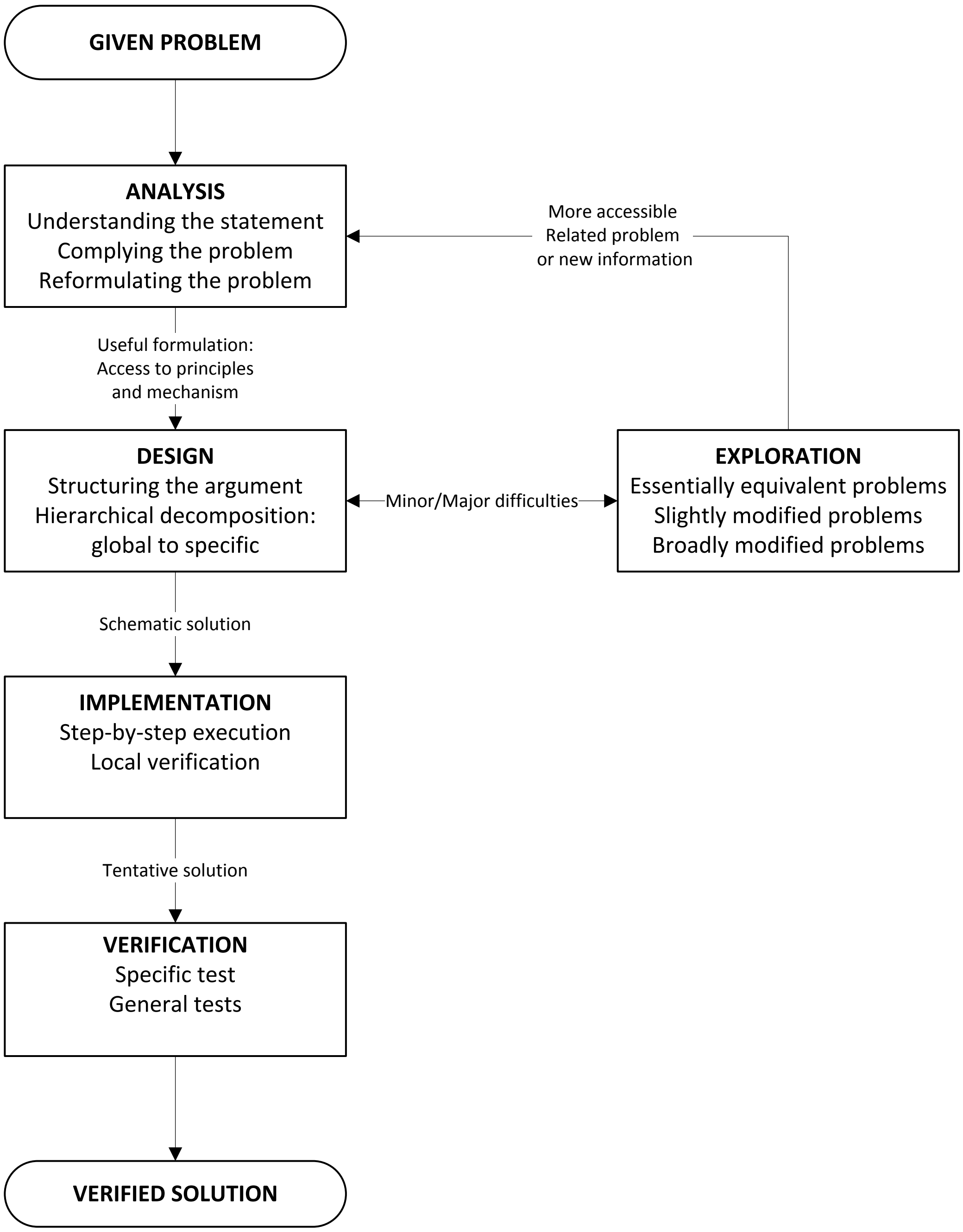
This classification of behaviour deals “with the way that individuals use the information potentially at their disposal” (p. 27). It involves “planning, monitoring and assessment, decision-making and conscious metacognitive acts” (p. 15). Schoenfeld (1985) submits the view that ‘good’ problem solvers metacognition differ significantly from ‘novices’ in the efficacy of their metacognitive strategies. He reinforces the position that: “One of the hallmarks of good problem solvers’ control behaviour is that, while they are in the midst of working problems, such individuals seems to maintain an internal dialogue regarding the way that their solutions evolve” (p. 140). In other words, they are more skilled at managing different mathematical resources. Lester (1994, p. 666) argues that “effective metacognitive activity during problem solving requires knowing not only and when to monitor, but also how to monitor”. In sum, metacognition plays a critical role in successful problem solving (e.g. Lester, 2013).

Belief systems

These signify an individual’s mathematical world view and the “perspective with which one approaches mathematical and mathematical tasks... Beliefs establish the context within which resources, heuristics and control operate” (p. 45). They shape the knowledge drawn upon and the mobilisation of that knowledge. Schoenfeld (1985) emphasised the need for future research on metacognition and beliefs. Main findings arising from his studies include: “Explicit heuristic instruction does (or can) make a difference with regard to problem-solving performance” (p. 215) and “students in a problem–solving course can learn to employ a variety of heuristic strategies” (p. 240).

Schoenfeld (1985) offers a schematic outline of a problem solving strategy used with his own students (Figure 2.8) and makes reference to exploration which he designates as "the heuristic heart of the strategy, for it is in the exploratory phase that the majority of problem-solving heuristics come into play" (p. 110).

Figure 2.8 Schematic outline of problem solving strategy (Adapted from Schoenfeld,1985)



However, two limitations are notable. Firstly, the participants worked in isolation thereby minimising social interactions. Secondly, all of the mathematical problems were supplied by the researcher which constrained the potential outcome as the objectives were established in advance. In short, Schoenfeld did not offer a theory of mathematical problem solving but a framework for analysing the success or failure of problem solving within a wide domain.

More recently, Schoenfeld (2011) provided an updated theoretical lens from which to explain how and why problem solvers make decisions that shape and guide their problem solving behaviours. The basic structure offered is recursive where individuals orient to situations and decide (on the basis of beliefs and available resources) how to pursue their goals. If the situation is familiar, they implement familiar routines; if mechanisms are unfamiliar or problematic, they reconsider. An interesting characteristic of this theoretical work is that although mainly designed for primary and mathematics teachers, it can be applied to analyse and predict the behaviours of other professions. I have summarised the framework used by Schoenfeld (2011) in Figure 2.9.

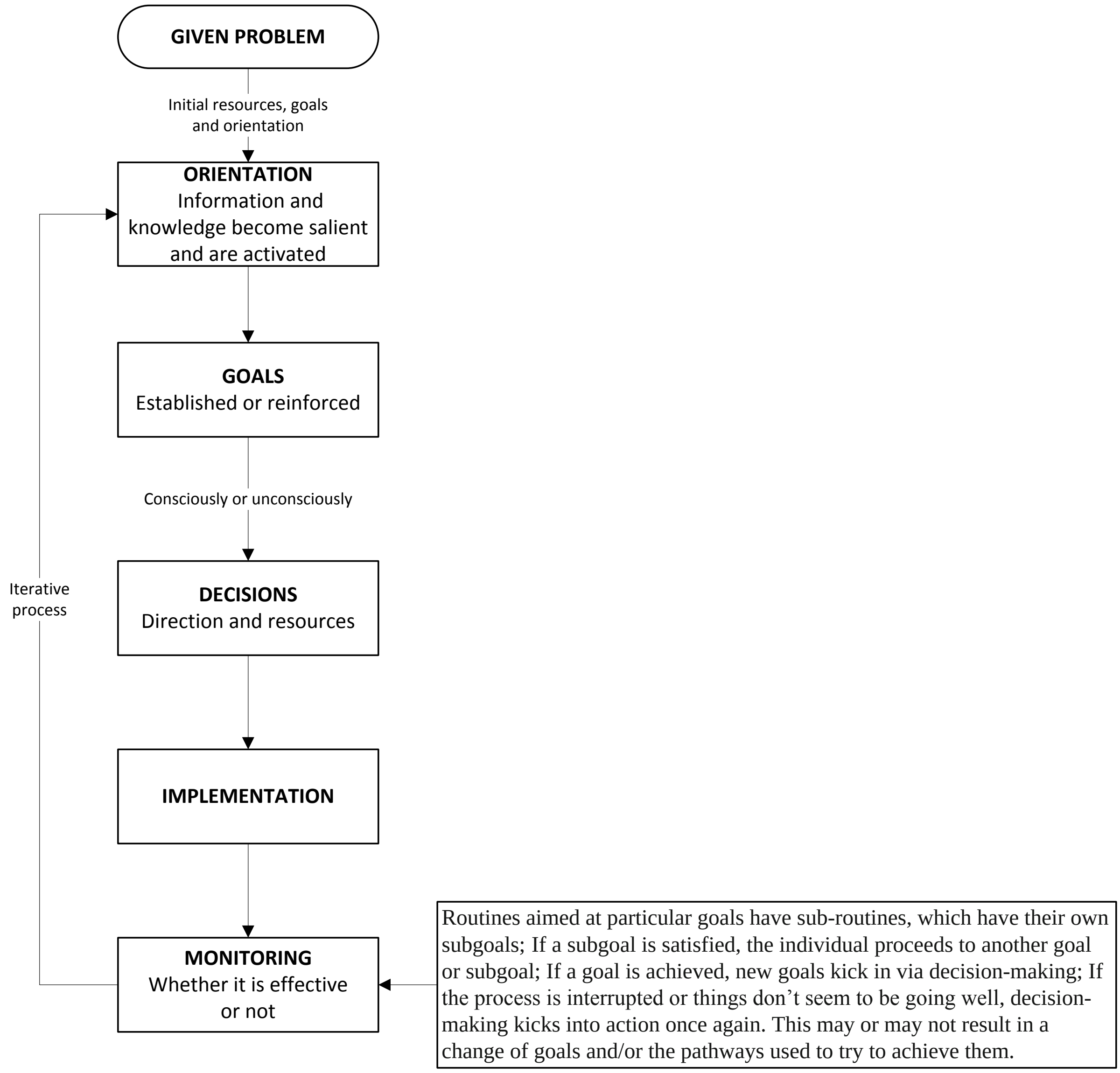
2.5.3 Does the teaching of heuristics improve pupil performance?

In this subsection, I make two assumptions. Firstly, pupil performance is concerned with conceptual understanding that leads to improved problem solving rather than procedural knowledge. In this case, Foong (1991, p. 45) argues that to become effective problem solvers, instructional activity must build "a repertoire of heuristics that are likely to be useful in a variety of problem situation, along with meta (cognitive) knowledge about situations in which specific heuristics are appropriate". Secondly, pupils cannot become successful problem solvers overnight (Hiebert, 2003; Lambdin, 2003; Lester & Cai, 2016). Considerable institutional investment is required throughout primary and secondary levels.

Whilst it may be plausible to anticipate that evidence supports the teaching of heuristics, the literature suggests only a weak correlation exists at best (Schoenfeld, 1979, 1985, 1992; Charles & Silver, 1988; Lester, 1994; Lesh & Zawojewski, 2007). According to Lester & Kehle (2003, p. 508): "Teaching students about problem solving strategies and heuristics and phases of problem-solving does little to improve students' ability to solve general mathematics problems". Previous reports concur with this claim. In his robust assessment of the research, Silver (1985) suggests that even in studies where some positive learning has been reported, the transfer of learning was insignificant. Likewise, Beagle (1979) noted that:

> A substantial amount of effort has gone into attempts to find out what strategies students use in attempting to solve mathematical problems... no clear-cut directions for mathematics education are provided... In fact, there are enough indications that problem solving strategies are both problem and student specific often enough to suggest that hopes of finding one (or few) strategies which should be taught to all (or most) students are far too simplistic (p. 145).

Figure 2.9 How things work (Adapted from Schoenfeld, 2011)



However, the success of any problem solving experience is interrelated to the pedagogical skills of the teacher involved. Although this has to be balanced against the retrievable nature of the research literature on mathematical problem solving. In practical terms, this translates to how effectively teachers can delineate generalisations in order to impact regular professional practice. A major caveat for teachers to consider is the link between theory and practice which is unclear (Lesh & Zawojeswki, 2007; Lester, 2013; Lester & Cai, 2016).

Irrespectively, in a study of 20 Israeli current teachers, Koichu, Berman & Moore (2003, p. 7) concluded that “heuristic training of students may be an effective tool in combination with heuristic training of their teachers, induced either by personal problem solving experience or by learning through teaching of regular curriculum with deliberate emphasis on heuristic approach in problem solving”. This pedagogical perspective may be well tender the remedy to improving the problem solving performance of learners. Though, given that a copious amount of research on learning and teaching of mathematical problem solving has already been conducted over the last 30 years or so, it would appear unlikely that the teaching of general heuristics has little effect on improving problem solving skills (Lester & Cai, 2016). Still, Ambrus & Barczi-Veres (2016) challenge this view based on their recent study of 16 Hungarian secondary pupils. The researchers noted that the implementation of open problems, questioning and cooperative techniques had a positive impact on student achievement.

2.6 Multiple solutions

The beauty of elegantly designed mathematical problems leads to the facilitation of an assortment of multiple solutions. Leikin (2013, p. 388) defines a multiple solution task as “an assignment in which a student is explicitly required to solve a mathematical problem in different ways”. According to Leikin (2011), solutions to the same problems are considered different when they involve (a) different representations of concepts (b) different theorems or mathematical relationships to support conjectures (c) different conceptual arguments and reasoning. All too often, pupils embrace the misconception that there is only one precise method to approach and solve a problem and fail to develop flexibility in formulating, selecting appropriate strategies and searching for alternative solutions (Cai & Nie, 2007). For example, to solve a system of two linear equations, a graphical solution can be provided. Alternatively, we can use algebra (elimination or substitution method), matrices or trial and error.

Engaging learners with problems that may be approached by employing different representations is widely accepted as fostering good practice (Tsamir et al., 2010) and entrenched within the looking back step of Polya’s (1957) heuristic. This is an essential constituent of any mathematics classroom as incorporation of these problems will deliver a vehicle for pupils to construct rich mathematical connections. Silver et al. (2005, p. 288) maintain that learners profit from comparing, reflecting on and discussing multiple solution methods and claim that “different solutions can facilitate connection of a problem at hand to different elements of knowledge with which a student may be familiar, thereby strengthening networks of related ideas”. When regularly exposed to problems that require multiple strategies, pupils learn different approaches and are more efficient in selecting appropriate methods to elucidate problems (Dowker, 1992) and can solve problems with greater ease and flexibility (Siegler, 2003). The NCTM (2000) designated that students should develop their “flexibility in exploring mathematical ideas and trying alternative solutions” (p. 21) and that educators should cultivate their students capacity “to link mathematical ideas and deeper understanding of how more than one approach to the same problem can lead to equivalent results, even though the approaches might look quite different” (p. 354). Similarly, the Scottish Government (2009, p. 2) assert that an important feature of effective learning and teaching of mathematics should ensure that primary and secondary pupils “explore alternative solutions” and aquire opportunities “presenting their solutions to others in a variety of ways”.

Not all mathematical problems offer multiple solutions but some branches present more rich opportunities than others. Whilst the power of algebra and calculus cannot be underestimated, Levav-Waynberg & Leikin (2012a) argue that geometry is a fertile ground to search for problems that encompass more than one solution. They maintain that “experience shows that almost any geometrical problem in a regular geometry textbook has multiple solutions” (p. 316) and “geometry contains a rich variety of problems with multiple solutions accessible to learners” (p. 329). However, within my professional practice I have employed problems from a wide spectrum of mathematics including many originating from other national syllabuses. My favourite is a combination problem involving ten people where everyone shakes hands with everybody else exactly once, where the objective is to determine how many handshakes take place. One of the solutions is a practical approach ensuring that this problem is within reach of all secondary pupils. In order to illustrate a task which offers multiple solutions, consider the following problem (McDonald, 2014), which is accessible to the majority of secondary pupils including well able primary pupils:

- <u>Example</u>

It is projected that the worth of a lump sum investment is 5% more than its value in the previous year. Find in as many ways as possible, the number of years that it will take for the investment to double.

Solution 1 Computationally

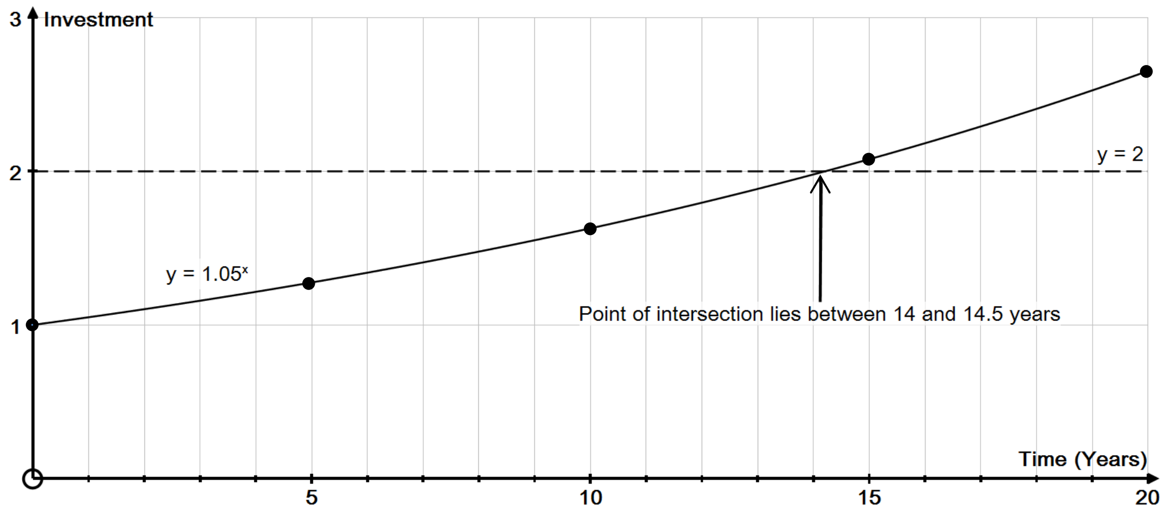
Since no value for the investment is specified, we can choose any number to represent this unknown amount. Selecting £100 and increasing by 5% each year will produce:

Year	Investment (£)
0	100.00
1	105.00
2	110.25
3	115.76
4	121.55
5	127.63
6	134.01
7	140.71
8	147.75
9	155.13
10	162.89
11	171.03
12	179.59
13	188.56
14	197.99
15	207.89

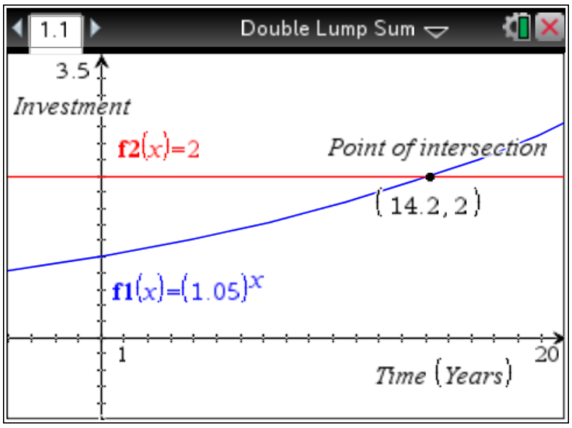
By considering only complete years, our answer is 15 years

Solution 2 Graphically

In order to find an approximate solution, we must draw the function $y = 1.05^x$ and $y = 2$. The intersection point is dependent on the degree of accuracy of the graph produced.



For a more accurate graphical solution, we can use a GDC (e.g. TI-Nspire):



Solution 3 Algebraically

Let x be the initial lump sum where n is time in years,

$\Rightarrow\ x\,(1\ +\ 0.05)^n =\ 2x$

$\Rightarrow\ \ (1.05)^n =\ 2$

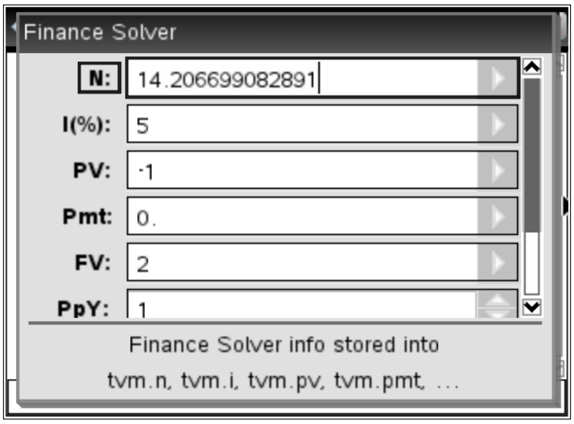
$\Rightarrow\ n\ \ln 1.05 =\ \ln 2$

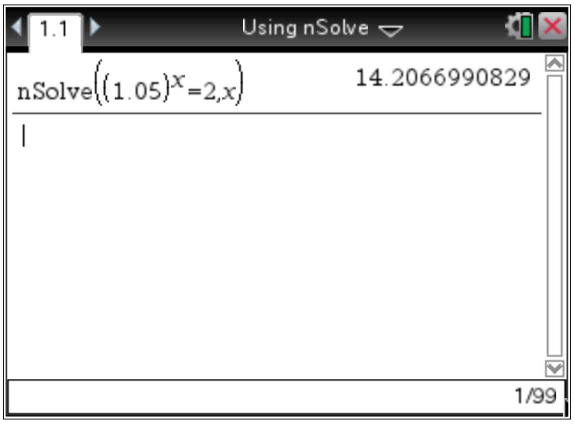
$$\Rightarrow\ n\ =\ \frac{\ln 2}{\ln 1.05}$$

$\Rightarrow\ n =\ 14.21$ years

Solution 4 ICT (e.g. TI-Nspire)

Whilst this approach is deemed a different strategy, it only serves to generate an algorithmic solution which does not augment the development of problem solving skills. Nevertheless, I have found this to be a valuable instrument during class discussions when comparing other solutions.





Solution 5 Trial and error

If we let $1.05^n = 2$, where n is the number of years, then we can determine the value of n by continually improving our guess, something like this:

n	$(1.05)^n$	Comment
20	2.653	Too big
10	1.639	Too small
15	2.079	Too big
14	1.980	Too small
14.5	2.029	Too big
14.25	2.004	Too big
14.22	2.001	Too big
14.21	2.000	Solution

2.6.1 Teachers and multiple solutions

Whilst there is a growing body of evidence that suggests that the engagement of multiple solutions can improve pupil learning (e.g. Stein & Lane, 1996; Silver et al., 2005; Rittle Johnson & Star, 2007; Star & Rittle-Johnson, 2008), considerably fewer studies exist that have examined the impact on teachers regarding the instructional process. For example, Ma (1999) found that Chinese teachers enhanced their mathematical curriculum knowledge while Stigler & Hiebert (1999) noted in their international comparative analysis of practitioners from America, Germany and Japan, the quality of teaching improved. Intriguingly, from a primary perspective, Shimizu (2003, p. 206) reports that “Japanese teachers in elementary schools often organize an entire mathematics lesson around multiple solutions to a single problem in a whole-class instructional mode. This organization is particularly useful when introducing a new concept or a new procedure during the initial phase of a teaching unit”. This pedagogical approach is common in other PISA high performing countries such as China and Hong Kong (Stigler & Hiebert, 1999; Cai & Nie, 2007).

In contrast, other studies have identified concerns from practitioners which have impacted on their engagement of multiple solutions. In their study of 12 American middle and secondary mathematics teachers, Silver et al. (2005) reported uncertainties regarding perceived time constraints and that more than one approach may coalesce to confuse less able learners. Equally, Leikin & Levav-Waynberg (2007) discovered that in their study of the professional development of 12 Israeli mathematics teachers, participants’ limited domain expertise coupled with weak pedagogical content knowledge prevented them from using multiple strategies. Based on a study of 92 American teachers' views about teaching with multiple strategies, Lynch & Star (2014) argue that a divide exists between the views of primary teachers and that of middle and secondary mathematics teachers. The researchers hypothesise there may be several contributing factors including different amounts of contact time with pupils in the sectors.

Solving mathematical problems using different methods can also serve as a didactical tool and as a diagnostic assessment technique (Levav-Waynberg & Leikin, 2009). Nevertheless, teachers are only too aware of the general apathy for some learners to disengage after securing a solitary outcome to a given problem. This disengagement may possibly be as a result of practitioners failing to encourage the habit of searching for multiple solutions or simply not regularly employing a variety of methods or approaches in the classroom. We are reminded by Polya (1957, p. 173) that "it should not be forgotten that a teacher of mathematics should know some mathematics and that a teacher wishing to impart the right attitude of mind towards problems to his students should have acquired that attitude himself". In offering his own experience, Schoenfeld (1994b) advocates that he prefers problems that can be solved, or at least approached, in a number of ways because:

> It's good for students to see multiple solutions, since they tend to think, on the basis of prior experience, that there is only one way to solve any given problem (which is usually the method the teacher has just demonstrated in class). I need for them to understand that the "bottom line" is not just getting an answer, but seeing connections. Moreover, on the process level, the possibility of multiple approaches lays open issues of executive decisions – what directions or approaches should we pursue when solving problems, and why? (p. 69).

However, in a study involving Turkish primary teachers, Bingolbali (2011) found that many participants during the implementation of a new mathematical curriculum did not value alternative solutions and reported experiencing difficulties in evaluating pupil's alternative solutions. His findings reveal a significant variation in the grading of different solutions resulting in practitioners conveying mixed messages to their pupils by promoting effort at the expense of mathematical accuracy. Many valid reasons may exist to explain why teacher nuances exist in this regard. One possible reason could be that some teachers do not share the same mathematical beliefs about problem solving as others do. Burton (1984, p. 23) warns that "if your pupils never see you engaged in problem solving, they will learn that despite what goes on in your classroom, it is not an activity which is important to you". Alternatively, some teachers may not support the notion that promoting multiple solutions in geometry is an effective use of their time in order to provide a benefit to their pupils learning of mathematics (Levav-Waynberg & Leikin, 2012a).

2.6.2 Mathematical creativity

The literature contains numerous contrasting structural and multifaceted definitions of mathematical creativity (Mann, 2006; Sriraman, 2009). Leikin et al. (2013) refer to the work of Guilford (1967) when distinguishing between convergent and divergent thinking of creativity. They argue that the act of convergent thinking involves seeking a single, precise solution to a problem, whereas divergent thinking is the creative product that generates multiple solution strategies. In the same vein, Leikin (2013) in her theoretical model of mathematical creativity expands the connection between creativity and divergent thinking based on an explicit requirement to solve mathematical problems in multiple ways. In this thesis, I will use the definition of mathematical creativity first suggested by Silver (1997, p. 75) who saw it as "an orientation or disposition towards mathematical activity that can be fostered broadly in the general school population". My motive for using this definition is centred on a belief that every child has the potential to be creative and that schools are obliged to fashion an educational environment to promote independent thinking to develop creativity skills (Education Scotland, 2013).

Mathematics educational research is plentiful with the influence of constructivist concepts that strongly value learner's individual knowledge building and independent development that perpetuate the solving of mathematical problems exercising multiple solutions (e.g. Schoenfeld, 1983; Silver, 1997; Leikin et al., 2006; Sriraman, 2009; Levav-Waynberg & Leikin, 2012a, 2012b). The significance of solving mathematical problems using multiple methods can promote advanced mathematical thinking amongst pupils. Krutetskii (1976) and Ervynck (1991) link the concept of mathematical creativity to multiple solutions. Leikin & Levav-Waynberg (2008, p. 234) argue that "solving problems in multiple ways contributes to the development of student's creativity and critical thinking".

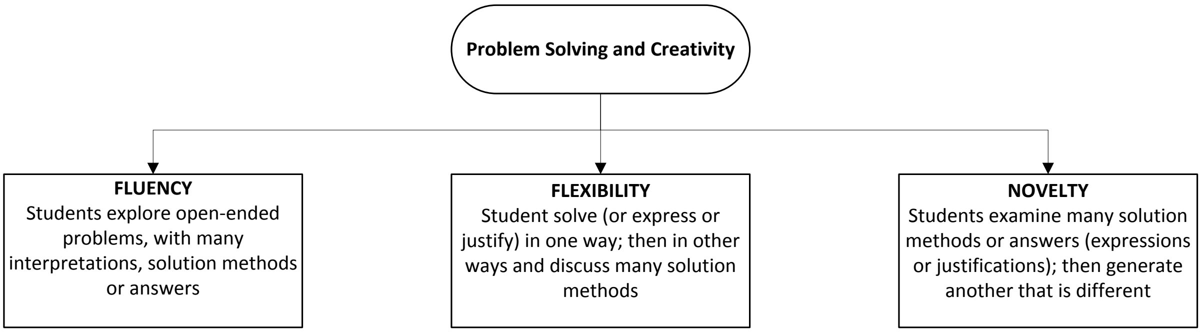
However, it is evident that the dynamic perspective surrounding mathematical creativity is not encouraged by the lack of formal evaluation in national examinations (e.g. Scotland). Chamberlin & Moon (2005, p. 42) lament that "the significance of creativity in school mathematics may be minimised because it is not formally assessed in standardized tests, which purport to thoroughly measure mathematical learning". Silver (1997, p. 75) counsels that whilst "genuine mathematical activity is intimately interwoven with creativity, schooling provides most students with little opportunity to experience this aspect of the domain of mathematics". Grounded on the work of Torrance (1974), he strongly advocates that teachers' nurture creativity through mathematical problem solving by promoting fluency, flexibility and novelty and illustrates (Figure 2.10) the relation of problem solving instructional activities to core components of mathematical creativity.

Teachers must encourage, support and cultivate the rich development of mathematical creativity by engaging learners in imaginative exploration during the integration of problem solving. Shriki (2010, p. 161-62) warns that: "Refraining from development of creativity in the classroom conveys the impression that mathematics is merely a set of skills and rules to memorize, and in doing so, many students' natural curiosity and enthusiasm for mathematics might vanish". Resonating with this advice is Nickerson (2011) who asserts that to enhance creativity in the classroom; practitioners need to nurture their learners, especially with ideas that are unconventional. He makes a poignant remark that I am sure all teachers can relate to:

> Failure to promote creativity in the classroom may well be due sometimes to recognition of the increased challenge that creatively expressive children represent to classroom order and teacher authority (p. 414).

In the course of teaching mathematical problem solving, I have observed children give up after a period of intense effort. As a response, I normally introduce an alternative approach such as 'time out' or provide an unrelated activity. Occasionally, on return to the problem, it has stimulated a breakthrough which has produced a creative piece of work.

Figure 2.10 Problem solving and creativity (Adapted from Silver, 1997)



Nevertheless, I suspect that some teachers do not accept the view that every pupil has the potential to demonstrate mathematical creativity. In a study of primary children, Kattou et al. (2013) found a strong positive correlation between mathematical creativity and mathematical ability. The researchers suggested that mathematical creativity is a subcomponent of mathematical ability. Likewise, in another study involving secondary pupils, Leikin & Lev (2013) noted that gifted pupils (high IQ) outperformed all other pupils on every measured task involving mathematical creativity. More recently in a further study of secondary pupils, Lev & Leikin (2017) assert that expertise in mathematics is a prerequisite for an individual to be creative. Together, these studies indicate that the imaginative promotion of multiple solutions during problem solving will stimulate and develop creativity skills.

2.6.3 Mathematical competence

In this section, I focus on three critical elements of mathematical competence which are intrinsic to generating multiple solutions within problem solving i.e. procedural knowledge, conceptual knowledge and procedural flexibility. Procedural knowledge is expressed as an integrated and functional grasp of mathematical ideas that can be utilised as an action sequence for solving problems (Kilpatrick, Swafford & Findell, 2001; Rittle-Johnson, Siegler & Alibali, 2001). In contrast, conceptual knowledge is considered explicit or implicit understanding of the principles that govern a domain and the interrelationships between parts of knowledge in a domain (Rittle-Johnson, Siegler & Alibali, 2001). More prosaically, it refers to the richness of the mathematical relationships and range of connections (Hiebert & Leferve, 1986). Over the years, much debate has taken place regarding the significance of drill and practice methods versus theoretical understanding. In their impressive review of the literature, Hiebert & Grouws (2007) concluded that both procedural and conceptual knowledge were crucial for successful mathematics instruction to take place. Procedural flexibility incorporates knowledge of multiple approaches and a propensity to select the most appropriate solution based on specific problem characteristics (Kilpatrick, Swafford & Findell, 2001; Star, 2005).

<u>Comparing different solutions</u>

During mathematical problem solving, it is highly likely that multiple solutions will be generated and thus launches a suitable platform for pupils to compare different solutions (assuming that they know that mathematical problems can have more than one solution). It is this action of comparing different solutions that will help extend knowledge by linking new knowledge to prior knowledge. Goldstone, Day & Son (2010, p. 103) note that comparison is one of the most fundamental components of human thought and "research has demonstrated that the simple act of comparing two things can produce important changes in our knowledge". Gentner (2005, p. 251) maintains that: "Comparison is a general learning process that can promote deep relational learning and the development of theory level explanations". Rittle-Johnson, Star & Durkin (2012) evaluated 198 American pupils learning of multi-step equation solving. They found that whilst comparing procedures had a limited impact on conceptual and procedural knowledge, procedural flexibility was significantly improved.

Thus far, a number of studies have explored the relationship between multiple solutions and mathematical competence. Rittle-Johnson & Star (2007) found that in their study of American children, comparing and contrasting solution methods was more effective than reflecting at same solution methods one at a time with respect to procedural knowledge and flexibility. However, the study would have been more interesting if it had incorporated non-algorithmic procedures. Likewise, in another experimental study, Rittle-Johnson & Star (2009) discovered that comparing solutions of the same problem augmented conceptual knowledge and procedural flexibility, than comparing solutions of similar problems with equivalent mathematical structure. Similar evidence suggested that presenting multiple solution strategies simultaneously is better than presenting them sequentially. The researchers draw attention to the role of prior knowledge and note that learning gains may be more beneficial for pupils with low prior knowledge.

Star & Rittle-Johnson (2008) showed that encouraging American elementary learners to solve linear equations using different methods improved procedural flexibility in problem solving. Schukajlow & Krug (2014) argue that teachers should support young people in developing multiple solutions during problem solving. In their study of German secondary pupils, the researchers investigated the influence of prompting learners to construct multiple solutions for real-world problems with vague conditions on pupils' interest in mathematics as well as on their experiences of competence and autonomy and the number of solutions developed. They revealed the positive influence of prompting pupils to find multiple solutions on individual interest in mathematics. In a further study of American pupils, Star, Rittle-Johnson & Durkin (2016) discovered that comparing different strategies for solving the same problem improved learning. Whilst the premise for this research was to simplify instructional methods for teachers, the main weakness of the study was the infrequent use of materials by the participants.

2.7 Problem solving as an overarching goal of mathematical learning

In many traditional classrooms an emblematic approach to introducing a new mathematical concept encompasses a teacher led demonstration of a computational algorithm, supported with the decontextualized treatment of some worked examples, followed by the repetitive setting of numerous analogous exercises for pupils to develop procedural knowledge. Typically, through a lack of challenge many children acquire weak conceptual understanding and attempt to subsist by memorising mechanical techniques. Whilst the requirement to enhance mathematical knowledge is recognised, a powerful opportunity to present pupils with the tools to discover the rich conceptual mastery is lost. We are reminded by the NTCM (2014, p. 17) that "learning is greatest in classrooms where the tasks consistently encourage high-level thinking and reasoning and least in classrooms where the tasks are routinely procedural in nature". Hiebert et al. (1997, p. 1) encourage the need for pupils to learn mathematics with understanding and argue that "things learned with understanding can be used flexibly, adapted to new situations, and used to learn new things. Things learned with understanding are the most useful things to know in a changing and unpredictable world". Grounded on research that teaching with a clear focus on understanding can cultivate the development of pupils' mathematical problem solving abilities (Hiebert, 2003; Lambdin, 2003), this section attempts to explore the instructional options for operationalising problem solving in schools.

2.7.1 Instructional approaches

Teachers have long been faced with a dilemma of how to coalesce mathematical problem solving into their professional practice. Given that problem solving is a complex and challenging mathematical enterprise (Lester, 2013), practitioners need to understand how to orchestrate an approach that will stretch and sustain the limit of pupil thinking. Various textbooks have 'story problems' isolated at the end of each instructional chapter, and thus concomitantly serve to perpetuate the notion that problem solving is a simple voluntary add-on task. Previous classroom encounters may remind educators of the difficulties that are apparent when catering for learners with an eclectic mix of mathematical abilities and experiences.

However, pupils must have a positive attitude towards the regular engagement of challenging activities (Lester & Charles, 2003; Schoen & Charles, 2003). Likewise, all primary and secondary mathematics teachers must be able to sustain a long-term pedagogical commitment to developing learners' abilities to solve problems. Such an obligation is necessary to ensure that every child, regardless of stage and capability, is taught using a method that fosters understanding of concepts, procedures and solving problems (Lester & Cai, 2016). Crucially, practitioners should be experienced problem solvers and should have a firm grasp of what successful problem solving involves (Lester, 2013; Chapman, 2015).

In their interesting analysis, Schroeder & Lester (1989) describe a theoretical framework outlining three distinct classroom instructional approaches to support teachers with mathematical problem solving:

1. Teaching mathematics for problem solving

In this approach, Schroeder & Lester (1989, p. 32) maintain "the teacher concentrates on ways in which the mathematics being taught can be applied in the solution of both routine and non-routine problems" and "students are given many instances of the mathematical concepts and structures they are studying and many opportunities to apply that mathematics in solving problems". That is, problem solving is undertaken after new mathematical concepts and procedures have been mastered. For example, in calculus pupils learn the rule for differentiation and then apply this technique to solve optimisation problems. Although, this method is engrained as the conventional instructional approach to problem solving, it requires that all learners have the necessary prior knowledge to understand new concepts. Typically, it involves a teacher presenting one method to perform a procedure which may disadvantage pupils who possess alternative solutions. Van de Walle, Karp & Bay-Williams (2014, p. 55) warn that this one dimensional tactic "can communicate that there is only one way to solve the problem, a message that misrepresents the rule of mathematics and disempowers students who naturally may want to try to do it their own way". Another drawback is that pupils may be afforded excessive help which will eliminate any cognitive demand and the necessity to 'struggle'. Hiebert et al. (1997) contend rich mathematical ideas are generated as a product of problem solving experiences that offer challenge opposed to the execution of standard algorithms. Learners need to explore problem situations and invent strategies to solve problems (Cai & Lester, 2016).

Nevertheless, it is purported that this method has merit and with the colossal pressure to prepare pupils for high stakes examinations, will probably ensure that this approach will continue for some time. For example, teaching for problem solving is the desired approach within CfE. The Scottish Government (2009, p. 2) emphatically state that: “Mathematics is at its most powerful when the knowledge and understanding that have been developed are used to solve problems”. Moreover, the Scottish Government (2011a, p. 4) emphasise development of higher-order thinking skills “that enable the learner to identify which particular mathematical techniques can be appropriately applied in order to progress towards a solution to a problem”. However, Siemon (1986, p. 35) cautions that to “spend the majority of one’s time “doing mathematics as it has always been done”, with “problem solving” added on as an interesting appendage, actively acts against encouraging a problem-solving approach”. This perspective resonates with Cai (2010) who warns that separating learning skills and concepts from problem solving does not contribute to improving pupil learning.

2. Teaching about mathematical problem solving

This process will seek to develop and encourage an awareness of mechanisms that will allow pupils to access a range of appropriate strategies to attempt to solve problems, at the expense of learning mathematics (English, Lesh, & Fennewald, 2008). For example, young people are taught Polya style heuristics such as draw a picture, make a table, organise a list, look for a pattern, write an equation, etc. Paradoxically, while this requires a significant investment of time to illuminate and demonstrate relevant processes, it is worthwhile as without problem solving skills, pupils need a prolonged period to solve problems successfully. Leong et al. (2016) maintain that the language of problem solving can be easily transferred and reinforced when solving future problems. However, Schroeder & Lester (1989, p. 34) caution that “instead of problem solving serving as a context in which mathematics is learned and applied, it may become just another topic, taught in isolation from the content and relationships of mathematics”. In order to circumvent such an undesirable outcome, Leong et al. (2016) argue that teachers should employ problems containing mathematical conditions that require mathematical solutions. This will allow pupils to link their conceptual and procedural knowledge to a cycle of thinking and asking questions, as a technique to augment their generic ability. One method to ensure that teachers have allotted time for this intervention is to include a structural change to the planned mathematics curriculum through the introduction of a formal component (Leong et al, 2016). Though, as indicated earlier, the main limitation of this approach is that pupils are unable to solve all types of mathematical problems (Lester, 1994).

3. Teaching mathematics through problem solving

In this rewarding approach, problem solving and learning mathematics are interdependent (Lambdin, 2003; Lesh & Jawojewski, 2007). Schroeder & Lester (1989) contend that:

> problems are valued not only as a purpose for learning mathematics but also as a primary means of doing so. The teaching of a mathematical topic begins with a problem situation that embodies key aspects of the topic, and mathematical techniques are developed as reasonable responses to reasonable problems (p. 33).

Similarly, individuals are forced into a state of needing to connect what they know with the problem at hand (Lambdin, 2003). For example, in exploring the vertex of a quadratic function, pupils are led to discover the procedure for completing the square and how to identify the axis of symmetry.

To date, a number of studies have suggested that this approach as an important linkage between theoretical research and effective practice which fosters learners problem solving abilities, reasoning skills and mathematical conceptual understanding (Cai, 2003; Lester & Cai, 2016; Leong et al., 2016). Lester & Lambdin (2004) draw a parallel with constructivism and maintain that pupils become active participants in the creation of knowledge rather than passive receivers of rules and procedures. Lester & Charles (2003, p. xi) argues that as young children attempt to solve problems, "they come to understand the mathematical concepts and methods involved, become more adept at mathematical problem solving, and develop mathematical habits of mind that are useful ways to think about any mathematical situation". What is consistently underpinned is the interplay between problem solving ability and mathematical understanding. Significantly, learners are afforded more chances to express their mathematical ideas and justify their answers verbally, including increased opportunities to engage in cognitively demanding tasks (Lampert, 1990; Hiebert & Wearne, 1993).

However, in my professional experience, solving mathematical problems is not perceived by teachers in the same light as computational skills required to find the equation of a circle, simplify an expression by applying the laws of logarithms, or using integration to find the area of a function below the x-axis. From a pedagogical perspective, teaching through problem solving requires a paradigm shift in the philosophical role of the teacher. Enhanced responsibility to select appropriate quality tasks that nurture mathematical knowledge blended with strategic questioning and an effective understanding of when to extend and formalise pupil thinking, will place an increase on the demand of the teacher (Van de Walle, Karp & Bay-Williams, 2014). Coordinating classroom discourse is multifaceted and requires high cognitive levels while pupils are learning and validating mathematical concepts (Smith, Hughes & Engle, 2009; Kilic et al., 2010). In his discussion on the Japanese school approach, Shimizu (2009, p. 100) concludes that: "In order to be successful, teachers have to understand well the relationship between the mathematics content to be taught and students' thinking about the problem to be posed. Anticipating students' responses to the problem is the critical aspect of lesson planning". Much encouragement and support is required for practitioners to learn this role which cannot be easily accomplished through attendance at training courses but primarily through professional interactions with colleagues and research.

Moreover, there may be occasions where this approach is not the preferred method to teach mathematics. Leong et al. (2016) argues that some 'definitions' can be more appropriately introduced by stating definitions with suitable examples and thus shift the emphasis on utilising the knowledge of these definitions in problem solving. For example, in the following problem it is more pragmatic to help pupils learn the prerequisite term 'median' which aligns with the first instructional approach (i.e. teaching mathematics for problem solving):

- <u>Example</u>

Given that the median is 5 for the data set: 2, 15, x, 6, 11, 10, 1, 7, 9, state the minimum value of x.

2.7.2 The role of problem solving in school mathematics

In their classic critique, Stanic & Kilpatrick (1989, p. 1) state emphatically that: "Problems have occupied a central place in the school mathematics curriculum since antiquity but problem solving has not. Only recently have mathematics educators accepted the idea that the development of problem solving ability deserves special attention". The authors highlight the historical limited view of learning and mathematical problem solving and challenge us to fully examine why we should teach problem solving. They promote the incorporation of problem solving as a vehicle for acquiring new mathematical knowledge by encouraging pupils to develop logical reasoning skills and take responsibility for their own learning. Stanic & Kilpatrick (1989) identify three different interactive themes about the role of problem solving in school mathematics.

In the first theme, Stanic & Kilpatrick (1989) describe problem solving as a ‘context’, when it is employed to reach and facilitate other valuable ends. In such cases, problem solving can be used to justify the teaching of mathematics, to motivate pupils and capture their interest in mathematics, to stimulate further this gained interest; problem solving can also be used as recreation to have fun with and even as a vehicle “through which a new concept or skill might be learned” (p. 14). Problem solving as a practice is included in this theme to describe situations in which problem solving is used to reinforce and practice previously taught concepts and skills. Schoenfeld (1992) maintains that problem solving itself is not usually seen as a goal but solving problems is seen as facilitating the achieving of other goals.

As a second theme, Stanic & Kilpatrick (1989) represent problem solving as a ‘skill’. The authors warn that employing problem solving in a hierarchy of competences to be gained by pupil’s leads to certain consequences for the role of problem solving in the mathematics curriculum. Stanic & Kilpatrick (1989) contend that:

> One consequence is that within the general skill of problem solving, hierarchical distinctions are made between solving routine and non-routine problems. That is, non-routine problem solving is characterized as a higher level skill to be acquired after skill at solving routine problems (which, in turn, is to be acquired after students learn basic mathematical concepts and skills) (p. 15).

As was mentioned in the previous section, this should be learned explicitly as part of the curriculum. Translating this into teaching terms, problem solving should be promoted as a set of explicit thinking routines, such as drawing a diagram, finding a pattern, logical reasoning, etc., which should be part of the repertoire of instructional practices.

In the final theme, Stanic & Kilpatrick (1989) refer to the rich work of Polya in portraying problem solving as a ‘highly creative process’. The authors express “problem solving as art as the most defensible, the most fair, and the most promising. But at the same time it is the most problematic theme because it is the most difficult to operationalize in textbooks and classrooms” (p. 17). Stanic & Kilpatrick (1989) underline the challenges for teachers to develop a practical artistic ability in pupils and cite Polya’s (1981, p. xi) comparison that problem solving should be deemed a practical art like “like swimming, or skiing, or playing the piano”. Though, it is necessary to point out that creative skills are often presented as separate entities to be learned didactically and applied without any theoretical justification (Lesh & Zawojewski, 2007; English & Sriraman, 2010; Lester, 2013).

In short, within an ever changing world, the function of mathematical problem solving is to empower pupils to manage the complexities and non-routine cognitive real life challenges that await them within the future workplace. Independent critical and creative thinking skills will help generate solutions to novel mathematical problems that cannot be solved by selecting previously learned concepts and rules. The OCED (2014) assert that:

> For students to be prepared for tomorrow's world, they need more than the mastery of a repertoire of facts and procedures; students' need to become lifelong learners who can handle unfamiliar situations where the effect on their intervention is not predictable. When asked to solve problems for which they have no ready-made strategy, they need to be able to think flexibly and creatively about how to overcome the barriers that stand in the way of a solution (p. 26).

In a study of Swedish primary teachers, Van Bommel & Palmer (2015) report that a collaborative professional development initiative influenced participants' awareness of the problem solving themes introduced by Stanic & Kilpatrick (1989), evidenced by the quality of produced lesson plans.

2.8 Assessment

There is a growing demand from employers and universities for school leavers to be able to apply their mathematical knowledge to problem solving in varied and unfamiliar contexts (Lesh & Zawojewski, 2007; English & Sriraman, 2010; OCED, 2014; Jones, Swan & Pollitt, 2014; ACME, 2016; English & Gainsburg, 2016). Assessment will impact on what is taught in the classroom and should be driven by mathematics that is valued and expected of a modern mathematics education (Suurtamm et al., 2016). Silver (2013, p. 273) reminds practitioners that "for students to become convinced of the importance of the sort of behaviors that a good problem-solving program promotes, it is necessary to use assessment techniques that reward such behaviors". Viewed in this way, the assessment of problem solving is essential in order to ensure the effective learning and teaching of problem solving throughout primary and secondary education (ACME, 2016). Lesh & Zawojewski (2007, p. 794) posit that "there is a growing recognition that a series mismatch (and is growing) between the low-level skills emphasized in test-driven curriculum materials and the kind of understanding and abilities that are needed for success beyond school". However, school mathematics examination instruments are typically dominated by short, structured questions that fail to assess problem solving (Kilpatrick, 1992; Jones & Inglis, 2015).

In Scotland, the centrality of problem solving is recognised as an intrinsic feature within the learning and teaching of mathematics (Scottish Government, 2009) although, illogically, discharged from any form of assessment accountability. Ironically, this delineated position was implicitly bolstered during a recent report established to transform the status of mathematics in Scotland by not appearing in any of the ten recommendations highlighted for change (Scottish Government, 2016b). It is important to consider how to interpret the common theme to emerge from narratives emphasising the indispensable role of problem solving along with the current assessment arrangements that are integral to CfE. As a practising teacher, I am cognisant of the issues of bureaucracy and lack of clarity which undermines our national assessment system but refuse to supplement any rhetoric to this topic. Instead, I will focus my attention briefly on exploring how mathematical problem solving can be evaluated within a suitable framework.

Kilpatrick (1992) suggested that to assess mathematical problem solving effectively, the narrowing effects of current testing practice and the continued pressure for efficient measurement must be addressed. Since this proposition, multinational comparative assessments such as TIMMS and PISA have influenced policy makers throughout the world leading to political agendas fueled with neoliberal ideologies. Increasing operation is being made of external assessments to gauge mathematical knowledge and continue to serve different purposes to the design goals enshrined within the multidimensionality of classroom assessments (Suutamm et al., 2016). In Scotland, I believe the functionality of data from external assessments ultimately serves to encourage practitioners to ‘teach to the test’ to the detriment of assessment for learning (Hodgen & Wiliam, 2006). Still, this scenario would not exist if national assessments aligned with curriculum goals and ironically may be held as a positive practice (Swan & Burkhart, 2012).

Notwithstanding the nuances that arise from assessing complex processes involved in solving mathematical problems, Szetela & Nicol (1992) present four categories that teachers can use as a marking rubric; answers, answer statements, strategy selection and strategy implementation. Though, it is argued that this method is unable to reliably capture the level of divergent thinking involved since thinking is not easily communicated to produce clearly formulated responses. Polya (1954, p. 154) highlights that: “The final form of the solution may be recorded, yet the changing plans and the arguments for and against them are mostly or entirely forgotten”. Since authentic problem solving tasks require an extended time period (since they are not suited to a timed examination) and observation to access evidence of process, the challenge is to design suitable mathematical problems that can be assessed within a controlled time.

Monaghan et al. (2009) argue that open-start mathematical problems offer a practical means to achieve this objective and encapsulate the type of problems involved:

- The mathematical knowledge needed to solve the problem must already be known securely: this is not about assessing curriculum content – it is about assessing the ability to deploy such knowledge.
- The problem-solver must not be familiar with a similar problem – the essence of ‘open-start’ is that it is not clear where to start and recall of a similar siltation would compromise this.
- It would not be clear at the outset whether the strategy will work, and it will have to be accepted by the problem-solver that further attempts may be needed (p. 26).

The authors suggest that much development work is required to implement this form of assessment. While no marking scheme can circumscribe all conceivable answers that examination candidates might offer, Monaghan et al. (2009) anticipate that this would not pose an issue for open-start problems. In my view, their contribution would have been more convincing if they had provided some empirical evidence.

In their study involving the design of a problem solving examination paper, Jones & Inglis (2015) administered a test to 750 English secondary pupils of varying mathematical ability. The participants work was assessed by experts using comparative judgement in addition to a specially designed resource intensive marking procedure. The construct of comparative judgment has an underlying theoretical basis grounded within a well-established psychological principle that people are more reliable when comparing outputs concurrently than when they are asked to judge something in isolation. In another English study, Jones, Swan & Pollitt (2014) demonstrated that comparative judgement was not a barrier to assessing mathematical problem solving. Results obtained from a review of a sample of examination scripts derived its validity from what is valued and expected by mathematics professionals, rather than what can be precisely captured in scoring rubrics. Both Jones & Inglis (2015) and Jones, Swan & Pollitt (2014) found that comparative judgment was successful and raise the possibility of a richer diet of mathematical assessments anchored on holistic relatively unstructured tasks being available to future Scottish pupils. However, if the goal of developing proficiently in mathematical problem solving is to be realised, its importance must be communicated to pupils, teachers and the general public through the assessments that are offered (Silver & Kilpatrick, 1989). Moreover, the main summative assessment challenge for stakeholders in Scotland is not novel planning or peripheral methodology concerns but a deviation from traditional measurements fixated by the recall of facts and fluency of procedural knowledge.

2.9 Factors contributing to successful mathematical problem solving

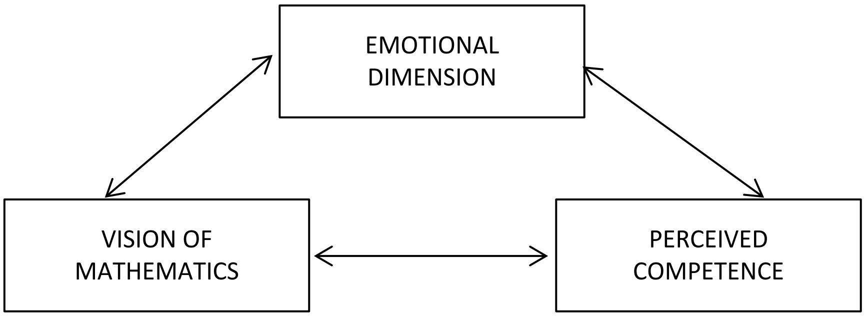
I commence this section by referring to the social construct of ‘attitude’. McLeod (1992) identified attitude along with beliefs and emotions as one of three key affective paradigms in mathematics education. All practitioners can relate to classroom experiences where pupils display a range of different behaviourisms towards mathematical problem solving which are generally construed across a continuum of positive and negative dispositions. They can have an affective and emotional character, while on the other hand, are of cognitive origin. For many years, this phenomenon was surprisingly neglected by a lack of a theoretical framework and new methods of inquiry.

A seminal study in this area is the work of Di Martino & Zan (2010) who collected and analysed autobiographical narratives written by 1,662 Italian pupils whose school levels ranged from early primary to the end of secondary. The results of the study showed that almost all of the participants describe their relationship with mathematics along at least one of the following three trajectories:

- emotional disposition towards mathematics
- vision of mathematics
- perceived competence in mathematics

Di Martino & Zan (2010) present a multidimensional model characterised by three strictly interconnected dimensions that pupils recognise as crucial in their development of their relationships with mathematics (Figure 2.11).

Figure 2.11 The three dimensional model for attitude (Adapted from Di Martino & Zan, 2010)



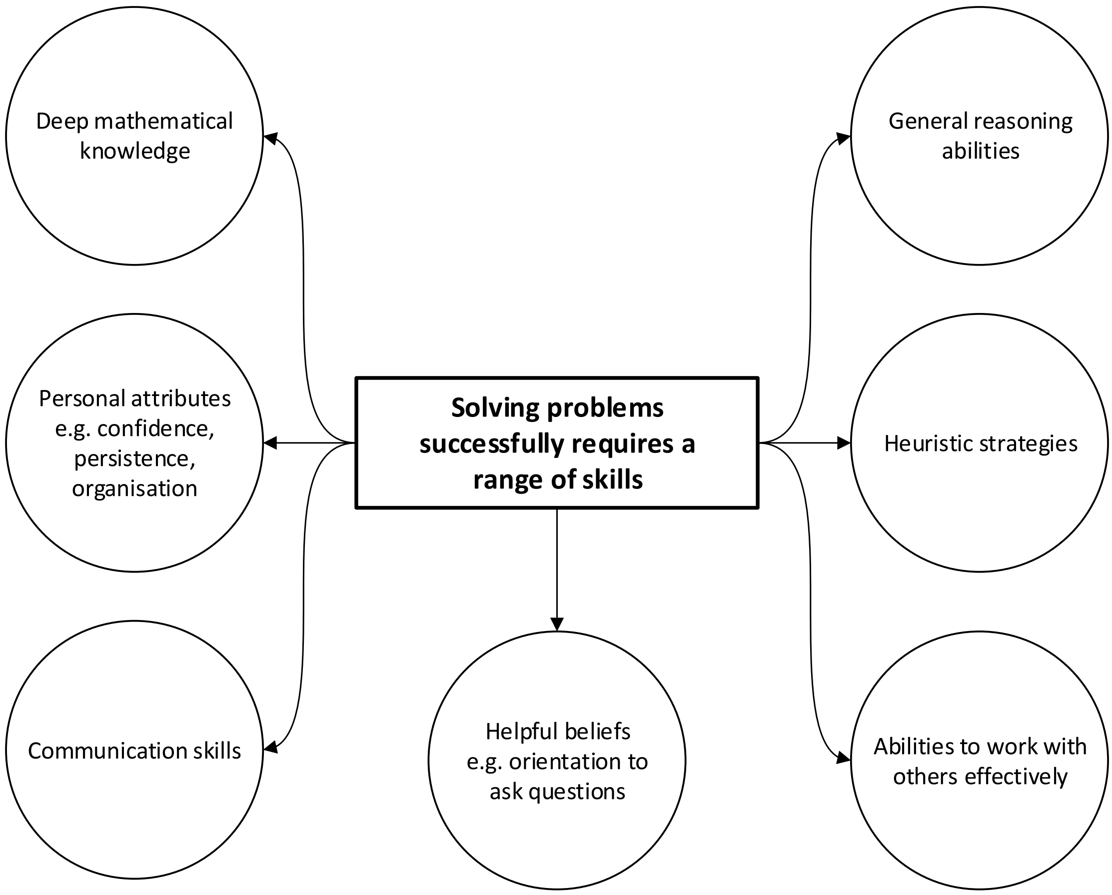
Di Martino & Zan (2010) suggest the need for a new approach about the positive/negative portrayal of attitude and offer a definition of 'negative attitude' aimed at supporting teachers:

> The multidimensionality of the model underlines the inadequacy of the positive/negative dichotomy for attitude referred to only to the emotional dimension (like/dislike), and rather suggests considering an attitude as negative, when at least one of the dimensions is negative. In this way, we can outline profiles of negative attitude, depending on the dimension that appears to be negative (p. 44).

In a study of 16 Belgium secondary children, Op't Eynde, De Corte & Vershaffel (2006) examined the relationship between mathematical related beliefs, emotions and problem solving behaviour. They found that the nature and intensity of emotion experienced during problem solving fluctuated between participants. One significant aspect to emerge from the results was the level of confidence. Guven & Cabakor (2013) investigated factors influencing mathematical problem solving achievement of 115 Turkish secondary pupils. The researchers discovered that self-efficacy, beliefs and mathematical anxiety were noteworthy. However, the study suffers from poor external validity. In a study of 20 Israeli primary children, Prusak, Hershkowitz & Schwarz (2013) explored the culture of problem solving. They noted the success of their findings heavily relied on five principles such as encouragement to produce multiple solutions, creating collaborative situations; social-cognitive conflicts, providing tools for checking hypothesis and inviting students to reflect on solutions.

In a review of the locus of problem solving within mathematics curriculums of Australia, UK, USA and Singapore, Stacey (2005) asserts that successful mathematical problem solving depends upon many factors which have distinctly different characters, illustrated in Figure 2.12. A more comprehensive paper would include Scotland (since a UK curriculum does not exist) and non-English speaking countries. A number of scholars argue that pupils should solve a wide range of types of problems and be regularly exposed systematically to planned problem solving instruction (e.g. Lester, Garafolo & Kroll, 1989; Lester, 1994, 2013; Boaler, 1998, Cai, 2003; Lesh & Zawojeswski, 2007). Schoenfeld (2011, 2013) maintains that learners require deep mathematical domain knowledge, heuristic strategies, metacognitive skills and relevant beliefs. Likewise, Goldin (1998) opines that beliefs systems are powerful facilitators of problem solving success, or otherwise, as obstacles to it. Finally, Lester (2013) points to the importance of intuition while Boaler (2016) advocates a growth mind set.

Figure 2.12 Factors contributing to successful problem solving (Adapted from Stacey, 2005)



I now turn my attention to the multifaceted role of the teacher (Lortie, 1975). From my professional experience, the selection of a mathematical problem is critical to the successful outcome of any lesson. To ensure equitable engagement of all levels of ability, suitable problems must present opportunities to be solved or at least partly attempted by low confident learners. Accessible problems should integrate enabling prompts for pupils experiencing difficulty and extending prompts for pupils who have completed the tasks (Hiebert et al., 1997; Sullivan, 2011; Van de Walle, Karp & Bay-Williams, 2014). An overarching ability to choose appropriate problems is interrelated to content knowledge and proficiency of solving mathematical problems including an understanding of how young people think about problem solving (Chapman, 2015). If teachers are unable to relate first hand to the tension and triumph of discovery engendered by solving problems, they are unlikely to be adept at fostering mathematical problem solving. During classroom discourse, practitioners should solicit questions that unpack pupils' thinking and press for them to explain their reasoning behind the process (Rigelman, 2007). Similarly, a resilient dexterity to identify common misconceptions is essential. Schoenfeld (1992) encapsulates widely recognised pupil beliefs about mathematics which without approach, form a barrier to the effective learning of mathematical problem solving:

- Mathematics problems have only one and only one right answer.
- There is only one correct way to solve any mathematics problems – usually the rule the teacher has most recently demonstrated to the class.
- Ordinary students cannot expect to understand mathematics; they expect simply to memorize it, and apply what they have learned mechanically and without understanding.
- Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.
- The mathematics learned in school has little or nothing to do with the real world.
- Formal proof is irrelevant to processes of discovery or invention (p. 359).

2.10 Summary

The centrality of problem solving in mathematics is incontrovertible. It can promote deep conceptual understanding, critical and independent thinking, habits of persistence and curiosity, confidence in unfamiliar situations that will serve pupils greatly in everyday life and in the future workplace (Lester, 1985; NCTM, 2000, Cai, 2010). No universally accepted definition of mathematical problem solving exists or the imminent prospect of a construct being agreed (English & Gainsburg, 2016). Mathematical problems encompass many characteristics and are classified in different ways. The learning of problem solving is extremely complex and multidimensional with much interplay rooted in the field of cognitive science. It can nurture creativity, flexibility and mental fluency (Silver, 1997; Guberman & Leikin, 2013). Considerable research has focussed around the theoretical framework introduced by Polya (1957). Schoenfeld (1985) established that resources, heuristics, metacognitive control and beliefs systems are fundamental mechanisms of successful mathematical problem solving.

The role of the teacher is instrumental in supporting learners to develop higher order thinking skills through generating multiple solutions and providing rich opportunities for comparing and reflection. Continued support exists for teaching mathematics using problem solving as a vehicle (Lester & Cai, 2016). There is little evidence to suggest that demonstrating heuristics to pupils leads to greater success in solving problems (Lester, 2013) although some strategies have merit. Although there is no obligation for practitioners to be expert problem solvers, a degree of mathematical proficiency blended with skilful pedagogical knowledge is required (Lester, 2013; Chapman, 2015). Concomitantly, tension of high-stakes national mathematics examinations which exclusively concentrate on assessing basic skills place educators in an undesirable position inconsistent with curriculum objectives (English & Sriraman, 2010). Such a misalignment with classroom practice suggests that a review of the philosophy of external assessment within CfE may be desirable.

New directions and perspectives emerging from the literature (e.g. English & Gainsburg, 2016) has proposed that future mathematical problem solving research be converged on modelling. Whilst I welcome such a move, it is debateable if modelling is a division of problem solving or a separate entity that requires a diverse set of skills. Likewise, there is a request for the recontextualisation of school mathematical problems so as to offer more cognitively challenging dynamic tasks that authentically simulate demands of 21st century work and life.

However, I believe that in order to advance the mathematical problem solving skills of all of our young people, research has to coalesce within two interrelated domains. Firstly, that of mathematical problem posing due to the valuable learning benefits that subsist. Secondly, teachers' beliefs since they appear to significantly impact on what takes place in classrooms.

The next chapter offers an introductory literature review of mathematical problem posing.

CHAPTER THREE

Mathematical Problem Posing

The aim of this chapter is to provide an introductory review of the interrelated multiple perspectives surrounding mathematical problem posing. First, it is useful to familiarise the reader since problem posing is not encompassed within the mathematical domain of Curriculum for Excellence. Second, an appropriate background is helpful in advance of the systematic literature review of mathematical problem posing arranged for chapter five.

3.1 Nature and definition

In Kilpatrick's (1987) landmark paper, he provides a valuable insight into our perception of the origin of good mathematical problems. More recently, Brown & Walter (2005) state that mathematical problems appear during schooling, predominantly from textbooks and to a much lesser extent from teachers, highlighting an issue with existing classroom practice. The authors encourage us to shift our thinking from solving predetermined problems to constructing and designing our own problems and argue that without engaging with this powerful form of mathematical inquiry, pupils will be unprepared in adapting to future workplace challenges. Unequivocally, without posed problems, there would be no mathematical problems to solve (Singer et al., 2011; Ellerton, 2013). As highlighted in chapter three, mathematical problems can be described as well-structured, structured, or ill-structured. Drawing on the work of Fredericksen (1984), Kilpatrick (1987) distinguishes between the categories as follows: Well-structured problems are clearly formulated, can be solved by the application of a known algorithm and have criteria available for testing the correctness of a solution; structured problems are similar to well-structured problems but require the solver to contribute in some way to the solution; ill-structured problems lack a clear formulation, a procedure that will guarantee a solution and criteria for determining when a solution has been achieved.

A number of researchers have reported that problem posing is a cognitive activity which encompasses both the generation of new problems and the reformulation of given problems (e.g. Silver, 1994; Silver & Cai, 1996; English, 2004; Whitin, 2006). Silver (1994) suggests that the generation of new problems can occur before or after the problem solving process and that reformulation follows when the original mathematical problem is formulated or transformed into a different version. Kilpatrick, Swafford & Findell (2001, p. 124) associate problem posing with strategic competence and express this as “the ability to formulate mathematical problems, represent them, and solve them”. Pirie (2002, p. 929) includes a classification of a problem type when she describes problem posing as “the creation of questions in a mathematical context and ... the formulation, for solution, of ill-structured existing problems”. This delineation is inadequate since is does not provide clarity on previous knowledge.

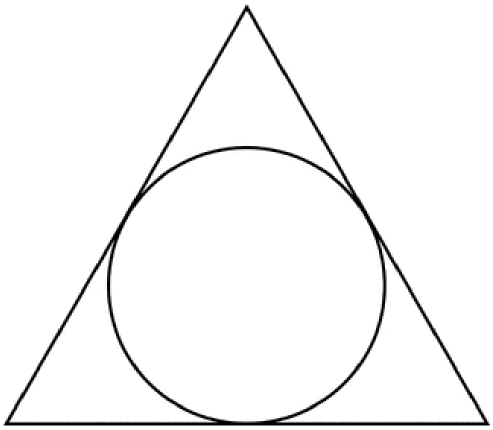
In this thesis, I will adopt the definition offered by Stoyanova & Ellerton (1996, p. 518) who refer to the practice of problem posing as “the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems”. What is appealing about this definition is that it clearly links constructivism to problem posing. The researchers offer a theoretical framework by classifying three categories of problem posing situations; free, semi-structured and structured. In free situations, pupils design problems from a real life context without restrictions (see Example 1 below). Semi-structured problem posing occurs when pupils are “given an open situation and are invited to explore the structure and to complete it by applying knowledge, skills, concepts and relationships from their previous mathematical experiences” (p. 520). I believe that this situation has the potential to maximise creative thinking (see Example 2 below). Finally, structured problem posing activities are centred on a specific problem that requires completion or reformulation (see Example 3 below). This approach resonates with Brown & Walter (2005) who introduced the “What-If-Not” strategy. All three examples are taken from my professional practice.

- <u>Example 1</u>

Heather has 145 marbles, Ruairidh has 114 marbles and Wallace has 220 marbles. Write and solve as many problems as you can using this information.

- <u>Example 2</u>

In the following diagram, there is an equilateral triangle and its inscribed circle. Make up as many problems as you can that are in some way related to this diagram.



- <u>Example 3</u>

The gradient of a straight line is ½ and passes through the point A (4, 12). Write and solve as many problems as you can using this information.

Nevertheless, problem posing is not an original concept. Eminent physicist, Albert Einstein (Einstein & Infeld, 1938) championed the notion when he famously stated:

> The formulation of a problem is often more essential that its solution, which may be merely a matter of mathematical or experimental skill. To raise new questions, a new possibility, to regard old problems from a new angle, requires creative imagination and marks real advances in sciences (p. 92).

Historically, this view has been shared by many others who have placed greater emphasis on the value of posing meaningful questions than on attempts to solve them. For example, Singer, Ellerton & Cai (2013, p. 2) reminds us that Socrates (470-399BC) "established an efficient method of learning through a continuous dialogue based on posing and answering questions to stimulate critical thinking and illuminate ideas".

In recent times, a focus on the idiosyncratic nature of critical thinking has continued, establishing this intrinsic feature as a highly desired characteristic. Prominent mathematicians and mathematics educationalists (e.g. Polya, 1954; Freudenthal, 1973, 1981; Halmos, 1980; Kilpatrick, 1987; Moses, Bjork & Goldenberg, 1990; Silver, 1994; Brown & Walter, 2005; Cai et al., 2015; Ellerton, Singer & Cai, 2015) consider problem posing to be an essential mathematical curriculum component and advocate that pupils are afforded extensive problem posing opportunities.

During the last three decades, research of mathematical problem posing has gained increasing awareness, although a lack of breadth may limit its future development. It has been suggested that problem posing as an emerging paradigm has been marginalised by the mathematics education community (English, 1998; Crespo, 2003; Leung, 2013) and be afforded similar research status as mathematical problem solving (Silver, Kilpatrick & Schlesinger, 1990; Pirie, 2002; Stoyanova, 2003; Silver & Cai, 2005). Concerns have been raised with the lack of opportunities provided to pupils (e.g. Ellerton, 1986; Silver et. al, 1996; Leung, 2013). However, recent studies have attempted to bestow further evidenced based strategies for classroom integration and within initial teacher education (e.g. Singer, Ellerton & Cai, 2015).

3.2 The role of problem posing in school mathematics

The literature has emphasised the importance of problem posing to the learning and teaching of school mathematics. For example, Polya (1957, p. 68) posited that “the mathematical experience of the student is incomplete if he never had an opportunity to solve a problem invented by himself.” Similarly, Kilpatrick (1987, p. 123) argued that problem posing should be a fundamental mathematical curriculum objective and stated that “the experience of discovering and creating one’s own mathematics problems ought to be a part of every student’s education”. Interestingly, both expressed views do not specify gender, age or ability which suggests this activity is accessible to all learners. Consequently, pupils at any stage may feel encouraged to develop their mathematical curiosity which can act as a motivational catalyst for further learning.

It has been established from a variety of sources that problem posing can offer valuable benefits for both teachers and pupils alike. Practitioners are able to create interesting problems for children which can shape and cultivate mathematical learning and help them develop into stronger problem posers (Crespo, 2003; Olson & Knott, 2013). The operationalisation of problem posing provides a lens through which teachers are able to assess learner’s conceptual understanding, problem solving and creativity (e.g. Ellerton, 1986; Kilpatrick, 1987; Silver & Cai, 1996; English, 1997a, 1997b; Silver, 1997; Cai & Hwang, 2002; Lowrie, 2002). It supplies rich opportunities for pupils to connect their own interest with all facets of mathematical education. Teachers can challenge learners to think deeply about what they are doing rather than mechanically respond to a set of questions with a prepared technique or algorithm. Other authors highlight the empowering aspect of problem posing which encourages pupils to construct knowledge (e.g. Ernest, 1991, English, 1997a) and decide on questions to be solved thus rejecting the assumption that there is only one method to solve a problem and that all problems have one correct answer (Fox & Surtees, 2010).

Based on fostering mathematics as a cognitive activity and grounded on a constructivist perspective (Silver & Cai, 1996; Cai, 1998), initiatives have recommended that problem posing play a pivotal role within the learning and teaching of school mathematics. This view is supported by Silver (1994, p. 19) who notes that “contemporary constructivist theories of teaching and learning require that we acknowledge the importance of student generated problem posing as a component of instructional activity”. Within my own professional practice, problem posing has created a dynamic learning environment where children are inspired to take more risks and are less afraid to make mistakes. Whitin (2004, p. 129) asserts that it can enhance the atmosphere of every mathematics classroom and portrays it as “a strategy that builds a spirit of intellectual excitement and adventure by legitimizing asking questions and freeing learners from the one-answer syndrome”.

Given its potential to augment the learning and teaching of mathematics, it is unsurprisingly that problem posing has featured within many curriculum reforms around the world. It has been shown that curriculum reform is a powerful driver for implementing instructional change within educational systems (e.g. Cai & Howson, 2013). For example, the NCTM (1989, p. 138) promulgate the importance of having secondary pupils immerse themselves in some of the problem posing aspects involved in the work of professional mathematicians by advocating that “students in grade 9-12 should also have some experience recognising and formulating their own problems, an activity that is at the heart of doing mathematics”. During a later reform, the NCTM (2000) declared that the function of the classroom teacher is to orchestrate opportunities for all learners to construct their own mathematical knowledge, emphasising that the formulation and modification of problems be within and outside mathematics.

Stoyanova & Ellerton (1996) reported that the Australian Education Council (1991) offers strong support for the use of open-ended problems in mathematics classrooms. In Asia, assimilating problem solving within Chinese schools has a long history and continues to be part of contemporary mathematical thinking (Cai & Nie, 2007). Since the start of the twenty first century, China has witnessed the development of using open-ended or real life problems and accordingly, of paying further attention to problem posing activities opposed to problem solving activities (e.g. Cai & Nie, 2007). The standards for problem solving in the Chinese National Curriculum Standards on Mathematics (Ministry of Education of Peoples Republic of China, 2001, p. 7) emphasise that pupils must be able to "pose and understand problems mathematically, apply basic knowledge and skills to solve problems and develop application awareness" (as cited by Chen et al., 2011). Within the Singapore Mathematics Curriculum Framework, pupils are explicitly encouraged to extend and generate problems (e.g. Ministry of Education, 2007). Leung (2013) testifies that curriculum standards and instruction in Taiwan emphasise the importance of problem posing. However, Leung (2013, p. 105) warns that in "the mathematics curriculum reform occurring in Taiwan, teachers are facing unprecedented challenges to change the way they teach, including incorporating problem solving and posing by children". She refers to previous research which highlights the inexperience of practitioners in posing activities and that such mathematical activities are difficult to implement (e.g. Leung, 1994) and counsels for the provision of problem posing training and access to suitable resources for teachers.

Various countries including Italy (e.g. Bonotto & Del Santo, 2015) and Turkey (e.g. Kilic, 2013) have introduced curriculum reforms to embed problem posing activities that develop conceptual understanding within different levels of mathematics education. This has challenged the capability of teachers to pose valid and interesting tasks for pupils, including refining their ability to pose better problems. Research has investigated the problem posing performance of prospective and current primary and secondary mathematics teachers (e.g. Crespo, 2003; Koichu & Kontorovich, 2013). Whilst, in general, it was found that practitioners are capable of posing worthy and quality problems, it appears that this may be connected to problem solving experience. Moreover, due to workload demands, teachers require accessible classroom resources such as sample problems in order to implement in practice. Though, it is possible for reliable problems to be generated from other sources. In their study of 70 Portuguese prospective primary teachers, Barbosa & Vale (2016) explored authentic contexts outside the classroom contributing to the posing of mathematical problems. Drawing on the work of Silver (1997) and Stoyanova (1998), the researchers analysed personal interpretations and formulations of real situations inspired by the local environment. They found that participants displayed a more positive attitude towards learning and teaching of mathematics by acquiring a wider view of the possible connections between the natural worlds. It is anticipated that this pedagogical change will help promote conditions for young children to discover and construct their own knowledge.

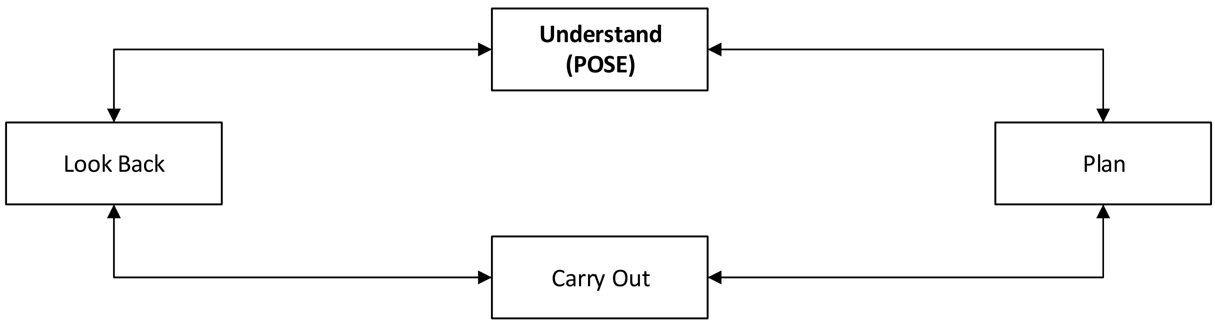
3.3 Relationship between problem posing and problem solving

Whilst it is acknowledged that problem posing and problem solving are not entirely diverse cognitive behaviours, they are nevertheless closely related (Lowrie, 2002). Fox & Surtees (2010) maintain that the two are “inextricably linked” whilst some scholars argue that problem posing is a “special case” (Kontorovich et al., 2012) or “important companion” (Kilpatrick, 1987; Bonotto, 2010) of problem solving. Brown & Walter (2005) illuminate the overarching connection by asserting that problem posing is deeply embedded in the activity of problem solving in two contrasting ways. Firstly, it is impossible to solve a new problem without reconstructing the task and secondly, understanding a solution is typically enhanced after the generation and analysis of a new problem. Regarding this latter point, Brown & Walter (2005, p. 122) highlight that “we need not wait until after we have solved a problem to generate new questions; rather, we may be logically obligated to generate a new question or pose a new problem in order to be able to solve a problem in the first place”. The authors skilfully illustrate the power of “What-If-Not” thinking by selecting a special case of the quadratic equation ($x^2 + x - 1 = 0$) and solving it by an unorthodox method to generate a continued fraction.

It may be reasoned that within the domain of problem posing, individuals have to productively engage in a higher level of intellectual or creative thought process. Appraised from a physics perspective, Mestre (2002, p. 15) contends that as a cognitively challenging undertaking, “it would not be very difficult to argue that posing meaningful, interesting problems is intellectually a more demanding task than solving problems”. Moreover, research suggests that problem posing activities help to diminish pupils’ mathematical anxiety while simultaneously foster a more positive disposition towards mathematics and may also improve learner’s conceptual understanding and problem solving ability (Silver, 1994; English, 1997a; NCTM, 2000; Brown & Walter, 2005).

In chapter two of this thesis, I refer to the problem solving framework offered by Polya (1957), which connects a journey back and forth through four phases. Leung (2013) presents a similar four phase cycle to incorporate problem posing and problem solving and argues that the decisions and actions of posing and solving can be interrelated (Figure 3.1). If an individual is solving their own generated problem, the initial 'Understand' phase is considered a 'Pose' phase. Consequently, problem posing can emerge at any time, before or after solving. A broader perspective has been adopted by Gonzalez (1998) who describes problem posing as the fifth phase of Polya's model and is dependent on practitioners providing worthwhile problems. However, it may be argued that the interaction between problem posing and problem solving is strongly influenced by the teachers' perception of what constitutes a suitable problem.

Figure 3.1 Four phases in problem posing and problem solving (Adapted from Leung, 2013)



Several studies have probed the interactions between problem posing and problem solving with mixed results (e.g. Ellerton, 1986; Silver & Mamona, 1989; Silver & Cai, 1996; Cai, 1998; Crespo, 2003; Chen et al., 2007). In this remainder of this section, centred on a theoretical argument presented by Kilpatrick (1987) that the quality of posed problems is directly linked to individual mathematical problem solving ability, I examine two famous studies involving pupils. In this first study, the problem posing and problem solving tasks are mostly unrelated. Whilst the tasks in the second study are considered to be identical in mathematical and contextual structure.

<u>Silver & Cai (1996)</u>

In this study, the researchers analysed the responses of 509 American secondary pupils who were asked to complete a problem posing task which consisted of generating three questions based on a driving situation. Posed problems were analysed by type, solvability and complexity. This outcome was compared with the results from eight open-ended problem solving tasks involving fractions, geometry, number theory, patterns and relationships, ratio/proportion and statistics. Silver & Cai (1996) discovered that pupils' problem solving performance was highly correlated with their problem posing performance. Moreover, compared to less successful problem solvers, good problem solvers generated more problems, and their problems were more mathematically complex. What is interesting about the study is that the participants were not selected on the basis of obvious mathematical ability.

Cai & Hwang (2002)

This cross-national comparative study examined American and Chinese primary pupils' mathematical performances. A total of 98 American and 155 Chinese children participated in the research which involved three pairs of problem solving and problem posing tasks. The results "showed differential relationships between problem posing and problem solving for US and Chinese students" (p. 419). The disparities appear to be related to learner's use of differing strategies. Chinese pupils tend to choose abstract and symbolic representations while American students favour concrete strategies and drawing images. In short, there was a stronger connection between problem posing and problem solving for the Chinese sample. Overall, the findings of this study are similar to Cai (1998) which located a positive correlation between problem posing and problem solving from a cross-national perspective.

Whilst both the studies of Silver & Cai (1996) and Cai & Hwang (2002) provide some evidence that a linkage exists between problem posing and problem solving, further research is required to explore this complex and multidimensional relationship in more detail. Recently, Silver (2013, p. 160) in his observation of previous research in the field, asserted that "progress has been stymied by the lack of an explicit, theoretically based explanation of the relationship between problem posing and problem solving that is consistent with existing evidence and that could be tested in new investigations". In particular, there is scarcity of research involving practising teachers (e.g. Silver & Mamoma, 1989; Silver et al., 1996; Chen et al., 2011).

3.4 Creativity

The operationalisation of creativity is a desired outcome within any mathematical educational setting. Sriraman (2009, p. 13) emphatically states that "mathematical creativity ensures the growth of the field of mathematics as a whole". No one can dispute technological innovations in modern society have been owed to the inspirational creativity of scientists and professional mathematicians (Nadjafikhan, Yaftian & Bakhshalizadeh, 2012). In Scotland, creativity has a high profile in education and fits very well within the broad framework of CfE, although ironically, the vast majority of pupils would not associate the domain of mathematics with creativity.

Nevertheless, numerous pupils' classroom experiences of mathematics entail working with practitioner-driven material and sequential tasks or being passive observers of mathematics (Boaler, 1997). Based on her ethnographic case studies of teaching approaches at two different English secondary schools, Boaler (1998, p. 59) cautions against the stereotypical limitations of using only standard mathematical methods when she warns "students developed an inert, procedural knowledge that was of limited use to them in anything other than textbook situations". Often, the creative side of mathematics education is neglected, as instruction normally has an imitative and reproductive character since it is focussed on rudimentary activities with a dependency on routine skills, where pupils are encouraged to think in narrow domains (Haylock, 1987).

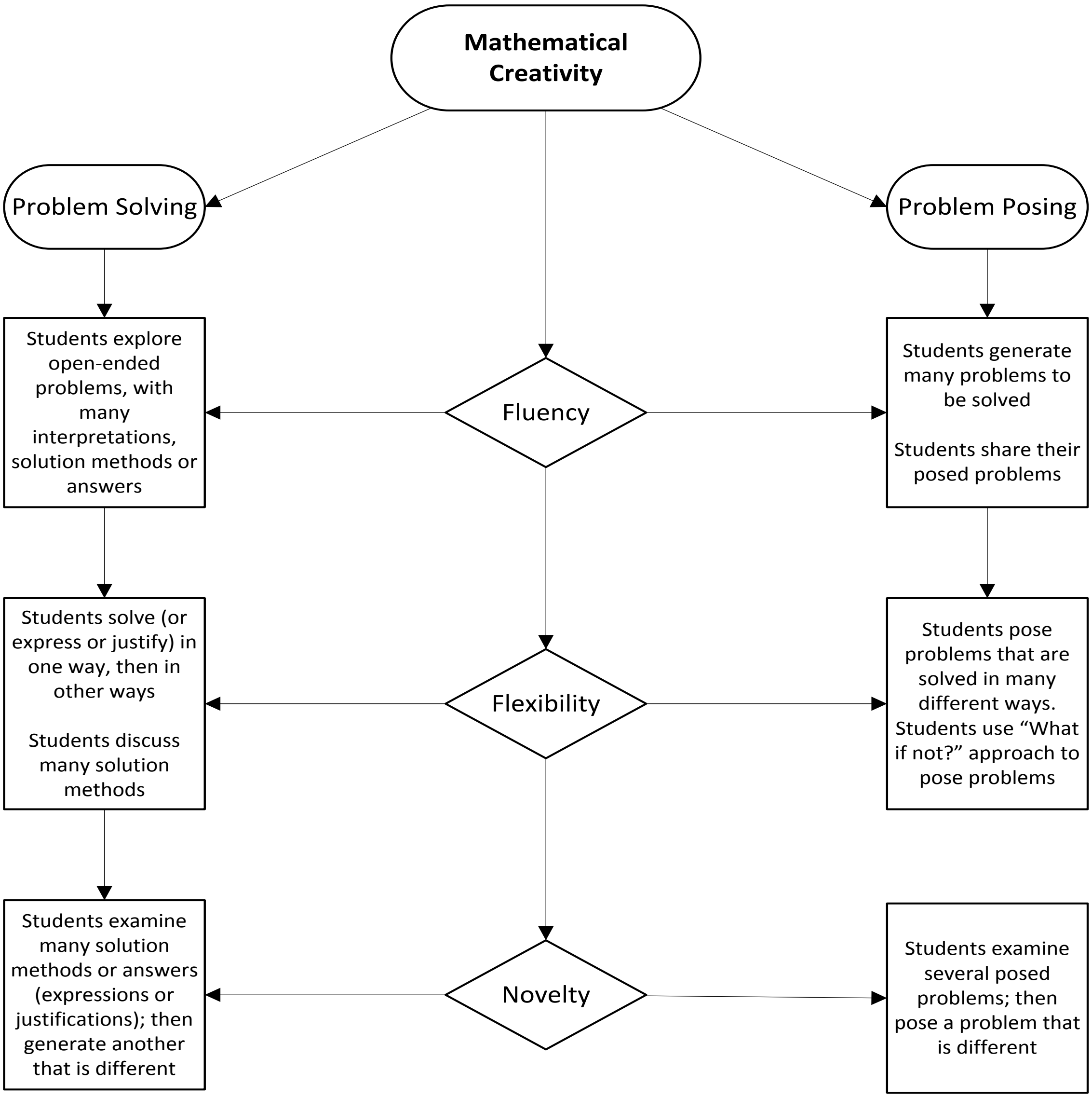
It is important to reflect on what is epitomised by mathematical creativity. Previous research has suggested that it may be confined to the employment of professional mathematicians when they formulate a problem that has not been solved before (Hadamard, 1945; Poincare, 1948). However, the conceptualisation of creative learning fluctuates due to the diversity of perspectives of creativity. Ervynck (1991) deems that mathematical creativity cannot occur in a vacuum and needs a context in which the individual moves forward through previous experiences which provide a suitable environment for creative development. Ervynck (1991) asserts that creativity plays a vital role in the full cycle of advanced mathematical thinking:

> It contributes in the first stages of development of a mathematical theory when possible conjectures are found as a result of individual experiences of the mathematical connects; it also plays a part in the formulation of the final edifice of mathematics as a deductive system with clearly defined axioms and formally constructed proofs (p. 42).

Silver (1997) views creativity as an orientation or disposition towards mathematical activity that can be fostered in the general school population. He proclaims the "connection to creativity lies not so much in problem posing itself, but rather the interplay between problem posing and problem solving. It is in this interplay of formulating, attempting to solve, reformulating, and eventually solving a problem that one sees creative activity" (p. 76). Silver discusses previous research by Getzels & Jackson (1962), Balka (1974) and Skinner (1991) amongst others which demonstrate valuable samples of problem posing. He proposes a didactical paradigm in which classroom practitioners can relate to three practical core assessment components of creativity i.e. fluency, flexibility and originality (novelty) as displayed in Figure 3.2. However, Kontorovich et al. (2011) argues that these indicators do not fully capture the essence of pupils' creativity and suggest that aptness be included as an additional quantitative evaluation. Silver (1997) presents a task which requires showing that the product of any four consecutive integers is divisible by 24. Whist this particular illustration is more emblematic of problem solving, it can be easily adapted to provide a problem posing activity. For example, generate as many problems as you can using the terms 'four', 'consecutive integers', 'divisible' and '24'. Silver (1997, p. 79) claims through the use of an inquiry based approach, "teachers can assist students to develop greater representational and strategic fluency and flexibility and more creative approaches to their mathematical activity".

At school level, Jenson (1973) maintains that mathematically creative pupils should be able to pose mathematical questions that extend and deepen the original problem as well as solve the problem using multiple methods. Likewise, Krutetskii (1976) portrayed creativity in the context of problem formation, invention, independence, originality and associates mathematical creativity with giftedness. In a study of 359 Cypriot pupils (aged 9-12 years) by Kattou et al. (2013), the researchers found a strong positive correlation between mathematical creativity and mathematical ability. In contrast, Skemp (1987, p. 64) argues that all learners have the ability to demonstrate mathematical creativity "since all new learning in mathematics by the method of concept-building consists of the formation by individuals of new ideas in their own minds, it is creative from their point of view". In the same vein, Mann (2006) warns that without providing for creativity in teaching mathematics, all learners are denied the option to appreciate the beauty of mathematics.

Figure 3.2 Core indicators of creativity (Adapted from Silver, 1997)



Logically, in order to cultivate mathematical creativity, teachers should select contexts that offer pupils opportunities to pose their own problems. Singer & Voica (2015) found that within the context of problem posing, mathematical creativity is a special type requiring abstraction and generalization. Jay & Perkins (1997, p. 257) maintain "the act of finding and formulating a problem is a key aspect of creative thinking and creative performance in many fields, an act that is distinct from and perhaps more important than problem solving". Another illustration of creativity is found in the work of Runco (1994, p. ix) when he expressed creativity as a multifaceted construct involving both "divergent and convergent thinking, problem finding and problem solving, self-expression, intrinsic motivation, a questioning attitude, and self-confidence". Alternatively, Torrance (1988) proclaimed that creativity is almost limitless and occurs whenever a solver has no learned solution for an existing problem.

While a number of researchers (e.g. Cai & Cifarelli, 2005; Singer et al., 2011; Siswono, 2011) have endorsed the connection between creativity and problem posing, this perspective is not universally shared (e.g. Haylock, 1997; Leung, 1997). Yuan & Sriraman (2011, p. 25) note "there might not be consistent correlations between creativity and mathematical problem-posing abilities or at least that the correlations between creativity and mathematical problem posing abilities are complex".

Within my professional practice, I have adapted two problem posing activities (Figure 3.3) from Christou et al. (2005b) that have stimulated the developmental growth of mathematical creativity between S1 and S2 pupils. Whilst I cannot verify the impact of such creativity on achievement, these tasks have promoted deep critical thinking and have generated many interesting and enjoyable learning experiences.

Figure 3.3 Examples of problem posing activities (Adapted from Christou et al., 2005b)

(a) Write a question to the following story so that the answer to the problem is '75 pounds':
Lachlann had 150 pounds. His mother gave him some more. After buying a book for 25 pounds he had 200 pounds.

(b) Write an appropriate problem for the following:
(2300 + 1100) – 790 = n

3.5 Technology

The integration of technology has a long and prominent history in mathematics education. Since the introduction of basic calculators in the 1970s, computers equipped with increasingly sophisticated software, graphics calculators that have morphed into 'all-purpose' hand-held devices assimilating graphical, symbolic manipulation, statistical and dynamic geometry packages, and web-based applications offering virtual learning environments have transformed the learning and teaching landscape (Goos, 2010). Concrete and virtual manipulatives reinforce mathematical concepts and can enhance mathematical sense making, communication, problem solving, reasoning and facilitate the tangible emergence of complex and abstract ideas.

The application of technology provides a range of rich and profound problem posing opportunities that allow learners to make conjectures, design their own explorations and create reinterpretations of existing concepts. More importantly, it can reliably provoke the stimulation of mathematical thinking due to the extensive assortment of ideas that can be instantaneously generated. Kilpatrick (1987) helped champion the future use of computers in problem formulation. However, the role of technology in problem posing has yielded few secondary studies, and even less involving primary pupils (Abramovich & Cho, 2015). Research has focussed on electronic spreadsheets (e.g. Abramovich, 2006; Abramovich & Cho, 2008), graphing software (e.g. Christou et al., 2005a; Lavy & Shriki, 2010; Leikin, 2012), computer algebra systems (e.g. Abramovich & Norton, 2006) and modelling (e.g. Abramovich & Cho, 2012).

Engagement in problem posing activities using dynamic geometry software can produce powerful learning environments where problems or relationships can be generalised or the validity of a new problem can be examined. This technology produces unique interactions between the software's interface and the users' actions and understandings, resulting from visual reasoning enhanced by dragging facilities (Lavy, 2015). In a study using this software, Contreras (2003) claimed that all mathematical problems contain some known information, some unknown information and sometimes explicit or implicit restrictions. By illustrating with parallelograms and angle bisectors, Contreras describes how to generate multiple geometric problems by varying the type of problem information and considering other types of problems. He maintains that such technology can show "not only how we can help students become better problem posers but also how the teacher can use a problem posing approach as an instructional tool to help students specialize, generalize, and extend problems" (p. 275).

Class discussions of problem posing activities using dynamic geometry software serve as a valuable mechanism for evaluating accurateness of generalisations. The exchange of ideas regarding the attributes and interrelations of mathematical objects under inspection may also stimulate the development of individual reflection by both teacher and pupil (Lavy, 2015).

In a later study using dynamic geometry software, Contreras (2007) advocated that all pupils should have extensive experiences posing proof problems. Proving is an essential feature intrinsic to understanding mathematics and provides the quintessential method of establishing propositions as results. Greeno (1994, p. 274) argues that “for students to learn mathematics without coming to appreciate the role of proof seems as impoverished as it would be for a student to learn science without coming to appreciate the role of empirical evidence”. Contreras (2007) underlined that reformulating a problem as a proof problem involves more than altering the syntactic structure of the problem. It requires an assumed degree of mathematical knowledge, since we either know that a proof exists or we can develop such a proof. In her study of 22 prospective Israeli mathematics teachers, Leikin (2015) highlighted the effectiveness of problem posing investigations in a dynamic geometry environment as a pedagogical instrument. She found that the majority of text book problems lead to performing mathematics fertile in surprises, discoveries and proofs. Nevertheless, Leikin (2015) cautions that the operationalisation of problem posing is dependent on the nature of teachers’ beliefs aligning with the suitability of such approaches and the critical provision of rich tasks. In the same vein, Abramovich & Cho (2015) illustrate the importance of future practitioners being equipped with conceptual understanding of didactic issues related to problem posing with technology.

3.6 Assessment

Although assessment is conducted for different reasons, it may be argued that its central purpose should be to support and enhance learning. Based on this premise, problem posing has been meaningfully employed to assess multiple mathematical constructs generated by pupils (e.g. Kantorovich et al. 2011; Van Harpen & Presmeg, 2013; Singer & Voica, 2015, Munroe, 2016) and prospective teachers (e.g. Crespo & Sinclair, 2008; Osana & Royea, 2011; Tisha & Hospesova, 2013; Singer, Voica & Pelczer, 2017). Pelczer & Rodriguez (2011) formulated criteria for assessing levels of creativity generated by a problem posing task on the topic of sequences. The lowest level was based on the application of a domain specific algorithm. A middle level was similar to the lowest level but was combined with some other form of knowledge. The highest level was categorised as using innovative knowledge from outside the topic. Kilic (2015) used semi-structured problem posing activities to determine prospective Turkish primary teachers’ knowledge structures of fractions.

Other researchers have designed frameworks or performance rubrics to support teachers in their assessment of problem posing tasks (e.g. Stoyanova & Ellerton, 1996; Leung, 1996;

Leung & Silver, 1997; Lowrie, 2002; Lin, 2004). Few studies exist that have focussed specifically on the assessment of children's problem posing skills (Cankoy & Ozder, 2017). Moreover, an extensive range of curriculum assessments remain underdeveloped for school mathematics (Rosli, Goldsby & Capraro, 2013). In this next section, I draw on two assessment tools to stimulate a discussion of characteristics and didactic competences.

Whilst promoting creative, flexible and higher-order thinking, it may be argued that a degree of subjectivity exists due to the open-ended nature of problem posing tasks. Shriki (2013) points out that creativity is dependent on a teacher's interpretation and is influenced by the mathematical abilities of a group. Likewise, Silver & Cai (2005, p. 131) warn that although considerable variability is common in the responses that pupils generate, "it can often present challenges from an assessment perspective". While the researchers make a valid point, it cannot be underestimated the significance of obtaining a diversity of problems from pupils.

Problem posing represents an essential form of authentic mathematical inquiry of which the basic tenet is the reformulation or generation of new problems. I believe that the main thrust of assessment should include early deduction of ill-structured and unsolvable problems, in addition to evaluating different levels of mathematical sophistication. Silver & Cai (2005) propose three criteria that can be coalesced for assessing problem posing ability within a classroom setting; quantity, originality and complexity. Quantity relates to the number of valid responses and fluency of generated problems can help to establish creativity. Originality is an obvious measure of creativity and a welcomed attribute. However, the emergence of originality may depend on working with large groups of pupils so as to distinguish between atypical responses. The complexity of pupils posed problems is a feature that is likely to be of interest to all teachers. Silver & Cai (2005) provide a good illustration of a problem posing task employed in a previous study (Silver & Cai, 1996) which can be used to evaluate complexity of pupil responses (Figure 3.4).

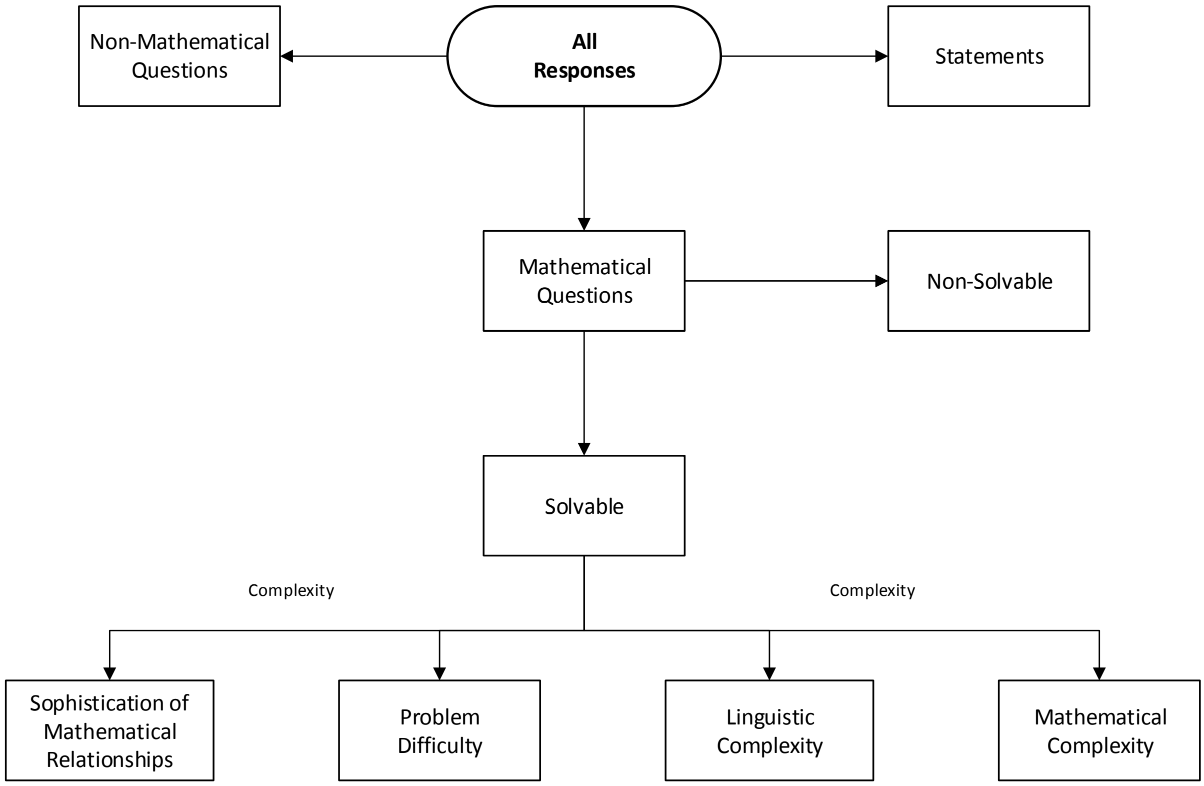
Figure 3.4 Task for evaluating pupil response (Adapted from Silver & Cai, 2005)

Write three different problems with the given situation:

Donald, Coinneach and Eilidh took turns driving home from a trip. Eilidh drove 80 miles more than Coinneach. Coinneach drove twice as many miles as Donald. Donald drove 50 miles.

Silver & Cai (2005) draw our attention to the aspect of linguistic complexity and suggest that this may be judged by focusing on linguistic structures, such as the presence of assignment, relational and conditional propositions in mathematical problem statements. The researchers extend this perspective to consider fundamental semantic structural relations in order to analyse complexity in more detail and offer a framework for assessing the complexity of pupil generated problems (Figure 3.5).

Figure 3.5 Framework for assessing complexity of pupil generated problems (Adapted from Silver & Cai, 2005)



In another study, Kwek (2015) explored the use of problem posing tasks as a formative assessment tool to examine thinking processes, understandings and competencies of secondary pupils. Her specific focus was on the cognitive demands of mathematical complexity which included aspects of knowing and doing mathematics, such as reasoning, performing procedures, understanding concepts or solving problems. Kwek employed a rubric (Table 3.1) to categorise three levels (low, moderate and high) of complexity of posed problems. Low complexity problems are usually solved by recalling and recognising facts or having a one-step solution. Problems that are categorised with moderate levels of complexity generally demand a combination of mathematics skills and knowledge. High complexity problems emphasise resourceful thinking by engaging solvers in a multitude of demands.

Table 3.1 Rubric for evaluating the complexity of posed problems (Adapted from Kwek, 2015)

	Low complexity	Moderate Complexity	High Complexity
Description	This category relies heavily on the recall and recognition of previously–learned concepts. Items typically specify what the solver is to do, which is often to carry out some procedure that can be performed mechanically. It leaves little room for creative solutions. The following are some, but not all, of the demands that items in the low–complexity category might make:	Items in the moderate– complexity category involve more flexibility of thinking and choice among alternatives than do those in the low–complexity category. They require responses that may go beyond the conventional approach, or require multiple steps. The solver is expected to decide what to do, using informal methods of reasoning and problem–solving strategies. The following illustrate some of the demands that items of moderate complexity might make:	High–complexity items make heavy demands on solver, who must engage in more abstract reasoning, planning, analysis, judgment, and creative thought. A satisfactory response to the item requires that the solver think in an abstract and sophisticated way. The following illustrate some of the demands that items of high complexity might make:
Cognitive demand	• Recall or recognize a fact, term, or property • Compute a sum, difference, product, or quotient • Perform a specified procedure • Solve a one–step word problem • Retrieve information from a graph, table, or figure	• Represent a situation mathematically in more than one way • Provide a justification for steps in a solution process • Interpret a visual representation • Solve a multiple-step problem • Extend a pattern • Retrieve information from a graph, table, or figure and use it to solve a problem • Interpret a simple argument	• Describe how different representations can be used to solve the problem • Perform a procedure having multiple steps and multiple decision points • Generalize a pattern • Solve a problem in more than one way • Explain and justify a solution to a problem • Describe, compare, and contrast solution methods • Analyse the assumptions made in solution • Provide a mathematical justification

3.7 Theoretical frameworks

During the last twenty years, a number of theoretical frameworks have emerged that conceptualise problem posing from an array of perspectives. For example, cognitive processes (e.g. Silver et al., 1996; Pittalis et al., 2004; Christou et al., 2005b; Chua & Wong, 2012; Kontorovich et al., 2012), assessment of problems posed (e.g. Silver & Cai, 2005; Kwek, 2015), strategic approaches (e.g. Silver, et al., 1996; Brown & Walter, 2005; Contreras, 2007), complexity in small groups (e.g. Kontorovich et al., 2012), connection to problem solving (English, 1997a, 1997b), learning opportunities (e.g. Lowrie, 2002; Crespo & Sinclair, 2008), mathematical modelling (e.g. Bonotto, 2010), creativity (e.g. Leung, 1997; Silver, 1997; Siswono, 2011), and situations of problems posed (e.g. Stoyanova & Ellerton, 1996).

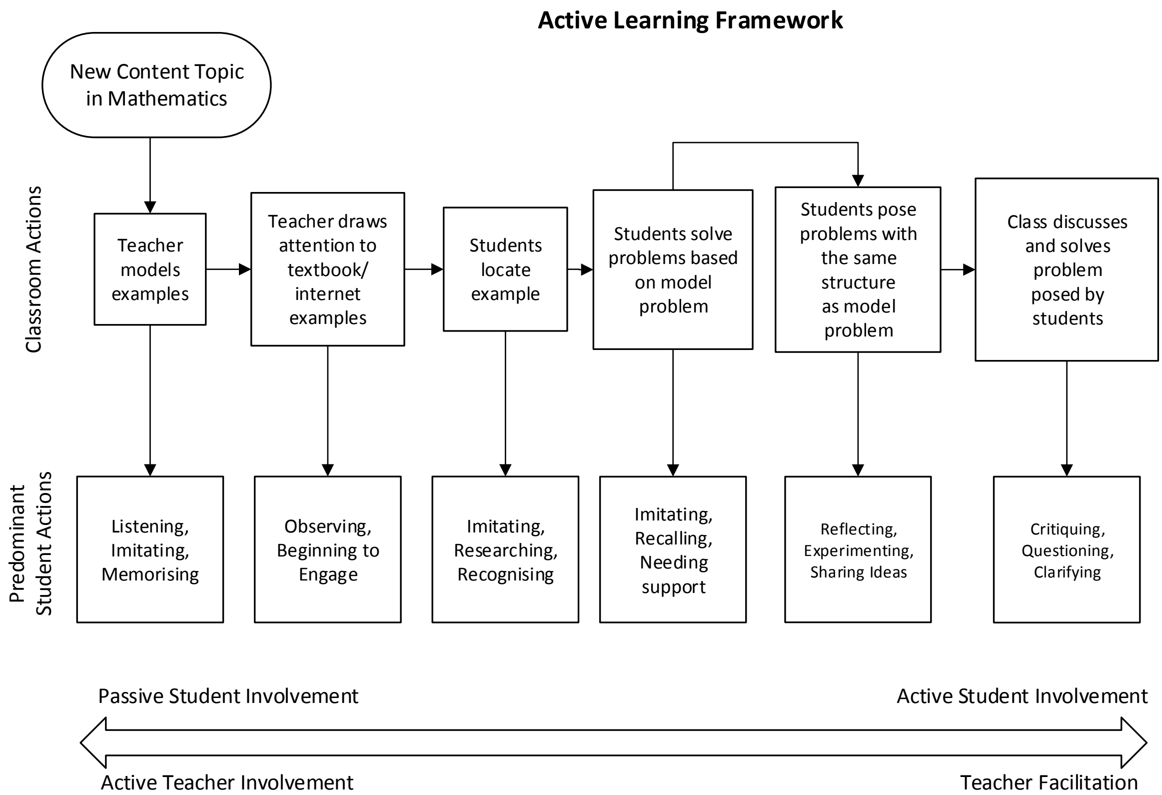
My main curiosity in problem posing rests with the professional development of teachers. In order for problem posing to be mobilised within classrooms, I believe that it is essential for practitioners to appreciate the pedagogical nature of problem posing and are competent in demonstrating the mathematical processes that define it. Although some research has highlighted teachers' difficulties with problem posing (e.g. Silver et al., 1996; Koichu, Harel & Manaster, 2013; Singer & Voica, 2015), several studies have noted that with intervention, teachers can lead to improved problem posing performances (e.g. Abu-Elwan, 2002; Crespo & Sinclair, 2008; Chapman, 2012).

In my personal view, the Active Learning Framework introduced by Ellerton (2013) aligns exactly with a constructivist orientation of learning and teaching mathematics. Central to this framework is the active engagement of pupils posing problems in parallel with corresponding problem solving activities (Figure 3.6). It is this seamless positioning of problem posing that Ellerton argues if excluded from school curricula, will deprive children of rich mathematical experiences. For problem posing to be introduced consistently into classrooms, teachers must acquire skills and confidence, which may be achieved by conceptualisation of problem posing within primary and secondary mathematics initial teacher education programmes or embedded within professional learning opportunities.

In a recent study employing the same framework, Ellerton (2015) derived that time consumed on posing mathematical problems should not be isolated from time expended on mathematics. She insists that "it should be seen by all stakeholders as time well spent on learning mathematics, and should not be seen as an imposition or an extra that somehow needs to be included in an already-busy curriculum" (p. 527).

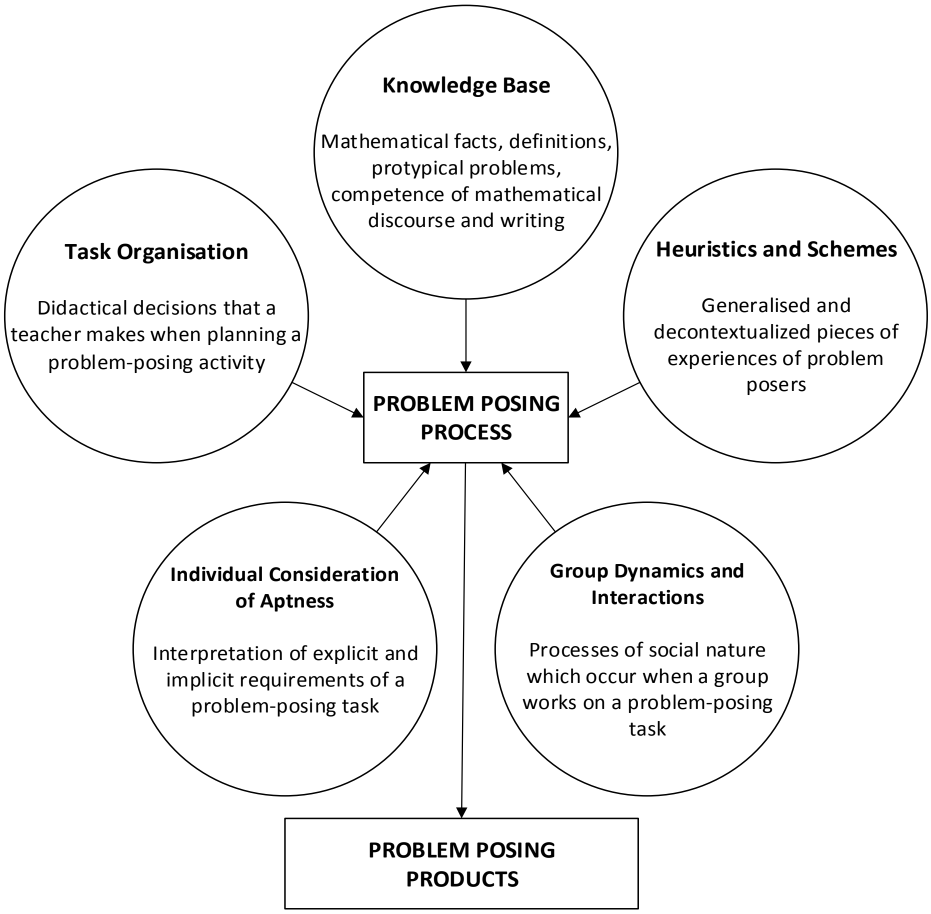
Teachers are continually required to engage learners in worthwhile mathematical experiences but such provision is influenced by the efficiency to select, create or pose appropriate problems. In a study of 40 Canadian student primary teachers, Chapman (2012) investigated methods of making sense of problem posing. By providing the participants with a range of assignments, she was able to analyse problem posing behaviour by task type. Chapman identified five perceptions on problem posing held by the teachers. Firstly, the paradigmatic perspective emphasises "creating a problem with a universal interpretation, a particular solution and an independent existence from the problem solver" (p. 140). Secondly, the objectivist perspective illustrates working backwards by beginning with a mathematical fact (e.g. 5 x 10 = 50) and then constructing a problem by providing context. Thirdly, the phenomenological perspective characterises problem posing that is meaningful from the individuals' point of view and which contains personalised interpretations and solutions. Fourthly, the humanistic perspective is similar to the phenomenological perspective but the context reflects the individuals' personal interests or experiences. (e.g. If the individual is interested in tennis, they might pose the following problem: A badminton club has 31 playing members. 27 play singles and 15 play doubles. How many play both singles and doubles?). Finally, the utilitarian perspective emphasises problem posing as an instrument to focus attention on the mechanics of mathematical thinking. Chapman states that the perspectives "provide a basis to compare and unpack their ways of problem posing. All five need to be explored in order to allow the teachers to understand how each could support or inhibit students' mathematical understanding and mathematical thinking" (p. 144).

Figure 3.6 Framework for locating problem posing in mathematics classrooms (Adapted from Ellerton, 2013)



I believe that, collectively, the frameworks offered by Ellerton (2013) and Chapman (2012) offer both primary and secondary mathematics teachers a suitable starting point in their development of problem posing knowledge. Kontorovich et al. (2012) present an insightful analysis of collaborative problem posing consisting of a framework which integrates five operationally defined facets (Figure 3.7) such as task organisation, knowledge base, problem posing heuristics and schemes, group dynamics and interactions, and individual considerations of aptness.

Figure 3.7 A confluence framework for handing the complexity of problem posing (Adapted from Kontorovich et al., 2012)



Kontorovich et al. (2012, p. 153) define considerations of aptness as “the posers comprehensions of explicit and implicit requirements of a problem–posing task within a particular context; they also reflect his or her assumptions about the relative importance of these requirements”. Interestingly, the researchers describe different types of individual considerations of aptness in problem posing such as aptness to potential evaluators, i.e. the poser’s assumptions about how other individuals would evaluate the problem poser’s skills and performance including aptness to group members i.e. one’s opinion about whether or not the idea suggested by the poser would be acknowledged by members of the group.

Kontorovich et al. (2012, p. 160) maintain that teachers should provide pupils with an opportunity to employ considerations of aptness to potential solvers, as this many lead to improving the quality of the problem posing product and suggest that “pedagogical effort should probably be invested in designing such situations, in which considerations of aptness will be addressed explicitly”. In practical terms, teachers can attempt to evaluate considerations of aptness based on observation of the interactions that takes place within groups and judge the quality of the problem posing ideas and resulting problems.

3.8 Professional reflection

This chapter has attempted to provide a summary of the works relating to mathematical problem posing from both a teacher and learner perspective. In doing so, it has presented a rationale for the conceptualisation and operationalisation of mathematical problem posing. In sum, it advocates the view that problem posing is of central importance in the domain of mathematics and to the nature of critical thinking (Silver & Cai, 1996; Silver et al., 1996) and endorses the stance from Ernest (1991, p. 265) that “school mathematics for all should be centrally concerned with human mathematical problem posing and solving”. According to Christou et al. (2005b, p. 149): “Problem posing is an important aspect of both pure and applied mathematics and an integral part of modelling cycles which require the mathematical idealization of real-world phenomena”.

Intertwined with problem solving, the reformulation of existing problems and the generation of new problems have the dynamic capability to increase conceptual mathematical understanding and to empower children to nurture their own innate creativity. Authentic problem posing activities can unleash a powerful connection between school mathematics and the real world, including being accessible to all learners irrespective of ability. In essence, problem posing with all its complexities, has the potential to redefine in a radical manner, independent learning, where pupils are energised to take a more active role in their mathematical development (Brown & Walter, 2005).

One criticism of much of the literature on problem posing is that it does not help to explain the dynamics of how teachers can support young people who reject or resist inquiry based pedagogy. Whilst problem posing activities promote autonomous learning and can empower pupils to interact more with mathematics, some children do not have the desire or motivation to engage with the cognitive demands of this instructional approach (Silver & Mamona, 1989; Silver, 1994). Characteristically, such pupils are products of previous mathematical success through the medium of teacher centred learning, which delimits the enactment of higher order thinking.

However, while problem posing is recognised as fundamental to the learning and teaching of mathematics, it remains on the periphery of school curricula (Ellerton, Singer & Cai, 2015). Arguably, without an official mandate, the injection of problem posing into classrooms appears to be fragmented. More concerning may be that all practitioners are not fully equipped to pose worthwhile problems (e.g. Koichu, Harel & Manaster, 2013; Singer & Voica, 2013).

I believe that if problem posing is to be interwoven within the fabric of mathematical instruction, the critical role of teachers needs to be examined. Since the enactment of any didactic vision is influenced by the beliefs of those charged with its implementation, practitioners must be robustly convinced of the theoretical merits and educational benefits of problem posing. Likewise, teachers need sufficient training to acquire the vital pedagogical skills to allow them to cultivate problem posing in practice. Crespo & Sinclair (2008, p. 412) contend "that in order for teachers to support student problem posing, they need to gain problem posing experience themselves". Abu-Elwan (2007) suggests that through technology guidance, it is possible to change the beliefs of teachers towards the role of problem posing in mathematics education.

Recently, as part of practitioner enquiry towards professional learning, I examined a rich problem posing activity known as the 'Billiard Task', with two experienced primary colleagues. Both individuals claimed no previous experience in mathematical posing problem and indicated a desire to collaborate in order to develop pedagogical knowledge of an innovative approach to teaching mathematics. The Billiard task has been utilised in previous studies involving prospective and practising mathematics teachers (Silver et al., 1996; Cifarelli & Cai, 2005; Koichu & Kontorovich, 2013) and can stimulate the generation of interesting problems and conjectures. Our interactions focused on considerations of aptness relating to individual understanding of an interesting problem and which problems would be suitable for potential learners (Kontorovich & Koichu, 2009; Kontorovich et al., 2012). Crucially, the outcome of the discourse prompted me to search studies reporting an element of ill-structured or cognitively undemanding problems created by teachers (e.g. Silver et al., 1996; Crespo & Sinclair, 2008). It is useful to consider what factors may inhibit the capability of teachers to pose authentic problems (e.g. poor conceptual knowledge of the underlying construct) and correspondingly, how to prepare teachers to produce interesting and challenging problems to cater for multiple levels of pupil abilities. Regarding this last point, in the absence of any recognised LEA mandate and within a climate of political and education insecurity, it is difficult to envisage a provision of learning opportunities for primary and secondary mathematics teachers to develop problem posing skills.

A more realistic window for achieving the future integration of problem posing in Scottish schools may have its origin in the education of new recruits, since the persistence of classroom norms operate against many teachers attempting to improve their professional practice. Ellerton (2013) maintains that:

> Perhaps the only way that problem posing has a chance of being seriously introduced into school mathematics curricula and classroom practices would be for young teachers to acquire problem-posing skills and confidence in problem posing themselves to the point where they would be capable and willing to help their students to pose problems. The simplest way to move towards achieving this would be to focus attention on this issue in early childhood, primary, and secondary mathematics teacher education programs (p. 100).

Naturally, the overarching goal for the international research community is to generate empirical evidence of improved pupil mathematical learning (e.g. English, 1997b, 1998) alongside other benefits such as nurturing creativity. The mandate for such evidence is similarly pertinent for engaging prospective and current teachers in order to strengthen knowledge and understanding that can be applied to raise professional standards. Harvesting empirical evidence of problem posing will help prompt relevant stakeholders and policy makers to take notice given that Scottish education is committed to drive forward improvements utilising evidence based research as an approach to classroom practice and curricula reform. Building on this professional reflection it seems important to move to a researcher's interrogation of the ideas.

The next chapter of this thesis presents a literature review of teachers' beliefs, which are at the epicentre of this research.

CHAPTER FOUR

Teachers' Beliefs

4.1 Introduction

Much societal and political rhetoric has positioned teachers as the critical variable in the successful development of learning of all children. Since I began researching my own practice, I have become increasingly aware of the richness attached to critiques of mathematics education by colleagues. Though, what continues to resonate is the diversity of beliefs held regarding mathematical problem solving and mathematical problem posing.

Over thirty years ago, Thompson (1985) highlighted the disproportionately small amount of attention that researchers had bestowed to the role of the teacher. Similarly, Grouws (1985) emphasised his concern with the lack of research on how practitioners conceptualise mathematical problem solving and how they attempt to teach it. In recent years, there has been an increasing interest in the potential of research to inform classroom practice. Pajares (1992, p. 307) asserts that "the beliefs of teachers should be a focus of educational research and can inform educational practice in ways that prevailing research agendas have not and cannot".

It is the premise of this thesis that individual teachers' deep rooted beliefs are a major influential factor in the concentration of learning and teaching of mathematical problem solving and mathematical problem posing. In other words, how a teacher conceptualises the nature of mathematics has a direct impact on what is delivered to pupils and therefore any changes will require an analysis of professed beliefs, actual beliefs and current practices. However, Forrester (2008, p. 25) points out that: "Whether or not a teacher's beliefs are successfully translated into practice, they give an important indication of the teacher's intentions for the future".

Within CfE, the mantra of increased autonomy to teachers as agents of change has been well broadcasted. Castle (2006) contends that autonomous practitioners learn to make enhanced instructional decisions by undertaking their own critical thinking about educational matters and opines that teachers, who are not autonomous, depend on others to communicate what to do. Scottish teachers theoretically enjoy the dynamic pedagogical freedom to exercise, reflect, research and develop their own perspectives on the learning and teaching of mathematics, thus connecting beliefs to professional practice. As was pointed out in chapter one, with no available empirical data to analyse it is unknown what the nature of this relationship is.

In this chapter, I will briefly review key theoretical perspectives and important contributions that have advanced research on teachers' beliefs. In doing so, I will provide a rational for the importance of teachers' beliefs with reference to recent educational policy reform in Scotland.

4.2 Defining the "belief" construct

Extensive academic debates attempting to define a precise universal conceptual definition on the belief construct has yet to be established within the research literature. Possibly as a result of researchers assuming that readers already know what beliefs are (Thompson, 1992) or the flexibility of the belief construct is accommodating to many (Goldin, Rosken & Torner, 2009). Alternatively, Leder & Forgasz (2002) contend that it is not easy to produce a precise definition because the belief concept is not directly observable and is inferred. However, Fives & Buehl (2012) argue that the difficulty lies not in the definition since several authors have provided so but instead encouraging researchers to consistently define and use such terms within and across interrelated fields. Nevertheless, a mosaic of overlapping constructs populated within a densely and uncertain world of interchangeable conceptualisations exist. Though Wilson & Cooney (2002) advise that it is more germane to be acquainted with the influence of teacher beliefs rather than seeking harmony on a definition. McLeod & McLeod (2002, p. 120) propose "there is no single definition of the term "belief" that is correct and true, but several types of definitions that are illuminative in different situations". For example, Pajares (1992) expresses the view that beliefs include:

> attitudes, values, judgments, opinions, ideology, perceptions, conceptions, conceptual systems, preconceptions, dispositions, implicit theories, explicit theories, personal theories, internal mental processes, action strategies, rules of practice, practical principles, perspectives, repertoires of understanding, and social strategy (p. 309).

According to Hermans, van Braak & Van Keer (2008, p. 128), beliefs are “a set of conceptual representations which store general knowledge of objects, people and events, and their characteristic relationships”. Cross (2015, p. 175) maintains that beliefs are “embodied conscious and unconscious ideas and thoughts about oneself, the world, and one’s position in it developed through membership in various social groups, which are considered by the individual to be true”. Tillema (1994) avows that beliefs serve as filters which screen new information and ultimately determine which elements are accepted and integrated in their knowledge base. Similarly, Clark & Peterson (1986) advocate beliefs act as a monitor which practitioners make their decisions rather than just relying on their pedagogical knowledge or curriculum guidelines. What is notable about this definition is that is recognises that teachers’ beliefs can influence classroom practice irrespective of the written curriculum. In contrast, Perry, Wong & Howard (2006) warn that beliefs are rooted and constrained by the culture of the society and educational systems in which the teachers are living and working.

Unsurprisingly, with so many different perspectives, Mason (2004, p. 347) calls for the research community “to work out what beliefs actually are, and where they fit into an entire alphabet of associated terms”. Skott (2013, p. 548) notes that belief research is notorious for its conceptual and methodological problems and laments that the “notion of beliefs, however, is still somewhat underspecified, and the discussion continues on how to distinguish it from knowledge, conceptions, emotions, and values”. For instance, in their study of American primary teachers’ pedagogical content beliefs, Peterson et al. (1989) describe practitioners as individuals who rely on their knowledge and beliefs to understand and interpret the rapid flow of events in a classroom, make decisions and act on their interpretations. Peterson et al. (1989) employ the term ‘knowledge’ which relates to ‘pedagogical content knowledge’ as illustrated by Shulman (1986) but appear to attempt to coalesce knowledge and beliefs into a common construct. While some support remains for this conception (e.g. Pajares, 1992; Calderhead, 1996) other researchers have at least attempted to distinguish between beliefs and other suppositions such as knowledge, affect, values, emotions, etc. (e.g. Nespor, 1987; Kagan, 1992; Thompson, 1992; Calderhead, 1996; Richardson, 1996; Handel, 2003; Philipp, 2007).

Beliefs have been extensively portrayed from a mathematical perspective. Goldin, Rosken & Torner (2009) argue that beliefs are fundamental to the discussion of problem solving approaches in mathematics education and are necessary components in the psychology of how mathematical problems are solved. Schoenfeld (1992, p. 358) interprets beliefs "as an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior". It is important to consider that the nature of the classroom environment that the teacher creates can shape learners beliefs about the nature of mathematics (Schoenfeld, 1992). Likewise, Lester, Garofalo & Kroll (1989, p. 77) articulate that "beliefs constitute the individuals subjective knowledge about self, mathematics, problem solving, and the topics dealt with in problem statements". In the same vein, Hersh (1986, p. 13) invites us to examine our mathematical pedagogy and alludes to the critical nature of a philosophy of mathematics when he questions: "The issue, then, is not, what is the best way to teach? But, what is mathematics really all about?".

In sum up, Skott (2015) in his analysis of the conceptualisation of teachers' beliefs concluded that there appears to be four key aspects. First, beliefs are used to describe individual mental constructs that are subjectively true for the person in question. Second, there are cognitive as well as affective aspects to beliefs. Third, beliefs are considered stable and may stem from schooling, life experiences, teacher education programmes and collaborations with colleagues. They tend to be resistant to change. Fourth, beliefs are expected to significantly influence classroom practice.

4.3 Teachers' belief systems

Within the literature, there is considerable agreement that teachers' beliefs are not regarded in isolation but consist of various substructures within a multidimensional system. Green (1971) identified three theoretical dimensions of belief systems, which have become fertile ground for researchers (e.g. Schoenfeld, 1985; Nespor, 1987; Pajares, 1992; Thompson, 1992; Cross, 2009; Braunling & Eichler, 2015). He postulated that beliefs are not compartmentalised but are in fact interrelated in elaborate ways. Firstly, there is the quasi-logical relation between beliefs which are depicted as either primary or derivative. Thompson (1992) illustrates this hierarchical dimension by considering a teacher who believes that it is important to present mathematics "clearly" (primary belief) and to obtain this outcome has to plan thoroughly and be readily prepared to answer pupil questions (both derivative beliefs). Secondly, based on their psychological strength, some beliefs are considered central or peripheral. Rokeach

(1968), as cited in Pajares (1992), maintains beliefs vary along a central-peripheral continuum, where the more central a belief is situated, the more resistant it is to change.

Thirdly, beliefs can be held in clusters and may coexist without conflict in isolation. This would explain why some individuals can hold incompatible or inconsistent views without any sense of divergence. Thus, beliefs are not considered mutually exclusive as teachers can hold overlapping views simultaneously and over time. Nonetheless, teachers' beliefs may change when they are afforded effective opportunities to reflect and challenge those beliefs (Wilson & Cooney, 2002).

In her case study of five American mathematics teachers, Cross (2009) highlights the diversity among practitioners' beliefs by presenting three hypothesised belief models. Firstly, she describes the parallel belief systems of Mr. Henry, Mr. Brown and Ms. Reid, unfolding that these individuals deem mathematical knowledge as an absolute established set of concepts that are rigid and infallible, with their classroom practices reflecting those beliefs. Secondly, in the example of Mr. Simpson, the researcher accounts that although his mathematical beliefs differed considerably from the other teachers, they did cluster in similar ways. Cross (2009) describes Mr. Simpson's mathematical views from a social constructivist perspective, not as fragmented groups of isolated facts and concepts but as an interconnecting and evolving set of relationships. Finally, in the case of Ms. Jones, the researcher conveys her mathematical beliefs as a conglomerate of viewpoints grounded on the importance of problem solving and critical thinking, coupled as a vast reservoir of knowledge rooted in numbers. Ms. Jones believed that it was vital she possessed an information base to teach pupils how to solve problems, identify errors, and demonstrate how to correct them. Cross (2009) concluded that her participants' beliefs were organised in a system such that theories about learning and teaching of mathematics were derived from their core mathematical beliefs. Furthermore, in the case of Ms. Jones, her opposing pedagogical views did not appear to present any internal conflict.

In another study, Braunling & Eichler (2015) investigated the belief systems of six recently qualified German primary and secondary mathematics teachers, which focused on the learning and teaching of arithmetic. Based on the analysis of Mrs. A, the researchers were able to distinguish her espoused central beliefs, peripheral beliefs and the assorted interrelationships between them. However, the study would have been more convincing if the authors had used a larger sample.

From my own collaborations of working with colleagues, two classifications of belief structures appear to dominate the teaching of mathematics within Scottish schools. Both orientations sit at the opposite ends of a spectrum. The first system is where teachers have a direct transmissionist view of learning, where knowledge is communicated in an explicit and structured way, where pupils are presented with demonstrations of correct solutions together with a provision of unsophisticated mathematical problems. Teachers resolutely believe that a quiet classroom is required for effective teaching. In contrast, the second system adopts a constructivist view of learning which focuses on children not as passive recipients but as active participants in the management of acquiring knowledge. Practitioners holding this perspective emphasise the facilitation of pupil inquiry and provide challenging mathematical problems to cultivate knowledge. Individuals offer the minimum of support in order to allow pupils more freedom to execute an active independent role in their thinking and reasoning processes.

A number of researchers have provided different classifications of teachers' mathematical beliefs systems (e.g. Skemp, 1987; Lerman, 1989, 1990; Askew et al, 1997; Chapman, 2002; Speer, 2005; Beswick, 2012). One well-known review that is often cited is Kuhs & Ball (1986) who draw our attention to the connection between teachers' mathematical conceptualisations and their instructional methods by identifying four overriding approaches to the teaching of mathematics. The first is a description of teaching as content focussed with an emphasis on performance, which has been expressed as instrumental learning (Skemp, 1978) and calculational orientation (Thompson, et al., 1994). Here, pupils are taught to follow and master rules and procedures without acquiring any conceptual understanding, where memorisation of mathematical facts is stressed. The second approach is a description of teaching as content focussed with an emphasis on ensuring conceptual understanding. The third arrangement is focussed on context where the classroom structure and organisation strongly influences student learning. Finally, the last approach is learner focused and is underpinned by a social constructivist view of learning (Thompson, 1992). This method is characterised by engaging the learner with activities that explore, discover, formulate and construct a wide range of mathematical ideas. This is similar to Skemp's (1978) relational concept of mathematics.

In his major contribution, Ernest (1989a) suggests that acceptance of teaching mathematics through problem solving depends fundamentally on profound changes to a teachers' belief system. He argues that mathematical instruction is dependent on several key elements, but in particular on the practitioners' conception on the nature of mathematics, including mental models of learning and teaching of mathematics. A teachers' conception of the nature of mathematics may be considered as conscious or subconscious views, perceptions, values, guidelines, mental images and preferences concerning the discipline of mathematics (Thompson, 1992).

Ernest (1989a) posits three distinct philosophies of the nature of mathematics that are held by teachers as individual beliefs systems. First, the instrumentalist view regards mathematics as an accumulation of facts, rules and skills in the pursuance of some external end. Thus mathematics is thought to be a set of unrelated but utilitarian rules and facts. Routinely, teachers expect pupils to listen, participate in didactic interactions and then replicate computational algorithms that have been demonstrated. Such a position has been the object of much criticism by mathematics educators (Thompson, 1992). Second, the Platonist view considers mathematics to be a static but unified body of certain knowledge. In this case, mathematics is discovered (not created) by humans through mathematical investigation. Third, the problem solving (or social constructivist) view deems mathematics as a dynamic, continually expanding field of human creation and invention, a cultural product. Mathematics is believed to be a process of inquiry and coming to know, not a finished product. Crucially, its results remain open to revision. Proficiency in mathematics is equated with autonomous problem solving and problem posing. The former two views assimilate within the domain of absolutism while the latter one within the domain of fallibilism (Thompson, 1992; Ernest, 2014).

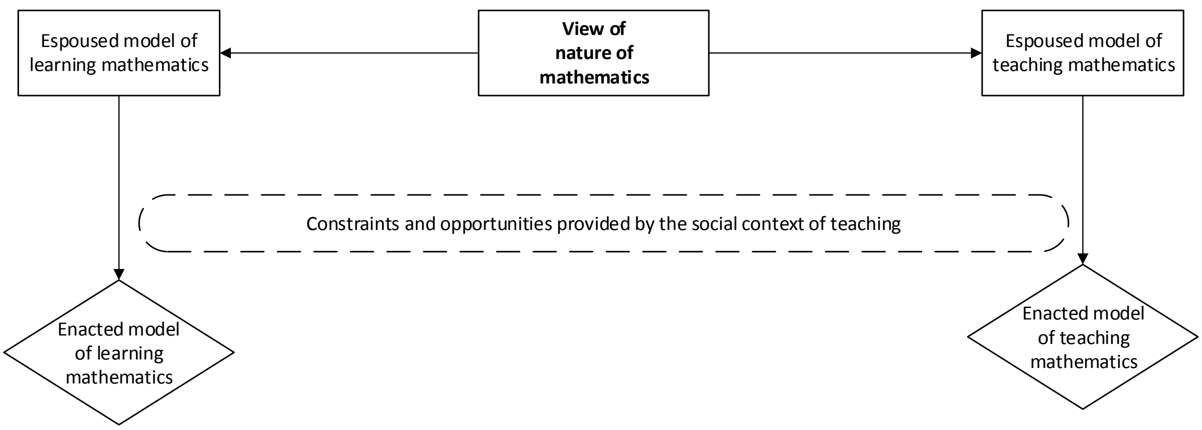
Ernest (1989a) proposes three instructional models to reflect the diverse roles a teacher might play within a classroom, which has been encapsulated by Leatham (2002) in his doctoral dissertation (Table 4.1). Both instrumental and Platonist views and their respective derived teaching strategies present narrow views of mathematics, compared to a problem solving perspective which promotes a child-centred and inquiry based approach (Schoenfeld, 1992).

Table 4.1 Ernest models as summarised by Leatham (2002)

Phenomena	Instrumentalist	Platonist	Problem Solving
Nature of mathematics	An accumulation of facts, rules, and skills	A static but unified body of certain knowledge	A dynamic continually expanding field of human creation and invention
Teacher's role	Instructor	Explainer	Facilitator
Intended outcome	Skills mastery with correct performance	Conceptual understanding with unified knowledge	Confident problem posing and problem solving
Use of curricular materials	Strict adherence to a text or scheme	Modification of the textbook approach, enriched with additional problems and activities	Teacher, student, or school construction of the mathematics curriculum

Ernest (1989a) outlines the relationship between beliefs and their impact on classroom practice by illustrating how teacher's views of the nature of mathematics provide a basis for mental modes of the learning and teaching of mathematics (Figure 4.1). However, the model proposed by Ernest (1989a) is not universally shared by all. For example, Skott (2013, p. 548) rejects such models and argues that this "line of research was and still is based on the assumption that teachers' beliefs are a main line to educational change, and that beliefs research may remedy what is generally referred to as the problems of implementation".

Figure 4.1 Relationship between beliefs and their impact on practice (Adapted from Ernest, 1989a)



Ernest (1991) defines three distinctive interpretations about the role of problem solving in the mathematics curriculum. First, he argues that problem solving is rejected by "Industrial trainers" as inappropriate and frivolous based on the perception that mathematics is a set of absolute decontextualized but utilirian truths and rules, and that its central function is to inculcate basic skills. Second, he suggests both "Old humanists" and "Technological pragmatists" consider problem solving as additional content and implemented as mechanical objects of inquiry used to enrich teaching. Finally, Ernest argues that "Progressive educators" and "Public educators" view problem solving as pedagogical approaches to the whole curriculum, and not just as an addition. Such ideologies arise from philosophies of mathematics which deem it a growing field of knowledge, if not as social constructivism and maintain full incorporation of these processes into the curriculum, including problem posing, leads to a problem solving and investigational pedagogy (Ernest, 1991).

4.4 The importance of teachers' mathematical beliefs

The thrust of research into mathematics related beliefs has centred on beliefs about the nature of mathematics and the learning and teaching of mathematics (Thompson, 1992; Beswick & Callingham, 2014). A growing number of empirical studies (e.g. Thompson, 1984; Cooney, 1985; Chapman, 1999; Aguirre & Speer, 2000; Beswick, 2004; Speer, 2008) have been propelled by the supposition that there subsists a positive correlation between espoused mathematical beliefs and instructional practices. In other words, there has been a plethora of research on teachers' beliefs based on the presumption that what teachers believe is a powerful indicator in selecting what mathematics is taught, how it is delivered and what is learned in the classroom (Wilson & Cooney, 2002; Beswick, 2006; Skott, 2015). The research has advocated that beliefs are one of the major components influencing pedagogical practice and should not be underestimated. Kilpatrick (2003) informs us that beliefs influence the choice of curriculum materials and therefore affect the type of mathematical ideas and opportunities offered to pupils. In the same vein, Pajaras (1992, p. 325) argues that "beliefs are instrumental in defining tasks and selecting the cognitive tools with which to interpret, plan, and make decisions regarding such tasks; hence, they play a critical role in defining behaviors and organizing knowledge and information".

In their research of 21 American primary teachers, Stipek et al. (2001) found a consistent association between mathematical beliefs and observed classroom practices. Likewise, Zakaria & Maat (2012) noted a positive connection between mathematical beliefs and reported pedagogical methods in their study of 51 Malaysian secondary mathematics teachers.

In her renowned case studies of three American junior high school mathematics practitioners, Thompson (1984) described a teacher named Kay who perceived mathematics as continuously expanding and changing to accommodate new developments. Kay's espoused views about mathematics were consistent with her espoused views about teaching mathematics and with Thompson's observations. According to Thompson (1984), she employed a variety of approaches to stimulate interest including encouraging pupils to ask questions, guess, theorise, and be wrong; used appropriate examples and non-examples; provided a variety of justifications; showed applications of the topics taught; and mobilised games and puzzles as motivational devices.

Other studies have been illuminating in comparable ways. For example, in an investigation of problem solving and problem posing ability and beliefs of 128 Chinese prospective and current primary teachers, Chen et al. (2011) found that their participants' mathematical beliefs strongly influenced their evaluation of pupils attempts to solve and pose mathematical word problems. Sivunen & Pehkonen (2009) analysed the mathematical beliefs on teaching problem solving of 42 Finish primary teachers. The researchers learned that practitioners held a limited knowledge of instructional techniques and were dependent on available resources. In an international comparative study, Zambo & Hong (1996) found that South Korean teachers held stronger views than American teachers regarding the importance of being a proficient problem solver prior to teaching problem solving. The research alluded that South Korean educators did not promote multiple solutions, much preferring single solution paths in order to minimise learner uncertainty. However, such an adverse belief is at the expense of fostering creative mathematical thinking.

By acknowledging the importance of beliefs in shaping teachers characteristic patterns of instructional behaviour, it is possible to formulate steps to improve the quality of mathematics education. Thompson (1984) forewarns that failure to recognise the role that teachers' beliefs might play in determining their professional practice is likely to result in misguided efforts to improve the standard of mathematics instruction in schools. The delivery of school mathematics has been compared with the work of professional mathematicians by several scholars (e.g. Ernest, 1991; Beswick, 2012; Boaler, 2015b). Boaler (2015b) argues that teachers' traditional beliefs of the nature of mathematics may adversely affect young people's image of the subject. She suggests that for pupils to appreciate and enjoy mathematics, they need to be regularly exposed to authentic ways in which mathematicians operate such as "posing problems, making guesses and conjectures, exploring with and refining ideas, and discussing ideas with others" (p. 31).

4.5 Inconsistencies between teachers' beliefs and practice

While miscellaneous studies have found consistencies between teachers' mathematical beliefs and instructional practices, more often research has revealed a misalignment between the two features (Thompson, 1992, Phillip, 2007). The correlation between mathematical beliefs and mathematics teaching are multifaceted, dialectical and can be influenced by a structure of reciprocal factors. These may be rationalised through the anxiety and unpredictability of classroom life, external pressures and constraints placed on teachers that compromise their contemporary views of education in place of more traditional methods. In clarifying such inconsistencies, Beswick (2006) draws on the notion of clustering, citing Green (1971). She maintains that "beliefs within a system can be held in groups that are isolated from other beliefs" and "a person may hold beliefs that contradict one another without being aware of the contradiction".

In her renowned case study of novice American primary teachers, Raymond (1997) described the case of Joanna who held traditional beliefs about mathematics but non-traditional beliefs about learning and teaching of mathematics. Raymond (1997) determined that this inconsistency arose from various factors and introduced a theoretical framework (Figure 4.2) towards understanding the complex nature of the interrelationship between mathematical beliefs and classroom practice. She warns stakeholders not to overlook multiple factors that teachers are frequently exposed to. Moreover, Thompson (1984, p. 124) maintains that: "Many factors appear to interact with the teachers' conceptions of mathematics and its teaching in affecting their decisions and behavior, including beliefs about teaching that are not specific to mathematics".

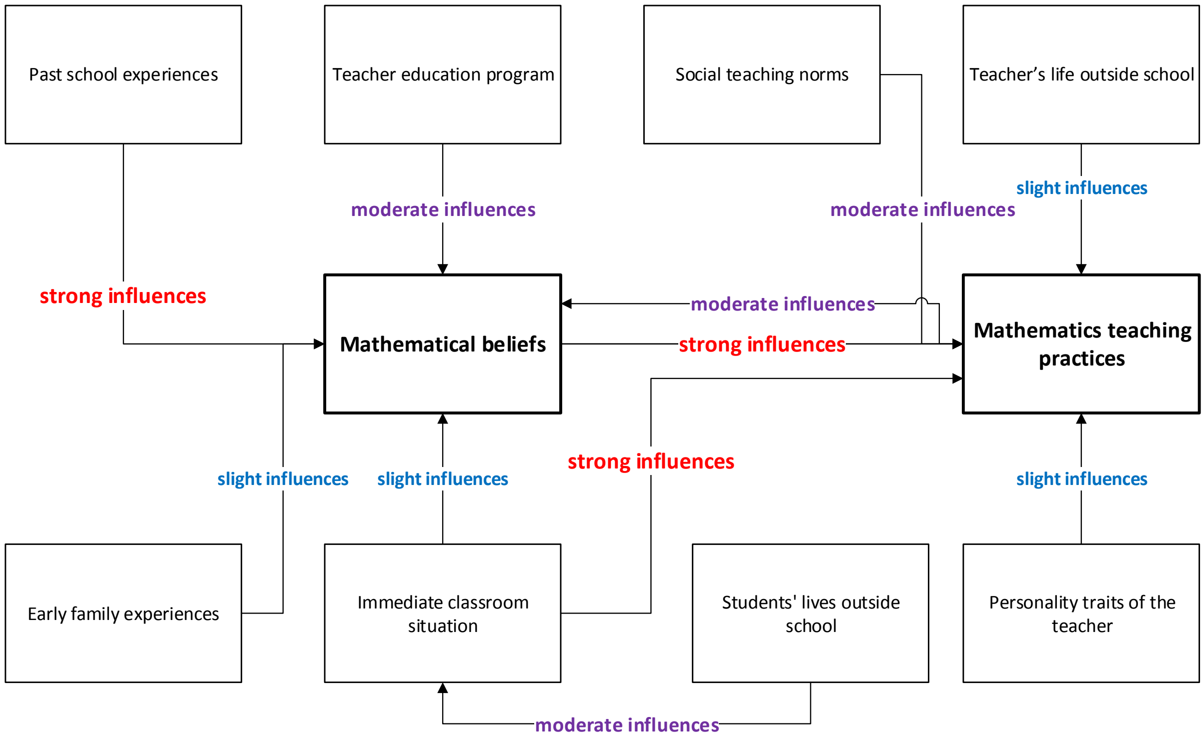


Figure 4.2 Model of relationships between mathematics beliefs and practice (Adapted from Raymond, 1997)

Mathematics beliefs:
About the nature of mathematics and mathematics pedagogy

Mathematics teaching practice:
Mathematical tasks, discourse, environment and evaluation

Immediate classroom situation:
Students (abilities, attitudes and behaviour), time constraints, the mathematics topic at hand.

Social teaching norms:
School philosophy, administrators, standardized tests, curriculum, textbook, other teachers, resources

Teachers life:
Day-to-day occurrences, other sources of stress

Students' lifes:
Home environment, parents' beliefs (about children, school and mathematics)

Teacher education program:
Mathematics content courses, methods courses, field experiences, student teaching

Past school experiences:
Successes in mathematics as a student, past teachers

Early family experiences:
Parents' view of mathematics, parents' educational background, interaction with parents (particularly regarding mathematics)

Personality traits:
Confidence, creativity, humour, openness to change

In a study of a novice Danish teacher, known as Christopher, Skott (2001) investigated the relationship between the beliefs of mathematics, learning and teaching of mathematics and that of classroom practice. He introduced the term 'school-mathematics images' to "describe teachers' idiosyncratic properties in relation to mathematics, mathematics as a school subject and the teaching and learning of mathematics in schools" (p. 6). Skott (2001) found that Christopher's school mathematics images were highly compatible with aspects of the reform discourse but this relationship with classroom practice was very different in contrasting situations. However, what is distinguishing about the study is that is challenged much of the underlying reasoning and evidence from previous research in the field on teachers' beliefs. In choosing not to instinctively exploit apparent inconsistencies, Skott attempted to rationalise the dissonance between beliefs and practice. By reflecting on his own position as a researcher, he was able to acknowledge that his initial assumption that Christopher's beliefs about mathematics would have been the driving force behind his pedagogical decisions. Instead, the more centrally held belief for Christopher was dominated by an emphasis on the individual learner, sometimes at the expense of his mathematical beliefs.

The outcome of the study helped influence Philipp (2007) to propose that as researchers, we must assume that contradictions between teachers' beliefs and practice do not exist. Taking this stance, Philipp (2007, p. 276) maintained "when we observe apparent contradictions, we would assume that the inconsistences exist only in our minds, not within the teachers, and would strive to understand the teachers' perspectives to resolve the inconsistencies". In his impressive analysis of teachers' beliefs, Leatham (2006, p. 92) anchored his sensible systems theoretical framework on the fundamental assumption that "teachers are inherently sensible rather than inconsistent beings". Put differently, individuals beliefs are organised in systems that make obvious sense to them. Leatham (2006) underlined the need for researchers to follow a process of exploring and explaining apparent inconsistencies rather than simply indicating conflicts so as to facilitate a deeper understanding of the nature of beliefs and how they are held.

Furinghetti & Morselli (2011) in their case studies of four Italian secondary mathematics teachers' treatment of proof, focussed on the detection of the reasons behind instructional practices. To unravel the dilemma of inconsistencies, the authors introduce the construct of leading beliefs which they define as "beliefs (whole nature may vary from teacher to teacher) that seem to drive the way the teachers treats proof" (p. 590). Furinghetti & Morselli (2011) claim that through the construct of leading belief they were able to divert attention away from inconsistencies.

Likewise, during a case study of two American elementary teachers, Cross (2015) skilfully looked beyond perceived inconsistencies to gain a better understanding of the nature of mathematics beliefs and how they were organised. Drawing on the tenets of a sensible systems framework (Leatham, 2006) and utilising both contextualised and de-contextualised data sources, Cross (2015) found that observed practices were aligned with other sets of beliefs. These aligned beliefs were personal and external factors including beliefs not directly related to the teaching of mathematics. To ensure that researchers better understand the complexities of individual beliefs systems, Cross (2015, p. 198) highlights the requirement "to expand the scope of their investigations to include multiple contexts, examining the role of macro-and mircofactors on instruction".

What is apparent from the case studies described in this section is the need for researchers to attempt to fully understand the complexities and interactions that manifest themselves within the context of school and classroom cultures. Teachers' espoused beliefs impact on their pedagogical practice but a direct causal relationship cannot be assumed. Multiple factors may influence both professional practice and the institutional context (Cooney, 1985; Hoyles, 1992).

4.6 Changing teachers' beliefs

It is a widely held view that teachers' beliefs are slow to form but once established are highly resistant to change. According to Schommer-Aikins (2004, p. 22), they "are like old clothes; once acquired and worn for a while, they become comfortable. It does not make any difference if the clothes are out of style or ragged. Letting go is painful and new clothes require adjustment." Furthermore, teachers may not be consciously aware of the underlying beliefs that underpin their practice (Schoenfeld, 2015).

Though it is asserted by Liljedahl (2010) that the trajectory of change in teachers' beliefs and practices can also be rapid and profound. In his research study of mathematics professional activities set within Canada, Liljedahl (2010) identifies five distinct mechanisms of belief change: (1) conceptual change (2) accommodating outliers (3) reification (4) leading belief change (5) push-pull rhythm of change. Within this chapter, I have referred to conceptual change and leading belief changes. Liljedahl (2010) illustrates a leading belief change by describing the case of Phil, a primary teacher and problem solving workshop participant. As a consequence of being deeply affected by one experience, Phil made a significant change to his belief system and evaluation practices. This transformation was expedited by Phil's ability to critically examine his professional practice.

In another study involving practising secondary mathematics teachers, Liljedahl (2011) strengthens his argument surrounding teacher change as conceptual change. All of the participants were situated within a professional learning environment and subjected over time to interventions designed to promote cognitive conflict within their core beliefs about various aspects of mathematics education. Based on the results, Liljedahl (2011) contends that the theory of conceptual change may act as a framework for changing teachers' beliefs. An intriguing outcome of this study revealed that participants not only rejected beliefs pertaining to their current practice but often did so without replacement. This can be exemplified by reference to an assignment tasked with reviewing Boaler (1997) and her dichotomous settings of Amber Hill and Phoenix Park. While many of the participants were quick to reject the teaching practices of Amber Hill, they were reluctant to embrace the paradigm extolled in the descriptions of Phoenix Park, which integrated problem posing.

I will now review two studies that feature specific mathematical domain beliefs about problem solving and problem posing i.e. Emenaker (1996) and Barlow & Cates (2006).

In the first study, Emenaker (1996) analysed the impact of a problem solving based mathematics course on 137 American prospective elementary teachers' beliefs about mathematics and the teaching of mathematics. Prior to launching the course, he found considerable support clustered around belief misconceptions listed as: (1) If a mathematics problem takes more than 5-10 minutes, it is impossible to solve (2) Mathematics is mostly memorisation (3) All problems can be solved using a step-by-step algorithm or a single equation (4) Only geniuses are capable of creating or understanding formulas and equations (5) There is only one correct way to solve any problem. On completion of the course, Emenaker (1996) detected positive alterations to participants beliefs manifested primarily through three underlying themes. First, certain problems contain multiple solutions and alternative answers. Second, conceptual understanding is more important than memorising procedures. Third, through independent thinking it is reasonable to expect children of average mathematical ability to discover some concepts on their own. However, the research would have been more convincing if the observed changes in beliefs were monitored over a longer time period to determine stability and resilience.

In the second study, Barlow & Cates (2006) investigated the impact of incorporating problem posing on the beliefs about mathematics and the teaching of mathematics of 61 American elementary teachers. The participants were exposed to a one year innovative professional learning programme featuring miscellaneous references to the literature complemented with pedagogical activities. The results suggested that working with teachers to incorporate problem posing into their classroom practice had a positive effect on changing teachers' beliefs and their instructional approaches.

In these two studies, it is evident that reflection on their individual beliefs and classroom practices contributed significantly to teacher change towards problem solving and problem posing respectively. Reflection is regarded as a critical factor for shifting beliefs, as teachers learn fresh ways to make sense of what they observe (Philipp, 2007). However, in both instances, participants volunteered for each experience implying that on entering each study may have had a degree of motivation and preconceived interest in changing their central or peripheral beliefs. In the following subsection, I consider the challenge of teacher belief change from the perspective of education reform, where practitioners may not be consciously aware of the beliefs that underpin their classroom strategies.

4.6.1 Teachers' beliefs and educational reform

Fundamental to successful implementation of any education reform is the teacher. Moreover, teachers are the key agents when it comes to transforming practice and curriculum enactment depends in great part on the capacity and will of the teachers involved (Spillane, 1999; Fullan, 2016). As a subset of this professional agency, teachers' beliefs play an essential role within this domain (Hargreaves, 1994; Biesta, Priestley & Robinson, 2015; Fullan; 2016)

Teachers' prevailing beliefs about mathematics and mathematics pedagogy have long been detected as one of the major obstacles to educational reform (Pajares, 1992; Cooney &

Shealy, 1997; Handel & Herrington, 2003; Aquirre, 2009). Chapman (1999) cautions that practitioners' beliefs about the nature of mathematics are a main factor in any movement to ameliorate the teaching of mathematics. Similarly, Goldin, Rosken & Toner (2009) warn that teachers' beliefs have been perceived as impediments to problem-solving based developments of the mathematics curriculum and of imaginative classroom teaching approaches. Ernest (1989a) upholds that adopting a problem solving approach to the teaching of mathematics depends on institutional reform but more essentially on individual teachers changing their entrenched philosophy to the learning and teaching of mathematics. He asserts that:

> It depends fundamentally on the teacher's system of beliefs, and in particular on the teacher's conception of the nature of mathematics and mental models of teaching and learning of mathematics. Teaching reforms cannot take place unless teachers' deeply held beliefs about mathematics and it teaching and learning change (p. 249).

Buzeika (1996) explored the relationship between beliefs and practices, as professed by three primary teachers implementing a new mathematics curriculum in New Zealand. The reform emphasised constructivist practices to incorporate problem posing. The participants expressed "difficulties in maintaining control over what was happening if children were left to explore an idea for themselves" and "confusion resulting from a document which they perceived as vague and lacking in direction" (p. 97). In her study of an urban American secondary school, Aguirre (2009) found that mathematics teachers' domain-specific beliefs did not entirely resonate with district-mandated progressive reforms aimed at increasing standards and accountability to learn mathematics. The main locus of attention focused around explicit algebraic beliefs. Aguirre found a perception by some participants that the learning of algebra was not necessary or practical for all pupils, raising significant content and equity issues. Charalambos & Phillippou (2010) investigated the concerns and efficacy beliefs of implementing a problem solving reform (five years after its introduction) of 151 primary teachers in Cyprus. In the study, participants highlighted a scarcity of information and training about planned educational changes and reported feeling more confident in teaching problem solving by employing previous methods with some harbouring a negative disposition towards the reform.

Another element inhibiting curriculum reform is teacher resistance to change. This may be fuelled by the opposition to restructure existing practices (Clarke, 1997) based on the conviction that reforms, as an alternative paradigm are implausible as they offer no obvious classroom improvements, are not compatible with established procedures and present no observable outcomes for stakeholders such as parents or senior management fixated on examination results. Experienced teachers can feel alienated when not empowered to participate in the reform conceptualisation but are required to take part in the implementation process, habitually at the expense of an increased workload (Hjelle, 2001).

Research undertaken by Doerr & Tinto (2000, p. 427) maintains that "teachers are asked frequently to change how they teach, but they seldom have meaningful data that encourages them to do so". Rather than radically transform their practice, teachers often tweak their approach to learning and teaching (Hughes, 2002) but this does not always match the intentions of the reform strategy. Fives & Buehl (2016) recommend that policy makers must attend to teachers' beliefs as part of any reform effort.

4.6.2 The Scottish context

Primary and secondary colleagues are reciprocally responsible for integrating the mechanism of problem solving and problem posing into the learning and teaching of mathematics. Simultaneously interwoven into the educational theoretical fabric is the challenge of facilitating learning from a constructivist perspective. The shift from employing a traditional (instrumentalist and Platonist) slant presents multiple pedagogical dilemmas for educators due to the transformational changes initiated by CfE. Due to its convolution, a problem solving approach demands that practitioners engage in agentic and proactive ways to prepare for its effective implementation.

However, little is known about individual teachers' beliefs and how they impact on professional practice or if they fully appreciate what is required or, more arguably, completely equipped to deliver to the implored standard. Priestley (2005, p. 36) warns that: "Teachers must clearly understand reform and have the pre-requisite skills to put it in place, if they are to enact it successfully". Paradoxically, Hayward, Priestley & Young (2004) state:

> ensuring that policies are coherent and grounded in research does not ensure their ready adoption in practice. Such an assumption implies a functionalist and social engineering conception of policy and research that ignores the heterogeneity of contexts within which policy has to take root, as well as the role of accumulated practitioner knowledge (p. 399).

Irrespectively, Donaldson (2011, p. 70) in his emphatic review of Scottish teacher education declares: "If we are to achieve the aspiration of teachers being leaders of educational improvement, they need to develop expertise in using research, inquiry and reflection as part of their daily skill set". Consequently, latest restructuring of national standards (GTCS, 2012, p. 8) prescribe that practitioners are expected to develop and apply their knowledge, skills and expertise through enquiry and sustained professional learning to "critically engage with a range of educational literature, research and policy to make meaningful links to inform and change practice". In spite of this ambitious doctrine, it is questionable if a suitable framework exists to allow Scottish teachers to successfully operationalise research literature to help execute this didactical requirement, forcing them to rely on their own, unexplored and possibly restricted cognitive past experiences (Ellis, 2010). In their ethnographic study of the beliefs of six Scottish primary and secondary teachers, Biesta, Priestley & Robinson (2015) delineates the existence of a narrow professional discourse about teaching and education. The researchers opine that the limiting nature of a weak set of orientations will prevent teachers from locating future beliefs within a wider intellectual dimension.

Tensions already exist between constructivist underpinnings of the curriculum and implicit transmissionist views of the teachers (Priestley & Minty, 2012). Priestley & Minty (2013, p. 50) argue that practitioners have different perceptions of the purposes and philosophy of the curriculum "which relate inherently to their prior experiences of the 5-14 Curriculum, the long tradition of subject specialism (in secondary schools), and to their own personal beliefs and values about education". It is inferred that the implementation of reform initiatives is compromised when teachers' beliefs are misaligned with the theoretical foundations of CfE.

In an autonomous environment with no prescriptive curriculum, the enactment of classroom practices is influenced and fashioned by conceptualisations of teachers' beliefs. Grouws (1996) contends if teachers executed a problem solving approach to mathematical learning, opposed to providing robotic "endless sets of exercises where each exercise has one answer and there is one set way of doing each exercise in the set" (p. 79), they would enrich pupils mathematical thinking. Moreover, he asserts that in such a terrain, "one would see lots of exploration of situations, hypothesis generation, problem posing, multiple solutions and solution methods, arguments followed by justifications and verifications" (p. 80). Though a common agreement exists that mathematics beliefs are personal philosophies and conceptions about the nature of mathematics and its learning and teaching (Thompson, 1992), such views encapsulate an array of perspectives which may be overtly influenced by the contextual nature of CfE.

4.7 Summary

Teachers' beliefs play an important role in the education landscape. They influence what is taught, how it is delivered and what is learned (Fives & Buehl, 2012). Teachers' beliefs are not held in isolation but are interrelated in complex ways known as a belief system (Rokeach, 1968; Green, 1971; Leatham, 2006). Mathematical beliefs are often classified as personal philosophies about the nature of mathematics, the learning of mathematics and the teaching of mathematics. The literature has highlighted inconsistences between teachers' espoused beliefs and enacted classroom practices (e.g. Thompson, 1984; Raymond, 1997; Skott, 2001). If we are to bridge the disjunction between educational policy, research and practice, it is essential that teachers' beliefs are explored and valued.

The next chapter presents and justifies the research methods and methodology selected to answer the research questions in this study.

CHAPTER FIVE

Research Methodology and Methods

The purpose of this chapter is to present the philosophical assumptions underpinning this research along with justifications and rationalisations for the methodology adopted. The first part outlines the objective of the study and the research questions. This is followed by an examination of the research design, consideration of ontology, epistemology and different research approaches. Next, the chapter reports on my own research positionality and assessment of the interdisciplinarity of the study. Succeeding this, I comment on the operationalisation of the research questions. This includes an explanation of the search strategy used in the systematic literature review of mathematical problem posing and an overview of the design of the instruments employed to measure the belief construct. Integrated is a narrative on the development of the questionnaire and interview schedule. The penultimate section provides analysis of the pilot and the role of reliability and validity in mixed methods research. Finally, ethical considerations are discussed.

5.1 Research questions

During the early planning stages of this inquiry process, I composed several general questions grounded on my pedagogical experience, knowledge of localised observations and interpretation of the mathematics guidelines within CfE. These preliminary queries generated a stimulus for personal reflection and developing more specific questions. On completion of the literature review (i.e. chapter two, three and four respectively), the questions were refined to capture my engagement with published works in the field that contributed to a conceptual framework. In particular, it explicitly underlined a requirement for me to coalesce with primary and secondary mathematics teachers in some form of cross-sector empirical study of beliefs.

The next phase involved framing the questions to take cognisance of data collection and analysis implications. In other words, it sensitised me to begin theorising about my contribution to existing knowledge, while simultaneously emphasising the practical nature of research methods and methodologies. Flick (2014, p. 146) notes that “reflecting on and reformulating the research questions are central points of reference for assessing the appropriateness of the decisions you take at several points”. Finally, the research questions emerged after reviewing theoretical advice on conceptualisation from several scholars (e.g. Bryman, 2012; Creswell, 2013a; Maxwell, 2013; Punch, 2014). Collective features included the construction of clearly stated questions that are researchable within the given time frame and location, including consideration of constraints due to my full-time teaching role.

The reflective and interrogative processes shaped and directed the development of the following three research questions:

Question 1
To what extent should mathematical problem posing be embedded within the framework of Curriculum for Excellence?

Question 2
Are there any differences in the mathematical beliefs of Scottish primary and secondary mathematics teachers?

Question 3
What factors impact on the mathematical beliefs of Scottish primary and secondary mathematics teachers?

From these questions and a further period of reflection, I derived five sub questions:

Q1(a) What would be the value for learners of emphasising mathematical problem posing in the curriculum?

Q1(b) What would be the implications for teachers’ professional practice of implementing mathematical problem posing in their pedagogy?

Q2(a) What is the importance of learning theory of shaping teachers’ mathematical beliefs?

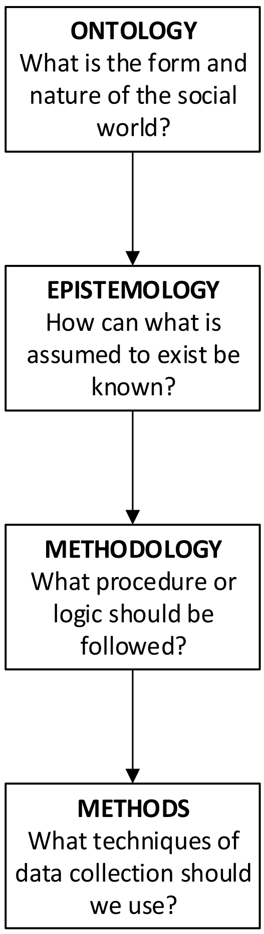
Q2(b) What are teachers’ beliefs about mathematical problem solving?

Q3(a) What are the practical characteristics that support or constrain the development of teachers’ beliefs?

5.2 Educational research

Cohen, Manion & Morrison (2011, p. 1) skilfully encapsulate many of the associated perspectives and complexities when they offer the following definition as the "systematic and scholarly application of the principles of a science of behaviour to the problems of teaching and learning within education and the clarification of issues having a direct and indirect bearing on those concepts". On further examination, it is reasonable to assume that educational research is undertaken within some paradigm or interpretive framework (Guba, 1990). For example, Denzin & Lincoln (2011, p. 13) maintain that interpretive research is "guided by a set of beliefs and feelings about the world and how it should be understood and studied. Some beliefs may be taken for granted, invisible, or only assumed, whereas others are highly problematic and controversial". Furthermore, Waring (2012) attests that all researchers need to understand that their research is encased by a series of related suppositions framed around four key questions, represented in Figure 5.1.

Figure 5.1 The relationship between ontology, epistemology, methodology and methods (Adapted from Waring, 2012)



Waring (2012) identifies a fundamental set of philosophical assumptions that underpin the research activity and describes their interrelationship and implications, in order that a researcher can appreciate the process and to locate their position. Crotty (1998) argues that a researcher can select any philosophical assumptions as a starting point. However, Grix (2010) contends that the need to understand the philosophical underpinnings that inform their choice of research questions, methodology, methods and intentions must begin with identification of ontological assumptions. He counsels:

> setting out clearly the relationship between what a researcher thinks can be researched (her ontological position) liking it to what we can know about it (her epistemological position) and how to go about acquiring it (her methodological approach), you can begin to comprehend the impact your ontological position can have on what and how you decide to study (p. 67).

I will now briefly attempt to make more sense of the assumptions of ontology, epistemology, methodology and methods as they apply to this research study.

5.2.1 Ontology

Ontology is the starting point of all research and enjoys a rich history in philosophy. It is governed by the form and nature of reality or a phenomenon. According to Guba & Lincoln (1994), ontology is centred on what we know about something. Likewise, Crotty (1998, p. 10) illustrates that it "is the study of being" and "is concerned with 'what is', with the nature of existence, with the structure of reality as such". Taken together, ontology encompasses our assumptions about how the world is made up and the nature of objects.

Therefore, it is only logical that different claims and assumptions exist, which in turn lead to the presence of a wide variety of human ontological perspectives. Consequently, this factor can produce multiple research results. Grix (2010, citing Lewis, 2002) points out that explicit reflection about ontological issues can help clarify the precise character of theoretical positions and arguments. For example, from a mathematical perspective, a simple ontological option is to postulate the existence of abstract mathematical objects like numbers or sets. In considering my own ontological position, I refer back to the three views of mathematics as characterised by Ernest (1989a) in chapter four. On this basis, I hold a problem solving perspective since I believe mathematics is aligned with a continually expanding field of human inquiry i.e. mathematics is not a finished product and its results remain open for revision.

5.2.2 Epistemology

Epistemology is an important branch of philosophy that investigates the origin, methods and limits of human knowledge and is concerned with the relationship between the researcher and the communities who are being examined. It deliberates over what constitutes valid information and how we can obtain it. Cohen, Manion & Morrison (2011) claim that useful knowledge is established by the societal and positional authority of the proponents of that knowledge. As Crotty (1998, p. 3) concludes, epistemology is "the theory of knowledge embedded in the theoretical perspective and thereby in the methodology" and additionally in shortened form, as "how we know what we know" (p. 8); this informs the research in different ways such as objectivism, subjectivism, etc.

All researchers are driven by particular epistemologies and all research is impregnated with epistemological beliefs. My own epistemological perspective, when faced with a proliferation of learning theories regarding the teaching of mathematics, is focused on what ideas can be practically adapted to my professional practice. What appeals to me is the position of Dewey and his support for a naturalistic approach and rejection of the dualistic epistemology and metaphysics of modern philosophy. In this view, inquiry does not consist of a passive observation of the world and ensuing inferences about reality, but rather as an active process which initiates human engagement such as testing hypotheses.

5.2.3 Methodology

The research methodology is the philosophy or general principles which guide the logic of scientific enquiry. It explains how we research complex and multiple realities (Cohen, Manion & Morrison, 2011). A methodology as a set of theories shows how research questions are articulated with questions asked in the field (Clough & Nutbrown, 2007). According to Crotty (1998, p. 3), methodology is defined as the "strategy, plan of action, process or design lying behind the choice and use of particular methods and linking the choice and the use of methods to the desired outcomes". In contrast, Cohen, Manion & Morrison (2011) argue that no single solution for planning research exists but that any design must be governed by the notion of 'fitness for purpose'. Moreover, as there are various ways of interpreting the world, a researcher must ensure that the objectives of the research drive the methodology and related design.

5.2.4 Methods

Research methods are specific instruments employed in collecting data. Denscombe (2014, p. 3) suggests that: “Methods are the equivalent of a microscope when used by a scientist, a thermometer when used by a medic, or a telescope when used by an astronomer. They do a job”. Although, the choice of methods will be influenced by ontological and epistemological assumptions and the range of research questions (Grix, 2010), caution is required to ensure that an appropriate instrument is selected. Recognition of the merits and limitations of each collecting device will assist the researcher, as will a foresight to change method if unsuccessful.

5.3 Paradigms in educational research

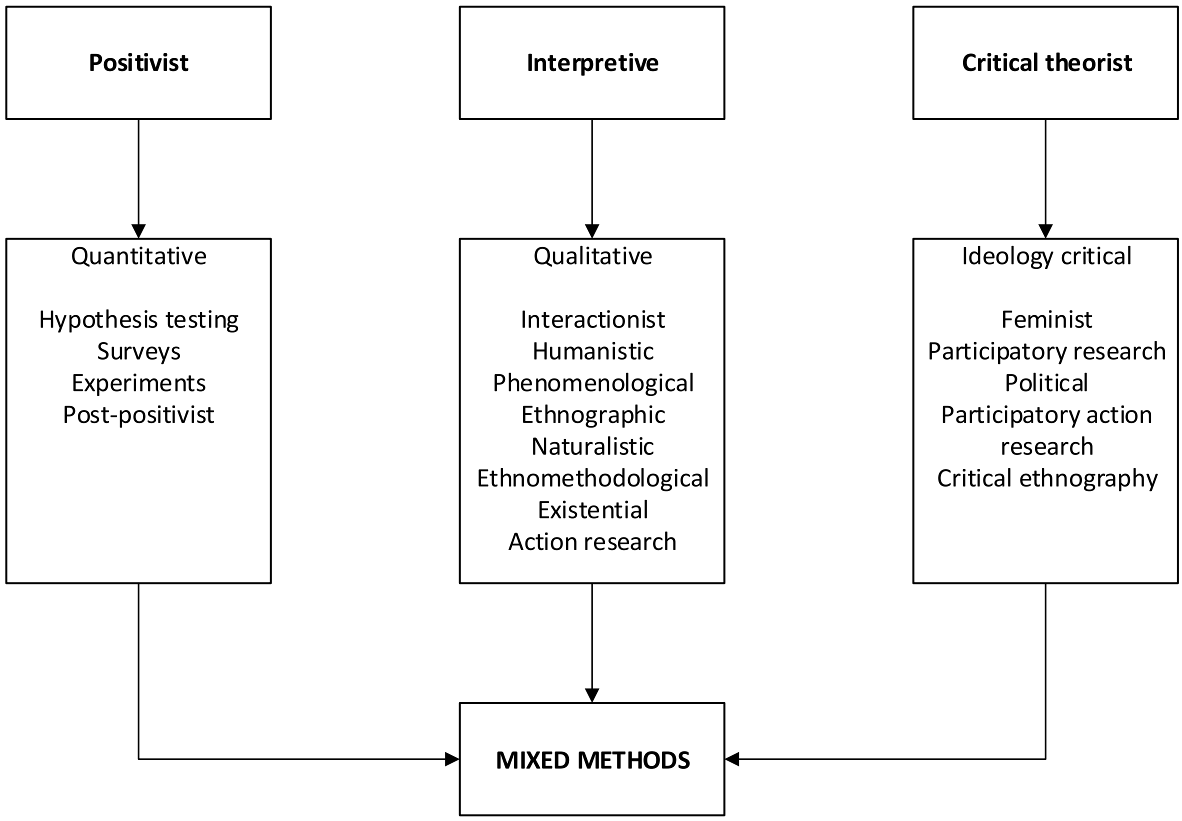
The concept of a paradigm is a worldview or a set of interrelated assumptions and beliefs about how objects work which is shared by members of a given community. Among the historiography of paradigms, perhaps the most well-known work is that of Kuhn (1996). He explained how paradigms represent ways of viewing the world and identified that “shared paradigms result in commitment to the same rules and standards for scientific practice” (p. 11). Conversely, Guba (1990) argues that a lack of consensus allows a paradigm to be reshaped as our perception of its many implications improves. Subsequent radical changes to our understanding are known as a ‘paradigm shift’ (Kuhn, 1996). Reinforcing the foundation of any paradigm is that an individual will contribute previous life experiences and knowledge to every research context.

Research paradigms represent a critical element in educational research as they strongly influence both the strategy and the method researchers construct and interpret the meaning of reality. It is important for new researchers to consider the philosophical underpinning and orientation that defines different perspectives in this regard. Therefore, advocates of any given paradigm should be able to summarise their beliefs relative to their responses to those ontological, epistemological, methodological and methods questions identified (e.g. Guba & Lincoln, 1994; Waring, 2012). Prior to finalising the research design for this study, my previous training and experience had the effect of hermetically sealing my choice of paradigm. Having recognised this at an early stage, I attempted to remain neutral throughout the process in order to justify all of my strategic decisions.

Executing a research project requires planning consideration as there are two main techniques on how to proceed. It is possible to begin with a paradigm, articulate it and develop research questions and methods from it. Alternatively, as in the case of this study, start with research questions that need answers and then select methods for solving them. Within education, this pragmatic approach often arises from practical problems and professional issues within the workplace (Punch, 2014).

Generally, there are three broad competing paradigms that guide the research process across a continuum from positivist to interpretivist positions, though, variations occur in the labelling of what sits between these two extreme positions. For example, Grix (2010) asserts that in the philosophy of the social and human sciences, there exist a positivist, post-positivist and interpretivist position. In the remainder of this section, I am directed by Cohen, Manion & Morrison (2011) who outline three major paradigms within educational research (Figure 5.2) in which two are of interest to me.

Figure 5.2 Paradigms in educational research (Adapted from Cohen, Manion & Morrison, 2011)



5.3.1 Positivism

Positivism is an epistemological position that endorses the application of the methods of the natural sciences to the study of social reality and beyond (Bryman, 2012). Developed in the early ninthteenth-century by French philosopher, Auguste Comte, it is represented as an acceptance of the natural sciences as the paradigm of human knowledge. Similarly, Punch (2014, p. 17) contends that positivism is “the belief that objective accounts of the world can be given, and that the function of science is to develop descriptions and explanations in the form of universal laws – that is, to develop nomothetic knowledge”.

Since positivism is heavily influenced by the role of theory in the natural world, I feel it apt to declare my understanding of this role. I believe that natural sciences provide reliable explanations or interpretations of phenomena that evolve after a hypothesis has accumulated sufficient evidence through testing; in such theories involving variables, the functioning relationships can be replicated independently. Entrenched within this premise is that scientific theories seek precision and objectivity through structured measurable observations. Positivism advocates that methodological procedures of natural science can be directly applied to the study of the complexities and interactions of human behaviour, the employment of normative questions such as value judgements are rejected. One position on this is that since there is a fundamental difference between facts and values, we cannot use empirical evidence in the construction of value judgements. Another position argues that values have a direct impact on research and that we should discontinue the fallacious dichotomy between facts and values (Lincoln & Guba, 1985).

Viewed from a classroom perspective, positivism is centred on the transmission of knowledge as an exchange of statements. Freire (1970) refers to this as the “banking theory” where a teacher deposits knowledge in the form of facts into empty receptacles, which are the students. Such a pedagogical approach only serves to devalue the educational experience of the learner since it rejects the intangible nature associated with human thoughts and feelings such as intuition and emotion. Kincheloe (2012, p. 12) opines that “human-created knowledge is conceptualized as a physical substance handed from one individual to another via the process of teaching”. Within the context of educational research, positivism presents many challenges; for instance, it is not possible to directly measure teachers’ beliefs.

Bryman (2012) describes positivism as compromising of five principles; firstly, only phenomena confirmed by the senses can be accepted as knowledge. Secondly, the testing of hypotheses can allow explanations of laws to be assessed. Thirdly, knowledge is arrived at through the gathering of facts that provide the basis of realities. Fourthly, science is conducted in a manner that is value free. Finally, there is a clear distinction between scientific statements and normative statements and a belief that the former is the true domain of the scientist.

5.3.2 Interpretivism

Interpretivism is a term assigned to a contrasting epistemology to positivism. Proponents of this paradigm argue that individual behaviour can only be identified by the researcher sharing their interpretations of the subjective world around them. Crotty (1998, p. 67) states that interpretivism searches for “culturally derived and historically situated interpretations of the social life-world”. Cohen, Manion & Morrison (2011, p. 15) attest that “the social world can only be understood from the standpoint of the individuals who are part of the ongoing action being investigated and that their model of a person is an autonomous one, not the plastic version favoured by positivist researches”. In other words, interpretivism maintains that each individual has their own interpretation of reality which cannot be measured and generalised. In contrast to positivism, which seeks objectivity, this position considers the social and natural sciences as being distinct from one another and thus the social world requires to be investigated from within and with different methods from those applied in the study of the natural sciences (Grix, 2010). Interpretivists believe in subjectivity where researchers are inextricably part of the social phenomenon being examined. The variable and personal nature of social constructions suggests that data can only be elicited through interaction between researcher and participant (Waring, 2012).

5.4 Research methods considered

Within this section, I will briefly examine approaches that I contemplated using to operationalise my research questions for this study.

5.4.1. Systematic literature review

According to Dixon-Woods (2016, p. 380), a systematic literature review is “a scientific process governed by a set of explicit and demanding rules oriented towards demonstrating comprehensiveness, immunity from bias, and transparency and accountability of technique and execution”. They are designed to locate, appraise and synthesis the best available support relating to a specific research question in order to provide informative and evidence-based answers (Boland, Cherry, & Dickson, 2014) and are characterised by being objective, efficient and replicable. Fundamentally, as a methodological approach, a systematic review seeks to promote impartiality in order to allow readers to evaluate the researchers' assumptions, procedures, evidence and suppositions, rather than accepting the researchers' findings on faith. Importantly, in comparison to traditional reviews, they are rigorous, follow prescribed protocols and employ overt methods to combine information from across different studies including carefully considered conclusions and recommendations (Cohen, Manion & Morrison, 2011; Punch, 2014).

5.4.2 Quantitative research

Quantitative research methods were originally developed in the sciences to study natural phenomena. The quantitative paradigm is more closely associated with positivism and distinguished by empirical research which purposely collects numerical data for mathematical analysis. Though, interpretative approaches, however, are equally applicable to the analysis of quantitative data as they can offer triangulation of research results from multiple perspectives, integration of measurements and modelling into a more holistic process of discovery and the ability to think reflexively about the manner in which data have come into existence (Babones, 2016). Even so, generalisability remains an issue with all aspects of interpretative approaches. Creswell (2009, p. 4) observes that quantitative research "is a means for testing objective theories by examining relationships among variables. These variables, in turn, can be measured, typically on instruments, so that numbered data can be analysed using statistical procedures". Non-quantitative data (e.g. teachers' beliefs) may be converted into quantitative form by using measurement instruments such as Likert scales. Another intention of quantitative research is to measure and analyse causal relationships between variables within an unrestricted framework (Denzin & Lincoln, 1994).

The ontological position is that there is only one truth, an objective reality that exists independent of human perception. Interestingly, Bryman (2012, p. 160) argues that: "The very fact that it has a distinctive epistemological and ontological position suggest that there is a good deal more to it than the mere presence of numbers". Epistemologically, it is assumed that the researcher is capable of studying a phenomenon without influencing it or being influenced by it and that inquiry takes place as through a one-way mirror (Guba & Lincoln, 1994). Nevertheless, it is misleading to suppose that quantitative research is entirely liberated from bias or error. Researchers have an element of discretion and choice throughout the process. For example, the development of 'standard questions' may produce data which reflects the view of the researcher opposed to the participating subject. Furthermore, from a statistical perspective, errors in the selection of procedures for determining statistical significance can result in erroneous findings.

5.4.3 Qualitative research

As a suite of interpretative activities, qualitative research is particularly useful in the investigation of educational settings and processes. Based on my professional experience, teachers operate within a multifaceted and interconnected social dimension that breeds the multiple interpretations and perspectives that qualitative research can help us to explore. Due to the complex overlapping family of terms, qualitative research is not easy to describe. Over twenty years ago, Denzin & Lincoln (1994, p. ix) reported that "the field of qualitative research is defined primarily by a series of essential tensions, contradictions and hesitations. These tensions work back and forth among competing definitions and conceptions of the field". More recently, Denzin & Lincoln (2011, p. 6) maintain that "qualitative research is difficult to define clearly" and argue: "It has no theory or paradigm that is distinctly its own ... Nor does qualitative research have a distinct set of methods or practices that are entirely its own". Nevertheless, Denzin & Lincoln (2011) present the following description:

> Qualitative research consists of interpretive, material practices that make the world visible. These practices transform the world. They turn the world into a series of representations, including field notes, interviews, conversations, photographs, recordings, and memos to the self. At this level, qualitative research involves an interpretive, naturalistic approach to the world. This means that qualitative researchers study things in their natural settings, attempting to make sense of or interpret phenomena in terms of the meanings people bring to them (p. 3).

From an ontological perspective, qualitative researchers are concerned with the changing nature of reality fashioned through human experiences, in which the researcher adopts an interactive, flexible and inseparable connection with the phenomena being investigated. Multiple methodologies exist to collect data within the field of qualitative research. Reflecting on the term 'reality' has made me think about the notion of whether reality changes according to human experiences. For example, it may be postulated that reality for someone living five hundred years ago is different to our reality of today. Likewise, it may be claimed the individualistic nature of reality, reflected by different life experiences has shaped each human perception of reality. I believe that our physical reality has not changed, but that multiple realities exist in the minds of people which have been manifested through idiosyncratic perspectives.

5.4.4 Mixed methods research

Over the years, mixed methods research has been conceptualised in various different ways. Based on their meticulous examination of researchers in the field, Johnson, Onwuegbuzie & Turner (2007) offer the following composite definition:

> Mixed methods research is the type of research in which a researcher or team of researchers combines elements of qualitative and quantitative research approaches (e.g., use of qualitative and quantitative viewpoints, data collection, analysis, inference techniques) for the broad purposes of breadth and depth of understanding and corroboration (p. 123).

Fundamentally, however, it is inadequate to suggest that mixed methods is simply the gathering of both quantitative and qualitative data. For example, Clark et al. (2008, p. 364) describe mixed methods research "as a design for collecting, analysing, and mixing both quantitative and qualitative data in a study in order to understand a research problem". Whilst this definition contains important qualities of mixed methods research, such as the integration of the two data sources, it does not make explicit reference to framing of the design within a philosophy or theory.

In searching for a definition that encapsulates the entire spectrum of basic features, I draw upon the robust contribution from Creswell & Clark (2007), which states that:

> Mixed methods research is a research design with philosophical assumptions as well as methods of enquiry. As a methodology, it involves philosophical assumptions that guide the direction of the collection and analysis and the mixture of qualitative and quantitative approaches in many phases of the research process. As a method, it focuses on collecting, analyzing, and mixing both quantitative and qualitative data in a single study or series of studies. Its central premise is that the use of quantitative and qualitative approaches, in combination, provides a better understanding of research problems than either approach alone (p. 5).

However, more recently, Creswell & Clark (2011) argue that a definition for mixed methods should incorporate many diverse viewpoints. The authors itemise several core characteristics to illustrate their perspective and whilst they compose a valid reason for the inclusion of a research design orientation, it is debatable if such a position is absolutely necessary given their well cited original version.

Naturally, a dilemma arises during the integration of numerical and text data; Researchers need to decide on the level of interaction between quantitative and qualitative strands in a study. Johnson, Onwuegbuzie & Turner (2007, p. 124) present contrasting approaches that incorporate overlapping types of research. Firstly,

> Qualitative dominant mixed methods research is the type of mixed research in which one relies on a qualitative, constructivist-poststructuralist-critical view of the research process, while concurrently recognizing that the addition of quantitative data and approaches are likely to benefit most research projects.

And secondly,

> Quantitative dominant mixed methods research is the type of mixed research in which one relies on a quantitative, postpositivist view of the research process, while concurrently recognizing that the addition of qualitative data and approaches are likely to benefit most research projects.

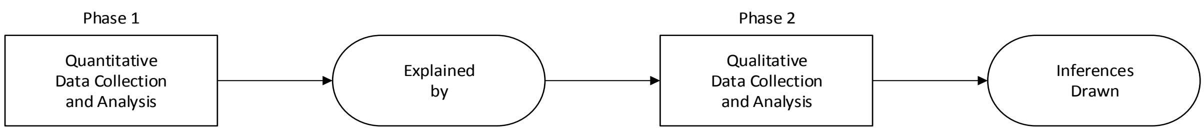
Other key decisions in choosing mixed methods design includes the timing of the strands and procedures for mixing the strands (Creswell & Clark, 2011). Timing can be classified by three different systems as follows; Firstly, concurrent timing occurs when both quantitative and qualitative stands are implemented during a single phase of the study. Secondly, sequential timing transpires in two distinct phases, with the collection and analysis of one strand occurring after the collection and analysis of the other. Finally, multiphase combination timing ensues when several phases of sequential and or concurrent timing over a programme of study.

Nevertheless, within the literature it is debatable if the rational for promoting mixed methods as a third paradigm, has assimilated sufficient merit, as underlying assumptions of the quantitative and qualitative paradigms result in differences which extend beyond philosophical and methodological debates. Tashakkori & Teddlie (2010) draw our attention to a variety of conceptual stances including the notion that quantitative and qualitative methods should not be mixed due to the incompatibility of the paradigms that underlie their methods. Conversely, both quantitative and qualitative methods may be used appropriately with any research paradigm (Guba & Lincoln, 1994). Notably, the logic for employing mixed methods is justified when the use of quantitative research or qualitative research alone is insufficient for gaining an understanding of the problem.

Notwithstanding the above positions, proponents of the tenets of mixed methods research hail it as an important and influential approach that is driven by pragmatism and directed by philosophical assumptions (e.g. Johnson, Onwuegbuzie & Turner, 2007; Greene, 2008; Creswell, 2009; Cohen, Manion & Morrison, 2011; Creswell & Clark, 2011). Whilst gaining in popularity, Bryman (2012) points out two key factors in promoting this development. First, he highlights a growing awareness of the consideration of research methods as data collection techniques or analysis which are not encumbered by epistemological and ontological 'baggage'. Second, he alludes to a 'softening' in the attachment towards quantitative research among feminist researchers, who had previously been highly resistant to its deployment. Regarding this latter factor, I suspect that Bryman is referring to the methodological paradigm wars of the 1970s where feminist researchers clearly positioned themselves firmly within the interpretative model.

Creswell & Clark (2011) outline six mixed methods designs, with contrasting worldviews, that provide an effective research framework. One such enterprise is an explanatory sequential design (Figure 5.3) which I particularly value because of my mathematical background and newness to the field of mixed methods research. The intention of this design is to begin with a quantitative strand followed by a qualitative stand to help explain the quantitative results. The strength of this strategy lies in the fact that the two phases build upon each other so that they are distinct and easily recognised stages. Although various challenges exist such as sampling considerations for phase two, this design offers many advantages. One such benefit is that the final report can be written with a quantitative component followed by a qualitative section, making it straightforward to write and providing a clear delineation for readers (Creswell & Clark, 2011). Thus far, a number of educational studies have employed this methodological design (e.g. Ivankova, Creswell & Stick, 2006; Palak & Walls, 2009; Lamb, 2011; Hung, 2012; Jetty, 2014).

Figure 5.3 Explanatory sequential design (Adapted from Creswell & Clark, 2011)



5.5 Researcher positioning and interdisciplinarity

Throughout my professional life as a teacher, I have always been an enthusiastic proponent of collaboration and teamwork in order to share insights and patterns of good practice. It has motivated me to look beyond subject domain boundaries to search for multiple perspectives. For example, I have applied an interdisciplinary approach to cross-circular projects involving colleagues from art and humanities subjects such as music, history and geography. With an open and curious mind, I approached this study with a willingness to develop and modify my own viewpoints so as to enhance personal scholarship. In particular, I have been cognisant of the requirement to guard against discipline entrapment (Grix, 2010).

Moreover, I have reflected on my beliefs that have strongly influenced this research journey thus far. During previous undergraduate and postgraduate study, I employed instruments to measure data which suggest that I possibly hold a deterministic or reductionist opinion associated with post positivism. Identifying and assessing causes that influence outcomes is of interest to me, as does the testing of a theory or a hypothesis. On the other hand, I have generated data using a focus group which may perhaps reflect a constructivist worldview of understanding several meanings; I believe that specific contexts in which practitioners work can help to comprehend the historical and cultural settings of participants. Nevertheless, both of these contrasting theoretical perspectives would not accurately represent my genuine philosophical orientation as I hold different viewpoints and favour a research approach that chooses methods of data collection that are fit for particular purposes. In education, problems require solutions and questions need answers. I therefore position myself with embracing a pragmatic view of the world and how knowledge is formed. Such a justification has been reinforced by copious classroom experiences interacting with theoretical and practical methods to learning and teaching.

At the same time, I have sought to actively acknowledge the influence of reflexivity and how this has impacted on my ontological and epistemological assumptions. Reflexivity recognises that researchers shape the research process and should seek to disclose their positionality on all matters that might affect the phenomena under study (Tashakkori & Teddlie, 2010; Cohen, Manion & Morrison, 2011). Foote & Bartell (2011, p. 46) argue that: “The positionality that researchers bring to their work, and the personal experiences through which positionality is shaped, may influence what researchers may bring to research encounters, their choice of processes, and their interpretation of outcomes”.

As a strategy for maintaining reflexivity, I kept a research journal which I used to convey my personal experiences throughout all phases of this study. I define this strategy as a mechanism that has fuelled a perpetual motion of reflection and critical self-evaluation. Significantly, it has propelled me back and forth along a continuum of objectivism and subjectivism. I believe that a researcher can strive to be objective but that an individual cannot completely detach themselves during the research process since they are part of the social world under investigation. Cohen, Manion & Morrison (2011, p. 225) refer to researchers in this basis as “actors” within “an already interpreted world”, of which undermines “the notion of objective reality”. Such an observation suggests that researchers should disclose their personal influences opposed to solely attempting to eliminate their positional affect. Assuming this to be universally true, I have viewed this research through the lens of an experienced and flexible Scottish secondary mathematics teacher who entered the initial process anchored with an empirical scientific philosophical position coupled with an intimate awareness of localised assessment concerns. Furthermore, I have come to realise that pragmatism is not driven by any particular dialectal stance and that research methods can be widespread and heterogeneous. Similarly, I fully recognise and value that educational research is underpinned by a socially constructed multidimensional subjective world, which is contrary to my own training and background.

I align my philosophical position on pragmatism with a mixed methods approach. Greene (2008) advocates that choosing mixed methods, as a way of answering research questions, is a natural paradigm for holding such a position. Likewise, interdisciplinarity is achieved as mixed methods integrate two different specialisms. Crossing disciplinary boundaries has helped me understand the wider impact of my research and enabled me to appreciate that interdisciplinarity can be utilised to address what Bridle et al. (2013) refer to as ‘complex problems’ that cannot be solved using a solitary disciplinary perspective.

Also, the interdisciplinarity of this study has impacted upon the selection of literatures that I have consulted and interrogated in several ways. Firstly, I have questioned my traditionally held assumptions and have attempted to think ‘outside the box’ by reviewing texts encompassing a proliferation of data collection methods. Secondly, I have progressed beyond the limitations of using theoretically driven empirical research to consider the usefulness of assessing articles underlined by social theories. Thirdly, I have evaluated writings concerned with symbolising ‘explanations’ opposed to emphasising measurements or causations. An illustration of this involves discourses that are specifically related to context in order to illuminate the manifestation of human variables on the situatedness of social experiences.

In short, while harnessing the creative tensions engendered by this research has developed my ability to be conversant with a wider repertoire of techniques, forging synergies across quantitative and qualitative disciplines stimulated a significant challenge for me as a new researcher. For instance, it required a considerable investment of time that I did not anticipate. Perhaps, the only ambiguity to resonate during this encounter is whether interdisciplinarity has inadvertently weakened my core discipline dependency.

5.6 Previous studies measuring teachers’ beliefs

Having justified my decision to employ a mixed methods research approach in this study, the next choice involved selecting appropriate instruments to collect the data. Various methods have been proposed to measure the construct of teachers’ beliefs. Supplementing the use of questionnaires and belief inventories, a researcher should consider open-ended interviews, responses to dilemmas and vignettes and observations including giving thought to metaphors, biographies and narratives (Pajares, 1992).

Consequently, in order to augment my knowledge, I reviewed a number of previous studies that reported different strategies of assessing teachers’ beliefs. Within the scope of my brief analysis, I searched for peer-reviewed empirically based articles of practising and student teachers published during the last fifteen years. To my surprise, I discovered that the dimension of teachers’ beliefs are purported to be measured in at least twenty different ways. For example, Ozgun-Koca & Sen (2006) used concept-mapping, journal writing and interviews to elicit the perspectives about effective teaching of 51 Turkish prospective mathematics and physics teachers. In another study, Khan & Begum (2012) employed a portfolio and interview approach to evidence the changes in beliefs of six Pakistani current practitioners undertaking professional development. Whilst this review of empirical studies generated practical information regarding methods of data collection, two critical interrelated features emerged as a result. First, it underlined the challenge of defining the belief construct and the methodological issue of how it can be operationalised successfully. Second, it revealed contrasting standards of research design within the literature.

Moreover, it highlighted an essential requirement for me to consider how to ensure the trustworthiness of any interpretations being made acceptable as worth. I reflected on what criteria are acceptable to evaluate the quality of any research. Eager to avoid any artificial categories of judgment or unrealistic frameworks, I referred to the influential and much cited work of Lincoln & Guba (1985). Guided by the conceptualisation of validity and reliability where appropriate, I prepared an overview of previous methods used to measure teachers' beliefs (Appendix A). This presented an excellent opportunity to compare and recognise various procedures utilised in the field. More significantly, it allowed me to identify studies which have demonstrated methodological and interpretative rigour, thus providing much needed design clarification as a new researcher.

To date, observations (e.g. Stipek et al., 2001; Beswick, 2007; Forrester, 2008) and case studies (e.g. Raymond, 1997; Cooney, Shealy & Arvold, 1998; Skott, 2001; Cross, 2009; Beswick, 2012) have been widely adopted as they allow a deeper insight into the complexities of teachers' mathematical beliefs within a contextualised setting. A distinct advantage of applying either of these methods is that they can be employed to corroborate or contrast teachers' espoused beliefs. However, I rejected both approaches due to time constraints.

In the remainder of this section, I briefly examine the suitability of one well-established quantitative and qualitative method that has been used extensively in the investigation of teachers' beliefs i.e. questionnaires and interviews respectively.

5.6.1 Questionnaires

Research on teachers' mathematical beliefs is dominated by questionnaires (e.g. Anderson, White & Sullivan, 2004; Beswick, 2004; Barkatas & Malone, 2005; Barlow & Gates, 2006; Perry, Wong & Howard, 2006; Yu, 2008; Wilkins, 2008; Chen et al., 2011; Roscoe &

Sriraman, 2011). The obvious advantage of using questionnaires is that they can be easily mobilised with large numbers of participants in multiple locations and can evaluate more than one construct. Many factors need to be addressed prior to successful operationalisation such as the time taken to complete the instrument, level of sensitivity of the questions, or the possible invasion of privacy (Cohen, Manion & Morrison, 2011). Crucially, in order for the resultant data to describe accurately what it is intended to describe, the design must combine the components of sampling, constructing questions and data collection (Fowler, 2014).

Various rating scales can be used to accommodate non-dichotomous questions such as the Likert scale. The Likert scale is uni-dimensional and normally provides an odd number of choice options. An illustration of a study employing an even number of response options is that by Stipek et al. (2001), which used a six-point Likert scale to measure teachers' mathematical beliefs and practices. It may be posited that the space between each response cannot possibly be equidistant, therefore failing to capture a true dimension of a respondents selection. Poorly articulated questions can attempt to force participants to condense their complex set of beliefs into pre-determined compartmentalised researcher statements that may or may not be mutually exclusive (Fang, 1996). Philipp (2007) probes the effectiveness of using Likert scales for measuring beliefs and notes that validity is reduced when questions provide little or no context. This view is supported by Speer (2005) who contends that beliefs are situated in contexts and interrogates the legitimacy of collecting data on beliefs separately from practices. However, Charalambous & Philippou (2010) suggest the ability to capture rich and accurate representations of teachers' beliefs using Likert scales is inherently restricted by design. In an attempt to address some of the limitations, Ambrose et al. (2004) designed a web based survey to assess prospective primary teachers' mathematical beliefs. The researchers claim that a major strength of their instrument is that video clips and learning episodes created contexts to which participants responded in their own words rather than choose from a set of pre-determined options. Denscombe (2014) offers a summary of the key criteria that researchers need to be aware of when designing and evaluating a questionnaire (Table 5.1).

Table 5.1 Criteria for the design of a questionnaire (Adapted from Denscombe, 2014)

Criteria	Research issue	Factors to be considered
Response rate Will a sufficient proportion of questionnaires be returned to avoid bias in the information collected?	Reliability	• Good design of questionnaires • Motivation of respondents • Follow-ups used
Full information Does the questionnaire gather information on all crucial areas necessary for the research?	Completeness	• Inclusion of all vital topics • All sections of the questionnaire answered
Accurate information Is the information free from errors arising (intentionally or accidently) from the nature of the questions being asked?	Validity	• Appropriate questions for the topic/concepts being investigated • Clear, precise and unambiguous questions • Honest answers
Ethical stance Has due consideration been given to the rights of those supplying the information?	Professional integrity	• Voluntary co-operation • Information given about the way data will be used • Respondents' identities not disclosed • Protection of sensitive data

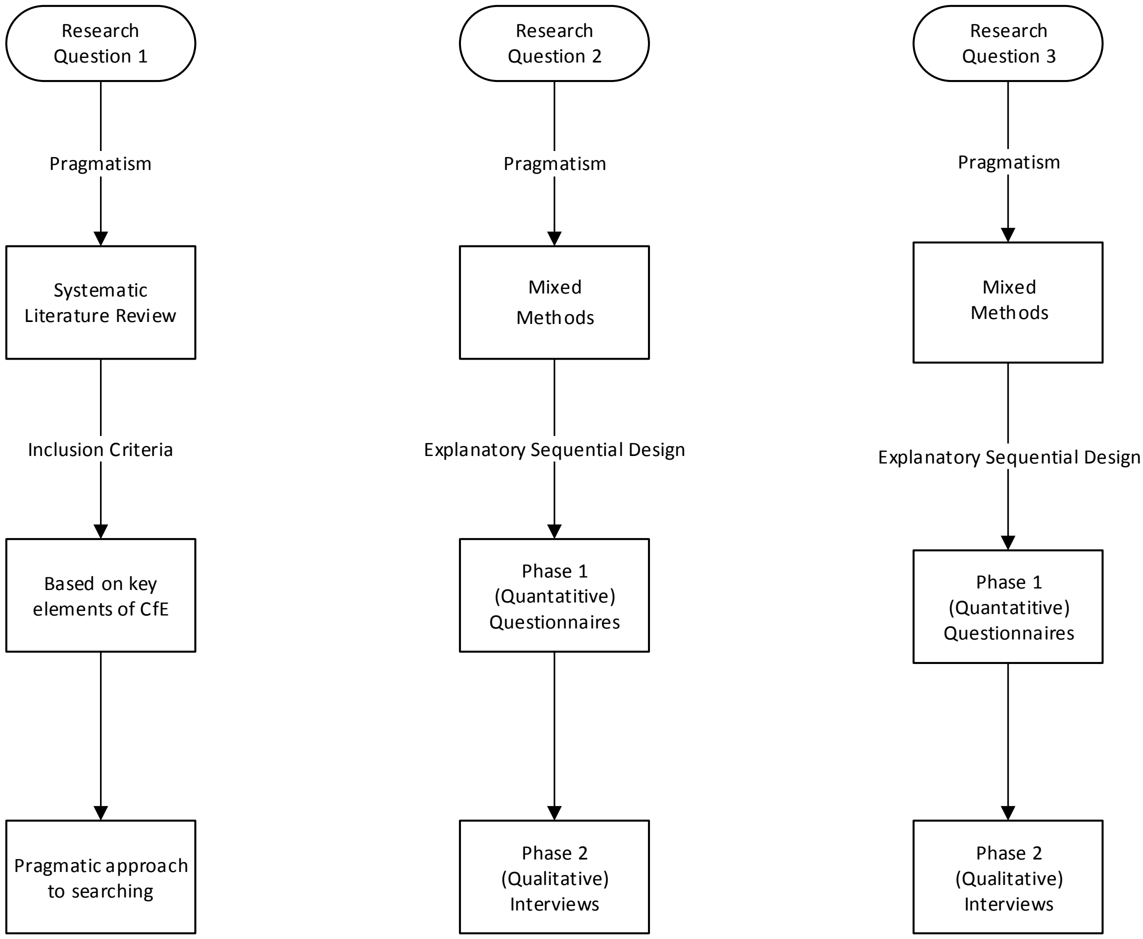
5.6.2 Interviews

An interview is a very powerful mechanism for understanding human interactions and has been widely employed within the research of teachers' mathematical beliefs (e.g. Aguirre & Speer, 2000; Anderson, Sullivan & White, 2004; Perry, Wong & Howard, 2006; Cross, 2009; Furinghetti & Morselli, 2011; Beswick, 2012; Skott, 2013; Xenofontos & Andrews, 2014). Interviews can empower participants to freely discuss their interpretations of the world in which they live, and to express how they understand situations from their own perspective (Cohen, Manion & Morrison, 2011). The selection of a particular interview type is based on the purposes of the research and nature of the questions. For example, Mosvold & Fauskanger (2013) used focus group interviews to investigate the beliefs about knowledge of mathematical definitions of 15 Norwegian secondary mathematics teachers. Consideration is accorded to practical aspects of the interview and how the process will be administered. It requires patience, demands considerable time and energy and involves coordinating between different timetables and establishments (Bryman, 2012). However, as with questionnaires, a drawback associated with this method is that there are no watertight ways of detecting false statements.

5.7 Research methods selected

The overarching goal of this thesis is to find an answer to each of the three research questions. Moreover, it is widely accepted that individual research questions should drive the range of research methods. That is, in this study, after due consideration and reflection, I believe that I have selected the most appropriate research method for each of the research questions. A summary mapping of the research methods to the research questions is shown in Figure 5.4.

Figure 5.4 Summary of mapping of research methods to research questions

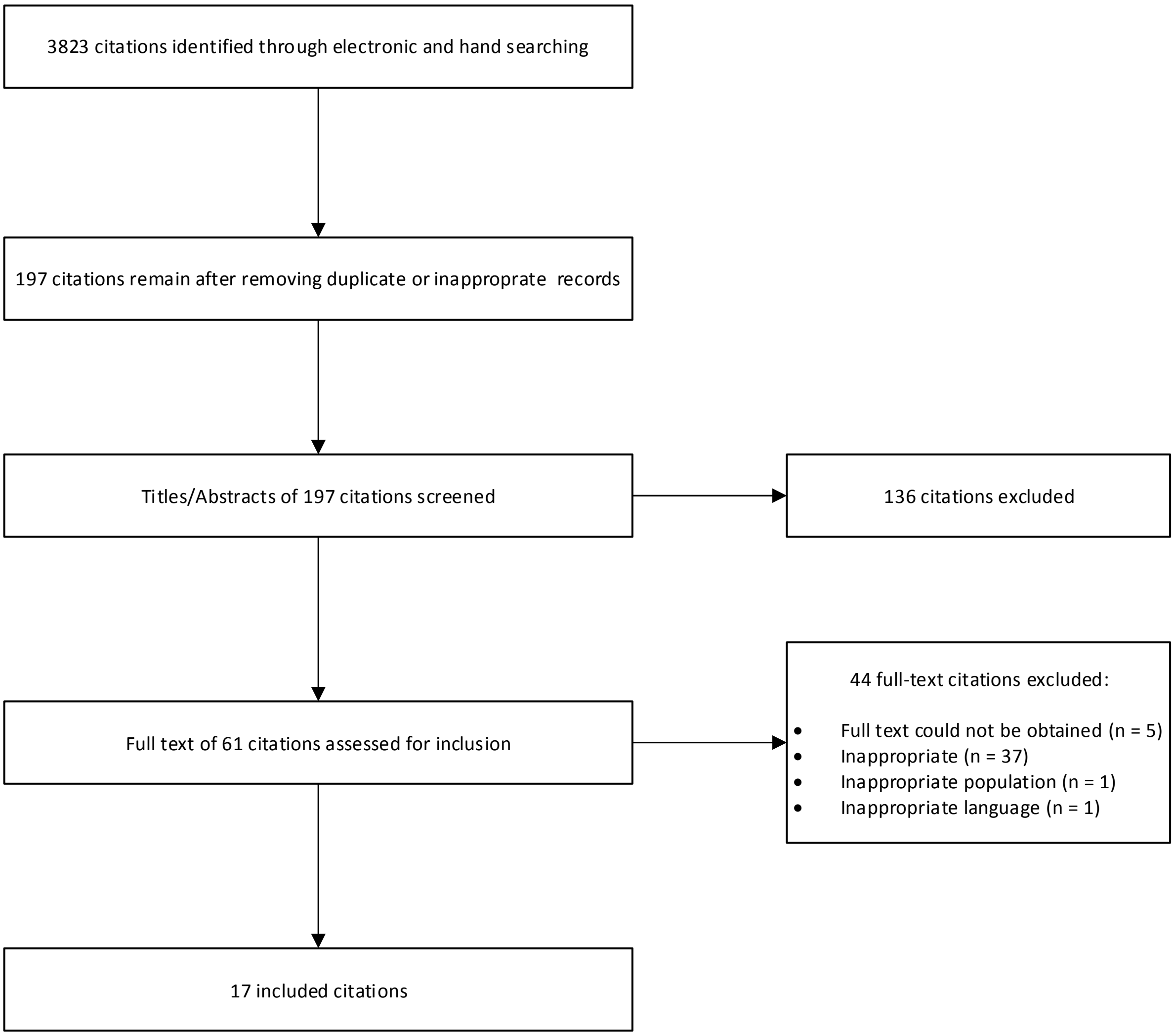


Research Question 1

A systematic literature review was chosen in order to establish a reliable evidence base for evaluating the effectiveness of embedding mathematical problem posing within the framework of Curriculum for Excellence. The purpose was to identify empirical research on problem posing which indicated educational benefits to the learning and teaching of mathematics. To ensure that the review was systematic, I followed the guidance recommended by Boland, Cherry & Dickson (2014).

The first step involved developing criteria for searching. I employed the keywords and phrases: “mathematical problem posing”, “problem posing intervention research”, “problem posing in mathematics education”, “experimental studies and mathematical problem posing” and “effect of problem posing on mathematics”. The next step concerned searching for publications. An electronic search of educational databases included JSTOR, ProQuest Dissertations and Theses (PQDT), PsycINFO and Educational Resources Information Centre (ERIC). Additional citations were located by inspecting the reference list of selected publications. Furthermore, a manual search was conducted of a number of well-respected academic journals such as *Journal of Mathematical Behavior*, *Journal of Mathematics Education*, *Journal for Research in Mathematics Education*, *International Journal of Mathematical Education in Science and Technology* including Advanced Google Scholar. The breadth of the literature yielded an initial pool of 3823 citations from these multiple searches. However, the vast majority of publications were duplicated across database searches or were found to be inappropriate and were discarded during this phase. Based on these searches, 197 unique sources from journals, conference papers, theses and ERIC documents related to mathematical problem posing were identified for inclusion (Figure 5.5).

Figure 5.5 Identification of included studies



In order to narrow the list of potential sources located in the initial database search, I employed specific eligibility criteria. First, only peer-reviewed works published in English between 1996 and 2016 were considered. Second, citations had to be related to primary and secondary pupils, university initial teacher education students or practising teachers. Third, only sources containing explicit details of the research design methodology with an appropriate level of statistical analysis were deemed suitable. Fourth, the search was restricted to empirical studies indicating a minimum sample size of 25.

After carefully screening the titles and abstracts, a further 136 records were removed. Thereafter, I strove to retrieve the full text of the remaining 61 sources to examine each citation in more depth. During this phase, 5 items were not available in full text or inaccessible, 10 were discarded because they adopted non-experimental research designs, while 27 were constrained to theoretical frameworks, literature reviews, pedagogical approaches or comparative studies. In one case, the source involved only 19 participants and in another example, the text language was not in English.

Results

A total of 17 citations were included in the systematic review (Appendix B). Eleven studies involved pupils and six centred on prospective primary or secondary mathematics teachers. Eight studies were published between 2010 and 2016 with the remainder between 1997 and 2009. The origin of the studies was diverse in nationality with four coming from Turkey, three deriving from Australia and Kazakhstan, two hailing from China and USA and one from Oman, Ireland and Iran respectively. The datasets contained 1,939 participants (i.e. 1,186 pupils and 753 ITE university students).

Application of the inclusion criteria to the results of the searches identified seventeen experimental studies for inclusion in this review. Surprisingly, no studies involved practising teachers from either sector. Nonetheless, piloting of the search strategy and supplementation of the results of the electronic search with hand searching and searching of reference publications allows a reasonable confidence in the deduction that all relevant research was included in this systematic review and that the conclusions arising from this review can be based on synthesis of all available empirical evidence.

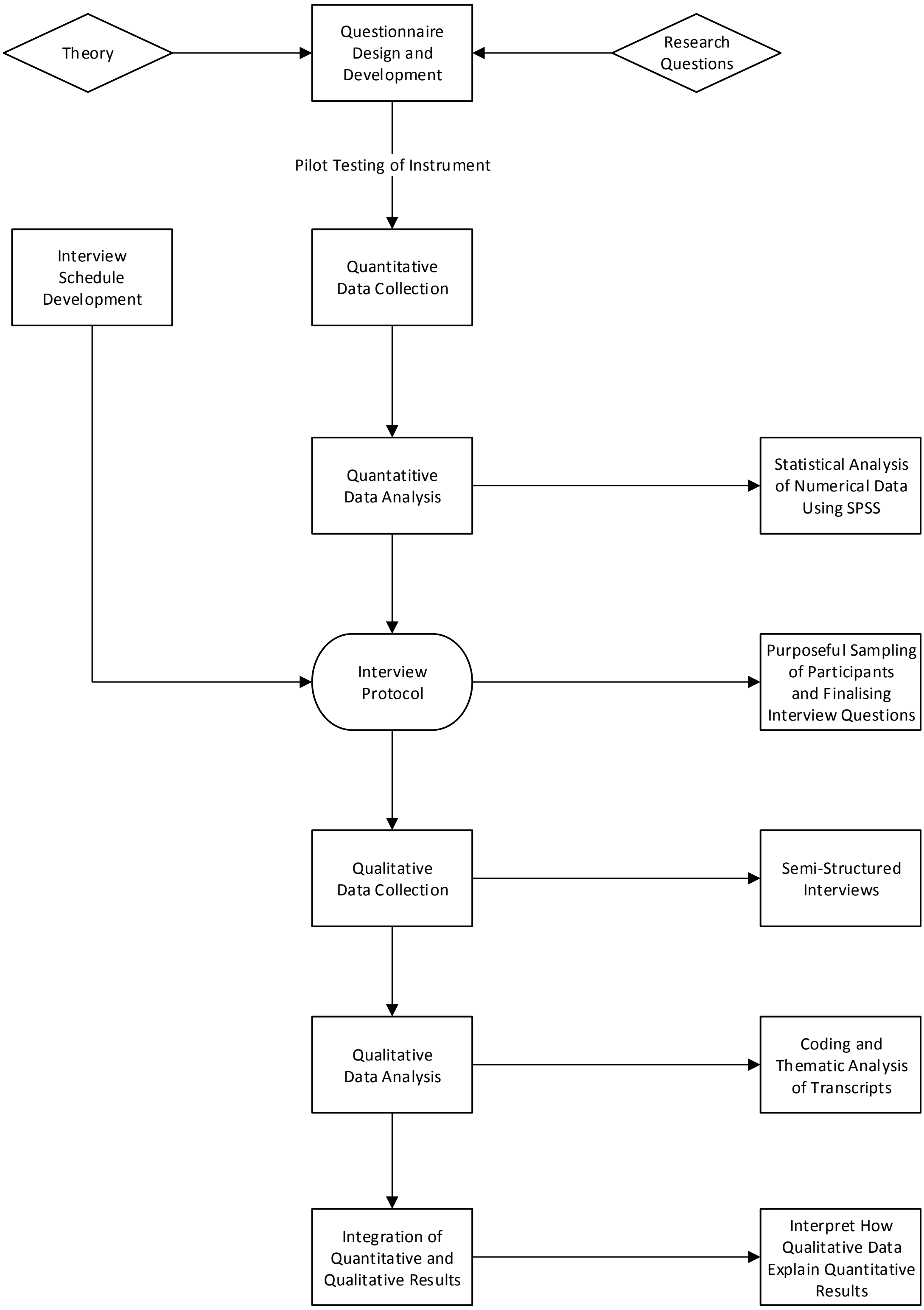
Overall, the methodological quality of the included experimental studies was of a high standard. No restriction was placed on geographical criteria which allowed the range of eligible worldwide literature to be maximised. All studies stated that participants were randomly allocated to treatment groups. The majority employed t-tests to analysis the statistical results. Ten are published in peer-reviewed international journals, five are individual university research degrees, one is an international conference paper and one is featured as a chapter in a renowned book. However, around one third of the studies involve a sample size of 40 or less.

Research Questions 2 and 3

A mixed methods explanatory sequential design was chosen to explore the mathematical beliefs of Scottish primary and secondary mathematics teachers. This decision was centred on the simplicity of the research strategy, my functioning knowledge of statistical processes and the challenges of conducting research as a full-time practitioner. Furthermore, as the construct of beliefs do not lend themselves easily to empirical investigation, I elected to incorporate more than one type of assessment to enhance reliability. Pajares (1992, p. 314) reminds us that beliefs "cannot be directly observed or measured but must be inferred from what people say, intend, and do – fundamental prerequisites that educational researchers have seldom followed".

The research design comprised of two distinct phases. In the opening phase, an internet based questionnaire was employed as the instrument to collect quantitative data. One practical advantage of using an online survey is that the data is instantly available and can easily be transferred into specialised statistical software for more detailed analysis. This was followed by a second phase, which used semi-structured interviews to gather qualitative data to help explain why certain variables and factors are important or non-significant. I opted for semi-structured interviews as I sought to remain flexible and establish an opportunity for any participant to develop ideas and speak more extensively on issues raised. An overview of the main procedures used is shown in Figure 5.6.

Figure 5.6 Main procedures for explanatory sequential design mixed methods study



5.8 Questionnaire

The design objective was to measure teachers' beliefs about mathematical problem solving and problem posing. Thoughtful consideration was afforded to a constellation of interrelated contextual beliefs rooted in the orientation of learning and teaching of mathematics, including theoretical constructivist and collaborative underpinnings projected within the framework of CfE.

Initial design

My strategy was centred on demographic information, belief statements and a procedure for recruiting interview participants. To begin with, I complied an inventory of potential belief statements utilising previously implemented research field instruments (e.g. Kloosterman & Stage, 1992; Ford, 1994; Emenaker, 1996; Perry, Howard & Conroy, 1996; Perry, Howard & Tracy, 1999; Hart, 2002; Kupari, 2003; Anderson, White & Sullivan, 2005; Beswick, 2005; Barlow & Cates, 2006; Memnun, Hart & Akkaya, 2012; Voss et al., 2013). Whilst these empirical studies presented a worthwhile starting point, several items on closer inspection required alteration to eliminate doubt regarding intelligibility. This triggered my awareness to assess the dimension of reliability and validity to ensure the quality of the ensuing data is acceptable. Fowler (2014) observes that good questions depend upon reliability (i.e. providing consistent measures in comparable situations) and validity (i.e. responses correspond to what they intend to measure).

The list of belief statements was lengthened in response to the literature (e.g. Polya, 1957; Thompson, 1992; Schoenfeld, 1985, 1992; Kilpatrick, 1987; Skemp, 1987; Ernest, 1989a, 1991; Stanic & Kilpatrick, 1989; Silver, 1994; Lester & Charles, 2003; Scheon & Charles, 2003; Brown & Walter, 2005; Lesh & Zawojewski, 2007; Mason, Burton & Stacey, 2010; Lester, 2013; Singer & Voica, 2013) including three items composed by myself. In sum, this component generated 65 items grouped under three main categories, considered most relevant to practice (Beswick & Callingham, 2014):

- Teachers' beliefs about the nature of mathematics
- Teachers' beliefs about the learning of mathematics
- Teachers' beliefs about the teaching of mathematics

A theoretical framework by Ernest (1989a) was used to create three subsets for teachers' beliefs about the nature of mathematics i.e. instrumentalist view, Platonist view and problem solving view. Juxtaposed with this perspective, personal epistemological philosophies of the teaching of mathematics as described by Ernest (1991) provided a further theoretical foundation; philosophies of mathematics included an absolutist and a fallibilist view, both of which sit at the opposing ends on the belief continuum. According to Ernest (2004), absolutist philosophies "view mathematics as an objective, absolute, certain and incorrigible body of knowledge, which rests on the firm foundations of deductive logic" (p. 12) whilst fallibility philosophies argue that mathematical knowledge is "understood to be fallible and eternally open to revision, both in terms of its proofs and its concepts" and "made up of many overlapping structures" (p. 14). As a Chartered mathematician, I find it difficult to accept uncritically the previous assertion about the validity of proofs or the truth of theorems but this position is valuable for this design. In completing this section, two hierarchal headings of teachers' beliefs about the learning of mathematics were dichotomised into traditional (transmissive) and constructivist (contemporary) orientations respectively.

Development

Having established a preliminary draft of variables, the next step involved critical assessment of the proposed instrument in terms of reliability, validity, length and unnecessary, difficult or ambiguous questions. Acting on practical advice from two Education Officers (located within different local education authorities), an internet-based survey program was introduced. It was advocated that as all potential participants were likely to have access to a computer and possession of the minimum degree of computer literacy required, administration costs would be significantly reduced. Cohen, Manion & Morrison (2011) point out other advantages of internet based surveys:

> it reduces the time to distribute, gather and process data (data entered onto a web-based survey can be processes automatically as soon as they are entered by the respondent rather than being keyed in later by the researcher); the computer can check incomplete or inconsistent replies; it enables a wider and much larger population to be accessed; human error is reduced in entering and processing online data (p. 280).

After deliberation of various options, https://www.surveymonkey.co.uk/ was selected as the internet based survey program to collect the data. Likewise, a five-point Likert scale ranging from 1 (Strongly Agree), 2 (Agree), 3 (Undecided), 4 (Disagree) and 5 (Strongly Disagree) was appointed as response options. With the assistance of five school colleagues and three external professionals, a pre-pilot was implemented to gather information. As a review group, the participants yielded rich feedback grounded on an accumulation of 170 years of primary and secondary teaching experience. Criticism focussed on significantly reducing the number of comparable items, extending the belief groupings to take a wider account of collaborative learning within CfE, together with a way of differentiating between a social constructivist approach and a problem solving viewpoint. Based on this critique, I conducted a short literature review.

As a consequence of constructive feedback and the acquisition of supplementary knowledge, I initiated a spectrum of changes to the original design. Oppenheim (1992, p. 47) argues that: “Questionnaires do not merge fully-fledged; they have to be created or adapted, fashioned and developed to maturity after many abortive test flights”. An augmented reconceptualised version emerged consisting of 41 belief statements featuring new items employed from empirical studies by Van Zoest, Jones & Thornton (1994) and Barkatas & Malone (2005). Moreover, the questionnaire structure was expanded to reflect five distinct mathematical belief classifications like so:

Under the heading of a constructivist (contemporary) orientation:

- A social constructivist view
- A problem solving view
- A collaborative view

Under the heading of a traditional (transmissive) orientation:

- A static (instrumentalist) view
- A mechanistic (Platonist) view

This modified design facilitated an anthology of information about the interrelationships among multiple teacher belief categories as follows:

Factor 1 A social constructivist orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics.

Factor 2 A problem solving orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics.

Factor 3 A static transmission orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics.

Factor 4 A mechanistic transmission orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics.

Factor 5 A collaborative orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics.

It is conjectured by me that by widening the groupings (i.e. factors) from three to five, it will engender a more accurate dimensional representation of teachers' mathematical beliefs systems, not previously contemplated in this study. Intrinsically, this should mobilise a richer diversity of belief constructs to be elicited during the research process. This is particularly worthwhile given the groupings are not mutually exclusive and that this study encompasses practitioners from both sectors, functioning within an expansive multiplicity of pedagogical and autonomous demands of CfE.

Finally, a second pre-pilot was conducted to evaluate the modified design version. This was undertaken by an accomplished secondary Principal teacher of mathematics in possession of recent online survey experience. The outcome of which prompted me to incorporate some minor word and layout refinements, including the rejection of a belief statement deemed to be theoretically inconsequential, thereby decreasing the total number of items to 40.

5.8.1 Pilot study

The purpose of a pilot study is to verify (although this can never be completely guaranteed) that the final design will operate in the field, by attempting to identify and amend problematic questions in advance. Creswell (2013) strongly advocates pilot testing to ensure that respondents share common understanding and meaning of the questions and to establish content validity of the instrument. Similarly, Bell & Waters (2014, p. 167) testify that the purpose of a pilot "is to get the bugs out of the instrument so that respondents in your main study will experience no difficulties in completing it". Oppenheim (1992) champions the benefits of making preparations for fieldwork and argues that pilot testing can be immensely rewarding for a researcher.

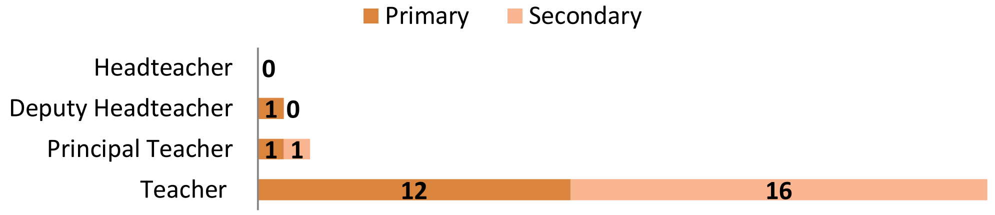
In offering guidance on selecting potential participants, Cohen, Manion & Morrison (2011, p. 403) advise researchers to pilot the questionnaire “using a group of respondents who are drawn from the possible sample but who will not receive the final, refined version”. Equally, Peat et al. (2002, p. 57) maintain that “an essential feature of a pilot study is that the data are not used to test a hypothesis or included with data from the actual study when the results are reported”. It is evident from the literature that researchers should not contaminate the main study by including flawed or inaccurate data from the pilot study.

During 2014, a two week pilot study was conducted with a sample of practising primary and secondary mathematics teachers from Dundee City Council, with the understanding that no teachers from this LEA would be invited to participant in the main study. The decision to approach Dundee City Council was based on a positive attitude towards educational research and its strong link with the University of Dundee. I am extremely grateful for the courtesy, advice and cooperation afforded to me by this LEA. The pilot was administered via an email and invitation link under the management of an Education Officer, and thus no information is available on the precise choice of sampling method employed or response rate. The main body of the instrument consisted of 40 belief statements (23 positive items and 17 negative items) during which respondents were encouraged to identify any problematic or ambiguous questions, including an opportunity to enter comments regarding any aspect of the survey design (e.g. clarity, appearance, relevance, limitations, etc.) and possible suggestions for improvement. In order to alleviate question order bias, ‘Question Randomization’ was selected on Survey Monkey to guarantee that each respondent answered the questions in a different order.

Results

The pilot questionnaire attracted 42 participants of which 11 entries were incomplete and consequently deleted. The remaining 31 complete entries included 14 primary teachers and 17 secondary mathematics teachers, consisting of 9 males and 22 females. All with the exception of one (TIS) were employed on a full-time basis. The estimated mean age of the participants was 37.3 years and the estimated mean length of teaching experience calculated to be 15.8 years. The distribution of grades is contained in (Figure 5.7). In terms of the highest level of qualification in the field of education, 2 participants indicated Masters level.

Figure 5.7 Bar graph of grade of pilot participants (N = 31)



The outcome of the pilot study provided valuable information summarised as follows:

- Participants offering constructive comments on belief statements (11)
- Participants offering critical comments on belief statements (1)
- Participants offering suggestions for design improvements (6)
- Participants agreeing to a telephone interview (2)

Telephone interviews

These were conducted within one week of the pilot and involved two experienced primary practitioners from different schools. Each participant was actively engaged in postgraduate study of mathematics education as part of a professional update process. The duration of the telephone interviews ranged between 15 and 20 minutes. Both responses underpinned the suitability of the overall design and guided the introduction of a handful of marginal adjustments (e.g. expansion of demographic information and enlargement of the space for volunteered comments). Furthermore, I utilised each opportunity to explain some of the quantitative results and to appraise a mixture of phase two interview open questions.

5.8.2. Trustworthiness of pilot study

It is essential to evaluate the credibility of any pilot to establish what modifications are required. In this section, I will briefly report on reliability and validity and will follow this up with a more critical examination of both concepts in the next section.

In order to determine reliability, Creswell (2009) advises that a researcher must obtain levels of internal consistency based on the correlations between various items. Such a concept measures the interrelatedness of the items and is expressed on a numerical scale between 0 and 1. In simple terms, it is used to evaluate the degree to which different test items that probe the same construct produce similar results. Reliability was calculated using Cronbach's alpha (Table 5.2). Pallant (2013) contends that the Cronbach alpha coefficient should be above 0.70, whereas Bryman (2012) reports that 0.80 is more typical to denote an acceptable level, although both authors caution that the value is sensitive to the quantity of items in the analysis.

Table 5.2 Pilot reliability statistics

Belief Factor	Number of items	Cronbach alpha coefficient
1	7	0.784
2	10	0.867
3	10	0.828
4	8	0.712
5	5	0.789

The overall Cronbach alpha for the 40 items was 0.940 indicating excellent internal consistency reliability. Although, one item was removed since its presence decreased this value. This item was part of belief factor 1 (i.e. Social constructivist orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics) and titled: “It is important for students to be provided with opportunities to reflect on and evaluate their own mathematical understanding”. Concurrently, it presented a chance to moderate the numerical imbalance of positive and negative items.

In terms of validity, the readability and clarity of the belief statements was reviewed by teachers from both sectors, producing only one adverse comment. This provided evidence that the questions can collect the intended data. Furthermore, the bulk of the items have been successfully employed in the field within previous empirical studies (e.g. Perry, Howard & Tracy, 1999; Nisbet & Warren, 2000; Hart, 2002; Anderson, Sullivan & White, 2005; Barkatsas & Malone, 2005; Barlow & Cates, 2006; Voss et. al, 2013). Arguably, the sample size of 31 may be considered small but I believe this quantity to be acceptable based on a forecasted main study population of 200 and that I had no control of the administration or sampling frame of the pilot. Whilst the exclusion of a HT is a peripheral concern, the mean teaching experience indicates that this sample has valuable knowledge of previous educational reforms.

Final design

Based on statistical analysis of the pilot including responses from the telephone participants, several minor design changes were inducted. As no additional belief statements emerged, it was deemed unnecessary to implement a second pilot. The final version of the questionnaire is divided into three main components (Appendix C) and is compiled of 39 belief statements (22 positive items and 17 negative items).

5.8.3 Interview schedule

A semi-structured approach was chosen because it allows a researcher to ask a list of formal questions but is flexible in order to probe key unique issues arising from the analysis of the quantitative results including unplanned discourses. To prevent interviewee bias, I sought to avoid the presence of any misleading question. An iterative process of revision and reflection fashioned the plan including constructive feedback from two colleagues used in the first pre-pilot. The final design of the interview schedule (Appendix D) is based on four stages. First, I will formalise the interview by explaining the purpose of the research project, what questions I want to explore and what my aims and objective are. This is followed by discussion of ethics such as the measures I will take to protect confidentiality and anonymity including the gathering of further demographic information. Second, I will ask ten identical questions focussed on the review of the literature relating to nature of mathematics, the learning of mathematics and the teaching of mathematics. Interrelated questions include reference to mathematical problem solving and problem posing within the context of Curriculum for Excellence. Third, quantitative analysis will guide which specific results from phase one require further explanation. Questions from this section will be determined by significant results, nonsignificant results, outliers and sector differences. Fourth, if relevant, an opportunity will be taken to probe emergent themes and ideas. Intrinsic to the overall design is control of the order based upon my perception of what seems most appropriate at the time. For example, question wording may be modified and explanations given; inappropriate questions for a particular interviewee may be omitted, or additional questions incorporated.

5.8.4 Issues for main study

Piloting testing focussed my attention towards design features that produce accurate, credible and replicable research. In doing so, it highlighted three main issues of which I will now discuss in turn. Firstly, sampling is a crucial component within research since it is impractical to access an entire target group. A variety of techniques (e.g. stratified where sampling occurs within groups of the population) are available to attempt to identify a representative sample of the population as a whole. Randomly selected samples will help reduce bias and permit generalisations (Bryman, 2012). Moreover, it has been reasoned that the quality of a piece of research relates in part to the suitability of the sampling strategy selected (Cohen, Manion &

Morrison, 2011; Fowler, 2014). Unfortunately, as in the case of the pilot study, it is anticipated that no scope for incorporating a strategy will be realised during phase one of the main study, as I will not be permitted direct access to the population. Though, it is speculated that with around two thirds of local education authorities granting permission to approach Headteachers (Appendix E), a suitable sample of 100 participants from each sector may be procured. With non-response an unknown factor, it is projected that this will be compensated by oversampling. More influence is envisaged with phase two of the main study where I will have full access to interview volunteers, thereby mobilising some form of purposeful sampling strategy. Based on my mixed methods design, the qualitative sample will be a subset of the quantitative sample, which requires fewer participants. In other words, the size of the samples for phase one and phase two will be unequal.

Secondly, I note that reliability is interrelated to the concept of validity and refers to the consistency to which research findings can be replicated or the trustworthiness by which the methods have been undertaken. Cohen, Manion & Morrison (2011, p. 199) maintain for research to be reliable, “it must demonstrate that if it were to be carried out on a similar group of respondents in a similar context (however defined), then similar results would be found”. Furthermore, the criteria of reliability in quantitative methodologies vary from those in qualitative methodologies, although some commonalities exist (Creswell, 2014). In the main study, I will seek to address quantitative reliability by statistical procedures of internal consistency and any test-retest comparisons while exploring the data. To ensure reliability in the qualitative element, I will provide an in-depth description of the inquiry process including checking transcripts for errors.

Thirdly, Creswell & Clark (2011) point out that validity differs in quantitative and qualitative research. For example, the matter of generalisability is problematic as positivists have no need for contextual variables, whilst for ethnographic researchers, human behaviour is infinitely complex, socially situated and unique (Cohen, Manion & Morrison, 2011). Within the literature, a plethora of validity classifications exist which is confusing to a new researcher. Onwuegbuzie & Johnson (2006) claim that because mixed methods research involves combining complementary strengths and nonoverlapping weaknesses of quantitative and qualitative research, assessing the validity of findings is particularly complex. However, Tashakkori & Teddlie (2010) argue that inferences are the overarching aspects or outcomes of any study. In their model, the authors state that the quality of inferences is assessed simultaneously by examining the process of reaching the results that they are based on and the attributes of the conclusions themselves. In the main study, I will use a convergent approach to assess validity. This is a strategy from the quantitative method such as content validity and another from the qualitative approach such as using audio recorded interviews to allow for repeated revisiting of the data and the use of rich, thick description to convey findings. Finally, triangulation of methods will enhance the strength of any conclusions drawn and will help to avoid the snare of confirmation bias.

5.9 Ethical considerations

Planning of any form of educational research has inescapably critical ethical considerations since the value of research depends as much on its ethical veracity as in the novelty of its discoveries (Walliman, 2006; Best, 2012). Researchers must ensure that they operate with honesty and integrity. Moreover, the research community and those using the findings have a right to expect that research is conducted vigorously, conscientiously and in an ethically defensible manner (Cohen, Manion & Morrison, 2011).

Without the assistance of other people, the operationalisation and robustness of this study would not exist. In this regard, Blaxter, Hughes & Tight (2010) offer the following advice for researchers:

> Ethical research involves getting the informed consent of those you are going to interview, question, observe or take materials from. It involves reaching agreements about the uses of this data, and how its analysis will be reported and disseminated. And it is about keeping to such agreements when they have been reached (p. 164).

My request to undertake this research was subject to the rigorous ethical procedures employed by the University of Glasgow. The research study was reviewed and approved by the College of Social Sciences Research Ethics Committee. Ethics committees play a crucial part in ensuring that no carelessly designed or harmful research is permitted (Bell & Waters, 2014) and to prevent misconduct such as fabrication, falsification or plagiarism in the research process (Best, 2012). In line with the regulatory framework provided by the University of Glasgow and SERA (2005), informed consent was sought from all those who participated in the study. Participants were informed that they would be able to withdraw from the study at any point during data collection. They were also informed that once the data collection stage was complete, they could ask for their data to be withdrawn, without the need to provide any reason for this. All research participants were furnished with my email address for this purpose and to request feedback on the outcomes of the research including any other matter. As a final protocol measure, all participants were provided with contact details of the Ethics Officer in order to direct any concerns regarding the conduct of this research project.

<u>Synopsis of procedure employed in main study</u>

In sum, I followed the ethical procedures within the regulatory framework as outlined in the statements of informed consent that each person associated with my research received. No complaints were made about my conduct of this research and so I am satisfied that the ethical considerations were fully respected.

- Phase one – All participants supplied with questionnaire information sheet (Appendix F). Informed consent realised by survey participation.

- Phase two – All participants supplied with interview information sheet (Appendix G). Informed consent achieved by written completion of consent form (Appendix H).

5.10 Summary

This chapter has allowed me to reflect on differing ontological and epistemological views. Key research paradigms and the role of theory have been briefly examined. Research methods have been introduced, propelling me to think deeply about the strengths and limitations of various approaches to measuring teachers' beliefs. I have expressed my research position and assessed the interdisciplinarity of this study. Significantly, I have expanded my appreciation that methods should follow from research questions (Grix, 2010). Moreover, carrying out the systematic literature review has been a great learning encounter. Validity, reliability and triangulation have been discussed. Finally, the influence of ethics has had a profound effect on my research experience.

The next chapter will report on the quantitative results of phase one of this study.

CHAPTER SIX

Quantitative Results

This chapter presents phase one of the empirical results derived from analysis of the online questionnaire. Consistent with a mixed methods explanatory sequential design, examination of the findings of the quantitative analysis will determine which results require further explanation during phase two of the study. However, outcomes of the analysis are reported within two distinctive sections. Firstly, statistical analysis of the responses to the 39 questionnaire items is presented, followed by qualitative analysis of the optional component which collected a variety of open-ended comments. Descriptive statistics are represented to display demographic information of the participants followed by the application of inferential statistics to address the second and third research questions. Throughout this chapter, IBM SPSS version 22 was employed as the computing software to generate all necessary statistical calculations with one exception.

6.1 Preliminary analysis

During the period from December 2014 to January 2015, a total of 543 practising teachers volunteered for the online questionnaire producing 475 fully completed entries and 68 incomplete entries. Of the incomplete entries, 3 fulfilled all elements of the questionnaire with the exception of responding to the interview option and were subsequently retained, thus producing a total of 478 completed entries. All of the remaining entries were deleted.

The online data were exported directly from Survey Monkey to SPSS. Nevertheless, data cleaning procedures were applied in line with advice offered by Pallant (2013, p. 44), who warns that “it is important to spend the time checking for mistakes initially, rather than trying to repair the damage later”. To reduce response bias, all negatively worded statements (i.e. questionnaire items 18 to 34 respectively) were reversed to allow computation of an overall total mathematical beliefs score for the 39 items and for summation of each of the five sub-domain belief factors. A five point Likert scale was used to offer a choice of responses from ‘strongly agree’ to ‘strongly disagree’. Correspondingly, a score of 1 was assigned to the 'strongly agree' option and a score of 5 to the 'strongly disagree' option respectively. This numerical system produced a theoretical range of total mathematical beliefs scores from 39 (most favourable) to 195 (least favourable).

Internal consistency reliability

The Cronbach's alpha coefficient was used to estimate how well the items that reflect the same construct yield similar results. This main study reports values for each of the five beliefs factors as 0.704, 0.759, 0.728, 0.789 and 0.699 respectively with an overall high coefficient value of 0.884. The 22 positive items (i.e. social-constructivist, problem-solving and collaborative orientation to the nature of mathematics, the learning of mathematics and the teaching of mathematics) measured 0.851. Likewise, the 17 negative items (i.e. static-transmission and mechanistic-transmission orientation to the nature of mathematics, the learning of mathematics and the teaching of mathematics) recorded a value of 0.817. Pallant (2013, p. 104) maintains that, "Values above .7 are considered acceptable; however, values above .8 are preferable". Though, Field (2013) argues that it is more germane for a researcher to think about what obtained values mean within the context of their own research, opposed to applying any 'general guidelines'. Accordingly, I identified similar studies of teachers' mathematical beliefs with comparable overall coefficient measurements (e.g. Peterson et al., 1989; Van Zoest, Jones & Thornton, 1994) leading me to conclude acceptance of the computed result obtained in this study.

Parametric or non-parametric?

For a study of this nature, it may be natural to select from a range of non-parametric statistical techniques such as the Mann-Whitney U-test or the Wilcoxon signed rank test on the basis of utilising ordinal data, since Likert Scales are coded accordingly. Previous studies of teachers' beliefs have employed these types of non-parametric tests (e.g. Jamieson-Proctor & Byrne, 2008; Rajabi, Kiany & Maftoon, 2011; Ampadu, 2014). Controversially, many authors promulgate conflicting statistical advice for researchers in this regard (e.g. Jamieson, 2004; Carifio & Perla, 2007; Norman, 2010; Brown, 2011). Though, what appears to be in harmony is that parametric tests are more powerful and exhibit additional applications than non-parametric tests (McCrum-Gardner, 2008; Field, 2013). Since the majority of previous studies involving teacher's mathematical beliefs have been statistically analysed using parametric methods (e.g. Van Zoest, Jones & Thornton, 1994; Stipek et al., 2001; Barkatas & Malone, 2005; Yates, 2006; Yu, 2008; Depaepe, De Corte & Verschaffel, 2010; Memnun, Hart & Akkaya, 2012), it suggests that the belief construct is normally distributed.

6.1.1 Exploratory factor analysis

Factor analysis seeks to reduce or summarise a compilation of variables into a smaller set of dimensions termed factors or components. In this study, 39 items of the positive and negative scale were subjected to principle components analysis (PCA) in order to explore the nature of previously unknown variables to seek underlying patterns, clusterings or groupings. Prior to performing PCA, the suitability of data for factor analysis was assessed as follows:

Sample size

This is determined by considering a minimum sample size or a ratio of subjects to variables. Comfrey & Lee (1992, p. 317) suggest that "the adequacy of sample size might be evaluated very roughly on the following scale: 50 – very poor; 100 – poor; 200 – fair; 300 – good; 500 – very good; 1000 or more – excellent". Whereas, Nunnally (1978) advises that the subject to item ratio should be at least 10:1, however this recommendation is not supported by published research. Irrespectively, exercising both distinctive approaches, the sample size of 478 is comfortably 'good' from a magnitude perspective and equally acceptable from an item ratio viewpoint.

Factorability of the correlation matrix

Inspection of the correlation matrix, as advocated by Tabachnick & Fidell (2014), revealed the presence of many coefficients of 0.3 and above. The Kaiser-Meyer-Olkin measure of sampling statistic was 0.903, generously exceeding the recommended minimum value of 0.6 (Kaiser, 1970, 1974) and Bartlett's Test of Sphericity (Bartlett, 1954) reached statistical significance [χ^2 (741) = 6057.958, $p < 0.001$], supporting the factorability of the correlation matrix.

Factor extraction

Factor extraction involves determining the smallest number of factors than can be used to best represent the interrelationships among a set of variables. Several techniques can be used to assist in this decision making process; Kaiser's criterion, scree test and parallel analysis (Pallant, 2013).

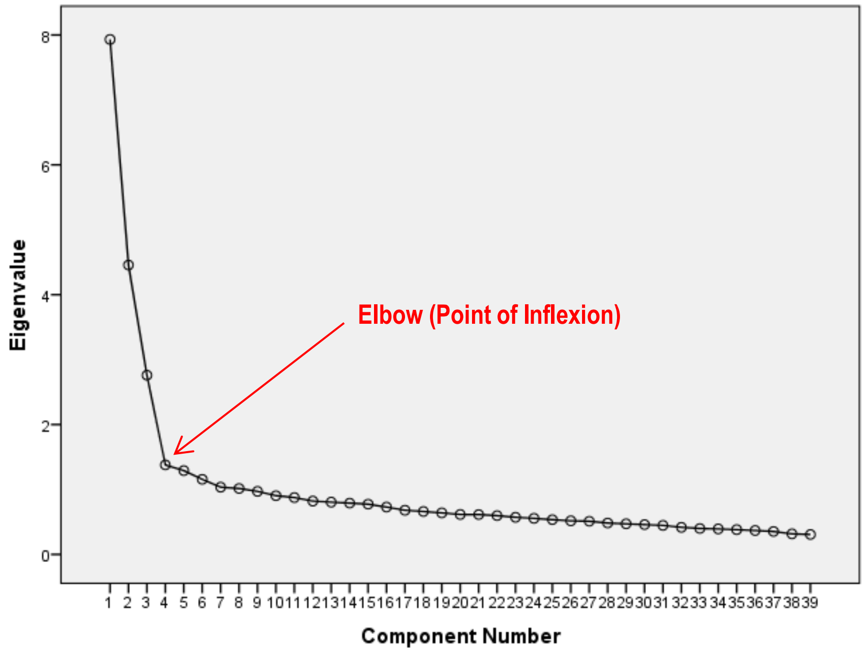
Kaiser's criterion

Kaiser (1960) recommended retaining all factors with eigenvalues greater than 1. According to Field (2013, p. 677), "This criterion is based on the idea that the eigenvalues represent the amount of variation explained by a factor and that an eigenvalue of 1 represents a substantial amount of variation". Principle component analysis revealed the presence of eight components with eigenvalues exceeding 1, explaining 20.338%, 11.429%, 7.078%, 3.536%, 3.311%, 2.957%, 2.657% and 2.606% of the variance respectively (Appendix G).

Cartell's Scree Test

Cartell's (1966) scree test is considered to be the best choice according to Field (2013) and involves plotting each of the eigenvalues of the factors and retaining all factors above the elbow. Conversely, Tabachnick & Fidell (2014, p. 697) caution that, "Unfortunately, the scree test is not exact; it involves judgment of where the discontinuity in eigenvalues occurs and researchers are not perfectly reliable judges". An inspection of the scree plot obtained (Figure 6.1) revealed a clear break after the third component, and it was decided to retain three components for further investigation.

Figure 6.1 Scree plot



Parallel analysis

This involves comparing the magnitude of the eigenvalues with those obtained from a randomly generated data of the same size. For this analysis, I employed Monte Carlo PCA software which showed only three components with eigenvalues exceeding the corresponding criterion values (please refer to Appendix H and Table 6.1 respectively) for a randomly generated data matrix of the same size (39 variables × 478 respondents). Therefore, the results of parallel analysis validate my decision from the scree plot to retain three factors for further investigation.

Table 6.1 Comparison of eigenvalues from PCA and criterion values from parallel analysis

Component number	Actual eigenvalue from PCA	Criterion value from parallel analysis	Decision
1	7.932	1.5790	**Accept**
2	4.457	1.5147	**Accept**
3	2.760	1.4693	**Accept**
4	1.379	1.4276	reject
5	1.291	1.3872	reject
6	1.157	1.3509	reject
7	1.036	1.3188	reject
8	1.016	1.2861	reject

Factor rotation and interpretation

To aid in the interpretation of these three components, direct oblimin rotation was performed. The rotated solution revealed the presence of a simple structure (Thurstone, 1947), with three components showing a number of fairly strong loadings and all variables loading substantially on only one component. This can be observed from the Pattern Matrix (Appendix I) and the Structure Matrix (Appendix J). To determine the strength of the relationship between the three factors, examination of the component correlation matrix was carried out. This revealed very weak positive affects between the three factors (r = 0.054, 0.147 and 0.140 respectively). The results of this analysis highlight the presence of three distinct mathematical belief systems as follows:

1. A social constructivist, problem solving and collaborative orientation;
2. A social constructivist, problem solving and static transmission orientation;
3. A static and mechanistic transmission orientation.

Further investigation will help to determine which belief system is associated with each sector and homogenous group.

6.2 Demographic information of the participants

The demographic information provides the rational for statistical analysis of the generated data guided by the research questions. The questionnaire responses involved teachers from 21 local education authorities (32 local education authorities exist in Scotland). Displayed by frequency tables, the data of the 478 participants is presented employing the following variables:

- Gender
- Sector
- Age (years)
- Mode of working
- Employment type
- Trained in or out with Scotland
- Grade
- Length of teaching experience (years)
- Highest level of qualification in the field of education

The frequencies and cumulative percentages associated with each variable is summarised in Table 6.2 to Table 6.10 respectively.

Table 6.2 Frequency table for gender (N = 478)

Characteristic	n	%	Cumulative %
Male	148	31.0	31.0
Female	330	69.0	100.0
Total	478	100.0	

Table 6.3 Frequency table for sector (N = 478)

Characteristic	n	%	Cumulative %
Primary	229	47.9	47.9
Secondary	249	52.1	100.0
Total	478	100.0	

Table 6.4 Frequency table for age in years (N = 478)

Characteristic	n	%	Cumulative %
21-25	31	6.5	6.5
25-34	102	21.3	27.8
35-44	118	24.7	52.5
45-54	140	29.3	81.8
55+	87	18.2	100.0
Total	478	100.0	

Table 6.5 Frequency table for mode of working (N = 478)

Characteristic	n	%	Cumulative %
Full-time	439	91.8	91.8
Part-time	39	8.2	100.0
Total	478	100.0	

Table 6.6 Frequency table for employment type (N = 478)

Characteristic	n	%	Cumulative %
Permanent	446	93.3	93.3
Temporary	14	2.9	96.2
Teacher Induction Scheme	18	3.3	100.0
Total	478	100.0	

Table 6.7 Frequency table for teacher training in Scotland (N = 478)

Characteristic	n	%	Cumulative %
Yes	450	94.1	94.1
No	28	5.9	100
Total	478	100.0	

Table 6.8 Frequency table for grade (N = 478)

Characteristic	n	%	Cumulative %
Teacher	321	67.2	67.2
Principal Teacher	75	15.7	82.8
Deputy Headteacher	30	6.3	89.1
Headteacher	52	10.9	100.0
Total	478	100.0	

Table 6.9 Frequency table for teaching experience in years (N = 478)

Characteristic	n	%	Cumulative %
0-5	87	18.2	18.2
6-10	101	21.1	39.3
11-15	67	14.0	53.3
16-20	50	10.5	63.8
20+	173	36.2	100.0
Total	478	100.0	

Table 6.10 Frequency table for highest level of qualification in the field of education (N = 478)

Characteristic	n	%	Cumulative %
BEd	162	33.9	33.9
PGCE/PGDE	274	57.3	91.2
Masters	42	8.8	100.0
Doctorate	0	0	100.0
Total	478	100.0	

As Table 6.2 shows, the ratio of male to female participants is approximately equal to 1:2. This is explained by the substantial presence of female primary teachers which dominate this sector. Unexpectedly, the proportion of primary and secondary teachers is reasonably similar (Table 6.3), given the superior quantity of primary teachers that are employed in Scotland. The overall estimated mean age of the participants is 42.8 years and the overall estimated length of teaching experience is 17.2 years. It is suggested that the most prominent characteristic is the grade distribution of the participants (Table 6.8). Whilst all levels are represented, one third are promoted teachers. In fact, it is argued that this study is unique in this respect since it incorporates such a wide continuum of grades, ranging from 18 probationers to 52 Headteachers, amalgamated between both primary and secondary establishments. Unfortunately, the questionnaire did not contain any participants holding a Doctorate in the field of education (Table 6.10). It is postulated that inclusion of such a unique group would have been of enormous interest to this study.

Since this study seeks to compare both primary and secondary mathematics teachers, I have compiled an overview of participants' demographic information based on sector (Table 6.11). Several similarities can be observed such as mode of working, employment type, Scottish teacher qualifying status, proportion of classroom teachers and Masters level graduates in the field of education. Whilst the age groups vary at both extremes, the estimated mean age of 41.5 years for primary teachers is comparable with the secondary mathematics counterparts of 44.0 years. Likewise, the length of teaching experience varies in the majority of categories but almost match when estimates are calculated; the estimated mean length of teaching experience for primary teachers is 17.0 years compared with 17.8 years for secondary mathematics participants. Sector differences exist regarding gender and grade but this is symptomatic of conducting research with participants from two separate systems. In general, analysis of the sample population supports the view that both sectors contain participants with similar characteristics.

Table 6.11 Demographic characteristics of participants by sector (N = 478)

Characteristic	Primary		Secondary	
	n	%	n	%
Gender				
Male	22	9.6	126	50.6
Female	207	90.4	123	49.4
Age (years)				
21-24	23	10.0	8	3.2
25-34	51	22.3	51	21.3
35-44	52	22.7	66	24.7
45-54	69	30.1	71	29.3
55 and over	34	14.9	53	18.2
Mode of working				
Full-time	208	90.8	231	92.8
Part-time	21	9.2	18	7.2
Employment type				
Permanent	212	92.6	234	94.0
Temporary	7	3.0	7	2.8
Teacher Induction Scheme	10	4.4	8	3.2
Teacher training in Scotland				
Yes	217	94.8	233	93.6
No	12	5.2	16	6.4
Grade				
Teacher	151	65.9	170	68.3
Principal Teacher	15	6.5	60	24.1
Deputy Headteacher	18	7.9	12	4.8
Headteacher	45	19.7	7	2.8
Teaching experience (years)				
0-5	59	25.8	28	11.2
6-10	38	16.6	63	25.3
11-15	22	9.6	45	18.1
16-20	26	11.3	24	9.6
Over 20	84	36.7	89	35.7
Highest qualification in the field of education				
BEd	130	56.8	32	12.7
PGCE/PGDE	84	36.7	190	76.3
Masters	15	6.5	27	10.8
Doctorate	0	0.0	0	0.0

Note: Totals of percentages are not 100.0 for every characteristic because of rounding.

6.3 Analysis of findings relevant to research question two

The second research question was designed to explore the espoused mathematical beliefs of primary and secondary mathematics teachers, irrespective of variable, and posed:

> Are there any differences in the mathematical beliefs of Scottish primary and secondary mathematics teachers?

In this section, I will begin by presenting descriptive statistics including a discussion of general observations of the two categorical groups, followed by a robust justification for using parametric tests, ending with statistical analysis of the differences in the mathematical beliefs between the sectors, using an independent samples t-test.

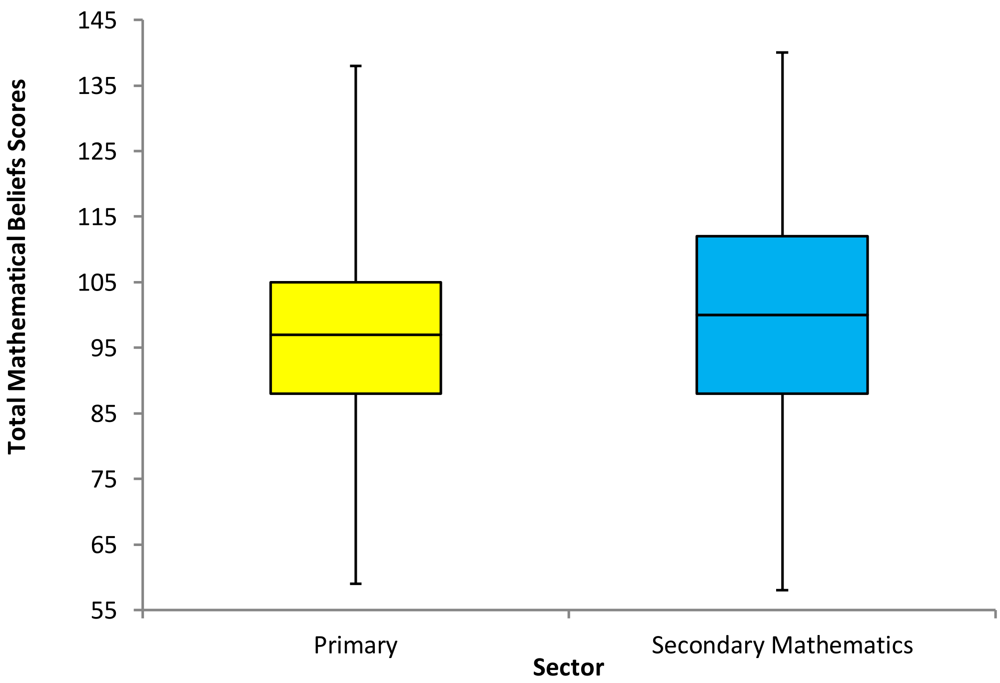
6.3.1 Descriptive statistics

Descriptive statistics of the total mathematics beliefs scores are displayed in Table 6.12. Inspection of the mean values of the two data sets suggest that primary teachers hold stronger mathematical beliefs than secondary mathematics teachers. Since both sector statistics for mean and 5% trimmed mean are similar, no extreme scores appear to have strongly influenced the original mean. The standard deviation measurements indicate that the mathematical beliefs of the secondary mathematics teachers are more dispersed around the mean. Comparing the median of both sectors confirms that primary teachers' mathematical beliefs are stronger than the secondary mathematics teachers. On inspection of the range, the secondary mathematics teachers' beliefs are more spread out suggesting a less homogenous group. This inference is confirmed when variability is considered, which noticeably reveals that primary teachers' mathematical beliefs are less inconsistent. This statistic is confirmed by visual inspection of the sector boxplots (Figure 6.2). Using SPSS, eight outliers were identified all associated with the primary data set. No extreme points were detected for each sector.

Table 6.12 Descriptive statistics of sector TMBS

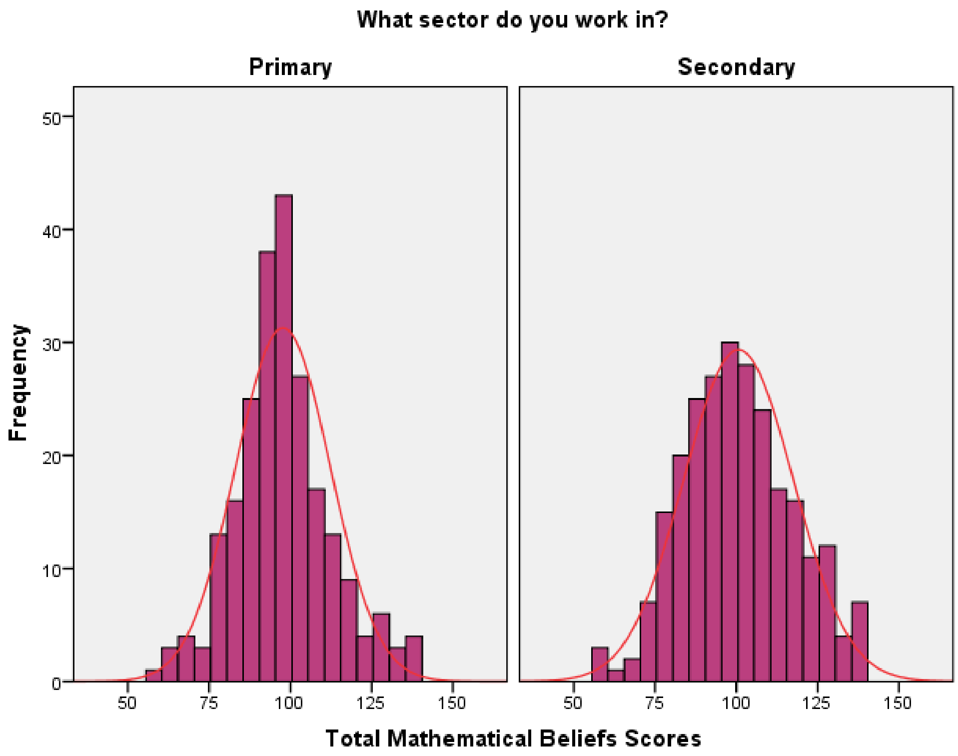
Characteristic		Primary		Secondary Mathematics	
		Statistic	Std. Error	Statistic	Std. Error
N		229		249	
Mean		97.71	.965	100.63	1.073
95% Confidence Interval for Mean	Lower Bound	95.81		98.52	
	Upper Bound	99.61		102.75	
5% Trimmed Mean		97.42		100.48	
Median		97		100	
Variance		213.224		286.418	
Std. Deviation		14.602		16.924	
Minimum		59		58	
Maximum		138		140	
Range		79		82	
Interquartile Range		17		24	
Skewness		.356	.161	.150	.154
Kurtosis		.576	.320	.324	.307

Figure 6.2 Boxplots of primary and secondary mathematics TMBS



Since measures of central tendency (i.e. mean and median values) for both sectors are similar, it suggests that each data set is symmetrical. Inspection of the appearance of both frequency histograms (Figure 6.3) reveal the majority of scores occurring in the centre, tapering out towards the extremes, indicating that the scores on each variable is normally distributed.

Figure 6.3 Frequency histograms of TMBS according to sector



<u>General comments</u>

Inspection of the results of the questionnaire (Appendix K) revealed that primary teachers, (Items 4 and 12) and secondary mathematics teachers (Items 2 and 12) failed to obtain a minimum belief response of 50% for Strongly Agree and Agree for two of the twenty two positive items. Surprisingly, both sectors simultaneously rejected problem solving belief construct (i.e. 'Teaching mathematics through problem solving is the best method to help students learn'). This statement also produced the highest number of combined undecided responses of the questionnaire, suggesting a degree of uncertainty surrounding this important concept.

However, more surprising were the results obtained for the seventeen negative beliefs constructs. Primary teachers (Items 18, 20, 21, 22, 25, 26, 27, 29, 31 and 34) remained unsuccessful in obtaining a minimum belief response of 50% for Strongly Disagree and Disagree for ten negative items, representing a lack of support for the rejection of almost three fifths of the belief statements aligning with a static transmission and a mechanistic orientation. Correspondingly, secondary mathematics teachers (Items 20, 21, 26, 27, 29 and 34) failed to achieve a consensus for rejecting one third of the negative belief statements. Whilst the mathematical beliefs of both sectors overlap with five negative items, differences remain apparent; results of two static transmission belief statements are particularly notable. Firstly, Item 20 (i.e. Mathematics is an accumulation of facts, rules and skills), which is based on an instrumentalist view of mathematics (Ernest, 1989a), is considerably favoured by both sectors. Secondly, an overwhelming majority of participants from both sectors strongly endorsed Item 21 (i.e. the primary purpose of teaching problem solving is to equip students with a collection of skills and processes). This belief statement was designed to extract the underlying philosophy of problem solving by establishing the priority between fostering learners computational and procedural abilities and the development of independent critical thinking skills. In his insightful research on mathematical problem solving, Lester (1985) reminds us that:

> The primary purpose of teaching problem-solving instruction is not to equip students with a collection of skills and processes, but rather to enable them to think for themselves. The value of skills and processes instruction should be judged by the extent to which the skills and processes actually enhance flexible, independent thinking (p. 66).

6.3.2 Assumptions

In order to implement any parametric test, it is essential to check that relevant assumptions have not been violated. Using the total mathematics beliefs scores as the dependent variable for the sample as a whole, the following five assumptions were checked for violation:

<u>Random sampling</u>

Samples have been randomly selected from the population. Within the subset of local education authorities granting research approval for me to contact their schools, each member of the subset has had an equal probability of being selected.

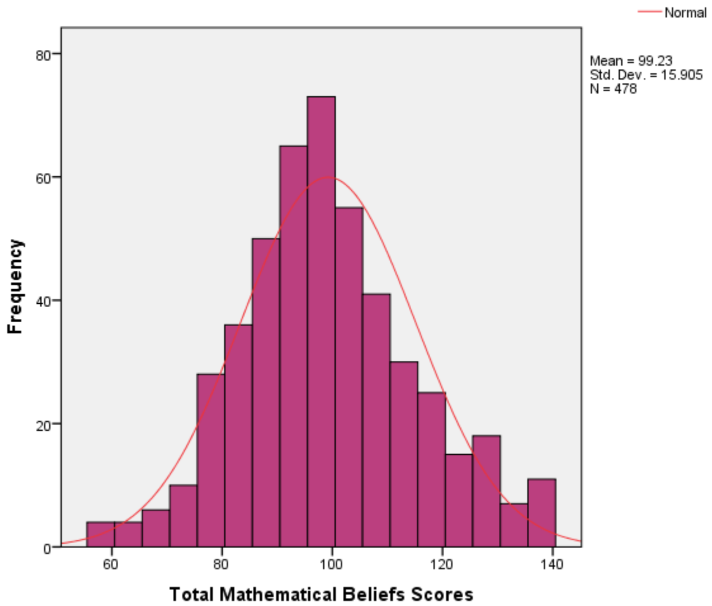
<u>Independence of observations</u>

Freedom of observations is a fundamental prerequisite for almost all hypotheses testing procedures (Stevens, 2009; Gravetter & Wallnau, 2012). According to Pallant (2013, p. 213-214), “[t]he observations that make up your data must be independent of one another; that is, each observation or measurement must not be influenced by any other observation or measurement”. In this study, data were collected from individual teachers employed from an assortment of primary and secondary schools, located throughout multiple local education authorities in Scotland. The instrument was administered and collected once. Furthermore, the design of the online questionnaire ensured that only one response per computer was possible, in an attempt to eliminate collusion between colleagues. As both sectors operate independently from each other and the autonomous nature of the profession, it is strongly intimated that there is no relationship between the observations in each group or between the groups themselves.

Normality

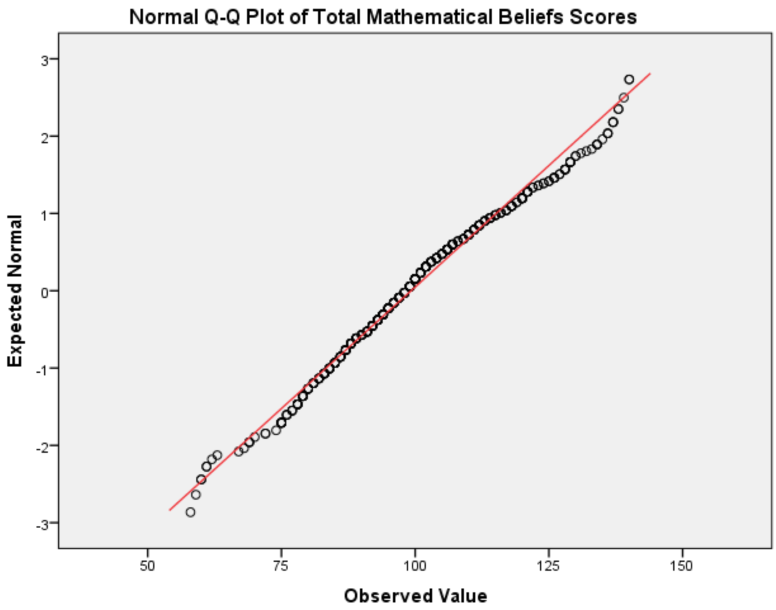
The total mathematical beliefs scores for the sample, D(478) = .069, $p < .001$, suggests violation of the assumption of normality. However, this is fairly common in large samples when using a Kolmogorov-Smirnov test (Field, 2013; Pallant, 2013) and is refuted by skewness of 0.265 (SE = 0.112) and kurtosis of 0.012 (SE = 0.223). Visual inspection of the shape of the associated frequency histogram (Figure 6.4) suggests that the data is normally distributed.

Figure 6.4 Frequency histogram of TMBS for whole sample



Normal distribution is further supported by inspection of the normal probability plots (labelled Normal Q-Q Plot and Detrended Normal Q-Q Plot respectively). Tabachnick & Fidell (2014, p. 115) maintain that “Frequency histograms are an important graphical device for assessing normality, especially with the normal distribution as an overlay, but even more helpful than frequency histograms are expected normal probability plots and detrended expected normal probability plots”. It can be observed that the Normal Q-Q plot (Figure 6.5) displays a reasonably straight line with some minor deviations indicating normal distribution. Likewise, no real clustering of points with most collecting around the zero line is exhibited in the Detrended Normal Q-Q plot (Figure 6.6).

Figure 6.5 Normal Q-Q plot of TMBS for whole sample

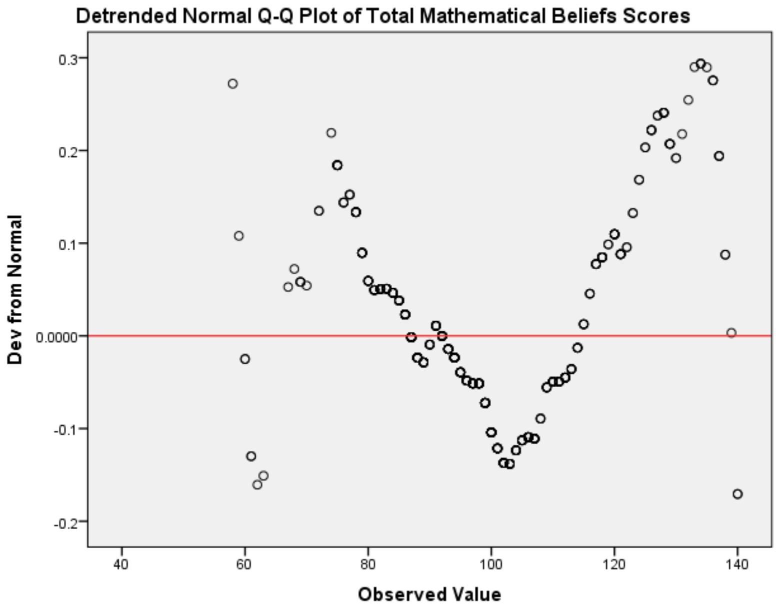


However, Yap & Sim (2011) argue that graphical methods are open to interpretation and require a researcher to be experienced in such matters:

> Even though graphical methods are useful in checking the normality of a sample data, they are unable to provide formal conclusive evidence that the normal assumption holds. Graphical method is subjective as what seems like a 'normal distribution' to one may not necessarily be so to others. In addition, vast experience and good statistical knowledge are required to interpret the graph properly (p. 2142).

Citing my familiarity as a chartered mathematician combined with that of a knowledgeable mathematics teacher, I am confident that my assessment of normality has been justified.

Figure 6.6 Detrended Normal Q-Q plot of TMBS for whole sample



No significant outliers

An outlier is an observation that lies an abnormal distance from other values in a random sample from a population. No extreme points or outliers were detected from inspection of a boxplot or by re-examining the tails of the distribution of the frequency histogram. Comparison of the 5% trimmed mean value of 99.01 with the similar numerical mean value of 99.23, confirms the absence of significant outliers.

Homogeneity of variance (Homoscedasticity)

For the total mathematics beliefs scores, variances were unequal for the primary and secondary mathematics teachers, $F(1, 476) = 8.383$, $p. = .004$. However, violations of this nature are common with unequal group sizes and large samples. Field (2013, p. 195) warns researchers that, "In large samples Levene's test can be significant even when group variances are not very different. Therefore, it should be interpreted in conjunction with the variance ratio". In this case, the variance ratio is calculated by dividing the largest variance (286.418) by the smallest variance (213.224), producing a resultant value of 1.343, strongly suggesting that the variances are approximately equal. Furthermore, this particular variance ratio is well within the parameters advised by Tabachnick & Fidell (2014, p. 120) who state that "If samples sizes are relatively equal (within a ratio of 4 to 1 or less for largest to smallest cell size), an F_{max} [ratio of largest cell variance to the smallest] as great as 10 is acceptable".

In short, all five assumptions have been satisfied.

6.3.3 Independent samples t-test

An independent samples t-test was conducted to compare the mean of the Total Mathematics Belief Scores (N = 478) between primary and secondary mathematics teachers. The results revealed a statistically significant difference between the two sectors, primary teachers (M = 97.71, SD = 14.602) and secondary mathematics teachers (M = 100.63, SD = 16.924), t (474.098) = -2.026, p = 0.043, two-tailed). The magnitude of the differences in the means (mean difference = 2.92, 95% CI [-5.758, -0.088] was significant. Therefore, we reject the null hypothesis that there is no difference in the TMBS between primary and secondary mathematics teachers.

In order to indicate a level of prominence of statistical significance, it is necessary to consider the effect size. A common used interpretation in social sciences is to refer to effect sizes as ‘small’ (d = 0.2), ‘medium’ (d = 0.5) and ‘large’ (d = 0.8), based on benchmarks suggested by Cohen (1988). Applying such a scale to this study, the effect size is considered ‘small’ (d = 0.185), although it may be argued that applying this generic descriptor to teachers’ beliefs is misleading. More relevant is an interpretation that contextualises the magnitude of an effect size (Lenth, 2001; Baguley, 2009). Therefore, in the absence of comparable Scottish educational research data, I have interpreted the effect size as having practical significance worthy of detailed further investigation.

6.3.4 Further investigation

In this section, I will report the outcome of the statistical analysis of three interrelated mathematical belief investigations of the following elements:

- Individual belief items
- Belief categories (i.e. Belief Factor 1, 2, 3, 4 & 5 respectively)
- CfE mathematical beliefs and Traditional mathematical beliefs (i.e. positive and negative items)

<u>Individual belief items</u>

In order to explore the previous findings in more detail, an independent samples t-test was conducted for each of the 39 questionnaire items. The objective of this exercise was to compare the mean of the individual belief construct scores (N = 478) between the primary and secondary participants. The results obtained from this analysis are displayed in Appendix M. From this data, it can be seen that when juxtaposed, 32 of the 39 belief statements produced statistically significant results i.e. the mathematical beliefs of primary and secondary mathematics teachers differed in their responses to more than fourth fifths of the questionnaire (82%). I will briefly comment on a result from each of the five belief categories:

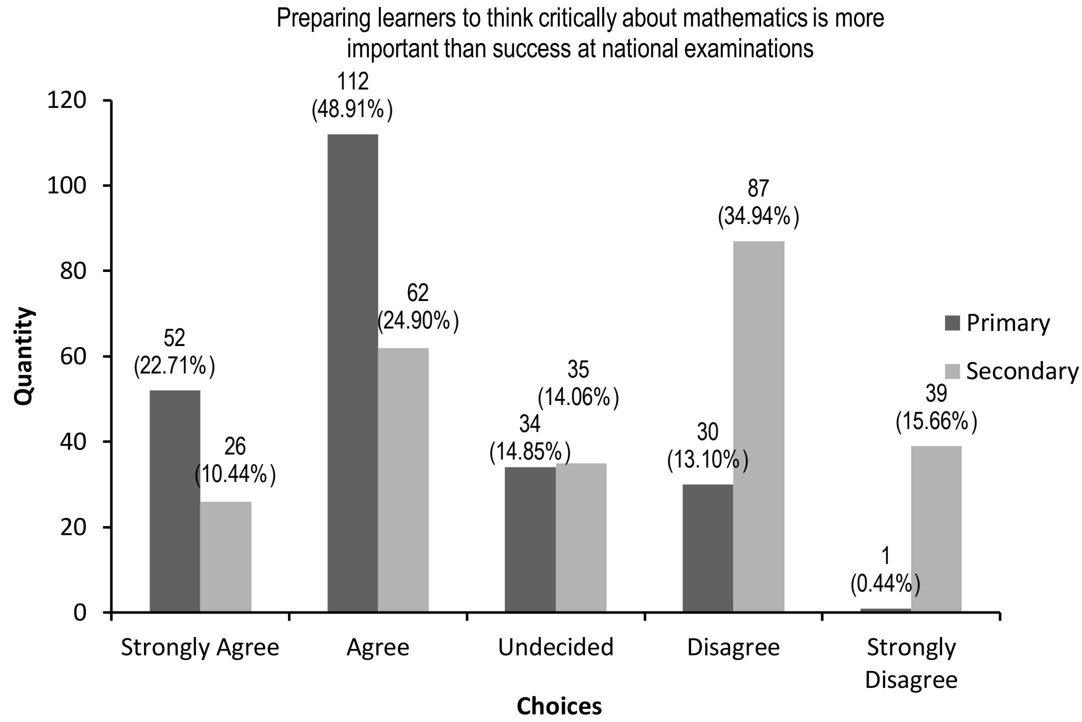
<u>Questionnaire Item 2 – ‘Preparing learners to think critically about mathematics is more important than success at national examinations’ (Figure 6.7)</u>

Responses to this social constructivist item divided the participants more than any other question. Almost three-quarters of primary teachers (72%) strongly agreed or agreed with this belief statement opposed to less than half of this amount by the secondary mathematics teachers (35%). A possible explanation for this disparity might be that primary teachers have a greater social constructivist freedom to exercise their mathematical beliefs, relatively unrestricted from the constraints of their intensely examination focused secondary counterparts. Another possible explanation for this is that secondary mathematics teachers may prefer to identify with the tangible nature of examination results, as critical thinking is an intrinsic element manifested within multiple aspects of mathematical learning such as the construction of arguments, reasoning and solving complex problems.

Questionnaire Item 10 – 'Teachers should be experienced problem solvers and should have a firm grasp of what successful problem solving involves' (Figure 6.8)

This problem solving belief statement is based on the theoretical work of Lester (2013) who maintains that teachers themselves need not be expert problem solvers, but have proficiency in teaching pupils how to solve mathematical problems. The vast majority of secondary mathematics teachers (95%) strongly agreed or agreed with this statement opposed to a significantly reduced quantity of primary teachers (74%). It seems possible that some primary practitioners lack enthusiasm or confidence in teaching mathematical problem solving. Thompson (1985) reminds us about Jeanne [one of her teacher participants] skipping some pages in a textbook containing story problems involving rates and proportions: "She then indicated that the reason for her skipping the pages involving problems was that the students did not enjoy working them and that problems caused them to experience a great deal of frustration with mathematics" (p. 288). This experience fuelled her argument for teachers "to experience mathematical problem solving from the perspective of the problem solver before they can adequately deal with its teaching" (p. 292).

Figure 6.7 Bar chart of sector responses to Questionnaire Item 2



<u>Questionnaire Item 18 – 'Mathematical problems can only have one final correct answer' (Figure 6.9)</u>

This negative item is a quintessential component of absolutism, which rejects the existence of multiple answers in mathematics. Such a belief system typifies a widely held misconception about mathematics that each problem contains a unique solution. The results reveal that more than three-quarters of secondary mathematics teachers (79%) strongly disagreed or disagreed compared to less than half of primary teachers (45%). Whilst it may be argued that secondary mathematics teachers are more naturally positioned to respond to this question due to their knowledge and understanding of negative numbers, trigonometric equations, graphs, etc., the sector divide is considerable given the critical nature of this belief statement within the operationalisation of mathematical problem solving. Worryingly, over one fifth of mathematics teachers (21%) were undecided or concurred with this item.

Figure 6.8 Bar chart of sector responses to Questionnaire Item 10

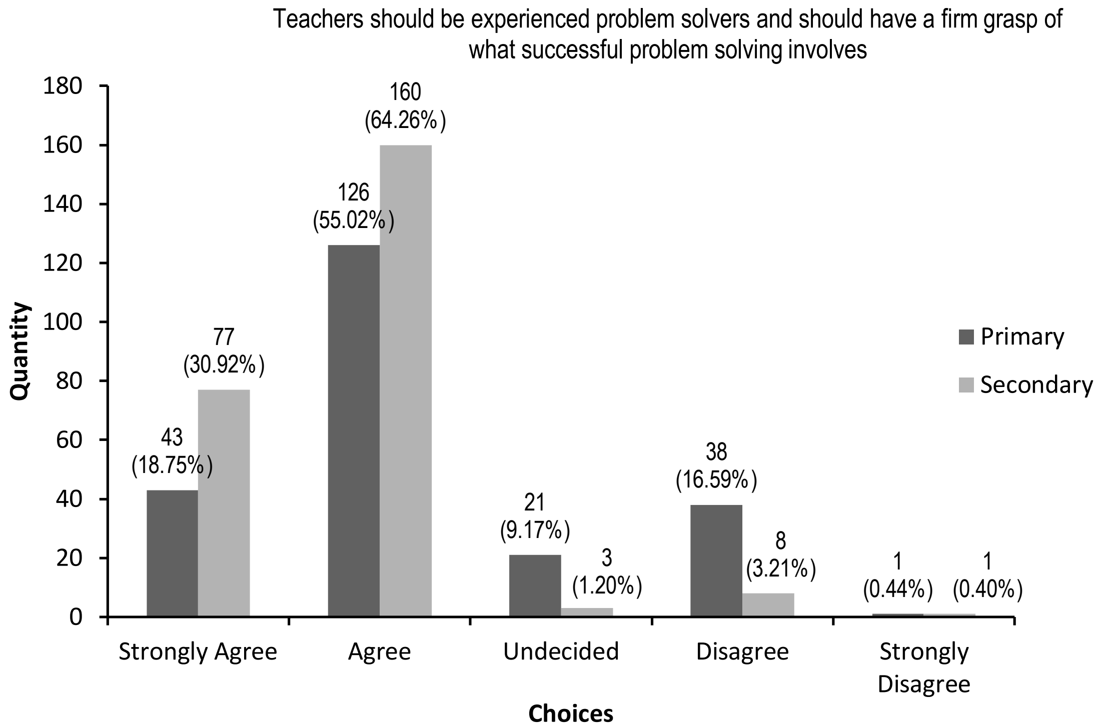
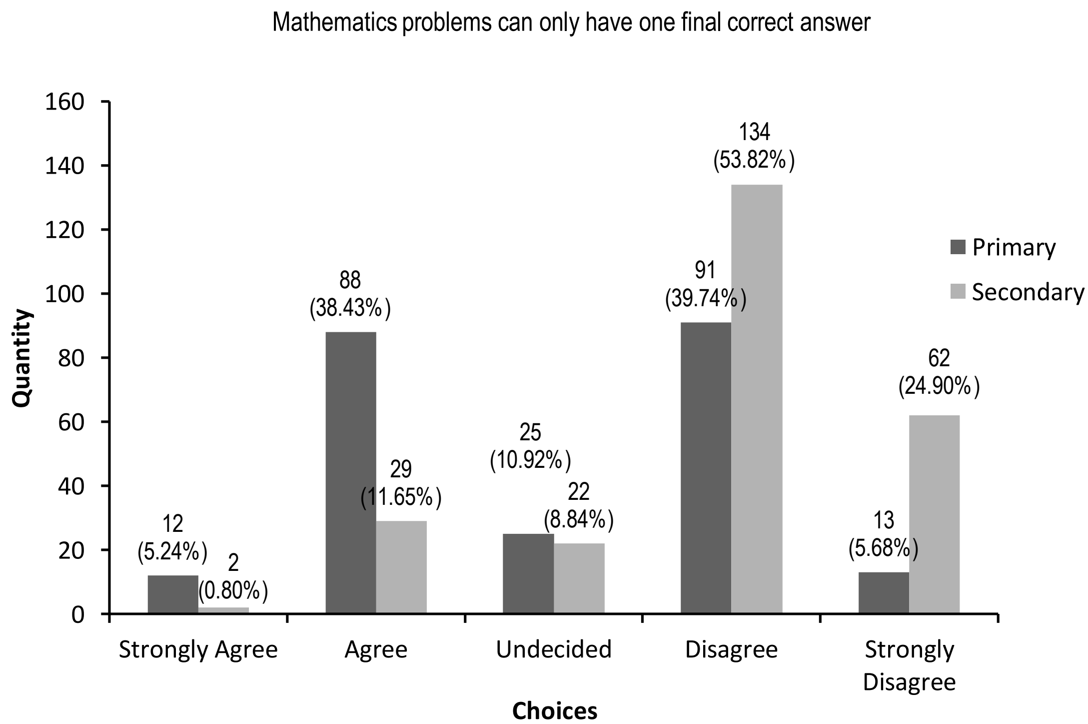


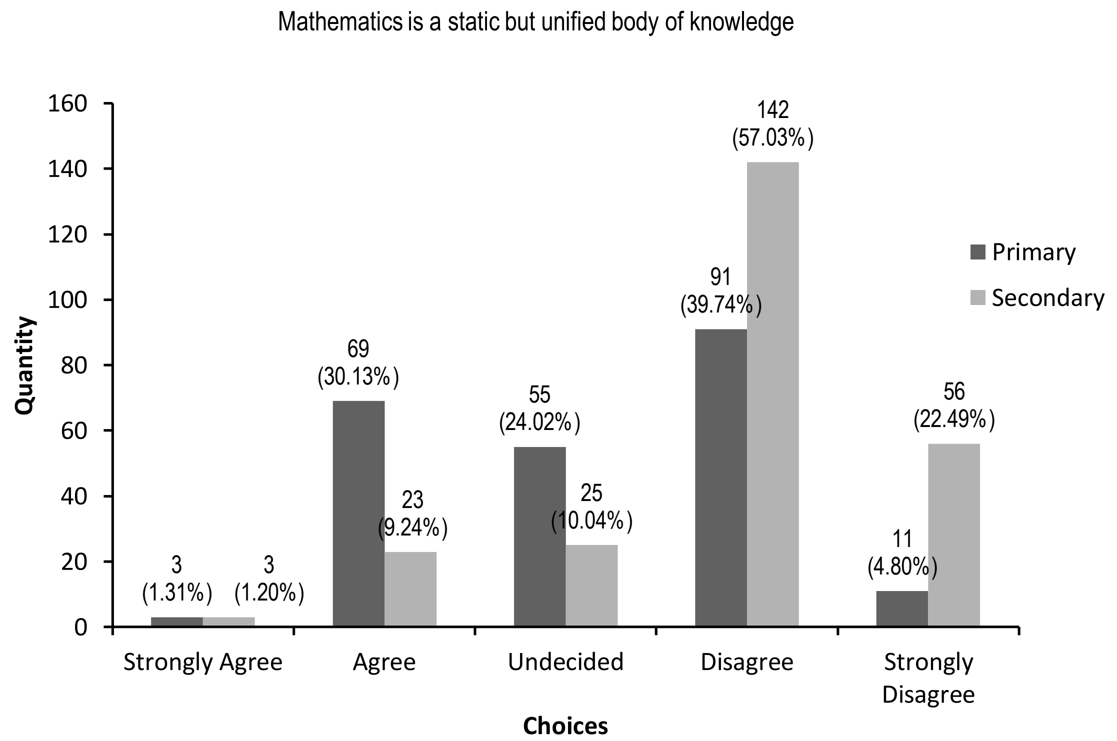
Figure 6.9 Bar chart of sector responses to Questionnaire Item 18



<u>Questionnaire Item 31 – 'Mathematics is a static but unified body of knowledge' (Figure 6.10)</u>

Proponents of this view postulate that mathematics is discovered by humans through investigations, is free from ambiguity and located within a hierarchical interconnected body of knowledge (Ernest, 1989a). The results indicate that less than half of primary teachers (45%) strongly disagreed or disagreed compared to fourth-fifths of secondary mathematics participants (80%). A possible explanation for this differential might be that primary teachers' beliefs align more with a Platonist philosophy compared with secondary mathematics teachers. Though, it is notable that almost a quarter (24%) of primary participants remained undecided about this item.

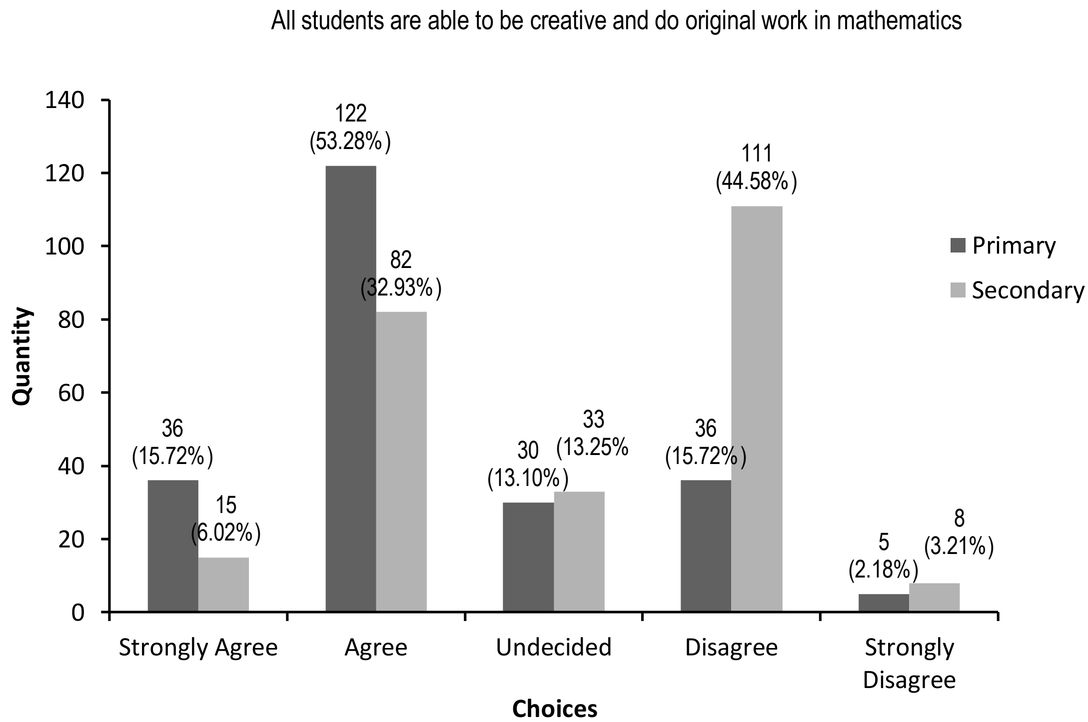
Figure 6.10 Bar chart of sector responses to Questionnaire Item 31



<u>Questionnaire Item 37 – 'All students are able to be creative and do original work in mathematics' (Figure 6.11)</u>

Creativity is considered essential for effective learning of mathematics regardless of age (Mann, 2006). Scottish practitioners have a duty to ensure that all pupils are supported in their development of creativity skills in ways which build on personal strengths whether as individuals or in group activities (Education Scotland, 2013). More than two-thirds of primary participants (69%) strongly agreed or agreed with this collaborative belief statement compared to less than two-fifths (39%) of secondary mathematics participants. This result suggests that primary teachers have a greater tendency to support pupils' ability to display flexible and divergent thinking along with generating unusual, novel or insightful answers or strategies. This may be explained by a belief system which aligns innately with a constructivist orientation. Perhaps the most surprisingly observation is the truncated provision of secondary participants, since paradoxically this has emerged when the level and sophistication of mathematical knowledge has unlocked centuries of beautiful and original work such as Pi, Apollonian circles, the golden ratio and Maclaurin series. It seems likely that many secondary mathematics teachers may not associate the learning of mathematics with creativity, possibly as a result of their own schooling since teachers tend to reproduce the kind of instruction they themselves received (Lortie, 1975; Pehkonen, 1997). Nevertheless, Lockhart (2009) argues that mathematics is an art, and should be taught by functioning artists, or if not, at least by individuals who appreciate the art form and can recognize it when they see it.

Figure 6.11 Bar chart of sector responses to Questionnaire Item 37



<u>Belief categories</u>

Independent samples t-tests were conducted for each of the five belief categories to compare the mean of the summation of the factor scores between primary and secondary mathematics participants. The results of this analysis are displayed in Table 6.13. From this data, we can see that statistically significant differences exist for two belief factors. Firstly, there was a significant difference in the mean belief scores of the social constructivist orientation to mathematics, mathematics learning and mathematics teaching (i.e. Belief Factor 1) between primary and secondary mathematics teachers. The effect size is deemed 'small' (Cohen, 1988). Secondly, there was a significant difference in the mean belief scores of the collaborative orientation to mathematics, mathematics learning and mathematics teaching between primary and secondary mathematics teachers (i.e. Belief Factor 5). However, in this case, the effect size is 'large' (Cohen, 1988).

Table 6.13 Results of independent samples t-tests by belief factors

Scale and Items	Primary		Secondary		Statistic			
	M	SD	M	SD	df	t	p	Cd
Belief Factor 1 (Items 1-7)	16.08	2.911	16.82	3.627	467.589	-2.484	**.013**	-.230
Belief Factor 2 (Items 8-17)	22.09	4.814	22.59	4.944	476	-1.134	.257	-.104
Belief Factor 3 (Items 18-26)	26.75	4.969	26.16	5.246	476	1.261	.208	.116
Belief Factor 4 (Items 27-34)	22.36	3.835	22.45	4.322	475.393	-.245	.807	-.022
Belief Factor 5 (Items 35-39)	10.44	2.347	12.61	3.240	451.793	-8.432	**.000**	-.793

Note: Cd is abbreviated for Cohen's d value (Cohen, 1988)

Having established that significant statistical differences exist between primary and secondary participants for BF1 and BF5, it was necessary to investigate the dynamics of both results in further detail. In order to explore more accurately where both sectors function within the continuum of any belief factor, I considered the concept of establishing descriptive parameters to produce such a judgement. On reviewing the uniformity of the questionnaire choices along with the overall distribution of responses for each item, I compiled a range of appropriate limits (Table 6.14).

The following analysis is offered using an additional variable (i.e. grade) to provide more data concerning the demographics of the participants.

<u>Belief Factor 1</u>

It can be shown from the line graph (Figure 6.12) that the mean of both sectors mathematical beliefs align with a weak social constructivist orientation towards the nature of mathematics, the learning of mathematics and the teaching mathematics. The most revealing aspect of the line graph is the variation between primary and secondary Headteachers, while both sectors share similar beliefs at the Deputy Headteacher grade.

Table 6.14 Belief factor response limits

BF		Agreement			Neutral	Disagreement		
		Strong	Moderate	Weak		Weak	Moderate	Strong
1	**Min**	**7.0000**	**12.6000**	**15.4000**	**18.2000**	**23.6000**	**26.5000**	**29.4000**
	Max	**12.5999**	**15.3999**	**18.1999**	**23.5999**	**26.4999**	**29.3999**	**35.0000**
2	Min	10.0000	18.0000	22.0000	26.0000	34.0000	38.0000	42.0000
	Max	17.9999	21.9999	25.9999	33.9999	37.9999	41.9999	50.0000
3	Min	9.0000	16.2000	19.8000	23.4000	30.6000	34.2000	37.8000
	Max	16.1999	19.7999	23.3999	30.5999	34.1999	37.7999	45.0000
4	Min	8.0000	14.4000	17.6000	20.8000	27.2000	30.4000	33.6000
	Max	14.3999	17.5999	20.7999	27.1999	30.3999	33.5999	40.0000
5	**Min**	**5.0000**	**9.0000**	**11.0000**	**13.0000**	**17.0000**	**19.0000**	**21.0000**
	Max	**8.9999**	**10.9999**	**12.9999**	**16.9999**	**18.9999**	**20.9999**	**25.0000**

Belief Factor 5

It can be shown from the line graph (Figure 6.13) that the mean of all grades of primary teachers' mathematical beliefs align with a moderate collaborative orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics. However, with the exception of Headteachers, the mean of all grades of secondary mathematics teachers' mathematical beliefs align with a weak collaborative orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics.

Figure 6.12 Line graph of mean Belief Factor 1 for sector and grade

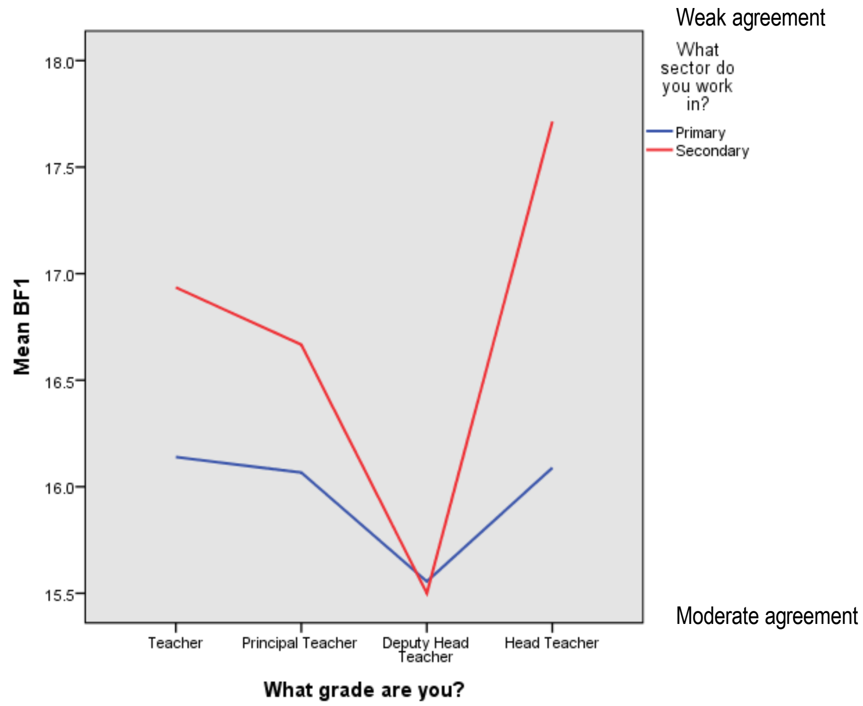
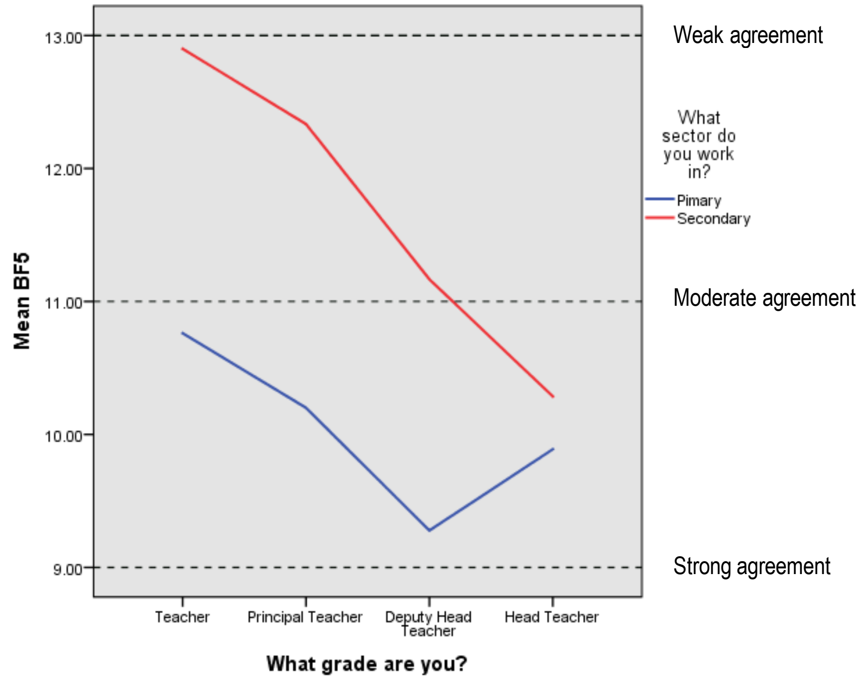


Figure 6.13 Line graph of mean Belief Factor 5 for sector and grade



<u>CfE and traditional mathematical beliefs</u>

An independent samples t-test was conducted for both CfE mathematical beliefs (i.e. BF1 + BF2 + BF5) and traditional mathematical beliefs (i.e. BF3 + BF4). On both occasions, the mean of the summation of the positive and negative scores respectively were compared between the primary and secondary participants. The results obtained from this analysis are displayed in Table 6.15. The most unexpected result to emerge was that there was a significant difference in the mean scores of CfE mathematical beliefs between primary and secondary mathematics teachers. The effect size is deemed ‘medium’ (Cohen, 1988). In other words, primary teachers hold statistically significantly stronger CfE mathematical beliefs than secondary mathematics teachers. However, the results did not find any statistical significant difference in the mean scores of traditional mathematical beliefs between the sectors.

Table 6.15 Results of independent samples t-test of CfE and traditional mathematical beliefs

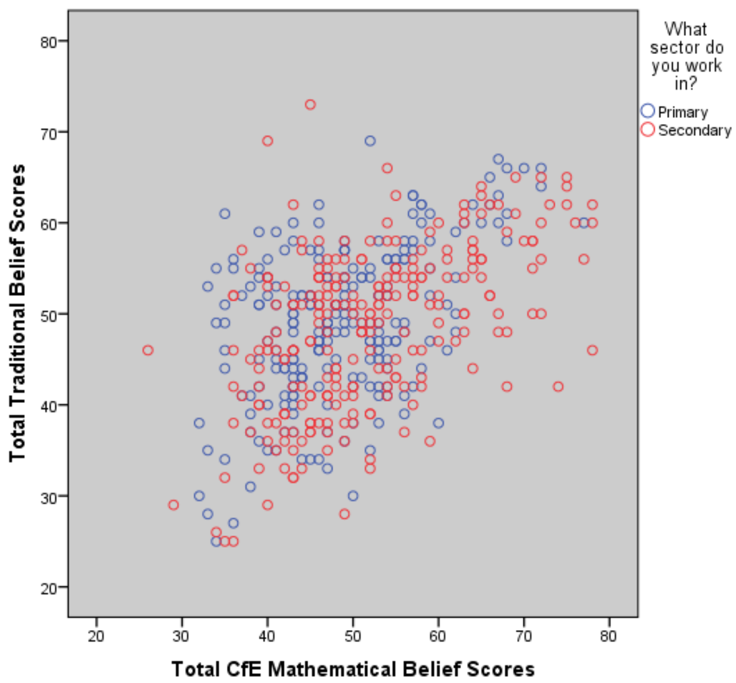
Scale and Items	Primary		Secondary		Statistic			
	M	SD	M	SD	df	t	p	Cd
Belief Factors 1, 2 & 5 (Items 1-17 & 35-39)	48.60	8.619	52.02	10.487	470.192	-3.909	**.000**	.633
Belief Factor 3 & 4 (Items 18-34)	49.11	8.928	48.61	8.872	476	.633	.527	-.361

Note: Cd is abbreviated for Cohen’s d value (Cohen, 1988)

6.3.5 Correlation between CfE and traditional beliefs

The relationship between CfE mathematical beliefs scores (i.e. total of the positive item scores) and traditional mathematical belief scores (i.e. total of the negative item scores) was investigated for both sectors, using Pearson product-moment correlation coefficient. A scatterplot (Figure 6.14) allows inspection of the graphical relationships. Preliminary analyses were performed to ensure no violation of the assumptions of normality, linearity and homoscedasticity. Prior to the interpretation of any correlation coefficient value, researchers need to be aware of four caveats such as causal relationships, Type I error, Type II error and effect size (Cohen, Manion & Morrison, 2011). Various authors recommend different techniques to translate the strength of linear relationships. Cohen (1988), as cited in Pallant (2013), suggests the following guidelines, which apply irrespective of sign: ‘small’ ($r = .10$ to $.29$), ‘medium’ ($r = .30$ to $.49$) and ‘large’ ($r = .50$ to 1.0). For the primary teachers, there was a modest positive correlation between the two variables, $r = .49$, $n = 229$, $p < .001$. For the secondary mathematics teachers, there was a strong positive correlation between the two variables, $r = .53$, $n = 249$, $p < .001$.

Figure 6.14 Scatterplot of sector CfE and traditional TMBS



However, since both correlation coefficients are similar, it was important to compare the strength of the correlation coefficients of the two groups. After satisfying appropriate assumptions, testing of this statistical significance produced an observed z value of -.362. Since, this value comfortably lies within the interval $-1.96 < z < 1.96$, the correlation coefficients for the sectors are deemed not statistically significantly different i.e. there is no difference between CfE mathematical beliefs scores and traditional mathematical belief scores for both sectors.

6.4 Analysis of findings relevant to research question three

The third research question investigated features contributing to teachers opinions about the nature of mathematics, the learning of mathematics and the teaching of mathematics, and posed,

> What factors impact on the mathematical beliefs of primary and secondary mathematics teachers?

In this section, five characteristics were explored i.e. gender, age, grade, length of teaching experience and highest level of qualification in the field of education. The objective was to determine if significant differences exist in the mean scores of total mathematical beliefs scores (continuous dependent variable) across five groups (categorical independent variables), where gender and sector are used as the main interaction effects. Two-way between groups analysis of variance (ANOVA) were used to compare the population means. One of the important advantages of this approach is that there are two sources of assignable causes of variation, and this helps to reduce the error variance thus making this design more efficient (Field, 2013).

6.4.1 Assumptions of ANOVA

The assumptions of ANOVA are analogous to the parametric assumptions presented earlier in this chapter. Nevertheless, the conditions for normality were reassessed given that various group sizes are involved. The results for skewness and kurtosis measurements are shown in Figure 6.16, where it can be observed that all values are comfortably positioned within acceptable guidelines (Field, 2013; Tabachnick & Fidell, 2014). Figure 6.17 displays an overview of normality tests with nine results suggesting violation of normality (i.e. $p < .05$). However, inspection of each corresponding sample size suggests that the population from which the samples are taken from are normally distributed. Pallant (2013, p. 214) asserts that: "With large enough sample sizes (e.g. 30+), the violation of this assumption [normality] should not cause any major problems". The condition of normal distribution is further corroborated for each of the five characteristics (i.e. gender, age, grade, length of teaching experience and highest level of qualification in the field of education) by inspection of each resultant histogram and Normal Q-Q plot of Total Mathematical Belief Scores (Appendix N).

Table 6.16 Skewness and Kurtosis values of TMBS for gender, age, grade, experience and highest level of qualification in the field of education

		Skewness			Kurtosis		
Group	Frequency	Statistic	Std. Error	z-score	Statistic	Std. Error	z-score
Gender							
Male	148	.244	.199	1.226	-.309	.396	-.322
Female	330	.253	.134	1.888	.186	.268	.694
Age (years)							
Under 25	31	.755	.421	1.793	-.127	.821	-.155
25 to 34	102	.051	.239	.213	-.088	.474	-.186
35 to 44	118	.530	.223	2.377	-.026	.442	-.059
45 to 54	140	.180	.205	.878	.419	.407	1.029
55 and over	57	.217	.258	.841	-.564	.511	1.104
Grade							
Teacher	321	.121	.136	.890	-.152	.271	.561
PT	75	.587	.277	2.119	.517	.548	.943
DHT	30	-.104	.427	-.244	-.469	.833	.563
HT	52	.308	.330	.933	-.307	.650	.472
Experience (years)							
5 and under	87	.515	.258	1.996	.574	.511	1.123
6 to 10	101	.216	.240	.900	-.070	.476	-.147
11 to 15	67	-.197	.293	-.672	.260	.578	.450
16 to 20	50	-.166	.337	-.493	.081	.662	.122
Over 20	173	.557	.185	3.011	-.297	.367	-.809
Qualification							
BEd	162	.507	.191	2.654	.006	.379	.016
PGCE/PGDE	279	.200	.147	1.361	.002	.293	.068
Masters	42	.086	.365	.236	-.286	.717	-.399

Table 6.17 Test of Normality

	Kolmogorov-Smirnov		
Group	Statistic	*df*	Sig.
Gender			
Male	.049	148	.200*
Female	.076	330	.000
Age (years)			
Under 25	.126	31	.200*
25 to 34	.037	102	.200*
35 to 44	.089	118	.022
45 to 54	.080	140	.030
55 and over	.084	87	.191
Grade			
Teacher	.056	321	.019
PT	.111	75	.023
DHT	.094	30	.200*
HT	.067	52	.200*
Experience (years)			
5 and under	.114	87	.007
6 to 10	.067	101	.200*
11 to 15	.085	67	.200*
16 to 20	.082	50	.200*
Over 20	.103	173	.000
Qualification			
BEd	.086	162	.005
PGCE/PGDE	.064	274	.009
Masters	.072	42	.200*

* This is a lower bound of the true significance

The results of Levene’s test of equality of error variances are set out in Table 6.18, where it can be observed that $p < .05$ for two measurements. Regarding this matter, Pallant (2013) advises:

> A significant result (Sig. value less than .05) suggests that the variance of your dependent variable across the groups is not equal. If you find this to be the case in our study, it is recommended that you use a more stringent significance level (e.g. .01) for evaluating the results of your two-way ANOVA (p. 279).

Table 6.18 Levene's Test of Equality of Error of Variances

	F	df1	df2	Sig.
Gender				
Age	1.901	9	468	.051
Grade	1.941	7	470	.062
Gender	1.082	9	468	.374
Experience	.947	5	472	.450
Sector				
Age	2.596	9	468	.006
Grade	2.591	7	470	.012
Gender	2.493	9	468	.090
Experience	2.028	5	472	.073

Consequently, in this study I have used a significant level of .01 for both sector and age and sector and grade respectively. Accordingly, I will consider the main effects and interaction effects significant only if the Sig. value is greater than .01.

In summary, all relevant assumptions have been inspected and satisfied.

6.4.2 Descriptive statistics of ANOVA

The mean and standard deviations of the participants total mathematical beliefs scores with respect to gender, sector, age, grade, experience and highest qualification in the field of education are summarised in Tables 6.19 to 6.26 respectively. The following interesting observations can be made:

- The group with the strongest mathematical beliefs are male teachers holding a Master's degree in the field of education.
- The overall group with the weakest mathematical beliefs are male teachers aged under 25 years.
- Female teachers have stronger mathematical beliefs in every age group and in each grade.
- The greatest disparity of mathematical beliefs occurs between primary and secondary mathematics teachers aged 55 and over.
- In terms of grade, primary Deputy Headteachers have the strongest mathematical beliefs. Unpromoted secondary mathematics teachers have the weakest mathematical beliefs.

- In terms of teaching experience, the group with the strongest mathematical beliefs are female teachers with 6-10 years of service. The group with the weakest mathematical beliefs are secondary mathematics teachers with 16-20 years of service.
- In terms of highest qualification in the field of education, the group with the weakest mathematical beliefs are male teachers with a Bachelor of Education degree. Primary teachers have stronger mathematical beliefs at every educational level.

Table 6.19 TMBS with respect to gender and age

	Male			Female			Total		
	M	SD	N	M	SD	N	M	SD	N
Under 25	105.00	7.92	9	103.18	17.90	22	103.71	15.55	31
25 to 34	98.84	16.42	31	96.28	15.94	71	97.06	16.05	102
35 to 44	101.87	16.77	39	100.06	13.59	79	100.66	14.67	118
45 to 55	98.78	16.92	37	96.77	14.79	103	97.30	15.35	140
Over 55	104.50	18.38	32	99.55	17.34	55	101.37	17.79	87
Total	101.22	16.69	148	98.34	15.48	330	99.23	15.91	478

Table 6.20 TMBS with respect to sector and age

	Primary			Secondary			Total		
	M	SD	N	M	SD	N	M	SD	N
Under 25	103.74	17.56	23	103.63	8.19	8	103.71	15.55	31
25 to 34	96.73	16.31	51	97.39	15.95	51	97.06	16.05	102
35 to 44	99.29	12.47	52	101.74	16.21	66	100.66	14.67	118
45 to 55	96.00	12.94	69	96.56	17.37	71	97.30	15.35	140
Over 55	96.18	15.50	34	104.70	18.49	53	101.37	17.79	87
Total	97.71	14.60	229	100.63	16.92	249	99.23	15.91	478

Table 6.21 TMBS with respect to gender and grade

	Male			Female			Total		
	M	SD	N	M	SD	N	M	SD	N
Teacher	102.20	16.90	101	100.10	16.44	220	100.76	16.59	321
PT	101.19	17.46	32	96.70	13.80	43	98.61	15.52	75
DHT	94.13	16.10	8	92.14	13.72	22	92.67	14.13	30
HT	95.43	7.93	7	94.33	11.03	45	94.48	10.61	52
Total	101.22	16.69	148	98.34	15.48	330	99.23	15.91	478

Table 6.22 TMBS with respect to sector and grade

	Primary			Secondary			Total		
	M	SD	N	M	SD	N	M	SD	N
Teacher	99.75	15.80	151	101.66	17.26	170	100.76	16.59	321
PT	95.00	8.58	15	99.52	16.75	60	98.61	15.52	75
DHT	90.44	13.96	18	96.00	14.32	12	92.67	14.13	30
HT	94.67	10.37	45	93.29	12.89	7	94.48	10.61	52
Total	97.71	14.60	229	100.63	16.92	249	99.23	15.91	478

Table 6.23 TMBS with respect to gender and experience

	Male			Female			Total		
	M	SD	N	M	SD	N	M	SD	N
5 or under	98.33	14.24	24	99.68	15.95	63	99.31	15.42	87
6 to 10	102.89	18.40	37	94.84	15.37	64	97.79	16.91	101
11 to 15	97.96	15.25	25	99.29	17.90	42	98.79	16.85	67
16 to 20	99.54	18.22	13	100.38	13.87	37	100.16	14.92	50
Over 20	103.49	16.69	49	98.54	14.84	124	99.94	15.56	173
Total	101.22	16.69	148	98.34	15.48	330	99.23	15.91	478

Table 6.24 TMBS with respect to sector and experience

	Primary			Secondary			Total		
	M	SD	N	M	SD	N	M	SD	N
5 or under	99.85	16.77	59	98.18	12.33	28	99.31	15.42	87
6 to 10	95.37	14.61	38	99.25	18.11	63	97.79	16.91	101
11 to 15	99.73	15.41	22	98.33	17.66	45	98.79	16.85	67
16 to 20	96.96	13.72	26	103.62	15.67	24	100.16	14.92	50
Over 20	96.98	13.03	84	102.74	17.22	89	99.94	15.56	173
Total	97.71	14.60	229	100.63	16.92	249	99.23	15.91	478

Table 6.25 TMBS with respect to gender and highest qualification in the field of education

	Male			Female			Total		
	M	SD	N	M	SD	N	M	SD	N
BEd	103.20	14.93	30	99.41	15.15	132	100.11	15.14	162
PGCE/PGDE	102.73	16.90	103	99.09	15.20	171	100.46	15.93	294
Masters	86.93	11.71	15	88.41	15.98	27	87.88	14.47	42
Total	101.22	16.69	148	98.34	15.48	330	99.23	15.91	478

Table 6.26 TMBS with respect to sector and highest qualification in the field of education

	Primary			Secondary			Total		
	M	SD	N	M	SD	N	M	SD	N
BEd	99.47	14.81	130	102.72	16.38	32	100.11	15.14	162
PGCE/PGDE	96.80	13.42	84	102.07	16.70	190	100.46	15.93	274
Masters	87.60	15.50	15	88.04	14.17	27	87.88	14.47	42
Total	97.71	14.60	229	100.63	16.92	249	99.23	15.91	478

6.4.3 Inferential statistics of ANOVA

A two-way between-groups of analysis of variance was conducted to explore the impact of gender, sector, age, grade, experience and highest level of qualification in the field of education on participants total mathematical beliefs scores. Included in the statistical hypothesis test measurements is power which is reported as a value between 0 and 1 (e.g. 0.997 relates to 99.7%). Cohen (1998, p. 4) asserts that: “The power of a statistical test of a null hypothesis is the probability that it will lead to the rejection of the null hypothesis i.e., the probability that it will result in the conclusion that the phenomenon exists”. The effect size statistic employed was partial eta squared which indicates the proportion of variance of the dependent variable that is explained by the independent variable. Guidelines for interpretation is based on Kirk (1996) who classifies .010 as ‘small’, .059 as ‘medium’ and .138 as ‘large’ respectively. I will now comment on each of the eight individual results, as follows:

1. <u>Exploration of gender and age</u>

Participants were divided into five groups, according to their age (Group 1: Under 25 years; Group 2: 25 to 34 years; Group 3: 35 to 44 years; Group 4: 45 to 54 years; Group 5: 55 years and over). As presented in Table 6.27, the interaction effect between gender and age group was not statistically significant, [F (4, 468) = 0.141, p = .967]. In other words, the total mathematical beliefs scores of male and female teachers are not moderated by age.

Table 6.27 Two-Way ANOVA between gender and age

Source	Type III Sum of Squares	df	Mean Square	*F*	Sig.	Partial Eta Squared	Observed Power *
Gender	528.614	1	528.614	2.105	.148	.004	0.305
Age	1891.873	4	472.968	1.883	.112	.016	0.570
Gender × Age	141.449	4	35.362	0.141	.967	.001	0.079
Error	117547.189	468	251.169				
Total	4827746.000	478					
Corrected Total	120665.757	477					

* Computed using alpha = .05

2. Exploration of sector and age

Participants were divided into five groups, according to their age (Group 1: Under 25 years; Group 2: 25 to 34 years; Group 3: 35 to 44 years; Group 4: 45 to 54 years; Group 5: 55 years and over). As presented in Table 6.28, the interaction effect between sector and age group was not statistically significant, [F (4, 468) = 0.847, p = .496]. In other words, primary and secondary mathematics teachers' total mathematical beliefs scores are not moderated by age.

Table 6.28 Two-Way ANOVA between sector and age

Source	Type III Sum of Squares	df	Mean Square	*F*	Sig.	Partial Eta Squared	Observed Power *
Sector	622.610	1	622.610	2.502	.114	.005	0.352
Age	1754.231	4	438.558	1.762	.135	.015	0.538
Sector × Age	843.001	4	210.750	0.847	.496	.007	0.271
Error	116481.494	468	248.892				
Total	4827746.000	478					
Corrected Total	120665.757	477					

* Computed using alpha = .05

3. Exploration of gender and grade

Participants were divided into four groups, according to their grade (Group 1: Teacher; Group 2: Principal Teacher; Group 3: Deputy Headteacher; Group 4: Headteacher). As presented in Table 6.29, the interaction effect between gender and grade was not statistically significant, [F (3, 470) = 0.131, p = .942]. There was a statistically significant main effect for grade, [F (3, 470) = 2.935, p = .033], as can be visually inspected by the line graph in Figure 6.15. However, the effect size was small (partial eta squared = .001).

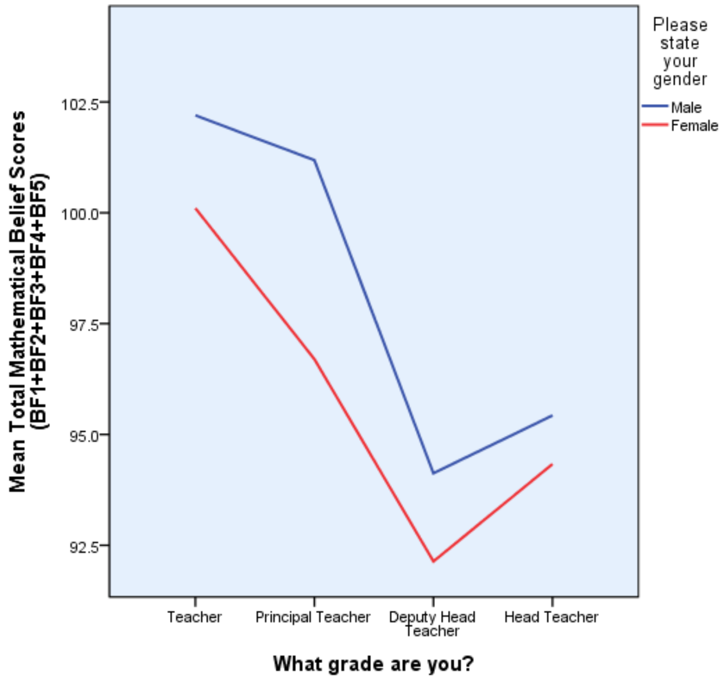
Post-hoc comparisons using the Tukey HSD test indicated than the mean score of the Teacher group (M = 100.76, SD = 16.59) was significantly different from the Deputy Headteacher group (M = 92.67, SD = 14.13) and from the Headteacher group (M = 94.48, SD = 10.61). The Principal Teacher group (M = 98.81, SD = 15.52) did not differ significantly from either of the other groups. The main effect for gender [F (1, 470) = 0.930, p = .335], did not reach statistical significance.

Table 6.29 Two-Way ANOVA between gender and grade

Source	Type III Sum of Squares	df	Mean Square	*F*	Sig.	Partial Eta Squared	Observed Power *
Gender	231.044	1	231.044	0.930	.335	.002	0.161
Grade	2186.466	3	728.822	2.935	**.033**	.018	0.697
Gender × Grade	97.470	3	32.490	0.131	.942	.001	0.074
Error	116713.760	470	248.327				
Total	4827746.000	478					
Corrected Total	120665.757	477					

* Computed using alpha = .05

Figure 6.15 Line graph of TMBS for gender and grade



4. <u>Exploration of sector and grade</u>

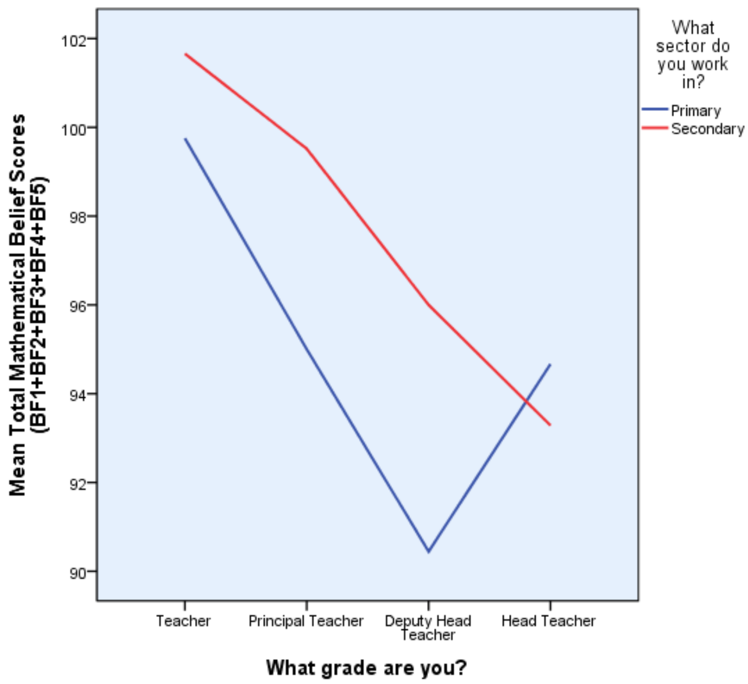
Participants were divided into four groups, according to their grade (Group 1: Teacher; Group 2: Principal Teacher; Group 3: Deputy Headteacher; Group 4: Headteacher). As presented in Table 6.30, the interaction effect between sector and grade was not statistically significant, [*F* (3, 470) = 0.309, *p* = .819]. There was a statistically significant main effect for grade, [*F* (3, 470) = 3.463, *p* = .016], as can be visually inspected by the line graph in Figure 6.16. However, the effect size was small (partial eta squared = .022). Post-hoc comparisons using the Tukey HSD test indicated than the mean score of the Teacher group (*M* = 100.76, *SD* = 16.59) was significantly different from the Deputy Headteacher group (*M* = 92.67, *SD* = 14.13) and from the Headteacher group (*M* = 94.48, *SD* = 10.61). The Principal Teacher group (M = 98.81, SD = 15.52) did not differ significantly from either of the other groups. The main effect for sector [F (1, 470) = 1.131, p = .288], did not reach statistical significance.

Table 6.30 Two-Way ANOVA between sector and grade

Source	Type III Sum of Squares	df	Mean Square	*F*	Sig.	Partial Eta Squared	Observed Power *
Sector	280.778	1	280.778	1.131	.288	.002	0.186
Grade	2578.384	3	859.461	3.463	**.016**	.022	0.775
Sector × Grade	229.754	3	76.585	0.309	.819	.002	0.110
Error	116649.002	470	248.189				
Total	4827746.000	478					
Corrected Total	120665.757	477					

* Computed using alpha = .05

Figure 6.16 Line graph of TMBS for sector and grade



5. Exploration of gender and experience

Participants were divided into five groups, according to their length of teaching experience (Group 1: 5 years and under; Group 2: 6 to 10 years; Group 3: 11 to 15 years; Group 4: 16 to 20 years; Group 5: Over 20 years). As presented in Table 6.31, the interaction effect between gender and experience was not statistically significant, [F (4, 468) = 1.483, p = .206]. In other words, the total mathematical beliefs scores of male and female teachers are not moderated by experience.

Table 6.31 Two-Way ANOVA between gender and experience

Source	Type III Sum of Squares	df	Mean Square	*F*	Sig.	Partial Eta Squared	Observed Power *
Gender	303.360	1	303.360	1.205	.273	.003	0.195
Experience	416.493	4	104.123	0.413	.799	.004	0.147
Gender × Exp.	1494.370	4	373.592	1.483	.206	.013	0.461
Error	117867.497	468	251.854				
Total	4827746.000	478					
Corrected Total	120665.757	477					

* Computed using alpha = .05

6. <u>Exploration of sector and experience</u>

Participants were divided into five groups, according to their length of teaching experience (Group 1: 5 years and under; Group 2: 6 to 10 years; Group 3: 11 to 15 years; Group 4: 16 to 20 years; Group 5: Over 20 years). As presented in Table 6.32, the interaction effect between gender and experience was not statistically significant, [F (4, 468) = 1.197, p = .311]. In other words, the total mathematical beliefs scores of primary and secondary mathematics teachers are not moderated by experience.

Table 6.32 Two-Way ANOVA comparing sector and experience

Source	Type III Sum of Squares	df	Mean Square	*F*	Sig.	Partial Eta Squared	Observed Power *
Sector	660.667	1	660.667	2.623	.106	.006	0.366
Experience	474.632	4	118.658	0.471	.757	.004	0.162
Sector × Exp.	1206.341	4	301.585	1.197	.311	.010	0.377
Error	117882.472	468	251.886				
Total	4827746.000	478					
Corrected Total	120665.757	477					

* Computed using alpha = .05

7. <u>Exploration of gender and highest level of qualification in the field of education</u>

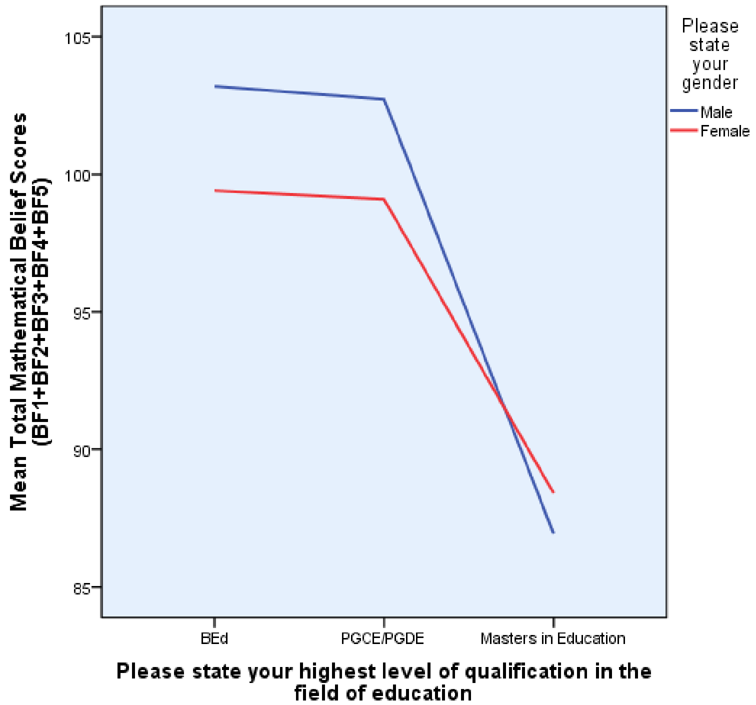
Participants were initially divided into four groups, according to their qualification (Group 1: BEd; Group 2: PCGE/PGDE; Group 3: Masters; Group 4: Doctorate. However, this was reduced to three groups as no entries were received for Group 4). As presented in Table 6.33, the interaction effect between gender and qualification was not statistically significant, [F (2, 472) = 0.482, p = .618]. There was a statistically significant main effect for qualification, [F (3, 470) = 2.935, p = .033], as can be visually inspected by the line graph in Figure 6.17. However, the effect size was small (partial eta squared = .052). Post-hoc comparisons using the Tukey HSD test indicated than the mean score of the Teacher group (*M* = 100.76, *SD* = 16.59) was significantly different from the Deputy Headteacher group (*M* = 92.67, *SD* = 14.13) and from the Headteacher group (*M* = 94.48, *SD* = 10.61). The Principal Teacher group (*M* = 98.81, *SD* = 15.52) did not differ significantly from either of the other groups. The main effect for gender [*F* (1, 470) = 0.930, *p* = .335], did not reach statistical significance.

Table 6.33 Two-Way ANOVA between gender and highest level of qualification in the field of education

Source	Type III Sum of Squares	df	Mean Square	*F*	Sig.	Partial Eta Squared	Observed Power *
Gender	221.572	1	221.572	0.921	.338	.002	0.160
Qualification	6208.626	1	3104.313	12.910	**.000**	.052	0.997
Gender × Qual.	231.618	2	115.809	0.482	.618	.002	0.129
Error	113494.234	472					
Total	4827746.000	478					
Corrected Total	120665.757	477					

* Computed using alpha = .05

Figure 6.17 Line graph of TMBS for gender and highest qualification in the field of education



8. Exploration of sector and highest level of qualification in the field of education

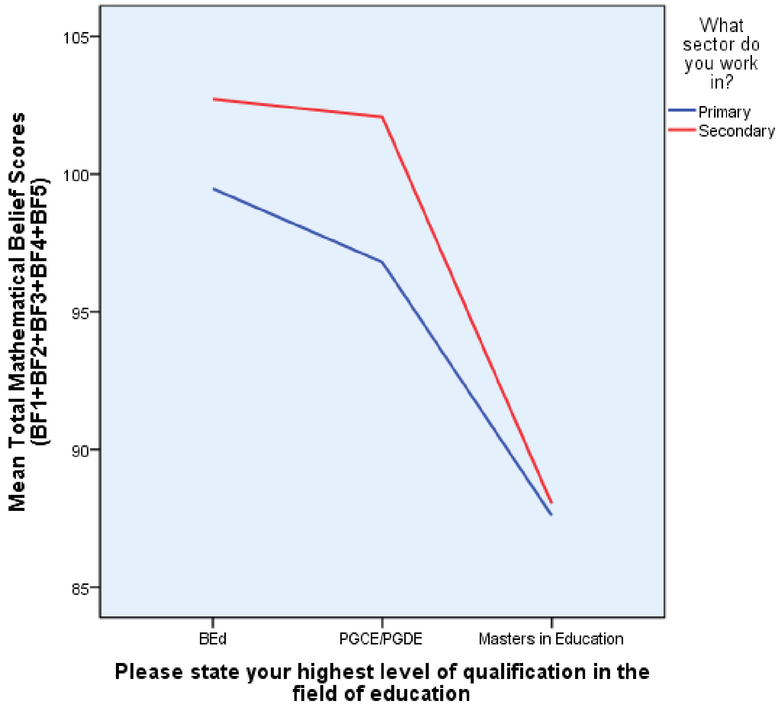
Participants were initially divided into four groups, according to their qualification (Group 1: BEd; Group 2: PCGE/PGDE; Group 3: Masters; Group 4: Doctorate. However, this was reduced to three groups as no entries were received for Group 4). As presented in Table 6.34, the interaction effect between sector and qualification was not statistically significant, [*F* (2, 472) = 0.482, *p* = .618]. There was a statistically significant main effect for qualification, [*F* (3, 470) = 2.935, *p* = .033], as can be visually inspected by the line graph in Figure 6.18. However, the effect size was small (partial eta squared = .052). Post-hoc comparisons using the Tukey HSD test indicated than the mean score of the Teacher group (*M* = 100.76, *SD* = 16.59) was significantly different from the Deputy Headteacher group (*M* = 92.67, *SD* = 14.13) and from the Headteacher group (*M* = 94.48, *SD* = 10.61). The Principal Teacher group (*M* = 98.81, *SD* = 15.52) did not differ significantly from either of the other groups. The main effect for sector [*F* (1, 470) = 0.930, *p* = .335], did not reach statistical significance.

Table 6.34 Two-Way ANOVA comparing sector and highest level of qualification in the field of education

Source	Type III Sum of Squares	df	Mean Square	*F*	Sig.	Partial Eta Squared	Observed Power *
Sector	502.638	1	502.638	2.103	.148	.004	0.304
Qualification	5281.875	2	2640.937	11.048	**.000**	.045	0.991
Sector × Qual.	227.145	2	113.572	0.475	.622	.002	0.128
Error	112823.937	472	239.034				
Total	4827746.000	478					
Corrected Total	120665.757	477					

* Computed using alpha = .05

Figure 6.18 Line graph of TMBS for sector and highest level of qualification in the field of education



6.5 Volunteer comments received

The final part of the questionnaire included an optional feature which generated a significant number of comments and provided a wide range of interesting viewpoints, which enriched the data collection method for this instrument. With the exception of a solitary contribution, 87 participants (18% of main study) submitted a coalesced text of over 6000 words relating to various aspects of the belief statements. In order to make sense of the narrative, I divided the comments into two categories based on sector and identified various perspectives. Several common and distinctive themes emerged which helped to illuminate some of the quantitative results.

6.5.1 Primary teachers

Primary teachers provided 42 (48%) of the contributions, from which five broad themes emerged from the analysis. The following comments were tendered by participants with a minimum of six to ten years teaching experience including two promoted staff, one of which holds a Masters level qualification in the field of education:

1. Time constraints

A number of participants offered support for mathematical problem solving but alluded that opportunities to implement in practice were limited due to workload demands and other variables. It appears that some participants consider problem solving to be separate from mathematics. Sivunen & Pehkonen (2009) reported a similar finding, inferring that ‘time’ was manipulated as an obstacle for implementing problem solving. The comments below illustrate two cases:

> The idea of learning through problem solving sounds fantastic in an ideal universe. However, this is nearly impossible to do in a busy classroom where the basic foundations of mathematics have to be put in place in order to give the pupils the tools to become problem solvers. In secondary school our whole exam system gets in the way of learning through problem solving. Where would the time come from in an already overcrowded curriculum?
>
> [Teacher A, Unpromoted, BEd, 11 to 15 years’ experience - December 2014]
>
> ... Many pupils have only a surface understanding of various topics and are unable to solve challenging problems without a lot of guidance - But with so many other curriculum subjects to deliver, I feel that as a primary teacher, I do not have enough quality time to spend on problem solving.
>
> [Teacher B, Unpromoted, BEd, 16 to 20 years’ experience - December 2014]

2. Teaching mathematics through problem solving

A variety of perspectives were expressed. Some participants reported that this teaching approach is an effective method of mathematical instruction but only possible to implement after pupils had acquired a solid base of mathematical knowledge. For example, one participant wrote, “in practice, it is not achievable to do until P7 as pupils had not yet developed any real algebraic skills”. There were some negative comments about the conceptualisation of teaching mathematics through problem solving, emphasising awareness for practitioners to incorporate various pedagogical approaches to accommodate the needs of all learners. One individual stated that:

> You cannot assume that there is only one effective way to teach maths as all children are different; they learn in different ways and require different styles and approaches to teaching. It doesn't mean that someone who is able to solve problems quickly is more of a mathematician than someone who can't ... it only means that they can transfer and apply their skills in that area.
>
> [Teacher C, Deputy Headteacher, Masters, Over 20 years’ experience - December 2014]

And another commented,

> I disagreed with the question – “Teaching mathematics through problem solving is the best method to help students learn.” The implication of best way to teach problem solving assumes that all children are identical. For example, some children thrive when doing problem solving

> in groups whilst others are completely switched off ... Are we not supposed to provide a range of teaching approaches when doing maths?
>
> [Teacher D, Unpromoted, PGCE/PGDE, 6 to 10 years' experience - January 2015]

3. Mathematics teaching self-efficacy

Some participants indicated they were incapable of operationalising the teaching of mathematical problem solving since in their view, they did not consider their individual mastery of the subject of mathematics to be of an acceptable standard. Another participant stated that in addition to "low confidence in solving problems", regularly experiences "a feeling of dread when manipulating numbers in everyday contexts but more so when under pressure", suggesting an association with mathematical anxiety. According to Richardson & Suinn (1972, p. 551), "mathematics anxiety involves feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations". If teachers who are anxious about mathematics are charged with delivering its content, their anxieties could have consequences for learners' mathematical achievement (Beilock, et al., 2010). In one case, a participant demonstrated self-evaluation and tentatively enquired about the availability of professional support to deepen their pedagogical knowledge:

> Coming from an Arts background, I have to admit that I do not really feel confident in teaching maths. It has always been a weak subject for me and all through school, I have never been able to solve problems and failed Int 2 in S4 and Higher in S6. I was OK at following some things but found it difficult to put 'everything together'... There is no doubt that maths is a key subject within the CfE and I would like to know how to go about teaching problem solving properly but would need a lot of training! Is there any such help available?
>
> [Teacher E, Unpromoted, PGCE/PGDE, 6 to 10 years' experience - December 2014]

4. Multiple solutions

A small number of participants were able to identify with this strategy as a valuable method of enhancing deeper conceptual understanding. However, one participant suggested a complete lack of awareness of what constitutes a mathematical problem, inferring that it is possible to teach pupils a procedure that can be repeated and later applied as an algorithm to every problem:

> When demonstrating problem solving, I regularly have issues with under confident children who say they just want one way that works for them ... Their thinking is not mature enough and constantly have to revisit stuff and trying alternative ways to solve problems undermines their confidence. In a class with a wide range of learning styles it is important to remember this small but significant group.
>
> [Teacher F, Principal Teacher, PGCE/PGDE, 11 to 15 years' experience - January 2015]

5. Pupil ability

A few participants felt that the nature of problem solving is complex and should exclude less mathematical able pupils, including those with poor literacy skills, unless additional help is provided. According to one teacher:

> In my experience, only more able pupils can solve problems. Less able pupils tend to become very frustrated and easily lose focus, unless they are given a lot of additional support which takes time with no guaranteed success.
>
> [Teacher G, Unpromoted, BEd, 16 to 20 years' experience - December 2014]

It was also suggested that the nature of classroom interactions during the implementation of problem solving is heavily influenced by the teacher but that successful problem solving can only occur when a class contains a nucleus of enthusiastic learners with a strong desire to collaborate with others. As one participant put it:

> I think teachers can make a big difference to what goes on throughout PS [problem solving] but what really works is having a few good children that are very keen and can help the others to keep going when stuck... My P6 have really struggled as a group in maths this year after the top two moved away.
>
> [Teacher H, Unpromoted, BEd, Over 20 years' experience – January 2015]

6.5.2 Secondary mathematics teachers

Secondary mathematics teachers provided 45 (52%) of the contributions, from which seven broad categories emerged from the analysis. The following comments were tendered by participants with a minimum of one year teaching experience including three promoted staff, one of which holds a Masters level qualification in the field of education:

1. National qualifications

A number of participants reported functioning within a 'results driven culture' in which their professional practice is inextricably linked to the enactment of traditional teaching methods i.e. emphasise on procedural fluency opposed to the development of critical thinking skills. Some participants referred to a deterioration of professional autonomy. For example, one participant stated:

> My mathematical beliefs help me to establish a basis for classroom teaching and given the freedom, I would happily fill every day with problem solving, problem posing, investigations, challenges, proofs, history of mathematics, etc... However, like everyone else, I am controlled by the intense pressure of exam targets which dictate what particular teaching methods can be used limiting the experiences of my pupils.
>
> [Teacher I, Principal Teacher, PCGE/PGDE, 16 to 20 years' experience - December 2014]

Another participant questioned the legitimacy of teachers' beliefs,

> Our first priority is to get students their qualifications as without these, their maths career comes to an end... Not getting their qualifications also damages confidence - Does it really matter what teachers believe as long as their pupils pass exams?
>
> [Teacher J, Unpromoted, PCGE/PGDE, Over 20 years' experience - December 2014]

2. Time constraints

Several participants expressed workload concerns regarding a lack of class time to allocate to mathematical problem solving, attributing liability to the administration of national assessments. The comment below illustrates a typical viewpoint:

> It would be great to approach the teaching of maths by problem solving but in many ways this is totally unrealistic in terms of time, when you are trying to get pupils through national exams, especially Nat.4/5 which drains a teacher's already limited time, especially when having to cater for absent pupils and organising re-assessments.
>
> [Teacher K, Unpromoted, PGCE/PGDE, 6 to 10 years' experience - December 2014]

While a different participant hinted towards a common conundrum faced by all teachers:

> We are continually faced with the choice of teaching for understanding or pass the test - Problem solving and posing questions to challenge thinking would be fantastic but the restriction of time when juggling so many other daily matters (e.g. behaviour) is a major issue.
>
> [Teacher L, Unpromoted, BEd, 5 or under years' experience - January 2015]

3. Teaching mathematics through problem solving

A number of participants expressed disapproval or rejection of this powerful mechanism for promoting conceptual understanding. Borko & Putnam (1996, p. 684) argue that teachers' views about education may serve as impediments to change and point out that "experienced teachers' attempts to learn in new ways also are highly influenced by what they already know and believe about teaching, learning, and learners". In one case, a teacher appeared to be unaware that a problem solving approach is rooted within the literature and characterised as a curricula objective:

> I have taught maths for over 40 years in a number of different schools and have yet to meet someone who teaches through problem solving. I imagine this would take a lot of preparation time and expertise, both something of which I don't readily have... I would also speculate that many teachers would be extremely anxious if this method was to be introduced!
>
> [Teacher M, Unpromoted, PGCE/PGDE, Over 40 years' experience - December 2014]

Another participant referred to the input of controversial New Zealand academic John Hattie and his ground breaking collection of evidence based research into learning in schools. This comment is the solitary contribution from both sectors that included explicit reference to the literature:

> John Hattie's findings show us that discovery teaching is a very ineffective way of learning, whereas direct teaching is shown to be the most effective. Problem solving cannot take place until a pupil has the range of mathematical skills required to solve the problem. If a range of skills and understandings are to be taught through posing problems this has major implications for planning LI and SC, and can overwhelm student and teacher. It can be very difficult to get and give effective feedback on where a pupils problems lie, is it the maths skills and understandings, is it the interpretation of the problem, or is it the modelling of the solution? I have read no literature or had any experience in the classroom that would suggest that problem solving (as in a question that requires an extended response across a range of mathematical areas or in an unfamiliar context) is the best way in which pupils' learn. Any question on a discrete area can be posed as a problem to pupils if their understanding or skills need to be developed in a particular area. The expectation in the response from the teacher in terms of showing understanding is the most important aspect.
>
> [Teacher N, Principal Teacher, PGCE/PGDE, Over 20 years' experience - December 2014]

This previous comment is grounded on a claim by Hattie & Yates (2014) who used meta-analysis to compare different learning strategies and concluded that, based on effect size, direct instruction is effective and discovery learning is not. Notwithstanding validity concerns when applying meta-analysis, the authors claim is seriously misleading as both techniques are pedagogical strategies that are utilised for divergent purposes. Besides, neither of these strategies should be exclusively employed as the sole means of instruction for teaching mathematics within a constructivist learning environment. Particularly notable is the relevance of the curricula argument regarding planning and evaluation of learning intentions and success criteria given that problem solving is infused into all aspects of mathematics learning. Moreover, Hattie & Yates (2014) advocate teaching problem solving as one of the top teaching practices having the biggest effect size.

4. <u>Primaries</u>

A common view amongst some participants included concerns with the recent level of mathematical competence of primary pupils entering secondary. In particular, a few teachers specifically criticised the quality of problem solving skills. One individual stated that:

> I believe that the primary curriculum to too diverse and not enough emphasis is given to basic numerical skills; we have observed deterioration in the standard of numeracy in the last few years e.g. some pupils do not know their times tables when starting first year of high school and problem solving skills are virtually non-existent.
>
> [Teacher O, Principal Teacher, Masters, 11 to 15 years' experience - January 2015]

This view was echoed by another participant who noted:

> ... we depend on the primaries to provide us with pupils with a basic set of mathematical skills but in recent years the quality of these skills has reduced with the very few being able to problem solve for themselves. Another thing I have noticed is that these pupils give up so easy and demand the answer instead of taking the time to work it out. They lack elementary problem solving strategies and perseverance.
>
> [Teacher P, Unpromoted, PGCE/PGDE, Over 20 years' experience - January 2015]

5. Pupil ability

Some participants questioned the necessity to engage all learners in problem solving activities, highlighting concerns with establishing inquiry oriented classrooms due to issues with pupils not participating in constructive dialogue and debate. One participant reported that:

> This questionnaire assumes the existence of a standard learner... They are anything but standard. Some are active problem solvers; some wouldn't have the sense to get in out of the rain. Some will throw themselves into your problem scenarios and some will stare at the paper for ten minutes before saying 'I don't get it' and hitting their neighbour with a rolled up jotter.
>
> [Teacher Q, Unpromoted, PGCE/PGDE, Over 20 years' experience - January 2015]

6. Mathematical problem posing

A small number of participants were particularly critical of the validity of engaging all ability levels with this important cognitive process. For example, one participant alluded to the notion that problem posing is academically elitist and should not be encouraged:

> While conjecturing, justifying, reasoning, proving, disproving, searching for patterns and tackling problems are an essential part of early mathematics training, posing your own problems is only suited for the more advanced pupil ... this is something that I don't really promote as it would exclude the vast majority of children.
>
> [Teacher R, Principal Teacher, PGCE/PGDE, 16 to 20 years' experience - December 2014]

Another participant referred to the sum of the angles in a triangle, suggesting a potential pitfall of collaborative problem posing:

> Problem solving and problem posing are fantastic tools for learning for capable pupils who are quick to grasp new topics and ideas. Problem posing for those who can't answer a basic times table question with any degree of consistency is the opposite of an effective approach... I've seen a pupil invent a series of problems regarding angles in a triangle where the two given angles always added to more than 360 degrees (let alone 180 degrees). I won't deny that it taught me something about their understanding, but it taught him and the poor boy he invented the problem for nothing at all.
>
> [Teacher S, Unpromoted, PGCE/PGDE, 16 to 20 years' experience - January 2015]

7. Solving mathematical problems quickly

A few participants suggested that in order to demonstrate mathematical competence, pupils must be able to solve problems within a very short time frame. Such views overlook the reality that engaging pupils in mathematical thinking demands awareness that some problems require a period of time and perseverance to solve. As one teacher recorded:

> The question – 'To be good at mathematics you must be able to solve problems quickly' – no indication of time is specified here but I understood this to be within minutes – Solving problems quickly is an obvious measure of strong mathematical ability.
>
> [Teacher T, Unpromoted, PGCE/PGDE, 11 to 15 years' experience - December 2014]

In general, the aforementioned comments appear to illustrate tension in teachers' beliefs about their existing role and how they perceive mathematical problem solving. A common theme permeating from both sectors is bureaucracy, which can suffocate attempts to innovate and implement intrinsic curricula objectives. For instance, this is encapsulated by 'Teacher I' who concomitantly alludes to a trend of de-professionalization by being "controlled by the intense pressure of exam targets". Such accountability suggests that what is taught in many schools is based on what is assessed nationally and therefore inevitably leads to a narrowing of the mathematics curriculum, to which Boaler (2015b, p. 2) symbolises as a "strange mutated version of the subject".

However, an underlying issue worthy of mention is in relation to the professional responsibility of teachers towards children and young people. Prevailing within the contributions was a repeated use of terms like 'able', 'advanced' or 'poor' to describe their pupils. Biesta, Priestley & Robinson (2015) reported a similar finding in their ethnographic research of Scottish teachers' beliefs. Such language suggests that many practitioners identify mathematical ability as a fundamental prerequisite for teaching problem solving and, arguably, justification for rebuffing specific individuals in favour of a localised selective engagement policy. Assuming this to be true, it raises serious concerns regarding issues of equality and equity, as the existing politically charged arena and societal impetus is to encourage every teacher to ensure that all learners are afforded similar opportunities and given the same chance of success to develop as fully rounded citizens. Moreover, it undermines the ideology that all primary and secondary mathematics teachers have a duty to expose their pupils to problems much more than to facts (Halmos, 1980).

In summary, inconsistencies appear to exist between the mathematical beliefs of both sectors including deviations from the mathematical philosophy underpinned by CfE. The optional element of the questionnaire generated significant interest and revealed that both sectors may have concerns that impact on their capability to implement mathematical problem solving and problem posing.

The next chapter moves on to report on the second phase of this empirical study.

CHAPTER SEVEN

Qualitative Results

This chapter describes the second phase of the mixed methods explanatory sequential design, which was driven primarily by the quantitative results along with due consideration of the outcome of the optional component, as reported in the previous chapter. The objective is to present the qualitative data to help explain the initial results from the questionnaire which addressed the second and third research questions of this study. Eleven semi-structured interviews were conducted with current teachers from the local education authorities of Edinburgh, Fife, North Lanarkshire and West Lothian respectively.

7.1 Nature of the semi-structured interviews

Arising from the analysis of the results of phase one, interviews were conducted during March and April 2015. Each interview was audio-recorded using a high-quality device and conducted at the participants' place of employment, lasting on average 40 minutes in duration. Field notes were used to gather demographic information and convey personal reflections. With one exception, all of the interviews were located in a quiet area free from interruption. Rapport was instantaneously established with 'social capital' afforded to me as a full-time practising teacher. All of the participants were cooperative allowing for a relaxed and informal atmosphere to dominant the interactions.

The protocol of each interview was similar and followed a prearranged sequence grounded in the quantitative results and optional comments from the first phase (Appendix O). The initial development of the interview questions were guided by significant results, non-significant results and group differences, followed by an iterative process of revisions and reflection, framed against the mathematical problem solving expectations of CfE. This procedure was later strengthened by other interrelated questions which focussed on teachers' beliefs about the nature of mathematics, the learning of mathematics and the teaching of mathematics. In essence, participants were posed a set of identical questions including follow-up questions corresponding to their individual responses to the online questionnaire. Due to the nature of the semi-structured arrangement, opportunities to capture unplanned conversations emerging from the interview were also examined.

All of the interviews were professionally transcribed ‘intelligent verbatim’ by an independent company, producing approximately 160 pages of transcripts. In terms of rigour, random samples of two transcripts were scrutinised by me against each audio-recording to determine the accuracy and reliability of the transcription service provided. As a further measure shortly after the interview, one individual transcript was examined in detail by the participant involved, to ensure some form of corroboration. With the exception of a few inconsequential punctuation errors and the misspelling of two words, no inconsistencies were detected in terms of accidental or intentional alterations. Subsequently, it is considered that the transcripts are trustworthy for analysis purposes.

7.2 Participant information

Purposeful sampling was undertaken (Teddlie & Yu, 2007) as the strategy to select participants from a sample of 63 volunteers, collected from the questionnaire. This sample did not yield individuals at or near extreme levels, although, a significant number of volunteers were typical or representative of different groups, including several that varied in their statistical results and others that contrasted in their scores on significant predictors. Principally based on total mathematical beliefs scores, intermixed with consideration of demographic characteristics, an interviewee list deemed capable of helping to explain the phase one results was formulated. Due to the unavailability of some individuals including limitations owing to my own full-time teaching commitments, the list was subsequently modified to produce a final array of participants with total mathematical beliefs scores ranging from 71 to 120. In terms of schools establishments, the participants worked in co-educational institutions classified as denominational and non-denominational, and located within urban and semi-rural areas. Where possible, considerable effort was made to ensure that the participants and their schools reflected the diversity of all schools. Pseudonyms are used throughout to ensure anonymity of the participants, of which relevant background information can be seen in Table 7.1.

<u>Integration</u>

At this point, I wish to clarify my position regarding the methodological issue of 'integration' which refers to the stage or stages in the research process where the mixing of the quantitative and qualitative methods occurs (Ivankova, Creswell & Stick, 2006). A number of scholars maintain that integration can dramatically enhance the value of mixed methods research (e.g. Bryman, 2006; Fetters, Curry & Creswell, 2013; Creswell, 2015). In this study, I connected the quantitative and qualitative phases during the intermediate stage in the research process, while selecting the participants for the qualitative follow-up interviews. Considerable attention was afforded to the selection of interview participants, in order to ensure a diverse representation of mathematical beliefs including a comparable range of mathematical beliefs between each sector. The second connecting point included development of the interview questions for the qualitative data collection, based on the results of the analysis of the first quantitative phase.

Table 7.1 Background information of the interview participants

Interview No.	Pseudonym	TMBS	Gender	Sector	Age group (Years)	Grade	Teaching experience (Years)	Highest level of qualification in the field of education
1	Iona	86	Female	Primary	45-54	HT	27	BEd
2	Alasdair	101	Male	Secondary	35-44	Teacher	14	PGCE
3	Grace	103	Female	Primary	35-44	Teacher	21	PGCE
4	Isabella	97	Female	Primary	Under 25	TIS	0.5	BEd
5	Skye	120	Female	Secondary	55 or over	Teacher	17	PGCE
6	Lorna	100	Female	Primary	45-54	PT	8	PGCE
7	Fraser	71	Male	Primary	45-54	DHT	21	PGCE
8	Morag	114	Female	Primary	35-44	Teacher	18	PGCE
9	Kirsty	106	Female	Secondary	55 or over	Teacher	40	PGCE
10	Hamish	83	Male	Secondary	45-54	PT	20	MSc
11	Cormac	116	Male	Secondary	25-34	Teacher	3	PGDE

Note: TIS (Teacher Induction Scheme) PT (Principal Teacher)
DHT (Deputy Headteacher) HT (Headteacher)

The estimated mean age of the group is 44.4 years and the mean length of teaching experience calculated as 17.6 years, comparable with the main study sample (N = 478). Similarly, the mean of the total mathematical beliefs scores for the participants measured 99.73, which is very close to the overall mean value of 99.23.

7.3 Qualitative analysis process

After careful consideration, an early decision was made to reject the use of qualitative data analysis software such as NVivo. The significant investment of time required to effectively master such a specialist system was not possible given my full-time teaching commitment. Irrespective of available time, Lambert (2012, p. 173) advises beginner researchers “to stick to more straightforward methods”. Although also potentially labour intensive, it was felt that a manual approach with a relatively small database, presented a rich opportunity for me to appreciate ‘first-hand’ the complexities of the interconnected analytical processes.

Four stages of analysis were undertaken in phase two of this study, all of which were iterative. The qualitative research questions designed for the interviews provided the development of a suitable thematic framework. I will explain each stage of the process separately.

The first stage involved carefully reading the transcripts several times in order to gain a general sense of the material. This allowed me to immerse myself in the details. Particular emphasis was made to ensure that the understanding of the transcripts was precise. Cohen, Manion & Morrison (2011, p. 537) warn of the controversy of interpretation by asserting: “one has to note that there are frequent multiple representations to be made of qualitative data - that is their glory and their headache!” Where necessary, I reviewed my interpretations on several different occasions. Principally to ensure that my original thinking about the data did not follow a stereotypical linear process as opposed to a desired contextual one. Patton (2002, p. 480) warns qualitative researchers about the temptation to “fall back on the linear assumptions of quantitative analysis”.

During the second stage, I highlighted all comments deemed ‘interesting’ and introduced a colour system to code the data. Creswell (2014, p. 267) states that “coding is the process of segmenting and labelled text to from descriptions and broad themes in the data”. As a large number of codes were generated, codes were examined for overlap and redundancy.

In the third stage, I annotated the emergent themes. By applying a critical perspective to the raw examination of the transcripts and field notes, I was able to identify reoccurring and new themes. Menter et al. (2011) sensitises researchers to:

> keep your mind open to what may emerge. Do not always assume that the first transcripts you read will provide you with all of the important insights you are hoping for and then all you are effectively doing when you read the later ones is looking for confirmation. Indeed, it is good to be looking for countervailing evidence that seems to contradict your previous judgement or at least represents a very different experience or perspective (p. 216).

I also identified and disregarded data that did not provide evidence for any theme.

The final stage involved tabulation of responses to questions and cross-participation analysis. This allowed me to re-read the data holistically and identify the key themes emerging from the interviews.

7.3.1 Emergent themes

On analysis of the qualitative data, eight interrelated broad themes emerged during the discourse that help to explain the results of the first phase one of this study:

1. Philosophies of mathematics

The most salient hierarchal feature of the participants' interactions is the multiple perspective philosophies held between individuals and sectors regarding the nature of mathematics, although one teacher did not articulate a coherent philosophy. Engrained within some primary teachers' beliefs is the personal view that mathematical knowledge is objective, unique, rational, inert, cold, abstract and logical. A pertinent illustration of this was offered by Grace, who firmly stated:

> Maths is about following a set of established rules that have been in place for a very long time and which don't change...The angles in a triangle have always added up to a hundred and eighty degrees have they not? Maths is a hard subject where your final answer is either right or wrong... You also need a good memory... When I doing maths at school, I always use to wonder why we needed to learn some things, like quadratic equations, since most people will never use them again... For me now, numeracy is just as important, if not more.

Similarly, Morag reported:

> ... If you need to solve a question, you need to know the correct procedure to follow... but sometimes in maths you just have to apply the rule even when it doesn't make sense to you... Yesterday, I got a question on dividing one by nothing and checked it on a calculator... still

> not really sure why it [calculator] gives you an error since one number is being divided by another.

Both positions reinforce an absolutist philosophical perspective of mathematics, which is succinctly defined by Ernest (1991, p. 7) as “consisting of certain and unchallengeable truths”. Conversely, this stance is contrasted with predominantly a fallibist philosophical perspective advocated by most of the secondary mathematics teachers, although support for a social constructivist approach is sporadic. Thompson (1992, p. 132) opines that “mathematics is considered fallible, and it is developed through conjectures, proofs, and refutations, and, uncertainly is accepted as inherent in the discipline”. Evidence of an underpinning fallibilist perception was noted in comments embracing historical and mathematical applications including the notion that mathematics as a discipline is open to revision. For example, Alasdair remarked that “maths is a beautiful subject that goes back to ancient times” and “it is used in every field of real life and is always evolving due to new research”. Likewise, this view was reinforced by Skye, who said:

> When I think about maths, I think about the contribution of Archimedes, Napier, Euler, Maxwell and wonder how these guys did what they did without the use of a computer! Maths has an infinite number of practical applications; you can explore almost anything. Numbers are used everywhere from building bridges, insurance premiums, sports, music, missiles, stock market, mobile phones, weather forecasting, voting methods, gaming, predicting the spread of a disease or population growths or even the likelihood of the next tsunami and so on...What other school subject can offer the same level of stimulus?

From my own professional experience, using mathematical applications and examples of antiquities as pedagogical tools for learning and teaching mathematics, can stimulate interest and enthusiasm, since it helps to humanize the subject. Bidwell (1993, p. 461) eloquently describes that teachers “can rescue students from the island of mathematics and relocate them on the mainland of life that contains mathematics that is open, alive, full of emotion, and always interesting”.

In expressing an equivalent notion, Kirsty, a veteran practitioner, emphasised a humanist philosophy of mathematics, based on her strong admiration of a pure mathematician. She reported that,

> I recently watched ‘The Imitation Game’ about the genius Alan Turing and his Enigma machine, which he used to break Nazi codes during the Second World War; totally amazing film! Cryptography is such a powerful branch of maths, and there’s so much more out there that has yet to be discovered!

What is particularly intriguing about Kristy's narrative is her use of the word 'discovered', intimating that mathematical truths are unearthed at some point in human evolution, which aligns with a Platonist view of mathematics. It also suggests refutation that mathematical knowledge is created by a community of mathematicians, which is the cornerstone of social constructivism. Ernest (1998) argues that mathematics is constructed by the mathematician and is not a pre-existing reality that is discovered. Assimilating to a platform focusing completely on school mathematics, three distinct belief dimensions emerged mostly in line with mathematical philosophies represented by Ernest (1989a). The mathematical beliefs of both primary and secondary mathematics teachers do not assimilate with one single belief structure but are reflected in a cluster of instrumentalist, Platonist and problem solving perspectives.

The effect of grade and highest qualification in the field of education was explored further and found to be noteworthy. This was suitably illustrated in the case of Fraser, a science graduate and experienced Deputy Headteacher (he holds the professional award of SQH and postgraduate diploma). His responses to the belief questionnaire were unambiguously consistent with a problem solving view of mathematics. For example, he strongly agreed or agreed with the following belief statements:

- The priority in teaching mathematics is to ensure students develop confidence in problem posing and problem solving.
- Teaching mathematics through problem solving is the best method to help students learn.
- Mathematics is a continually expanding field of human creation and invention.

Consistent with these views, Fraser also strongly disagreed or agreed with the following belief statements:

- Mathematics is an accumulation of facts, rules and skills.
- You explain in detail what the students have to do to solve problems.
- Mathematics is a collection of procedures and rules that specify how to solve problems.

Fraser noted that his enthusiasm for a problem solving philosophy of mathematics was fuelled during his time spent as a former LEA development officer for science. He dismissed the notion that the nature of mathematics is all about numbers, but that it can be appreciated by considering different types of "scientific relationships". In particular, Fraser expressed a need to understand why a given formula is true, in order that the formula can be internalised without memorisation. He said,

> Although I learned maths at school the traditional way by following procedures, I could still pass exams but never really understood what I was doing... When I had to plan science lessons, I realised that a problem solving approach could be applied to other areas maths... To me, understanding is the key opposed to reciting facts or formulas... Maths is an incredibly useful tool in the real world, it's at the core of the curriculum and without it, we can't solve problems. The fact that it can be adapted to so many different applications, not just science, means that it's flexible and essential for everyone to learn.

Equally, in the case of Hamish, a Principal Teacher and Masters graduate in 'Professional Education and Leadership', his responses to the belief questionnaire were unmistakably consistent with a problem solving view of mathematics. He strongly agreed or agreed with the following belief statements:

- Teaching mathematics through problem solving is the best method to help students learn.
- Mathematics is a continually expanding field of human creation and invention.
- Problem posing is beneficial for developing students' mathematical skills and investigating their understanding of mathematics.

Consistent with these views, Hamish also strongly disagreed or disagreed with the following belief statements:

- Mathematics is computation.
- You explain in detail what the students have to do to solve problems.
- Mathematics is a static but unified body of knowledge.

Hamish conveyed mainly a formalist view, emphasising that mathematics is characterised by logic, intuition and proof. However, he maintained that within its subject domain, mathematics embodies infinite creativity because it perpetuates the requirement to solve new and exciting real life problems using innovative techniques. Hamish said:

> I love everything about maths... logic, proofs, theorems, complex numbers, geometric shapes, fractals, modelling... It's available to everyone, male, female and is all around us in the natural world. It's always been a creative and exciting subject for me because of its use in solving real life problems... in fact, maths is really all about solving problems... and of course we have the future, which will need our youngsters to use imaginative ways to solve new problems...

In general, only Fraser and Hamish explicitly referred to mathematics as a "scientific relationship". Both participants highlighted the link to solving real life problems and for interpreting the natural world. This view was shared by three other secondary participants. A common theme amongst primary participants was that mathematics is viewed as a numerical tool for managing daily life scenarios such as household accounts. This is comparable with Archer's (1999) Australian cross-sector study who found that primary teachers tended to view mathematics as linked to everyday functions while secondary mathematics teachers were inclined to regard it as self-contained, orderly and logical.

2. Image of mathematics

Emerging directly from the philosophies about the nature of mathematics is the interrelated theme concerning 'images of mathematics' as espoused classroom practices. An image of mathematics is conceptualised as a mental representation or view of mathematics, presumably constructed as a result of previous social or personal experiences. All of the secondary mathematics participants portrayed a positive image of mathematics. This was clearly manifested by their promotion of multiple solutions or elegant solutions to mathematical problems. In one case, Alasdair commented that:

> Teachers must encourage pupils to look for alternative ways to solve problems if they want to raise attainment in maths... The solutions don't have to be fancy, simple is better with no need to include a long list of calculations.

This comment follows closely with views articulated by participants in a study of Israeli mathematics teachers' conceptions of multiple solutions, by Leikin & Levav-Waynberg (2009). One illustration is centred on a similar awareness that solving problems in different ways may "contribute to the development of students' mathematical understanding" (p. 12). In contrast, a variety of perspectives were expressed by the primary participants. For example, two primary teachers communicated a negative image of mathematics by their support of the over use of routine mathematical tasks in order to 'repeat newly acquired algorithmic and memorised procedures'. Grace and Lorna claimed that pupils need sufficient time to develop self-assurance and that low confidence in mathematics is a major barrier to successful learning. Grace insisted that:

> It's really important that time is set aside to allow children to practice their new maths knowledge without the need for questions [problem solving] that will put them under too much pressure... If they don't practice what they have been taught, for instance number bonds, they may not remember things later on...

A possible explanation from Grace's narrative is that memorisation is instrumental in learning mathematics (a view shared by many participants from both sectors during phase one). Assuming this to be correct, it may be reasonable to speculate that such practitioners wish to prevent the creation of misconceptions, although misunderstandings can never be fully circumvented in teaching mathematics (Swan, 2001).

Within the first two themes, all of the participants explicitly referred to their previous school experiences as former pupils and offered evidence to what instigated and shaped their early mathematical beliefs. Lortie (1975) contends many of the beliefs held by teachers about the profession originate from personal schooling experiences, gained through an ‘apprenticeship of observation’. For example, Hamish positively said:

> My first memories were fantastic thanks to an inspirational primary teacher who took the time to help me try new things and made numbers fun and meaningful... I was never frightened of making a mistake... she taught me algebra and always encouraged me to learn more advanced stuff including solving difficult problems... Her enthusiasm for the subject was infectious... she was the main reason why I became a teacher. It was certainly not due to much of my secondary education, as the teachers there were obsessed with handing out the belt [former school corporal punishment] opposed to helping people achieve... not until fifth year [S5] doing proofs and calculus did I begin to really enjoy maths again... even after all these years, because of one teacher, algebra is still my favourite topic.

There were some negative comments about early mathematical experiences that appear to be unaffected by the impact of initial teacher education including years of professional experience. The comment below from Morag is a typical illustration of such a tension:

> ... I remember when I was young trying to learn my times tables and always getting shouted at, even though I could recite Burns... One time, Dad was angry with me for failing to pass an entrance exam that included lots of long division... I cried a bit as a teenager because I couldn’t follow all the rules, especially equations... Disliked maths at secondary because of the oppressed and repetitive way it was taught and that we were always set... The teachers were impatient because they knew I was really good at other subjects... I even struggled during my BEd and almost failed the final placement because of fractions and decimals [laughter]. Even with CPD, I still feel indifferent towards maths... I will never forget the feeling of dread as a wee girl been forced to stand up and repeat the twelve times table... This is the same wee lassie that could easily stand up during an assembly and perform ‘Ae fond kiss’.

It is apparent by her emotive language that Morag was and still is upset by her early school experiences of learning mathematics. This finding is consistent with that of Hudson, Henderson & Hudson (2015) who describe the case of Angela, a Scottish primary teacher, who had been traumatised by mathematics during secondary schooling. Many primary teachers lack confidence in their mathematical abilities (Winteridge, 1989) and often exhibit feelings of anxiety and emotion relating to their negative experiences of school mathematics (Buxton, 1981). Moreover, Chalke (2007) warns that practitioners who have themselves been exposed to poor teaching of mathematics may, as a result, lack confidence in teaching it or have a negative image of the subject, falling back and relying on the way they were taught and thereby transmitting their negative attitudes towards the subject to their own pupils.

3. Learning and teaching of mathematics

Some primary participants offered evidence to support the view that their beliefs are consistent with a social constructivist and problem solving orientation and that the strength of these beliefs may be influenced by grade. However, two of the participants hold conflicting prescriptive views that are incompatible with these belief orientations. To illustrate this, consider the case of Morag. For example, she strongly agreed or agreed with the following belief statements:

- Ignoring the mathematical ideas generated by the students can seriously limit their learning.
- After solving a problem, students should be encouraged to search for alternative solutions.
- Mathematical learning is enhanced when students are encouraged to take part in challenging activities.

Consistent with these views, Morag also strongly disagreed or disagreed with the following belief statements:

- The most effective way to learn mathematics is by listening carefully to the teacher.
- Teachers are the authority for what is right or wrong.
- The primary purpose of teaching problem solving is to equip students with a collection of skills and processes.

However, inconsistent with these views, Morag also strongly disagreed with the following belief statements:

- Preparing learners to think critically about mathematics is more important than success at national examinations.
- Teaching mathematics through problem solving is the best method to help students to learn.
- It is important for students to create and solve their own problems.

Continuing in the same vain, Morag also strongly agreed or agreed with the following belief statements:

- Students learn best by doing lots of exercises and practices.

- When there is more than one way of solving a problem, it is generally safer to practice just one of the approaches.
- You explain in detail what the students have to do to solve problems.

Morag's resonance with a social constructivist and problem solving orientation to the learning and teaching of mathematics is undermined with a number of conflicting beliefs aligned to a static transmission or mechanistic transmission orientation. For instance, when asked about the promotion of critical thinking and inquiry in mathematics, she said:

> I agree with this to a certain extent but definitely not at the expense of passing national assessments, as they must always come first. Otherwise there would be no point... Being able to think can be a good thing but only if the thinking is similar to the teacher... Sometimes when we are building plastic 3D shapes, I might leave them to it, but usually I tell them how to get started...

Further evidence that substantiates this traditional position is contained in Morag's justification for strong agreement with the statement - 'You explain in detail what students have to do to solve problems'. This negative statement which at its root, eliminates the intellectual challenge for the learner, deemed an inherent strategic ingredient of mathematical problem solving. Also, it rejects the pivotal role of the teacher as a facilitator in emphasising pupils' active involvement doing mathematics e.g. exploring, making mathematical conjectures, stimulating learners to think, etc. Irrespectively, Morag expressed a firm belief in the importance of achievement and indicated an implicit detachment with the need to promote the construction of deep conceptual understanding. In her words:

> It's important that children can achieve success with the right answer... By clearly explaining what they have to do, it makes it easier for them to understand, this will allow them, well most of them, not to fail... which in turn will boost their confidence... and not think of maths as a hard subject that they can't do. It also helps them the next time they come across the same problem. Hopefully, they will remember what to do.

Morag's last comment resonates with the promotion of instrumentalism (Skemp, 1978). With this approach, it is usually easier to understand, offers instant rewards and allows learners to obtain the right answer quickly and reliable. However, it impedes pupils from monitoring their thinking and using adaptive reasoning during mathematical problem solving (Schoenfeld, 1992). On closer examination, it may be apparent that Morag's adverse experience as a pupil (as highlighted earlier in this chapter) may illuminate why she does not want children to fail. Moreover, by lowering the level of challenge during a lesson will eliminate the productive struggle that is essential for developing an understanding of what to do and why (Hiebert & Grouws, 2007; Schoenfeld, 2014).

With respect to the secondary mathematics teachers, an assortment of beliefs exists between the participants. In one case, Cormac, a young inexperienced practitioner, expressed a multidimensional perspective that impacted on his own pedagogical practice. He described an unsolicited culture of target setting at his current school, which resulted in him reluctantly adopting an overwhelming support for integrating national marking guidelines within the teaching of S4 classes and above. Furthermore, he alluded to the controversial notion that older pupils are unperturbed with relinquishing conceptual knowledge and other enrichment experiences, in order to focus on enhancing procedural fluency. Cormac testified,

> I feel under intense pressure to teach not my subject [mathematics] but the exam techniques that the SQA are looking for... In fact, everything is geared towards the final exam, so much so that my pupil's don't really care about learning content as long as I show them how to pass the exam.

I have reviewed the data from all of the secondary participants regarding a social constructivist orientation. What has transpired is a mixed view, influenced by grade or the highest level of qualification in the field of education, exists (it is not possible to distinguish between the effect of these two characteristics). It is suggested that class teachers hold little support for social constructivism, in contrast with the beliefs of promoted teachers. Interestingly, with respect to a problem solving orientation, a spread of beliefs exists, comparable to the espoused beliefs of primary teachers. Again, the strength of these beliefs is influenced by grade and the highest level of qualification in the field of education.

Several common features exist for both sectors. For instance, the level of support for encouraging multiple solutions during mathematical problem solving, promoting mathematical problem posing, importance of memorising mathematical facts during learning and that mathematical problem solving should not be considered a separate element within CfE. Likewise, a number of conflicting discourses emerged from the data. To illustrate this, I have selected a participant from each sector with a similar TMBS, grade and qualification. Using the responses from Grace and Alasdair respectively, with regard to three random belief statements:

(a) <u>'Preparing learners to think critically about mathematics is more important than success at national examinations'</u>

This positive belief statement encompasses desirable skills such as mathematical problem solving, analogical reasoning, independent thinking, generalising and verbal and written communication of mathematical ideas. In the questionnaire, Grace agreed with this statement while Alasdair strongly disagreed. According to Grace, she deemed that it was more imperative that teachers make the distinction between short and long term learning implications. She explained:

> In the short term, it is sometimes easier just to prepare children to pass an exam... some of them might retain the knowledge but most of them will probably forget... developing critical thinking will have a longer positive effect on the overall needs and progress of all children...

Conversely, Alasdair inferred that regardless of what theoretical didactic options he is presented with, nothing will override the foremost priority of his school which is to produce individuals that can achieve success at national examinations. After all, in his words,

> We [teachers] are only measured on how well pupils perform in exams as you cannot assess critical thinking... I am grilled big time if my results don't meet a certain target... It happened to me last year and I am still suffering from the fallout from the Headteacher... so this is why I normally teach to the test... It also helps to prevent parents writing in to try and rubbish you.

It is particularly notable that Alasdair's espoused mathematical beliefs are inconsistent with his professed conventional classroom practice, due to the significant influence of an external contextual factor.

(b) '<u>Teachers should encourage their students to strive for elegant solutions when they solve problems'</u>

This positive belief statement encourages teachers to promote originality and imagination in mathematical problem solving. Digging beyond the surface of a problem can help develop characteristics such as interest, versatility and perseverance. Regarding this belief statement, Grace disagreed with this item during the questionnaire. She maintained that by encouraging pupils to practice algorithmic tasks, it was possible for learners to "remember shortcuts" but that her specific interest lies only with a desirable outcome. Grace affirmed:

> It really doesn't matter what a solution to a problem looks like as long as it's correct... If a child has obtained the right answer then this is much more important than how they have solved it... if they can make it neater then it might look better visually but it is still worth the same... sometimes it is not easy to judge when I give out practical tasks like tangrams or during strategy games on the computer.

This comment suggests that Grace has not considered the possibility of a pupil obtaining a correct answer by using a wrong approach or by guessing. Furthermore, she appears not to support abstract thinking and creativity during mathematical problem solving, which hampers the intellectual curiosity of pupils (Scottish Government, 2009, 2010a, 2011a). In sharp contrast, Alasdair concurred with this belief statement and upheld the view that practitioners should actively engage pupils to look for "simple but clever" or "original" solutions, commenting that:

> A short cut version is normally an indication of stronger understanding... it takes good imagination to produce a novel type solution... teachers should encourage this because it can help pupils to think deeper... I sometimes ask pupils to solve a problem first and then challenge them to come up with an improved version. Usually most of them are unable to do this but at least it makes them aware that some solutions may be better than others...

It is advocated that by encouraging pupils to participate in mathematical discourse about different solutions to problems, Alasdair has concomitantly fostered the rational for promoting multiple solutions during mathematical problem solving.

(c) <u>'Mathematical problems can only have one final correct answer'</u>

This negative belief statement highlights one of the most widely held misconceptions that mathematical problems have a unique correct answer (Schoenfeld, 1985, 1992; Devlin, 2003). In upholding a decision to strongly agree with this questionnaire item, Grace related her belief about the nature of mathematics as "an exact subject that can only have a right answer and lots of wrong answers". In expanding further she pronounced,

> Two times three is the same as three times two... you get the same answer every time... If you're hinting that there is the possibility that more than one answer can exist then this questions everything I believe about maths... it is simply not possible to have more than one answer...

In robustly opposing this viewpoint, Alasdair endorsed his decision to strongly disagree with this questionnaire item. He claimed throughout his mathematical learning he had encountered many non-computational problems that contained multiple answers and that "it's a matter of training the mind to be open to more than one response". Alasdair explained as follows:

> Maybe it's because I have a maths degree that I know that for some problems, more than one answer might be possible... in fact some problems cannot be solved at all since they do not contain enough information... or it's my experience teaching negative numbers, quadratic equations, trigonometric functions...

Overall, perhaps the most troubling finding is that some participants from both sectors do not advocate the orchestration of challenging mathematical tasks that require cognitive and metacognitive demands of their pupils. Instead, it is proposed that direct instruction will ensure procedural fluency which will help override the requirement for profound conceptual understanding and circumvent the need for pupils to ‘struggle’ during the learning process. In one case, Skye thought that:

> When most youngsters fail at something, they tend to give up immediately and you can lose them forever so there’s no point in making things hard for you or for them. Whether it’s right or wrong, I provide all the steps so they can answer the questions and progress further since you don’t get them moaning there’re not learning anything... I have tried making them think in the past when I did a weekly problem but it ended up you’re the one being criticised because there’re stuck and you’re accused of not helping them...

The above comment raises a plethora of pedagogical issues. However, I do not wish to attempt to unpack all of the possible reasons but prefer to speculate briefly on why Skye does not appear to employ rich mathematical tasks that promote discussion, foster challenge and develop higher order thinking. In her narrative, she alludes to her teaching being criticised for not helping pupils when interacting as a ‘Public Educator’ (Ernest, 1991). If we assume that Skye has performed this function to the best of her ability, and in doing so, has received complaints. Based on human nature, it is expected that people will incorporate adjustments to their professional practice to avert unwarranted attention. Though, I suspect in this case, it is more plausible that Skye holds an absolutist view of mathematics and lacks confidence when having to align her beliefs with a fallibilist philosophy rooted in social constructivism, creativity and critical thinking. Jaworski (2010, p. 13) asserts that a fallibilist perspective can be “threatening for teachers who feel insecure without an authority to sanction their judgements”.

4. Nature of a mathematical problem

In general, both sectors expressed similar and contrasting beliefs of the nature of a mathematical problem. I will now discuss the comments expressed by each group separately.

Beginning with the primary teachers, two of the six participants were unable to offer a meaningful definition of a mathematical problem. For example, Isabella, a probationer stated: “I can’t really explain what it is but I know when I see one in a textbook”. All of the remaining participants indicated that mathematical problems are contextualised word problems associated with a real world scenario and solvable by application of one or more of the basic arithmetical functions. Furthermore, three of the four participants alluded to the inclusion of a human presence. According to Lorna, mathematical problems “comprise of an account of someone performing an everyday function like spending money, sharing fruit or measuring a quantity, followed by a sum question about what they are doing”. Likewise, Grace described a mathematical problem as “a wee story set within a familiar real life situation involving people and numbers”. This finding is consistent with that of Xenofontos & Andrews (2014) who found that prospective Cypriot primary teachers regarded mathematical problems having both a connection to reality and human actors.

Absent from any narrative is explicit reference to the level of cognitive challenge, time required to obtain a solution or consideration of different strategies. Embedded in such perspectives is the tacit acceptance that mathematical problems must be linguistically accessible to all learners, require no higher order thinking skills and that every problem is solvable by the execution of a simple algorithm. Moreover, there is no distinction between various types of problems. Though, Iona, a primary Headteacher, intimated that “a problem which is easy for someone might be really hard for someone else”. This point resonates with Hiebert et al. (1996, p. 16) who maintain that the issue of ‘problematic interpretation’ is subjective, noting that tasks “which are neither problematic nor routine, whether they become problematic depends on how teachers and students treat them”.

Four of the six participants were able to present manipulatives of problems employed during the school term. A representative peer example of a second level mathematics problem used with P5 pupils was offered by Morag:

> A bus driver travels 75 km on the first day of a five day school trip. On the second day he travels 246 km, on the third day he travels 103 km and on the fourth day, 398 km. If the total bus journey is 1000 km, how many kilometres has the bus driver got to travel?

However, this is not characteristic of a genuine mathematical problem, since a previously learned algorithm exists to solve it (Hiebert, 1997). More accurately, it is an illustration of a routine word problem that is incapable of eliciting a trajectory of cognitive and metacognitive behaviours, typically manifested by solvers tackling unfamiliar mathematical tasks (Schoenfeld, 1985, 1992). In other words, although grounded on reality, this task is focused on producing a correct answer opposed to developing mathematical understanding.

The data generated from the secondary teachers yielded three sub-themes concerning the nature of mathematical problems. These were:

- The solitary source of mathematical problems is textbooks.
- Problems can have different structures and different cognitive demands.
- Problems are designed to test procedural knowledge and be solved quickly.

<u>The solitary source of mathematical problems is textbooks</u>

Three of the five participants referred to the origin of mathematical problems deriving from official curriculum textbooks. This factor appeared to justify their decision to incorporate such exemplars in lessons. For instance, Skye stated that mathematical problems “are real life application questions that we use from the textbooks”. Equally, Kirsty explained that “good problems usually have a relevant context such as the Olympic games... I use a couple of different textbooks to ensure that I can offer variety”. These narratives coalesce to suggest that practitioners are dependent on extracting their supply of mathematical problems from school textbooks (Kilpatrick, 1987).

<u>Problems can have different structures and different cognitive demands</u>

With one exception, secondary participants alluded that a mathematical problem may be represented in multiple ways such as encompassing text, visuals, symbols and numbers, none of which should be mutually exclusive. Furthermore, it was considered that problems should engage learners in a suitable challenge but not anything requiring complex thinking or considerable cognitive effort. An emblematic comment capturing this attachment was produced by Hamish:

> Maths problems can include pictures, symbols, words, geometric shapes or just numbers on their own... They should offer various challenges but never be too difficult so that the majority of pupils can still solve them by using what [mathematics] they already know and importantly, without any real help.

<u>Problems are designed to test procedural knowledge and be solved quickly</u>

Three secondary teachers intimated that mathematical problems require the application of previously learned facts, rules and formulas, without the need to provide explanations or logical reasoning. It may be the case that these participants do not agree that learners should critically engage with conceptual ideas or explore the nature of processes, connections and complex relationships. Perhaps the most unexpected finding to surface is the lack of reference to any heuristics (e.g. Polya, 1957). Moreover, it was intimated that problems should be solvable within a short period of time. For example, Cormac declared mathematical problems “are normally not too demanding so that some pupils like the fast finishers can practice them after completing a textbook chapter”.

In sum, comparing the sectors revealed that secondary mathematics teachers are more informed about the nuances of mathematical problem solving, using terminology like "interesting" and "challenging". Nevertheless, the evidence suggests that both sectors exhibit a fragmented perspective surrounding the conceptualisation of a mathematical problem, inextricably linked to a series of familiar algorithmic procedures isolated of appropriate cognitive challenge. It seems plausible that without an association with specific desirable characteristics (e.g. a mechanism to foster critical thinking), the operationalisation of mathematical problem solving is impeded.

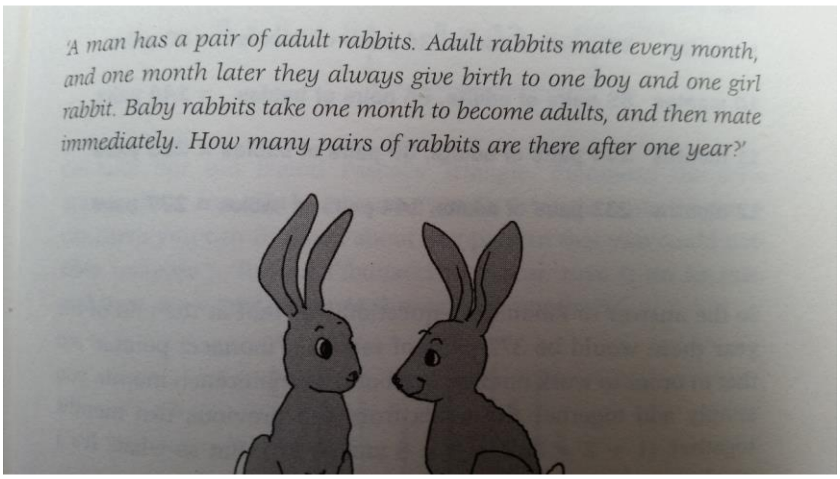
5. Mathematics teaching self-efficacy

Some primary participants felt that their ability to deliver meaningful mathematical problem solving instruction, is inhibited by not having obtained SQA Higher Mathematics. Arguably, their perception of self-efficacy may well be socially acceptable within the context of Scottish educational culture. For instance, Isabella explained:

> I didn't do well in maths at school because I always struggled to understand what was being asked. I failed Higher [Mathematics] because it contained lots of wordy questions... I think to be honest this is why I don't know how to teach problem solving... Sorry to say this but at the end of the day, English [Higher] is much more important as you cannot get into teacher training in Scotland without it [laughter]...

Isabella's low self-esteem and insecurities regarding problem solving is shared to a lesser extent with Grace and Morag. Collectively, all appear to be circumscribed with an ineffective didactical knowledge base of problem solving skills and heuristics. Morag made explicit reference to a colleagues "maths challenge" (Figure 7.1), presented to a P7 class. She disclosed that "during lunch we all had a wee go at this but only one of us could solve it. This is the type of thing that I can't do because I never know what rule to use".

Figure 7.1 Example of "maths challenge" used by a colleague of Morag with a P7 class



Two points are noteworthy here. Firstly, Morag refers to this good exemplar of a rich mathematical problem (Fibonacci sequence) as a "maths challenge" and when probed about the difference in terminology between a mathematical problem and a mathematics challenge, responded:

> A maths problem can be worked out fairly quickly on your own whereas a maths challenge is something we would always give to a group or to pairs because it is much harder and takes longer to answer...

Intriguingly, this last comment reaffirms a previous theme concerning the misconception of what constituents a 'mathematical problem'. Secondly, with promoting an instrumentalist view that a "rule" must be applied, it is evident that Morag is unaware that mechanical algorithms do not exist for the solving of every mathematical problem.

It may be plausible to suggest that one method to attempt to enlighten the narrated self-efficacy dilemma is that primary teachers believe mathematical subject knowledge and the teaching of mathematics to be mutually exclusive. The dichotomy of addressing these two distinctive bodies of knowledge was first introduced by Shulman (1986). He proposed the term 'pedagogical content knowledge' (PCK) as a form of practical knowledge used by teachers to guide their actions in highly contextualised settings.

Returning briefly to the issue of 'academic entry standard' raised by Isabella, a similar view was expressed by Iona, who questioned the rationality of contrasting admission requirements for core CfE subjects, by stating:

> I believe that something is wrong in the recruitment of primary teachers that it's not essential that they have Higher Maths. Maybe not every Higher Maths student is the best... you know, there are maybe some people that don't have Higher Maths but can still teach the subject really well. But I think it devalues the subject by saying, well, you need a Higher in English but you don't need one in maths.

Nevertheless, in a study of Scottish student primary teachers' levels of mathematics competence and confidence, Henderson & Rodrigues (2008) advocate that it is perhaps not the level of mathematics that requires to be changed but the nature of mathematics taught and learned at that level, needs to be ameliorated. The researchers assert that:

> Our findings suggest that simply raising the entry qualification does not make the student primary teacher more competent with respect to the mathematics required at the primary

> school level. The results... show that students who have Higher Mathematics are no more competent than their peers who hold Standard Grade Credit or Intermediate 2 [equivalent to SCQF Level 5]. Furthermore, having a higher level mathematics qualification does not necessarily improve their confidence (p. 103).

Similarly, McKechan & Day (2015) found that Scottish prospective primary teachers with Higher Mathematics did not perform significantly better in a subject knowledge assessment of mathematics, when compared with counterparts holding a Standard Grade Credit pass. However, their findings affirm concerns raised by Donaldson (2011) that current entry requirements relating to qualifications in mathematics do not seem to provide a sufficient guarantee of competence required for primary teaching. It is of interest that in both studies, no assessment of mathematical problem solving or problem posing skills took place.

6. Workload issues

The main thrust of this theme centred on an external driven assessment culture, which is thought to be undermining the professional autonomy, judgement and responsibility of both primary and secondary sectors, irrespective of grade. As a consequence of perceived idealistic target setting and excessive bureaucracy by local education authorities, all participants reported some form of 'reprioritising' their mathematical beliefs to gratify their respective stakeholders. In other words, with additional teaching time assigned to more assessment and related administration tasks, a significant incongruence has grown between teachers' espoused beliefs and their enacted pedagogical practices. Unexpectantly, the intensity of the participants' frustration was comparable between both sectors, irrespective of the fact that secondary mathematics teachers have responsibility for external national examinations. Representing the views of primary teachers, Fraser made explicit reference to the first three national curriculum levels. He commented that:

> The classroom teacher is now under relentless pressure to have children at the level expected for 'the norm' regardless of real mathematical ability, as we are driven by assessment and written results... In terms of a management perspective, as a school we are judged by how many children we have at certain levels at one, four and seven... we are forever told from the authority, 'okay right, now you need to get better than that'... The upshot is that we have less unstructured time to make maths more fun like playing chess, interactive strategy games or exploring websites like NRICH...

Fraser elaborated further on this theme, captured as follows:

Interviewer:	Are you hinting that primary teachers are unable to be agents of change as they are controlled entirely by an external results driven system?
Fraser:	Yes, but it's not something we are all happy with as we have no choice but to follow whatever the attainment targets set by our local authority are. This is regardless of

what our children's ability is like in reality... In effect, we're all being undermined in this issue... our professional autonomy has been watered down which has seriously affected staff morale and what we can we actually do in class...

Interviewer: Do you think it would it be possible to present your authority with an alternative to attainment targets? Say, instead of pushing through X percent this year, we have used the extra time to enrich mathematical learning by energising everyone with research and have produced a group of confident children that can demonstrate creative and critical thinking skills that can be applied to solve real world problems?

Fraser: [Laughter]... No, I could not say that if I wanted to keep my job. That just would not be accepted.

Interviewer: Do you think it should be accepted?

Fraser: Yes, of course most definitely! But I am only one teacher, one voice. However, I do think it's all about beliefs. If you sincerely believe that creativity and critical thinking skills are key, then we need to stand up for them. I firmly believe that as a profession we need to say, look, we have pushed these bairns through hoops to get 85 percent last year, 86 percent this year, and we'll do 87 percent next year. But what real good is it doing them? What values are we actually teaching them? ... All we are really doing is showing them how to pass an exam without having real understanding. Surely there is something more important than government statistics?... They do not provide a true picture of ability.

Interviewer: How does your school prepare pupils for such assessments?

Fraser: Some genuine revision but mostly we have to provide lots of similar examples, which the children practice... For the vast majority though, it's pretty much all rote to be honest, as we are under severe pressure to achieve.

Interviewer: Do you know of any other schools who adopt a similar approach?

Fraser: Oh yes, our fellow cluster would definitely say the same. In fact, I have been told this is fairly common in maths by our local secondary school. They are also under huge pressure to achieve their targets.

Interviewer: In your view, what is the nature of an effective maths assessment?

Fraser: Well, something much more useful to the one being used at the moment. It should offer variety like practical tasks like say solving a Sudoku puzzle cube, magic squares or completing a fractions jigsaw but definitely where the child is at the centre of the whole process, opposed to being exploited for political gain.

This extract highlights that Fraser would welcome a revision to the method of reporting mathematical ability for P1, P4 and P7 respectively. He advocates the embodiment of 'variety' in official assessments. This perspective is shared with the NCTM (2000, p. 24) who testify that "assembling evidence from a variety of sources is more likely to yield an accurate picture of what each student knows and is able to do". However, what is disturbing is the apparent widespread use of a behaviourist methodology, in responding to high-stakes testing. It does not promote conceptual understanding of mathematical ideas that allow knowledge to flourish and be further developed. This may help to explain why Fraser is dismissive of summative assessment scores. In their research of learning trajectories in mathematics, Daro, Mosher & Corcoran (2011, p. 30) state of such testing: "As teachers have found through hard experience, these scores and associated inferences are not of much help in designing instructional interventions to help students stay on track and continue to progress".

Equivalent feelings were expressed by the secondary mathematics teachers. For example, Kirsty asserted that the lack of opportunities to engage pupils in problem solving was attributed to "the excessive amount of assessments" which she also confessed to having a negative impact on her motivation to teach. She explained,

> Reassessments are the real bugbear. You might be in the middle of a run of really interesting lessons and stop; someone demands an update and hey, reassessments take over... They are outrageously time consuming, not just for teachers... it results in a lot of unnecessary wasted periods for the pupils that passed it the first time... I strongly believe this dead time could be spent on other activities, such as the UKMT problem solving stuff...

A similar view was conveyed by Skye, who insisted that her approach to the learning and teaching of S3/S4 mathematics is compromised by the imperatives of national assessments. She commented:

> With Nat four and five, I have to have to follow a regimented text book route, not because of my PTC, but because the CfE has created an assessment monster which is controlled by evidence... So in order to survive, I revert to rote learning with virtually no time given to solve problems.

It was also suggested that another barrier to implementing mathematical problem solving is the contextual nature of the school. In the case of Cormac, he reported feeling restricted in his professional enquiry, due to the social dynamics of his working relationships with colleagues. Inherent within his faculty was a tacit expectation that all practitioners adopt a uniform classroom approach. Cormac stated:

> Even if I had any spare time for doing problem solving, the other teachers in the department would probably not want to follow as this would involve lots of extra work... They are much older than me and some are due to retire soon... In fact, they are really set in their ways and just want an easy life [laughter]... Maybe they don't like the fear of the unknown? Who knows? But for me, I can do change but only when we all do it... This is my first full-time position after my induction and I don't want to stand out as being different because they're all really good teachers and very helpful.

A study which resonates with the contextual issue raised by Cormac, is the case study by Priestley & Miller (2012). The researchers illustrate the situation of Drew, an ambitious Scottish mathematics teacher working within a department reluctant to implement change. They describe that, "Drew believed that he was swimming against the tide in his department, which he described as being 'stuck' and very resistant to change" (p. 112). Teacher resistance to educational reform is nothing new. Consequently, when teachers do not support reform, the successful execution of curriculum change is unlikely (Elmore, 1995; Fullan, 2016). In particular, this tendency is more prevalent among older and experienced teachers. Hargreaves (1991) points out that:

> … in mid-to late career, if they [teachers] have not been promoted out of the classroom, or become disenchanted and disengaged by dispiriting conditions, blocked careers and lack of recognition, teachers are still committed to change and improvement, but on a more modest and gradualistic scale with their own classrooms that they can control. [They are] unlikely to invest fashionable innovations with unmitigated enthusiasm, especially when they have seen so many come and go in the past (p. 249).

7. Collaborative mathematical problem solving

Whilst collaborative learning in mathematics has the capability to promote deeper conceptual understanding amongst other benefits (Swan, 2006), teachers' beliefs of this pedagogical approach diverged between the sectors. To illustrate the significant variations in beliefs, I have compiled participants extracts (Table 7.2 and Table 7.3 respectively) relating to two items belonging to Belief Factor 5.

Both tables are quite revealing in several ways. The primary participants appear to embrace collaborative mathematical problem solving as a method that can enrich the development of positive independence. A possible explanation for this could be that younger pupils cooperate well together in a heterogeneous system where all members of a group feel valued as having equal worth. Furthermore, it is suggested that synergy can help generate the growth of alternative problem solving strategies and stimulate mathematical engagement. A derivative of this can ameliorate personal responsibility and social skills. For example, Fraser expressed the view that pupils “share strategies that can lead to different solutions”. Likewise, Morag highlighted the efficiency of communication between pupils noting “they can share success equally with no one child to blame for failing to solve a task”. Moreover, it is alluded that primary teachers have higher expectations of pupils and therefore respond better to diversity enabling more children to feel empowered to learn mathematics (Rosenthal & Jacobson, 1968; Rubie-Davies, 2015).

In contrast, the secondary participants conveyed a disjunctive and apathetic view pointing towards the existence of a hierarchical orientation within older children that is centred on competition. A tension permeates in relation to noise levels generated by non-mathematical interactions, since from a traditional perspective, excessive sound generally indicates a lack of discipline or teacher control. However, more likely bolstering this position is a reluctance to accept a child-centred approach including knowledge of how to design an organised classroom structure to promote active learning. Such a barrier to collaborative learning may be manifested by a lack of support for social constructivism, which is illustrated in the following comment from Skye:

> For practical reasons, this [collaborative learning] is not an easy thing to manage. Apart from not being in control of the class, many pupils are incapable of debating problem solving at a mature level because they don’t like to admit that they unable to do something or don’t understand... Since maths is a subject where things are either right or wrong, then I am not convinced there is a legitimate need for any discussion anyway... Surely, a much better use of everyone’s time is to have pupils working on their own, so they can quietly practice things and receive the individual help they need to progress.

Taken together, these results provide important insights into teachers’ beliefs of collaborative mathematical problem solving and highlight the emergence of conflicting discourses between the sectors.

Table 7.2 Overview of Item 35 – ‘An effective way to teach mathematics is to provide pupils with interesting problems to investigate in small groups’

Primary Teachers			Secondary Mathematics Teachers		
Participant	Questionnaire Response	Interview Extract	Participant	Questionnaire Response	Interview Extract
Iona	Strongly Agree	This approach can be very productive as children learn a lot from each other through different ideas... pupils are free to walk about and offer help... they can explain some things better than teachers as they use a similar language.	Alasdair	Agree	I only do this type of thing once in a while as it can get very noisy and in the past I’ve had behavioural problems with people disrupting the lesson and others not being mature enough to properly debate together...
Grace	Agree	Good way to allow children to express themselves and feel part of a group...The key is to choose a task that cannot be solved by just one pupil but needs input from others... Using a poster to display the results is very effective...	Skye	Strongly Disagree	I don’t do problem solving that often and prefer to have pupils working individually when I do... Many youngsters use group work as an excuse to talk about other stuff or take out their phone and do absolutely nothing...
Isabella	Agree	It can help to generate lots of different answers which is useful for the teacher... the tactic is to select groups with a mixture of abilities... I often use pentominoes because the children really enjoy the challenge.	Kirsty	Agree	When I have the time, I like to do this because it’s good experience for them having to work in a team... but normally I have to force some of them to sit together as they prefer to work on their own.
Lorna	Undecided	Not convinced about this for problem solving... Sometimes children can easily lose focus or find it hard to work together. I already have children that I have to keep apart... Some still look to me for immediate help...	Hamish	Agree	I should be doing more of this but don’t have the time due to exams and staff absence... not always easy to assess individual progress... We always had team-teaching here but budget cuts have restricted our approach with groups...
Fraser	Strongly Agree	Great way for children to liaise together... they share strategies that can lead to different solutions... with a really good problem it can stimulate creativity and increase motivation... It allows teachers to stand back and observe, rather than jump in with offering help straight away...	Cormac	Disagree	This activity requires lots of planning and finding an interesting problem that can keep thirty bairns entertained for almost an hour is not easy... I prefer whole class teaching as you can explain things at the start which cuts down on mistakes... To be honest, the main drawback is noise because classes in this corridor are quiet and anytime there’s a din, the PT marches in to see what’s going on!
Morag	Agree	Children are natural at sharing information... They communicate well together and can all help each other... One main advantage is they can share success equally with no one child to blame for failing to solve a task...			

Table 7.3 Overview of Item 37 – 'All students are able to be creative and do original work in mathematics'

Primary Teachers			Secondary Mathematics Teachers		
Participant	Questionnaire Response	Interview Extract	Participant	Questionnaire Response	Interview Extract
Iona	Strongly Agree	In groups, practical work with shape and space is a fantastic way to witness creativity and all children are capable of demonstrating this by coming up with individual ways to solve problems...	Alasdair	Undecided	Not sure I really understand how to judge creativity in problem solving... I have a third year bottom set who struggle to tell the time let alone can produce something which would be considered creative or original...
Grace	Agree	With group work, its importance to allow children to convey their own thoughts about the subject... They can express things in original ways.	Skye	Disagree	This is definitely not possible with any of the classes I have this year as they all need me to direct them. They are not good at thinking things through on their own...
Isabella	Agree	During one of my placements, I observed two pupils spontaneously create a 3D sketch of a pizza to demonstrate their understanding of fractions.	Kirsty	Disagree	Not convinced this is possible for every student... some are switched off and don't like maths full stop... they also give up so easily...
Lorna	Agree	I believe that all children are capable of producing something imaginative if a positive [classroom] climate is in place... with the right project, some weaker children can produce brilliant work.	Hamish	Agree	Some pupils thrive in pairs and can achieve much more than working on their own, so this might be a creative use of their time... A few of my S6 are creative because they can explore, make conjectures, verify and prove things without much help from me... Pupils can all do something special but there're some who have reached their peak by S4 and our job really is to help maintain their basic numeracy skills... I tend to think of original work being like Gauss and his method for the sum of the first n integers.
Fraser	Strongly Agree	Teachers should encourage groups to search for different ways to solve problems... this will help encourage innovative and independent thinking. I really believe that all pupils have something unique to offer no matter how small it might be... It also helps if children enjoy maths...	Cormac	Disagree	I dinnae think that all pupils can achieve this unless the really poor ones are spoon fed all the way and even then it will not really be creative or original as they have been helped.
Morag	Undecided	Some can struggle to be creative even with lots of help...			

8. Mathematical problem posing

Within mathematics education, teachers have a critical role in cultivating problem posing activities within lessons (Gonzales, 1996). Such a requirement has been recognised by a growing body of research which underlines multiple learner benefits such as enhanced creativity (e.g. Silver, 1997; Shriki, 2013; Singer, Ellerton & Cai, 2013). In this study, whilst peripheral support for this practice appears to prevail in the implicit beliefs of the majority of participants, no evidence was uncovered that operational knowledge exists of a theoretical framework to underpin its centrality and effective pedagogical implementation. For instance, Lorna reported that: “I have never read anything about it in CfE guidelines or came across anything that explains what we have to do or what we have to follow”. This perspective was reinforced by Alasdair who said:

> I’ve never heard of problem posing until your questionnaire... I think I know what it means but I am not exactly sure... Since it’s not included as an experience or an outcome then obviously it’s not that important to what we already do in maths as it would be part of our assessment procedure... and everybody would know about it... but it does sound interesting.

A variety of perspectives were expressed by other participants. It was suggested by Morag that problem posing is considered to be a voluntary mathematical topic, independent of problem solving. She stated:

> You can do problem posing work with the children anytime I suppose... It’s not linked to problem solving; they are two completely different activities...

In another case, Cormac rejected the notion that practitioners should invest time with problem posing activities, claiming that such classroom endeavours offer no intrinsic merit. He claimed the generation of new mathematical problems only serve to replenish available inventories that already exist. In his words, he affirmed:

> Problem posing is when I put a factorising or equation type question on the board when I don’t have a resource handy... I have never asked pupils to make up questions or problems before... not sure what they can gain from this apart from everyone wasting valuable teaching time...

Conversely, Grace declared the construction of new mathematical problems to be a creative exercise for children that required “a good imagination”. She concurred that learners should try to create and solve their own problems but was unable to articulate a meaningful rational for this viewpoint, although consented that it had something to do with “improved thinking”. She revealed:

> I sometimes write three numbers on the board like 3, 6 and 12, then ask the children to produce as many different numbers from this as possible... they really enjoy this type of challenge and it's good for their confidence... I then split them into pairs and ask them to make up their own set of numbers... This approach has led to the creation of other problems involving the likes of money, weight, and so on...

In general, participants from both sectors indicated that they hold a constellation of positive and negative beliefs of mathematical problem posing, which appear to contradict the favourable outcome from phase one of this study. To illustrate this, consider the case of Iona and Hamish. Both participants strongly agreed or agreed with the following belief statements:

- The priority in teaching mathematics is to ensure students develop confidence in problem posing and problem solving.
- Problem posing is beneficial for developing students' mathematical skills and investigating their understanding of mathematics.
- It is important for students to create and solve their own problems.

Nevertheless, whilst both promoted teachers were able to identify with the theoretical value of problem posing within the stimulus of learning and teaching of mathematics, they voiced concerns with problem posing as an officially sanctioned method of promoting classroom inquiry and suggested that it may never be regarded in the same vain as problem solving. For example, Iona explained:

> I'm sure problem posing has benefits in maths but I'm not 100% I know what these are but I think some form of improved thinking which is great... and I imagine that some teachers are already doing this but the bottom line is that it's not a LI in the same way we are expected to do problem solving... I consider this type of thing 'ideal' in a world where we had the time to choose all our own activities without the constant pressure from the authority to perform.

Hamish alluded to the notion that problem posing could be employed as a diagnostic tool within mathematics education as it has the potential to uncover deficits in pupils' knowledge and that it is a work intensive process. He commented:

> If pupils make up problems, this can be used to test their knowledge and understanding and may throw up their mistakes. But this would have to be done in class because they could easily look up a textbook or search the internet... Although I think my department would not be too happy, as this would be time consuming and it's not even in the CfE... It's hard enough trying to get them to do problem solving right now, which is more relevant.

The participants on the whole demonstrated less intense support for ensuring pupils develop confidence in problem posing, compared with the promising results (i.e. 70% of both sectors strongly agreed or agreed) obtained from the questionnaires.

In general, the primary participants revealed a willingness to recognise the conceptualisation of problem posing within the echelons of real world mathematics education, compared with a polarised position adopted by the secondary mathematics participants. Significantly, primary practitioners exhibited a more enthusiastic disposition towards fostering creative experiences for their learners.

In summary, the results in this chapter help to explain the context of the statistical results from the quantitative first phase of this study. Thematic analysis revealed interrelated themes which provide sufficient breadth and depth of understanding to help distinguish between the mathematical problem solving and problem posing beliefs of both primary and secondary participants.

The next chapter moves on to discuss the findings of the results with reference to each of the research questions in more detail.

CHAPTER EIGHT

Discussion

8.1 Introduction

The purpose of this thesis is to contribute to the field of mathematics education through increasing our understanding of teachers' beliefs of mathematical problem solving and problem posing. Teachers' beliefs of mathematical problem solving and problem posing belong to a larger totality that includes teachers' beliefs about the nature of mathematics, the learning of mathematics and the teaching of mathematics. It is postulated that by researching teachers' mathematical beliefs, it may be possible to determine current levels of classroom practice (Thompson, 1992; Richardson, 1996; Leder, Pehkonen & Torner, 2002; Philip, 2007; Speer, 2008; Beswick, 2012). However, it is a widely held view from the literature that researching teachers' beliefs is problematic, due to a lack of consensus about an explicit conceptual definition including significant methodological issues surrounding how to operationalise the belief construct (Pajares, 1992; Skott, 2015). In this chapter, I will respond to the research questions, synthesising the findings, followed by a discussion of the key aspects of my study.

I addressed three main research questions as follows:

1. To what extent should mathematical problem posing be embedded within the mathematical framework of Curriculum for Excellence?

2. Are there any differences in the mathematical beliefs of Scottish primary and secondary mathematics teachers?

3. What factors impact on the mathematical beliefs of Scottish primary and secondary mathematics teachers?

By considering each research question in turn, I will formulate an appropriate response followed by a discussion where I review my research in relation to the wider context in which it is located.

8.2 Research question one

To what extent should mathematical problem posing be embedded within the mathematical framework of Curriculum for Excellence?

The empirical evidence from the systematic literature review amalgamated with the theoretical and empirical evidence from my research leads me to argue that mathematical problem posing should be embedded within the mathematics framework of Curriculum for Excellence. While problem posing is considered an inseparable part of mathematical problem solving (Kilpatrick, 1987), it should be compartmentalised as a unique cognitive activity (Cai & Hwang, 2002), since it resonates with a social constructivist paradigm. The evidence also leads me to argue that there is a theoretical gap in the mathematics framework of CfE, and that this gap has been exacerbated as a consequence of the incorporation of nebulous guidelines throughout the curriculum itself.

The systematic literature review provided favourable results for implementing problem posing within the learning and teaching of mathematics at both primary and secondary school level including the training and development of university students at initial teacher education institutions. The requirement to consider prospective teachers is grounded on a growing body of research which has highlighted problem posing as a valuable tool in developing mathematics teaching at all levels (e.g. Pittalis et al., 2004; Singer et al., 2011; Cai et al., 2015; Ellerton, 2015; Osana & Pelczer, 2015). Hospesova & Ticha (2015) argue that problem posing within initial teacher education is an effective way of enhancing subject didactic competence. Equally, Crespo (2015) maintains that without substantial work on problem posing during teacher preparation, prospective teachers will enter the profession with limited vision and strategies for mathematics teaching.

There is strong evidence from nine studies (English 1998; Dickerson, 1999; Demir, 2005; Xia, Lu & Wang, 2008; Priest, 2009; Kesan, Kaya & Guvercin, 2010; Guvercin, Cilavaroglu & Savas, 2014; Guvercin & Verbovskiy, 2014; Haghverdi & Gholami, 2015) of a significant impact on pupils’ mathematical attainment. Likewise, there is reasonable evidence from three studies (English, 1997b; Kesan, Kaya & Guvercin, 2010; Guvercin & Verbovskiy, 2014) of increased levels of pupil motivation, cognition and flexible mathematical thinking. Furthermore, there is strong evidence from five studies (Demir, 2005; Xia, Lu & Wang, 2008;

Guvercin, Cilavarouglu & Savas, 2014; Guvercin & Verbovskiy, 2014; Chen, Dooren & Vershaffel, 2015) of improved levels of pupil interest and positive attitudes towards mathematics. Moreover, the study by Priest (2009) is particularly noteworthy since it found that a problem posing intervention facilitated the mathematical re-engagement of disengaged middle-year pupils.

Secondly, there is strong evidence from four studies (Abu-Elwan, 2002; Akay & Boz, 2009a; Toluk-Ucar, 2009; Walsh, 2016) of a significant impact of prospective teachers' mathematical achievement such as improved problem solving performance and conceptual knowledge. Toluk-Ucar (2009) makes a valuable contribution with regard to problem posing changing prospective primary teachers traditionally held beliefs on the nature of mathematics. Similarly, there is reasonable evidence from two studies (Akay & Boz, 2010; Fetterly, 2010) that problem posing can help to enhance mathematical creativity and self-efficacy, foster positive mathematical beliefs and reduce mathematical anxiety for prospective teachers.

Discussion

My findings for this question are in line with previous literature advocating that problem posing can enhance the mathematical experiences of learners and prospective teachers. For example, Stoyanova (2003, p. 39) encapsulates the essence of many scholars when she affirms that "problem posing activities provide environments that seem to engage students in reflective mathematical abstraction in a natural way. Such activities nurture students' attempts to explore problems and solutions structures rather than to focus only on finding solutions". In their study of 81 Australian primary pupils, English & Watson (2015) investigated the impact on developing statistical literacy. They found that the participants worked creatively and critically on tasks and that problem posing has the power to develop diverse mathematical thinking and improve confidence. Cai et al. (2013) employed problem posing tasks as a tool to investigate the long term effect on mathematical learning of 390 American secondary pupils. Using a system of linear equations, the researchers found a strong relationship between the participants' ability to solve a problem and their capacity to pose valid problems within the same mathematical context. Also, Cai et al. (2013) opine that problem posing can engender augmented conceptual understanding and bolster the growth of problem solving skills. In the same vein, it has been argued that problem posing can help young people to stimulate diverse and flexible reasoning (e.g. Silver, 1994; Leung, 2013; Kwek, 2015), foster creativity (e.g. Silver, Kilpatrick & Schlesinger, 1990; Silver, 1997; Leung, 2013), eliminate textbook dependency (e.g. Brown & Walter, 2005) and support the promotion of independent learning and critical thinking skills (e.g. Kilpatrick, 1987; Silver, 1994; Silver & Cai, 1996; Brown & Walter, 2005; Mamona-Downs & Downs, 2005), which are the cornerstones of CfE. Brown & Walter (2005, p. 1) argue that "problem posing can help students to see a standard topic in a sharper light and enable them to acquire a deeper understanding of it as well". Interestingly, they maintain that problem posing can be applied to "encourage the creation of new ideas from any given topic - whether a part of the standard curriculum or otherwise."

Grundmeier (2003) found that in his study of 19 American prospective K-8 teachers, the instruction of problem posing has a positive impact of participants' beliefs about the nature of mathematics, the learning of mathematics and the teaching of mathematics. Akay & Boz (2009b) investigated the views of 41 prospective Turkish primary teachers on completion of a problem posing training course. They reported that participants expressed numerous benefits of using a problem posing approach such as the encouragement of creative thinking and a connection to real life mathematics. In another study, Ticha & Hospesova (2013) examined the didactic competence of 56 prospective Czech primary teachers. The researchers discovered that problem posing provided a motivational influence which resulted in participants acquiring a deeper conceptual understanding of fractions. Likewise, in their study of 25 prospective Israeli secondary mathematics teachers, Lavy & Shriki (2010) found that by engaging in geometric problem posing activities, participants increased their mathematical knowledge and expressed a curiosity and enthusiasm towards learning mathematics.

So far, two previous systematic literature reviews have been implemented on mathematical problem posing. In their comprehensive meta-analysis, Rosli, Capraro & Capraro (2014) reported that problem posing activities have important benefits for mathematical achievements of learners from primary and secondary levels such as improved problem solving skills and positive attitudes towards mathematics. Their inclusion criteria identified fourteen individual experimental studies published between 1989 and 2011. Though, the researchers fail to offer an adequate explanation of why the study by Xia, Lu & Wang (2008) is presented as two distinct studies. Likewise, Zuya (2017) found similar valuable educational benefits of problem posing, based on sixteen single experimental studies published up to

2016. However, his study suffers from a lack of clarity in defining a search strategy and makes no attempt to discuss the strengths and limitations of the review process. Collectively, the systematic literature reviews by Rosli, Capraro & Capraro (2014) and Zuya (2017) played an important role in helping me decide to create a clear record keeping system. Moreover, the reviews spawned twenty empirical studies, fifteen of which are respectively featured in this study.

My research evidence suggests that mathematical problem posing is a powerful agent for raising attainment. The main educational benefits include the autonomous promotion of higher levels of mathematical thinking, reasoning, creativity, engagement and enhanced problem solving performances (Stoyanova & Ellerton, 1996; Silver, 1997; Cai et al., 2013; Singer, Ellerton & Cai, 2013; Chen, Dooren & Vershaffel, 2015). Nevertheless, my teaching experience and knowledge of the literature recognises that mathematical tasks with high cognitive demands are difficult to implement and are often converted into less challenging tasks during instruction (Stein, Grover & Henninsen, 1996; Stigler & Hiebert, 2004). This factor alludes to the effectiveness of the classroom practitioner which ultimately requires examination of their mathematical beliefs since change may be required to induce professional practice (Wilson & Cooney, 2002; Sowder, 2007). Since problem posing tasks are nurtured by their pedagogical actions, it is essential that practising teachers can be trained accordingly (Lowrie, 2002; Leung, 2016).

From professional experience, and considering my own research, there appears to be a mismatch between the holistic values and principles advocated by the mathematical research community and those implemented in Scottish classrooms. For instance, it is argued that there is an overplaying of examination techniques, which consequently, have suppressed the cultivation of young people’s creativity. I believe that the Scottish Government has a responsibility to recalibrate how they measure mathematical success in primary and secondary schools. To enhance pupil learning, the curriculum should be centred on rich mathematical tasks which consistently encourage higher levels of cognitive thinking and reasoning opposed to a saturation of routine procedural or computational activities (Hiebert & Wearne, 1993; Stein & Lane, 1996). Mathematical problem posing yields such tasks.

I would argue that the overarching current curriculum structure does not possess the intended flexibility to sustain the coalescing of new research perspectives within classrooms. Although it may be perceived by many that it is straightforward for someone to initiate a change in professional practice, it is another matter to navigate the trajectory of a transformational change in educational policy. One method to achieve this is to combine both a descriptive and prescriptive approach to the mathematics framework that will ensure conceptualisation and operationalisation of mathematical problem posing throughout all schools in Scotland. Pragmatically, such an intervention can only be promulgated by expanding the mathematics guidelines of Curriculum for Excellence. Cai et al. (2015, p. 17) maintain that if "problem-posing activities are to play a more central role in classrooms, they must be more prominently represented in curricula". Similarly, Bonotto & Del Santo (2015, p. 121) concluded from their exploratory study of Italian primary pupils, that "the presence of problem-posing activities should not emanate from a specific part of the curriculum but should permeate the entire curriculum".

In summary, this research study has found that robust empirical evidence exists which shows that mathematical problem posing can improve pupil learning by deepening conceptual understanding, fostering problem solving skills, transforming attitudes towards mathematics, cultivating creativity and promoting critical and independent thinking at various echelons of school education. It supports the view that mathematical problem posing can provide a springboard for connecting school settings with real life situations and offers a constellation of educational benefits that assist practitioners in promoting social constructivism and collaborative learning. Moreover, it has also emerged from this study that strong empirical evidence exists that mathematical problem posing is effective in the pedagogical development of prospective primary and secondary mathematics teachers. These remarks provide responses to each of the sub questions of the first research question.

8.3 Research question two

Are there any differences in the mathematical beliefs of Scottish primary and secondary teachers?

The results of my research indicate that whilst both sectors share particular commonalities, a number of significant differences exist between the espoused mathematical beliefs of primary and secondary mathematics teachers. Narrative evidence of enacted mathematical beliefs has engendered multiple conjectures of classroom practices.

Discussion

The evidence suggests that both primary and secondary mathematics teachers hold similar clusters of mathematical beliefs, of which are strongly influenced by their perception about the nature of mathematics, the learning of mathematics and the teaching of mathematics. These aspects have been previously highlighted in chapter five as fundamental topographies of mathematical beliefs (Stipek et al., 2001; Barkatas & Malone, 2005). Indeed, three distinct, but not mutually exclusive, mathematical beliefs systems emerged as follows:

1. A social constructivist, problem solving and collaborative orientation;
2. A social constructivist, problem solving and static transmission orientation;
3. A static and mechanistic transmission orientation.

This finding that teachers' beliefs systems are not in isolation but are grouped together in clusters is consistent with the literature (Rokeach, 1968; Green, 1971; Leatham, 2006). Furthermore, some teachers appear to hold a mixture of fallibilistic and absolutist beliefs. Such an incompatible or inconsistent arrangement suggests that not all teachers' beliefs systems are logically structured, which accords with other research perspectives (e.g. Thompson, 1992; Ernest, 2004; Beswick, 2012).

Previous empirical studies have identified a range of similar mathematical belief orientations as are found in my research. For example, in a study of Australian secondary mathematics teachers, Beswick (2005) located three clusters of beliefs that she identified as 'content and understanding', 'content and clarity' and 'relaxed problem solvers'. Curiously, Beswick did not uncover a cluster representing teachers with an instrumentalist view of mathematics and associated views of learning and teaching of mathematics. Barkatsas & Malone (2005) found that Greek secondary mathematics teachers held two main beliefs; a contemporary-constructivist orientation (consisting of a social constructivist view, problem solving view and a collaborative view) and a traditional-transmission information processing orientation (consisting of a static view and a mechanistic view).

Similarly, in a study of 249 Australian secondary mathematics teachers' beliefs, Howard, Perry & Lindsay (1997) also discovered two main categories i.e. constructivist beliefs and transmission beliefs; the results of which are intriguing given that it involved an identical sample size of secondary mathematics participants and a comparable range of grades to this study. Analysis of several individual questionnaire items revealed a number of similarities and contrasts. For example, comparable quantities of participants from both countries agreed with the item: 'Teachers should provide instructional activities which result in problematic situations for learners.' However, more than double the amount of Scottish participants disagreed with the negative belief statement: 'Mathematics is computation'. Overall, it is suggested from a cross-national perspective, mathematical problem solving is equally supported and that Scottish mathematics teachers hold less instrumentalist views about the nature of mathematics. However, this analysis is superficial given the Australian sample was confined to teachers only from the south western suburbs of Sydney.

In another Australian study, Nisbet & Warren (2000) surveyed 389 primary teachers' beliefs of mathematics, the teaching of mathematics and the assessment of mathematics. The researchers found that beliefs about the nature of mathematics reflected only two of Ernest's (1989a) three categories of views of mathematics. Nisbet & Warren (2000, p. 44) noted "it seems that primary teachers hold limited views of what mathematics is – static and mechanistic views, rather than a dynamic problem-driven ever-expanding field of human creation".

In a study involving English practising primary teachers undertaking postgraduate training to become primary mathematics specialists, Allen (2010) uncovered little evidence of practitioners viewing mathematics in terms of problem solving. She argued that in order for them to become effective teachers of mathematics, they need to shift their beliefs of mathematics from Platonist or instrumentalist to one of problem solving (Ernest, 1989a; Nisbet & Warren, 2000).

In short, it is postulated that for both sectors, the operationalisation of mathematical problem solving and problem posing is restricted in practice. Even although substantial advice is readily available for practitioners to teach problem solving skills and to employ problems as a focus of learning in mathematics education (Wilson & Cooney, 2002; Lester & Cai, 2016)

There are six possible explanations which help to illuminate this posit:

- Conceptualisation of mathematical problem solving is misunderstood
- Struggle is not widely supported
- Time pressure due to assessments
- No curricula requirement
- Mathematics self-efficacy
- Misconception of the construct of mathematical problem posing

Each will now be considered in turn:

Firstly, the evidence suggests that the theoretical conceptualisation of mathematical problem solving is considered to be misunderstood within significant divisions of both sectors. The comprehension of what constitutes a mathematical problem is a particularly troubling finding. In my research, more than half of the primary teachers and over a third of the secondary mathematics teachers strongly agreed or agreed with the misconception that ‘a mathematical problem is the description of a situation involving stated quantities, followed by a question about some relationship among the quantities’. As a practising teacher, I have concerns from my experience that the ‘everyday function problem’ (illustrated by Lorna) may be indicative of a shortcoming among some Scottish teachers. The reason for this is not obvious but may have something to do with the misinterpretation of the principles underlying their professional development. Such a situation resonates with Cohen’s (1990) well-known case study of a teacher named Mrs Oublier. According to Cohen (1990), Mrs. Oublier was open to new curriculum ideas but that the change initiated by her professional development remained dormant. Cohen (1990, p. 312), concluded that while her teaching reflected the new framework in many innovative ways, “Mrs. O seemed to treat new mathematical topics as though they were part of traditional school mathematics”. Perhaps, another possible reason on the theme of CPD, might be the case that the professional development provided in mathematics education has not met the individual needs of teachers (Sowder, 2007). Overall, the results from this study were more disappointing than Thompson (1989) obtained where she reported that under one third of American primary teachers' concurred with a similar definition of a mathematical problem.

As an interrelated issue, participants from both cohorts in this study were unable to articulate the structure and purpose of different types of mathematical problems (e.g. open-ended). In order to facilitate pupil learning, practitioners must be able to identify what kind of problems exist in order to address the issue of solving them (Polya, 1981). Recent studies have bolstered teachers' knowledge of rich problems and highlighted the importance of diverse types of problems in implementing problem solving (e.g. Crespo & Sinclair, 2008; Guberman & Leikin, 2013). Furthermore, without a coherent understanding of the conceptualisation of problem solving, it is impractical for teachers to evaluate the success of any corresponding classroom experience.

Secondly, a finding of this study suggests that a third of primary teachers and half of secondary mathematics teachers do not endorse the view that 'struggle' is a natural component of the learning mechanism during problem solving. While making reference to socio-mathematical norms, both Morag and Skye conveyed the message that a child struggling during the learning of mathematics is an adverse and sterile corollary. Subsequently, such a teacher dominated instructional approach to mathematics eliminates an essential ingredient for engendering pupils to think independently by taking responsibility for their own learning, developing new thoughts and ideas, making choices and learning from their mistakes and most importantly, persevering. Such a representation erroneously conveys to young people that solving mathematical problems is achieved by solely following an algorithm formulated by someone else, with no obligation to actively engage in any strategic and metacognitive process. It will not promote desirable characteristics such as fluency, flexibility and creativity. The notion of challenge is underlined by the NCTM (2014, p. 7) who opine that "[s]tudent learning is greatest in classrooms where the tasks consistently encourage high-level student thinking and reasoning and least in classrooms where the tasks are routinely procedural in nature".

In the interest of clarity, the phenomenon of 'struggle' (also known as 'productive struggle') discussed here refers to the intellectual effort pupils expend to make sense of mathematical problems (Hiebert & Grouws 2007). It does not refer to needless frustration or extreme levels of challenge created by nonsensical or overly difficult problems but tasks that fall within the pupils' reasonable capabilities (Hiebert & Grouws, 2007), given appropriate time and support (Warshauer, 2015). Teachers have a professional obligation to ensure that they encourage pupils to persevere when solving mathematical problems irrespective of struggle, since putting pupils in such situations where they will experience difficulties and frustrations serve as rich learning opportunities (Schoenfeld, 2014; Star, 2015). Likewise, it should also be acknowledged that pupils need to harmonise with the systematic engagement of challenging mathematical activities (Lester, 1994; Lester & Charles, 2003).

This is a rather worrying finding given that secondary mathematics teachers, by the nature of their undergraduate training, possess higher levels of mathematical problem solving expertise and experience compared to their primary colleagues. A possible explanation for this might be that the majority of primary teachers have superior mathematical pedagogical knowledge and thus are able to facilitate pupils learning through various strategies. Another alternative cause to illuminate my finding is the contextual factor of curriculum accountability. For instance, Alasdair indicated reluctance to promote critical thinking because he wanted to circumvent another adverse examination results review. Also, Skye defends her decision to teach by rote learning based on the presence of an "assessment monster". In their study of teacher characteristics, Clark et al. (2014) maintain that elements such as fast curriculum planning, heightened accountability pressures and prevalent tracking policies may negatively influence teachers' mathematical beliefs.

Thirdly, it is suggested that time restrictions due to a statutory requirement to incorporate national assessments has prevented participants from both sectors in engaging learners in regular employment of mathematical problem solving, although this is more notable with secondary mathematics teachers. It is postulated that time demands have resulted in an acceptance of traditional forms of instruction with an emphasis on high-stakes testing in 'order to survive'. Consequently, there is a de-emphasis on aspects of mathematical learning such as reasoning, representation, problem solving, communication and making connections, since these strands are not tested (Schoenfeld, 2001). The tension between teaching and assessment has been widely examined in the literature. In speaking of this custom, Sacks (1999) writes:

> Schools and teachers, under intense pressure to boost achievement scores, have discovered the educationally dubious practice of teaching to tests. That, in turn, has narrowed [what is] taught... teaching to tests has a dumbing effect on teaching and learning, as worksheets, drills, practice tests, and similar rote practices consume greater amounts of teaching time (p. 156).

In this study, Cormac makes disapproving reference to the pressure of not being liberated to teach mathematics, “but the exam techniques that the SQA are looking for”. This narrative is commensurate with a recent critical evaluation about the situation of English school examinations by Anthony Seldon, who perceptively avowed: “Schools have come under unbearable pressure to teach not the academic subject but the exam techniques that markers look for” (Thunderer, 2016). In their exploratory study of one Scottish local education authority carried out prior to the implementation of CfE, Hayward et al. (2008) report a similar assertion:

> The pressure of the examination syllabus is frequently offered as a reason why many teachers are reluctant to move away from traditional patterns of continuous summative assessment and examination rehearsal that could be said to have dominated the upper stages of secondary schools in Scotland for many years (p. 1).

Likewise, the Scottish Government (2010c, p. 2) acknowledged “there is a consensus that we focus too much on preparation for examinations as learner’s progress through schools”. It is ironic that the same administration has spawned a proliferation of CfE documentation that has led to an inexorable development of assessment bureaucracy.

Whilst teaching to the test has the potential to produce improved examination results, such a pedagogical approach focuses on rote memorisation exercises which do not promote deep conceptual understanding or the fabrication of creative and critical thinking skills. The evidence implies a systemic perspective on the explicit driving force stimulating this didactical tradition in Scottish education; namely that each school is controlled by the assessment culture of their respective LEA. It is conjectured that minimum scope exists for teachers to exercise freedom to regularly employ problem solving with mathematics.

Fourthly, and interrelated to the previous reason, a general apathy subsists amongst both sectors that as problem solving is not formally assessed within CfE, there is no requirement to engage pupils with this enterprise. For example, Alasdair argues that it is not feasible to “assess critical thinking”. However, the Scottish Government (2011b) oppose this viewpoint since they state that “assessment will place a greater emphasis on... higher order skills including, creativity” (p. 7) and “Assessment should probe the ability to apply the learning in more challenging tasks and in unfamiliar situations” (p. 14). In order for pupils to become convinced of the importance of problem solving within mathematics, it is necessary to use assessment techniques that encourage and reward the development of higher order thinking skills (Lester, 2013).

Nevertheless, it is possible that this finding is not an accurate representation and a more plausible concomitant explanation might be secondary practitioners feel ‘unsupported’ when engaging pupils in problem solving. The manifestation of which is explicit reference to other agents that may have rebuffed previous attempts to integrate a culture of problem solving within classroom practice. Based on the accounts of Alasdair and Skye, it is conjectured that both individuals may have been undermined at some stage in their professional enquiry by authoritarian leadership fixated with school improvement agendas. Tension propagates in situations where teachers find that their colleagues do not share their zeal for the inquiry or feel threatened by it (Drew, Priestley & Michael, 2016). For instance, it may be the position for some stakeholders, that procedural knowledge including memorisation of facts is more valued than conceptual understanding gained through problem solving. Lester & Cai (2016, p. 127) assert that “many people, parents and teachers alike, worry that the development of students higher-order thinking skills in teaching problem solving comes at the expense of the development of basic mathematical skills”. Alternatively, the hierarchical structure of many educational establishments may serve to counteract the criticality of research informed practice. Theoretically, it is conceivable for an unpromoted teacher to enact positive mathematical beliefs of problem solving and still encounter unfounded criticism of their enhanced classroom practice by less informed senior contemporaries. It is important to consider such an eventuality and how to safeguard the growth of teacher professional enquiry.

Fifthly, some primary participants reported possessing weak mathematical content knowledge resulting in feelings of inadequacy in teaching how to solve mathematical problems. Having low mathematics teaching self-efficacy may serve as a barrier to teacher effectiveness and tends to be characterised by totalitarian and teacher centred approaches. The important construct of self-efficacy has been explored by many researchers. Grounded on the seminal theoretical contribution to social cognitive theory by Bandura (1977), Tschannen-Moran, Woolfolk Hoy & Hoy (1998) observe that:

> Teaching self-efficacy has been defined as both context and subject matter specific. A teacher may feel very competent in one area of study or when working with one kind of student and feel less able in other subjects or with different students (p. 215).

For example, it is possible for a practitioner to have high self-efficacy while teaching estimation and rounding but maintain low self-efficacy for teaching expressions and equations. Within the domain of mathematics, Hackett & Betz (1989, p. 262) express mathematics self-efficacy as “a situational or problem-specific assessment of an individual’s confidence in her or his ability to successfully perform or accomplish a particular [mathematics] task or problem”. In other words, mathematics self-efficacy may be conceptualised as a belief of a teacher for successfully implementing problem solving. If a teacher does not believe they are capable of implementing problem solving, they have minimum incentive to act (Bandura, 1986). Other indicators of low self-efficacy of mathematics teaching include an emphasis on decontextualised conceptual understanding such as memorising rules and repeated practice of similar questions, opposed to searching for multiple solutions, exploring patterns and formulating conjectures. Arguably, these symptoms may also be associated with an instrumentalist philosophy of mathematics.

However, it is not clear from the results in this study if primary teachers’ self-efficacy relates jointly to solving mathematical problems and to teaching mathematical problem solving. Irrespectively, self-efficacy is a critical factor for the effective delivery of mathematics teaching and is reflected within a growing body of empirical evidence. For instance, Charalambous & Philippou (2010) examined the connection between 151 Cypriot elementary mathematics teachers’ concerns about curriculum reform, problem solving and their teaching efficacy beliefs. The researchers discovered that teachers’ efficacy beliefs were found to be complex and influenced the nature of classroom practice. In her study of mathematics teaching self-efficacy beliefs of 33 Turkish primary teachers, Nurlu (2015) reported that teachers with higher self-efficacy beliefs were associated with increased effort, persistence, openness to new pedagogical approaches and building stronger pupil relationships.

Finally, with respect to mathematical problem posing, the initial results from the questionnaires suggest that both sectors similarly espouse to hold strong beliefs of problem posing, in spite of this central activity being excluded from curricula guidelines. However, during the interviews, participants were generally unable to articulate the intrinsic abstract link to problem solving or convey the positive impact of problem posing on pupils' enriched mathematical learning, higher order thinking, creativity and confidence. This rather unexpected result was compounded by a common misconception of the problem posing construct, including minimum support for the orchestration of problem posing within classroom practice. It is likely that this inconsistency between the results for stage one and stage two of this study may be attributed to the small sample size of interviewee participants. Another possible explanation for this discrepancy is that participants were dependent on regurgitating established curricula guidelines and were confused by the non-availability or acted in defence of the non-availability of curricula guidelines.

Reflecting on the rich data generated in my research, it is suggested that the mobilisation of mathematical problem posing in primary and secondary schools may be restricted. Furthermore, it is conjectured that teachers' espoused beliefs are inconsistent with their enacted beliefs. Such a finding is in accordance with previous research (e.g. Thompson, 1992; Raymond, 1997; Cooney, 1999; Barkatsas & Malone, 2005).

As a consequence of these aforementioned reasons, it is believed that this may produce a detrimental effect on the quality and nature of the mathematical learning experiences of pupils, as teachers' beliefs have been found to influence their instructional practices (Thompson, 1985; Nespor, 1987; Stipek et al., 2001; Wilson & Cooney, 2002). Ebert & Risacher (1996, p. 5) argue that: "Teachers' beliefs about how students learn mathematics, their beliefs about mathematics itself, and their knowledge of teaching in general, are likely to affect how they design and teach lessons".

Statistical evidence

The interpretation of statistical significant differences has to be grounded in a meaningful context. In juxtaposing a core curriculum subject delivered by non-mathematical specialists and mathematics experts, it may be natural to assume that diverse mathematical belief systems feature intrinsically. Though, theoretically this view is rejected by the linear mechanism of CfE which advocates that the transformation of learning is the responsibility of every practitioner, irrespective of sector. In other words, primary and secondary mathematics teachers are challenged to deliver a uniform standard of mathematical experiences. Furthermore, both sectors are compelled to operationalise mathematical problem solving within their continuum of classroom practice.

The first evidence to emerge of significant statistical differences between the mathematical beliefs of primary and secondary mathematics teachers is highlighted by findings of the questionnaires, which indicated sector agreement with less than one fifth of the thirty-nine individual belief statements. This is a troubling result of which, due to time restrictions, I was only able to partially explore during the interviews. As a strategy, I concentrated on accumulating data from each participant, regarding one or two identical belief statements from each of the five belief factors.

There is a notable paucity of cross-sector empirical studies focussing specifically on investigating the differences between current primary and secondary mathematics teachers' beliefs about the nature of mathematics, the learning of mathematics and the teaching of mathematics (Anderson, 2005; Perry, Wong & Howard, 2006; Dede, 2015). In their comparative study, Perry, Wong & Howard (2006) compared the beliefs of primary and secondary mathematics teachers from Hong Kong and Australia. For comparison purposes, I reviewed one statistically significant positive item (Table 8.1) which involved merging the results from this study for the response options of 'strongly agree' and 'agree' along with 'strongly disagree' and 'disagree'. Whilst this comparison only serves to highlight one example, it has provoked me to consider the perpetual internationalisation of mathematics education and Scotland's position within it.

Table 8.1 Comparison of one positive belief statement with Perry, Wong & Howard (2006)

Belief Statement	Teacher Group	Sample	Agree (%)	Undecided (%)	Disagree (%)
Teachers should provide instructional activities which result in problematic situations for learners	Australian Primary	252	87	9	4
	Australian Secondary Mathematics	249	82	15	2
	Hong Kong Primary	377	10	42	48
	Hong Kong Secondary Mathematics	179	12	73	15
	Scottish Primary	**229**	**65**	**14**	**21**
	Scottish Secondary Mathematics	**249**	**73**	**13**	**14**

Further evidence to emerge of significant statistical differences between the mathematical beliefs of primary and secondary mathematics teachers is highlighted by the interesting findings of two belief factors.

Social constructivism

Primary teachers hold stronger social constructivist beliefs towards the nature of mathematics, the learning of mathematics and the teaching of mathematics, compared with secondary mathematics teachers. There are several possible explanations that may help to illuminate this finding.

The evidence from my study suggests that primary and secondary mathematics teachers' beliefs are not homogeneously compatible with a social constructivist orientation, and reject the tacit assumption that teachers' beliefs were 'appropriately consonant' prior to implementation of CfE. In the absence of any previous empirical research, it is not possible to determine to what extent curricula reform has impacted on Scottish teachers' beliefs. However, it is widely acknowledged that pedagogical reforms cannot take place unless teachers' beliefs about mathematics, the learning of mathematics and the teaching of mathematics change (Ernest, 1989a). Simultaneously, accepting that teachers' mathematical beliefs are notoriously difficult to change (e.g. Pajares, 1992; Goldin, 2002; Handel, 2003), it cannot be supposed that all changes in beliefs translate directly into transformations in professional practice (Richardson, 1996). In other words, it may be conceivable that a dissonance between primary and secondary mathematics teachers' beliefs has perpetually subsisted and this study has only served to reveal this reality.

My own professional experience informs me that teachers are to some extent institutionalised by the contextual constraints of their societal working environment, which due to external influences such as colleagues, senior management or national priorities can internalise the enactment of individual didactic approaches. Ernest (1989b) states that:

> The socialization effect of the context is so powerful that despite having differing beliefs about mathematics and its teaching, teachers in the same school are often observed to adopt similar classroom practices... The social context clearly constrains the teacher's freedom of choice and action, restricting the ambit of the teacher's autonomy (p. 252-253).

In other words, teachers' enacted beliefs may be inconsistent with their espoused beliefs due to the context they find themselves working in. In my research, it was found that teachers mathematical beliefs are not regarded in insolation but belong to a cluster which form a belief system (Rokeach, 1968; Green, 1971; Leatham, 2006). Strongly held beliefs are considered more central and further resistant to change, while less firmly held beliefs are deemed peripheral (Green, 1971). As was mentioned in chapter five, the relative centrality of beliefs varies with context (Green, 1971). From this perspective, teachers' espoused beliefs are not considered indicators of classroom practice. Conversely, classroom practice is determined through interactions with context and shifts along a continuum of diverse beliefs. According to Beswick (2007, p. 97), the "relative centrality of an individual's beliefs will vary from context to context. Failure to enact a particular belief evident (via words and/or actions) in one context or another, can thus be seen as the result of different beliefs taking precedence in the different situations". Likewise, Skott (2009, p. 44) in his instrumental study of a Danish novice mathematics teacher called Larry, underlines the existence of multiple, possible conflicting, authentic and virtual communities of practice that help to contextualise "the act of teaching in intersubjectively established and continually re-generated settings". Consequently, it is argued that the interpretation of enacted beliefs is meaningless without due consideration of context (Skott, 2009). In this study, it is robustly contended that assessment demands of national examinations have restricted the flexibility of secondary mathematics teachers to adopt learner-centred approaches in place of traditional instructional practices.

In addition, I postulate that primary teachers are more able to assimilate a social constructivist approach to the learning of mathematics. It is argued that the structure, ethos and philosophy of primary education is more likely to encourage the active involvement of all pupils, where feedback is more forthcoming and the multitude of learning activities help to motivate and promote relevance. Primary practitioners contribute to the holistic development of children and perhaps, are adept to recognise the uniqueness and complexity of learners' needs, which allows for a more accurate diagnostic assessment of prior knowledge. For example, it was notable in the case of Morag that she was concerned with children's confidence for learning. Sustaining motivation to learn is strongly dependent on pupils' confidence (Von Glaserfeld, 1989).

Curiously, only primary participants in my research appear to promote the employment of manipulatives in the learning and teaching of mathematics. Manipulatives are physical or virtual objects that pupils and teachers use to illustrate and discover mathematical concepts (Van de Walle, Karp & Bay-Williams, 2014) and can be utilised to connect ideas and integrate knowledge during mathematical problem solving and problem posing (Rosli, Goldsby & Capraro, 2013). Much of the literature surrounding manipulatives is supportive as they offer powerful visual representations to intangible mathematical concepts (Golafshani, 2013). In a study of 503 American primary teachers, Uribe-Florez & Wilkins (2010) found that teachers' beliefs of manipulatives were related to the level of classroom use. Interestingly, the researchers concluded that "teachers who tend to believe that the use of manipulatives with older students is less necessary were found to use manipulatives less often" (p. 370).

Nevertheless, another possible factor may be that secondary mathematics teachers' beliefs are more resistant to change due to concerns about classroom management. For example, Skye defends her teaching approach regarding the need to struggle to avoid pupils "moaning" and her professional practice "being criticised". Thompson (1989) on the theme of mathematical problem solving maintains:

> While secondary teachers tend to be stronger than elementary [primary] teachers in their knowledge of the subject matter, I have found secondary teachers generally more resistance to introducing changes into their teaching. Elementary teachers, for the most part, tend to act more enthusiastically to new techniques, but their generally weaker mathematics background, and feelings of inadequacy to handle mathematical problem solving, become serious obstacles (p. 234).

However, in the Scottish context, the construction of mathematical proofs is no longer the driving force behind the expansion of mathematical knowledge in schools. Central to the proof discourse is the use of appropriate language and diagrams. My research leads me to question whether secondary practitioners are more disadvantaged by this pedagogical propensity to underuse one of the fundamental constructs within mathematics. From my perspective as a practising teacher, I am concerned that pupils are deprived of rich enculturation opportunities to actively engage together in conjecturing, exploring, reasoning, justifying, verifying and critiquing. Besides, it is unlikely that secondary teachers holding instrumentalist or Platonist beliefs about the nature of mathematics will advocate the employment of proofs to construct ideas or to extend mathematical thinking within their classroom practice.

Collaborative learning

Primary teachers hold stronger collaborative beliefs towards the nature of mathematics, the learning of mathematics and the teaching of mathematics, compared with secondary mathematics teachers. There are two possible explanations that may help to illuminate this result.

Firstly, it is suggested that primary teachers are more committed to orchestrating learning activities that offer challenge through which pupil understanding is achieved by mathematical discussion. It is evident that primary teachers' beliefs are consistent with the need to promote dialogue, which is a critical feature to enhance understanding. For example, Grace asserts that a mathematical task should not be decomposable by urging teachers to "choose a task that cannot be solved by just one pupil" and complements this by highlighting positive interdependence with the observation, "but needs input from others". Likewise, Morag expressed that pupils should not readily accept an opposing view without verification by inclusion of the word "equally" in the comment, "they can share success equally with no one child to blame for failing to solve a task". Swan (2006, p. 85) refers to the seminal work of Piaget when he reminds us "the most effective form of social interaction is cooperation between equals in which each tries to understand and modify the other's point of view". This perspective resonates with Boaler (2008) and her induction of the term 'relational equity'. In her study of American secondary mathematics teaching approaches, Boaler (2008) uncovered excellent societal relations that developed in classrooms among pupils at one particular school. In defining the construct, she outlines three important strands:

1. respect for other people's ideas, leading to positive intellectual relations;
2. commitment to the learning of others;
3. learned methods of communication and support (p. 174).

What is fascinating about her findings is that it involved mixed ability groups and challenges the conventional belief held by many Scottish mathematics teachers that a heterogeneous approach to secondary school mathematics is counterproductive to effective teaching. Interestingly, in this study, Isabella advised that collaborative groups should contain "a mixture of abilities", prompting me to acknowledge another possible difference between the sectors.

Secondly, secondary mathematics teachers appear to hold weak collaborative beliefs towards the learning of mathematics and the teaching of mathematics. The main thrust of justification for this apathy is grounded on overarching concerns about time constraints and classroom management issues. Moreover, there appears to be a lack of cohesion with the principles underlying collaborative learning. For example, participants made no reference to an interpersonal activity in which pupils are challenged to think for themselves and to arrive at understanding through open discussion (Askew et al., 1997) or the potential rich construction of multiple solutions or multiple answers during problem solving. Instead, participants conveyed an unambiguous message of low expectations of pupils to actively engage in fertile discussion during collaborative learning. This was salient in the case of Skye who claimed that young people use it as “an excuse to talk about other stuff or take out their phone and do absolutely nothing”. Perhaps the most disappointing finding in this matter is derived from the narrative of Cormac, who articulates that his pedagogical approach is inhibited by the contextual nature of his school. He critically draws attention to an archaic expectation that mathematics learning must be overtly conducted within a vacuum isolated from social discourse by asserting that a key shortcoming “is noise because classes in this corridor are quiet and anytime there’s a din, the PT marches in to see what’s going on!”. The literature is replete with hazards of teachers’ overcontrol of interactive classroom discourses. The upshot of which is that children’s opportunities of expressing, sharing and communicating mathematical ideas about problem solving and problem posing are compromised, producing superficial learning cameo roles.

Though, what appears to dominate the previous points per se is a disparaging perception by participants that not all learners possess an equitable capacity to demonstrate creativity or originality, which only serves to perpetuate the myth that imagination is interlinked with intelligence or mathematical ability. Furthermore, such a pervasive disjunction will not foster motivation or cultivate confidence but will permeate to marginalise pupils within their own learning communities. More prosaically, classroom practices may insidiously polarise pupils into accepting that equity is not the universal entitlement for all. Schoenfeld (2014) demonstrates that one of the dimensions of mathematically powerful classrooms is the active engagement of all pupils. In short, it is posited that secondary mathematics teachers may not value the contribution of every young person.

It is unsurprising that when coupled together, the results for a social constructivist, problem solving and collaborative orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics, show that primary teachers hold statistically significant stronger CfE mathematical beliefs than secondary mathematics teachers. It is of interest that both sectors share similar ‘neutral’ beliefs of a static transmission and mechanistic transmission towards the nature of mathematics, the learning of mathematics and the teaching of mathematics. Furthermore, whilst statistically significant social constructivist differences exist between the sectors, it is notable that the strength of both orientations is considered ‘weak’.

These findings are concerning, since from a hierarchical perspective secondary mathematics teachers are generally assumed to form a hegemonic group within mathematics education. In their exploration of the dynamics of teachers’ beliefs about mathematics, Burroughs & Schmidt (2014) illuminate differences between the sectors as follows:

> Secondary mathematics instructors tend to be specialists in their fields. Given their more intensive exposure to mathematics, their attitudes about mathematics instruction may be quite different from those of primary school teachers, who are responsible for giving basic instruction in many subjects. As generalists with what may be only a smattering of math courses during their preparation to become teachers, the cultural background of primary school teachers could play an especially large role in shaping their beliefs. In addition, as their first exposure to formal mathematics, students attitudes about math may be powerfully influenced by the beliefs of their elementary school teachers (p. 280).

The above statement highlights two main themes that require further examination. To begin with, it infers that primary practitioners’ prescriptive espoused mathematical beliefs act as a catalyst for the implementation of didactic instructional practices. This perspective resonates with previous studies (e.g. Stipek et al, 2001; Beswick, 2005; Cross, 2009). For example, Nisbet & Warren (2000) observed that:

> primary teachers hold limited views of what mathematics is – static and mechanistic views, rather than the view as a dynamic problem-driven ever-expanding field of human creation (a view more aligned with the constructivist model of learning) – and this impacts on their approach to teaching (p. 45).

However, such a position is not the exclusive domain of primary teachers. Within this study, several mathematics participants professed to enact teacher-centred approaches, in contrast to their problem solving orientation towards the nature of mathematics. Perry, Wong & Howard

(2006) feature a compelling extract from a secondary teacher, which encapsulates a comparable approach:

> Sometimes it is easier and quicker to use a transmission approach to get through the maths content, particularly in Years 11 and 12 [S4 and S5]. The amount of content affects the style that we use to teach. You have to have planning – time and effort – to use child-centred approaches while you cover a lot more and it is a lot easier to use transmission approaches but the long-term results are not as good (p. 445).

The second theme is reference to the influence of primary teachers' mathematical beliefs on classroom practices of future primary teachers; although I suspect that secondary education experiences will also profoundly feature in this regard (Handel, 2003). It is evident from my study, that teachers' implicit and explicit beliefs have been fashioned by prior school experiences, which is consistent with previous research (e.g. Hudson, Henderson & Hudson, 2015). Such a finding suggests that multiple long term emotional and cognitive attachments to patterns of thinking act as a barrier to the enactment of new techniques of working. To stimulate a change in beliefs, practitioners need to critically reflect on their classroom practice through professional enquiry. Engaging in research will energise teachers to make sense of the misalignment between beliefs and practice. For example, inconsistencies (Cross, 2015), problem solving (Andrews & Xenofontos, 2015), multiple solutions (Guberman & Leikin, 2013), problem posing (Barlow & Cates, 2006) and collaborative learning (Swan, 2006).

Notwithstanding any of the previous comments, I now turn my attention to what I perceive to be the fundamental issues that may explain the tension between teachers' beliefs and the conceptualisation and operationalisation of mathematical problem solving and problem posing. Few studies have examined the role of Scottish teachers' mathematical beliefs and no previous study has investigated the mathematical beliefs of Scottish current primary and secondary mathematics teachers. Such a paucity of national research may well enlighten the lack of awareness and impetus for raising the profile of teachers' beliefs within the Scottish educational landscape. No formal attempt has been made by relevant stakeholders to disseminate any of the localised contributions (e.g. Henderson, 2012b; Priestley & Minty, 2013; Hudson, Henderson & Hudson, 2015) to school establishments. Surprisingly, Bryce et al. (2013), throughout their various editions of work detailing our national educational system, fail to acknowledge the important role of teachers' beliefs, despite the growing body of literature including substantive reviews (e.g. Kagan, 1992; Pajares, 1992; Thompson, 1992; Calderhead, 1996; Fang, 1996; Richardson, 1996; Philipp, 2007; Fives & Buehl, 2012).

While practitioners from both sectors appear to successfully promote numeracy, confidence and progression, a political undercurrent overshadows the enactment of mathematical beliefs designed to promote relational understanding. Classroom practices are circumscribed by an over dominate national assessment regime which mechanically restricts the conceptualisation and operationalisation of mathematical problem solving and problem posing. In his review of Australian education, Clarke (1987, p. 9) noted that, “schools continue to succeed in the teaching of routine computation and to fail in the teaching of such skills as problem solving... the maintenance of current assessment procedures serves only to maintain the illusion that significant learning is taking place”. It is important to consider to what extent teachers have been empowered by the theoretical underpinnings of CfE, while reflecting analytically on a formidable capacity to de-professionalise the same teachers by an oppressive culture of high stakes testing.

8.4 Research question three

What factors impact on the mathematical beliefs of Scottish primary and secondary mathematics teachers?

The results of my research indicate that two factors impact on the mathematical beliefs of Scottish primary and secondary mathematics teachers i.e. grade and highest qualification in the field of education. In order to provide a more comprehensive overview, beliefs have been examined from a cumulative perspective (i.e. TMBS) followed by the impact from each of the five belief factors respectively (i.e. BF1, BF2, BF3, BF4 & BF5).

Discussion

Grade and TMBS

This study indicates that primary deputy headteachers hold significantly stronger mathematical beliefs towards the nature of mathematics, the learning of mathematics and the teaching of mathematics, than unpromoted primary teachers. What is surprising is that primary headteachers’ mathematical beliefs did not differ significantly from any other grade. It is difficult to explain the variance between the result for primary deputy headteacher and primary headteacher (Figure 6.16), but is might be related to the length of time already served by primary headteachers in their current management role. It is important to consider the actions of primary headteachers prior to the introduction of GTC professional standards.

Secondary mathematics teachers beliefs were found to be unaffected by grade, in particular, the position of headteacher. This rather unexpected result might be explained by the professionalism and expertise required for leadership and management does not require domain subject competence although this is arguable given that headteachers have to evaluate effective learning and teaching. Although the small sample size of secondary headteachers may explain this statistical anomaly.

Grade and Belief Factors

Firstly, the results suggest that primary headteachers have a stronger problem solving orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics, compared with primary teachers. A possible explanation for this might be that primary headteachers are more able to critically engage with literature, research and policy. There are, however, other possible reasons. For example, it seems plausible that mathematics teaching self-efficacy beliefs may be inconsistent among primary teachers. Alternatively, primary teachers' mathematical beliefs may align with the American practitioner represented by Ms. Perry (Aguirre & Speer, 2000). In their study, the researchers describe Ms. Perry's beliefs about the learning mathematics being preoccupied on explaining a new concept and ensuing classroom interactions, opposed to focussing on developing her pupils' mathematical thinking.

Secondly, my research indicates that primary teachers have a stronger static transmission orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics, compared with primary deputy headteachers. The reason for this is not clear but if may have something to do with primary teachers maintaining a prescriptive belief of the teaching of mathematics. For example, Morag maintains that it is important to explain in detail what pupils have to do to solve problems. This view resonates with the instructional beliefs of the practitioner, Lynn (Thompson, 1984, p. 117), who contends that pupils "learn mainly by attentively watching the teacher demonstrate procedures and methods for performing mathematical tasks and by practicing those procedures". Another possible explanation for this is that some primary teachers may encourage an instrumentalist approach underlined by a repetition of arbitrary rules and procedures that reward learners with instant success by allowing them to solve similar undertakings without little or no conceptual understanding (Ernest, 1991). Morag stressed the importance of children achieving success with the right answer in order to boost their confidence. This mechanical perspective of teaching does not promote making sense of mathematics. In this respect, mathematics is considered to be uninspiring at best, and mentally and emotionally crushing at worst (Schoenfeld, 1992). Perhaps, another possible alternative explanation of this finding is that primary teachers lack confidence in teaching mathematics and are drawn to a set of beliefs and practices that require relatively less teacher judgment and decision-making (Stipek et al. 2001).

Thirdly, the results allude that secondary principal teachers of mathematics have a stronger problem solving orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics, compared with mathematics teachers. It seems possible that this result may be due to the philosophy of mathematics held by principal teachers. In this study, Hamish expressed that mathematics is fundamentally about solving problems, which is a comparable belief articulated by Jim, an Australian senior mathematics teacher included in a study by Beswick (2007). In her research, Beswick (2007, p. 108) reported that Jim was particularly happy “when he and the students were engaged in a genuine problem to which neither he nor they know the answer”, accentuating a philosophy associated with a problem solving view of mathematics (Ernest, 1989a). This result differs from Barkatas & Malone (2002), but is broadly consistent with Perry, Howard & Tracey (1999), who found that head mathematics teachers (equivalent to curriculum leaders) held stronger learner-centred beliefs and weaker transmissive beliefs than mathematics teachers. Though, the study would have been more relevant if the authors had selected participants from a wider geographic area.

Highest qualification in the field of education and TMBS

My research suggests that primary teachers holding a Masters in the field of education hold significantly stronger mathematical beliefs towards the nature of mathematics, the learning of mathematics and the teaching of mathematics than primary participants holding a BEd. However, this study indicates that mathematics teachers holding a Masters in the field of education hold significantly stronger mathematical beliefs towards the nature of mathematics, the learning of mathematics and the teaching of mathematics than participants holding a BEd or a PGCE/PGDE. This rather intriguing finding to emerge between the sectors might be explained by the fact that primary PGCE/PGDE participants have accumulated Masters credits.

Highest qualification in the field of education and Belief Factors

This study indicates that primary teachers holding a Masters in the field of education have significantly stronger mathematical beliefs for three of the five belief factors. Correspondingly, mathematics participants, have significant stronger mathematical beliefs for all five belief factors. The findings reported here suggest that a higher educational degree does not impact on primary teachers' mathematical beliefs aligning to a social constructivist and collaborative orientation to mathematics, the learning of mathematics and the teaching of mathematics. Instead, these findings provide support for the premise that primary teachers, in general, hold stronger CfE mathematical beliefs than mathematics teachers.

Collectively, these results support the view that a Masters level qualification in the field of education is positively linked to more robust mathematical beliefs. Although, there is already a wealth of anecdotal evidence that undertaking postgraduate research and study at Masters level significantly changes how teachers interrelate with aspects of learning and teaching (Edwards, 2008). However, this finding is consistent with previous research. For example, in a study of South Korean primary teachers, Kim, Sihn & Mitchell (2014) found that practitioners holding a Master's degree in mathematics education had significantly stronger mathematics teaching efficacy beliefs than colleagues with a Batchelor degree. Likewise, Beswick (2004) illustrates the case of Andrew, a mathematics school coordinator with a MEd, who held a problem solving view of mathematics and a social constructivist view of learning. Moreover, reflecting on my own Masters experience, it activated me to critically engage with ideas and debates in mathematics education research, which reinvigorated my professional practice by forcing me to question my existing beliefs and pedagogical approaches.

Factors which did not impact on teachers' mathematical beliefs

The results for both sectors show gender, age and teaching experience did not influence the dimension of teachers' mathematical beliefs. I will now discuss each characteristic in more detail.

Gender

This finding broadly supports the work of other studies but that inconsistencies exist within the literature. For example, in a study of 39 Malaysian mathematics teachers' beliefs about the nature of learning and teaching of mathematics, Zikre & Eu (2016) found no significant differences between male and female participants - although, the study suffers from limited sample size. Gender was not found to be a feature in the study carried out by Marshall et al. (2009) of 1,222 American primary and secondary science and mathematics teachers' beliefs about the use of inquiry in the classroom. Similarly, in their study of 521 Norwegian primary mathematics participants, Thorndsen & Turmo (2012) found no significant differences between male and female teachers' beliefs about instruction. In a cross-national investigation of mathematics teachers from England and China, Yu (2008) observed that gender did not have a significant influence on teachers' beliefs regarding the nature of mathematics, the teaching of mathematics and the purposes of mathematics education. Bayaga, Wadesango & Wadesango (2015) revealed that gender did not impact on the beliefs on mathematics education in their study of 183 South African prospective teachers. Likewise, gender was not significant in a study of mathematical problem solving beliefs of 138 Turkish prospective primary teachers by Bal (2015). Previous Turkish studies involving teachers' beliefs have described similar results (e.g. Memnun, Hart & Akkaya, 2012; Saglam & Dost, 2014).

In contrast, Zakaria & Musiran (2010) reported in their study of 100 Malaysian prospective mathematics teachers, that gender was a significant factor regarding beliefs about the nature of mathematics and the learning of mathematics but not involving the teaching of mathematics. In their study of Australian primary and secondary mathematics teachers beliefs about the learning and teaching of mathematics, Tracey, Perry & Howard (1998) found that female teachers' beliefs were more 'child-centred' regarding the teaching of mathematics. The researchers indicated that male teachers "may well base their teaching on their past experiences in classes where power relationships and the delivery of information (transmission of information) was the preferred mode of delivery by their teachers" (p. 619). However, the study makes no attempt to distinguish the effect of gender between sectors. In another study, Barkatas & Malone (2002) found that Greek female mathematics teachers' placed more emphasis on a social constructivist view of the nature of mathematics, the learning of mathematics and the teaching of mathematics. Yuzici & Ertekin (2010) observed that Turkish prospective primary male teachers had stronger instrumentalist beliefs concerning the learning of mathematics. In their study of 72 Mexican secondary mathematics teachers' beliefs, Canto-Herrera & Salazar-Carballo (2010) noted significant differences between male and female 'teaching styles' but not significant for any of the four belief categories under investigation. In a cross-national study of 181 Australian and Israeli mathematics teachers perceptions of mathematics, Forgasz et al. (2015) reported no differences in the views of Australian teachers but that Israeli males appeared to hold more traditionally gender-stereotyped views than their female counterparts. In her seminal review, (Li, 1999) suggested that female teachers are more likely to promote a collaborative learning environment than male teachers but this was not supported by empirical evidence.

Age

This finding is consistent with that of Yates (2006) who found that age was not a factor in her study of 127 Australian primary teachers' beliefs about the nature mathematics, the learning of mathematics and the teaching of mathematics. This result corroborates the outcome of Bayaga, Wadesango & Wadesango (2015). Similarly, Canto-Herrera & Salazar-Carballo (2010) reported that age did not impact on teachers' beliefs. Moreover, in a study of 390 Latvian mathematics teachers' beliefs of effective teaching, Sapkova (2011) stated that although constructivist beliefs of teachers' grew more distinctly with age, her results did not produce any significant differences.

Teaching experience

This finding is in line with those of previous studies (e.g. Nisbet & Warren, 2000; Yates, 2006; Yu, 2008; Marshall et. al, 2009; Zakaria & Maat, 2012). Nisbet & Warren (2000, p. 41) found that "beliefs about teaching mathematics are not significantly influenced by the number of years a teacher has been teaching". However, this outcome is contrary to other studies such as Barkatsas & Malone (2002) who observed that experience did significantly influence teachers' mathematical beliefs for one of five belief factors. The researchers concluded that a social constructivist orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics was related to experience at junior high school level but that the results were found not significant for experience at senior high school level for all five belief factors. The study would have been more convincing if the authors had employed more than three categories to delineate a range of 45 years of experience. In their study of 258 Turkish primary teachers, Isiksal-Bostan, Sahin & Ertepihar (2015), found that participants with more than 16 years' experience held significantly more favourable beliefs on using inquiry-based instructional approaches than participants with 6 to 10 years. However, the researchers found that experience was not a factor in other relationships, suggesting this particular finding was not archetypal.

This outcome reinforces the conjecture that teachers' deep rooted mathematical beliefs are constructed early, well established prior to entering the profession and highly resistant to change during their careers (Kagan, 1992; Richardson, 1996; Lerman, 2002; Wilson & Cooney, 2002). Pajares (1992) offers a myriad of fundamental assumptions including the view that:

> Beliefs are formed early and tend to self-perpetuate, preserving even against contradictions caused by reason, time, schooling, or experience... The earlier a belief is incorporated into the belief structure, the more difficult it is to alter... Belief change during adulthood is a relatively rare phenomenon, the most common cause being a conversion from one authority to another or a gestalt shift... Beliefs about teaching are well established by the time a student gets to college (p. 324-326).

The narratives of Hamish and Morag exemplify this supposition, irrespective of the contrasting nature of their early school experiences as pupils. In the case of Hamish, he revealed his beloved topic of algebra was established by an inspiration primary teacher and that a subsequent mixture of education and work experiences failed to alter his preference. Similarly, in the case of Morag, she described her adverse school mathematical encounters, including reference to an unpleasant childhood memory. Morag identified that her apathy towards mathematics was fashioned by anxiety as a young learner being coerced into responding to indiscriminate questions in front of her peers, despite having the confidence to sing to large audiences. This outcome is consistent with the study by Uusimaki & Nason (2004) who illustrate the case of Rose, an Australian prospective primary teacher. The researchers describe that Rose felt most anxious about mathematics when she had to verbally communicate her mathematical knowledge in some way. Uusimaki & Nason (2004) suggest that negative beliefs and anxiety about mathematics are most often shaped in primary school, as a result of negative experiences as learners, and that the main contributory factor for the dislike and fear of mathematics is the teacher. An intriguing point highlighted by Morag is criticism of the praxis of setting in secondary school mathematics, which resonates with the view expressed earlier by Angela (Hudson, Henderson & Hudson, 2015), who describes being traumatised by the unfavourable effect of labelling. It is conceivable that other practitioners' beliefs may have been adversely affected by this practice, triggering an evolution of negative images of mathematics.

Lortie (1975) portrays the influence of early school experiences on the belief construct as a formidable contributory source of the formation of teachers' mathematical beliefs. As Ball (1988, p. 40) points out: "Long before they enrol in their first education course or math methods course, they have developed a web of interconnected ideas about mathematics, about teaching and learning mathematics, and about schools". Likewise, Beswick (2012) notes from her case studies of two secondary mathematics practitioners that more attention needs to be paid to the beliefs about the nature of mathematics that teachers have constructed as a result of the cumulative experience of learning mathematics in primary and secondary schools. Teachers in this study have spent thousands of hours as pupils, and, through observation of and participation in the educational process, formed beliefs about the nature of mathematics, the learning of mathematics and the teaching of mathematics. It is therefore important to consider the impact of early school experiences on teachers' deep rooted mathematical beliefs. Though, Biesta, Priestley & Robinson (2015, p. 626) argue that it is more precise to focus on "influences from the past, orientations towards the future and engagement with the present".

Although several participants indicated that they suffered adverse school encounters with mathematics as former pupils, it was reassuring that they were determined not to perpetuate their anxiety within their own classes. I would argue that such a desire to instil more positive notions of mathematics than afforded to themselves is a testimony to their professionalism and commitment. It is possible, therefore, that having negative childhood experiences might not have a detrimental long lasting effect on practising teachers' beliefs about the nature of mathematics, the learning of mathematics and the teaching of mathematics. This finding is consistent with that of Gujarati (2013) who investigated the inverse relationship between the mathematical identities and classroom practices of three American early career primary practitioners. Gujarati (2013) describes the formative experiences of Andrea, Lisa and Melody. For instance, in the case of Andrea, it is reported that her school experiences with mathematics were all unconstructive. However, Gujarati (2013, p. 641) states that "despite negative experiences, she [Andrea] put extra effort into her mathematics practices to ensure that her students would not dread mathematics as she did, have a positive image of it, and would be successful at it". Gujarati (2013) argues that the key to changing teachers' negative beliefs is to invite teachers to reflect on their own histories. In order to begin evaluating the complex factors that influences their classroom decisions, she points out that it is critical to explore mathematics identities as early as possible in teachers' careers to potentially impact learners more positively.

8.5 Summary

This chapter has discussed findings from each of the three research questions. Mathematical problem posing can help improve pupil learning and is supported by a growing body of empirical evidence at different educational levels. Teachers' beliefs are shaped by previous school mathematical experiences (Lortie, 1975; Thompson, 1992; Handel, 2003) and constrained by the context in which they are situated within. Inconsistencies exist between espoused beliefs and professed classroom practices. Changing classroom practices will depend on changing teachers' beliefs (Lerman, 2002; Wilson & Cooney, 2002). Primary teachers hold stronger positive mathematical beliefs than secondary mathematics teachers. The conceptualisation and operationalisation of mathematical problem solving appears to be compromised by an incoherent theoretical mathematical framework coupled with an over dominant national assessment culture. Teachers' beliefs of mathematical ability are illustrated by association with examination performance opposed to levels of critical thinking. Characteristics of gender, age and teaching experience do not positively impact on teachers' beliefs. Whilst grade has a modest influence on both sectors, a Masters qualification in the field of education has a considerable impact on primary teachers' beliefs and an overarching impact on secondary mathematics teachers' beliefs.

The next chapter will present the implications and recommendations for current policy and practice. Furthermore, it will list relevant limitations of this study and suggest themes for future research.

CHAPTER NINE

Conclusions

9.1 Introduction

This study was motivated by two aims. The first aim was to examine critically a reconceptualization of the existing mathematics curriculum by addressing a policy gap originated by the omission of mathematical problem posing. This study set out to explore to what extent mathematical problem posing should be embedded within the current mathematics policy of Curriculum for Excellence. The second aim was to obtain empirical evidence of teacher's beliefs and espoused classroom practices of mathematical problem solving and problem posing. This study set out to provide the first cross-sector account of Scottish teachers' beliefs and professed classroom practices to determine levels of enactment of mathematical problem solving and problem posing.

Several tentative conclusions can be associated with the research reported within this thesis. The use of a theoretical framework initially formulated using Ernest (1989a) and later expanded with reference to Barkastas & Malone (2005) has helped to shape the facilitation of multiple perspectives in interpretation of the data. In summary, this study has produced five main findings.

Firstly, this study appears to be the first study to examine the legitimacy of infusing mathematical problem posing within the national curricula of Scotland. The result of this research is relevant to both practitioners and policy makers and supports the idea that to improve pupil learning, mathematical problem posing should be embedded within Curriculum for Excellence. This finding resonates with previous research which advocates that mathematical problem posing should be an integral component of school mathematics (Stoyanova, 2003; Bonotto, 2013; Singer, Ellerton & Cai, 2013; Leung, 2013, 2016; Cai et al., 2015).

Secondly, factor analysis has identified that teachers' beliefs align to three distinct but not mutually exclusive, mathematical belief systems as follows:

1. A social constructivist, problem solving and collaborative orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics;

2. A social constructivist, problem solving and static transmission orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics;

3. A static and mechanistic transmission orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics.

This finding strengthens the idea that teachers' mathematical beliefs are not held in isolation but are embedded within clusters which coalesce to form an interconnected and structured belief system (Rokeach, 1968; Green, 1971; Leatham, 2006). Teachers' beliefs are extremely complex. Some are more central and influential than others. They are formed early (e.g. during schooling) and are highly resilient and resistant to change. Belief systems include a compendium of positive and negative cognitive positions towards the nature of mathematics, the learning of mathematics and the teaching of mathematics. Likewise, belief systems align with conceptualised personal philosophies designated as instrumentalist, Platonist and problem solving (Ernest, 1989a).

Thirdly, this study indicates that primary and secondary teachers hold significantly different mathematical beliefs. As both sectors share mutual responsibility for the learning and teaching of mathematics, this finding is particularly noteworthy. That is, if the cross-sector relationship is disconnected, such an imbalance may adversely affect continuity and progression of the educational experience afforded to all learners. Moreover, limited support for teaching mathematics through problem solving exists including a widespread belief that problems can be solved by using standard algorithms. Some practitioners from both sectors conceive of problem solving as an irregular follow on step after learners have acquired mastery of basic numerical and computational skills coupled with procedural understanding. Modest encouragement is present for the promotion of multiple solutions. Overall, it was found that, when judged against the mathematical philosophy as championed by CfE, primary teachers hold stronger beliefs than secondary mathematics teachers.

Fourthly, it was discovered that gender, age and teaching experience had no discernible influence on teachers' mathematical beliefs. While the evidence indicates that grade was significant in some cases, it is possible that these results merely reflect a selection effect. However, one factor to emerge that significantly impacted on both primary and secondary teachers' beliefs is post-graduate study at Masters level within the field of education, although the exact domain is unknown.

Fifthly, it is suggested that the conceptualisation and operationalisation of mathematical problem solving and problem posing is restricted in practice. A number of variables impinge on the process which inhibits teachers' functioning capabilities such as a scarcity of a shared understanding of what constitutes a mathematical problem and the theoretical structure of problem posing. In other words, emergent beliefs have provided a practical awareness of the professional contexts within how teacher operate and help to make sense of the multifaceted terrain of classroom situations. Inconsistencies between teachers' espoused beliefs and reported practices, are manifested by an over dominant national assessment culture, which promotes attainment of localised and national targets at the expense of the development of critical and independent thinking. Such contextualised inconsistencies are similar to previous research studies (e.g. Cooney, 1985; Raymond, 1997; Skott, 2001; Beswick, 2012).

In guiding the research process, teachers' espoused beliefs have an overwhelming influence on the degree of enactment of the mathematical philosophy advocated by CfE. Prior mathematical experiences have strongly impelled teachers' perspectives on the nature of mathematics, which have fashioned the disposition of mathematical orientations surfacing in classrooms.

Given the international dimension and theoretical significance of mathematical problem solving and problem posing, it is a major concern that the implementation of both elements appear to be restricted in practice. Whilst a number of constraints have been identified that help to explain this phenomenon, it will require more than rhetoric to reverse this situation. In Scotland, the accountability for mathematics education firmly rests with local education authorities and schools, whilst taking account of national guidelines and advice.

A particular strength of my study is the high rate of volunteered comments obtained during phase one. A total of 87 participants (18% of main study) generated a combined narrative of over 6000 words providing a rich constellation of perspectives about the belief statements. This contribution included cross-sector representation from all grades which facilitated the triangulation of the results of phase two, thus improving the overall reliability of this study.

In short, the evidence provided in this study allows me to justify the claim that I have made a contribution to knowledge. Firstly, I have undertaken a systematic literature review which has explored changes in contemporary knowledge and emerging research that has engendered empirical evidence for the inclusion of mathematical problem posing within Curriculum for Excellence. Secondly, I have acquired an understanding of the cross-sector beliefs and espoused professional practices of Scottish teachers regarding mathematical problem solving and problem posing.

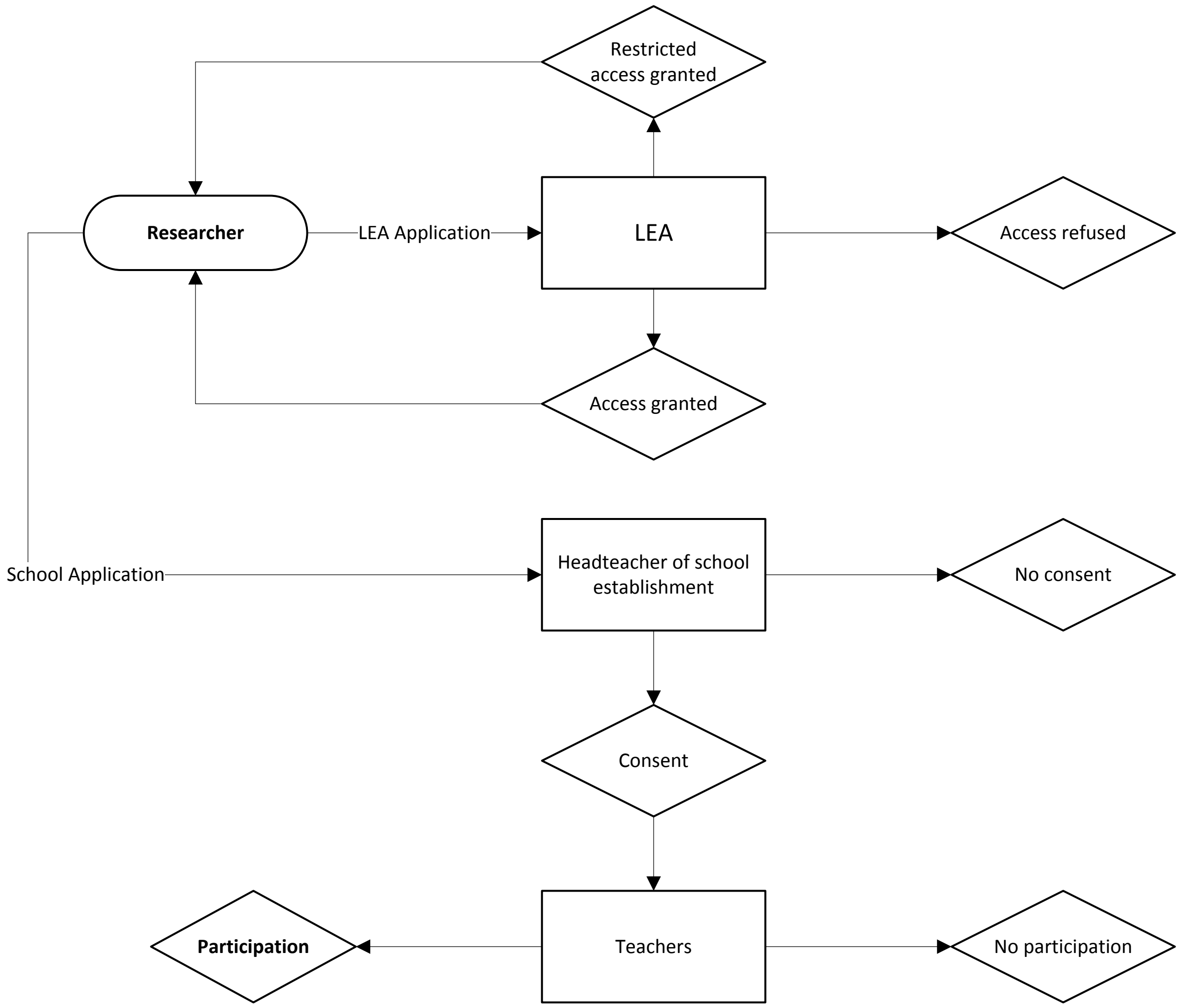
9.2 Limitations of the study

This study was subject to a number of potential methodological limitations which need to be considered. First, the decision to restrict the pilot study to an individual LEA may have affected the measurements obtained during the initial design phase. In retrospective, I should have employed more than one LEA to enrich the data collection process thus simultaneously increasing the potential to obtain a larger sample size.

Second, during phase one of the study it was not possible to implement any probability sampling technique in the field. Research requests involving practising teachers are initially controlled by individual LEA's, who subsequently grant or refuse approval for researchers to contact corresponding school establishments. Nevertheless, upon conferring consent an additional layer of administration rests with each respective Headteacher who govern if staff can participate. Ultimately, the final decision rests firmly with the individual practitioner. An overview of the research engagement process is shown in Figure 9.1. The concern raised here is that direct access to the research field is negated as this route is managed by the Headteacher. In this study, all 32 LEA's were formally approached for access to the research field. The outcome of this process is as follows:

- Pilot study access granted by LEA (1)
- Access granted by LEA and quantity of participants over 25 (7)
- Access granted by LEA and quantity of participants between 10 and 25 (12)
- Access granted by LEA and no participants (2)
- Restricted access granted to a single school establishment selected by LEA (1)
- Restricted access granted to a single participant selected by LEA (1)
- Access refused by LEA (3)
- No engagement by LEA (4)

Figure 9.1 Process for gaining research access to teacher population



In two instances, it is not possible to explain why LEA access was granted but no participants followed after a number of school establishments were approached. Likewise, I am unable to provide any reason why four LEA's did not interact with numerous research request communications, other than by recognising the enormous workload demands made on teachers in all roles within our education system. Speculating on both concerns does not outweigh the reflection on my positionality and identity as a researcher in the field.

With no influence over the target population, I depended on a suitable random sample size being generated to produce a range of participants with the desired demographic characteristics. As a consequence, the determination of a response rate was not feasible. On reflection, a sample size of 478 is respectable given the circumstances, although arguably this quantity produced a mean of 23 participants per accessed LEA, which may be considered modest at best. However, it was not possible to attract any teachers' holding a doctorate in the field of education. Whilst it is unknown the extent of individuals with this qualification, it is conjectured that such practitioners may have enriched the process due to their multifaceted theoretical knowledge and understanding of contemporary issues. Furthermore, the grade distribution of participants reveals that the balance of Principal Teacher is skewed towards the secondary sector. Conversely, the balance of Headteacher is skewed towards the primary sector. Although the equilibrium of unpromoted teachers to promoted teachers is comparable for both sectors, it is deliberated that an unequal spread of both Principal Teacher and Headteacher may affect the generalisability of the results.

Third, during phase two, with the exception of one case, each interview was restricted to a maximum of 40 minutes due to the workload commitments of the participants. In fact, of the eleven interviews conducted, eight were organised during lunchtimes or between teaching periods. In one case, a participant had to reschedule the interview to a different date due to a last minute 'please take'. Unfortunately, I had already travelled sixty miles to the venue and was only informed of this change on arrival. Given the complexity of the nature of this research, it is reasoned that an extended time period would have allowed for a more in-depth discussion of the factors influencing teachers' mathematical beliefs and the underlying philosophy of CfE. Whilst recognising that access to the field was granted to me by the majority of LEA's, it is suggested that Headteachers could have afforded their staff more flexibility to engage professionally with the practical mechanism of educational research, free from time restrictions, where possible.

Fourth, phase two of this study shared a common problem with other qualitative research, in that it was not free from being “impressionistic and subjective” (Bryman, 2012, p. 405). Personal bias that the researcher imposes can be present during the interview and data analysis process (Creswell & Clark, 2011). In this study, although considerable effort was made to minimise bias, it is acknowledged that complete objectivity is impossible. It is conceivable that my secondary teaching background and research inexperience may have influenced the formulation of interview questions and the interpretation of thematic codes emerging from analysis of the qualitative data.

Fifth, concerning the reported inconsistencies between teachers’ espoused mathematical beliefs and professed instructional practices, it may be plausible that as a novice researcher, I have not searched profoundly enough to locate the intricacies manifested within the context of classroom life (Fang, 1996). Grounded on the premise that teachers are ‘inherently sensible’, Leatham (2006) underlines the notion that potential inconsistencies should not lead directly to a supposition but instead be inferred as opportunities for further investigation. He warns researchers to probe deeper “for we must have either misunderstood the implications of that belief, or some other belief took precedence in that particular situation” (p. 95).

Sixth, it is speculated that since English was the sole language employed by me during the research process, I may have inadvertently marginalised a minor group of teachers working exclusively within Gaelic medium education. It is important to acknowledge that in Scotland, mathematics in primary and secondary schools is delivered either in English or in the national idiom of Gaelic.

Finally, due to restrictions of combining part-time doctoral study with full-time employment, the phase two interviews were conducted within a relatively short period with only transcription time in between. It is purported that a longer reflection period to deliberate between interviews would have been useful. For example, increased cognisance of the ‘interviewer effect’ may have ameliorated the quality of the data (Denscombe, 2014).

9.3 Implications for policy and practice

The learning and teaching of mathematics is a politically charged arena. This thesis purports that there is a need for a transformational change in the mathematical beliefs of primary and secondary mathematics teachers. One factor which has precipitated this notion is the prescriptive framework supplied by the Scottish Government which does not adequately offer a suitable theoretical template from which teachers can easily advance the conceptualisation and operationalisation of mathematical problem solving and problem posing. What is required is not a particular methodology but a sustainable set of pedagogical practices that will improve the standard of mathematical interactions and experiences for all learners. The initial step to ensure teacher change is to provide practitioners with valuable professional development opportunities to reflect and absorb theoretical knowledge that will help underpin their enactment of any new instructional approach (Lerman, 2002; Wilson & Cooney, 2002; Hudson, Henderson & Hudson, 2015; Fullan, 2016). However, the key is not merely providing courses, as Clark et al. (2014, p. 275) point out: “simply giving teachers more mathematics or mathematics education courses may improve their mathematical and pedagogical knowledge, yet these courses will not necessarily influence teachers’ beliefs and awareness”. For professional development of teachers to be considered effective, Day (1999) argues that individual modifications must be internalised and teachers must ensure participation and ownership of the decision-making change process. Otherwise, as Handel & Herrington (2003, p. 62) indicate, that without shifting teachers’ beliefs, change can “be cosmetic, that is, a teacher can be using new resources, or modify teaching practices, without accepting internally the beliefs and principles underlying the reform”.

Concurrently, a governmental change of focus is required encompassing the indoctrinated high stakes assessment culture that currently permeates both sectors. Our examination system, saturated with uncompromising targets, places teachers under unrealistic pressures of time and pupil performance. Such demands make it difficult to engage learners in rich mathematical experiences. Mason (2016, p. 110) counsels that “[t]he force of tests and examinations is to impel teachers to get students to practise routine procedures in an attempt to score highly, even though such ‘learning’ may not be robust or stable over time”. Teachers should be enabled to reconceptualise the educational priority of inculcating in all pupils the growth of critical and independent thinking skills. Solving and posing mathematical problems is an indispensable life skill that must be encouraged and nurtured from the early years through to when young people leave school. It should be embedded as a creative endeavour and core national outcome in which practitioners contribute to ameliorating pupils' deep learning of mathematics.

Innovative methods of assessing mathematical achievement present alternatives to rewarding the recall of routine algorithmic procedures. Local education authorities should support teachers in a systematic development of new curriculum materials aimed at fostering and assessing mathematical problem solving and problem posing. Logically, having ready-made resources available would facilitate teachers' engagement of problem solving and problem posing activities in their classrooms. Though, it would also be important for teachers to have sufficient understanding on how to employ these resources effectively. Moreover, it is worthy to consider if current CfE textbooks embody the intended curriculum.

<u>Initial Teacher Education</u>

With any proposed change of practice for current teachers, it is necessary to review the framework for initial teacher education to reflect such a stimulus. ITE is the first phase of a career-long continuum which provides theoretical and practical pedagogical training for prospective teachers. According to Green (1971, p. 48), teaching involves the development of beliefs and "is an activity which has to do, among other things, with the modification and formation of belief systems". It is hoped that my research might help to influence this provision, in order to augment the future quality of learning and teaching of mathematics that can be offered to pupils.

Within this domain in Scotland, a paucity of empirical studies exists that explore the nature of prospective primary or secondary mathematics teachers' beliefs. Such studies (e.g. Macnab & Payne, 2003; Henderson & Rodrigues, 2008; Henderson & Hudson, 2011) help to collectively enrich the knowledge of the views of teacher candidates which may in time assist in identifying and confronting misconceptions and negative beliefs, prior to entering the profession (Raymond, 1997). For instance, one main obstacle to overcome is challenging the widely held hierarchal ‘building block metaphor’ image of mathematics (Hewitt, 1987). However, by exploring conceptualisations of the nature of mathematics, teacher educators can anticipate possible implications for the learning and teaching of mathematics. Naturally, such a strategic process is futile without gathering beliefs from all relevant students.

In the same vein, complementing the empirical evidence presented earlier in this thesis, is growing endorsement from the research community advocating for prospective primary and secondary mathematics teachers to be exposed to and supported through mathematical problem posing experiences (e.g. Leung & Silver, 1997; Crespo, 2015; Ellerton, 2015; Hospesova & Ticha, 2015; Osana & Pelczer, 2015; Rosli et al., 2015). For prospective teachers to prepare for future school communities, they require representative practical experiences (Putman & Borko, 2000). Typically, such experiences must empower teachers to deliver mathematics in an effective way. That is, simultaneously promoting mathematical problem solving and problem posing within all stages and ability.

9.4 Recommendations

The findings of this study have a number of practical implications for policy, professional learning and initial teacher education. Whilst I am confident that these findings enable me to make useful recommendations, at the very least I hope that my research will help to stimulate discussion at various levels. I draw on the contribution by Atkinson (2000, p. 328) who reminds us that: “[t]he purpose of educational research is surely not merely to provide ‘answers’ to the problems of the next decade or so, but to continue to inform discussion, among practitioners, researchers and policymakers, about the nature, purpose and content of the educational enterprise”.

<u>Policy</u>

The mathematical framework of Curriculum for Excellence could be restructured to introduce the interplay between problem solving and problem posing as an overarching feature of mathematical learning. This can be realised by combining three distinct elements as follows:

Firstly, I would recommend embedding specific theoretical references to mathematical problem solving in curricular documents. For example, an opportunity should be taken to provide a conceptual definition in order to ensure a transition to more authentic and cognitively demanding tasks. Derived from my research, I would propose the following as a description:

> Mathematical problems can be posed in various formats such as words, shapes, graphs, multiple solutions, alternative answers and should be unfamiliar to the learner. It offers challenge and cannot be solved quickly by simple computational or algorithmic procedures. Several key ingredients differentiate a mathematical problem from routine textbook exercises; it requires critical thinking and perseverance, whilst provoking originality.

This may be supported by citations to the literature and illustrations of problem solving strategies (e.g. Polya, 1957; Schoenfeld, 1985; Mason, Burton & Stacey, 2010). Moreover, any definition should make explicit what the structural difference is between computational exercises commonly found in curriculum materials such as textbooks with the typical characteristics of non-standard problems. Also, an emphasis of the importance of encouraging teachers to promote multiple solutions should be presented along with samples of suitable problems that allow for this facilitation.

Secondly, if mathematical problem posing is to be woven smoothly into the fabric of CfE then this may require the enculturation of practitioners. Initially, the conceptualisation of problem formulation, including the connection to problem solving, could be disseminated to all schools by highlighting the endorsement of Kilpatrick (1987) and bolstered with the assurance of future professional development activities. Thereafter, this may be supported by citations to the literature and listing examples of mathematical problem posing activities and theoretical frameworks that illustrate seminal research contributions (e.g. Freudenthal, 1973; Silver et al., 1996; Stoyanova & Ellerton, 1996; Cai, 1998; Cai & Hwang, 2002; Brown & Walter, 2005; Ellerton, 2013). The interaction between problem solving and problem posing should be made explicit. Likewise, it is essential that mathematical problem posing is perceived by teachers as a mechanism to nurture creativity, independence and originality.

Thirdly, while the above two recommendations are designed to raise the profile and increase national awareness of both constructs, they are of little use as leverage tools without a change to current assessment procedures. I would argue that our national examinations and internal assessments do not test individual mathematical problem solving and problem posing abilities. It is therefore proposed that for SCQF level 6 and 7 (i.e. Higher Mathematics and Advanced Higher Mathematics), the existing internally assessed component should be revised. Currently, this component consists of three unit assessments within which a candidate has to achieve a minimum competency standard, the success of which has no connection with the final overall grade. Whilst accepting the merit of such tasks as a practical method of gathering evidence and of ongoing progress, in practice it is arguable if whether such a model has the ability to strengthen or enrich a candidates understanding and appreciation of mathematics which is any different to that already offered by traditional external examination.

An alternative approach for how we elicit and interpret evidence of mathematical learning is required. Time-restricted objective tests that demand recall of previously learned facts and rehearsed procedures is out of alignment with the principles of a social constructivist conceptual framework (Goos, 2014). In fact, such tests only serve to perpetuate the institutionalised societal notion that higher order thinking and creativity are fruitless, or in some ways unnecessary tenets of mathematics.

I have argued throughout this thesis that the operationalisation of mathematical problem solving and problem posing is restricted without the accompaniment of a corresponding assessment system that is consistent with their goals. It is necessary to give recognition to all valued learning experiences as this communicates most clearly to learners which proficiencies are important. My preferred model of assessment would be a formative portfolio evaluation designed to contain a diverse assortment of rich productions that can demonstrate factors such as critical thinking, active learning, achievement, creativity, engagement and mathematical literacy. Suitable exemplars include a mathematical problem solving and problem posing journal to formally encourage pupils to reflect on their personal development and on their critique of the work of their peers. This may refer to heuristics, multiple solutions, unsolvable problems, investigations, conjectures, arguments, problems with more than one answer, interdisciplinary collaborations, creation of interesting problems, reformulation of previous problems, observations, discussions, etc. Portfolio work should infuse the entire curriculum. It could be internally assessed and externally moderated by the SQA. In terms of feedback, teachers should provide written comments on pupils work as the sole vehicle of offering advice and setting targets (Hodgen & Wiliam, 2006). Naturally, to reflect the significance of this recommendation, it is essential to attach an appropriate weighting. To begin with, I would suggest 20%, therefore generating a revised external assessment element weighting of 80%. For example, assuming no alteration to paper 1 and paper 2 for Higher (130 marks), the portfolio would be scored out of 32½. In regulating this result, intersubjectivity is necessary for successful negotiation of teacher consistency.

Finally, for mathematical problem posing to play a prominent role in primary and secondary classrooms, teachers must have straightforward access to relevant resources. In particular, mathematics curriculum materials should feature a wide representation of problem posing activities (Cai et al., 2016).

Professional learning

Beliefs are a fundamental construct in teachers' professional development, particularly during educational reform (Roesken, 2011). The success of recent national initiatives such as those orchestrated by the University of Dundee (Hudson, Henderson & Hudson, 2012a, 2015) has helped to develop mathematical thinking in Scottish primary classrooms, whilst challenging the existence of adverse beliefs. Based on my findings, that negative mathematical beliefs impregnate both sectors, a similar enterprise would stimulate mathematical thinking in secondary classrooms. Leatham (2006) advocates that:

> Teacher educators should provide teachers with opportunities to explore their beliefs about mathematics, teaching and learning. Teacher education strategies such as critiquing tradition, demonstrating by case and example, and encouraging rigorous discussion take on new meaning when beliefs are explicitly examined. In the process, teachers acquire terms and expressions requisite for ongoing, meaningful reflection on their beliefs and practice (p. 100).

The effective delivery of mathematical problem solving and problem posing is multifaceted and cannot be mastered instantly. It demands interrelated abilities such as an awareness of the structure of problems, heuristics, metacognition, robust pedagogical content knowledge and general didactical skills. Equally, adoption of any instructional strategy may only realistically surface if practitioners possess a theoretical understanding of the principles that underpin those approaches (e.g. Beswick, 2012; Singer, Ellerton & Cai, 2013).

It is therefore proposed that, to influence teachers' beliefs and to prepare individuals from both sectors for delivering mathematical problem solving and problem posing, a Masters level module be designed to advance this critical objective. Prendergast & Roche (2017, p. 214) argue "the provision of high-quality CPD is essential so that teachers are supported in keeping abreast of the on-going changes to their profession". Perhaps, infused within the theoretical content could be a practical way of instructing mathematics via problem solving, exploring issues of national and international mathematics education and how to encourage learners to pose mathematical problems. Together, it is expected that this module would create a professional opportunity to invigorate thinking towards a change of mathematical beliefs by energising teachers to develop and evaluate their own practice.

Initial Teacher Education

It is recommended on entry, that an investigation is undertaken to determine the domain specific beliefs of the nature of mathematics and how the subject could and should be delivered, for each prospective primary and secondary mathematics practitioner. This would contribute to analytically identifying at an early stage, variations in individual beliefs, which may allow respective institutions ample opportunity to engage in ways to attempt to reverse any imbalance by at least instigating explanation and critical thinking. However, it is arguable if this can be successful within the duration of any PGDE programme and may well require further strengthening during later years. Thompson (1992, p. 135) emphasises that the "task of modifying long held, deeply rooted conceptions of mathematics and its teaching in the short period of a course in methods teaching remains a major problem in mathematics teacher education".

An underlying requirement of any mathematics education course is to allow future teachers to engage critically with the literature on problem solving. This will foster debate and lead to an evaluation of theory, research and current curriculum policies. Based on the assumption that this is already occurring in Scotland, it is suggested that this provision be extended to embed the literature on mathematical problem posing which features the interplay with problem solving (e.g. Polya, 1954; Kilpatrick, 1987; Brown & Walter, 2005; Singer, Ellerton & Cai, 2013; Cai et al., 2015). Moreover, it is proposed that the treatment of mathematical problem posing is afforded equivalent status to that of problem solving to ensure that both components are viewed as subsets of each other.

9.5 Further research

There is a universal requirement to expand our understanding of teachers' mathematical beliefs and their relationship to classroom practice. With increased knowledge of teachers' beliefs, we can enrich the quality of our educational system. It is recommended that further research be undertaken in the following areas:

1. Further cross-sector research could be carried out to collect individual observational data (e.g. longitudinal studies or case studies) in order to explore the relationship between Scottish teachers' espoused beliefs and enacted beliefs. Thompson (1992, p. 135) argues that "investigation of teachers' mathematical beliefs should examine teachers' verbal data along with observational data of their instructional practice or mathematical behavior; it will not suffice to rely solely on verbal data". One particular focus of enquiry is to investigate the constructivist praxis of teaching mathematics via problem solving (Schroeder & Lester, 1989).

2. Teachers' beliefs have a critical role in the integration of mathematical investigations. More research could be undertaken at all stages of Scottish education to determine the nature of mathematical problem posing as a source of mathematical inquiry.

3. Further work is needed to examine the mathematics teaching self-efficacy beliefs of Scottish primary teachers, in order to fully understand the underlying barriers to orchestrating mathematical problem solving.

4. As a common resource, textbooks present learning trajectories that heavily influence how Scottish teachers interpret and implement mathematical curricula understanding (Scottish Government, 2014, 2016). CfE caters for all stages from ages 3-18. It would be worthwhile to investigate what kind of knowledge is prioritised at the different stages and to what extent mathematical problem solving strategies and mathematical problem posing tasks are embodied within commercially produced classroom textbooks.

5. More research could be undertaken with Scottish teachers to establish the composition of their Masters qualifications in the field of education. My research has shown that these qualifications have a significant impact on teachers' beliefs. The influence of specific courses and research led critical enquiry could be investigated. This could also be extended to compare degrees offered in different institutions.

6. The Scottish Government (2008, p. 11) contend that CfE encapsulates "the totality of experiences which are planned for children and young people through their education, wherever they are being educated". However, with autonomy extended to each LEA, it is questionable whether uniformity can subsist between pedagogical approaches to learning, teaching and assessment. Teachers' beliefs may already be contextualised by the conceptualisation and operationalisation of mathematical problem solving adopted by their corresponding LEA. A future study should assess the validity of this claim.

7. Internationalisation and globalisation has projected awareness that other countries have diverse mathematics curricula which may yield fruitful rewards on inspection. Such research into teachers' beliefs underlines the contextual nature as a factor that may influence the enactment of teachers' beliefs (e.g. Andrews, 2007; Cai & Wang, 2010). A natural progression of this work is to conduct cross-cultural comparative studies of Scottish teachers' mathematical beliefs.

9.6 Autobiographical reflection

Undertaking this thesis has been a rich, engaging but lonely experience. I have developed a firm understanding of the interconnected nature of educational research and of the cyclical, sometimes frustrating, multilayered landscape of the investigation process. For example, I have discovered that it is not always possible to establish methodological control. Retrospectively, I have considered how different my research would have been if I had selected an alternative paradigm to describe my research topic. Unquestionably, this would have modified my research design and produced different outcomes in fieldwork, findings and conclusions.

During the research I came to recognise the value of practitioner research. In particular, the empowerment of possessing increased levels of criticality and scholarship with regards my own professional practice together with enhanced self-confidence in debating hierarchical policy issues. Being able to manage the acquisition of current knowledge as it emerges has stimulated the growth of my pedagogical expertise. Conversely, this has highlighted my previous acceptance of a transmissive structure underpinned by the filtering down of fragmented elements of undisclosed information.

What surprised me about mixed methods research was the diversity of knowledge required to analyse both quantitative and qualitative components. As a result of this study, I have learned to appreciate the synergy that can be gained by the additional work of employing both quantitative and qualitative techniques. This experience has motivated me to develop a broad set of methodological skills. Likewise, it has fuelled an increase in the rigour of my own conceptual thinking by widening awareness of further areas of challenge and potential research questions in mathematics education and in education more generally.

Viewing mathematical problem solving and problem posing from a wider perspective has validated my commitment towards teaching in a way that would have been previously impossible. I have been propelled through a continuum of conceptual transformations that have enlightened my classroom practice. Consequently, I believe that mathematics should not be characterised as a fixed body of knowledge requiring memorisation, mastery of algorithms and available only to a selected few. In contrast, it is a dynamic learner-centred activity built on a premise that success is grounded on critical and independent thinking interspersed with varying levels of creativity, originality and collaboration that is accessible to all.

During the research, I uncovered the acute nature of teachers’ beliefs which are fundamental to educational reform and classroom practice. Initially, I considered teachers’ beliefs to be compartmentalised within a study of mathematical problem solving and problem posing. However, with further reading I eventually recognised that teachers’ beliefs are an important theme in their own right. Furthermore, as an unexpected derivative of rummaging through the literature, I expanded my abstract understanding of Vygotskian ideas on enriching mathematical thinking. In particular, theoretical underpinnings connected to how learners develop higher mental functions and reasoning through societal and collaborative activities. Juxtaposed with my findings of teachers' beliefs, is the tacit recognition that the success of any whole class mathematical activity is entirely dependent on the individual in charge. Almost certainly, practitioners choose to strengthen pupils' mathematical thinking or, otherwise, hinder and obstruct opportunities for its growth. Within my own professional setting, I have enhanced my sensitivity towards advancing young people's mathematical thinking by attending more to their strategies. By transferring further attention to producing sophisticated interpretations of their existing understandings, I have been able to respond more effectively.

The challenges such as a continuous shift in academic writing presented by part-time doctoral study have led me to assert that I would not easily advocate this approach to practising teachers. Moreover, factors impinging heavily on personal time and space, a perpetual need to reappraise priorities and review accepted patterns of work and home life, require careful consideration.

Nevertheless, my major transformation is in the positionality of my epistemological beliefs and ontological world views about learning and teaching. My current thinking was provoked during the phase two interviews, when I began intensely reflecting on my own accumulated experience and previously held paradigms. Comparing the beliefs of other practitioners helped me appreciate the multiple barriers that have to be overcome to allow the implementation of theoretical perspectives in Scottish classrooms.

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APPENDIX A

OVERVIEW OF PREVIOUS RESEARCH METHODS USED TO MEASURE TEACHERS BELIEFS

Instrument(s)	Focus of Inquiry	Participants	Reliability	Validation	Study
Action Research	Teachers' beliefs about teacher identity	50 Canadian practising elementary teachers	Credibility by description of length of study	Researcher immersed in research setting	Goodnough (2010)
Blogging & Social Networking	Teachers' beliefs about reform-based science practices	15 American practising secondary science teachers	Inter-rater	Online inspection	Luehmann & Tinelli (2008)
Case Study	Inconsistency between teachers' mathematical beliefs and classroom practice	6 American practising elementary teachers (Main focus on single teacher)	Peer review	Triangulation	Raymond (1997)
Case Study (Multiple)	Teachers beliefs structures and their influences on practice	5 American practicing secondary mathematics teachers	Dependability audit	Triangulation through multiple data sources	Cross (2009)
Classroom Artefact	Teachers' beliefs about documentable practices	57 American pre-school teachers		Triangulation	McMullen et al. (2006)
Concept-Mapping, Journal Writing & Interview	Teachers' beliefs about effective teaching.	51 Turkish student mathematics & physics teachers	Member checking	Triangulation through multiple data sources	Ozgun-Koca & Sen (2006)
Drawing & Interview	Teachers' beliefs about field-based teaching experiences	16 USA student secondary science teachers	Closed questions	Triangulation	Hancock & Gallard (2004)
Essay & Interview	Teachers' beliefs about school practices	48 Turkish student social studies teachers		Open-ended questions	Güven (2004)
Ethnography	Teachers' beliefs about mathematics teaching and learning	7 Norwegian practising mathematics teachers		Focus group	Kleve (2009)

Instrument(s)	Focus of Inquiry	Participants	Reliability	Validation	Study
Grounded Theory	Teachers' beliefs about source and stability of teaching knowledge	110 American student & practising teachers	Audit trial	Coded for emerging concepts	Buehl & Fives (2009)
Interview	Teachers' treatment and beliefs of mathematical proof	10 Italian practising secondary mathematics teachers		Coded for emerging concepts	Furinghetti & Morselli (2011)
Life History	Impact of life experiences on teachers' science beliefs and practices	2 American practising elementary teachers			Smith (2005)
Longitudinal	Teachers' beliefs about the role of teachers	80 Estonian student teachers	Member checking	SPSS analysis	Löfstöm & Poom-Valickis (2013)
Narrative Research	Teacher beliefs about teaching pupils identified as having a disability	1 American practising secondary English teacher		Inductive analytic approach	Del Rosario (2006)
Observation, Interview & Video Scenario	Teachers' beliefs about teaching and learning of mathematics	3 Scottish practising secondary mathematics teachers		Triangulation	Forrester (2008)
Phenomenology	Change in teachers' beliefs after first year of teaching	12 Turkish practising middle grade mathematics teachers	Clear outline and defining of steps involved	Purposeful drawn sample; Themes selected as they emerged	Haser & Star (2009)
Portfolio & Interview	Teachers' beliefs about environmental education and its relationship with classroom practices	6 Pakistani practising secondary science teachers	Document analysis	Peer review & Triangulation	Khan & Begum (2012)

Instrument(s)	Focus of Inquiry	Participants	Reliability	Validation	Study
Portraiture	Relationship between teachers' mathematics identities and classroom practices	3 Amercian primary teachers	Cross-checking Member checking	Triangulation	Gujarati (2013)
Questionnaire	Teachers' beliefs about mathematical problems and the nature of problem solving	244 Turkish student primary mathematics teachers	Cronbach's alpha	Peer review, 2 Pilot studies & SPSS analysis	Kayan Fadlelmula & Cakiroglu (2011)
Questionnaire & Interview	Teachers' beliefs about the role of problem solving in learning mathematics	162 Australian practising primary teachers	Previous study	SPSS analysis Triangulation	Anderson, Sullivan & White (2004)
Questionnaire & Case Study	Teachers' beliefs about learning and teaching mathematics	465 Greek practising mathematics teachers	Cronbach's alpha	SPSS analysis Triangulation	Barkatas & Malone (2005)
Questionnaire, Interview & Observation	Teachers' mathematical beliefs and their connection to practice	25 Australian practising secondary mathematics teachers	Cronbach's alpha	Pilot study SPSS analysis Triangulation	Beswick (2005)
Scenario, Interview & Observation	Teachers' belief structure and teacher growth	2 Canadian practising mathematics teachers		Triangulation	Chapman (2002)
Tests & Questionnaire	Teachers' problem posing and problem solving beliefs	128 Chinese student and practising elementary teachers	Previous study	SPSS analysis Multiple researchers	Chen et al. (2011)

Instrument(s)	Focus of Inquiry	Participants	Reliability	Validation	Study
Videotaping & Interview	Teachers' beliefs about the use of metacognition and heuristics related to problem solving	2 Flemish practising sixth grade teachers	Inter-rater Member checkijng	Content analysis	Depaepe, De Corte & Verschaffel (2010)
Vignette	Teachers' beliefs about key issues in teaching of mathematics	30 American practising pre-school teachers		SPSS analysis	Lee & Ginsburg (2007)

APPENDIX B

SUMMARY OF EXPERIMENTAL STUDIES

Study	Title	Source	Country	Sector	Participants	Research Method	Major Finding(s)
Abu-Elwan (2002)	Effectiveness of problem posing strategies on prospective mathematics teachers' performance	Journal of Science and Mathematics Education in S.E. Asia	Oman	Secondary	50 ITE mathematics students	Experimental design	Significant improvement in problem solving and problem posing performance for the experimental group compared with the control group.
Akay & Boz (2009a)	The effect of problem posing oriented calculus-II instruction on academic success	Journal of the Korea Society of Mathematical Education Series D: Research in Mathematical Education	Turkey	Primary	79 ITE students	Experimental design	Significant improvement in mathematics performance for the experimental group compared with the control group.
Akay & Boz (2010)	The effect of problem posing orientated analyses-II course on attitudes towards mathematics and mathematics self-efficacy of elementary prospective mathematics teachers	Australian Journal of Teacher Education	Turkey	Primary	82 ITE students	Experimental design	The attitude toward mathematics was significantly more positive for the experimental group than for the control group. Furthermore, mathematics self-efficacy beliefs were significantly stronger for the experimental group than for the control group.
Chen, Dooren & Vershaffel (2015)	Enhancing the development of Chinese fifth-graders' problem-posing and problem-solving abilities, beliefs, and attitudes: A design experiment	Book chapter	China	Primary	69 pupils (5th grade)	Experimental design	The originality of the problems posed by the experimental group was significantly better than for the control group. Further evidence included significantly better problem solving performances and more positive beliefs and attitudes towards problem posing and problem solving.

Study	Title	Source	Country	Sector	Participants	Research Method	Major Finding(s)
Demir (2005)	The effect of instruction with problem posing on tenth grade students' probability achievement and attitudes towards probability	Unpublished Masters' thesis	Turkey	Secondary	82 pupils (10th grade)	Experimental design	Significant improvement in probability attainment for the experimental group compared with the control group. Further evidence noted improved attitude towards probability and mathematics.
Dickerson (1999)	The impact of problem posing intervention on the mathematical problem solving achievement of seventh graders	Unpublished doctoral thesis	USA	Secondary	200 pupils (7th grade)	Experimental design	Significant improvement in problem solving achivement for the experiemtal group compared with the control group.
English (1997b)	The development of fifth-grade children's problem-posing abilities	Educational Studies in Education	Australia	Primary	27 pupils (5th grade)	Experimental design	In comparison to the control group, the experimental group appeared to show substantial development of (a) recognition and utilisation of problem structures (b) perceptions of, and preferences for different problem types (c) diverse mathematical thinking.
English (1998)	Children's problem posing within formal and informal contexts	Journal for Research in Mathemaitcs Education	Australia	Primary	54 pupils (3rd grade)	Experimental design	The experimental group demonstrated a significant improvement in the ability to generate mathematical problems compared with the control group.

Study	Title	Source	Country	Sector	Participants	Research Method	Major Finding(s)
Fetterly (2010)	An exploratory study of the use of a problem-posing approach on pre-service elementary education teachers' mathematical creativity, beliefs, and anxiety	Unpublished Doctoral thesis	USA	Primary	32 ITE students	Experimental design	Problem posing can foster and sustain mathematical creativity. Problem posing had a significant positive impact on mathematical beliefs and reducing mathematical anxiety for the experimental group compared with the control group.
Guvercin, Cilavaroglu & Savas (2014)	The effect of problem posing instruction on 9th grade students' mathematical academic achievement and retention	The Anthropologist	Kazakhstan	Secondary	60 pupils (9th grade)	Experimental design	Significant increase in mathematical academic achievement of the experimental group compared with the control group. Further evidence included significant visual effect on retention and a positive attitude towards mathematics.
Guvercin & Verbovskiy (2014)	The effect of problem posing tasks used in mathematics instruction to mathematics academic achievement and attitudes towards mathematics	International Online Journal of Primary Education	Kazakhstan	Secondary	54 pupils (8th grade)	Experimental design	Significant increase in mathematical academic achievement of the experimental group compared with the control group. Further evidence included positive attitude towards mathematics and increased levels of motivation and cognitive thinking.
Haghverdi & Gholami (2015)	A study of the effect of suing "what if not" strategy in posing geometry problems	Conference Paper	Iran	Secondary	29 pupils (unspecified stage)	Experimental design	Significant increase in the quantity of relevant problems posed by the experimental group compared with the control group. Futhermore, problem posing strengthened the connections between geometric concepts.

Study	Title	Source	Country	Sector	Participants	Research Method	Major Finding(s)
Kesan, Kaya & Guvercin (2010)	The effect of problem posing approach to the gifted student's mathematical abilities	International Online Journal of Educational Sciences	Kazakhstan	Secondary	40 pupils (8th grade)	Experimental design	Enhanced motivation and improved flexible thinking of the experimental group compared with the control group. Futhermore, greater classroom interaction resulting in increased mathematical performance.
Priest (2009)	A problem-posing intervention in the development of problem-solving competence of underachieving middle-year students	Unpublished Doctoral thesis	Australia	Primary	31 (Year 7)	Experimental design	The intervention facilitated the re-engagement of pupils from the experimental group compared with the control group. Further evidence included improved problem solving competence and the facilitatation of developmental learning.
Toluk-Ucar (2009)	Developing pre-service teachers understanding of fractions through problem posing	Teaching and Teaching Education	Turkey	Primary	95 ITE students	Experimental design	The experimental group demonstrated a positive impact on the understanding of fractions and on views about what it means to know mathematics compared with the control group.
Walsh (2016)	Pre-service primary teachers' understandings of mathematical problem posing and problem solving: Exploring the impact of a study intervention	Unpublished Masters' thesis	Ireland	Primary	415 ITE students	Experimental design	The intervention greatly improved the conception of what constituted a mathematical problem for the experimental group compared with the control group.

Study	Title	Source	Country	Sector	Participants	Research Method	Major Finding(s)
Xia, Lu & Wang (2008)	Research on mathematics instruction experiment based problem posing	Journal of Mathematics Education	China	Secondary	540 pupils (Junior high & 9th grade)	Experimental design	Significant effect on interest in learning mathematics, posing problems and ability to learn mathematics was discovered for the experimental group compared with the control group.

APPENDIX C

PHASE TWO QUESTIONNAIRE INSTRUMENT

TEACHERS' MATHEMATICAL BELIEFS QUESTIONNAIRE

Please read carefully before completing

My name is Paul Argyle McDonald and I am a fellow teacher and part-time PhD student within the College of Social Sciences, School of Education, University of Glasgow.

I am very interested in the mathematical beliefs of primary and secondary mathematics practitioners (including promoted staff). Your participation is most appreciated and will help to enrich the knowledge of what is considered a critical area of educational research within Curriculum for Excellence. Additional information regarding this research can be found in the accompanying participant phase one information sheet.

This questionnaire has been designed to take around 15-20 minutes to complete and is in three parts as follows:

Part A Demographic information

Part B Teachers' mathematical beliefs

Part C Interview option

Part A Demographic information

1. Please state your gender

- Male
- Female

2. What sector do you work in?

- Primary
- Secondary

3. How old are you?

- Under 25
- 25 to 34
- 35 to 44
- 45 to 54
- 55 or over

4. What is your mode of working?

- Full-time
- Part-time

5. Please state your employment type

- Permanent
- Temporary
- Teacher Induction Scheme

6. Did you complete your teacher training in Scotland?

- Yes
- No

7. What grade are you?

- Teacher
- Principal Teacher
- Deputy Headteacher
- Headteacher

8. Which LEA are you employed with?

- Inverclyde
- Renfrewshire
- West Dunbartonshire
- East Dunbartonshire
- Glasgow
- East Renfrewshire
- North Lanarkshire
- Falkirk
- West Lothian
- Edinburgh
- Midlothian
- East Lothian
- Clackmannanshire
- Fife
- Dundee
- Angus
- Aberdeenshire
- Aberdeen
- Moray
- Highland
- Na h-Eileanan Siar
- Argyll and Bute
- Perth and Kinross
- Stirling
- North Ayrshire
- East Ayrshire
- South Ayrshire
- Dumfries and Galloway
- South Lanarkshire
- Scottish Borders
- Orkney Islands
- Shetland Islands

9. To the nearest year, how long have you been teaching?

- 5 and under
- 6 to 10
- 11 to 15
- 16 to 20
- Over 20

10. Please state your highest level of qualification in the field of education

- BEd
- PGCE/PGDE
- Masters in Education
- Doctorate in Education

Part B Teachers' mathematical beliefs

Item	Statement	SA	A	U	D	SD
1.	Ignoring the mathematical ideas generated by the students can seriously limit their learning.	1	2	3	4	5
2.	Preparing learners to think critically about mathematics is more important than success at national examinations.	1	2	3	4	5
3.	After solving a problem, students should be encouraged to search for alternative solutions.	1	2	3	4	5
4.	Teachers should encourage their students to strive for elegant solutions when they solve problems.	1	2	3	4	5
5.	Justifying the mathematical statements that a person makes is an important part of mathematics.	1	2	3	4	5
6.	A person who does not understand why an answer to a mathematics problem is correct, has not really solved the problem.	1	2	3	4	5
7.	Mathematical knowledge is the result of the learner interpretation and organising the information gained from experiences.	1	2	3	4	5
8.	The priority in teaching mathematics is to ensure students develop confidence in problem posing and problem solving.	1	2	3	4	5
9.	Mathematics learning is enhanced when students are encouraged to take part in challenging activities within a supportive environment.	1	2	3	4	5
10.	Teachers should be experienced problem solvers and should have a firm grasp of what successful problem solving involves.	1	2	3	4	5
11.	Teachers should provide instructional activities which result in problematical situations for learners.	1	2	3	4	5
12.	Teaching mathematics through problem solving is the best method to help students learn.	1	2	3	4	5
13.	I provide opportunities for the development of students' mathematical creativity.	1	2	3	4	5
14.	Teachers always need to hear students' mathematical explanations before correcting their errors.	1	2	3	4	5
15.	Mathematics is a dynamic continually expanding field of human creation and invention.	1	2	3	4	5
16.	Problem posing is beneficial for developing students' mathematical skills and investigating their understanding of mathematics.	1	2	3	4	5
17.	It is important for students to create and solve their own problems	1	2	3	4	5

Item	Statement	SA	A	U	D	SD
18.	Mathematical problems can only have one final correct answer.	1	2	3	4	5
19.	The most effective way to learn mathematics is by listening carefully to the teacher explaining a mathematics lesson.	1	2	3	4	5
20.	Mathematics is an accumulation of facts, rules and skills.	1	2	3	4	5
21.	The primary purpose of teaching problem solving is to equip students with a collection of skills and processes.	1	2	3	4	5
22.	Mathematics is computation.	1	2	3	4	5
23.	Good mathematics teachers are the ones who show students the exact way to answer the mathematics questions they will be tested on.	1	2	3	4	5
24.	You explain in detail what the students have to do to solve problems.	1	2	3	4	5
25.	Students learn best by doing lots of exercises and practice.	1	2	3	4	5
26.	A mathematical problem is the description of a situation involving stated quantities, followed by a question about some relationship among the quantities.	1	2	3	4	5
27.	The memorisation of mathematics facts is important in mathematics learning.	1	2	3	4	5
28.	Teachers or the textbook - not the students - are the authority for what is right or wrong.	1	2	3	4	5
29.	Mathematics is a collection of procedures and rules that specify how to solve problems.	1	2	3	4	5
30.	Problem solving should be a separate distinctive part of the curriculum.	1	2	3	4	5
31.	Mathematics is a static but unified body of knowledge.	1	2	3	4	5
32.	When there is more than one way of solving a problem, it is generally safer to practice just one of the approaches.	1	2	3	4	5
33.	To be good at mathematics you must be able to solve problems quickly.	1	2	3	4	5
34.	Any problem can be solved if you know the right steps to follow.	1	2	3	4	5
35.	An effective way to teach mathematics is to provide students with interesting problems to investigate in small groups.	1	2	3	4	5
36.	Students are rational decision makers capable of determining for themselves what is right or wrong.	1	2	3	4	5

Item	Statement	SA	A	U	D	SD
37.	All students are able to be creative and do original work in mathematics.	1	2	3	4	5
38.	It is critical for students to view themselves as mathematical problem posers rather than to wait for problems from external sources such as a textbook or teacher.	1	2	3	4	5
39.	Students should share their problem solving thinking and approaches with other students.	1	2	3	4	5

Part C Interview option

Phase two of my research study is to conduct a sample of individual interviews.

It is expected that each interview will last between 30 and 45 minutes. In any publication arising from this research, participants will be referred to by a pseudonym.

Please note that you can withdraw from the interview stage at any point and without providing a reason.

Would you be willing to take part in a follow up interview to this questionnaire?

- o Yes
- o No

If yes, please provide a contact email:

Comments are welcome here regarding any aspect of this questionnaire

Many sincere thanks for taking the time to complete my questionnaire!

APPENDIX D

PHASE TWO INTERVIEW SCHEDULE

Interview Schedule

Part A Introduction and background information

- Formal introduction, rapport and outline of interview plan
- Discussion of phase two information sheet
- Completion of ethics consent form
- Determination of age, length of teaching experience, range of qualifications, ITT & CPD.

Part B Standard questions

- What is the nature of mathematics?
- Describe your personal experiences of school mathematics as a pupil
- What is the definition of a mathematical problem?
- What are important characteristics of a good problem?
- How do you rate your own ability to solve mathematical problems?
- Do you incorporate mathematical problem solving into your practice?
- How do you assess pupils' problem solving ability?
- What do you understand by mathematical problem posing?
- Do you ask pupils to pose mathematical problems?
- Are there any constraints imposed upon your professional practice that affect any aspect of the learning and teaching of mathematics?

Part C Questionnaire responses

Participant will be requested to elaborate on a small sample of individual responses (a copy of each participants' responses will be made available). The objective is to provide a rational to help illuminate the choice of response.

Part D Exploration of emergent themes and ideas

If relevant, appropriate questions will be posed and responses probed.

APPENDIX E

RESEARCH REQUEST TO HEADTEACHER OF INDIVIDUAL EDUCATIONAL ESTABLISHMENT

Subject: CfE: Relationship between teachers' mathematical beliefs and their engagement of problem solving and problem posing (Provisional title)

Sent: Date

To: Email of Headteacher/School Establishment

Attachments Phase One Information Sheet

Dear Headteacher,

As a fellow teacher, I am hoping to enlist your support with my professional learning (PhD research, College of Social Sciences, School of Education, University of Glasgow) of which I obtained written approval to contact you from [Head of Service and Name of LEA inserted here].

Teachers' mathematical beliefs are widely considered to play a central role in their practice and have been long regarded as critical to the reform of mathematics education. My research will attempt to enrich the knowledge of this important facet of educational research within Scotland.

Participation (see the link below) is **voluntary** and should take each of your [mathematics inserted here for secondary Headteachers] staff around 15-20 minutes to complete. All promoted members are encouraged to take part.

Please note that I am fully committed to strict confidentiality and anonymity (i.e. no staff or individual school will be identified).

Further participant information is attached (Phase One Information Sheet) for your interest.

Your help will be very much appreciated.

Warm regards,

Paul Argyle McDonald

https://www.surveymonkey.com/s/teacherq

APPENDIX F

PHASE ONE INFORMATION SHEET FOR QUESTIONNAIRE PARTICIPANTS

University of Glasgow | College of Social Sciences

PHASE ONE INFORMATION SHEET (Questionnaire Participants)

Study title (Provisional)
CfE : Relationship between teachers' mathematical beliefs and their engagement of problem solving and problem posing.

Researcher details
My name is Paul Argyle McDonald and I am a full-time teacher and part-time PhD student within the College of Social Sciences, School of Education, University of Glasgow.

I would like to invite you to take part in the first phase of a research study for my own doctoral thesis.

Purpose of the study
Research indicates that teachers' mathematical beliefs can have a significant impact on classroom practices. In Scotland, both primary and secondary mathematics practitioners have responsibility for delivering mathematics within Curriculum for Excellence. Understanding the nature of teachers' mathematical beliefs may help to explain why different experiences exist.

Research criteria
I am keen to recruit primary and secondary teachers who have responsibility for delivering the subject of mathematics within Curriculum for Excellence to participate in my study. If you would like to receive a summary of the findings of this research, please email me at the address below.

Voluntary participation
Please note that participation is completely voluntary. Even after deciding to participate, you are still free to withdraw at any time and without giving a reason. This includes withdrawing any data previously supplied.

Research involvement
Your participation will involve completing one online questionnaire (web link). The time taken to fill the questionnaire will be approximately 15-20 minutes. An opportunity to take part in an individual interview at a later stage is optional and completely voluntary (Part C of questionnaire).

Strict confidentiality
All information obtained from you as a research participant during the course of the research will be carefully safeguarded, dealt with anonymously and will be destroyed upon completion and award of degree.

Research results
Your data will be collected and used to calculate a range of statistical results. The findings of these results will help to establish possible connections involving other groups of teachers. Any subsequent conclusions will be included in the thesis produced at the end of the research.

Ethics review and further contact details
This research study has been approved by the College of Social Sciences Research Ethics Committee. Ethics contact details - School of Ethics Forum, Dr Muir Houston: email - Muir.Houston@glasgow.ac.uk

Research supervisors – Dr Catherine Fagan: email - Catherine.Fagan@glasgow.ac.uk and Dr Fiona Patrick: email - Fiona.Patrick@glasgow.ac.uk

If you have any concerns regarding the conduct of this research project, please contact the College of Social Sciences Ethics Officer by contacting Dr Muir Houston, College of Social Sciences Ethics Officer: email - Muir.Houston@glasgow.ac.uk or Telephone 0141 330 4699.

For any queries and summary of findings, please contact Paul Argyle McDonald: email - p.mcdonald.1@research.gla.ac.uk

APPENDIX G

PHASE TWO INFORMATION SHEET FOR INTERVIEW PARTICIPANTS

University of Glasgow | College of Social Sciences

PHASE TWO INFORMATION SHEET (Interview Participants)

Study title (Provisional)
CfE : Relationship between teachers' mathematical beliefs and their engagement of problem solving and problem posing.

Researcher details
My name is Paul Argyle McDonald and I am a practising full-time teacher and part-time PhD student within the College of Social Sciences, School of Education, University of Glasgow.

I would like to invite you to take part in the second phase of a research study for my own doctoral thesis.

Purpose of the study
Research indicates that teachers' mathematical beliefs can have a significant impact on classroom practices. In Scotland, both primary and secondary mathematics practitioners have responsibility for delivering mathematics within Curriculum for Excellence. Understanding the nature of teachers' mathematical beliefs may help to explain why different experiences exist.

Research criteria
I am keen to recruit primary and secondary teachers who have responsibility for delivering the subject of mathematics within Curriculum for Excellence to participate in my study. If you would like to receive a summary of the findings of this research, please email me at the address below.

Voluntary participation
Please note that participation is completely voluntary. Even after deciding to participate, you are still free to withdraw at any time and without giving a reason. This includes withdrawing any data previously supplied.

Research involvement
Your participation will involve taking part in a face to face interview with the researcher. The duration of the interview is expected to last between 30 and 45 minutes. A consent form is required to be completed by you prior to taking part.

Strict confidentiality
All information obtained from you as a research participant during the course of the research will be carefully safeguarded, dealt with anonymously and will be destroyed upon completion and award of degree.

Research results
Your data will be collected and used to calculate a range of statistical results. The findings of these results will help to establish possible connections involving other groups of teachers. Any subsequent conclusions will be included in the thesis produced at the end of the research.

Ethics review and further contact details
This research study has been approved by the College of Social Sciences Research Ethics Committee. Ethics contact details - School of Ethics Forum, Dr Muir Houston: email - Muir.Houston@glasgow.ac.uk

Research supervisors – Dr Catherine Fagan: email - Catherine.Fagan@glasgow.ac.uk and Dr Fiona Patrick: email - Fiona.Patrick@glasgow.ac.uk

If you have any concerns regarding the conduct of this research project, please contact the College of Social Sciences Ethics Officer by contacting Dr Muir Houston, College of Social Sciences Ethics Officer: email - Muir.Houston@glasgow.ac.uk or Telephone 0141 330 4699.

For any queries and summary of findings, please contact Paul Argyle McDonald: email - p.mcdonald.1@research.gla.ac.uk

APPENDIX H

PHASE TWO RESEARCH ETHICS INTERVIEW CONSENT FORM

University of Glasgow | College of Social Sciences

PHASE TWO RESEARCH ETHICS INTERVIEW CONSENT FORM

Study title (Provisional)

CfE : Relationship between teachers' mathematical beliefs and their engagement of problem solving and problem posing.

Researcher details

Paul Argyle McDonald, part-time PhD student within the College of Social Sciences, School of Education, University of Glasgow.

Please initial box

1. I confirm that I have read and understand the information sheet for the above study and have had the opportunity to ask questions.
2. I understand that my participation is voluntary and that I am free to withdraw at any time, without giving reason.
3. I agree to take part in the above study.
4. I agree to the interview being audio-recorded.
5. I confirm to the use of anoymised quotes in publications.

Please complete below:

Full name of participant	Date	Participant signature
Paul Argyle McDonald		
Full name of researcher	Date	Researcher signature

APPENDIX I

TABLE OF TOTAL VARIANCE EXPLAINED

Total Variance Explained (Extraction Method: Principal Component Analysis)

Component	Initial Eigenvalues			Extraction Sums of Squared Loadings		
	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	7.932	20.338	20.338	7.932	20.338	20.338
2	4.457	11.429	31.768	4.457	11.429	31.768
3	2.760	7.078	38.845	2.760	7.078	38.845
4	1.379	3.536	42.381	1.379	3.536	42.381
5	1.291	3.311	45.692	1.291	3.311	45.692
6	1.157	2.967	48.658	1.157	2.967	48.658
7	1.036	2.657	51.315	1.036	2.657	51.315
8	1.016	2.606	53.921	1.016	2.606	53.921
9	.971	2.490	56.411			
10	.905	2.321	58.732			
11	.876	2.246	60.977			
12	.821	2.106	63.083			
13	.805	2.064	65.147			
14	.790	2.025	67.172			
15	.774	1.986	69.158			
16	.729	1.870	71.028			
17	.681	1.745	72.773			
18	.661	1.696	74.469			
19	.640	1.640	76.109			
20	.616	1.578	77.687			
21	.614	1.574	79.261			
22	.597	1.531	80.792			
23	.571	1.465	82.257			
24	.556	1.425	83.682			
25	.536	1.374	85.056			
26	.517	1.327	86.383			
27	.511	1.311	87.694			
28	.484	1.241	88.935			
29	.472	1.209	90.144			
30	.459	1.177	91.321			
31	.447	1.147	92.468			
32	.417	1.069	93.537			
33	.399	1.024	94.562			
34	.393	1.007	95.569			
35	.382	.979	96.547			
36	.369	.945	97.493			
37	.354	.907	98.400			
38	.317	.813	99.213			
39	.307	.787	100.000			

APPENDIX J

TABLE OF OUTPUT FROM PARALLEL ANALYSIS

Output from Parallel Analysis

```
27/03/2015   22:27:22
Number of variables:     39
Number of subjects:     478
Number of replications: 100

Eigenvalue       Random Eigenvalue     Standard Dev
      1              1.5790                .0423
      2              1.5147                .0284
      3              1.4693                .0225
      4              1.4276                .0222
      5              1.3872                .0222
      6              1.3509                .0206
      7              1.3188                .0181
      8              1.2861                .0188
      9              1.2537                .0167
     10              1.2250                .0176
     11              1.1958                .0163
     12              1.1687                .0153
     13              1.1423                .0154
     14              1.1167                .0147
     15              1.0905                .0143
     16              1.0681                .0141
     17              1.0443                .0134
     18              1.0196                .0133
     19              0.9965                .0126
     20              0.9739                .0132
     21              0.9505                .0125
     22              0.9305                .0132
     23              0.9089                .0136
     24              0.8883                .0135
     25              0.8658                .0125
     26              0.8450                .0128
     27              0.8221                .0140
     28              0.8013                .0134
     29              0.7818                .0136
     30              0.7615                .0136
     31              0.7414                .0129
     32              0.7204                .0125
     33              0.6961                .0132
     34              0.6723                .0135
     35              0.6496                .0143
     36              0.6231                .0146
     37              0.6001                .0168
     38              0.5733                .0164
     39              0.5393                .0199
27/03/2015   22:27:45
Monte Carlo PCA for Parallel Analysis
©2000 by Marley W. Watkins. All rights
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APPENDIX K

THE PATTERN MATRIX

Item	Description	Loading
Belief System 1: A social constructivist, problem solving and collaborative orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics.		
1	Ignoring the mathematical ideas generated by the students can seriously limit their learning.	.662
17	It is important for students to create and solve their own problems.	.661
38	It is critical for students to view themselves as mathematical problem posers rather than to wait for problems from external sources such as a textbook or teacher.	.654
12	Teaching mathematics through problem solving is the best method to help students learn.	.620
2	Preparing learners to think critically about mathematics is more important than success at national examinations.	.617
16	Problem posing is beneficial for developing students' mathematical skills and investigating their understanding of mathematics.	.599
35	An effective way to teach mathematics is to provide students with interesting problems to investigate in small groups.	.583
39	Students should share their problem solving thinking and approaches with other students.	.575
Belief System 2: A social constructivist, problem solving and static transmission orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics.		
15	Mathematics is a continually expanding field of human creation and invention.	.669
5	Justifying the mathematical statements that a person makes is an important part of mathematics.	.644
10	Teachers should be experienced problem solvers and should have a firm grasp of what successful problem solving involves.	.637
18	Mathematical problems can only have one final correct answer.	.594
6	A person who does not understand why an answer to a mathematics problem is correct, has not really solved the problem.	.577
11	Teachers should provide instructional activities which result in problematical situations for learners.	.574
Belief System 3: A static transmission and mechanistic transmission orientation towards the nature of mathematics, the learning of mathematics and the teaching of mathematics.		
20	Mathematics is an accumulation of facts, rules and skills.	.708
26	A mathematical problem is the description of a situation involving stated quantities, followed by a question about some relationship among the quantities.	.588
29	Mathematics is a collection of procedures and rules that specify how to solve problems.	.562
27	The memorisation of mathematics facts is important in mathematics learning.	.541
21	The primary purpose of teaching problem solving is to equip students with a collection of skills and processes.	.528

Extraction Method: Principal Component Analysis.
Rotation Method: Oblimin with Kaiser Normalization. Rotation converged in 18 iterations.

APPENDIX L

THE STRUCTURE MATRIX

	Component		
	1	2	3
Item 38	.677	.272	.202
Item 17	.673	.316	.109
Item 12	.653	.259	.271
Item 1	.648	-.041	.013
Item 2	.601	-.272	.053
Item 16	.599	.399	.003
Item 32	.584	.156	.215
Item 35	.568	.066	-.025
Item 37	.567	-.205	.175
Item 39	.561	.249	-.071
Item 36	.560	-.159	.207
Item 13	.548	.079	.146
Item 19	.542	.031	.396
Item 23	.539	.041	.438
Item 3	.521	.418	.132
Item 33	.520	-.187	.103
Item 28	.481	-.021	.360
Item 9	.476	.263	.080
Item 7	.455	-.068	-.053
Item 14	.424	-.073	.047
Item 8	.386	.283	.021
Item 15	.118	.699	.298
Item 5	.129	.648	.097
Item 10	.092	.636	.065
Item 18	-.071	.629	.369
Item 6	.059	.585	.130
Item 11	.288	.582	.075
Item 31	-.140	.539	.497
Item 4	.020	.431	.286
Item 20	.105	.227	.726
Item 26	.029	.342	.615
Item 29	.257	.284	.614
Item 27	.248	-.134	.536
Item 22	-.135	.400	.532
Item 25	.407	-.240	.512
Item 24	.366	.013	.510
Item 34	.017	.222	.506
Item 21	-.023	-.002	.504
Item 30	.097	.106	.448

Extraction Method: Principal Component Analysis.
Rotation Method: Oblimin with Kaiser Normalization.

APPENDIX M

THE RESULTS OF THE QUESTIONNAIRE

Item	SA f (%)	A f (%)	U f (%)	D f (%)	SD f (%)	Mean	Standard Deviation
1.	132 (27.62)	278 (58.16)	29 (6.07)	39 (8.16)	0 (0.00)	1.95	.814
2.	78 (16.32)	174 (36.40)	69 (14.44)	117 (24.48)	40 (8.37)	2.72	1.234
3.	71 (14.85)	277 (57.95)	59 (12.34)	71 (14.85)	0 (0.00)	2.27	.891
4.	42 (8.79)	179 (37.45)	91 (19.04)	158 (33.05)	8 (1.67)	2.81	1.045
5.	94 (19.67)	313 (65.48)	42 (8.79)	29 (6.07)	0 (0.00)	2.01	.727
6.	57 (11.92)	267 (55.86)	47 (9.83)	103 (21.55)	4 (0.84)	2.44	.984
7.	59 (12.34)	293 (61.30)	69 (14.44)	55 (11.51)	2 (0.42)	2.26	.835
8.	54 (11.30)	279 (58.37)	72 (15.06)	70 (14.64)	3 (0.63)	2.35	.886
9.	147 (30.75)	298 (62.34)	17 (3.56)	15 (3.14)	1 (0.21)	1.80	.669
10.	120 (25.10)	286 (59.83)	24 (5.02)	46 (9.62)	2 (0.42)	2.00	.852
11.	50 (10.46)	281 (58.79)	63 (13.18)	83 (17.36)	1 (0.21)	2.38	.898
12.	22 (4.60)	173 (36.19)	110 (23.01)	164 (34.31)	9 (1.88)	2.93	.980
13.	40 (8.37)	317 (66.32)	74 (15.48)	44 (9.21)	3 (0.63)	2.27	.768
14.	89 (18.62)	261 (54.60)	39 (8.16)	87 (18.20)	2 (0.42)	2.27	.981
15.	105 (21.97)	253 (52.93)	77 (16.11)	40 (8.37)	3 (0.63)	2.13	.870
16.	111 (23.22)	273 (57.11)	45 (9.41)	49 (10.25)	0 (0.00)	2.07	.856
17.	96 (20.08)	276 (57.74)	44 (9.21)	61 (12.76)	1 (0.21)	2.15	.895
18.*	14 (2.93)	117 (24.48)	47 (9.83)	225 (47.07)	75 (15.69)	2.52	1.110
19.*	9 (1.88)	106 (22.18)	62 (12.97)	262 (54.81)	39 (8.16)	2.55	.985
20.*	36 (7.53)	269 (56.28)	43 (9.00)	112 (23.43)	18 (3.77)	3.40	1.043
21.*	103 (21.55)	301 (62.97)	33 (6.90)	41 (8.58)	0 (0.00)	3.97	.793
22.*	8 (1.67)	119 (24.90)	76 (15.90)	226 (47.28)	49 (10.25)	2.60	1.022
23.*	6 (1.26)	75 (15.69)	33 (6.90)	268 (56.07)	96 (20.08)	2.22	.982
24.*	24 (5.02)	185 (38.70)	43 (9.00)	204 (42.68)	22 (4.60)	2.97	1.096
25.*	33 (6.90)	202 (42.26)	67 (14.02)	163 (34.10)	13 (2.72)	3.17	1.060
26.*	10 (2.09)	196 (41.00)	97 (20.29)	153 (32.01)	22 (4.60)	3.04	.999
27.*	77 (16.11)	300 (62.76)	34 (7.11)	65 (13.60)	2 (0.42)	3.81	.882
28.*	3 (0.63)	81 (16.95)	36 (7.53)	282 (59.00)	76 (15.90)	2.27	.946
29.*	21 (4.39)	236 (49.37)	75 (15.69)	133 (27.82)	13 (2.72)	3.25	.998
30.*	14 (2.93)	85 (17.78)	55 (11.51)	254 (53.14)	70 (14.64)	2.41	1.034
31 *	6 (1.26)	92 (19.25)	80 (16.74)	233 (48.74)	67 (14.02)	2.45	1.034
32.*	10 (2.09)	106 (22.18)	30 (6.28)	260 (54.39)	72 (15.06)	2.42	1.056

Item	SA f (%)	A f (%)	U f (%)	D f (%)	SD f (%)	Mean	Standard Deviation
33.*	0 (0.00)	113 (23.64)	42 (8.79)	268 (56.07)	55 (11.51)	2.45	.976
34.*	30 (6.28)	254 (53.14)	63 (13.18)	116 (24.47)	15 (3.14)	3.35	1.015
35.	90 (18.83)	305 (63.81)	28 (5.86)	54 (11.30)	1 (0.21)	2.10	.842
36.	28 (5.86)	255 (53.35)	83 (17.36)	108 (22.59)	4 (0.84)	2.59	.929
37.	51 (10.67)	204 (42.68)	63 (13.18)	147 (30.75)	13 (2.72)	2.72	1.093
38.	47 (9.83)	233 (48.74)	95 (19.87)	99 (20.71)	4 (0.84)	2.54	.955
39.	214 (44.77)	247 (51.67)	7 (1.46)	9 (1.88)	1 (0.21)	1.61	.637

Note: SA – Strongly Agree
A – Agree
U – Undecided
D – Disagree
SD – Strongly Disagree

Totals of percentages are not 100.00 for every item because of rounding.

* These items are negatively stated and have been reversed in scoring. Therefore, a higher mean value indicates participants disagree with the statement.

** The minimum possible mean value is 1 and the maximum possible mean value is 5.

APPENDIX N

THE RESULTS OF THE QUESTIONNAIRE BY SECTOR

Item	Sector	SA f (%)	A f (%)	U f (%)	D f (%)	SD f (%)
1.	Primary	88 (34.4)	127 (55.5)	12 (5.2)	2 (0.9)	0 (0.0)
	Secondary	44 (17.7)	151 (60.6)	17 (6.8)	37 (14.9)	0 (0.0)
2.	Primary	52 (22.7)	112 (48.9)	34 (14.9)	30 (13.1)	1 (0.4)
	Secondary	26 (10.4)	62 (24.9)	35 (14.1)	87 (34.9)	39 (15.7)
3.	Primary	36 (15.7)	139 (60.7)	24 (10.5)	30 (13.1)	0 (0.0)
	Secondary	35 (14.1)	138 (55.4)	35 (14.1)	41 (16.5)	0 (0.0)
4.	Primary	9 (3.9)	54 (23.6)	63 (27.5)	97 (42.4)	6 (2.6)
	Secondary	33 (13.3)	125 (50.2)	28 (11.2)	61 (24.5)	2 (0.8)
5.	Primary	42 (18.3)	131 (57.2)	33 (14.4)	23 (10.0)	0 (0.0)
	Secondary	52 (20.9)	182 (73.1)	9 (3.6)	6 (2.4)	0 (0.0)
6.	Primary	26 (11.4)	106 (46.3)	27 (11.8)	69 (30.1)	1 (0.4)
	Secondary	31 (12.5)	161 (64.7)	20 (8.0)	34 (13.7)	3 (1.2)
7.	Primary	40 (17.5)	148 (64.6)	32 (14.0)	9 (3.9)	0 (0.0)
	Secondary	19 (7.6)	145 (58.2)	37 (14.9)	46 (18.5)	2 (0.8)
8.	Primary	27 (11.8)	132 (57.6)	35 (15.3)	34 (14.9)	1 (0.4)
	Secondary	27 (10.8)	147 (59.0)	37 (14.9)	36 (14.5)	2 (0.8)
9.	Primary	91 (39.7)	121 (52.8)	10 (4.4)	7 (3.1)	0 (0.0)
	Secondary	56 (22.5)	177 (71.1)	7 (2.8)	8 (3.2)	1 (0.4)
10.	Primary	43 (18.8)	126 (55.0)	21 (9.2)	38 (16.6)	1 (0.4)
	Secondary	77 (30.9)	160 (64.3)	3 (1.2)	6 (3.21)	1 (0.4)
11.	Primary	22 (9.6)	126 (55.0)	31 (13.5)	49 (21.4)	1 (0.4)
	Secondary	28 (11.2)	155 (62.3)	32 (12.9)	34 (13.6)	0 (0.0)
12.	Primary	10 (4.4)	94 (41.1)	54 (23.6)	69 (30.1)	2 (0.9)
	Secondary	12 (4.8)	79 (31.7)	56 (22.5)	95 (38.2)	7 (2.8)
13.	Primary	21 (9.2)	176 (76.9)	25 (10.9)	6 (2.6)	1 (0.4)
	Secondary	19 (7.6)	141 (56.6)	49 (19.7)	38 (15.3)	2 (0.8)
14.	Primary	60 (26.2)	124 (54.2)	19 (8.30)	26 (11.4)	0 (0.0)
	Secondary	29 (11.7)	137 (55.0)	20 (8.0)	61 (24.5)	2 (0.8)
15.	Primary	37 (16.2)	107 (46.7)	53 (23.1)	30 (13.1)	2 (0.9)
	Secondary	68 (27.3)	146 (58.6)	24 (9.6)	10 (4.0)	1 (0.4)

Item	Sector	SA f (%)	A f (%)	U f (%)	D f (%)	SD f (%)
16.	Primary	60 (26.2)	126 (55.0)	28 (12.2)	15 (6.6)	0 (0.0)
	Secondary	51 (20.5)	147 (59.0)	17 (6.8)	34 (13.7)	0 (0.0)
17.	Primary	58 (25.3)	132 (57.6)	20 (8.7)	18 (7.9)	1 (0.4)
	Secondary	38 (15.3)	144 (57.8)	24 (9.6)	43 (17.3)	0 (0.0)
18.	Primary	12 (5.2)	88 (38.4)	25 (10.9)	91 (39.7)	13 (5.7)
	Secondary	2 (0.8)	29 (11.7)	22 (8.8)	134 (53.8)	62 (24.9)
19.	Primary	4 (1.8)	33 (14.4)	21 (9.2)	143 (62.4)	28 (12.2)
	Secondary	5 (2.0)	73 (29.3)	41 (16.5)	119 (47.8)	11 (4.4)
20.	Primary	17 (7.4)	145 (63.3)	29 (12.7)	32 (14.0)	6 (2.6)
	Secondary	19 (7.6)	124 (49.8)	14 (5.6)	80 (32.1)	12 (4.8)
21.	Primary	55 (24.0)	143 (62.5)	18 (7.9)	13 (5.7)	0 (0.0)
	Secondary	48 (19.3)	158 (63.5)	15 (6.0)	28 (11.2)	0 (0.0)
22.	Primary	4 (1.8)	82 (35.8)	46 (20.1)	89 (28.9)	8 (3.5)
	Secondary	4 (1.6)	37 (14.9)	30 (12.1)	137 (55.0)	41 (16.5)
23.	Primary	3 (1.3)	15 (6.6)	14 (6.1)	141 (61.6)	56 (24.5)
	Secondary	3 (1.2)	60 (24.1)	19 (7.6)	127 (51.0)	40 (16.1)
24.	Primary	12 (5.2)	68 (29.7)	23 (10.0)	109 (47.6)	17 (7.4)
	Secondary	12 (4.8)	117 (47.0)	20 (8.0)	95 (38.2)	5 (2.0)
25.	Primary	7 (3.1)	75 (32.8)	34 (14.9)	104 (45.4)	9 (3.9)
	Secondary	26 (10.4)	127 (51.0)	33 (13.3)	59 (23.7)	4 (1.6)
26.	Primary	7 (3.1)	109 (47.6)	57 (24.9)	51 (22.3)	5 (2.2)
	Secondary	3 (1.2)	87 (34.9)	40 (16.1)	102 (41.0)	17 (6.8)
27.	Primary	38 (16.6)	138 (60.3)	22 (9.6)	31 (13.5)	0 (0.0)
	Secondary	39 (15.7)	162 (65.1)	12 (4.8)	34 (13.7)	2 (0.8)
28.	Primary	1 (0.4)	21 (9.2)	12 (5.7)	146 (63.8)	48 (21.0)
	Secondary	2 (0.8)	60 (24.1)	23 (9.2)	136 (54.6)	28 (11.2)
29.	Primary	11 (4.8)	118 (51.5)	42 (18.3)	54 (23.6)	4 (1.8)
	Secondary	10 (4.0)	118 (47.4)	33 (13.3)	79 (31.7)	9 (3.6)
30.	Primary	6 (2.6)	45 (19.7)	27 (11.8)	124 (54.2)	27 (11.8)
	Secondary	8 (3.2)	40 (16.1)	28 (11.2)	130 (52.2)	43 (17.3)

Item	Sector	SA f (%)	A f (%)	U f (%)	D f (%)	SD f (%)
31.	Primary	3 (1.3)	69 (30.1)	55 (24.0)	91 (39.7)	11 (4.8)
	Secondary	3 (1.2)	23 (9.2)	25 (10.0)	142 (57.0)	56 (22.5)
32.	Primary	4 (1.8)	31 (13.5)	13 (5.7)	135 (59.0)	46 (20.1)
	Secondary	6 (2.4)	75 (30.1)	17 (6.8)	125 (50.2)	26 (10.4)
33.	Primary	0 (0.0)	23 (10.0)	19 (8.3)	155 (67.7)	32 (14.0)
	Secondary	0 (0.0)	90 (36.1)	23 (9.2)	113 (45.4)	23 (9.2)
34.	Primary	11 (4.8)	139 (60.7)	45 (19.7)	32 (14.0)	2 (0.9)
	Secondary	19 (7.6)	115 (46.2)	18 (7.2)	84 (33.7)	13 (5.2)
35.	Primary	62 (27.1)	141 (61.6)	10 (4.4)	15 (6.6)	1 (0.4)
	Secondary	28 (11.2)	164 (65.9)	18 (7.2)	39 (15.7)	0 (0.0)
36.	Primary	15 (6.6)	150 (65.5)	36 (15.7)	26 (11.4)	2 (0.9)
	Secondary	13 (5.2)	105 (42.2)	47 (18.9)	82 (32.9)	2 (0.8)
37.	Primary	36 (15.7)	122 (52.3)	30 (13.1)	36 (15.7)	5 (2.2)
	Secondary	15 (6.0)	82 (32.9)	33 (13.3)	111 (44.6)	8 (3.2)
38.	Primary	34 (14.9)	117 (51.1)	44 (19.2)	34 (14.9)	0 (0.0)
	Secondary	13 (5.2)	116 (46.6)	51 (20.5)	65 (26.1)	4 (1.6)
39.	Primary	123 (53.7)	103 (45.0)	2 (0.9)	1 (0.4)	0 (0.0)
	Secondary	91 (36.6)	144 (57.8)	5 (2.0)	8 (3.2)	1 (0.4)

Note: SA – Strongly Agree
A – Agree
U – Undecided
D – Disagree
SD – Strongly Disagree

Totals of percentages are not 100.00 for every item because of rounding.

APPENDIX O

GROUP DIFFERENCES BY SECTOR

	Primary		Secondary					
Item	M	SD	M	SD	df	t	p	Cohen's d
1.	1.69	.612	2.19	.898	439.635	-7.206	**.000**	.687
2.	2.20	.951	3.20	1.268	457.863	-9.886	**.000**	.924
3.	2.21	.863	2.33	.914	475.645	-1.473	.141	.135
4.	3.16	.948	2.49	1.028	476	7.359	**.000**	.675
5.	2.16	.840	1.88	.572	397.358	4.314	**.000**	.433
6.	2.62	1.047	2.27	.890	449.330	3.977	**.000**	.375
7.	2.04	.687	2.47	.907	459.436	-5.765	**.000**	.538
8.	2.34	.888	2.35	.887	476	-.104	.917	.010
9.	1.71	.693	1.88	.636	462.877	-2.821	**.005**	.262
10.	2.25	.962	1.78	.663	400.377	6.168	**.000**	.617
11.	2.48	.949	2.29	.840	457.042	2.324	**.021**	.217
12.	2.82	.945	3.02	1.004	476	-2.273	**.023**	.208
13.	2.08	.583	2.45	.870	436.146	-5.456	**.000**	.523
14.	2.05	.895	2.48	1.012	475.257	-4.928	**.000**	.452
15.	2.36	.933	1.92	.749	437.162	5.684	**.000**	.544
16.	1.99	.806	2.14	.897	476	-1.858	.064	.170
17.	2.00	.835	2.29	.927	475.797	-3.533	**.000**	.324
18.*	2.98	1.106	2.10	.933	447.682	9.382	**.000**	.887
19.*	2.31	.925	2.77	.989	475.864	-5.221	**.000**	.479
20.*	3.59	.911	3.23	1.126	468.551	3.819	**.000**	.353
21.*	4.05	.739	3.91	.835	476	1.940	.053	.089
22.*	2.93	.978	2.30	.968	471.831	7.107	**.000**	.654
23.*	1.99	.830	2.43	1.061	464.216	-5.150	**.000**	.478
24.*	2.78	1.111	3.14	1.053	476	-3.710	**.000**	.340
25.*	2.86	1.022	3.45	1.015	476	-6.368	**.000**	.584
26.*	3.27	.916	2.83	1.027	475.554	4.990	**.000**	.458
27.*	3.80	.875	3.81	.889	476	-.150	.881	.014
28.*	2.04	.821	2.49	1.004	469.583	-5.289	**.000**	.488
29.*	3.34	.949	3.16	1.036	475.993	1.938	.053	.178
30.*	2.47	1.020	2.36	1.046	476	1.207	.228	.111
31.*	2.83	.959	2.10	.893	464.824	8.683	**.000**	.805
32.*	2.18	.963	2.64	1.092	475.212	-4.888	**.000**	.448
33.*	2.14	.779	2.72	1.055	455.169	-6.862	**.000**	.643
34.*	3.55	.824	3.17	1.135	452.121	4.136	**.000**	.389
35.	1.92	.682	2.27	.860	475.940	-4.741	**.000**	.435
36.	2.34	.800	2.82	.981	469.268	-5.812	**.000**	.537

	Primary		Secondary					
Item	M	SD	M	SD	df	t	p	Cohen's d
37.	2.35	.996	3.06	1.070	475.928	-7.474	**.000**	.685
38.	2.34	.907	2.72	.963	475.729	-4.470	**.000**	.410
39.	1.48	.543	1.73	.693	476	-4.377	**.000**	.401

Note: * These items are negatively stated and have been reversed in scoring. Therefore, a higher mean value indicates participants disagree with the statement.

** The minimum possible mean value is 1 and the maximum possible mean value is 5.

APPENDIX P

HISTOGRAMS AND NORMAL Q-Q PLOTS FOR TOTAL MATHEMATICAL BELIEFS SCORES

A. Histograms and Normal Q-Q Plots for TMBS with respect to gender

Figure 10.1 Histogram of TMBS for male participants

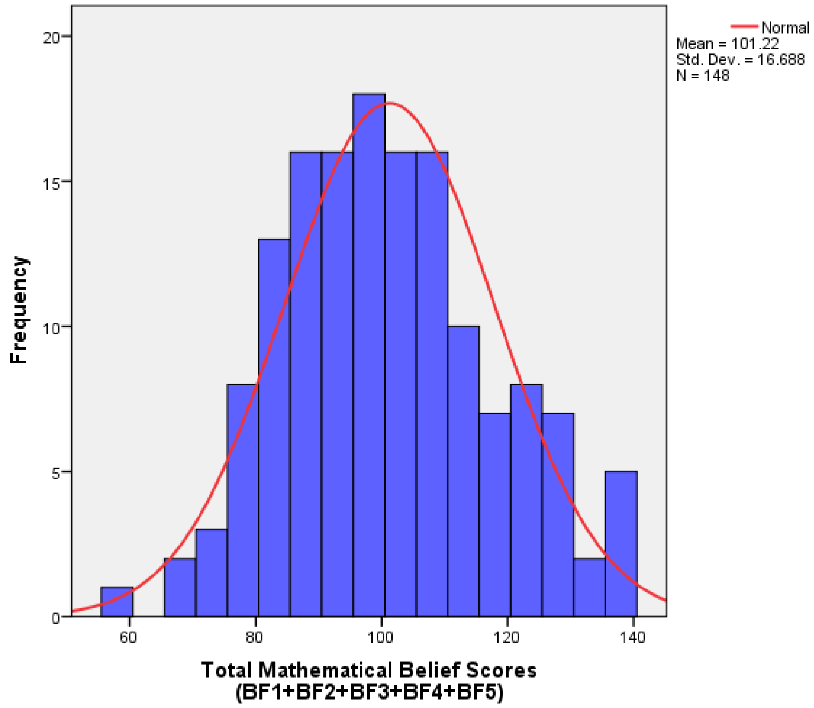


Figure 10.2 Histogram of TMBS for female participants

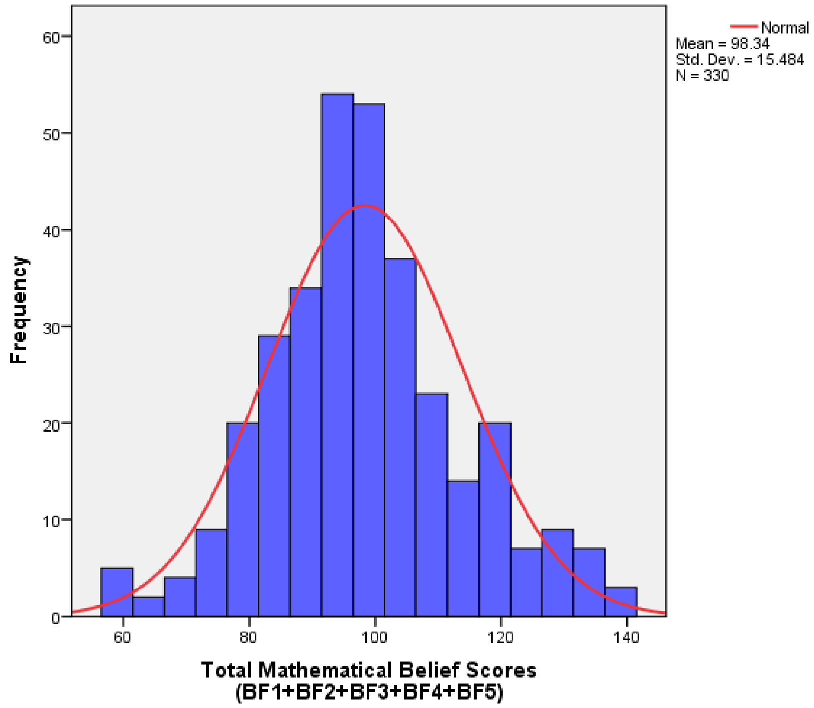


Figure 10.3 Normal Q-Q Plot of TMBS for male participants

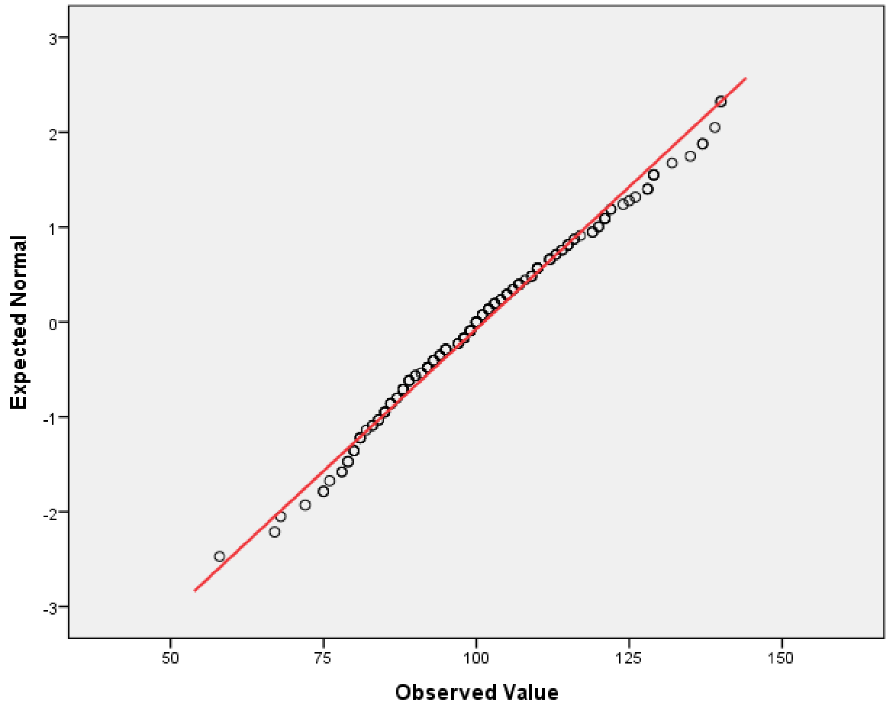
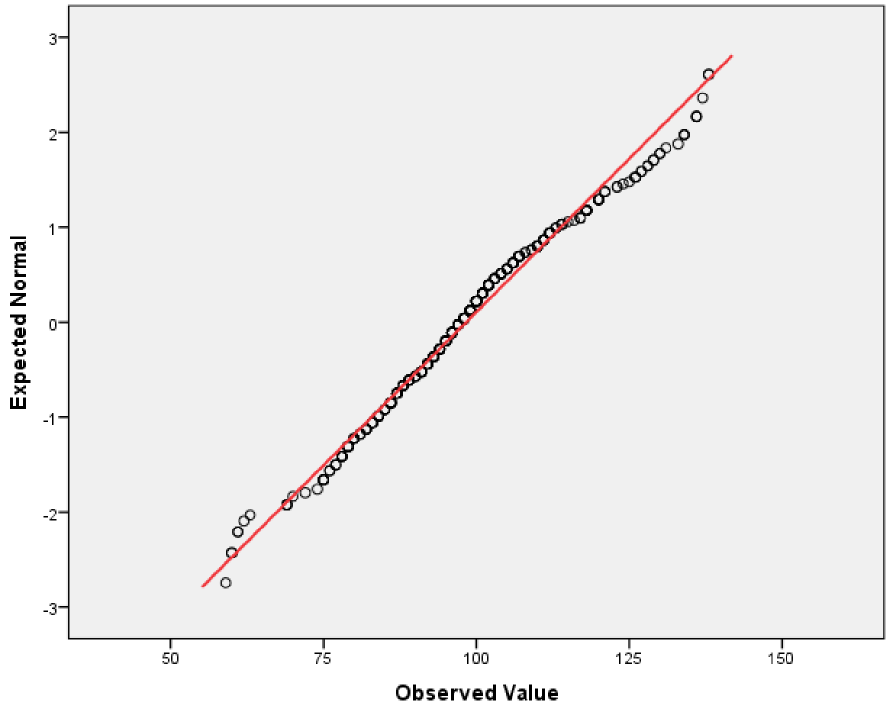


Figure 10.4 Normal Q-Q Plot of TMBS for female participants



B. Histograms and Normal Q-Q Plots for the TMBS with respect to age (years)

Figure 10.5 Histogram of TMBS for participants under 25

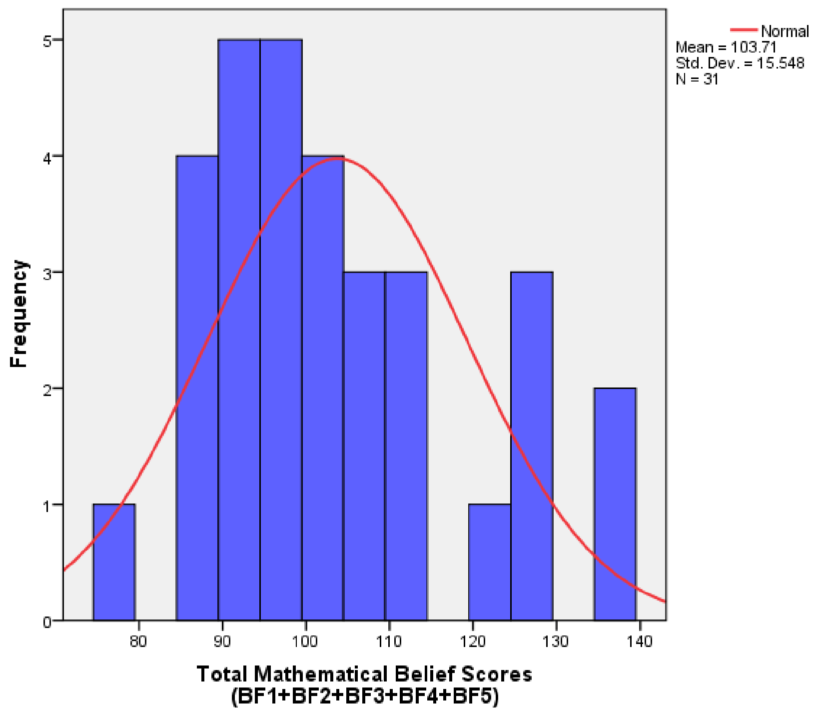


Figure 10.6 Histogram of TMBS for participants 25 to 34

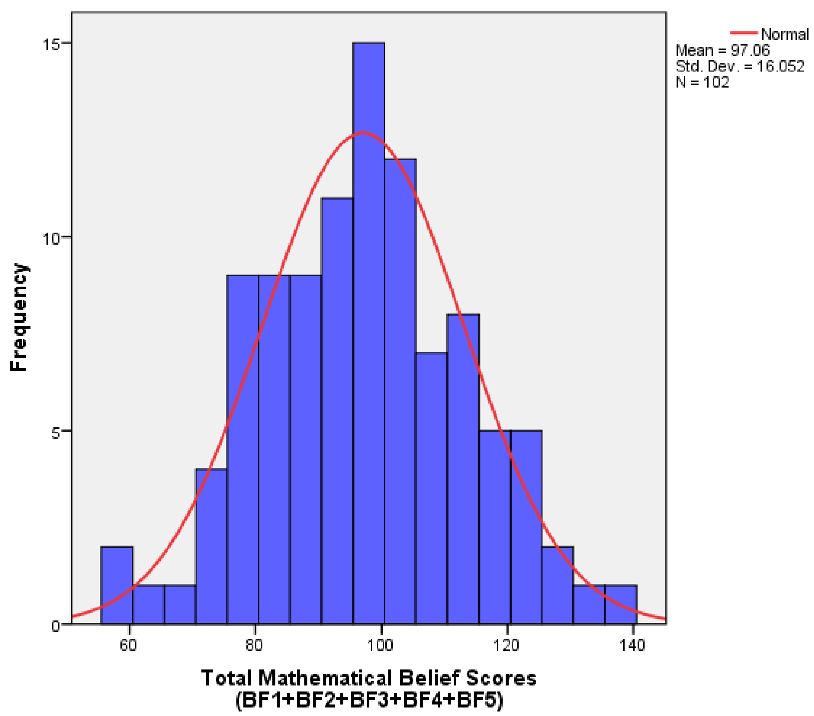


Figure 10.7 Histogram of TMBS for participants 35 to 44

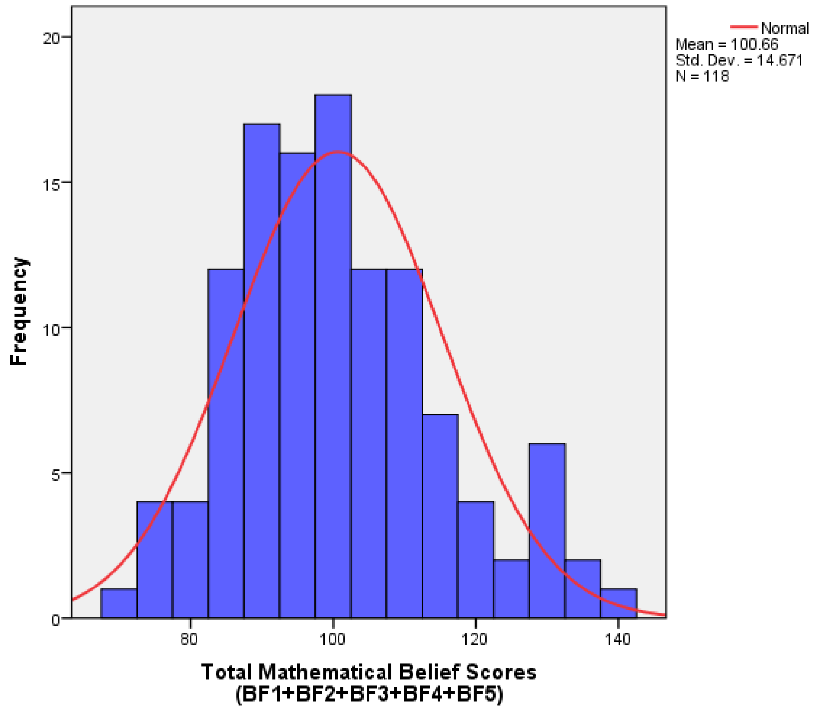


Figure 10.8 Histogram of TMBS for participants 45 to 54

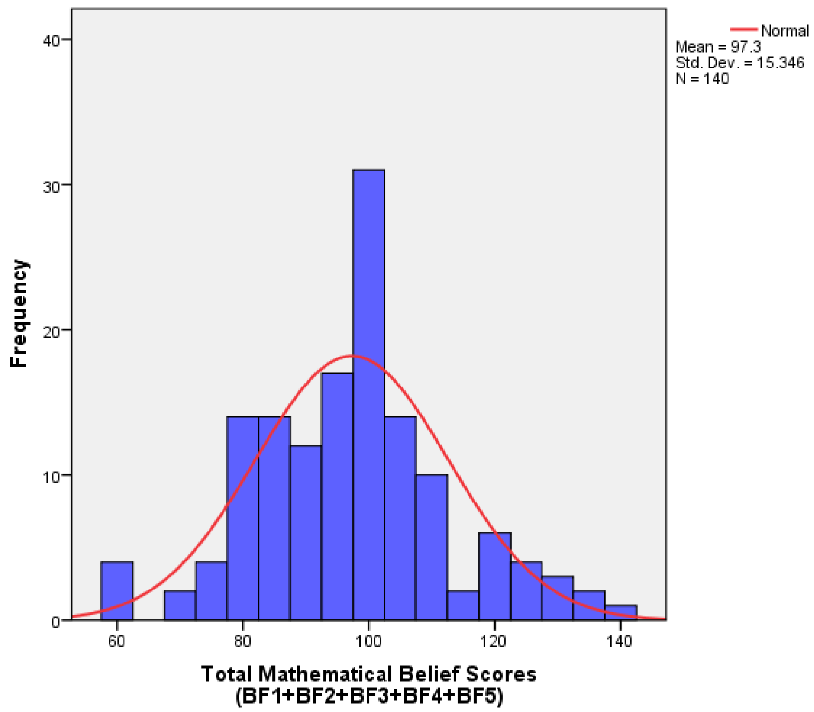


Figure 10.9 Histogram of TMBS for participants 55 or over

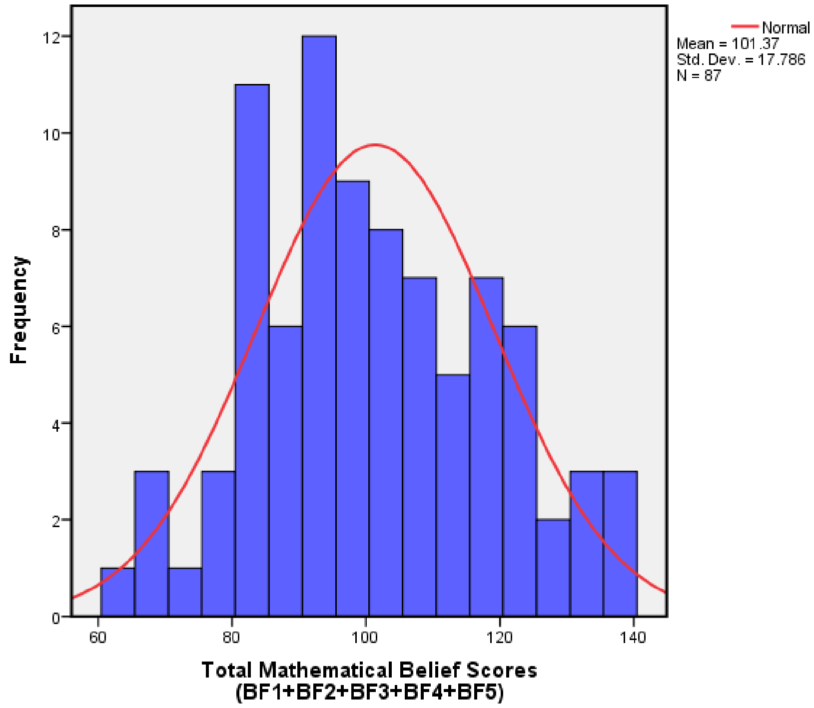


Figure 10.10 Normal Q-Q Plot of TMBS for participants under 25

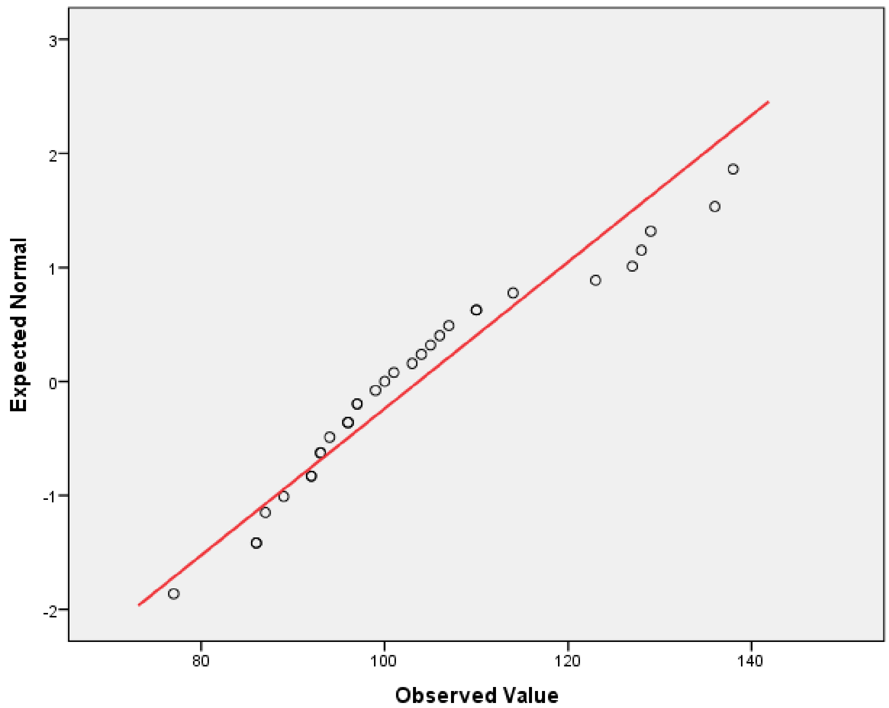


Figure 10.11 Normal Q-Q Plot of TMBS for participants 25 to 34

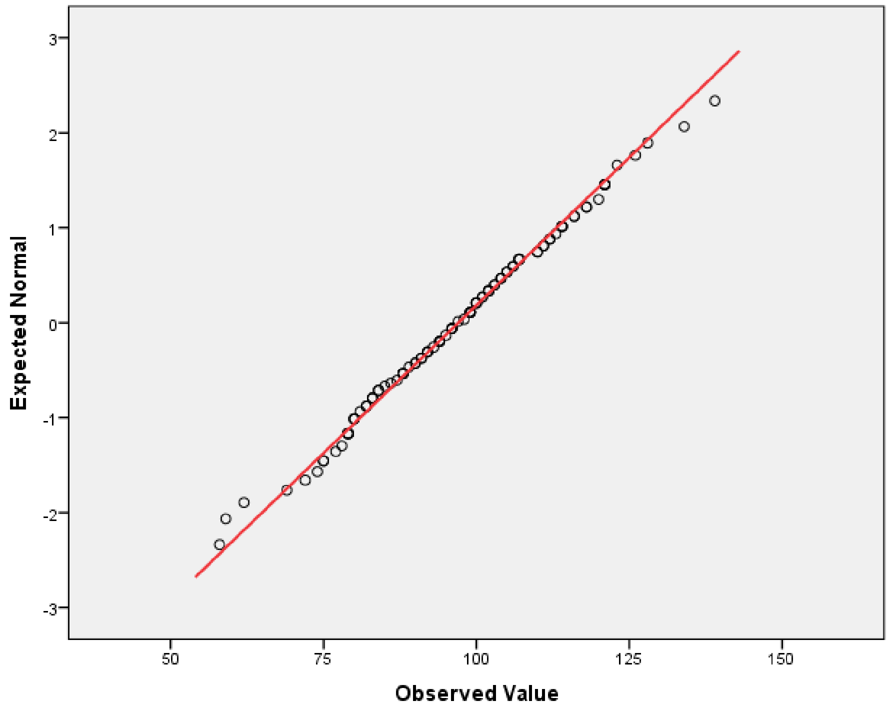


Figure 10.12 Normal Q-Q Plot of TMBS for participants 35 to 44

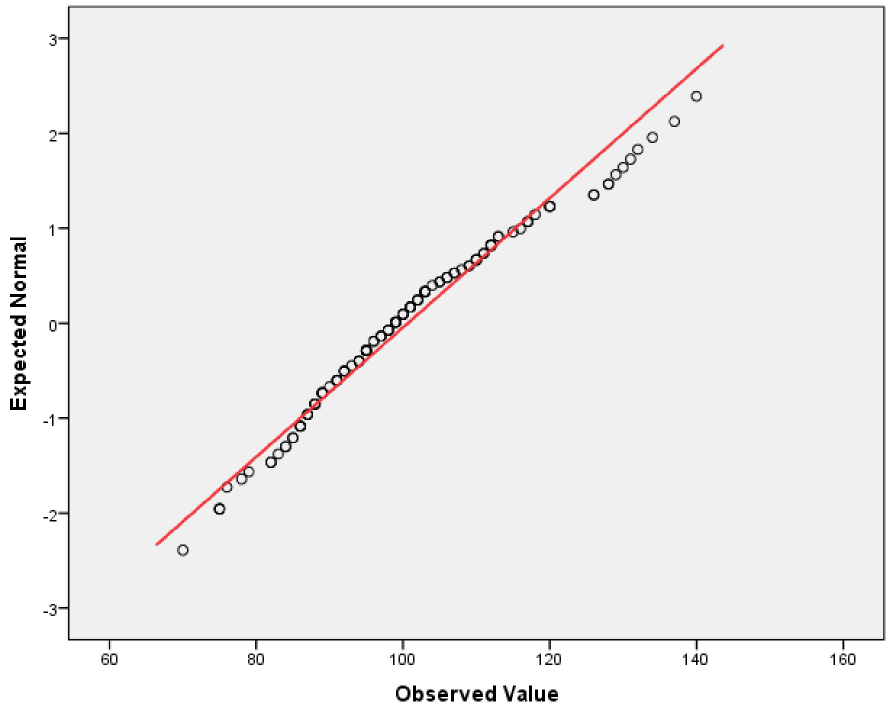


Figure 10.13 Normal Q-Q Plot of TMBS for participants 45 to 54

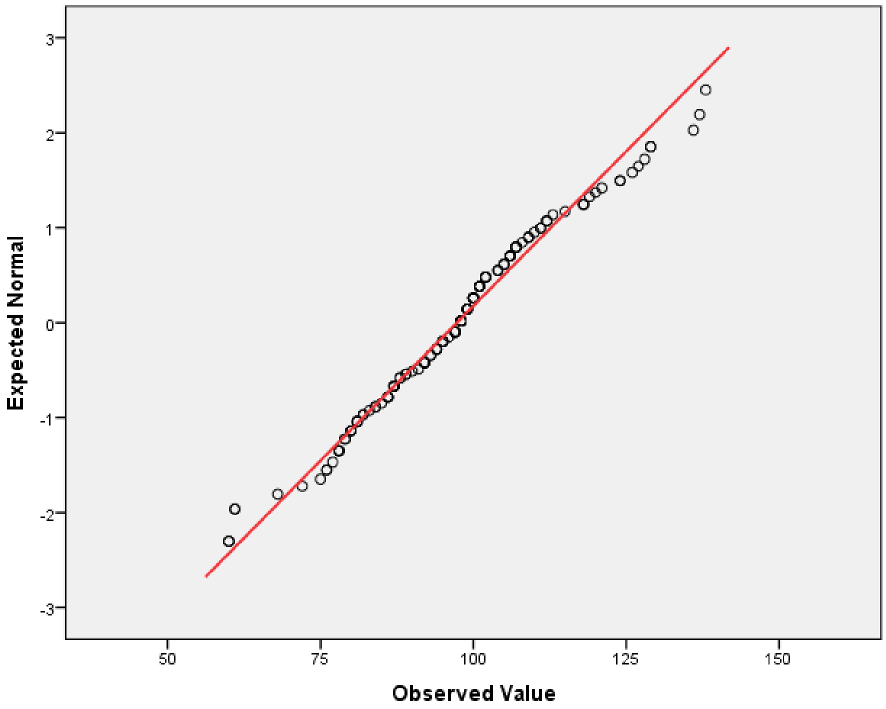
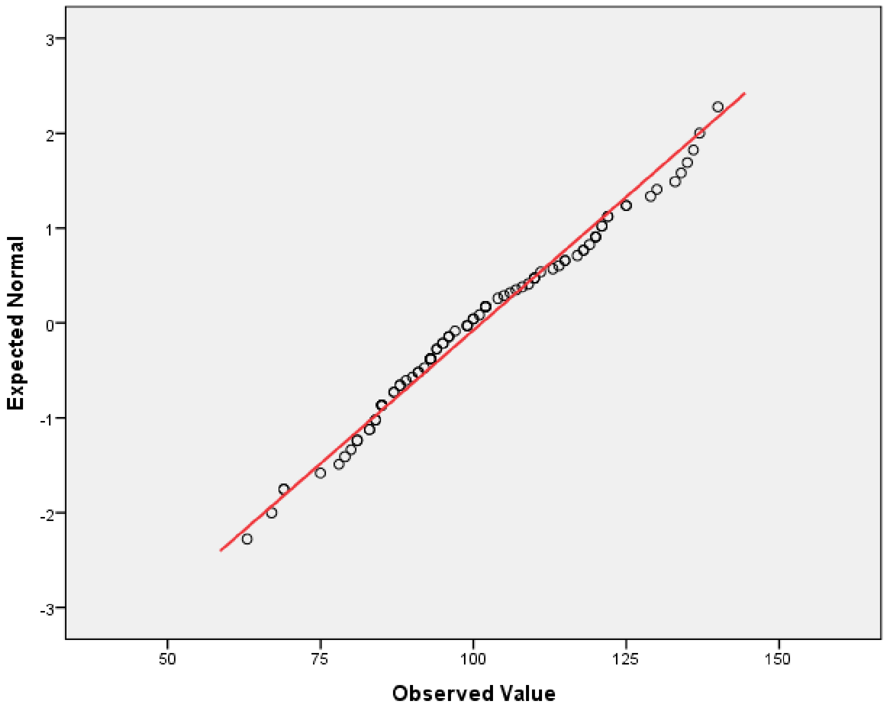


Figure 10.14 Normal Q-Q Plot of TMBS for participants 55 and over



C. Histograms and Normal Q-Q Plots for the TMBS with respect to grade

Figure 10.15 Histogram of TMBS for teacher grade participants

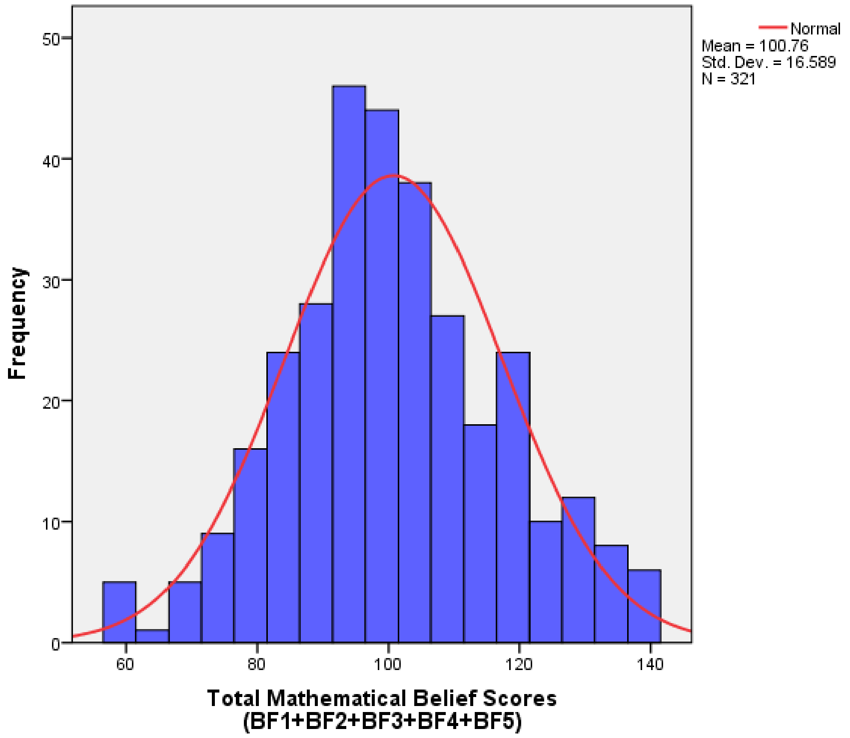


Figure 10.16 Histogram of TMBS for PT participants

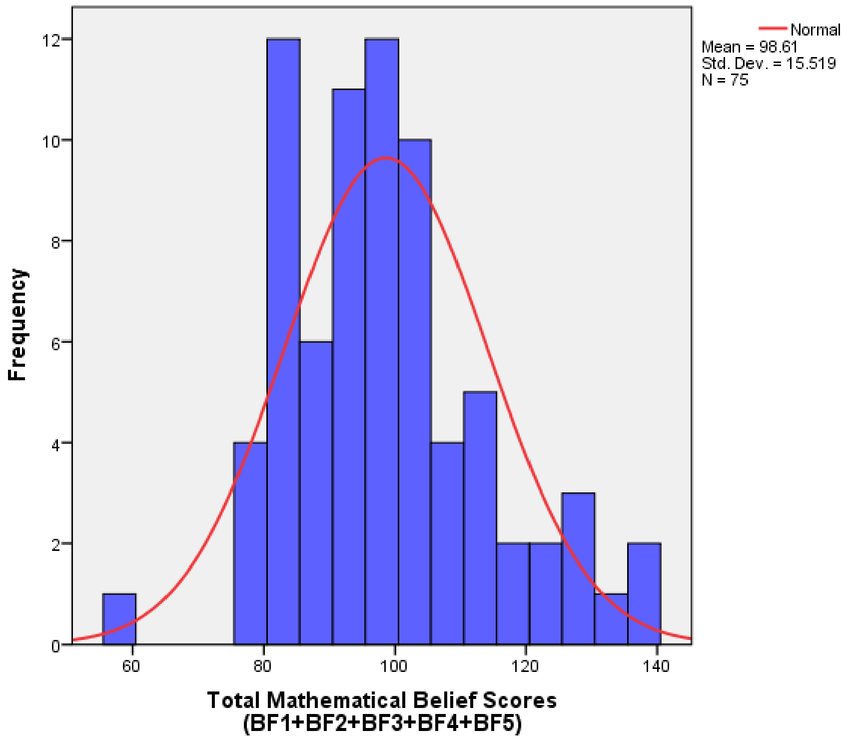


Figure 10.17 Histogram of TMBS for DHT participants

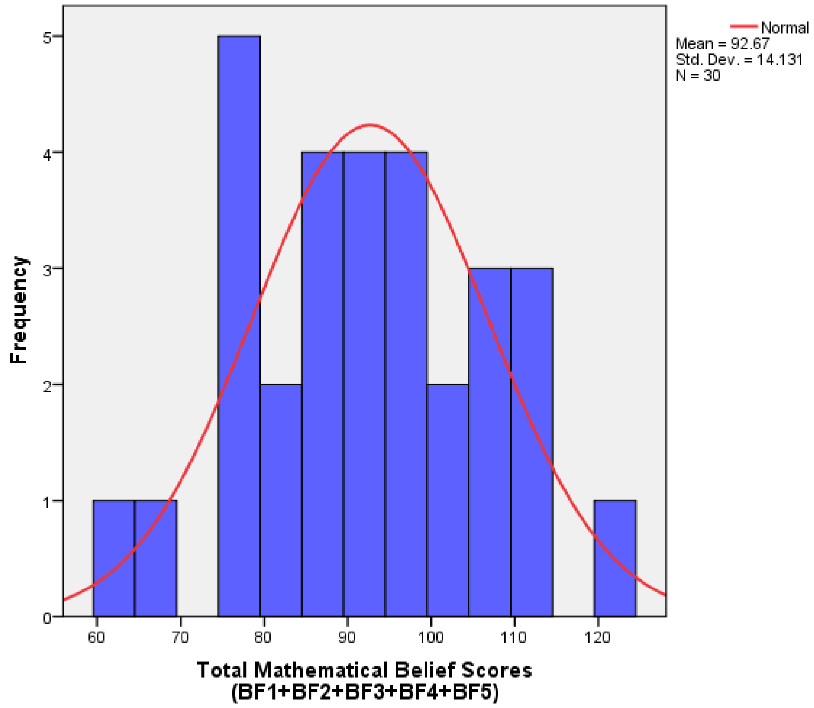


Figure 10.18 Histogram of TMBS for HT participants

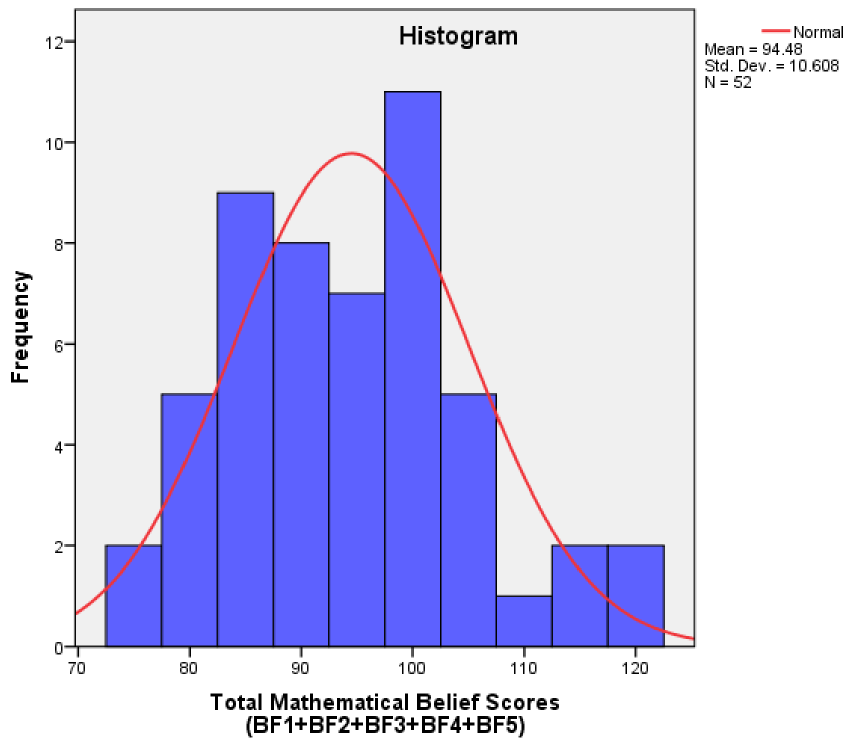


Figure 10.19 Normal Q-Q Plot of TMBS for teacher participants

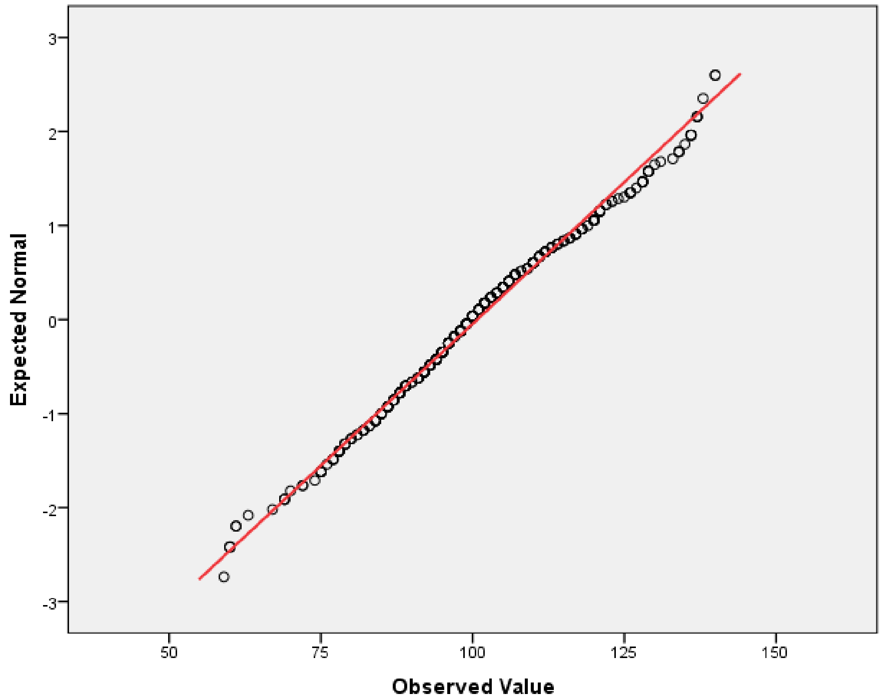


Figure 10.20 Normal Q-Q Plot of TMBS for PT participants

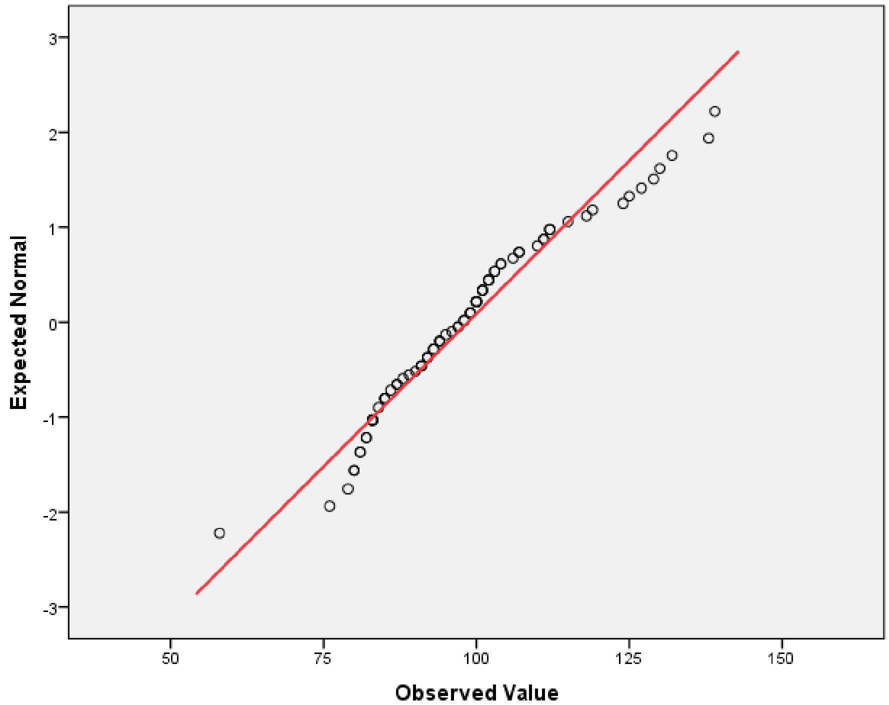


Figure 10.21 Normal Q-Q Plot of TMBS for DHT participants

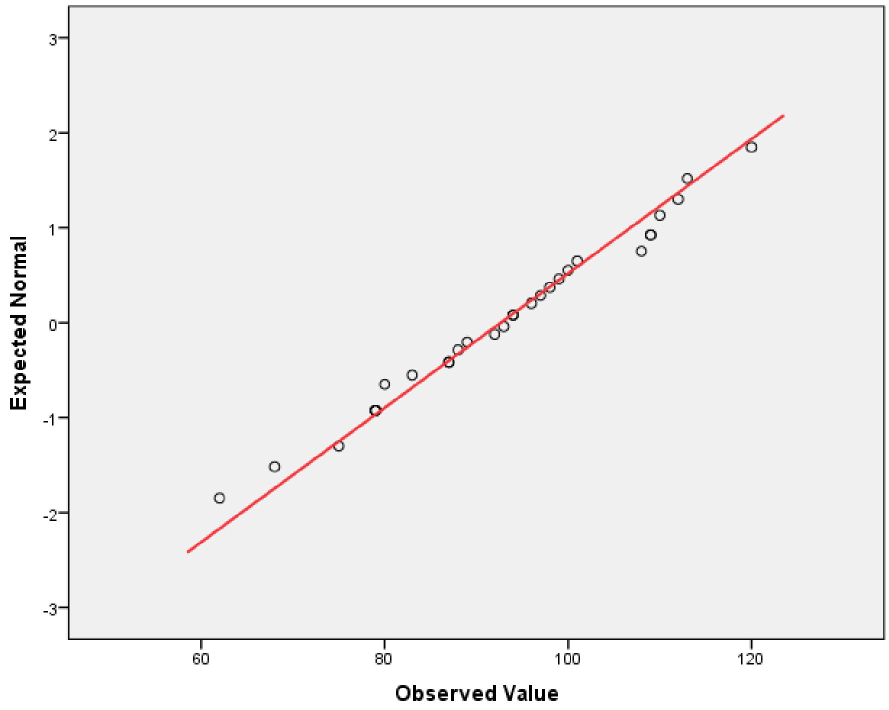
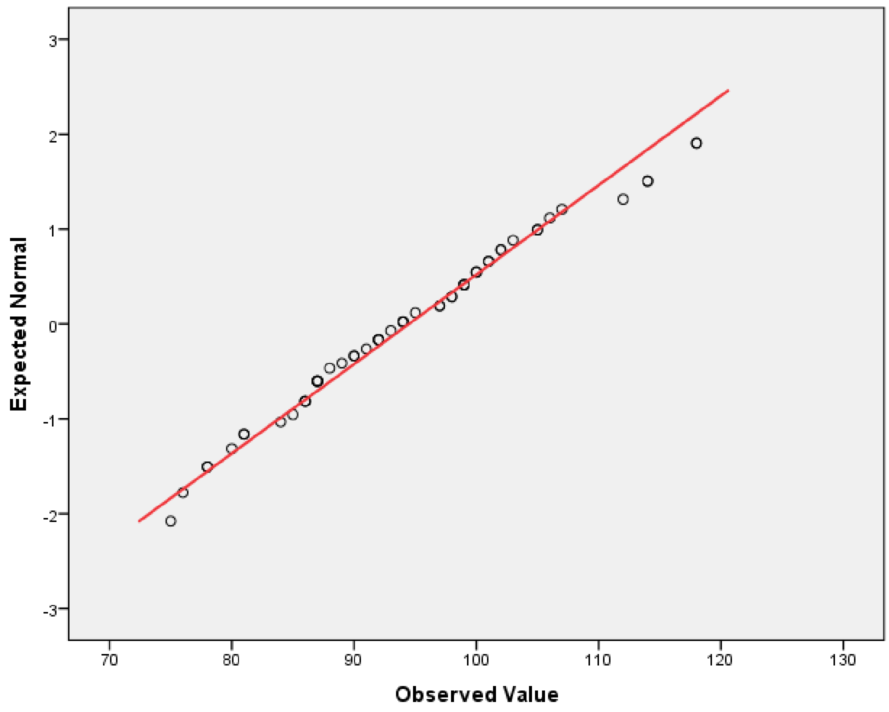


Figure 10.22 Normal Q-Q Plot of TMBS for HT participants



D. Histograms and Normal Q-Q Plots for the TMBS with respect to teaching experience

Figure 10.23 Histogram of TMBS for participants with 5 years and under teaching experience

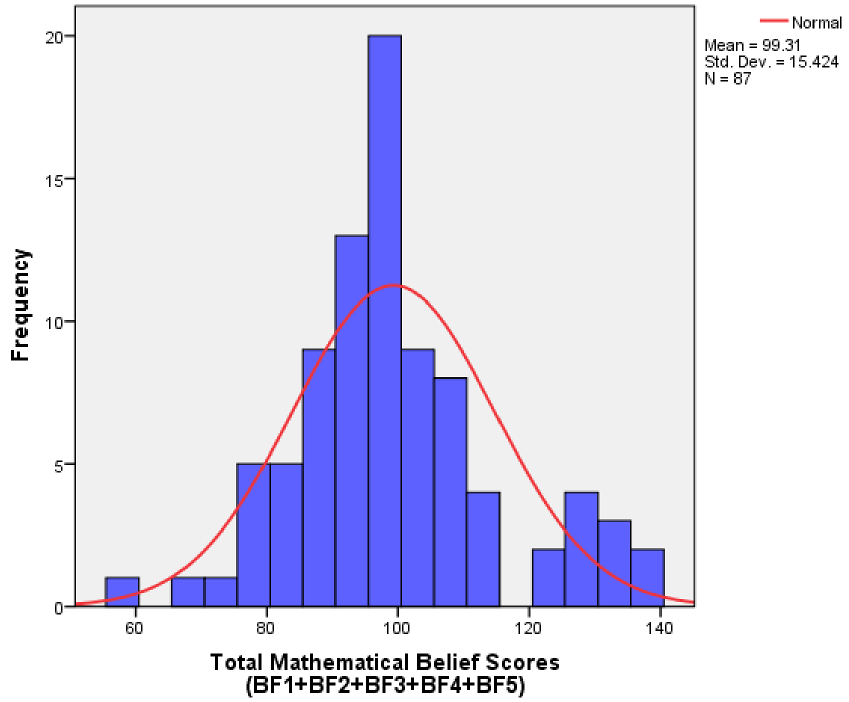


Figure 10.24 Histogram of TMBS for participants with 6 to 10 years under teaching experience

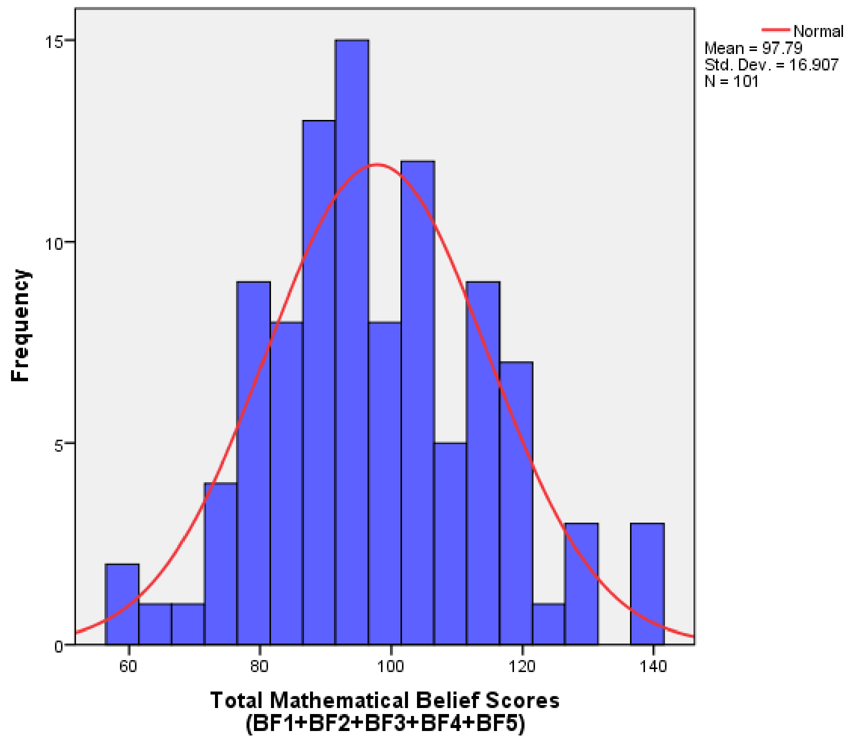


Figure 10.25 Histogram of TMBS for participants with 11 to 15 years teaching experience

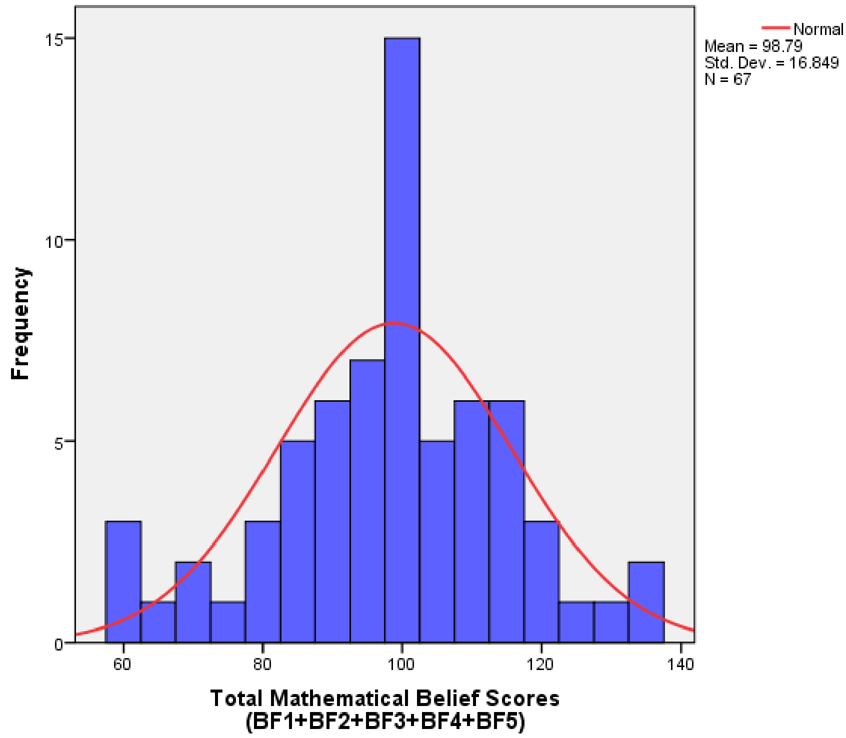


Figure 10.26 Histogram of TMBS for participants with 16 to 20 years teaching experience

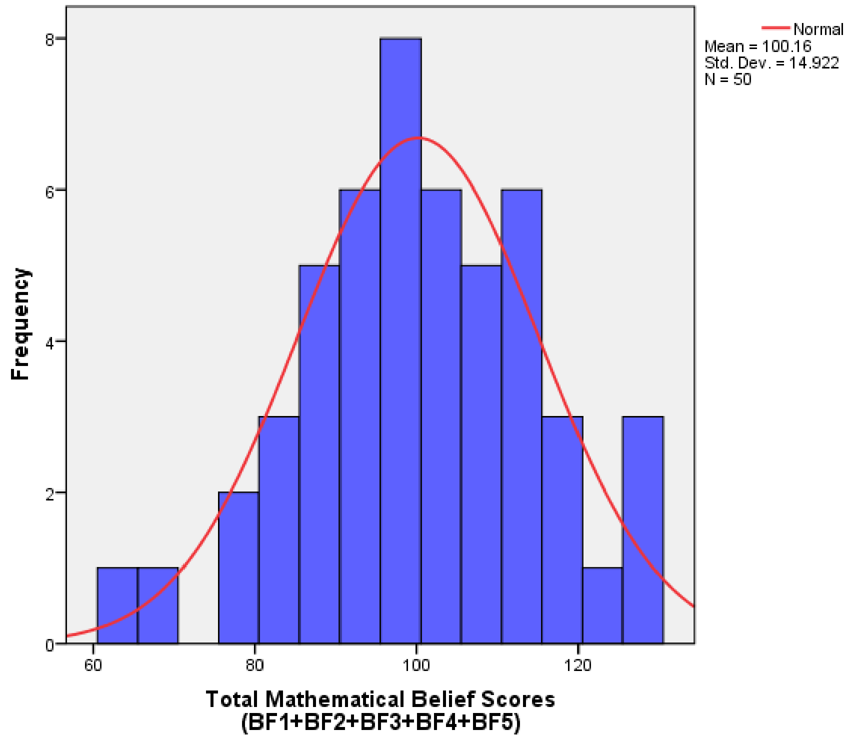


Figure 10.27 Histogram of TMBS for participants with over 20 years teaching experience

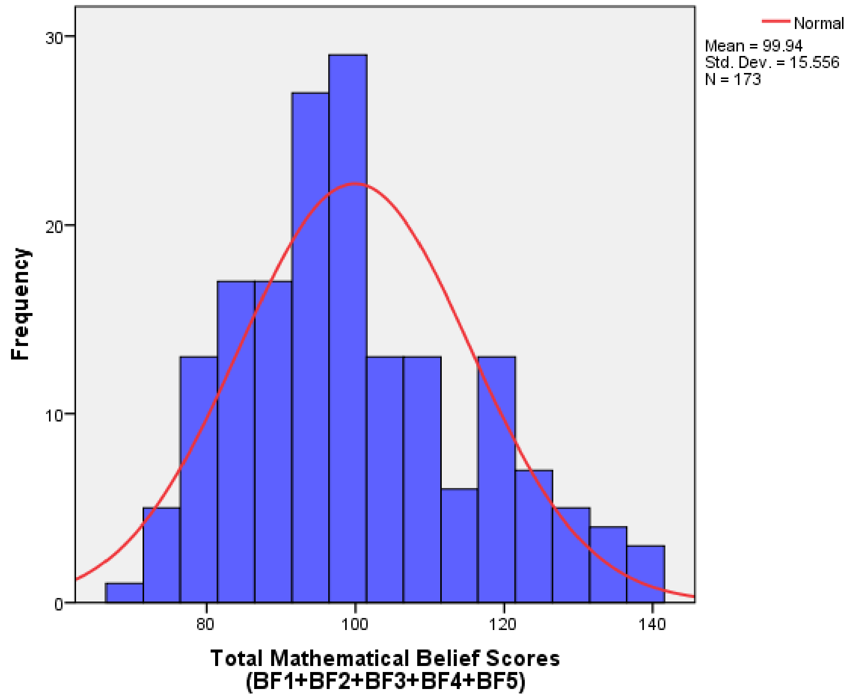


Figure 10.28 Normal Q-Q Plot of TMBS for participants with 5 years or under teaching experience

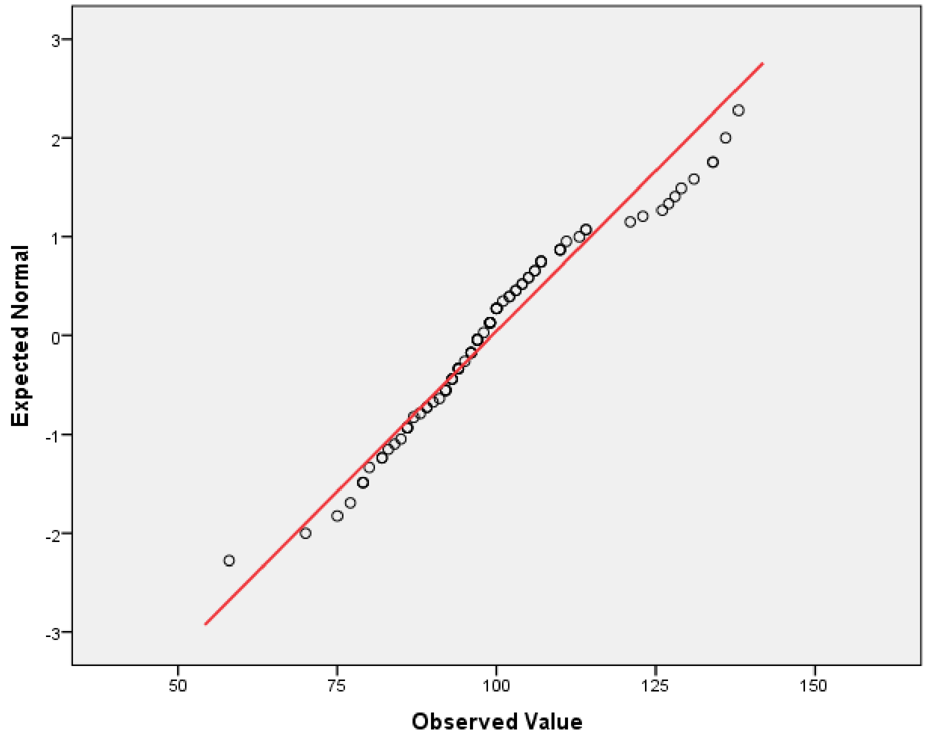


Figure 10.29 Normal Q-Q Plot of TMBS for participants with 6 to 10 years teaching experience

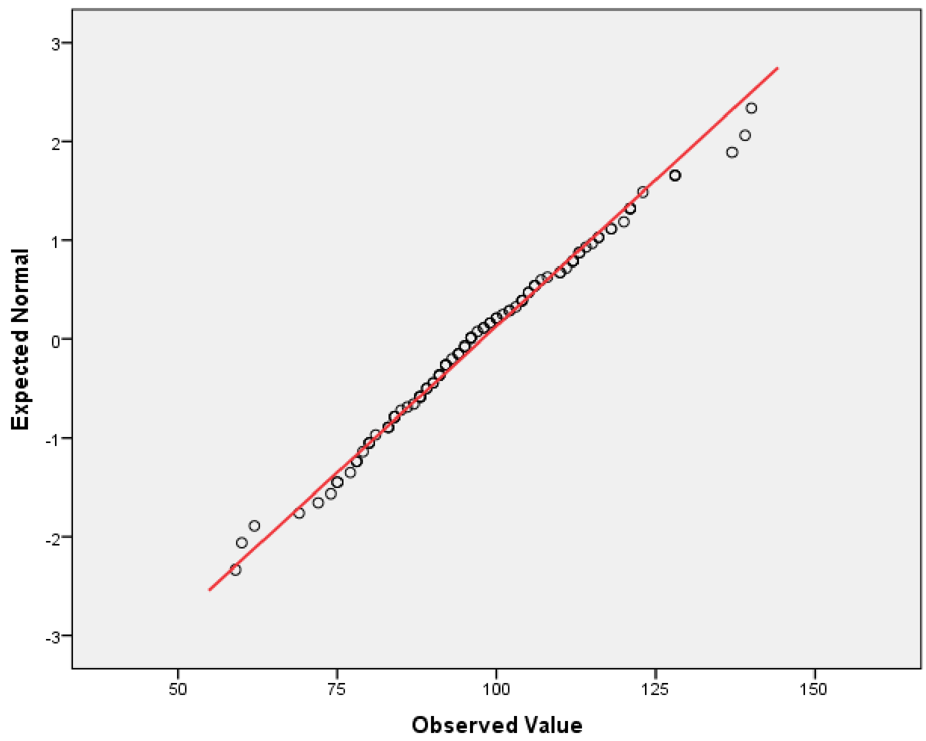


Figure 10.30 Normal Q-Q Plot of TMBS for participants with 11 to 15 years teaching experience

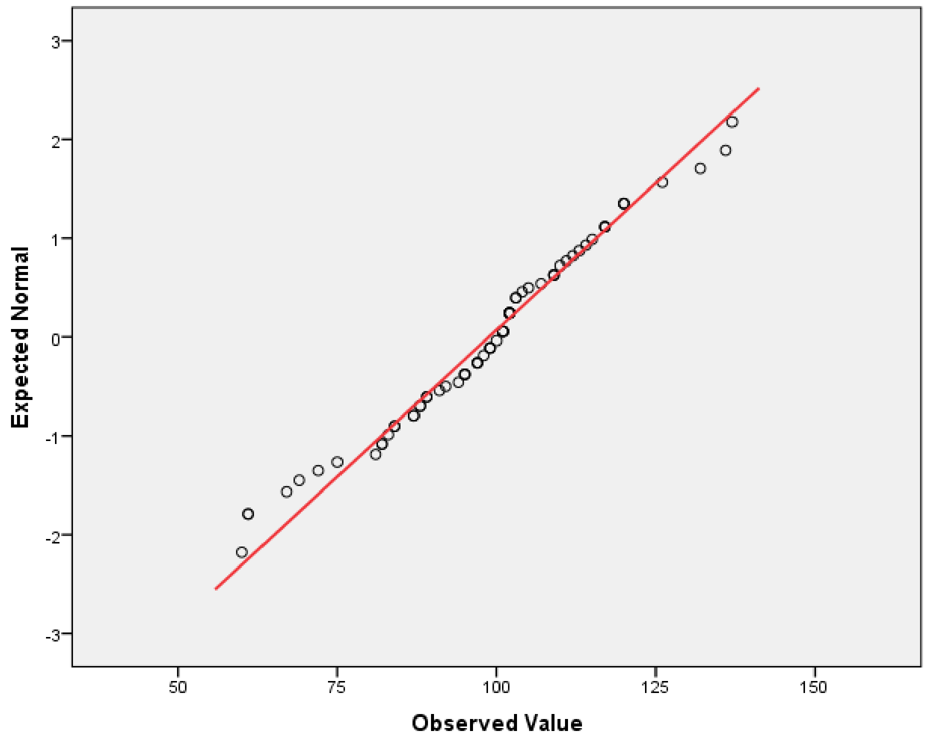


Figure 10.31 Normal Q-Q Plot of TMBS for participants with 16 to 20 years teaching experience

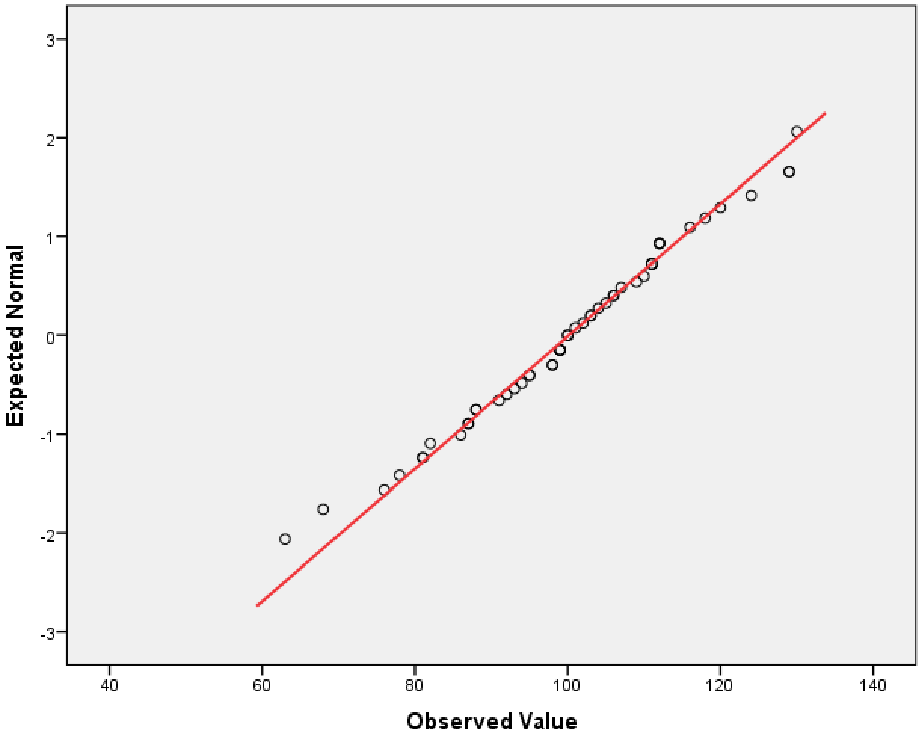
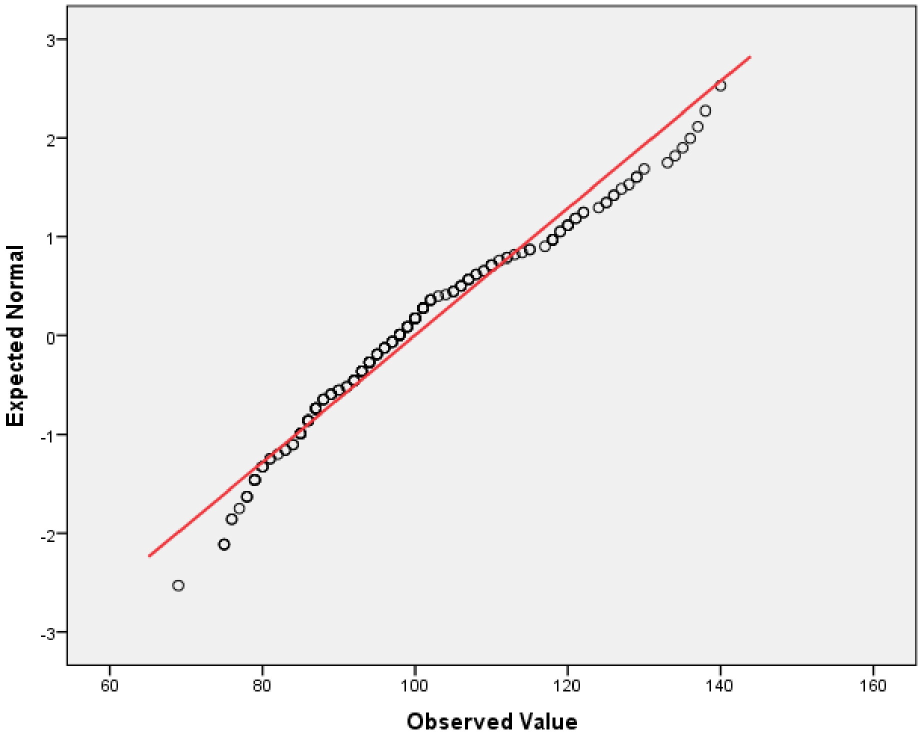


Figure 10.32 Normal Q-Q Plot of TMBS for participants with over 20 years teaching experience



E. Histograms and Normal Q-Q Plots for the TMBS with respect to Highest Level of Qualification the field of Education

Figure 10.33 Histogram of TMBS for participants with BEd

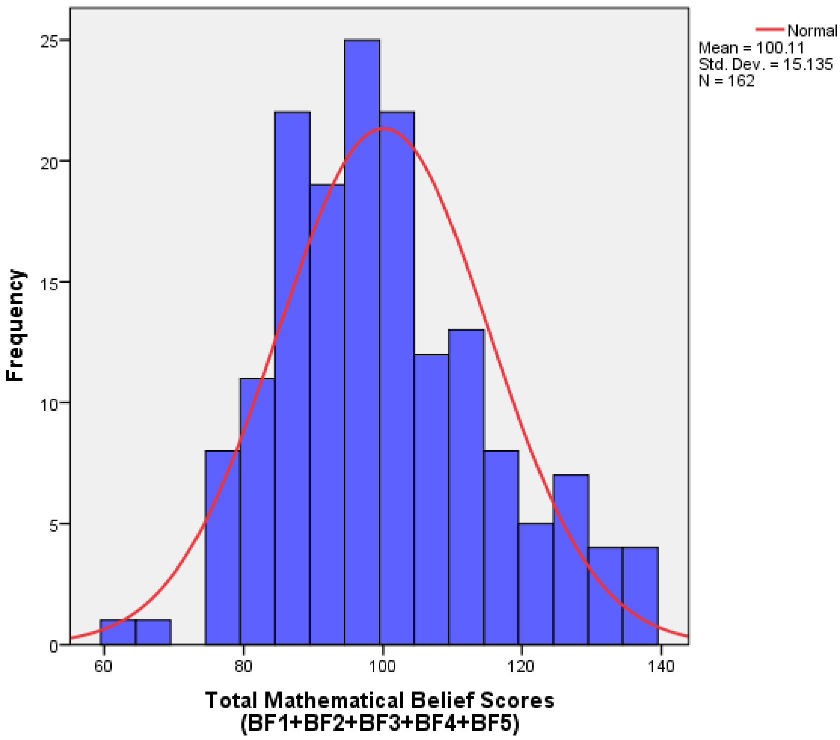


Figure 10.34 Histogram of TMBS for participants qualified to PGCE/PGDE

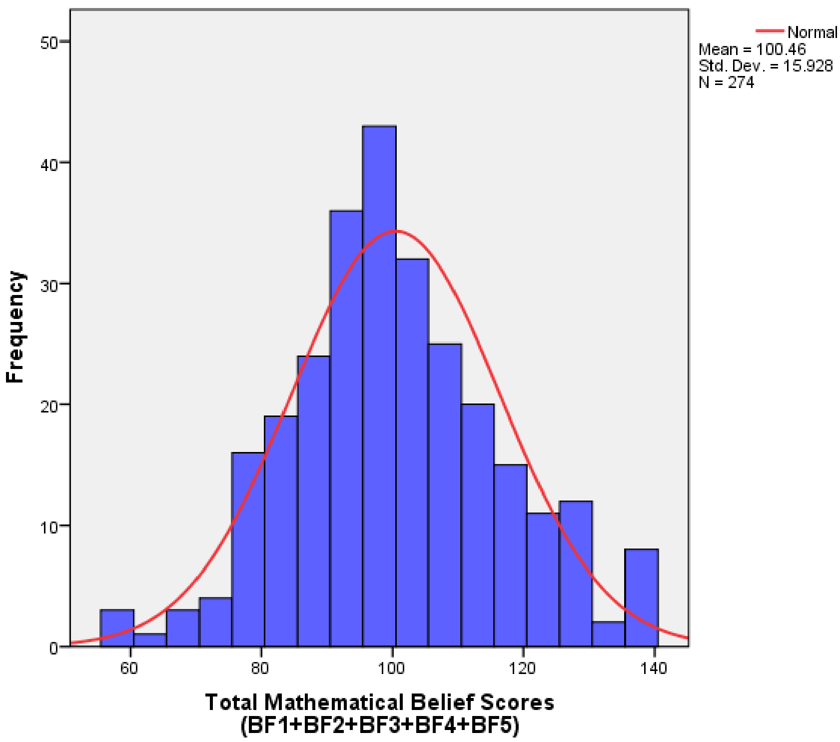


Figure 10.35 Histogram of TMBS for participants qualified to Masters

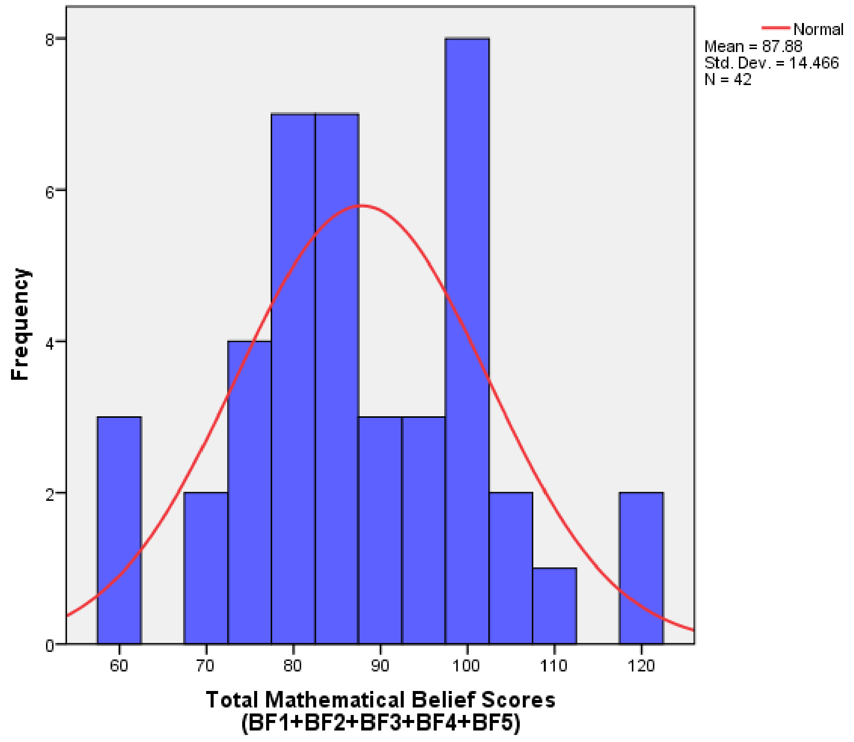


Figure 10.36 Normal Q-Q Plot of TMBS for participants qualified to BEd

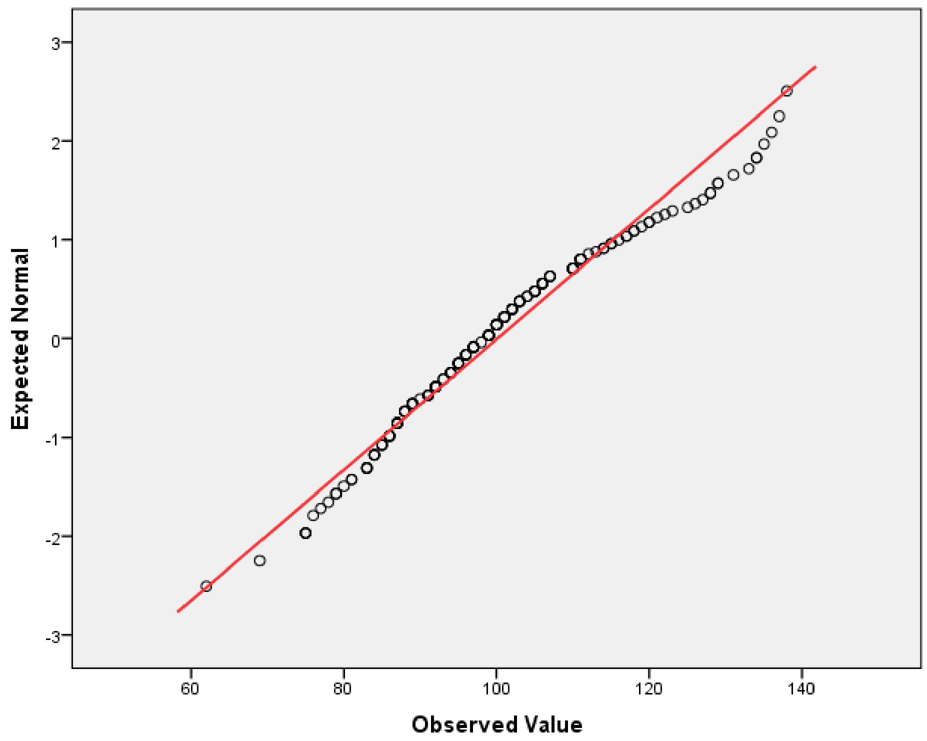


Figure 10.37 Normal Q-Q Plot of TMBS for participants qualified to PGCE/PCDE

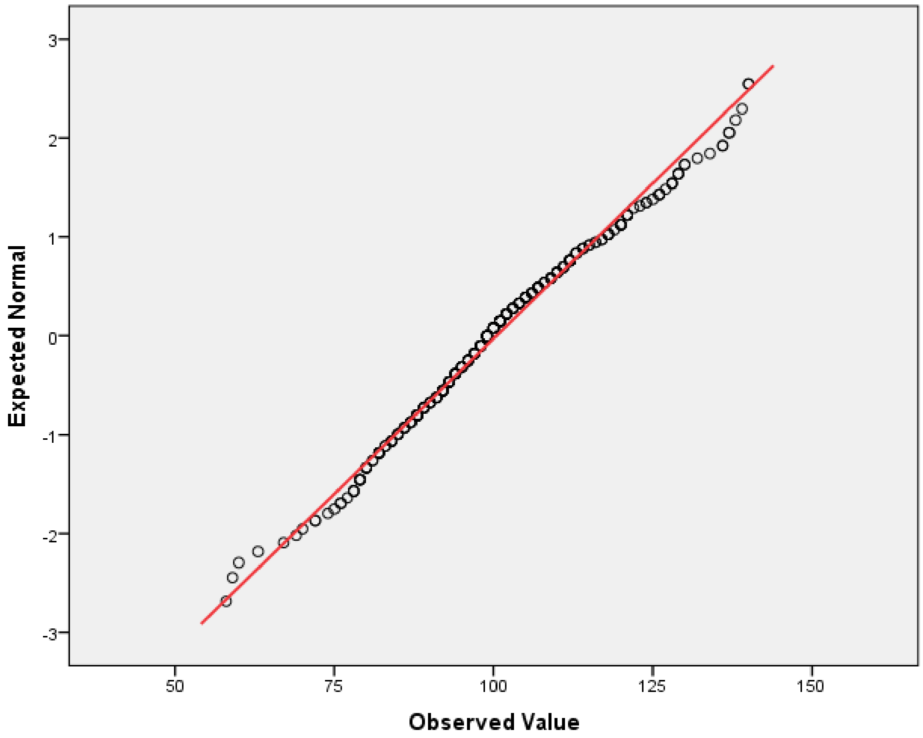


Figure 10.38 Normal Q-Q Plot of TMBS for participants qualified to Masters