The double copy and classical solutions

Andrés Luna Godoy

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School of Physics and Astronomy
College of Science and Engineering
University of Glasgow

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Abstract

The Bern-Carrasco-Johansson (BCJ) double copy, which relates the scattering amplitudes of gauge and gravity theories has been an active area of research for a few years now. In this thesis, we extend the formalism of BCJ to consider classical solutions to the field equations of motion, rather than scattering amplitudes.

One first approach relies on a family of solutions to the Einstein equations, namely Kerr-Schild metrics, which linearise the Ricci tensor. Using them we propose a simple ansatz to construct a gauge theory vector field which, in a stationary limit, satisfies linearised Yang-Mills equations. Using such ansatz, that we call the Kerr-Schild double copy, we are able to relate, for example, colour charges in Yang-Mills with the Schwarzschild and Kerr black holes. We extend this formalism to describe the Taub-NUT solution (which is dual to an electromagnetic dyon), perturbations over curved backgrounds and accelerating particles, both in gauge and gravity theories.

A second, more utilitarian approach consists on using the relative simplicity of gauge theory to efficiently compute relevant quantities in a theory of perturbative gravity. Working along this lines, we review an exercise by Duff to obtain a spacetime metric using tree-level graphs of a quantum theory of perturbative gravity, and repeat it using a BCJ inspired gravity Lagrangian. We find that the computation is notably simplified, but a new formalism must be developed to remove the unwanted dilaton information, that naturally appears in the double copy.

Contents

To Natalia, with love.
To my parents, with gratitude.

Para Natalia, con cariño.
Para mis padres, con gratitud.

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Declaration

I declare that this thesis is the result of my own original work and has not been previously presented for a degree. In cases where the work of others is presented, appropriate citations are used. Chapters 1 through 3 and 7 serve as an introduction to the research topics presented in the rest of the thesis. Chapters 4 to 6 and 8 through 10 are a result of my own original work, in collaboration with the authors listed below. Work appearing in this thesis was presented in the following publications [1–4]:

- A. Luna, R. Monteiro, D. O'Connell, C. D. White, "The classical double copy for Taub-NUT spacetime", Phys.Lett. B750 (2015) 272-277, arXiv: 1507.01869 [hep:th]
- A. Luna, R. Monteiro, I. Nicholson, D. O'Connell, C. D. White, "The double copy: Bremsstrahlung and accelerating black holes", JHEP 1606 (2016) 023, arXiv: 1603.05737 [hep:th]
- A. Luna, R. Monteiro, I. Nicholson, A. Ochirov, D. O'Connell, N. Westerberg, C. D. White, "Perturbative spacetimes from Yang-Mills theory". JHEP 1704 (2017) 069, arXiv: 1611:07508 [hep:th]
- N. Bahjat-Abbas, A.Luna, C. D. White, "The Kerr-Schild double copy in curved spacetime", arXiv:1710.01953 [hep:th]

Related topics published in [5]:

- A. Luna, S. Melville, S. Naculich, C. D. White, "Next-to-soft corrections to high energy scattering in QCD and gravity", JHEP 1701 (2017) 052, arXiv:1611:02172 [hep:th]

have not been covered in this text.

Chapter 1

Introduction

The recent years in physics have been marked by a couple of major discoveries, performed using two of the biggest experiments ever created. Last year, the Laser Interferometer Gravitational Observatory (LIGO) collaboration announced the detection of gravitational waves [6], while five years ago, the ATLAS and CMS collaborations announced the discovery of the Higgs boson particle in the Large Hadron Collider (LHC) at CERN [7, 8]. Besides being wonders of engineering, a great deal of the importance of these experiments comes from the fact that they provide validation of General Relativity (GR) in the case of LIGO, and the Standard Model (SM) of particle physics in the case of LHC, which are two of the most successful theories ever formulated to describe the universe we live in. Such breakthroughs would not have been possible without the continuous effort of theoretical physicists[1] devoted to coming up with theories that better describe Nature. The SM is an example of a gauge theory, and as such, it is formulated in the language we call Quantum Field Theory (QFT), where one important object used to describe particles and their interaction is the scattering amplitude. In the textbook approach to QFT[2], a theory is defined by its Lagrangian. Then, the computation of scattering amplitudes (where a perturbative expansion is allowed) can be performed using Feynman diagrams. The philosophy of this is simple: a set of rules (called Feynman rules) describing particle interaction and propagation is to be deduced from the Lagrangian. Then, we have to draw diagrams representing every single way that the process we are interested in can occur, and follow our Feynman rules to assign every diagram a value. The sum of all diagrams yields the scattering amplitude. Although this is in principle all we need, in practice this is only possible for the simplest examples, since more complicated problems become swiftly too complex to manage, even using computers. In the now classic review "Calculating

[1] Proof of this is that the hard work of hundreds of physicists (thousands of people) at LHC resulted in a Nobel prize for two theoretical physicists (Englert and Higgs) who had no involvement whatsoever in the experiment. A milder result can be seen in LIGO, as one of the Nobel prize laureates (Thorne) was involved with the experiment, but is a theoretical physicist nonetheless.

[2] See, for example, [9, 10]

scattering amplitudes efficiently" [11], where Dixon shows a collection of tricks to avoid computing amplitudes with Feynman diagrams, he cites, among other reasons for the inefficiency of the Feynman diagram approach, that there are too many such diagrams, and that a single diagram can have many terms. Let us comment briefly on this now. First, imagine we want to compute the amplitude for the process of scattering of four gluons, two of them with negative helicity, and two of them positive[3]. To do this, we need to consider four Feynman diagrams. The result can be cast into the nice and compact formula[4]

$$A(1^-2^-3^+4^+) = ig^2 \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{1.1}$$

Now, consider we were to compute the amplitude for eight gluons, instead of four, where again two of them have negative helicities, while the rest have positive polarization. Instead of four diagrams, we would need to look at (literally) millions of them, yet, the amplitude wouldn't grow that much in complexity, as it can be described by

$$A(1^-2^-3^+4^+5^+6^+7^+8^+) = ig^6 \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 67\rangle\langle 78\rangle\langle 81\rangle}. \tag{1.2}$$

Actually, we could consider any number of gluons, such that two (say, the first two) of their helicities are negative, while all the other are positive. Tha amplitude is given by

$$A(1^-2^-3^+\dots n^+) = ig^{n-2} \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n-1\ n\rangle\langle n1\rangle}. \tag{1.3}$$

The last equation is called the Parke-Taylor formula [12], and it has played a major role in shifting the paradigm of the computation of scattering amplitudes. It hints that (at least in this case) it doesn't matter that there are too many diagrams, because in the end you can obtain a simple compact result due to lots of terms cancelling out. It goes to show that a Lagrangian approach, i.e. using Feynman diagrams, is very inefficient in this problem, as it would be as well in many others. The Parke-Taylor formula celebrated its 30th birthday last year. In that time, several brilliant scientists have developed many ingenious tricks and techniques to study scattering amplitudes avoiding the Lagrangian approach. Recursion relations [13, 14] to build trees out of (smaller) trees, generalized unitarity to build loops out of trees [15–19], novel mathematics to deal with integration [20], exploiting supersymmetry [21], or other symmetries of the theories [22–24], Wilson loops [25, 26], or even more mathematical ideas like describing amplitudes in terms of twistors [27] or getting rid also of the spacetime and interpreting amplitudes like geometric objects [28], are some

[3] This is, in some sense, the simplest non-trivial amplitude for definite helicities we can compute. Amplitudes with all or all but one equal helicities vanish. See [11] for a nice argument of why this happens.

[4] We will not explain here what the notation means, as we're not interested in the value itself, but just want to make a point. We refer the reader to the aforementioned Dixon [11].

of the most emblematic areas of research in the amplitudes community. Some of these subjects have been reviewed in [29–31]. After this intermission, we can go back to our main tread. There is another example we can explore that relates to Dixon's second point. Let us compute the three graviton scattering amplitude.

In 1967, Bryce DeWitt wrote three papers on perturbative quantization of gravity [32–34]. In the third of them, he constructs Feynman rules for a graviton field.

To understand a bit more about this, we may (as he does) review the case for the three-point Feynman rule for Yang-Mills theory. He expresses it in the form[5]

$$\frac{\delta S^3}{\delta A^a_\mu \delta A^b_\nu \delta A^c_\rho} = f^{abc}\left((p^\rho - q^\rho)\eta^{\mu\sigma} + (q^\mu - r^\mu)\eta^{\sigma\rho} + (r^\sigma - p^\sigma)\eta^{\rho\mu}\right), \tag{1.4}$$

where particle 1 comes in with momenta p and colour index a, particle 2 comes in with momenta q and colour index b and particle 3 comes in with momenta r and colour index c. The action S^3 refers to those terms in the Yang-Mills Lagrangian that contain exactly three gluon fields.

In an analogous fashion, DeWitt then gives the three-point Feynman rule for three gravitons

$$\begin{aligned}\frac{\delta S^3}{\delta h_{\mu\nu}\delta h_{\sigma\tau}\delta h_{\rho\lambda}} = \text{sym}\Big[&-\frac{1}{4}P(p\cdot q\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}) - \frac{1}{4}P(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda}) + \frac{1}{4}P(p\cdot q\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda})\\ &+\frac{1}{2}P(p\cdot q\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}) + P(p^\sigma p^\lambda\eta^{\mu\nu}\eta^{\tau\rho}) - \frac{1}{2}P(p^\tau q^\mu\eta^{\nu\sigma}\eta^{\rho\lambda}) - \frac{1}{2}P(p^\rho q^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})\\ &+\frac{1}{2}P(p^\rho p^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}) + P(p^\sigma q^\lambda\eta^{\tau\mu}\eta^{\nu\rho}) + P(p^\sigma q^\mu\eta^{\tau\rho}\eta^{\lambda\nu}) - P(p\cdot q\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\Big],\end{aligned} \tag{1.5}$$

where the action S^3 refers to the terms in the Einstein-Hilbert action[6] that, when expanded in powers of the graviton $h_{\mu\nu}$, contain exactly three such fields. The "sym" standing in front of these expressions indicates a symmetrization is to be performed on each index pair $\{\mu,\nu\}$, $\{\sigma,\tau\}$ and $\{\rho,\lambda\}$. The symbol P indicates a summation is to be carried out over all distinct permutations of the momentum-index triplets $\{p,\mu,\nu\}$, $\{q,\sigma,\tau\}$ and $\{r,\rho,\lambda\}$.

This expression may not seem that intimidating, but a full expansion of it contains 171 separate terms, in contrast with the six terms coming from expanding eq. (1.4) for the three-gluon vertex in Yang Mills. Moreover, the number of terms for the four-graviton vertex (that we do not show here) is 2580. This renders the approach of computing physical quantities using Feynman diagrams rather prohibitive[7]. The approach we'll take is a different one. To introduce it, let us look at possibly the simplest example, i.e. the

[5] c.f. eq. (1.27).

[6] c.f. eq. (1.74).

[7] Although some people actually made heroic efforts to compute things in this framework.

three-graviton scattering amplitude. Again, before doing this, we will look at the Yang-Mills case.

The first issue with using Feynman diagrams is that they are obtained from rules which depend on the gauge, and hence are not directly physical. To obtain any physical observable we need the external particles to be "on-shell". For gluons or gravitons this means being massless and it will impose (for the three particles scattering case)

$$k_i^2 = 0, \qquad k_i \cdot k_j = 0, \tag{1.6}$$

where we use the notation $k_1^\mu \equiv p^\mu$, $k_2^\mu \equiv q^\mu$, $k_3^\mu \equiv r^\mu$, and the second equality follows from momentum conservation. To get the scattering amplitude, for external gluons, we must contract eq. (1.4) with three transverse polarization vectors. Gluons can have positive or negative helicity polarization satisfying $\epsilon_i^\pm \cdot k_i = 0$. We will also be able to choose our polarization vector in such a way that $\epsilon_i^\pm \cdot \epsilon_j^\pm = 0$. If we consider gluons 1 and 2 to have negative helicity, and the third one to have positive helicity, the contraction yields

$$\epsilon_1^-{}_\mu \epsilon_2^-{}_\sigma \epsilon_3^+{}_\rho \frac{\delta S^3}{\delta A_\mu^a \delta A_\sigma^b \delta A_\rho^c} = -2if^{abc}((k_1 \cdot \epsilon_2)(\epsilon_1 \cdot \epsilon_3) - (k_2 \cdot \epsilon_1)(\epsilon_2 \cdot \epsilon_3)). \tag{1.7}$$

Here we have also used the conservation of momentum $k_3 = -k_1 - k_2$. We may read from this last expression an "on-shell Feynman rule" for Yang-Mills theory as

$$\left\langle \frac{\delta S^3}{\delta A_\mu^{-a} \delta A_\sigma^{-b} \delta A_\rho^{+c}} \right\rangle_{\text{on-shell}} = -2if^{abc}(k_1{}^\sigma \eta^{\mu\rho} - k_2{}^\mu \eta^{\rho\sigma}). \tag{1.8}$$

We can perform an analogous procedure now for perturbative gravity. Gravitons are also massless so they must also be in one of two states (in four dimensions), which can be taken to have helicity ± 2. Their polarization tensors then factorize into a product of spin-1 polarization vectors: $\epsilon_i^{\pm\pm\mu\nu} = \epsilon_i^{\pm\mu} \epsilon_i^{\pm\nu}$. The constraints from dotting the graviton Feynman rule into polarizations allow us to write

$$\epsilon_1^-{}_{\mu\nu} \epsilon_2^-{}_{\sigma\tau} \epsilon_3^+{}_{\rho\lambda} \frac{\delta S^3}{\delta h_{\mu\nu} \delta h_{\sigma\tau} \delta h_{\rho\lambda}} = 4\left((\epsilon_1^- \cdot \epsilon_3^+)(k_1 \cdot \epsilon_2^-) - (\epsilon_2^- \cdot \epsilon_3^+)(k_2 \cdot \epsilon_1^-)\right)^2, \tag{1.9}$$

so we may also obtain the on-shell graviton Feynman rule at three-points as

$$\left\langle \frac{\delta S^3}{\delta h_{\mu\nu}^- \delta h_{\sigma\tau}^- \delta h_{\rho\lambda}^+} \right\rangle_{\text{on-shell}} = 4(k_1^\sigma \eta^{\mu\rho} - k_2^\mu \eta^{\rho\sigma})(k_1^\tau \eta^{\nu\lambda} - k_2^\nu \eta^{\lambda\tau}) \tag{1.10}$$

There are two lessons we can gather from this. The first one, which we had already seen in the Parke-Taylor example, is that using off-shell objects like Feynman diagrams may complicate computations in intermediate stages, while the on-shell physical information may actually be very simple. The second one is that the scattering amplitude in gravity is the square of that in gauge theory. This is by no means a coincidence, but one of the simplest manifestations of the paradigm gravity=gauge^2, which hints at the theories being intimately related. This relation is far from clear when inspecting the Lagrangian of the theories in question. While the Yang-Mills Lagrangian has three-gluon and four-gluon vertices, the Einstein-Hilbert Lagrangian contains an infinite number of terms (that describe the interaction of an infinite number of gravitons simultaneously). In spite of these differences, some insight on dualities between gauge and gravity thories has been gained from string theory. The most famous[8] installment of a relation between gauge and gravity theories is the AdS/CFT correspondence conjectured by Maldacena [35][9] in 1997. However, more relevant for us is an older result by Kawai, Lewellen and Tye [36]. They showed that the scattering amplitude for closed strings in a string theory can be expressed as a sum of products of (two) scattering amplitudes of open strings. As it has been shown that the low energy limit of scattering amplitudes for open and closed strings in string theory yields scattering amplitudes in $\mathcal{N}=4$ super Yang Mills (sYM) and N=8 supergravity (Sugra) [37], a low energy limit of the KLT relations directly relates the tree-level scattering amplitudes in (perturbative) gauge and gravity theories. These relations have been useful in the search for a consistent theory of quantum gravity. They have even been used (along with generalized unitarity methods) to investigate the divergences of Sugra to three-loops [38]. Recently[10], Bern, Carrasco and Johansson (BCJ) conjectured that it is always possible to cast the kinematic information in a gauge theory in such a way that it satisfies algebraic relations analogous to those imposed by the gauge (colour) group from a Jacobi relation [39]. This property is referred to as "colour-kinematics duality". By assuming its validity, BCJ deduced relations between colour-ordered amplitudes in the gauge theory, that further reduces the set of independent amplitudes. There is a most important byproduct to colour-kinematics duality. BCJ also conjectured that by substituting the colour information in the amplitude with a second copy of the kinematic information (the so-called numerators), the result immediately yields a scattering amplitude in a gravity theory [40]. This is called the BCJ double copy, or simply the double copy. We want to make precise some of the statements that we found in the last couple of pages, in particular, those referring to KLT, colour-kinematics, and the BCJ double copy. In order to do this, we will now review the definitions and notations of the theories we will work with.

[8] This is the most cited paper in hep-th, with ∼13000 citations.
[9] ∼13000+1.
[10] We keep saying this, but it's been almost ten years now.

1.1 Gauge theory

To set some of the notation we are about to use we may start by reviewing the theory called Yang-Mills, which describes the interaction of gluons (this is like Quantum Chromodynamics (QCD), which describes the strong interactions, but without quarks). We consider the theory defined by the Lagrangian

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}\text{Tr}(F^{\mu\nu}F_{\mu\nu}), \tag{1.11}$$

where the field strength tensor is defined as

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu - i\frac{g}{\sqrt{2}}[A_\mu, A_\nu]. \tag{1.12}$$

We may note that the Yang-Mills Lagrangian and the field-strength tensor look a lot like those of Maxwell electromagnetic theory. Both the novelties, namely, the trace in the Lagrangian and the commutator in the field strength tensor are a consequence of considering a theory with a non-abelian symmetry group, instead of the abelian $U(1)$ of electromagnetism. The trace in eq. (1.11) is taken with respect to the gauge (colour) degrees of freedom. The gauge field $A_\mu(x)$ is a traceless hermitian matrix of fields, and we can expand it in the following way

$$A_\mu(x) = A_\mu^a(x)T^a. \tag{1.13}$$

The matrices T^a are the generator of the gauge group $SU(N)$. There are N^2-1 generators T^a, and they are hermitian and traceless. These properties follow immediately from the special unitarity of $SU(N)$. The generator matrices obey commutation relations of the form

$$[T^a, T^b] = i\tilde{f}^{abc}T^c, \tag{1.14}$$

and the numerical factors $\tilde{f}^{abc}$ are called the structure constants of the group. If they do not vanish, the group is non-Abelian. We can choose the generator matrices to obey the normalization[11]

$$\text{Tr}(T^aT^b) = \delta^{ab}. \tag{1.15}$$

[11] Another normalization that occurs in literature is $\text{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$ and $[T^a, T^b] = if^{abc}T^c$, so $\tilde{f}^{abc} = \sqrt{2}f^{abc}$ but we choose instead the convention eq. (1.15) since it is more convenient for the colour ordered amplitudes discussed in a later section.

In a similar fashion to eq. (1.13), we can expand the field strength tensor eq. (1.12) as

$$F_{\mu\nu} = F^a_{\mu\nu} T^a, \tag{1.16}$$

in terms of the generator matrices T^a. If we also consider the explicit form of eq. (1.12) we can write an expression for the components of the field strength tensor

$$F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + \frac{g}{\sqrt{2}} \tilde{f}^{abc} A^a_\mu A^b_\nu. \tag{1.17}$$

Then, the trace in the Lagrangian eq. (1.11) amounts to

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F^{\mu\nu e} F^e_{\mu\nu}, \tag{1.18}$$

and plugging eq. (1.17) we get the Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{YM}} = &- \frac{1}{2}\partial^\mu A^{\mu e}\partial_\mu A^e_\nu + \frac{1}{2}\partial^\mu A^{\nu e}\partial_\nu A^e_\mu \\ &- \frac{g}{\sqrt{2}} \tilde{f}^{abc} A^{\mu a} A^{\nu b} \partial_\mu A^e_\nu - \frac{1}{8} g^2 \tilde{f}^{abe} \tilde{f}^{cde} A^{\mu a} A^{\nu b} A^c_\mu A^d_\nu. \end{aligned} \tag{1.19}$$

In order to compute physical meaningful quantities, we need to get rid of the gauge freedom. This process is commonly referred to as fixing the gauge, and it was proved by Faddeev and Popov that we may do this by defining a function $G^a(x)$ and adding to the Lagrangian the so-called gauge fixing term[12]

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2}\xi^{-1} G^a G^a. \tag{1.22}$$

One popular choice is the function

$$G^a = \partial^\mu A^a_\mu, \tag{1.23}$$

[12] One consequence of doing this is that we need to consider the auxiliary fields called ghosts, defined by the Lagrangian

$$\mathcal{L}_{\text{gh}} = \bar{c}^a \frac{\partial G^a}{\partial A^b_\mu} D^{bc}_\mu c^c, \tag{1.20}$$

which also needs to be added to the Lagrangian of our theory. In the last equation,

$$D^{bc}_\mu = \delta^{bc}\partial_\mu + g f^{abc} A^a_\mu \tag{1.21}$$

is the covariant derivative in the adjoint representation, and c and $\bar{c}$ are the ghost and anti-ghost field. The effect of the ghosts fields only appears when we go to loop level. However, we will restrict ourselves to tree-level computations, so we effectively ignore this part.

which yields the gauge-fixing term

$$\mathcal{L}_{\rm gf} = -\frac{1}{2}\xi^{-1}\partial^\mu A^e_\mu \partial^\nu A^e_\nu. \tag{1.24}$$

Adding eqs. (1.19) and (1.24) and doing some integration by parts in the quadratic terms, we find

$$\begin{aligned}\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm gf} =& \frac{1}{2}A^{\mu e}(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{\nu e} + \frac{1}{2}\xi^{-1}A^e_\mu\partial^\mu\partial^\nu A^e_\nu \\ & - \frac{g}{\sqrt{2}}\tilde{f}^{abc}A^{\mu a}A^{\nu b}\partial_\mu A^e_\nu - \frac{1}{8}g^2\tilde{f}^{abe}\tilde{f}^{cde}A^{\mu a}A^{\nu b}A^c_\mu A^d_\nu.\end{aligned} \tag{1.25}$$

This is known as the R_ξ gauge. The first line of eq. (1.25) yields the gluon propagator,

$$\Delta^{ab}_{\mu\nu}(k) = \frac{i\delta^{ab}}{k^2 - i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{1.26}$$

The second line of eq. (1.25) yields three-gluon and four-gluon vertices. The three-gluon vertex factor is

$$V^{abc}_{\mu\nu\rho} = \frac{g}{\sqrt{2}}\tilde{f}^{abc}[(q-r)_\mu\eta_{\nu\rho} + (r-p)_\nu\eta_{\rho\mu} + (p-q)_\rho\eta_{\mu\nu}]. \tag{1.27}$$

The four-gluon vertex factor is

$$\begin{aligned}V^{abcd}_{\mu\nu\rho\sigma} = -i\frac{g^2}{2}\Big[& \tilde{f}^{abe}\tilde{f}^{cde}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \\ & + \tilde{f}^{ace}\tilde{f}^{dbe}(\eta_{\mu\sigma}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}) \\ & + \tilde{f}^{ade}\tilde{f}^{bce}(\eta_{\mu\nu}\eta_{\sigma\rho} - \eta_{\mu\rho}\eta_{\sigma\nu})\Big].\end{aligned} \tag{1.28}$$

The amplitudes constructed from these rules can be organized into different group theory structures. For example, the colour factors for the s, t and u channel diagrams of the four-gluon tree amplitudes are

$$c_s = \tilde{f}^{a_1a_2b}\tilde{f}^{ba_3a_4}, \qquad c_t = \tilde{f}^{a_1a_3b}\tilde{f}^{ba_4a_2}, \qquad c_u = \tilde{f}^{a_1a_4b}\tilde{f}^{ba_2a_3}, \tag{1.29}$$

while the 4-point diagram will contain a sum of terms with all of the previous factors, as it can be seen from eq. (1.28). We can express this information in terms of traces of the generators T^a by noting that

$$i\tilde{f}^{abc} = \mathrm{Tr}(T^aT^bT^c) - \mathrm{Tr}(T^bT^aT^c). \tag{1.30}$$

Then, we can use the Fierz identities (completeness relation)

$$(T^a)_i{}^j (T^a)_k{}^l = \delta_i{}^l \delta_k{}^j - \frac{1}{N}\delta_i{}^j \delta_k{}^l, \tag{1.31}$$

to reduce a product of generator traces. For example, for the 4-gluon s-channel diagram we have

$$\tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4} \propto (\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + (2 \leftrightarrow 4)) - (4 \leftrightarrow 3). \tag{1.32}$$

Similarly, the other three diagrams can also be written in terms of single-trace group theory factors. So the amplitude can be written as

$$\mathcal{A}_4^{\text{tree}} = g^2 (A_4(1,2,3,4)\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \text{permutations of } (234)), \tag{1.33}$$

where the amplitudes $A_4(1,2,3,4)$ are called partial or colour ordered amplitudes and we'll have much more to say about them. This single-trace colour structure is a common feature, as we see next.

1.1.1 Colour ordered amplitudes

We could repeat similar analysis for every amplitude, but instead, we take a short cut using another, more amplitude-friendly gauge called the Gervais-Neveu gauge. To do this, we first introduce the matrix-valued[13] complex tensor

$$H_{\mu\nu} \equiv \partial_\mu A_\nu - \frac{ig}{\sqrt{2}} A_\mu A_\nu. \tag{1.34}$$

Then, $F_{\mu\nu}$ is the antisymmetric part of $H_{\mu\nu}$

$$F_{\mu\nu} = H_{\mu\nu} - H_{\nu\mu}. \tag{1.35}$$

The Yang-Mills Lagrangian can now be written as

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2}\mathrm{Tr}(H^{\mu\nu}H_{\mu\nu} - H^{\mu\nu}H_{\nu\mu}) \tag{1.36}$$

To fix the gauge, we choose a matrix-valued gauge-fixing function $G(x) = H^\mu{}_\mu$, and add

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2}\mathrm{Tr}(H^\mu{}_\mu H^\nu{}_\nu) \tag{1.37}$$

[13] This means that we are considering the field itself (e.g. A_μ) instead of the coefficients of the field when expanded in terms of the generators T^a (e.g. A^a_μ).

to $\mathcal{L}_{\rm YM}$. Here we have set the parameter $\xi = 1$ in eq. (1.22) and ignored ghost Lagrangians (since we'll focus on tree-level diagrams). This is called the Gervais-Neveu gauge, and it is particularly well suited for tree-level scattering amplitudes computations because the Lagrangian now yields directly colour-ordered Feynman rules. Combining eqs. (1.36) and (1.37) we get the total

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(H^{\mu\nu}H_{\mu\nu} - H^{\mu\nu}H_{\nu\mu} + H^{\mu}{}_{\mu}H^{\nu}{}_{\nu}), \tag{1.38}$$

and integration by parts leads to the simple expression

$$\mathcal{L} = \mathrm{Tr}\left(-\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - i\sqrt{2}g\partial^\mu A^\nu A_\nu A_\mu + \frac{1}{4}g^2 A^\mu A^\nu A_\mu A_\nu\right). \tag{1.39}$$

This is the Lagrangian for $SU(N)$ Yang-Mills theory in Gervais-Neveu gauge. Using this, we obtain the scattering amplitude for any n-point tree-level gluon[14] in the form

$$\mathcal{A}_n^{\rm tree} = g^{n-2}\sum_{\text{perms }\sigma} A_n(1,\sigma(2),...,\sigma(n))\mathrm{Tr}(T^{a_1}T^{a_{\sigma(2)}}\dots T^{a_{\sigma(n)}}), \tag{1.40}$$

where each A_n is a colour-ordered or partial amplitude, and is gauge invariant. The sum is over the $(n-1)!$ elements that take into account the cyclic nature of the traces. We may note that it has a single-trace colour structure. This is an important feature of tree-level amplitudes. For loop amplitudes we should encounter multi trace structures.

We can compute the colour ordered amplitudes appearing in eq. (1.40) directly by considering an effective set of Feynman rules (that we call colour-ordered) for planar diagrams (i.e. with no line crossing) with a fixed, counter-clockwise ordering $i_1, \dots, i_n$ of the external lines which corresponds to the colour factor $\mathrm{Tr}(T^{a_{i_1}}\dots T^{a_{i_n}})$. From the first term in eq. (1.39), we see that the gluon propagator is simply

$$\Delta_{\mu\nu}(k) = \frac{i\eta_{\mu\nu}}{k^2 - i\epsilon}. \tag{1.41}$$

The second and third terms in eq. (1.39) yield three- and four gluon vertices

$$V_{\mu\nu\rho}(p,q,r) = -i\sqrt{2}g(p_\rho\eta_{\mu\nu} + q_\mu\eta_{\nu\rho} + r_\nu\eta_{\rho\mu}), \tag{1.42}$$

$$V_{\mu\nu\rho\sigma} = ig^2\eta_{\mu\rho}\eta_{\nu\sigma}, \tag{1.43}$$

respectively. Using eqs. (1.41-1.43), we can compute the colour-ordered amplitudes.

[14] Actually, this applies to any amplitudes involving only particles that transform in the adjoint representation of the gauge group

1.1.2 Relations between colour-ordered amplitudes

The $(n-1)!$-dimensional basis we summed over in eq. (1.40) is actually over-complete. This is a consequence of a series of relations obeyed by tree-level colour-ordered scattering amplitudes. We will mention those relations now.
1. Cyclic Symmetry.

$$A_m^{\text{tree}}(1,2,\ldots,m) = A_m^{\text{tree}}(2,\ldots,m,1). \tag{1.44}$$

2. Reflection (anti)-Symmetry.

$$A_m^{\text{tree}}(1,2,3,\ldots,m-1,m) = (-1)^m A_m^{\text{tree}}(1,m,m-1,\ldots,3,2). \tag{1.45}$$

These first two relations correspond to properties of the traces that accompany the partial amplitudes in eq. (1.40).
3. Photon Decoupling.

$$\sum_{\sigma\in\text{CP}(2,\ldots,m)} A_m^{\text{tree}}(1,\sigma) = 0, \tag{1.46}$$

where we sum over every cyclic permutation σ of $(2,\ldots,m)$. As an example, the five-point case takes the form

$$A_5^{\text{tree}}(1,2,3,4,5) + A_5^{\text{tree}}(1,3,4,5,2) + A_5^{\text{tree}}(1,4,5,2,3) + A_5^{\text{tree}}(1,5,2,3,4) = 0. \tag{1.47}$$

This relation might be understood as substituting one gluon with a photon. Since it does not couple to the other gluons, the amplitude should vanish, thus yielding eq. (1.46) (see, for example [10]).
4. Kleiss-Kuijf (KK) relations. These relations imply that we can express any m-point colour-ordered amplitude as a linear combination of the $(m-2)!$ colour-ordered scattering amplitudes that result from fixing the position of two of the leg labels (say 1 and m). The coefficients can actually only be $\{-1,0,1\}$. The relations are

$$A_m^{\text{tree}}(1,\{\alpha\},m,\{\beta\}) = (-1)^{|\beta|} \sum_{\sigma\in\text{OP}(\{\alpha\},\{\beta^R\})} A_m^{\text{tree}}(1,\sigma,m) \tag{1.48}$$

where the sum is over ordered permutations (OP): all permutations merging the sets $\{\alpha\}$ and $\{\beta^R\}$ that maintain the order of the individual elements belonging to each set within the merged set. The notation $\{\beta^R\}$ represents the set $\{\beta\}$ with inverted order, and $|\beta|$ is the number of elements of $\{\beta\}$. These relations were conjectured in [41] and proven in [42].
5. Bern-Carrasco-Johansson (BCJ) relations. It is possible to further reduce the basis

to $(m-3)!$ elements by means of these relations. We will then express every m-point colour-ordered amplitude as a linear combination of the amplitudes arising from fixing the position of three external legs (say 1,2,3). Each one of the $(m-3)!$ colour-ordered tree-amplitudes will have a coefficient that is a function of external momentum Lorentz invariants. The formula is

$$A_m^{\rm tree}(1,2,\{\alpha\},3,\{\beta\}) = \sum_{\sigma\in{\rm POP}(\{\alpha\},\{\beta\})} A_m^{\rm tree}(1,2,3,\sigma)\prod_{k=4}^{n}\frac{\mathcal{F}_k(3,\sigma,1)}{s_{2,4,\ldots,k}}, \tag{1.49}$$

where the momentum invariants are given by

$$s_{i,j,\ldots,l} = (k_i,k_j,\ldots,k_l)^2, \tag{1.50}$$

$n = |\{\alpha\}|+3$ is the position in the list $\{1,2,\{\alpha\},3,\{\beta\}\}$ of k_3, and the sum runs over partially ordered permutations (POP) of the merged $\{\alpha\}$ and $\{\beta\}$ sets. This gives all permutations of $\{\alpha\}\cup\{\beta\}$ consistent with the order of the $\{\beta\}$ elements. The function $\mathcal{F}_k$ associated with leg k is given by,

$$\mathcal{F}_k(\{\rho\}) = \begin{cases} \sum_{l=t_k}^{m-1} S_{k,\rho_l} & \text{if } t_{k-1}<t_k \\ -\sum_{l=1}^{t_k} S_{k,\rho_l} & \text{if } t_{k-1}>t_k \end{cases} + \begin{cases} s_{2,4,\ldots,k} & \text{if } t_{k-1}<t_k<t_{k+1} \\ -s_{2,4,\ldots,k} & \text{if } t_{k-1}>t_k>t_{k+1} \\ 0 & \text{else} \end{cases}, \tag{1.51}$$

where t_k is the position of leg k in the set $\{\rho\}$, except for t_3 and t_{n+1} which are always defined to be,

$$t_3\equiv t_5, \qquad t_{n+1}\equiv 0. \tag{1.52}$$

The expression $S_{i,j}$ is given by,

$$S_{i,j} = \begin{cases} s_{i,j} & \text{if } i<j \text{ or } j=1 \text{ or } j=3 \\ 0 & \text{else} \end{cases}. \tag{1.53}$$

These relations were first conjectured in ref. [39] and then proven as a low-energy limit of string theory relations in ref. [43, 44], and then directly using BCFW relations in field theory in refs. [45–47].

The origin of the last set of relations is closely related to the concept known as colour-kinematics duality, that we review next.

1.1.3 Colour-kinematics duality

An alternative way to write the full colour-dressed n-point tree amplitude of Yang-Mills theory is as a sum over all cubic diagrams, labelled by i,

$$\mathcal{A}_n^{\text{tree}} = \sum_{i \in \text{cubic}} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \tag{1.54}$$

The denominator of each term is given by the product of all propagators (labeled by α_i) of a given diagram. The numerator factorizes into a (group-theoretic) colour-part c_i, and a kinematic part n_i, which is a polynomial of Lorentz-invariant contractions of polarization vectors ϵ_i and momenta p_i. Let us now consider as an example, the 4-point amplitude

$$\mathcal{A}_4^{\text{tree}} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \tag{1.55}$$

where

$$c_s \equiv \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}, \qquad c_t \equiv \tilde{f}^{a_1 a_3 b} \tilde{f}^{b a_4 a_2}, \qquad c_u \equiv \tilde{f}^{a_1 a_4 b} \tilde{f}^{b a_2 a_3}. \tag{1.56}$$

The numerators n_i can be straightforwardly constructed using Feynman rules. The Yang-Mills 4-point contact terms can be blown-up into s-, t- or u-channel 3-vertex pole diagrams. There will not be a unique prescription for how to assign a given contact term into the cubic diagrams, so the numerators are not uniquely defined. We can actually deform the numerators n_i in several ways without changing the result of the amplitude. For example, one can trivially shift the polarization vectors as $\epsilon_i(p_i) \to \epsilon_i(p_i) + \alpha_i p_i$. This changes the numerator, but has no effect in the amplitudes, since they are gauge invariant. A more non-trivial deformation uses the colour factor Jacobi identity

$$c_s + c_t + c_u = 0. \tag{1.57}$$

The transformation

$$n_s \to n_s + s\Delta, \qquad n_t \to n_t + t\Delta, \qquad n_u \to n_u + u\Delta, \tag{1.58}$$

where Δ is an arbitrary function, leaves the amplitude invariant. In general, for any set of three cubic diagrams whose colour factors are related through a Jacobi identity,

$$c_i + c_j + c_k = 0, \tag{1.59}$$

the following numerator-deformation leaves the amplitude invariant

$$n_i \to n_i + s_i\Delta, \qquad n_j \to n_j + s_j\Delta, \qquad n_k \to n_k + s_k\Delta \tag{1.60}$$

Here $1/s_i$, $1/s_j$ and $1/s_k$ are the unique propagators that are not shared among the three diagrams. Because Δ is arbitrary it is reminiscent of a gauge transformation, except that it doesn't transform the field but the numerator. Because of this, the freedom is usually called a generalized gauge transformation. This freedom to transform the numerators was used by Bern, Carrasco and Johansson to propose the colour-kinematics duality in ref. [39]. They state that the scattering amplitudes of Yang-Mills theory, and its supersymmetric extensions, can be given in a representation where the numerators n_i have the same algebraic properties as the corresponding colour factors c_i. More precisely, using the representation (1.54), the BCJ proposal is that one can always find a representation such that the following parallel relations hold for the colour and kinematic factors

$$c_i = -c_j \Leftrightarrow n_i = -n_j \tag{1.61}$$

$$c_i + c_j + c_k = 0 \Leftrightarrow n_i + n_j + n_k = 0 \tag{1.62}$$

The duality does not state that the numerators need be local.

The first consequence of this colour-kinematics duality, is that by imposing it, we deduce relations among colour ordered amplitudes. For example, using the fact that the colour-ordered tree amplitudes can be expanded in a convenient representation in terms of the poles that appear, we have at four points,

$$A_4(1,2,3,4) \equiv \frac{n_s}{s} + \frac{n_t}{t}, \tag{1.63}$$

$$A_4(1,3,4,2) \equiv -\frac{n_u}{u} - \frac{n_s}{s}, \tag{1.64}$$

$$A_4(1,4,2,3) \equiv -\frac{n_t}{t} + \frac{n_u}{u}. \tag{1.65}$$

Then, imposing the relation $n_u = n_s - n_t$, we can obtain the relations

$$tA_4(1,2,3,4) = uA_4(1,3,4,2), \qquad sA_4(1,2,3,4) = uA_4(1,4,2,3), \tag{1.66}$$

$$tA_4(1,4,2,3) = sA_4(1,3,4,2), \tag{1.67}$$

which can be combined and in turn imply the photon decoupling relation (c.f. eq. (1.46))

$$A_4^{\text{tree}}(1,2,3,4) + A_4^{\text{tree}}(1,3,4,2) + A_4^{\text{tree}}(1,4,2,3) = 0, \tag{1.68}$$

For 5-point amplitudes, we can get for example

$$A_5^{\text{tree}}(1,3,4,2,5) = \frac{-s_{12}s_{45}A_5^{\text{tree}}(1,2,3,4,5) + s_{14}(s_{24}+s_{25})A_5^{\text{tree}}(1,4,3,2,5)}{s_{13}s_{24}}. \tag{1.69}$$

and three other similar relations. These are both examples of BCJ relations in the sense of eq. (1.49). There is actually a simple type of such relations (sometimes called fundamental

BCJ relations) that can be nicely condensed in the form

$$\sum_{i=3}^{n}\left(\sum_{j=3}^{i}s_{2j}\right)A_n(1,3,\ldots,i,2,i+1,\ldots,n)=0 \tag{1.70}$$

which is equivalent to eq. (1.49).

Now that we have understood the meaning of colour-kinematics duality, the application of these ideas to compute gravity amplitudes will be a simple one. However, we must start by stating what we mean by gravity theories. We will do that next.

1.2 Perturbative gravity

The idea of developing a quantum theory of gravitation is an old one, and there have been several attempts[15]. Because our subject is directly linked to classical solutions, the approach we are interested in is the quantization of a classical theory of gravity. So we will focus first on our traditional way to study gravity, i.e. using general relativity (GR). This relies on constructing a tensor differential equation (known as the Einstein equation) whose solution is the metric tensor for the spacetime. Examples of this are the Schwarzschild and Kerr metrics for black holes. There are actually several known solutions [49] to the classical equations of motion and they will play a central role in this work. However, for the time being, we are interested in a different approach, namely that of the scattering of perturbative states at weak coupling. We know that scattering amplitudes are obtained after the quantization of the field theory. We should start with the Lagrangian, extract Feynman rules and calculate the scattering effects perturbatively. Although this approach will be almost prohibitive (this will be clear in a couple pages), it is important to know what we mean by perturbative gravity. The Einstein equation is

$$G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi G T_{\mu\nu}. \tag{1.71}$$

The tensor $G_{\mu\nu}$ is commonly referred to as the Einstein tensor, while $T_{\mu\nu}$ is the stress-energy tensor, and G is the Newton gravitational constant. The curvature scalar is given by $R=g^{\mu\nu}R_{\mu\nu}$, and the Ricci tensor $R_{\mu\nu}$ takes the form

$$R_{\mu\nu}=\partial_\rho\Gamma^\rho_{\mu\nu}-\partial_\nu\Gamma^\rho_{\rho\mu}+\Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu}-\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\rho}. \tag{1.72}$$

In the previous equation, the Christoffel symbol $\Gamma^\alpha_{\beta\gamma}$ is given by

$$\Gamma^\alpha_{\mu\nu}=g^{\alpha\delta}(\partial_\mu g_{\delta\nu}+\partial_\nu g_{\delta\mu}-\partial_\delta g_{\mu\nu}). \tag{1.73}$$

[15]See, for example [48], for an inclusive review.

The Einstein equation (1.71) could also have been derived from the variational principle corresponding to the so called Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{\kappa^2} \int d^d x \sqrt{-g} R + S_{\text{matter}}, \tag{1.74}$$

where the gravitational coupling constant is given by $\kappa^2 = 16\pi G$. The action is written in d spacetime dimensions. It is worth saying that Newton's constant G has a dependence on the dimension of the spacetime on which we are working. Most of the work we will perform is naturally in $d = 4$. However, we will also consider some extensions to spaces of higher dimension.

The metric $g_{\mu\nu}(x)$ then becomes the field whose excitations we describe with the theory defined by the action in eq. (1.74). Taking the functional derivative with respect to $\delta g_{\mu\nu}$, the $\sqrt{-g}R$ part of the action gives the Einstein tensor[16], while the metric variation of the matter action gives the energy-momentum tensor

$$T_{\mu\nu} = \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{1.75}$$

We will now focus on applying quantum field theory on flat spacetime to the scattering of the particles associated with the quantization of the gravitation field $g_{\mu\nu}$. We will consider the metric tensor $g_{\mu\nu}$ to take the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{1.76}$$

so we have the field $h_{\mu\nu}$ as the fluctuation over the flat spacetime background $\eta_{\mu\nu}$ (which we usually refer to as Minkowski). It is this fluctuation $h_{\mu\nu}$ that we will usually refer to as the graviton.

We can now insert eq. (1.76) into the EH action[17] eq. (1.74) to obtain an expansion in powers of the coupling constant κ

$$S_{\text{EH}} = S_{\text{EH}}^{(0)} + \kappa S_{\text{EH}}^{(1)} + \kappa^2 S_{\text{EH}}^{(2)} + \dots \tag{1.78}$$

Where the ellipsis ... represents a finite number of higher orders in κ. The explicit expressions for the first terms are not given here, since they are not very illuminating, but can be found in [34]. To obtain Feynman rules for the three-graviton interaction vertex,

[16] See, for example [50]

[17] Since gravity is invariant under diffeomorphisms, we need to add a term that breaks such invariance in order to fix the gauge. We'll refer to that term as S_{gf}. We should then consider the action

$$S = S_{\text{EH}} + S_{\text{gf}}, \tag{1.77}$$

However, since we are not pursuing this method, we will not specify this. This will be different in a later chapter.

we use DeWitt's [34] method of taking functional derivatives of the action with respect to the interacting fields. This is of course easier and more useful to express in momentum space, where it takes the explicit form

$$\begin{aligned}\frac{\delta S^3}{\delta h_{\mu\nu}\delta h_{\sigma\tau}\delta h_{\rho\lambda}} = \mathrm{sym}\Big[&-\frac{1}{4}P(p\cdot q\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}) - \frac{1}{4}P(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda}) + \frac{1}{4}P(p\cdot q\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}) \\ &+\frac{1}{2}P(p\cdot q\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}) + P(p^\sigma p^\lambda\eta^{\mu\nu}\eta^{\tau\rho}) - \frac{1}{2}P(p^\tau q^\mu\eta^{\nu\sigma}\eta^{\rho\lambda}) - \frac{1}{2}P(p^\rho q^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}) \\ &+\frac{1}{2}P(p^\rho p^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}) + P(p^\sigma q^\lambda\eta^{\tau\mu}\eta^{\nu\rho}) + P(p^\sigma q^\mu\eta^{\tau\rho}\eta^{\lambda\nu}) - P(p\cdot q\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\Big],\end{aligned} \tag{1.79}$$

where the action S^3 refers to the terms in the Einstein-Hilbert action eq. (1.74) that, when expanded in powers of the graviton $h_{\mu\nu}$, contain exactly three such fields. Following this path, we can also obtain an expression for the propagator, and for higher order interaction vertices. Using them, it would be possible (although extremely tedious) to compute graviton scattering amplitudes

$$\mathcal{M}_n(h_{\mu_1\nu_1}, h_{\mu_2\nu_2}, \ldots h_{\mu_n\nu_n}). \tag{1.80}$$

However, this is not the way things are usually done. Instead, some of the modern methods to compute amplitudes, like on-shell recursion relations have been extended also to gravity theories [51]. More along the lines of this thesis is a set of relations coming from string theory, which are exploited to simplify the computation of graviton scattering amplitudes, as we review now.

1.2.1 KLT relations

One important set of relations between gravity and gauge theory amplitudes was derived in string theory by Kawai, Lewellen and Tye (so they are referred to as the KLT relations [36]), and they state that the n-point, tree-level scattering amplitude for closed strings is related to a sum over products of n-point tree-level open strings partial amplitudes, with coefficients that depend on the kinematic variables, as well as on the string tension. In the limit of infinite string tension the open-string partial scattering amplitudes with spin 1 massless external string states become the colour-ordered gluon amplitudes A_n, while in this same limit, the closed string amplitudes with massless spin-2 string external states become the regular graviton scattering amplitudes M_n. In gauge theory, the partial amplitude differs from the full amplitude by the colour structure, as shown in eq. (1.40), while the graviton scattering amplitude M_n is simply given by

$$\mathcal{M}_n^{\text{tree}} = \left(\frac{\kappa}{2}\right)^{n-2} M_n^{\text{tree}}. \tag{1.81}$$

We may note that, since there is no colour-structure in gravity, there is no canonical sense of ordering of the external states, unlike the case of the gauge theory amplitudes. In this infinite string tension limit, KLT offers a relationship between tree-level M_n and A_n for each n. For $n = 4, 5, 6$, the field theory KLT relations are

$$\begin{aligned}
M_4^{\text{tree}}(1,2,3,4) &= -\,is_{12}A_4^{\text{tree}}(1,2,3,4)\tilde{A}_4^{\text{tree}}(1,2,4,3), &(1.82)\\
M_5^{\text{tree}}(1,2,3,4,5) &= -\,is_{12}s_{34}A_5^{\text{tree}}(1,2,3,4,5)\tilde{A}_5^{\text{tree}}(2,1,4,3,5)\\
&\quad + is_{13}s_{24}A_5^{\text{tree}}(1,3,2,4,5)\tilde{A}_5^{\text{tree}}(3,1,4,2,5), &(1.83)\\
M_6^{\text{tree}}(1,2,3,4,5,6) &= -\,is_{12}s_{45}A_6^{\text{tree}}(1,2,3,4,5,6)[s_{35}\tilde{A}_6^{\text{tree}}(2,1,5,3,4,6)\\
&\quad + (s_{34}+s_{35})\tilde{A}_6^{\text{tree}}(2,1,5,4,3,6)]\\
&\quad + \mathcal{P}(2,3,4), &(1.84)
\end{aligned}$$

where $s_{ij} = (k_i + k_j)^2$. In the $n = 6$ case, $\mathcal{P}(2,3,4)$ stands for the sum of all the permutations of legs 2, 3 and 4. At 7-point and higher, the KLT relations are more complicated. Note that there is no specification of helicities of the external states in eqs. (1.82-1.84). Furthermore, the KLT relations are valid in d-dimensions.

1.3 The double copy

While the KLT relations follow the gravity=(gauge)^2 storyline, it is unsatisfactory in some respects. First, the formula becomes tangled at higher points, as it involves nested permutation sums and rather complicated kinematic invariants. Second, it involves products of different colour-ordered amplitudes, so it is not really a squaring relation. Finally, it is only valid at tree-level. There is, however, a more direct squaring relation that has been proposed to be valid at both tree- and loop-level, namely the BCJ double copy, that we review in the following section.

1.3.1 The BCJ double copy

It turns out that the colour-kinematics duality renders the squaring relation immediate. We start with a gauge theory full amplitude expressed in the form

$$\mathcal{A}_n^{\text{tree}} = \sum_{i\in\text{cubic}} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}. \tag{1.85}$$

It was proposed by Bern, Carrasco and Johansson [39] that once the colour-kinematics duality-satisfying n_i are obtained, the formula

$$\mathcal{M}_n^{\text{tree}} = \sum_{i\in\text{cubic}} \frac{n_i^2}{\prod_{\alpha_i} p_{\alpha_i}^2} \tag{1.86}$$

calculates the n-point tree amplitude in the (super)gravity whose spectrum is given by squaring the (super) Yang-Mills spectrum. That is, we take the Yang-Mills amplitude and replace each colour factor c_i with the corresponding duality satisfying numerator n_i to get gravity. This is called the BCJ double copy relation. It can be shown that KLT relations can be derived from them. The BCJ double copy actually goes beyond this. It has been shown that if we have two different sets of numerators where only one set of numerators, say n_i, satisfies the colour-kinematics duality while the other can be an arbitrary representation of the Yang-Mills amplitude:

$$\mathcal{A}_n^{\text{tree}} = \sum_{i\in\text{cubic}} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \qquad \mathcal{A}_n^{\text{tree}} = \sum_{i\in\text{cubic}} \frac{c_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \tag{1.87}$$

the squaring relation can be generalized to

$$\mathcal{M}_n^{\text{tree}} = \sum_{i\in\text{cubic}} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \tag{1.88}$$

where the gravity numerators are given as the products of the two possibly distinct Yang-Mills numerators. To understand this is so, let us assume that n_i respects the duality while $\tilde{n}_i$ does not. We define the difference

$$\Delta_i = n_i - \tilde{n}_i. \tag{1.89}$$

It follows from (1.87) that

$$\sum_{i\in\text{cubic}} \frac{c_i \Delta_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0. \tag{1.90}$$

Since we haven't specified the group, the only relation that can underlie this are the Jacobi identities. But the set of colour-dual numerators n_i satisfies the same Jacobi identities. We are then able to substitute the colour factor with the BCJ-dual numerators and the following equation is to hold

$$\sum_{i\in\text{cubic}} \frac{n_i \Delta_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0. \tag{1.91}$$

This establishes the equivalence of eqs. (1.86) and (1.88). The consequences of this relation are far reaching, since the distinct sets of numerators may come from different Yang-Mills theories, thus allowing us to construct scattering amplitudes for gravity theories that contain different amounts of supersymmetry. Although such relations are themselves remarkable, one of the most powerful features they have is that it is possible to generalise them to loop level. We briefly describe this in the next section.

1.3.2 Loop-level double copy

To explore the generalisation of the double copy to loop level, we begin by considering the full L-loop colour-dressed Yang-Mills amplitude. This can be written as

$$\mathcal{A}_n^{L\text{-loop}} = i^L g^{n-2+2L} \sum_{i\in\text{cubic}} \int \left(\prod_{l=1}^{L} \frac{d^d\ell_l}{(2\pi)^d}\right) \frac{1}{S_i} \frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \tag{1.92}$$

where S_i is the symmetry factor of each diagram. It was conjectured [39] that there exist representations where the kinematic numerators n_i satisfy the same algebraic conditions as c_i. If this is actually the case (and there is mounting evidence of it), the gravity amplitude is given by the double-copy formula

$$\mathcal{M}_n^{L\text{-loop}} = i^{L+1} \left(\frac{\kappa}{2}\right)^{n-2+2L} \sum_{i\in\text{cubic}} \int \left(\prod_{l=1}^{L} \frac{d^d\ell_l}{(2\pi)^d}\right) \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \tag{1.93}$$

where only one copy of the n_i's is required to be duality-satisfying. The validity of this loop-level double copy can be justified through the generalized unitarity method [17,18]. In this method higher loop level integrands are constructed by taking the product of lower-loop or tree amplitudes and imposing on-shell conditions on intermediate legs. Then assuming that gauge theory numerators n_i satisfy the duality, the gravity integrand built by taking double copies of numerators has the correct cuts in all channels and thus gives the correct answer.

1.3.3 Zeroth copy

Similarly, one may start with eq. (1.92) and replace the kinematic numerator n_i with a second set of colour factors $\tilde{c}_i$. This process yields the amplitude

$$\mathcal{T}_n^{L\text{-loop}} = i^L y^{n-2+2L} \sum_{i\in\text{cubic}} \int \left(\prod_{l=1}^{L} \frac{d^d p_l}{(2\pi)^d}\right) \frac{1}{S_i} \frac{\tilde{c}_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}. \tag{1.94}$$

where y is the appropiate coupling constant. The particle content of this theory is a set of scalar fields $\Phi^{aa'}$, which transform in the adjoint representation of two (possibly different)

Lie algebras. This is an example of a biadjoint scalar theory [52–55]. The equation of motion of such a theory is explicitly given by

$$\partial^2\Phi^{aa'} - y f^{abc} f^{a'b'c'} \Phi^{bb'} \Phi^{cc'} = 0, \tag{1.95}$$

where the second term arises from a cubic interaction involving both sets of structure constants.

1.3.4 A (partial) bird's-eye view of the double copy

The BCJ double copy was first hinted in [39] but it is more formally stated in [40]. At tree level, it was proven by Bern et. al. in 2010 [56], assuming that colour-kinematics satisfying numerators exist. BCJ relations were proven in string theory by monodromy [43, 57, 58] (see also [59]) and in field theory using BCFW recursion relations [45, 47] and they can be used with an inverted logic as a proof of the existence of the BCJ-dual numerators. It has since been understood that at tree level, the double copy is equivalent to the KLT relations. However, since it is formulated in terms of the numerators, the BCJ double copy represents a great improvement with respect to KLT for the purpose of obtaining loop level results.

The validity of the loop level double copy relies on the existence of duality-satisfying numerators. Unlike tree level, it hasn't been formally proved that there always exists such a representation, nor are there ways to systematically construct one. We have explicit duality-satisfying numerators for some theories. For example, for $\mathcal{N} = 4$ SYM: up to 4-loops, 2-loops and 1-loop for 4-points, 5-points and 7-points respectively, and for pure Yang Mills: up to 2-loops for 4-point in the all plus theory, 1-loop for 4-point in arbitrary dimension, 1-loop for n-points in the all plus or all minus sectors. These and other examples of numerators at loop-level for specific theories are in [40, 60–79].

The formalism of the double copy has been extended to theories like Einstein-Yang-Mills [80–83], biadjoint scalars [54, 55], the non-linear sigma model (NLSM) [84, 85], conformal gravity [86], QCD-like theories [87–90], and string theory [44, 91–100]. Some other works include understanding supergravities as products of Yang-Mills theories [101–110], the application to form factors [111, 112] and looking for the corresponding kinematic algebra [52, 113–115]. (See also [116–129] for related studies).

Another field of research that bears a direct relation with this thesis is the use of the double copy to simplify the action of gravity theories. This was pioneered by Bern and Grant in [54], where they build an action for gravity which shows a factorization that makes it compatible with KLT. Later Bern et. al. introduced a Lagrangian for Yang-Mills that produces colour-kinematics duality satisfying numerators [56] up to six-points (later, Weinzierl and Tolotti systematized the construction to any order [130]). Using this BCJ-compliant Lagrangian, it is straightforward to obtain a simpler Lagrangian for a gravity theory. Other works on Lagrangian approaches include [113,131,132]. Cheung also studied the double copy at the level of a redefined action in [133, 134].

It is important to mention that there exist orthogonal approaches to study the relation between gauge and gravity theories. For example, the scattering equations approach of Cachazo, He and Yuan (CHY) [53, 135–137] (this formalism has been extended to include fermions [138, 139]), and the ambitwistor string [140–142] (the study of scattering amplitudes from an ambitwistor point if view has been pioneered by Adamo et. al. in [143]).

Excellent reviews of double copy and related topics that have been written recently include [144–147].

One iconic computation using the double copy formalism, is the investigation of the divergences of $\mathcal{N} = 8$ Sugra to four loops [70]. The difficult problem, however, gets replaced by the task of obtaining loop level, colour-kinematics dual numerators. State of the art results correspond to five loop numerators [77], that took more than five years (and a great deal of new methods [78]) to compute.

Because of the difficulty of obtaining multi-loop numerators, an interesting problem has been that of obtaining all-order results. This has been possible only in some kinematic scenarios like the Regge limit [5, 148–152] or soft limits [153, 154]. It is in this spirit that we will study the double copy in the context of classical solutions.

The classical double copy is studied in [1–3, 155–158] (see also [159, 160] for classical solutions of biadjoint theories, and [161–164] for related gravity topics), and it will be reviewed thoroughly in this thesis.

1.4 Overview

This thesis consists of two parts. We devote part I to studying a class of special solutions in general relativity called Kerr-Schild spacetimes (or metrics), and we relate them to the BCJ double copy. First, in chapter 2, we review the results for the double copy of the self-dual sectors in Yang-Mills and general relativity, as they provide the simplest example of a double copy between solutions of the equation of motion. Then, in chapter 3, we review the double copy relation between some black hole examples in Kerr-Schild form, and gauge theory solutions obtained from a simple ansatz. We extend this formalism in chapter 4 to also consider solutions which exhibit multiple Kerr-Schild form. Such is the case with the Taub-NUT metric, whose single copy is an electromagnetic dyon. In chapter 5, we extend this formalism to consider the process when gravitons are defined over curved backgrounds, opening a possible pathway to cosmological applications. Finally, chapter 6 deals with solutions representing accelerated radiating particles, and it also relates via the double copy the radiation emitted by the particles in both theories.

The focus in part II moves to perturbative solutions of the equation of motion. In chapter 7 we review a computation by Duff to obtain the Schwarzschild metric using a quantum perturbative gravity computation at tree level. We then repeat the procedure in chapter 8, using a gravity Lagrangian obtained as a double copy of Yang-Mills instead of the Einstein-Hilbert one. A consequence of this is the appearance of a dilaton degree of freedom. A formalism to extract the graviton and the dilaton information from the double copy result is developed in chapter 9.

Finally, we present our conclusions in chapter 10. Some computations are given in the appendix.

Part I

The Kerr-Schild Double Copy

Chapter 2

Invitation: Double copy and self-dual sectors

Beside some specific examples like those mentioned in the last chapter, the existence of BCJ duality-satisfying numerators has not been proven for any arbitrary theory. Given that the double copy is naturally defined in a perturbative way, it is fair to say that our lack of understanding of the origin of the double copy is in great part caused by the technical difficulties that arise when trying to obtain multi-loop numerators that obey colour-kinematics duality. We may then consider the possibility of the double copy being applied away from the perturbative scattering amplitudes context. We focus our efforts in the first part of this thesis to explore such an idea in the framework of classical field theory.

Instead of looking at scattering amplitudes, the natural object to study in classical field theory are the solutions of the equations of motion of the fields. These two items, though, are directly related as we will see below. In simple terms, tree-level scattering amplitudes can be obtained by taking a limit of a series expansion of an appropriate classical solution.

2.1 Classical solution as a generating functional

To make the previous statement more precise, we can consider as starting point, the functional path integral approach in quantum field theory. The generating functional is given by

$$Z[J] \equiv \int \mathcal{D}\phi \; e^{i\int d^4x \;\, [\mathcal{L}+J\phi]} = e^{iW[J]}, \tag{2.1}$$

and the functional $W[J]$ generates connected correlation functions. The scattering amplitudes are then computed from these connected correlators by the LSZ procedure. Since we will focus here on the classical limit (which corresponds to tree-level graphs), we take

$\hbar \to 0$[1]. In this limit, the saddle point approximation allows us to consider

$$Z[J] = \int \mathcal{D}\phi \; e^{i(S[\phi]+\int d^4x \;\; J\phi)} \simeq e^{i(S[\phi_{\rm cl}]+\int d^4x \;\; J\phi_{\rm cl})}, \tag{2.2}$$

where $\phi_{\rm cl}$ is the solution to the field equations in the presence of the source J. We thus identify

$$W[J, \phi_{\rm cl}] \equiv S[\phi_{\rm cl}] + \int d^4x \; J\phi_{\rm cl}, \tag{2.3}$$

so the functional derivative with respect to the current J yields

$$\begin{aligned} \frac{\delta W[J,\phi_{\rm cl}]}{\delta J(x)} &= \frac{\delta S[\phi_{\rm cl}]}{\delta \phi_{\rm cl}} \frac{\delta \phi_{\rm cl}}{\delta J(x)} + \phi_{\rm cl} + J \frac{\delta \phi_{\rm cl}}{\delta J(x)}, \\ &= \left(\frac{\delta S[\phi_{\rm cl}]}{\delta \phi_{\rm cl}} + J \right) \frac{\delta \phi_{\rm cl}}{\delta J(x)} + \phi_{\rm cl} = \phi_{\rm cl}, \end{aligned} \tag{2.4}$$

where, in the last line, we have applied the equations of motion. This implies also that we can use $\phi_{\rm cl}$ as a generating functional, where in order to obtain an n-point Green function, we need to differentiate $n-1$ times with respect to the source. On the other hand, since $W[J]$ is the generating functional for connected Green functions, the vacuum expectation value is computed by the means of

$$\langle 0 \,| \phi(x) |\, 0 \rangle_J = \frac{\delta W[J,\phi_{\rm cl}]}{\delta J(x)}. \tag{2.5}$$

Using eqns. (2.4) and (2.5), we have

$$\langle 0 \,| \phi(x) |\, 0 \rangle_J = \phi_{\rm cl}. \tag{2.6}$$

In conclusion, the vacuum expectation value of a field in presence of a classical source $J(x)$ corresponds to the solution to the classical field equation.

These ideas were exploited in ref. [113] where solutions of the self-dual sectors in gravity and gauge theories were considered and linked via the BCJ double copy. We briefly review this in the following section.

[1] This statement looks strange, since there is no $\hbar$ as we are working as usual with $\hbar = c = 1$. The planck constant would appear in

$$Z[J] = e^{\frac{i}{\hbar} \int d^4x \;\; [\mathcal{L}+J\phi]}$$

.

2.2 Self-dual sectors

There are many reasons why the self-dual sector of Yang-Mills theory is very interesting. In particular, it is known that MHV scattering amplitudes can be obtained directly using solutions of self-dual Yang-Mills theory as a generating functional [165]. It has also been conjectured that the simplicity of the Parke-Taylor formula, is somehow related to integrability properties of the self-dual theory [166]. These ideas have been extended to loop level [167, 168]. Thus, in the same sense that MHV represents the simplest possible set up in scattering amplitudes, the self-dual sectors of Yang-Mills and gravity may provide the simplest place to start studying classical solutions. Let us now review the equations of motion and perturbative solutions to both theories. We follow mostly [167, 168] (see also [169, 170])

2.2.1 Self-dual Yang-Mills

The self-dual Yang-Mills (SDYM) equations are most commonly studied in Euclidean space, or in $(2+2)$ dimensional spacetime, so that the solutions are real. We instead consider them directly in Minkowski spacetime, where they read

$$F_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{2.7}$$

and the gauge field solving the equation is necessarily complex. Such complex self-dual configurations have a physical interpretation as waves of positive helicity. Thus, it is useful to use the light-cone coordinate system

$$u = t - z, \qquad v = t + z, \qquad w = x + iy, \tag{2.8}$$

where the flat-space Minkowski metric is given by

$$ds^2 = -du\,dv + dw\,d\bar{w}. \tag{2.9}$$

Choosing the light-cone gauge, where $A_u = 0$, the self dual equation (2.7) implies

$$A_w = 0, \qquad A_v = -\frac{1}{4}\partial_w\Phi, \qquad A_{\bar{w}} = -\frac{1}{4}\partial_u\Phi. \tag{2.10}$$

The Lie-algebra valued scalar field Φ is determined by the equation

$$\partial^2\Phi - ig[\partial_w\Phi, \partial_u\Phi] = 0, \tag{2.11}$$

where $\partial^2 = -4(\partial_v\partial_u - \partial_w\partial_{\bar{w}})$ is the wave operator. The double copy is most easily understood in momentum space, so we perform a Fourier transform. This yields

$$\Phi^a(x) = \frac{1}{2}g\int \text{đ}p_1\text{đ}p_2\frac{F_{p_1p_2}{}^k f^{abc}}{k^2}\Phi^b(p_1)\Phi^c(p_2), \tag{2.12}$$

where we have defined

$$F_{p_1p_2}{}^k \equiv \delta\!\!\!{}^-(p_1+p_2-k)X(p_1,p_2), \qquad X(p_1,p_2) \equiv p_{1w}p_{2u} - p_{1u}p_{2w}. \tag{2.13}$$

Also, to simplify our notation, we use

$$\int \text{đ}^d p F(p) \equiv \int \frac{d^dp}{(2\pi)^d}F(p), \qquad \delta\!\!\!{}^-{}^d(p) \equiv (2\pi)^d\delta^{(d)}(p). \tag{2.14}$$

Thus, in order to obtain solutions to the self-dual sector of Yang-Mills theory, it suffices to solve a scalar equation with cubic coupling.

2.2.2 Self-dual gravity

The equations of motion for self-dual gravity (SDG) can be written as

$$R_{\mu\nu\lambda\delta} = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\lambda\delta}. \tag{2.15}$$

Considering we expand the metric in the usual form

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{2.16}$$

and exploiting gauge (diffeomorphism) freedom, the non-vanishing components of the graviton can be written in terms of a single scalar field ϕ:

$$h_{vv} = -\frac{1}{4}\partial_w^2\phi, \qquad h_{\bar{w}\bar{w}} = -\frac{1}{4}\partial_u^2\phi, \qquad h_{v\bar{w}} = h_{\bar{w}v} = -\frac{1}{4}\partial_w\partial_u\phi, \tag{2.17}$$

The SDG equations imply that this scalar field obeys

$$\partial^2\phi - \kappa\left((\partial_w^2\phi)(\partial_u^2\phi) - (\partial_w\partial_u\phi)^2\right) = 0, \tag{2.18}$$

where ∂^2 denotes the Minkowski wave operator. This equation was first derived by Plebanski, and it gives the most general way to represent SDG. Introducing the Poisson bracket $\{f,g\} \equiv (\partial_w f)(\partial_u g) - (\partial_u f)(\partial_w g)$ we may rewrite

$$\partial^2\phi - \kappa\{\partial_w\phi, \partial_u\phi\} = 0, \tag{2.19}$$

so that the resemblance between SDYM eq. (2.11) and SDG becomes even more striking.

2.2.3 Perturbative solutions

We will review now a technique to recursively construct solutions to the equation of motion in a quantum field theory. To illustrate this, let us consider the simple setting of an interacting (cubic) scalar field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{3!}g\phi^3 + J\phi. \tag{2.20}$$

The last term in the Lagrangian correspond to a source for the field. The classical equation of motion is given by

$$\partial^2\phi + \frac{1}{2}g\phi^2 - J = 0. \tag{2.21}$$

Performing a Fourier transform, we have the equation in momentum space[2]

$$k^2\phi(k) - \frac{1}{2}g\int d\!\!{}^-\!{}^4p_1 d\!\!{}^-\!{}^4p_2 \delta\!\!{}^-\!{}^4(p_1+p_2-k)\phi(p_1)\phi(p_2) = -J(k). \tag{2.22}$$

We will solve this equation order by order in perturbation theory. Thus, we write

$$\phi(k) = \phi^{(0)}(k) + g\phi^{(1)}(k) + g^2\phi^{(2)}(k) + \ldots, \tag{2.23}$$

In this expansion, the perturbative coefficients $\phi^{(i)}$ are assumed to have no dependence on the coupling g. Inserting the expansion eq. (2.23) in the equation of motion we obtain, at zeroth order, the free equation

$$k^2\phi^{(0)}(k) = -J(k). \tag{2.24}$$

It is useful to solve it for the zeroth order in the expansion $\phi^{(0)}$, thus yielding the equivalent expression

$$\phi^{(0)}(k) = -\frac{J(k)}{k^2}. \tag{2.25}$$

At next to leading order, we have the equation of motion

$$k^2\phi^{(1)}(k) = \frac{1}{2}\int d\!\!{}^-\!{}^4p_1 d\!\!{}^-\!{}^4p_2 \delta\!\!{}^-\!{}^4(p_1+p_2-k)\phi^{(0)}(p_1)\phi^{(0)}(p_2), \tag{2.26}$$

[2]Throughout this thesis, rather than denoting Fourier coefficients with a tilde, we use the argument of the function to make this clear.

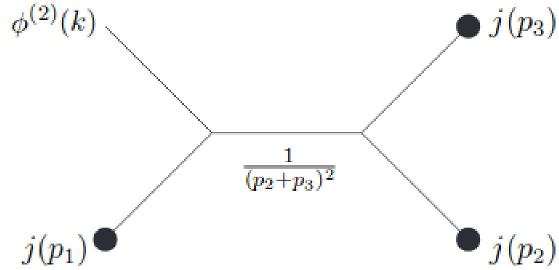

Figure 2.1: The second order correction $\phi^{(2)}(k)$ consists of an interaction between two particles, creating a disturbance which propagates before scattering against a third particle

or, inserting eq. (2.25), we have the equivalent

$$\phi^{(1)}(k) = \frac{1}{2}\int đ^4p_1 đ^4p_2 \delta\!\!\!{}^-{}^4(p_1+p_2-k)\frac{1}{k^2}\frac{J(p_1)}{p_1^2}\frac{J(p_2)}{p_2^2}. \tag{2.27}$$

Note that we can express the results in terms of the current $J(k)$, as well as in terms of the zeroth order $\phi^{(0)}(k)$. This will play an important role in chapters to come. Just for the sake of comparison with later results, if we continue one more order in perturbation theory, the next correction satisfies

$$k^2\phi^{(2)}(k) = \int đ^4p_1 đ^4p_2 \delta\!\!\!{}^-{}^4(p_1+p_2-k)\phi^{(0)}(p_1)\phi^{(1)}(p_2) \tag{2.28}$$

and inserting eqs. (2.24) and (2.27) we find

$$\phi^{(2)}(k) = \int đ^4p_1 đ^4p_2 đ^4p_3 \delta\!\!\!{}^-{}^4(p_1+p_2+p_3-k)\phi^{(0)}(p_1)\phi^{(0)}(p_2)\phi^{(0)}(p_3)\frac{1}{k^2}\frac{1}{(p_2+p_3)^2}. \tag{2.29}$$

The Feynman diagram corresponding to this expression is shown in Fig. 2.1. We are now ready to apply this technology to gauge and gravity theories.

Gauge and gravity

Recall that the equation of motion for the self-dual Yang-Mills sector reduces to

$$\partial^2\Phi - ig[\partial_w\Phi, \partial_u\Phi] = 0, \tag{2.30}$$

that we can express in the equivalent form

$$\Phi^a(x) = -\frac{1}{2}g\int đp_1 đp_2 \frac{F_{p_1p_2}{}^k f^{abc}}{k^2}\Phi^b(p_1)\Phi^c(p_2), \tag{2.31}$$

Then, if we consider the expansion

$$\Phi^a(k) = \Phi^{(0)a}(k) + g\Phi^{(1)a}(k) + g^2\Phi^{(2)a}(k) + \dots \tag{2.32}$$

Repeating the procedure from last section, we can show that it has the iterative solution

$$\Phi^{(0)a}(k) = j^a(k), \tag{2.33}$$

$$\Phi^{(1)a}(k) = -\frac{1}{2}\int đp_1 đp_2 \frac{F_{p_1p_2}{}^k f^{b_1b_2a}}{k^2} j^{b_1}(p_1)j^{b_2}(p_2), \tag{2.34}$$

$$\Phi^{(2)a}(k) = -\frac{1}{2}\int đp_1 đp_2 đp_3 \frac{F_{p_1q}{}^k F_{p_2p_3}{}^q f^{b_1ca} f^{b_2b_3c}}{k^2(p_2+p_3)^2} j^{b_1}(p_1)j^{b_2}(p_2)j^{b_3}(p_3), \tag{2.35}$$

where we have defined

$$j(k) \equiv -\frac{J(k)}{k^2} = \phi^{(0)}(k). \tag{2.36}$$

In a similar fashion, the equation for self-dual gravity is given by

$$\partial^2\phi - \kappa\{\partial_w\phi, \partial_u\phi\} = 0, \tag{2.37}$$

which can be brought to the form

$$\phi(k) = -\frac{1}{2}\kappa\int đp_1 đp_2 \frac{F_{p_1p_2}{}^k X(p_1,p_2)}{k^2}\phi(p_1)\phi(p_2). \tag{2.38}$$

Again, proposing the expansion

$$\phi(k) = \phi^{(0)}(k) + \kappa\phi^{(1)}(k) + \kappa^2\phi^{(2)}(k) + \dots \tag{2.39}$$

This has the iterative solution

$$\phi^{(0)}(k) = j(k), \tag{2.40}$$

$$\phi^{(1)}(k) = -\frac{1}{2}\int đp_1 đp_2 \frac{F_{p_1p_2}{}^k X(p_1,p_2)}{k^2} j(p_1)j(p_2) \tag{2.41}$$

$$\phi^{(2)}(k) = -\frac{1}{2}\int đp_1 đp_2 đp_3 \frac{X(p_1,q)F_{p_1q}{}^k X(p_2,p_3)F_{p_2p_3}{}^q}{k^2(p_2+p_3)^2} j(p_1)j(p_2)j(p_3) \tag{2.42}$$

2.3 Relation with BCJ double copy

It is clear now that is possible to deduce solutions to SDG from SDYM by replacing $SU(N_c)$ structure constants f^{abc} by appropiate factors of $X(p_1, p_2)$. These factors are related to the $F_{p_1p_2}{}^k$, but the relationship is a bit more subtle than the substitution $f \to$

F as this would involve squaring a delta function. Instead, the algorithm to deduce the gravitational expression from the Yang-Mills formulae involves extracting the overall momentum-conserving delta function, and then following the BCJ procedure of identifying a kinematic numerator which is to be squared. We will detail this now.

2.3.1 Link with BCJ

We had already obtained the equation

$$\phi_{\text{cl}} = \frac{\delta W[J, \phi_{\text{cl}}]}{\delta J(x)}, \tag{2.43}$$

meaning that, we can use ϕ_{cl} as a generating functional. We will use this to obtain scattering amplitudes from the perturbative solution to SDYM theory. Then, using a BCJ double copy, we find the proper numerator for SDG. To fully illustrate things, let us consider the second-order perturbation of eq. (2.35), where we substitute eq. (2.13)

$$\Phi^{(2)a}(k) = -\frac{1}{2}\int đq\, đp_1 đp_2 đp_3 \delta\!\!\!{}^{-}(p_1+q-k)\delta\!\!\!{}^{-}(p_2+p_3-q) \frac{X(p_1,q)X(p_2,p_3)f^{b_1ca}f^{b_2b_3c}}{k^2(p_2+p_3)^2} j^{b_1}(p_1)j^{b_2}(p_2)j^{b_3}(p_3), \tag{2.44}$$

and we have used an integral Einstein convention for the contraction of indices of $F_{p_1p_2}{}^{k}$, i.e. we have the relation

$$F_{p_1p_2}{}^{q}F_{qp_3}{}^{p_4} \equiv \int đq\, \delta\!\!\!{}^{-}(p_1+q-k)X(p_1,q)\delta\!\!\!{}^{-}(p_2+p_3-q)X(p_2,p_3).$$

Now, performing the integration of the variable q, we obtain

$$\Phi^{(2)a}(k) = -\frac{1}{2}\int đp_1 đp_2 đp_3 \delta\!\!\!{}^{-}(p_1+p_2+p_3-k) \frac{X(p_1,p_2+p_3)X(p_2,p_3)f^{b_1ca}f^{b_2b_3c}}{k^2(p_2+p_3)^2} j^{b_1}(p_1)j^{b_2}(p_2)j^{b_3}(p_3). \tag{2.45}$$

To obtain the amplitude, we differentiate with respect to j and amputate the final leg. The expression becomes

$$\mathcal{A}(p_1,p_2,p_3,-k) = -\frac{1}{2}\frac{X(p_1,p_2+p_3)X(p_2,p_3)f^{b_1ca}f^{b_2b_3c}}{(p_2+p_3)^2}\delta\!\!\!{}^{-}(p_1+p_2+p_3-k) + \ldots \tag{2.46}$$

where the dots indicate other channels. The BCJ procedure now identifies the kinematic numerator as

$$n_t = X(p_1, p_2 + p_3)X(p_2, p_3). \tag{2.47}$$

This is the object we substitute in place of the colour factors. Note that this reproduces the solution for the gravity case. This first attempt to study classical solutions, not only provides a precedent to our work, but will be neatly related to the formalism we are about to develop.

2.3.2 Bonus: Kinematic algebra

Some of the ideas that we have reviewed in this chapter were originally explored by Monteiro and O'Connell in [113]. However, their objective was a bit different. They noticed that the fact that the kinematic numerators satisfy Jacobi identities strongly suggests that there is a genuine infinite dimensional kinematic Lie algebra.

To understand this, note that using the integral Einstein convention from eq. (2.45), and using $\delta^{pq} = \bar{\delta}(p+q) = \delta_{pq}$, such that $\delta_{pk}\delta^{kq} = \delta_p{}^q = \bar{\delta}(p-q)$, to raise and lower indices, it is easy to see that $F^{p_1p_2p_3} = F_{p_1p_2p_3}$ is totally antisymmetric, and also possible to show that they satisfy a kinematic Jacobi identity

$$F_{p_1p_2}{}^q F_{p_3q}{}^k + F_{p_2p_3}{}^q F_{p_1q}{}^k + F_{p_3p_1}{}^q F_{p_2q}{}^k = 0. \tag{2.48}$$

It is then clear that the coefficients $F^{p_1p_2p_3}$ have the same algebraic properties as the structure constants f^{abc} and are in fact structure constants for an infinite-dimensional Lie algebra. To understand what algebra this is, we need to look at the gravity case.

For the Poisson bracket we can construct the algebra

$$\{e^{-ik_1\cdot x}, e^{-ik_2\cdot x}\} = -X(k_1, k_2)e^{-i(k_1+k_2)\cdot x}, \tag{2.49}$$

which is the kinematic algebra discussed before. It is the Poisson version of the area-preserving diffeomorphisms of w and u. The infinitesimal generators of such transformations are

$$L_u = e^{ik\cdot x}(-k_w\partial_u + k_u\partial_w), \tag{2.50}$$

and they obey the Lie algebra

$$[L_{p_1}, L_{p_2}] = iX(p_1, p_2)L_{p_1+p_2} = iF_{p_1p_2}{}^k L_k. \tag{2.51}$$

This infinite dimensional Lie algebra turns out to be the kinematic analogue of the Yang-Mills Lie algebra. Let us remark that this result is restricted to the self-dual case, and a kinematic algebra is not known for the full Yang-Mills theory.

This formalism was extended in ref. [52], where the construction of amplitudes that satisfy BCJ relations allows to learn something about the kinematic algebra, and in ref. [114] where the relation to scattering equations is explored. Also, quite recently, Cheung and Shen [171] found the kinematic-algebra underlying the flavour-kinematics duality in a non-linear sigma model (NLSM).

In this chapter, we have seen how the close relation between scattering amplitudes and perturbative classical solutions allows us to understand the BCJ double copy in the context of the self-dual sectors of both theories. The idea of studying solutions in a perturbative fashion will be central in the second part of the thesis. However, in the next chapter, we will study a family of exact solutions to the Einstein equations. This may feel like a detour after the things we reviewed in this chapter, but we will eventually show a neat relation between them.

Chapter 3

Stationary Kerr-Schild solutions

After seeing a double copy of classical solutions in the self-dual sectors of general relativity and Yang-Mills, we move on to analysing a family of corresponding solutions in both theories.

We start using an inverse approach compared to the usual in BCJ, i.e. we take the single copy of solutions to the classical equations of motion of gravity theory. We mentioned before that the solutions of the Einstein equation

$$G_{\mu\nu} = \frac{\kappa^2}{2} T_{\mu\nu}, \tag{3.1}$$

are spacetime metric tensors $g_{\mu\nu}$. It turns out, that the double copy process is actually quite transparent for solutions that can be cast into a particular choice of coordinates, namely Kerr-Schild solutions. We will next describe such systems.

3.1 The Kerr-Schild ansatz

We will refer to Kerr-Schild solutions as those with the property that their metric tensor can be written in the form

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \kappa h_{\mu\nu}, \\ &= \eta_{\mu\nu} + \kappa k_\mu k_\nu \phi, \end{aligned} \tag{3.2}$$

where ϕ is a scalar function and the vector field k_μ is null with respect to both the flat metric $\eta_{\mu\nu}$ and the spacetime metric $g_{\mu\nu}$. This is

$$k_\mu \eta^{\mu\nu} k_\nu = 0 = k_\mu g^{\mu\nu} k_\nu. \tag{3.3}$$

Also, the vector field k_μ defines a geodetic congruence, i.e. it can be parallel transported along the curves in the congruence. This statement is equivalent to the identity

$$k^\mu \partial_\mu k_\nu = 0. \tag{3.4}$$

An immediate consequence to eq. (3.3) is the fact that the inverse metric takes the simple form

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa k^\mu k^\nu \phi. \tag{3.5}$$

Note that this implies we can raise and lower the index on the vector field k using the flat spacetime metric η, instead of the whole metric g. Using this property of the Kerr-Schild metrics, we can compute the Christoffel symbol (eq. (1.73))

$$\begin{aligned}
\Gamma^\rho_{\mu\nu} &= \frac{1}{2}(\eta^{\rho\delta} - k^\rho k^\delta \phi)((k_\mu k_\delta \phi),_\nu + (k_\nu k_\delta \phi),_\mu - (k_\mu k_\nu \phi),_\delta), \\
&= \frac{1}{2}((k_\mu k^\rho \phi),_\nu + (k_\nu k^\rho \phi),_\mu - (k_\mu k_\nu \phi),^\rho) + k^\rho k^\delta \phi (k_\mu k_\nu \phi),_\delta),
\end{aligned} \tag{3.6}$$

as well as the Ricci tensor and the curvature scalar

$$R^\mu{}_\nu = \frac{1}{2}\left(\partial^\mu \partial_\alpha(\phi k^\alpha k_\nu) + \partial_\nu \partial^\alpha(\phi k_\alpha k^\mu) - \partial^2(\phi k^\mu k_\nu)\right), \tag{3.7}$$

$$R = \partial_\mu \partial_\nu (\phi k^\mu k^\nu), \tag{3.8}$$

where $\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$. It is a remarkable fact that the Ricci tensor with mixed indices (this is, one index up and one down) for a metric that can be cast into Kerr-Schild form is linear in the graviton $h_{\mu\nu}$. Interestingly, the Ricci tensor only acquires this linear compact form when the indices are in that position (note that unlike the vector k^μ, we use the whole metric $g^{\mu\nu}$ to raise its indices). This linear behaviour is a consequence of the multiple identities satisfied by the vector k^μ (eqs. (3.3) and (3.4)) and it will play a central role in our study.

Thus far this section should feel like a detour of our previous discourse. However, we will argue that this is directly related to the double-copy formalism of BCJ. To do this, we want to show that if we start with a solution in Kerr-Schild form, eq. (3.2), and construct the vector field

$$A^\mu \equiv \phi k^\mu. \tag{3.9}$$

the latter will be a solution to abelianised Yang-Mills equations (these are effectively Maxwell equations). To illustrate this, let us first consider the stationary case (this means that all time derivatives vanish). We'll also use the fact that without loss of generality we may set the value of the first component in the Kerr-Schild vector to unity ($k^0 = 1$) by absorbing this component into the scalar function ϕ. Considering such conditions, the components of the Ricci tensor eq. (3.7) and the curvature scalar eq. (3.8) take the form

$$R^0{}_0 = \frac{1}{2}\partial_i\partial^i\phi, \tag{3.10}$$

$$R^i{}_0 = -\frac{1}{2}\partial_j\left[\partial^i(\phi k^j) - \partial^j(\phi k^i)\right], \tag{3.11}$$

$$R^i{}_j = \frac{1}{2}\partial_l\left[\partial^i(\phi k^l k_j) - \partial_j(\phi k^l k^i) - \partial^l(\phi k^i k_j)\right], \tag{3.12}$$

$$R = \partial_i\partial_j(\phi k^i k^j). \tag{3.13}$$

Then, using the vector field A_μ we can construct the field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.14}$$

It is then straightforward to show that vacuum Einstein equations $R_{\mu\nu} = 0$ imply in the stationary case

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0. \tag{3.15}$$

Actually, we only needed eqs. (3.10) and (3.11) to construct eq. (3.15). More generally, we could consider a non-abelian gauge field of the form

$$A^a_\mu = c^a\phi k_\mu, \tag{3.16}$$

and this would still be a solution to the Maxwell equations (note that the simple dependance on the colour vector c^a would cancel the nonlinear commutator in the Yang-Mills equations). The fact that this non-abelian gauge field satisfies abelian equations seems to be closely related to the linear character of the Ricci tensor in Kerr-Schild coordinates.

We may also interpret ϕ in the spirit of the zeroth copy. The Kerr-Schild ansatz for the gauge field A_μ is obtained by removing a factor of k_ν from the (non-perturbative) graviton $h_{\mu\nu}$. Repeating this, we find that the Kerr-Schild scalar function ϕ is the field that survives upon taking the zeroth copy. This field then satisfies the equation of motion[1]

$$\nabla^2\phi = 0. \tag{3.17}$$

Thus, we see that eq. (3.17) is an abelianised version of the biadjoint field equation of eq.(1.95). It is important to note that the scalar field ϕ plays a role analogous to the propagators in the amplitudes story. It is present and unchanged, in the scalar, gauge and

[1] Here, the Laplacian symbol means $\nabla^2 \equiv \partial_i\partial^i\phi$.

gravitational cases. This gives a direct physical interpretation of the scalar field ϕ. In the zeroth copy theory, considering the general case in which a source term is also present, the field ϕ will be the Green's function (scalar propagator) integrated over the source. This is the same idea as in the amplitudes double copy.

3.2 Relation to self-dual sectors

One way to extend this idea to particle scattering, is to try to apply a similar ansatz in momentum space, which corresponds to the vector k_μ becoming a differential operator in momentum space $k_\mu \to \hat{k}_\mu$. To this end, let us suppose we can write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa \hat{k}_\mu \hat{k}_\nu(\phi), \tag{3.18}$$

where $\hat{k}_\mu$ is a linear differential operator. Because we want $g_{\mu\nu}$ to be a metric, it must be symmetric, so we assume the operator $\hat{k}$ commutes with itself i.e. $[\hat{k}_\mu, \hat{k}_\nu] = 0$. We also restrict our attention to double copies with no dilaton field, so we want $h_{\mu\nu}$ to be traceless. This implies $\eta^{\mu\nu}\hat{k}_\mu \hat{k}_\nu(\phi) = 0$. We will go beyond and require $\hat{k}_\mu(\psi)\eta^{\mu\nu}\hat{k}_\nu(\phi) = 0$. Using this, it is possible to show that the inverse metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa \hat{k}_\mu \hat{k}_\nu(\phi). \tag{3.19}$$

The solutions that underlie the double copy in the self-dual sector are precisely of this form. Considering the light cone coordinates of eq. (2.8), we can express the relevant linear operator $\hat{k}^\mu$ for the self-dual theory as

$$\hat{k}_u = 0, \qquad \hat{k}_v = \frac{1}{4}\partial_w, \qquad \hat{k}_w = 0, \qquad \hat{k}_{\bar{w}} = \frac{1}{4}\partial_u \tag{3.20}$$

Note that $\hat{k} \cdot \partial \equiv 0$, consistent with the geodetic condition. The Christoffel symbols (analogous to eq. (1.73)) are

$$\Gamma^\rho_{\mu\nu} = \frac{\kappa}{2}(\partial_\mu \hat{k}^\rho \hat{k}_\nu \phi + \partial_\nu \hat{k}^\rho \hat{k}_\mu \phi - \partial_\rho \hat{k}_\mu \hat{k}_\nu \phi + \kappa(\hat{k}^\rho \hat{k}^\sigma \phi)(\partial_\sigma \hat{k}_\mu \hat{k}_\nu \phi)). \tag{3.21}$$

Using this expression we can compute the Ricci tensor (analogous to eq. (1.72) and we find the vacuum Einstein equation to be

$$R_{\mu\nu} = \frac{\kappa}{2}\left[-\hat{k}_\mu \hat{k}_\nu \partial^2 \phi + \kappa(\hat{k}_\mu \hat{k}_\nu \partial_\rho \partial_\sigma \phi)(\hat{k}^\rho \hat{k}^\sigma \phi) - \kappa(\hat{k}_\mu \hat{k}_\rho \partial^\sigma \phi)(\hat{k}_\nu \hat{k}_\sigma \partial^\rho \phi)\right] = 0 \tag{3.22}$$

Note that unlike the usual Kerr-Schild Ricci tensor eq. (3.7), this is no longer linear in the graviton $h_{\mu\nu}$ (nor in ϕ).

It is then possible to show that Einstein vacuum equation (3.22) is equivalent to the single scalar equation

$$\partial^2\phi - \frac{\kappa}{2}(\hat{k}^\mu\hat{k}^\nu\phi)(\partial_\mu\partial_\nu\phi) = 0. \tag{3.23}$$

Considering the explicit form of the linear operator $\hat{k}_\mu$ (given in eq. (3.20)), one finds that this is none other than the Plebanski equation for self-dual gravity eq. (2.19) which, as we discussed before, provides an example of a classical double copy.

The next step is, of course, to apply an analogous procedure to the gauge field

$$A^a_\mu = \hat{k}_\mu\phi^a, \tag{3.24}$$

which has been defined in analogy with the vector field eq. (3.9). Here, ϕ^a are Lie-algebra-valued scalars, and the linear differential operator $\hat{k}^\mu$ is that of eq. (3.20). The Yang-Mills equation for the field (3.24) is simply

$$\hat{k}_\nu\partial^2\phi^a + 2gf^{abc}(\hat{k}^\mu\phi^b)(\hat{k}_\nu\phi_\mu\phi^c) = 0. \tag{3.25}$$

Multiplying by the $SU(N)$ generator T^a and expanding using eq. (3.20) we get the equation

$$\hat{k}^\nu\left(\partial^2\Phi - ig[\partial_w\Phi, \partial_u\Phi]\right) = 0, \tag{3.26}$$

where $\Phi = \phi^a T^a$. This equation is equivalent to the standard self-dual Yang-Mills equation (2.11). We therefore see the self-dual double copy from section 2 arise from a momentum space Kerr-Schild description.

3.3 Stationary Kerr-Schild examples

We saw before that there is a class of gravitational Kerr-Schild solutions which map to solutions of the abelian Yang-Mills equation upon a single copy procedure. We can sum this up as follows. Let eq. (3.18) be a stationary solution of the Einstein equation with $k^0 = 1$. Then the gauge vector field

$$A^a_\mu = c^a\phi k^\mu \tag{3.27}$$

is a solution of the Yang-Mills equations, for an arbitrary choice of constants c^a (since this ansatz linearises the Yang-Mills equations). This constitutes a large general class of solutions that can be identified between gauge and gravity theories. We then refer to the gauge theory solution as a single copy (or "square root") of the gravity solution. In the following parts we will analyse some examples of these, as well as a couple of extensions.

Let us begin with the simplest one.

3.3.1 The Schwarzschild black hole

The Schwarzschild black hole is the most general spherically symmetric static solution of the vacuum Einstein equations. The usual form in which we encounter this solution is given by

$$ds^2 = -f(r)dt'^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{3.28}$$

where $f(r) = 1 - \psi(r)$. The scalar function $\psi(r)$ is given by $\psi(r) = r_s/r$ and the so-called Schwarschild radius r_s, which defines the position of the horizon, is given by $r_s = 2GM$, where G is Newton's constant. We may now use a transformation to cast the solution in Kerr-Schild form. The change of basis we are looking for is simply

$$t' = t - r_s \log\left(\frac{r}{r_s} - 1\right), \tag{3.29}$$

which yields the expression

$$ds^2 = -\left(1 - \psi(r)\right)dt^2 + 2\psi(r)dtdr + \left(1 + \psi(r)\right)dr^2 + r^2 d\Omega^2. \tag{3.30}$$

We may now use this to extract the metric tensor in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2GM}{r}k_\mu k_\nu, \tag{3.31}$$

where we have used the explicit form of $\psi(r)$, and the four-vector k^μ takes the form

$$k^\mu = (1, \hat{r}), \tag{3.32}$$

in spherical coordinates (t, r, θ, φ). Note that, due to the use of spherical coordinates, we need to consider covariant differentiation, even if we remain in a flat metric. One then finds that consistent with eq. (3.2), the graviton for the exterior Schwarzschild solution is given by

$$h_{\mu\nu} = \frac{\kappa}{2}\phi k_\mu k_\nu, \qquad \phi = \frac{M}{4\pi r}. \tag{3.33}$$

The field ϕ is exactly what we would expect it to be based on a weak field limit. The single copy of such graviton is given by

$$A^\mu = \frac{g c_a T^a}{4\pi r}k^\mu. \tag{3.34}$$

To obtain this we take the replacements

$$\frac{\kappa}{2} \to g, \qquad M \to c_a T^a, \qquad k_\mu k_\nu \to k_\mu \qquad \frac{1}{4\pi r} \to \frac{1}{4\pi r} \tag{3.35}$$

These replacements make perfect sense from a double copy perspective. The first corresponds to the usual BCJ identification of the coupling constants in the theories. The second one replaces a charge in the gravity theory (a mass) with a corresponding colour charge. The final replacement corresponds to a scalar propagator that remains unchanged on the gravity side.

Physical interpretation of the single-copy

To fully appreciate the power of this procedure, we need to physically interpret the solution we obtained. We exploit here the freedom to perform a gauge transformation to our vector A^a_μ. Since we are working in an abelianised theory, this transformation amounts to adding the derivative of a scalar function

$$A^a_\mu \to A^a_\mu + \partial_\mu \chi^a(x). \tag{3.36}$$

We choose the scalar function in such a way that we get rid of the spatial components of the vector. The trick is done by the function

$$\chi^a = -\frac{g c_a}{4\pi} \log\left(\frac{r}{r_0}\right). \tag{3.37}$$

Note that the constant r_0 is insubstantial, since we are taking its derivative. However, we include it as an arbitrary length scale to make the argument of the logarithm dimensionless. In this new gauge, the vector field is given by

$$A_\mu = \left(\frac{g c_a T^a}{4\pi r}, 0, 0, 0\right) \tag{3.38}$$

This is nothing but a Coulomb solution for a charge consisting of a superposition of static colour located at the origin. This is consistent with the notion of the most general spherically-symmetric solution in electromagnetism being Coulomb plus a radiation field. Since we are working in a stationary limit, the radiation part cannot exist, so we would indeed expect a Coulomb-like solution.

Another tool we have to analyse our double (or single copy) process, is the inspection of the sources that produce the fields we are identifying. This will actually become much more important when we consider non-stationary solutions, but to learn about it we start applying it to the task at hand. It is well known that a Schwarzschild solution is sourced by a point-like mass M via the energy momentum tensor

$$T^{\mu\nu} = M u^\mu u^\nu \delta^3(\bar{x}). \tag{3.39}$$

The time-like vector

$$u^\mu = (1, 0, 0, 0), \tag{3.40}$$

is a common feature when working with point particles. On the other hand, substituting the single-copy vector field into (abelian) Maxwell equation, one finds[2]

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{3.41}$$

where

$$j^\nu = -g(c_a T^a) u^\nu \delta^3(\bar{x}). \tag{3.42}$$

We can see that the single copy, applied to the fields, makes natural an identification of the required sources in gravity and gauge theory. This correspondence of the energy momentum tensor and the current will play a central role in chapter 6, where we study radiation solutions. For now, let us recall that an important feature of the BCJ double copy is being valid in arbitrary dimensions. We want to investigate if such a feature is shared by this construction. To do this, we look at generalisations of the Kerr-Schild solutions living in a higher dimensional space.

Higher-dimensional generalization

The d-dimensional generalization of the Schwarzchild black hole was first found by Tangherlini, and in Kerr-Schild form, the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{\mu}{r^{d-3}} k_\mu k_\nu, \tag{3.43}$$

where k^μ is a simple generalisation of the lower dimensional vector field eq. (3.32) and the parameter μ is related to the mass M via

$$M = \frac{\Omega_{d-2}}{8\pi G}\mu, \qquad \Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}. \tag{3.44}$$

[2]We started our treatment considering vacuum Einstein and Maxwell equations. However, given that the sources include delta functions, we may consider it a vacuum "almost everywhere". If you are to discuss with someone a bit more pedantic, you may argue that the sources being zero-measure objects, our analysis should hold.

Here, Ω_{d-2} is the area of a unit $(d-2)$-sphere. The Newtonian potential is given by

$$\phi = \frac{4\pi}{\Omega_{d-2}} \frac{GM}{r}. \tag{3.45}$$

Taking the single copy one obtains the vector field

$$A^{\mu} = \frac{gT^a}{\Omega_{d-2} r^{d-3}} k^{\mu}, \tag{3.46}$$

which solves the d-dimensional Maxwell equations. It is important having found that the corresponding solutions for higher dimension satisfy the proposed process of single copy, since the amplitudes double copy holds in principle for any dimension. However, this doesn't add much insight to our analysis. We will next study a much less trivial example.

3.3.2 The Kerr black hole

In 1963 (after several attempts that had among other consequences the development of the Kerr-Schild ansatz [172]), Kerr found a solution to describe an uncharged, rotating black hole. In Kerr-Schild coordinates, the graviton field is

$$h_{\mu\nu} = \frac{\kappa}{2} \phi(r) k_{\mu} k_{\nu}, \tag{3.47}$$

where the scalar function is given by

$$\phi(r) = \frac{M}{4\pi} \frac{r^3}{r^4 + a^2 z^2}, \tag{3.48}$$

while the Kerr-Schild vector takes the form

$$k_{\mu} = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right), \tag{3.49}$$

in cartesian coordinates (t, x, y, z). In this case r is not simply the modulus of the vector (x, y, z). It is instead defined implicitly via the equation

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1, \tag{3.50}$$

except for the region $x^2 + y^2 \leq a^2,\ z = 0$ (i.e. a disc of radius a about the origin in the (x, y) plane), where $r = 0$. We may immediately note that the limit with zero angular-momentum $a \to 0$ reproduces a Schwarzschild vector. In such limit, we could identify the scalar function from eq. (3.48) with that of eq. (3.33), as well as the vectors in eqs. (3.49) and (3.32), by noticing that we may write a radial vector as $\hat{r} = r^{-1}(x, y, z)$, for $r^2 = x^2 + y^2 + z^2$. Actually, although the solution was given in this Cartesian Kerr-Schild system in the original paper [173], this set of coordinates will be inconvenient for our purposes. Everything will look more natural, though, after we define a suitable coordinate system.

In order to perform computations in the most natural way, we introduce here the spheroidal coordinate system (t, r, θ, ϕ). This is related to Cartesian coordinates via

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\phi, \qquad y = \sqrt{r^2 + a^2}\sin\theta\sin\phi, \qquad z = r\cos\theta, \tag{3.51}$$

where r, θ and ϕ play the role of a radial, polar angular and azimuthal angular coordinate respectively. Surfaces of constant $r = R$ are ellipsoids, such that $R \to 0$ converges to the disk of radius a in the (x, y)-plane. We may use eq. (3.51) to construct a change of basis matrix. We can then express the Minkowski metric as

$$ds^2 = -dt^2 + \frac{\rho^2}{a^2 + r^2}dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\varphi^2, \tag{3.52}$$

where we have used the definition

$$\rho^2 \equiv r^2 + a^2\cos^2\theta. \tag{3.53}$$

We can also express the vector k_μ and the scalar function $\phi(r)$ as

$$k_\mu dx^\mu = dt + \frac{\rho^2}{a^2 + r^2}dr - a\sin^2\theta d\varphi, \qquad \phi(r) = \frac{M}{4\pi}\frac{r}{\rho^2}. \tag{3.54}$$

We may now, follow the Kerr-Schild single copy procedure as we did for the Schwarzschild black hole, and construct the gauge field

$$A^a_\mu = \frac{g}{4\pi}\phi(r)c^a k_\mu. \tag{3.55}$$

It is not difficult to verify that this vector field is a solution to the abelian Maxwell equations in the vacuum region described above. The novelty with respect to the Coulomb solution (single copy of the Schwarzschild black hole) is that the rotation associated with the Kerr solution, introduces a magnetic component to the Maxwell field eq. (3.55). We interpret the sources in the following section.

Physical interpretation of the single-copy

Just as in the Schwarzschild black hole case, we may interpret the gauge and gravity solutions further by determining the sources that create them (unlike the last example, the identification will not be as transparent and some subtleties will arise). We need to consider the minimal source that will generate the Kerr metric. This actually is not a trivial problem, but it was showed by Israel [174] that the required source turns out to be a disk whose mass distribution exhibits a ring singularity at $x^2 + y^2 = a^2$ (which generates the well known curvature singularity of the metric there). To find the corresponding source, we used an approach different to that of Israel. Based on the fact that the Ricci tensor can be written as a divergence (c.f. eq. (3.7)), we can compute the energy-momentum tensor by integrating the flux of the tensor field through a Gaussian surface (this is simply using the divergence theorem). The reason we bother doing this is that we later use an analogous computation to find the current in the single copy case.

In the spheroidal coordinates introduced above, the energy momentum tensor we found for the Kerr Metric is

$$T^{\mu\nu} = \sigma(w^\mu w^\nu + \zeta^\mu \zeta^\nu), \qquad \sigma = -\frac{M}{8\pi a^2 \cos\theta}\delta(z)\Theta(a-\rho) \tag{3.56}$$

where we have introduced the radial and space-like 4-vectors (in the spheroidal coordinate system (t, r, θ, ϕ))

$$w^\mu = \tan\theta(1, 0, 1/(a\sin^2\theta), 0), \qquad \zeta^\mu = (0, 1/(a\cos\theta), 0, 0), \tag{3.57}$$

and also we defined $\rho = a\sin\theta$. This energy momentum tensor has the form of a negative proper surface density (given by σ). This surface density is rotating about the z-axis with superluminal velocity, as may be seen from the vector w^μ, which has a spatial component in the polar angle direction and no component in the azimutal angle direction. There is also a balancing term which is a radial pressure (we read this from ζ^μ being radial).

We now turn to the problem of interpreting the source that creates the vector field which corresponds to the single copy of the Kerr solution. Substituting the gauge field into the abelian Maxwell equations, one finds a source current

$$j^\mu = -\delta(z)\Theta(a-\rho)\frac{1}{a^2\cos\theta}(\sec^2\theta, 0, (\sec^2\theta)/a, 0) \tag{3.58}$$

where we have taken the replacements described in eq. (3.35). In order to write things in a more convenient way, we introduce the vector

$$\xi^\mu = (1, 0, a^{-1}, 0). \tag{3.59}$$

Using this, the current may be cast into the form

$$j^\mu = q\xi^\mu, \qquad q = -\delta(z)\Theta(a-\rho)\frac{g c_a T^a}{4\pi a^2}\sec^3\theta \tag{3.60}$$

Note that this has the form of a colour charge (as given in eq. (3.60)) rotating about the z-axis[3]. Unlike the Schwarzschild case, it is not obvious that the (Kerr black hole sourcing) energy momentum tensor eq. (3.56) is a double copy of the current eq. (3.58), which corresponds to the single copy vector field. However, we can use the vector ξ to write the tensor eq. (3.56) as

$$T^{\mu\nu} = \delta(z)\Theta(a-\rho)\left(-\frac{M\sec^3\theta}{8\pi a^2}\right)\left[\xi^\mu\xi^\nu - \cos^2\theta\tilde{\eta}^{\mu\nu}\right] \tag{3.61}$$

where

$$\tilde{\eta}^{\mu\nu} = \mathrm{diag}(-1,1,1,0). \tag{3.62}$$

The first term indeed corresponds to a double copy of the current eq. (3.60). The additional term acts as a pressure needed to stabilize the system, in order to maintain the stationary behaviour. This seemingly unnatural property has been further studied in [1].

Higher-dimensional generalization

The higher-dimensional extension of the Kerr black hole is the so-called Myers-Perry black hole [175]. A major difference is that, in d spacetime dimensions, there are $(d-1)/2$ independent rotation planes if d is odd, and $(d-2)/2$ if d is even; this is the dimension of the Cartan subgroup of $SO(d-1)$. Multiple angular momenta are allowed, one per rotation plane, which makes the problem challenging. It is possible to write such solutions in Kerr-Schild form, where the scalar field is given by:

$$\phi(r) = \begin{cases} \frac{\mu r^2}{\Pi F}, & \text{if d is odd.} \\ \frac{\mu r}{\Pi F}, & \text{if d is even.} \end{cases} \tag{3.63}$$

Furthermore, the Kerr-Schild vector is

$$k_\mu dx^\mu = \begin{cases} dt + \sum_{i=1}^{(d-1)/2} \frac{r(x^i dx^i + y^i dy^i) + a_i(x^i dy^i - y^i dx^i)}{r^2+a_i^2} & \text{if d is odd,} \\ dt + \sum_{i=1}^{(d-2)/2} \frac{r(x^i dx^i + y^i dy^i) + a_i(x^i dy^i - y^i dx^i)}{r^2+a_i^2} + \frac{zdz}{r} & \text{if d is even.} \end{cases} \tag{3.64}$$

For each rotation plane, there is a rotation parameter a_i and a pair of coordinates (x^i, y^i). We have used the functions

$$\Pi = \prod_i^{(d-2)/2} (r^2 + a_i^2), \qquad F = 1 - \sum_{i=1}^{(d-1)/2} \frac{a_i^2({x^i}^2 + {y^i}^2)}{(r^2+a_i^2)^2}. \tag{3.65}$$

[3] Interestingly, although we have an accelerated charge, this system is rather static since the energy flows around the disk, instead of going out to infinity. This can be understood by a direct computation of the Poynting which circulates around the disk.

Finally, the radial variable r is defined via

$$\sum_{i=1}^{(d-1)/2} \frac{a_i^2({x^i}^2+{y^i}^2)}{(r^2+a_i^2)^2} = 0 \text{ if } d \text{ is odd}, \qquad \frac{z^2}{r^2} + \sum_{i=1}^{(d-2)/2} \frac{a_i^2({x^i}^2+{y^i}^2)}{(r^2+a_i^2)^2} = 0 \text{ if d is even.} \tag{3.66}$$

The Myers-Perry black holes provide a straightforward extension of our discussion on the Kerr black hole. They allow for solutions to the Maxwell equation in higher dimensions based on the Kerr-Schild ansatz, $A_\mu = \phi k_\mu$.

3.4 Time dependent examples

Having analysed two examples of stationary solutions with a well defined double copy, we will now briefly discuss a couple of explicitly time-dependent solutions, and analyse the significance of their single copy.

3.4.1 Plane waves

Plane wave (pp-waves) solutions are arguably the simplest time-dependent solutions in either gauge or gravity theories [176]. They are Kerr-Schild solutions, so they can be written as in eq. (3.2). Using light-cone coordinates $x^\mu = (u, v, x^i)$, with $i = 1 \ldots d-2$, we can express a pp-wave using

$$k_\mu dx^\mu = du = dz - dt, \qquad \phi = \phi(u, x^i). \tag{3.67}$$

Then, the Einstein equations are simply

$$\partial_i \partial^i \phi = 0. \tag{3.68}$$

Non-abelian plane wave solutions also have the form [177]

$$A^a_\mu = k_\mu \phi^a(u, x^i), \tag{3.69}$$

where ϕ^a fulfils the propagator equation (3.17). As in previous cases, the Kerr-Schild language makes the double copy explicit.

3.4.2 Shockwave solutions

The shockwave solution corresponds to an infinitely boosted particle, whose field (in both gauge and gravity) is Lorentz contracted so that it lies in a flat plane transverse to the particle direction. In gravity, shockwaves are described by the Aichelburg-Sexl metric. In four dimensions, the metric takes the pp-wave form from eq. (3.68) with

$$\phi(u, x^i) = C\delta(u) \log |\vec{x}|. \tag{3.70}$$

In their original paper, they showed explicitly how one may obtain the shockwave metric by a coordinate transformation, namely an infinite boost, of the Schwarzschild black hole.

The perturbative relationship between the shockwave solutions in QCD and gravity was recently discussed extensively in [148], which used Feynman diagram arguments in the Regge limit (corresponding to the scattering of two highly boosted particles) to perturbatively construct the shockwave solution in both Yang-Mills theory and gravity, making clear the double copy relationship. Note that this is entirely consistent with the results found for a Schwarzschild metric, which is not unexpected, since the shockwave itself should reduce to that solution in a non-boosted limit.

Closing remarks

In this chapter, we have reviewed the relation between a number of classical gauge and gravity solutions that can be related by a double copy procedure. In particular, on the gravity side, we considered Kerr-Schild metrics and an associated ansatz for the gauge field. We found that for an infinite family of Kerr-Schild solutions, namely those with the graviton (and hence the corresponding gauge field) stationary, the gauge field obtained by the (Kerr-Schild) single copy ansatz is a solution of the linearised Yang-Mills (effectively Maxwell) equations. The sources needed to generate these solutions also relate in a double-copy manner, involving the replacement of colour charge by mass, as was explicitly shown in the examples of the Schwarzschild and Kerr black holes.

Although this result is remarkable on its own the Kerr-Schild ansatz described in eq. (3.2), along with the condition of being stationary may be, in general, too restrictive to capture the full meaning of this double copy. In the following chapters we will study extensions to the Kerr-Schild ansatz that will allow us to study some physical systems with neat physical interpretations.

Chapter 4

Multiple Kerr-Schild solutions

Having understood the double copy process applied to stationary Kerr-Schild solutions, we now consider the classical single copy of the Taub-NUT metric. Since the approach of last section was successful, the first question to ask is if it is possible to cast a Taub-NUT solution in Kerr-Schild form. A quick[1] browse through the literature, leads us to ref. [178], where a Kerr-Taub-NUT-de Sitter solution in higher (than four) dimensions is considered. This can be cast in the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa\,(\phi\,k_\mu\,k_\nu + \psi\,l_\mu\,l_\nu)\,.$$

We will not show yet the explicit form of the elements of this solution, but we will note that there are two major novelties in this metric with respect to the kind of solutions approached in the previous chapter, namely, this is cast into a double Kerr-Schild form, i.e. there are now two Kerr-Schild vectors k_μ and l_μ, and they will solve a set of null and geodetic conditions. The second one is that the background metric $\bar{g}_{\mu\nu}$ is actually curved. In this case it is de Sitter space. We will address the first aspect in the current chapter, but most of the curved background treatment will be deferred until the following chapter.

4.1 Double Kerr-Schild ansatz

Let us explore double Kerr-Schild metrics. This is, there are two geodetic null congruences over some background. For the sake of generality, we consider a non-flat background, so we turn now to metrics of the form

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \\ &= \bar{g}_{\mu\nu} + \kappa\,(\phi\,k_\mu\,k_\nu + \psi\,l_\mu\,l_\nu)\,, \end{aligned} \tag{4.1}$$

[1] Google aided.

where the vectors k_μ and l_μ are two linearly independent, mutually orthogonal affinely parametrised null geodesic congruences. The null conditions are the same as in the regular Kerr-Schild form,

$$k_\mu \bar{g}^{\mu\nu} k_\nu = 0 = l_\mu \bar{g}^{\mu\nu} l_\nu, \tag{4.2}$$

but we also need to add the mutual orthogonality of the congruences [2]

$$k_\mu \bar{g}^{\mu\nu} l_\nu = 0. \tag{4.3}$$

The geodetic nature of the vectors is codified in

$$k^\mu D_\mu k_\nu = 0 = l^\mu D_\mu l_\nu,$$

where D^μ is the covariant derivative compatible with the background metric $\bar{g}_{\mu\nu}$. The first thing we notice is that, because k_μ and l_μ satisfy eqs. (4.2)-(4.3), we'll have the usual form for the inverse metric

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \kappa h^{\mu\nu}. \tag{4.4}$$

In a metric of single Kerr-Schild form eq. (4.1), the Ricci tensor of the full metric, is shown in [49] to be given by

$$R^\mu{}_\nu = \bar{R}^\mu{}_\nu - h^\mu{}_\rho \bar{R}^\rho{}_\nu + \frac{1}{2} D_\rho \left(D_\nu h^{\mu\rho} + D^\mu h^\rho{}_\nu - D^\rho h^\mu{}_\nu \right).$$

We would like to obtain this same (or some similar) result for the double Kerr-Schild form. However, because of the weaker conditions satisfied by k_μ and l_μ (e.g. $(k \cdot D) l_\mu \neq 0$), some of the cancellations that led to eq. (4.5) won't happen, so we get the former expression, corrected by a non-linear factor, this is

$$R^\mu{}_\nu = \bar{R}^\mu{}_\nu - h^\mu{}_\rho \bar{R}^\rho{}_\nu + \frac{1}{2} D_\rho \left(D_\nu h^{\mu\rho} + D^\mu h^\rho{}_\nu - D^\rho h^\mu{}_\nu \right) + R_{\text{nl}}{}^\mu{}_\nu \,, \tag{4.5}$$

[2] The conditions in eqs. (4.2) and (4.3) are inconsistent with real components and a Lorentzian signature. However, throughout this (and later) chapters, we'll find examples with either $(2,2)$ signatures, or complex vectors.

where the non-linear terms of the Ricci tensor are given by

$$R_{\rm nl}{}^{\mu}{}_{\nu} = -\frac{\kappa^2}{2}\bigg[\frac{1}{2}D^{\mu}h(k)^{\rho}{}_{\delta}D_{\nu}h(l)^{\delta}{}_{\rho} + h(l)^{\mu\delta}D_{\rho}D_{\nu}h(k)^{\rho}{}_{\delta} \\ + D_{\rho}\Big(h(l)^{\rho\delta}D_{\delta}h(k)^{\mu}{}_{\nu} + 2h(l)^{\rho\delta}D_{(\nu}h(k)^{\mu)}{}_{\delta} - 2h(l)^{\mu\delta}D^{[\rho}h(k)_{\delta]\nu}\Big)\bigg] + (k \leftrightarrow l), \tag{4.6}$$

and we have defined the shorthand notation

$$h(k)_{\mu\nu} = \phi k_{\mu}k_{\nu}, \quad h(l)_{\mu\nu} = \psi l_{\mu}l_{\nu}. \tag{4.7}$$

Our first approach coming to this point was to investigate conditions for the Kerr-Schild vectors k_μ and l_μ which led to $R_{\rm nl}{}^{\mu}{}_{\nu}$ to vanish, so we proposed as candidates the relations

$$(k \cdot D)l_{\mu} = 0 = (l \cdot D)k_{\mu}, \tag{4.8}$$

along with

$$k^{\mu}D_{\nu}l_{\mu} = 0 = l^{\mu}D_{\nu}k_{\mu}. \tag{4.9}$$

However, we found such conditions to be neither sufficient nor necessary, as we'll see in the next section. Some efforts have been made trying to determine conditions to simplify eq. (4.6), none successful thus far.

4.2 The Taub-NUT solution

The Taub-NUT metric was first derived in ref. [179] by Taub, and was later extended to a more general manifold by Newman, Unti and Tamburino in ref. [180]. It is a stationary, axisymmetric vacuum solution, but unlike Schwarzschild, it is not asymptotically flat. It can be sourced by a pointlike object at the origin, which besides its mass, has an extra parameter that is conventionally referred to as NUT charge. The latter is associated with the lack of spherical symmetry and asymptotic flatness. It has been shown that the NUT charge is related to a monopole-like behaviour at spatial infinity (see e.g. [181] for a review). We will see in this section that the Taub-NUT solution provides an interesting example of a double Kerr-Schild double copy.

In a series of papers written during the 1970's, of which the most important for our purposes is ref. [182], Plebanski studied the most general solutions of Einstein-Maxwell equations. Following his study, a formulation of the Taub-NUT-Kerr-de Sitter metric can be given in the form

$$ds^2 = \frac{q^2 - p^2}{\Delta_p} dp^2 + \frac{q^2 - p^2}{\Delta_q} dq^2 - \frac{\Delta_p}{q^2 - p^2}(d\tau + q^2 d\sigma)^2 - \frac{\Delta_q}{q^2 - p^2}(d\tau + p^2 d\sigma)^2 \quad (4.10)$$

where

$$\Delta_p = \gamma - \epsilon p^2 + \lambda p^4 - 2Np, \qquad \Delta_q = -\gamma + \epsilon q^2 - \lambda q^4 - 2Mq. \quad (4.11)$$

Here the parameter M represents the mass of the solution, and N the NUT charge, while ϵ is a constant, and γ is related to the angular momentum. This is, actually, an analytic continuation (in practice $p \to ip$) removed from the original form showed in Plebanski's paper, so it has $(2,2)$ signature. The ranges of the coordinates are given by

$$p \in [-a, a], \quad q \in [0, \infty], \quad \sigma \in [0, 2\pi/a], \quad \tau \in [-\infty, \infty]. \quad (4.12)$$

The metric is a solution to the Einstein equation with non-zero cosmological constant λ. It was shown in ref. [178] that by means of the change of coordinates

$$d\tilde{\tau} = d\tau + \frac{p^2 dp}{\Delta_p} - \frac{q^2 dq}{\Delta_q}, \qquad d\tilde{\sigma} = d\sigma - \frac{dp}{\Delta_p} + \frac{dq}{\Delta_q}, \quad (4.13)$$

this metric exhibits a double Kerr-Schild form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa \left(\phi\, k_\mu\, k_\nu + \psi\, l_\mu\, l_\nu\right).$$

The explicit form of the background line element is

$$d\bar{s}^2 = -\frac{1}{q^2 - p^2}\left[\bar{\Delta}_p(d\tilde{\tau} + q^2 d\tilde{\sigma})^2 + \bar{\Delta}_q(d\tilde{\tau} + p^2 d\tilde{\sigma})^2\right] - 2(d\tilde{\tau} + q^2 d\tilde{\sigma})dp - 2(d\tilde{\tau} + p^2 d\sigma)dq, \quad (4.14)$$

where

$$\bar{\Delta}_p = \gamma - \epsilon p^2 + \lambda p^4, \qquad \bar{\Delta}_q = -\gamma + \epsilon q^2 - \lambda q^4. \quad (4.15)$$

The Kerr-Schild vectors are given in the $(\tilde{\tau}, \tilde{\sigma}, p, q)$ coordinate system (which has (2,2) signature) by

$$k_\mu = (1, q^2, 0, 0), \qquad l_\mu = (1, p^2, 0, 0), \quad (4.16)$$

and satisfy the conditions stated in eqs. (4.2)-(4.4). The accompanying scalar functions are given by

$$\phi = \frac{2Np}{q^2 - p^2}, \quad \psi = \frac{2Mq}{q^2 - p^2}. \quad (4.17)$$

It is remarkable that the non-linear terms in the Ricci tensor, which are written in eq.

(4.6), vanish. This means that we again have linear Einstein equations, so this system is a natural candidate to have a single copy according to a Kerr-Schild ansatz, that satisfies Maxwell-like equations.

Let us now obtain and interpret the single copy of this solution, where we will eventually turn to the case of vanishing angular momentum ($\gamma = 0$), which leads to a pointlike source. For the double Kerr-Schild case, the natural generalisation to the Kerr-Schild ansatz of eq. (3.27) is to construct the gauge field

$$A^a_\mu = c^a \left(\phi k_\mu + \psi l_\mu\right). \tag{4.18}$$

That is, the double copy of this solution proceeds term-by-term, analogously to how the BCJ double copy for amplitudes is applied separately to terms involving different scalar propagators. We have verified that the gauge field of eq. (4.18) satisfies the Yang-Mills equations (which linearise),

$$D^\mu F^a_{\mu\nu} = 0, \quad F^a_{\mu\nu} = D_\mu A^a_\nu - D_\nu A^a_\mu. \tag{4.19}$$

Note that the Yang-Mills equations are satisfied even for a non-Minkowski background as we discussed in the previous section. We will also make the replacements

$$\frac{M\kappa}{2} \to (c_a T^a) g_s, \quad \frac{N\kappa}{2} \to (c_a T^a) \tilde{g}_s. \tag{4.20}$$

These are analogous to the coupling constant replacement appearing in eq. (3.35), and will play an important role that we explain in the next section.

4.2.1 Physical interpretation of the single copy

For simplicity, we restrict ourselves now to the flat background case $\lambda = 0$, for which $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. We have taken the single copy in Plebanski coordinates[3] because it is in this system that the double Kerr-Schild form is manifest. However, this unusual set of coordinates make difficult any physical interpretation of the gauge theory solution we obtained as a single copy. We will then transform to a more suitable choice of coordinates. We will do this in two stages. First, following refs. [178, 182], we transform to spheroidal coordinates using the transformation

$$\tau = t + a\varphi, \qquad \sigma = \frac{\varphi}{a}, \qquad q = r, \qquad p = a\cos\theta, \tag{4.21}$$

[3]We adopted this name for the set of coordinates where we worked. However, Plebanski himself proposed calling them Boyer coordinates, in memory of his late friend.

where the coordinates τ and σ were defined earlier as

$$d\tilde{\tau} = d\tau + \frac{p^2 dp}{\Delta_p} - \frac{q^2 dq}{\Delta_q}, \quad d\tilde{\sigma} = d\sigma - \frac{dp}{\Delta_p} + \frac{dq}{\Delta_q}. \tag{4.22}$$

In spheroidal coordinates, the Kerr-Schild vectors take the explicit form

$$k_\mu dx^\mu = dt + \frac{\rho^2}{a^2 + r^2} dr - a \sin^2\theta d\varphi, \qquad l_\mu dx^\mu = dt + \frac{i\rho^2 \csc\theta}{a} d\theta - \frac{a^2 + r^2}{a} d\varphi. \tag{4.23}$$

One may note that, as expected, the first vector corresponds to that of the Kerr black hole (c.f. eq. (3.54)). Next, one may take the parameter $a^2 \equiv \gamma$ (which is related to the angular momentum) to zero, so that the spheroidal radius becomes a spherical one. This coordinate transformation is subtle, in that the vector l^μ becomes singular as $a \to 0$. The prefactor ψ entering the gauge field, however, is $\mathcal{O}(a)$, such that gauge field A^a_μ itself is well-defined. In the spherical polar coordinate system (t, r, θ, ϕ), the field strength tensor then becomes

$$F_{\mu\nu} = \frac{(c_a T^a)}{4\pi} \begin{pmatrix} 0 & -\frac{g_s}{r^2} & 0 & 0 \\ \frac{g_s}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\tilde{g}_s \csc\theta}{r^4} \\ 0 & 0 & \frac{\tilde{g}_s \csc\theta}{r^4} & 0 \end{pmatrix}. \tag{4.24}$$

We can use the language of differential forms to write this as

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = -\frac{(c_a T^a)}{8\pi} \left(\frac{g_s}{r^2} dt \wedge dr + \tilde{g}_s \sin\theta d\theta \wedge d\varphi \right), \tag{4.25}$$

where it is now evident that the contributions from the constants g_s and $\tilde{g}_s$ split easily. The first term on the right-hand side of eq. (4.25) gives a pure electric field, corresponding to a Coulomb solution. Thus, the mass in the Taub-NUT metric single copies to a static colour charge, exactly as in the Schwarzschild case of ref. [183]. This must in fact be the case, given that the Taub-NUT metric becomes the Schwarzschild metric as $N \to 0$. This explains our choice of factors in eq. (4.20).

The NUT charge contribution to the field strength tensor is a pure magnetic field, and we can interpret this in more detail by expressing eq. (4.25) as

$$F = -\frac{(c_a T^a)}{8\pi} \left(\frac{g_s}{r^2} dt \wedge dr + \star \frac{\tilde{g}_s}{r^2} dt \wedge dr \right), \tag{4.26}$$

where $\star$ denotes the Hodge dual of a 2-form, say $\omega_{\mu\nu}$

$$\star\omega_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \omega^{\alpha\beta}, \qquad \star^2 = -1. \tag{4.27}$$

Thus, the dual tensor for the NUT-charge term contains a pure electric field corresponding to a point charge of strength $\tilde{g}_s$. It follows that the magnetic field in the original field strength tensor corresponds to a magnetic monopole, where the NUT charge in the gravity theory single copies to the monopole charge in the gauge theory. This is perhaps to be expected, given that the NUT charge in the Taub-NUT metric is known to be associated with monopole-like behaviour, since a requirement of the metric having a periodic time coordinate can be interpreted as the source of the field including a semi-infinite massless source of angular momentum (a Dirac string-like object) [181]. This analogy has now been turned into an exact statement under the classical double copy. We have then chosen the constant $\tilde{g}_s$ in eq. (4.20) to obey the same normalisation as g_s in the (non-dual) field strength tensor.

Note that the transformation from the Plebanski coordinate system to the spherical coordinate system involves a change of signature (from (2,2) to (1,3)), and thus a Wick rotation. In the Plebanski system itself, the two charges M and N appear on an equal footing, as is clear from eqs. (4.14-4.17). In other words, in this signature one cannot tell the difference in the gauge theory between an electric and a (dual) magnetic charge. For the (anti-) self-dual case, the gauge and gravity solutions can be interpreted as instantons (see also [181]). As is well known, consistency of the monopole gauge field leads to the quantisation condition (in the present notation)

$$g_s\tilde{g}_s = \frac{n}{2}, \quad n \in \mathbb{Z}, \tag{4.28}$$

relating the electric and magnetic charges. This has an analogue in the gravity theory, as discussed in ref. [184, 185]. There, recovery of spherical symmetry demands a periodic time coordinate. This corresponds to quantisation of the energy of the dyon, or its mass in the non-relativistic approximation. There is then a quantisation condition relating the mass and NUT charge, which is the equivalent of eq. (4.28) from a double copy perspective.

As in the standard Kerr-Schild case, we may take the zeroth copy, which produces a biadjoint scalar field

$$\Phi^{aa'} = c^a \tilde{c}^{a'} (\phi + \psi) \,. \tag{4.29}$$

Similarly to the results of ref. [183], this is a solution of the linearised biadjoint eq.(1.95). In fact, both ϕ and ψ satisfy that equation separately. They have the interpretation of a scalar propagator integrated over the source charges, and are analagous to the scalar propagators that are not modified when double-copying scattering amplitudes. As has already been mentioned above, another property that links the generalised Kerr-Schild double copy to the corresponding story for amplitudes is that each Kerr-Schild term (involving a different scalar propagator) is copied individually, with no mixing between these terms on the gravity side.

4.2.2 Higher-dimensional generalization

One important feature of the BCJ double copy is that it works on arbitrary spacetime dimensions. This property seems to be present also for the double copy of Kerr-Schild solutions, as evidenced by the Tangherlini and Myers-Perry solutions of the previous chapter. It is an interesting question if this is also possible for multi Kerr-Schild solutions, like Taub-NUT. To investigate this, we consider an extension to our formalism using a higher-dimensional Kerr-Taub-NUT-de Sitter solution. This was obtained in [186], and can be viewed as a generalisation of the metric studied by Plebanski in [182].

This solution is parametrised by the mass, multiple NUT charges and arbitrary orthogonal rotations. It was later shown [187] that it is possible to cast the d-dimensional metric into multi-Kerr-Schild form, where the mass and all of the NUT parameters enter the metric linearly.

To do this, it is first necessary to perform a Wick-rotation of the metric to $([d/2],[(d+1)/2])$ signature, so it admits $[d/2]$ linearly-independent, mutually-orthogonal and affinely parametrised null geodesic congruences. It is possible then to introduce $[d/2]-1$ NUT parameters, from which $(d-5)/2$ are non-trivial in odd dimensions, whilst $(d-2)/2$ are non-trivial in even dimensions. This significant difference is the reason why odd-dimensional and even-dimensional cases are usually treated separately. We will show here the general form for even dimensions, and the explicit form for $d=6$, since this is the simplest non-trivial example, as $d=5$ has no non-trivial NUT parameters.

For $d=2n$, after performing Wick rotations, the Kerr-Taub-NUT-de Sitter metric with (n,n) signature is given by

$$ds^2 = \sum_{j=1}^{n}\left[\frac{dx_j^2}{Q_j} - Q_j\left(\sum_{i=0}^{n-1} A_j^{(i)} d\psi_i\right)^2\right], \tag{4.30}$$

where the various functions that appear are given by

$$\begin{aligned} Q_j &= \frac{X_j}{U_j}, \qquad X_j = \sum_{i=0}^{n} c_i x_j^{2i} + 2b_j x_j, \\ U_j &= {\prod_{k=1}^{n}}'(x_k^2 - x_j^2), \qquad A_j^{(i)} = {\sum_{k_1<k_2<\ldots<k_i}}' x_{k_1}^2 x_{k_2}^2 \ldots x_{k_i}^2. \end{aligned} \tag{4.31}$$

The prime on the summation and product symbols in the definition of U_j and $A_j^{(i)}$ indicate that the value j is omitted on the iteration of the k index. The constants c_i and b_j are arbitrary, except for c_n, which is fixed by the value of the cosmological constant. Then, after performing the coordinate transformation

$$d\hat{\psi}_i = d\psi_i + \sum_{j=1}^{n} \frac{(-x_j^2)^{n-i-1}}{X_j} dx_j, \qquad i = 0, \ldots, n-1, \tag{4.32}$$

we can write the metric (4.30) in the form

$$ds^2 = d\bar{s}^2 - \sum_{j=1}^{n} \phi_{(j)} \left[k_{(j)\mu} dy^\mu\right]^2 . \tag{4.33}$$

This is an n-tuple Kerr-Schild form, since it possesses n linearly independent mutually orthogonal null geodetic congruences. The corresponding vectors and scalars are given by

$$k_{(j)\mu} dy^\mu = \sum_{i=0}^{n-1} A_j^{(i)} d\hat{\psi}_i, \qquad \phi_{(j)} = \frac{2b_j x_j}{U_j}. \tag{4.34}$$

The fiducial metric in eq. (4.33) is given by

$$d\bar{s}^2 = -\sum_{j=1}^{n} \left[\frac{\bar{X}_j}{U_j} \left(\sum_{i=0}^{n-1} A_j^{(i)} d\hat{\psi}_k \right)^2 - 2 \left(\sum_{i=0}^{n-1} A_j^{(k)} d\hat{\psi}_i \right) dx_j \right], \tag{4.35}$$

and the $\bar{X}_j$ function appearing there is defined as

$$\bar{X}_j = \sum_{i=0}^{n} c_k x_j^{2i}. \tag{4.36}$$

It is straightforward to verify that the fiducial metric corresponds to de Sitter. These expressions are rather lengthy, and not particularly enlightening, but we can try to get to understand them by checking one example. As we discussed earlier, the five dimensional case is somewhat trivial, so we explore the six dimensional case in the following section.

Example: six-dimensional metric

For the case $d = 6$ we may use coordinates $(x_1, x_2, x_3, \psi_0, \psi_1, \psi_2)$ to write the metric (4.33) (with the value $n = 3$) in the form

$$g^{\alpha\beta} = \bar{g}^{\alpha\beta} - \sum_{i=1}^{3} \phi_i k_i^\alpha k_i^\beta, \tag{4.37}$$

where the scalar functions ϕ_i are given by

$$\phi_i = \frac{2b_i x_i}{U_i}, \qquad i = 1, 2, 3, \tag{4.38}$$

while the vectors k_i and the functions U_i take the form

$$\begin{aligned}
k_{1\,\alpha} &= (0,0,0,1,x_2^2+x_3^2,x_2^2x_3^2), \qquad & U_1 &= (x_2^2-x_1^2)(x_3^2-x_1^2),\\
k_{2\,\alpha} &= (0,0,0,1,x_1^2+x_3^2,x_1^2x_3^2), \qquad & U_2 &= (x_1^2-x_2^2)(x_3^2-x_2^2),\\
k_{3\,\alpha} &= (0,0,0,1,x_1^2+x_2^2,x_1^2x_2^2), \qquad & U_3 &= (x_2^2-x_3^2)(x_1^2-x_3^2).
\end{aligned}$$

Using this, we may define the vector field

$$A^{\alpha}_{\text{6d}} = \phi_1 k_1^{\alpha} + \phi_2 k_2^{\alpha} + \phi_3 k_3^{\alpha}, \tag{4.39}$$

which is a term-by-term single copy of the graviton entering the metric (4.37). We have explicitly verified that this metric linearises the Ricci tensor (in the sense of (3.7)), so we expected the vector field to solve linearised Maxwell equations

$$D_{\alpha} F^{\alpha\beta}_{\text{6d}} = 0 \tag{4.40}$$

where D is the covariant derivative compatible with the fiducial metric (4.35), and we have again considered an Abelian version of the field strength tensor

$$F^{\alpha\beta}_{\text{6d}} = D^{\alpha} A^{\beta}_{\text{6d}} - D^{\beta} A^{\alpha}_{\text{6d}}. \tag{4.41}$$

The parameters b_i (for $i = 1, 2, 3$) are identified with the mass, and two non-trivial NUT-charges which, analogously to the $d = 4$ case, we interpret as electric and two classes of magnetic charge in the gauge theory. Following this same procedure, we have verified the validity of this Single Copy up to $d = 8$, though we expect it to hold for all dimensions.

Although it was straightforward to show that a vector field obtained as a term by term single copy of the multiple Kerr-Schild form of eq. (4.37) satisfies Maxwell equations, the physical interpretation of such vector field is not without subtlety. Indeed, when interpreting the result for the field in $d = 4$, we took into account the specific behaviour of the electric and magnetic fields under a Hodge dual transformation, but this does not extend straightforwardly to arbitrary dimensions. Furthermore, when working in Plebanski coordinates, the mass and the NUT charge appear on equal footing in the metric. It is not until after an analytic continuation that that we are able to distinguish the mass from the NUT charge, and we're able to interpret the single copy counterparts as electric and magnetic charge, respectively. In the higher dimensional case, since we have mass and multiple NUT charges, we would be tempted to interpret them as electric and multiple magnetic charges in the gauge theory. However, the identification might be delicate, and deserves further investigation.

Closing remarks

In summary, the results of this section constitute an interesting generalisation of the Kerr-Schild double copies of ref. [183], in that a double Kerr-Schild form is used. It is highly non-trivial that the particular double and multiple Kerr-Schild results for the solutions we considered here linearise the Einstein equations. This allowed us to study the single copy effectively in aMaxwell theory, so we could give a neat physical interpretation of the system as an electromagnetic dyon.

One interesting feature of the double Kerr-Schild form of the Taub-NUT metric is that it can be understood as a single Kerr-Schild metric, by considering a point NUT charge over a Schwarzschild metric (In a symmetric fashion, we could consider also the Kerr-Schild system that consists in putting a point mass over a background perturbed by a NUT charge). This can be generalised to any multiple Kerr-Schild solution, so we can study it as a single Kerr-Schild form over a curved background. We will further develop these, among other ideas in the next chapter.

Chapter 5

Non-flat backgrounds

In the previous chapter, we already hinted at the possibility of obtaining the single copy of a Kerr-Schild gravity solution when the background is not a flat metric. Our interest in this kind of process is twofold. On the one hand, this potentially has astronomical or cosmological applications. On the other hand, recent work by Adamo et. al. [143] has studied an amplitudes double copy process over a curved spacetime. In this chapter we address this issue in a thorough manner, and we will show interesting links with both fronts.

The extension of the Kerr-Schild ansatz to a non-flat background is straightforward. It is reviewed, for example in ref. [49]. We will only go through some particular aspects important for our treatment. Let us start by examining a metric in (non-flat) Kerr-Schild form

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \\ &= \bar{g}_{\mu\nu} + \frac{\kappa^2}{2} k_\mu k_\nu \phi, \end{aligned} \tag{5.1}$$

where $\bar{g}_{\mu\nu}$ (which we sometimes call the fiducial metric throughout this work) is not necessarily Minkowski, ϕ is still a scalar function of the coordinates, and k_μ is null both with respect to the fiducial and the whole metric:

$$k_\mu g^{\mu\nu} k_\nu = 0 = k_\mu \bar{g}^{\mu\nu} k_\nu. \tag{5.2}$$

We will also have the Kerr-Schild vector behave as a geodetic congruence:

$$k^\mu D_\mu k_\nu = 0, \tag{5.3}$$

where D_μ is the covariant derivative associated with the fiducial metric, i.e. $D_\rho(\bar{g}_{\mu\nu}) = 0$. The null property implies that the index of the Kerr-Schild vector k_μ can be raised with either $\bar{g}_{\mu\nu}$ or $g_{\mu\nu}$. A consequence of eq. (5.2) is the fact that the inverse metric takes the simple form

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \frac{\kappa^2}{2} k^\mu k^\nu \phi. \tag{5.4}$$

Substituting the ansatz of eq. (5.1) into the Einstein equations, one finds the (mixed-index) Ricci tensor of the full metric to be given by

$$R^\mu{}_\nu = \bar{R}^\mu{}_\nu - \kappa h^\mu{}_\rho \bar{R}^\rho{}_\nu + \frac{\kappa}{2} D_\rho \left(D_\nu h^{\mu\rho} + D^\mu h^\rho{}_\nu - D^\rho h^\mu{}_\nu \right). \tag{5.5}$$

If we consider the case where the background metric is flat space (Minkowski) i.e. $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, the fiducial Ricci tensor from eq. (5.5) vanishes ($\bar{R}^\mu{}_\nu = 0$), and the (fiducial) covariant derivative reduces to the partial ($D = \partial$) so we recover the result for a flat background. We can summarize this as

$$R^\mu{}_\nu \xrightarrow{\bar{g}_{\mu\nu} \to \eta_{\mu\nu}} \frac{\kappa}{2} \partial_\rho \left(\partial_\nu h^{\mu\rho} + \partial^\mu h^\rho{}_\nu - \partial^\rho h^\mu{}_\nu \right) \tag{5.6}$$

The next step in the flat case was showing that vacuum Einstein equations imply vacuum Maxwell equations for a vector field constructed as

$$A^a_\mu = c^a \phi k_\mu, \tag{5.7}$$

that is, as a single copy of the graviton. In that case, the two assumptions that were used to simplify the analysis were the stationary limit (i.e. no explicit dependence on the time coordinate $\partial_0 = 0$) as well as setting $k^0 = 1$, so the Maxwell equations emerge from Einstein ones (c.f. eqs. (3.10-3.11)).

We would like to have an analogous proof for the non-flat background, but it turns out this is not straightforward. A number of factors account for this difference. First, even if the metric has no explicit dependence on the time coordinate, we may not have the simplification $D_0 = 0$. Also, even if we have $k^0 = 1$ (which doesn't occur naturally in the examples we'll give), there is a difference in the non-flat case, since covariant derivatives behave according to the object they are acting on. Consider, for instance, the derivative $\partial^\mu (k^\alpha k^\nu \phi)$. In the stationary limit, taking the component $\nu \to 0$ we loosely write

$$\partial^\mu (k^\alpha k^0 \phi) = \partial^\mu (k^\alpha \phi),$$

since $k^0 = 1$. However in the non-flat case we'll have

$$\begin{aligned} D_\mu (k^\alpha k^\nu \phi) &= k^\nu D_\mu (k^\alpha \phi) + k^\alpha \phi D_\mu k^\nu \\ &= k^\nu D_\mu (k^\alpha \phi) + k^\alpha \phi (\partial_\mu k^\nu + \bar{\Gamma}^\nu_{\mu\beta} k^\beta). \end{aligned}$$

If we then consider the $\nu = 0$ component we'll have the "relation"

$$D_\mu(k^\alpha k^0 \phi) = D_\mu(k^\alpha \phi) + k^\alpha k^\beta \phi \bar{\Gamma}^0_{\mu\beta}, \tag{5.8}$$

which is rather counter intuitive, and thus renders difficult an approach analogous to that of last section.

However, these factors won't be important for a number of examples that we will study in the following, since cancellations occur, and the field still satisfies the Maxwell-like equation

$$D^\mu F^a_{\mu\nu} = 0. \tag{5.9}$$

In particular, we will turn our attention to black hole solutions living over non-flat backgrounds.

5.1 Black holes over (Anti-)de Sitter background

Schwarzschild (Anti-)de Sitter black hole

The easiest possible example of a black hole over a curved background combines the simplest black hole, i.e. the Schwarzschild metric and the simplest non-flat background, this is, (Anti-)de Sitter which has a constant non-zero curvature. An explicit expression for this metric in four dimensions is known (see, for example ref. [188]), and is already in the Kerr-Schild form of eq. (5.1), where the background metric is given by

$$\bar{g}_{\mu\nu} = \mathrm{diag}\left(-1 + \lambda r^2, \frac{1}{1 - \lambda r^2}, r^2, r^2 \sin\theta\right), \tag{5.10}$$

in a usual spherical coordinate system (t, r, θ, φ). A straightforward computation shows that this background is the (Anti-)de Sitter metric (whose constant curvature is 12λ). Of course, the sign of λ determines if the metric eq. (5.10) corresponds to de Sitter or Anti-de Sitter. Both backgrounds result of physical interest, since they are relevant for cosmological and holography applications, respectively. The difference between them are notorious when looking at the global structure of the spacetime, so this should be taken into account when studying applications of this formalism.

The Kerr-Schild vector k_μ is defined as

$$k_\mu = \left(1, \frac{1}{1 - \lambda r^2}, 0, 0\right), \tag{5.11}$$

while the scalar function is $\phi = \frac{2M}{r}$, same as in the flat background case. It is not difficult to show that the vector eq. (5.11) satisfies the null and geodetic conditions of eqs. (5.2) and (5.3). Note that if we take the limit $\lambda \to 0$ we recover the usual Schwarzschild black hole from section 3.3.1.

The vector gauge field A^a_μ is constructed as a single copy of the graviton, so it is given by

$$A^a_\mu = \phi c^a k_\mu, \tag{5.12}$$

and it solves a Maxwell-like equation over a non-flat background

$$D_\mu F^{\mu\nu}_{\text{non flat}} = 0. \tag{5.13}$$

where, we define the field strength tensor as

$$F^{\mu\nu\,a}_{\text{non flat}} \equiv D^\mu A^{\nu\,a} - D^\nu A^{\mu\,a}. \tag{5.14}$$

Note that the non-abelian term that should appear in the last equation vanishes due to the trivial dependence on the colour vector.

The zeroth copy does not work in a trivial way here, as the equation is

$$D^2\phi \equiv D_\mu D^\mu \phi = \partial_\mu \partial^\mu \phi = -2\lambda\phi. \tag{5.15}$$

We will further analyse this result in terms of a biadjoint field equation coupling to background curvature. However, to draw a parallel with the previous chapter, let us analyse first the example of the rotating black hole over a (Anti-)de Sitter background.

Kerr (Anti-)de Sitter black hole

In the same way that it is possible to generalize the Kerr black hole over a flat background to higher dimensions, one can write the general Kerr-(Anti-)de Sitter metric in arbitrary spacetime dimension [188]. In particular, in four spacetime dimensions, the metric has the Kerr-Schild structure from eq. (5.1) with the background metric

$$d\bar{s}^2 = -\frac{(1-\lambda r^2)\Delta_\theta}{1-\lambda a^2}dt^2 + \frac{\rho^2}{(1-\lambda r^2)(r^2+a^2)}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{(r^2+a^2)\sin^2\theta}{1+\lambda a^2}d\varphi^2. \tag{5.16}$$

The functions entering the metric are defined as

$$\rho^2 \equiv r^2 + a^2\cos\theta, \qquad \Delta_\theta \equiv 1 + \lambda a^2 \cos^2\theta. \tag{5.17}$$

Despite the formidable look of the fiducial metric eq. (5.16), this is nothing but (Anti-)de Sitter space in spheroidal coordinates (as defined in eq. (3.51)). Note that if we take $a \to 0$, this is the metric eq. (5.10). The scalar function is

$$\phi = \frac{2Mr}{\rho^2}. \tag{5.18}$$

The vector k_μ is defined by

$$k_\mu dx^\mu = \frac{\Delta_\theta}{1+\lambda a^2}dt + \frac{\rho^2 dr}{(1-\lambda r^2)(r^2+a^2)} - \frac{a\sin^2\theta d\varphi}{1+\lambda a^2}. \tag{5.19}$$

The vector eq. (5.19) is indeed null with respect to both the fiducial and the full metric, and we have also verified that it satisfies the geodetic condition. If we now construct a gauge field as a single copy of the graviton, this is again a solution to the non-flat Maxwell equation (5.13), and considering the zeroth copy, we get an equation identical to eq. (5.15). Having verified that this procedure works let us focus now on (Anti-)de Sitter space.

5.1.1 Zeroth copy over (Anti-)de Sitter background

There is one very interesting aspect of the zeroth copy over a (Anti-)de Sitter background. For the Schwarzschild (Anti-)de Sitter and Kerr (Anti-)de Sitter metrics, we found that the scalar field of eq. (5.1) satisfies

$$D^2\phi = -2\lambda\phi. \tag{5.20}$$

Actually, should we have considered instead the biadjoint field $\Phi^{aa'} = c^a\tilde{c}^{a'}\phi$, we would have found that it satisfies the equation of motion

$$D^2\Phi^{aa'} = -2\lambda\Phi^{aa'}. \tag{5.21}$$

It is interesting that there appears a new term in the right-hand side, and that it is actually proportional to the curvature scalar. Although we are not sure about the physical meaning of this term, we may note that eq. (5.21) can be effectively obtained from the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(D^\mu\Phi^{aa'})(D_\mu\Phi^{aa'}) - \frac{y}{6}f^{abc}\tilde{f}^{a'b'c'}\Phi^{aa'}\Phi^{bb'}\Phi^{cc'} - \frac{R}{12}\Phi^{aa'}\Phi^{aa'}, \tag{5.22}$$

and this corresponds to a non-minimal coupling of the biadjoint scalar to the gravity background. What is remarkable is that the coefficient of the extra term coincides with a conformally coupled scalar in four spacetime dimensions. This perhaps can be explained as a consequence of classical Yang-Mills theory being conformally invariant in four spacetime dimensions, and that the zeroth copy somehow preserves this invariance in the free scalar theory. We have however found that in higher dimensions, where the Yang-Mills theory is not conformally invariant, the relevant coefficient of the $R\Phi^{aa'}\Phi^{aa'}$ term does not coincide with the conformal coupling.

5.1.2 (Anti-)De Sitter as a Kerr-Schild metric

A different way to approach a double copy over a (Anti-)de Sitter background is to note that its metric can also be cast into the Kerr-Schild form [187]

$$g_{\mathrm{dS}\,\mu\nu} = \eta_{\mu\nu} + \lambda\, r^2 k_\mu k_\nu, \quad k_\mu = (1, \hat{r}). \tag{5.23}$$

We may then construct a gauge field via the Kerr-Schild single copy. This yields

$$A_\mu = \rho\, r^2 k_\mu, \tag{5.24}$$

where we have replaced $\lambda \to \rho$. We can interpret the latter parameter by noting that the electrostatic potential in the Kerr-Schild gauge satisfies

$$\nabla^2 A_0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_0}{\partial r}\right) = 6\rho. \tag{5.25}$$

Thus, ρ plays the role of a uniform charge density, filling all space. This is exactly what one expects from the single copy of the cosmological constant, given that the latter is a uniform energy density. If one chooses the fiducial metric to be Minkowski rather than (Anti-)de Sitter space, the conformal coupling in the biadjoint scalar theory would be absent (due to the vanishing Ricci scalar), but one must then include the uniform charge density explicitly.

5.2 Kerr-Schild solutions in curved space

A motivation to extend our study to non de Sitter backgrounds comes from the recent work in ref. [143], which considered generalising the double copy for amplitudes to include a non-trivial background metric in the gravity theory. In particular, it considered the case of so-called *sandwich plane waves*, namely plane wave solutions whose deviation from Minkowski space has a finite extent in space and time[1]. One may consider such waves in either gauge theory or gravity, and the authors demonstrate explicitly that a three-point amplitude for a graviton defined as the deviation from a gravitational sandwich wave can be obtained as the double copy of a gauge theory three-point function, where the gauge field

[1]More precisely, such waves are confined to a finite region of the lightcone coordinate $u = z - t$, for a wave travelling in the $+z$ direction.

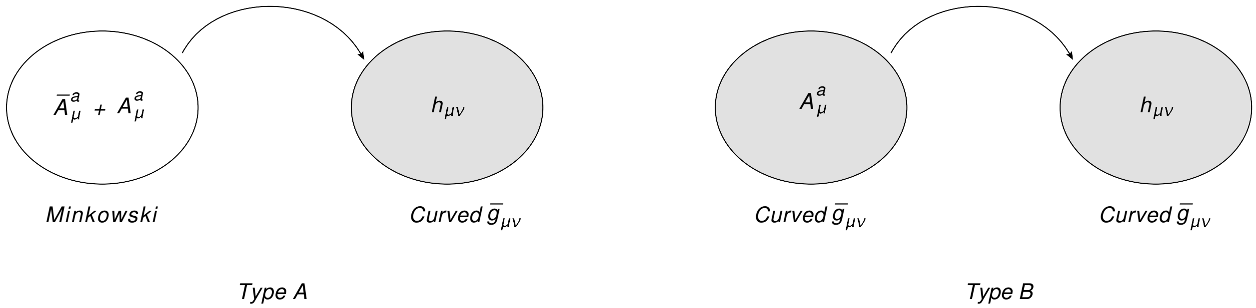

Figure 5.1: Two possible interpretations of a double copy in curved space: in *type A*, a gauge field has a non-trivial background field $\bar{A}^a_\mu$ in Minkowski space, and copies to a graviton defined on a curved background $\bar{g}_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ and $\bar{A}^a_\mu$ are themselves related by a double copy relationship. In *type B*, a gauge field on a non-dynamical curved background $\bar{g}_{\mu\nu}$ double copies to a graviton defined around the same background.

is defined around a gauge theory sandwich wave. They further note that this procedure is obtainable from ambitwistor string theory, which would in principle provide a general framework for formulating a similar procedure for different types of background [140,189].

The results of ref. [143] suggest that some analogue of the curved space amplitude double copy should also be possible for classical solutions. The aim of the following sections is to study this issue, and we will present a number of examples. Firstly, we will construct Kerr-Schild solutions on a curved background by trivially rewriting single Kerr-Schild solutions. We will be able to interpret such solutions as double copies of gauge fields with non-trivial backgrounds, and we will call this relationship a *type A* curved space double copy. We will also find an alternative interpretation, namely that one may regard the graviton as being the double copy of a gauge field living on a non-dynamical curved spacetime background, which we will refer to as *type B*. The difference between these two double copies is shown schematically in figure 5.1, and the second of these is perhaps at odds with what one normally means by the double copy, which relates entire gravity solutions to gauge theory counterparts in flat space. It is then presumably the case that the type B map is not fully general, but exists only in special cases. That does not however, reduce its usefulness, where it applies.

After examining simple Kerr-Schild examples, we will generalise our findings to multiple Kerr-Schild solutions, including a re-examination of the Taub-NUT spacetime considered in the previous chapter. Finally, we will show a family of non-trivial examples of the type B double copy map, in which the background spacetime is conformally flat, without a Kerr-Schild form. This illustrates that this second type of double copy map may be more applicable than naïvely thought, and can also provide a double copy in cases in which it is not known how to construct a double copy of type A.

As stated before, our examination of curved space instances of the classical double copy is motivated by the results of ref. [143], which are a double copy of type A. This associates a gauge theory amplitude in the presence of a non-trivial background field, with a gravity amplitude defined with respect to a non-Minkowski background metric, where the gauge and gravity background fields should be related. In this section, we will see that Kerr-Schild solutions indeed provide a natural framework for constructing such double copies for exact field solutions, rather than perturbative amplitudes.

5.2.1 Single Kerr-Schild solutions

The simplest such examples can be constructed, albeit rather artificially, by starting with single Kerr-Schild metrics around Minkowski space. We may split up such solutions according to

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \phi k_\mu k_\nu \\ &= \eta_{\mu\nu} + \phi_1 k_\mu k_\nu + \phi_2 k_\mu k_\nu, \end{aligned} \tag{5.26}$$

where we have introduced

$$\phi_1 = \xi\phi, \quad \phi_2 = (1-\xi)\phi, \quad 0 \leq \xi \leq 1. \tag{5.27}$$

Thus, any given single Kerr-Schild metric can always be thought of as a double Kerr-Schild metric. It is straightforward to single copy eq. (5.26) term-by-term, resulting in the gauge field

$$A^a_\mu = c^a \Big[\phi_1 k_\mu + \phi_2 k_\mu\Big]. \tag{5.28}$$

This is itself a rewriting of the regular single copy, that is ultimately possible due to the linearity of the field equations in the Kerr-Schild double copy. However, we can reinterpret eqs. (5.26) and (5.28) as follows. By defining

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \phi_1 k_\mu k_\nu, \tag{5.29}$$

we may rewrite eq. (5.26) as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu}, \quad \tilde{h}_{\mu\nu} = \phi_2 k_\mu k_\nu, \tag{5.30}$$

so that the solution of eq. (5.26) may be regarded as containing a graviton field involving only the field ϕ_2, defined with respect to the non-Minkowski background $\bar{g}_{\mu\nu}$. On the gauge theory side, we can define

$$\bar{A}^a_\mu = c^a \phi_1 k_\mu, \tag{5.31}$$

so that the solution of eq. (5.28) becomes

$$A^a_\mu = \bar{A}^a_\mu + \tilde{A}^a_\mu, \quad \tilde{A}^a_\mu = c^a \phi_2 k_\mu. \tag{5.32}$$

This is thus our first example of a type A curved space double copy, for classical solutions rather than amplitudes. A gauge field defined with respect to a non-trivial background field copies to a graviton field with a non-trivial background, where the two backgrounds are themselves related (i.e. they are themselves Kerr-Schild, so we know how to double copy them).

As indicated in figure 5.1, there is another way to consider double copies in curved space (type B). Namely, it may be possible to single copy a graviton defined with respect to a non-Minkowski background, to a gauge field living on the same background. To this end, one may consider the graviton field $\tilde{h}_{\mu\nu}$ of eq. (5.30), which single copies to the field $\tilde{A}^a_\mu$ of eq. (5.32). On the gauge theory side, one may impose the same background $\bar{g}_{\mu\nu}$, and examine the curved space Maxwell equations

$$D^\mu \tilde{F}^a_{\mu\nu} = j_\nu, \tag{5.33}$$

where $\tilde{F}^a_{\mu\nu}$ is the field strength tensor formed from the gauge field $\tilde{A}^a_\mu$. For a consistent double copy of type B, one requires that the source current is somehow related to the energy-momentum tensor in a recognisable way, so that the two solutions are related. Let us give two examples. Firstly, one may consider the Schwarzschild metric, for which

$$\phi = \frac{2M}{r}, \quad k^\mu = (1, 1, 0, 0), \tag{5.34}$$

where we adopt spherical polar coordinates (t, r, θ, φ). Writing the graviton as

$$h_{\mu\nu} = \frac{2M_1}{r} k_\mu k_\nu + \frac{2M_2}{r} k_\mu k_\nu, \quad M_1 + M_2 = M, \tag{5.35}$$

we may define the background field

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}, \quad \bar{h}_{\mu\nu} = \frac{2M_1}{r} k_\mu k_\nu, \tag{5.36}$$

and then single copy the graviton

$$\tilde{h}_{\mu\nu} = \frac{2M_2}{r} k_\mu k_\nu \tag{5.37}$$

to get a gauge field

$$\tilde{A}^a_\mu = \frac{c^a}{r} k_\mu. \tag{5.38}$$

The curved space Maxwell equations of eq. (5.33) then yield[2]

$$j^a_\mu = 0, \tag{5.39}$$

which is indeed consistent: the Schwarzschild metric is a vacuum solution in General Relativity. Here we find that its curved space single copy is also a (gauge theory) vacuum solution, on the curved space defined by $\bar{g}_{\mu\nu}$.

A second example is given by de Sitter spacetime, which has the Kerr-Schild form

$$\phi = \lambda r^2, \quad k_\mu = (-1, 1, 0, 0), \tag{5.40}$$

where λ is the cosmological constant. Splitting this similarly to eq. (5.35) gives

$$h_{\mu\nu} = \lambda_1 r^2 k_\mu k_\nu + \lambda_2 r^2 k_\mu k_\nu, \quad \lambda_1 + \lambda_2 = \lambda. \tag{5.41}$$

We can then define the graviton

$$\tilde{h}_{\mu\nu} = \lambda_2 r^2 k_\mu k_\nu, \tag{5.42}$$

whose single copy gauge field

$$\tilde{A}^a_\mu = c^a \lambda_2 r^2 k_\mu \tag{5.43}$$

satisfies the curved space Maxwell equation with

$$j^a_\nu = (6\lambda_2, 0, 0, 0). \tag{5.44}$$

Again this makes sense: the graviton is sourced by a constant energy density filling all space which, in the gauge theory, becomes a constant charge density. The single copy has thus turned momentum degrees of freedom into colour degrees of freedom, precisely as in the flat space case examined in previous chapters.

We have not been able to prove in general that the curved space Maxwell equations are satisfied for arbitrary single Kerr-Schild solutions that are rewritten in the form of eq. (5.30). However, we have at least shown for some special – and, indeed, astrophysically relevant – cases, a type B double copy map is possible. The question then arises

[2]Here, we do not include the delta function source at the origin, corresponding to the point charge (mass) sourcing the field $\tilde{A}^a_\mu$ ($h_{\mu\nu}$).

of how general this map is. The conventional double copy, in its simplest form, relates a gauge theory to a gravity theory. A gauge theory on a curved background (even if this is non-dynamical) would appear to involve gravity, and thus this type of double copy map seems to relate a coupled Einstein-gauge theory system to itself. One does not then expect this map to be fully general, or to apply to arbitrary supersymmetric generalisations of gauge and gravity theories.

Evidence towards this viewpoint can be obtained by examining the zeroth copy. As discussed in section 3.1, the Kerr-Schild field ϕ is found to satisfy the linearised biadjoint scalar field equation, and can be interpreted as a scalar propagator. In the type B double copy, we can take the zeroth copy of the gauge field $\tilde{A}^a_\mu$ to generate a scalar field

$$\tilde{\Phi}^{aa'} = c^a \tilde{c}^{a'} \phi_2, \tag{5.45}$$

and consider the curved space linearised biadjoint equation

$$D^\mu D_\mu \Phi^{aa'} = c^a \tilde{c}^{a'} \xi, \tag{5.46}$$

which defines ξ. For the Schwarzschild and de Sitter examples, we find

$$\xi_{\rm SWC} = -\frac{4M_1 M_2}{r^4}, \quad \xi_{\rm dS} = 6\lambda_2 - 10r^2\lambda_1\lambda_2 \tag{5.47}$$

respectively, which we can not straightforwardly interpret as being related to the source current in the gauge theory. It thus seems that the type B double copy can indeed associate a gauge theory solution in curved space with a gravity counterpart, at the expense of not having a consistent zeroth copy. This also sheds light on our previous speculation that the zeroth copy for a curved background may result in a biadjoint scalar theory conformally coupled to gravity (eq. (5.22)). The results of eq. (5.47) provide a simple counter-example to this conjecture, showing that the situation is more complex than previously thought.

5.2.2 Multiple Kerr-Schild solutions

In the previous section, we used single Kerr-Schild solutions to provide some first examples of curved space double copies, of both type A and type B. Here, we study whether such conclusions also apply to more complicated solutions. As a first generalisation, we may consider multiple Kerr-Schild solutions in Minkowski space, namely those of form

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_i \phi_i k^{(i)}_\mu k^{(i)}_\nu, \tag{5.48}$$

where each vector $k_\mu^{(i)}$ is null and geodetic with respect to both the Minkowski and full metric, and the set of Kerr-Schild vectors obeys the mutual orthogonality relations

$$\eta^{\mu\nu} k_\mu^{(i)} k_\nu^{(j)} = g^{\mu\nu} k_\mu^{(i)} k_\nu^{(j)} = 0, \quad \forall i, j. \tag{5.49}$$

In certain cases, as seen in the previous chapter, this ansatz linearises the mixed Ricci tensor R^μ_ν, and thus provides an exact solution of the Einstein equations. We further consider the general class of multi-Kerr-Schild solutions in which each term in the graviton is itself a solution of the linearised Einstein equations. In the static case, we may then single copy eq. (5.48) to produce a gauge field

$$A^a_\mu = c^a \sum_i \phi_i k_\mu^{(i)}. \tag{5.50}$$

Given that each term in the graviton constitutes a static Kerr-Schild solution, the results of ref. [183] immediately imply that each term in eq. (5.50) satisfies the linearised Yang-Mills equations. Linearity then implies that the complete field of eq. (5.50) is also a solution, and thus a well-defined single copy of the gravity result.

As for the solution of eq. (5.26), we can use any multi-Kerr-Schild solution of the form of eqs. (5.48, 5.50) to construct a type A curved space double copy. To do this, one may partition the terms in eq. (5.48) into two sets Γ_1 and Γ_2, before defining

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \sum_{i\in\Gamma_1} \phi_i k_\mu^{(i)} k_\nu^{(i)}, \quad \bar{A}^a_\mu = c^a \sum_{i\in\Gamma_1} \phi_i k_\mu^{(i)}, \tag{5.51}$$

and

$$\tilde{h}_{\mu\nu} = \eta_{\mu\nu} + \sum_{i\in\Gamma_2} \phi_i k_\mu^{(i)} k_\nu^{(i)}, \quad \tilde{A}^a_\mu = c^a \sum_{i\in\Gamma_2} \phi_i k_\mu^{(i)}. \tag{5.52}$$

The full gravity and gauge fields may now be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu}, \quad A^a_\mu = \bar{A}^a_\mu + \tilde{A}^a_\mu. \tag{5.53}$$

This is indeed an example of the type A double copy shown in figure 5.1: the gauge field $\tilde{A}^a_\mu$ defined with respect to the background field $\bar{A}^a_\mu$ double copies to the graviton $\tilde{h}_{\mu\nu}$, defined with respect to the background metric $\bar{g}_{\mu\nu}$.

Furthermore, the zeroth copy is also well-defined, as for the flat space classical double copy: from eq. (5.50), we may define the biadjoint field

$$\Phi^{aa'} = c^a \tilde{c}^{a'} \sum_i \phi_i. \tag{5.54}$$

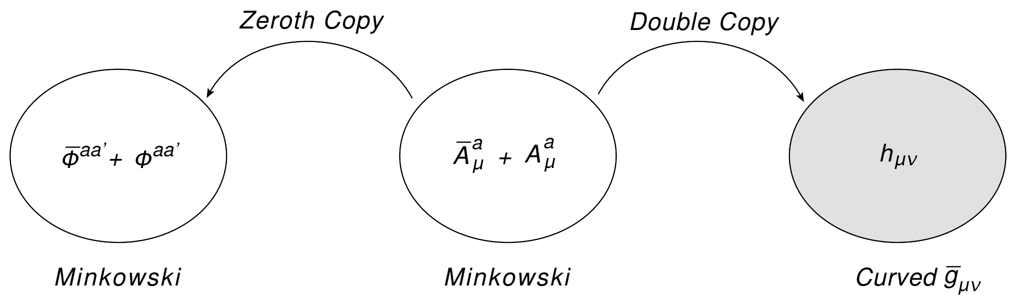

Figure 5.2: Generalisation of the type A double copy of figure 5.1 to include the zeroth copy, which relates the gauge field defined with a non-trivial background to similar solutions in a biadjoint scalar theory.

The fact that each term in the gauge field satisfies the linearised Yang-Mills equations implies, again from ref. [183], that each term in eq. (5.54) satisfies the linearised biadjoint scalar equation. Similarly to eq. (5.51), we may then define

$$\bar{\Phi}^{aa'} = c^a \tilde{c}^{a'} \sum_{i\in\Gamma_1} \phi_i, \quad \tilde{\Phi}^{aa'} = c^a \tilde{c}^{a'} \sum_{i\in\Gamma_2} \phi_i, \tag{5.55}$$

so that the full biadjoint field can be written

$$\Phi^{aa'} = \bar{\Phi}^{aa'} + \tilde{\Phi}^{aa'}. \tag{5.56}$$

This is a direct analogue of the type A double copy between gauge theory and gravity: a classical field defined with respect to a background copies between biadjoint scalar and gauge theory. The relationship between the three theories is shown in figure 5.2. Given that we will always be talking about solutions of the linearised Yang-Mills equations from now on, we will omit colour indices and vectors in what follows.

We may also examine whether or not it is possible to construct a type B double copy for multi-Kerr-Schild solutions, by considering specific examples. In section 5.2.1, we saw that this was indeed possible for the Schwarzschild and de Sitter solutions, split according to eqs. (5.26, 5.27). More generally, we can take either of these gravitons as part of the background metric $\bar{g}_{\mu\nu}$, and allow either of them to be the perturbation $\tilde{h}_{\mu\nu}$. The full list of possibilities is enumerated in table 5.1, where the full metric is given by eq. (5.26), with $k^\mu = (1, 1, 0, 0)$ in spherical polar coordinates. The first two rows contain the pure Schwarzschild (denoted as SWC in the table) and de Sitter (dS) metrics, while the third and fourth rows the cases already considered in the previous section. Finally, the fifth and sixth rows contain the metric formed by perturbing the Schwarzschild solution with a de Sitter Kerr-Schild graviton, and vice versa. For each metric, we give an expression for the timelike component j^t of the source current that appears in the curved space Maxwell

Metric	ϕ_1	ϕ_2	j^t	ξ
SWC	0	$2m_2/r$	0	0
dS	0	$\lambda_2 r^2$	$6\lambda_2$	$6\lambda_2$
SWC+SWC	$2m_1/r$	$2m_2/r$	0	$-4m_1m_2/r^4$
dS+dS	$\lambda_1 r^2$	$\lambda_2 r^2$	$6\lambda_2$	$6\lambda_2 - 10r^2\lambda_1\lambda_2$
SWC+dS	$2m_1/r$	$\lambda_2 r^2$	$6\lambda_2$	$6\lambda_2 - 8m_1\lambda_2/r$
dS+SWC	$\lambda_1 r^2$	$2m_2/r$	0	$4\lambda_1 m_2/r$

Table 5.1: Table of type B single and zeroth copies of Kerr-Schild metrics of the form of eq. (5.26), where ϕ_1 and ϕ_2 are allowed to be different. Here X+Y denotes a Kerr-Schild graviton for metric Y considered as a perturbation on background metric X, where SWC and dS represent the Schwarzschild and de Sitter gravitons respectively.

equation of eq. (5.33) (the spacelike components are found to vanish in all cases), as well as the quantity ξ that appears on the right-hand side of the curved space linearised biadjoint equation (eq. (5.46)).

In all cases, the type B single copy indeed holds. That is, the gauge theory contains a source current consistent with the perturbation term in the gauge field: zero in the Schwarzschild case[3], and a uniform charge density in the de Sitter case, whose counterpart in gravity is the cosmological constant. There are no terms in the source current which are sensitive to the field ϕ_1, which would invalidate the picture of figure 5.1. The zeroth copy holds only in the cases of a pure single Kerr-Schild solution (i.e. the cases considered in the original classical double copy of refs. [1,183]). For all of the double Kerr-Schild solutions, the source includes a position-dependent term that has no immediately evident counterpart in the gauge or gravity theory.

In the above examples, the full metric contains two Kerr-Schild terms, each of which has the same vector k^μ, corresponding to a spherically symmetric system. We can then ask what the most general results for j^t and ξ are, for unspecified functions $\phi_1(r)$ and $\phi_2(r)$. The results are

$$j^t = \frac{2\phi_2'(r)}{r} + \phi_2''(r) = \nabla^2_{\mathrm{M}}\phi_2, \quad \xi = \nabla^2\phi_2 = j^t(1-\phi_1(r)) - \phi_1'(r)\phi_2'(r). \tag{5.57}$$

Here ∇^2 is the Laplacian operator associated with the full background metric, and ∇^2_{M} the corresponding operator in Minkowski space. We thus conclude that if ϕ_2 is associated with a vacuum solution in Minkowski space, the type B single copy is well-defined, in that it is also a vacuum solution. However, the source for the zeroth copy involves the background field ϕ_1 and thus does not seem to have a meaningful interpretation. Of course, the fields

[3]As earlier, we do not bother showing the delta function source at the origin.

ϕ_1 and ϕ_2 in eq. (5.57) are not arbitrary, but must be fixed by the Einstein equations. For the case of spherically symmetric (and stationary) vacuum solutions up to the presence of a cosmological constant, the only possible solutions are the Schwarzschild and de Sitter cases examined already in table 5.1. Nevertheless, the general form of the current in eq. (5.57) does not rule out that there may be non-trivial solutions with extended sources, such that one may still find a consistent single copy interpretation. It is not known even in the flat space case how to construct such maps (see e.g. refs. [2, 156] for discussions of source terms in various contexts).

Above we have discussed cases in which the background scalar field ϕ_1 is spherically symmetric. Our results are more general than this, however. We have explicitly checked that our conclusion that the type B single copy is a vacuum solution if ϕ_2 is associated with a vacuum solution in Minkowski space, holds true even if ϕ_1 has an arbitrary spatial and temporal dependence.

It is furthermore useful to note that, as in the flat space cases considered in ref. [183], one may transform the type B single copy gauge field into a more recognisable form. Starting with the gauge field in spherical polar coordinates,

$$A_\mu = \phi_2(-1, 1, 0, 0),$$

one may perform a gauge transformation

$$A_\mu \to A'_\mu = A_\mu + D_\mu \chi(r) = A_\mu + \partial_\mu \chi(r), \tag{5.58}$$

where

$$\chi(r) = -\int^r dr'\, \phi_2(r'), \tag{5.59}$$

so that eq. (5.58) implies

$$A'_\mu = (-\phi_2, 0, 0, 0). \tag{5.60}$$

Thus, ϕ_2 indeed has the interpretation of an electrostatic potential.

As implied above by the above results, the type B double copy is not necessarily expected to be a fully general map between exact solutions in gauge and gravity theories in curved space. However, it is interesting to examine whether or not it shares the property of the type A (and amplitude) double copies, in being independent of the number of spacetime dimensions d. Indeed, one may show that for a d-dimensional background metric $\bar{g}_{\mu\nu}$ of the form of eq. (5.29), the gauge field $\tilde{A}_\mu$ of eq. (5.32) satisfies the Maxwell equations, with a current density given by [4]

$$j^\mu = (\nabla^2_{\rm M}\phi_2, 0, 0\ldots, 0), \tag{5.61}$$

where the Minkowski-space Laplacian on the right-hand side is in $(d-1)$ space dimensions. Thus, our above discussion generalises for any d.

Having examined multiple Kerr-Schild solutions where each term contains the same Kerr-Schild vector k^μ, it is instructive to instead consider an example in which these vectors can be different. One such example is the Taub-NUT solution, for which the metric takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu + \psi l_\mu l_\nu. \tag{5.62}$$

The Minkowski line element can be written as

$$ds^2 = -dt^2 + \frac{\rho^2}{a^2+r^2}dr^2 + \rho^2 d\theta^2 + (r^2+a^2)\sin^2\theta d\varphi^2 \tag{5.63}$$

in spheroidal coordinates, where

$$\rho^2 \equiv r^2 + a^2\cos^2\theta. \tag{5.64}$$

The vectors k_μ and l_μ are defined by

$$l_\mu dx^\mu = dt + \frac{\rho^2}{a^2+r^2}dr - a\sin^2\theta d\varphi \tag{5.65}$$

$$k_\mu dx^\mu = dt - \frac{i\rho^2}{a\sin\theta}d\theta + \frac{r^2+a^2}{a}d\varphi, \tag{5.66}$$

while the scalar functions ϕ and ψ are given by

$$\psi = \frac{2mr}{\rho^2}, \qquad \phi = \frac{2la\cos\theta}{\rho^2}. \tag{5.67}$$

As for the various metrics considered in table 5.1, in considering the type B single copy, we can take either of the Kerr-Schild terms to be part of the background metric, resulting in two possibilities:

$$\text{Case 1:} \qquad g_{\mu\nu} = \bar{g}_{\mu\nu} + \phi k_\mu k_\nu, \qquad \bar{g}_{\mu\nu} = \eta_{\mu\nu} + \psi l_\mu l_\nu,$$

$$\text{Case 2:} \qquad g_{\mu\nu} = \bar{g}_{\mu\nu} + \psi l_\mu l_\nu, \qquad \bar{g}_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu.$$

[4]Equation (5.61) also turns out to be true when the field ϕ_1 depends on time and the non-radial spatial coordinates.

The gauge fields obtained from the single copy of the perturbation term in both cases satisfy homogeneous Maxwell equations

$$j^{\nu} = 0, \tag{5.68}$$

so that the single copy is indeed consistent (n.b. the Taub-NUT solution is a vacuum solution). The zeroth copy factor is given in both cases by

$$\xi = \frac{4\phi\psi(\rho^2 - 2r^2)}{\rho^4}, \tag{5.69}$$

so that, consistently with our previous results, the type B single copy does not appear to be meaningful.

We have thus found a procedure, based on Kerr-Schild solutions, to construct a double copy for classical solutions that mimics the result found for amplitudes in ref. [143]. In this picture, a gauge field defined with respect to a non-trivial background field copies to a graviton defined with respect to a background metric, where the background fields in the two theories are related, since they obey the original Kerr-Schild double copy by themselves. We call this procedure a type A curved space double copy, to distinguish it from the type B in which the gauge field lives on a non-dynamical curved spacetime, and copies to a graviton field defined with respect to the same spacetime.

In the above cases, we knew how to construct a type A double copy due to the fact that we could relate the background gauge field with its gravitational counterpart. The type B double copy, however, does not require such a relationship, as the same curved metric appears in both the gauge and gravity theories. It is then interesting to look for examples of this relationship in which the background metric is not of Kerr-Schild form, and thus cannot be single-copied according to the procedure of refs. [1, 2, 183]. We have indeed found such examples, which we describe in the following section.

5.3 Conformally flat background metrics

In this section, we consider conformally flat spacetimes. More specifically, we consider spacetimes whose metrics can be written (in some coordinate system) as a conformal transformation of Minkowski space:

$$\bar{g}_{\mu\nu} = \Omega^2(x^{\mu})\eta_{\mu\nu}. \tag{5.70}$$

As the bar notation on the left-hand side already suggests, we will use such metrics as background metrics for Kerr-Schild solutions. This will work for any conformally flat metric, given that if k^μ is null and geodetic with respect to the Minkowski metric, it is straightforward to show that it is also null and geodetic with respect to $\bar{g}_{\mu\nu}$.

As a warm-up, let us examine the case where the background is the well-known Einstein static universe. For convenience, we will adopt the coordinates and conventions of ref. [190], such that the line element is

$$d\bar{s}^2 = -dt^2 + dr^2 - 2a\sin^2\theta d\varphi dr + \frac{|\beta|^2}{\mathcal{D}^2}d\theta^2 + (|\beta|^2 + a^2\sin^2\theta)\sin^2\theta d\varphi^2, \tag{5.71}$$

where

$$\mathcal{D} = 1 - (a^2/R_0^2)\sin^2\theta,$$
$$\beta = (R_0^2 - a^2)^{1/2}\sin\frac{r}{R_0} + ia\cos\theta.$$

The Ricci tensor and scalar for this metric take the particularly simple forms

$$\bar{R}_{\mu\nu} = \frac{2}{R_0^2}(\bar{g}_{\mu\nu} + \bar{u}_\mu\bar{u}_\nu), \qquad \bar{R} = \frac{6}{R_0^2}, \tag{5.72}$$

respectively, where u_μ is the unit timelike vector given by

$$\bar{u}_\mu = (1, 0, 0, 0), \qquad \bar{u}^\mu = (-1, 0, 0, 0). \tag{5.73}$$

We can construct a solution

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2Hk_\mu k_\nu \tag{5.74}$$

of single Kerr-Schild form, where for the Kerr-Schild term we adopt the notation of ref. [190] for ease of comparison. The Kerr-Schild vector k_μ is defined by

$$\sqrt{2}k_\mu = (-1, -1, 0, a\sin^2\theta), \tag{5.75}$$

while the scalar function

$$H = mD_\mu k^\mu, \tag{5.76}$$

with D_μ the covariant derivative associated with $\bar{g}_{\mu\nu}$. The solution defined by eqs. (5.74-5.76) corresponds to a rotating black hole over the Einstein static universe. In order to further examine the effect of this perturbation, we note that the mixed-index Ricci tensor of the full metric takes the form

$$R^{\mu}{}_{\nu} = -\frac{2}{R_0^2}(1-H)(\delta^{\mu}_{\nu} + u^{\mu}u_{\nu}), \tag{5.77}$$

where we have introduced the vectors

$$u^{\mu} = \frac{\tilde{u}^{\mu}}{\sqrt{1-H}}, \qquad u_{\mu} = \frac{1}{\sqrt{1-H}}(\tilde{u}_{\mu} + \sqrt{2}Hk_{\mu}). \tag{5.78}$$

The Einstein equations become

$$R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R = -8\pi((\rho+p)u^{\mu}u_{\nu} + p\delta^{\mu}_{\nu}) + \Lambda\delta^{\mu}{}_{\nu}. \tag{5.79}$$

That is, the matter content of the theory is that of a perfect fluid, whose energy density ρ and pressure p are given in this case by

$$8\pi\rho = \frac{3}{R_0^2}(1-H) - \Lambda, \tag{5.80}$$

$$8\pi p = -\frac{1}{R_0^2}(1-H) + \Lambda. \tag{5.81}$$

We see that the presence of the Kerr-Schild term acts to redefine the parameters associated with the background metric, reminiscent of the split Kerr-Schild metrics we considered in section 5.2.1. A number of other such solutions are presented in ref. [190].

We may single copy the graviton appearing in eq. (5.74) by defining the gauge field

$$A^a_{\mu} = c^a H k_{\mu}, \tag{5.82}$$

which we find satisfies the homogeneous linearised Yang-Mills equation

$$D_{\mu}F^{\mu\nu} = 0, \tag{5.83}$$

where D_{μ}, as above, is the covariant derivative for the Einstein static universe. This provides an example of the type B double copy of figure 5.1: on the gravity side, a fluid is needed to source the background metric. There is no corresponding source current in the gauge theory, as there is no background gauge field, unlike in the type A double copy.

In the previous examples of the type B double copy, we saw that the zeroth copy did not appear to have a meaningful interpretation. Interestingly, in the present example the field H satisfies the homogeneous linearised biadjoint scalar equation

$$D^2 H = 0, \tag{5.84}$$

which indeed leads to a well-defined zeroth copy for this case.

Having seen a particular example of the type B single copy for non-Kerr-Schild backgrounds, let us now consider the general case of background metrics of the form of eq. (5.70), where the Minkowski metric is given in spherical polar coordinates (t, r, θ, φ), so that the conformally transformed metric takes the form

$$\bar{g}_{\mu\nu} = \Omega^2(x^\mu)\text{diag}(-1, 1, r^2, r^2 \sin^2\theta), \tag{5.85}$$

Upon constructing the gauge field

$$A_\mu = k_\mu \phi_2(r), \quad k_\mu = (-1, 1, 0, 0), \tag{5.86}$$

we find that this satisfies the curved space Maxwell equation (in the spacetime whose metric is $\bar{g}_{\mu\nu}$)

$$D_\mu F^{\mu\nu} = j^\nu, \quad j^\nu = \left(\frac{\nabla^2_{\rm M}\phi_2}{\Omega^4(x^\mu)}, 0, 0, 0).\right), \tag{5.87}$$

where $\nabla^2_{\rm M}$ is the Minkowski space Laplacian operator. Note that this result does not require the conformal factor Ω to have spherical symmetry - it may be a general function of (t, r, θ, φ). From eq. (5.87), we see that if the gauge field of eq. (5.86) satisfies a vacuum Maxwell equation in Minkowski space, it also does so in the conformally transformed metric. Thus, the Minkowski space single copy extends to a type B curved space double copy, even though the background metric $\bar{g}_{\mu\nu}$ does not have a Kerr-Schild form, and thus is not amenable to a type A single copy. We may also examine the type B zeroth copy, and one finds the curved space linearised biadjoint equation

$$D^\mu D_\mu \phi_2 = \frac{\nabla^2_{\rm M}\phi_2}{\Omega^2(x^\mu)} + \frac{2\phi_2'(r)\partial_r\Omega(x^\mu)}{\Omega^3(x^\mu)}. \tag{5.88}$$

The second term on the right-hand side involves a spatial derivative of the conformal factor, which is not present in the gauge theory source. Thus, it does not seem possible to interpret the zeroth copy in general, in line with our previous conclusions for the type B procedure.

Closing remarks

Let us briefly summarise the results of this chapter. We started by studying Kerr-Schild solutions where the background is given by the (Anti-)de Sitter metric, since this was our first encounter with curved backgrounds. We found that by considering a graviton field defined with respect to the (Anti-)de Sitter metric, its single copy interpreted as a gauge field living over (Anti-)de Sitter satisfies Maxwell equations in such a background. The zeroth copy does not satisfy a propagator equation directly, but we saw that its equation of motion was compatible with that of a scalar conformally coupled to the gravity background. This result does not seem to be a general feature, as we have observed that for other backgrounds (or in different numbers of dimensions), this is no longer the case, and it might relate to the fact that Yang-Mills has conformal symmetry in $d = 4$, or to one of several special features of (Anti-)de Sitter. One of these interesting properties is that the (Anti-)de Sitter metric can itself be expressed in Kerr-Schild form.

After that, we found a double copy procedure for classical solutions that mimics the result found for amplitudes in ref. [143], based on the Kerr-Schild solutions that worked on a flat space. In this picture, a gauge field defined with respect to a non-trivial background field copies to a graviton defined with respect to a background metric, where the background fields in the two theories are related, due to the fact that they obeyed the original Kerr-Schild double copy by themselves. Furthermore, there is a well-defined zeroth copy, in which the resulting biadjoint field is also defined with respect to a background, where the latter is the zeroth copy of the background gauge field. We call this procedure a *type A curved space double copy*, to distinguish it from the alternative procedure (*type B*) in which the gauge field lives on a non-dynamical curved spacetime, and copies to a graviton field defined with respect to the same spacetime. In this picture, the zeroth copy does not appear to be meaningful, in that the biadjoint field appears not to be physically related to its gauge theory counterpart, due to the presence of unwanted source terms. Note that the process described for (Anti-)de Sitter backgrounds in section 5.2 is a type B double copy, while the treatment in section 5.3 shows how to single copy the background in a type A process.

We considered a number of cases where we knew how to construct a type A double copy due to the fact that we could relate the background gauge field with its gravitational counterpart. The type B double copy, however, does not require such a relationship, as the same curved metric appears in both the gauge and gravity theories. We then looked at examples of this relationship in which the background metric is *not* of Kerr-Schild form, and thus cannot be single-copied according to the usual procedure. In particular, we focused on conformally flat metrics.

Thus far, all of our efforts have focused on working with a family of gravity solutions whose single copies solve Abelian Maxwell equations. Although we have argued that this process is analogous to the BCJ double copy from amplitudes, we would like to have a link between them beyond the heuristic level. As we'll see next, one strong candidate to give us that link comes from a neat geometrical interpretation of the Kerr-Schild ansatz, which results in a set of solutions for accelerated point particles.

Chapter 6

Kerr-Schild double copy and radiation

This chapter was motivated by a neat geometrical interpretation of the Kerr-Schild ansatz. In an attempt to study the asymptotic behaviour of the metric and Riemann tensors at spatial infinity, in spaces which are asymptotically flat, Newman and Unti [191] introduced a coordinate system, intrinsically attached to an arbitrary time-like world-line. Our interest in gravity solutions describing accelerated particles is twofold. First, this constitutes a probe into non-trivial time-dependent effects, unlike most of the examples studied in the previous chapter, where the stationary condition of the metric is essential for the Kerr-Schild double copy to hold. A second, and related, point comes from the realization that in electromagnetism, accelerated particles can be described by the Liénard-Wiechert potential, which encodes the radiation emitted by the particle. Since we expect the solutions to behave according to the double copy, we are in principle interested in the mapping between the radiation in both theories. To develop these ideas, let us start looking at the solution for accelerated particles in the gravity theory.

6.1 Gravity solution

Instead of using the metric given by Newman and Unti, we will follow a somewhat different approach which leads more directly to a Kerr-Schild like solution, namely that of Kinnersley [192–195]. We consider an arbitrary smooth world-line L in Minkowski space that is everywhere time-like. Let τ be the proper time along the curve and $\lambda^\mu(\tau)$ denotes the unit tangent vector at any point, directed toward the future i.e.

$$\lambda^\mu(\tau) = \frac{dy^\mu(\tau)}{d\tau} \tag{6.1}$$

For an arbitrary point x^μ outside the curve, there are two null lines intersecting the worldline L. As we show in figure 6.1, the past null cone of x intersects L exactly once. Hence there exists a unique retarded null vector connecting x^μ with the curve. Let the

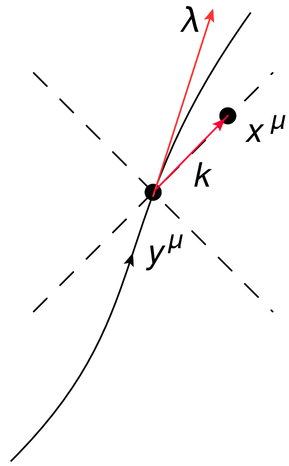

Figure 6.1: Geometric interpretation of the Kerr-Schild solution for an accelerated particle.

null vector be $k^\mu(x)$ and let y be the point of contact. Now, the definitions of τ and $\lambda^\mu(\tau)$ may be extended off the world line by setting

$$\tau(x) = \tau(y) \tag{6.2}$$

Thus the fields τ, k^μ and λ^μ are well defined everywhere and may be differentiated. With this in mind, we may consider the proper time τ to be a function of x^μ if we consider the relation

$$\eta_{\mu\nu}(x^\mu - y^\mu(\tau))(x^\nu - y^\nu(\tau)) = 0. \tag{6.3}$$

Then, taking the derivative with respect to x^ρ, we have

$$2\eta_{\mu\nu}(x^\mu - y^\mu(\tau))(\delta^\nu{}_\rho - \frac{dy^\nu}{d\tau}\tau_{,\rho}) = 0. \tag{6.4}$$

Solving this for $\tau_{,\rho}$ we get

$$\tau_{,\rho} = \frac{x_\rho - y_\rho(\tau)}{r}, \tag{6.5}$$

where we have defined the retarded distance

$$r \equiv (x^\mu - y^\mu(\tau))\,\lambda_\mu. \tag{6.6}$$

We use this to define the vector k^μ as

$$k_\mu \equiv \tau_{,\mu} = \frac{\partial \tau}{\partial x^\mu}. \tag{6.7}$$

This vector will play a major role, as we will show it is indeed the Kerr-Schild vector for this metric. We may immediately note that since it is proportional to $x_\mu - y_\mu(\tau)$, eq. (6.3) implies k^μ is a null vector. We may also note that from eqs. (6.5)-(6.7), it follows immediately that $k \cdot \lambda = 1$, with λ^μ defined as in (6.1). Another quantity it is convenient to define is the vector

$$R^\mu \equiv x^\mu - y^\mu(\tau). \tag{6.8}$$

so using eqs. (6.5)-(6.7), we have the relation

$$R^\mu = rk^\mu. \tag{6.9}$$

6.1.1 Kinnersley's photon rocket

We will now consider a point-particle following an arbitrary timelike worldline, and sourcing a gravitational field. There exists an exact solution corresponding to this situation, known as Kinnersley's photon rocket [192–195]. This contains an additional pure radiation field stress-energy tensor, hence the name. The solution is presented in Kerr-Schild form[1]

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \kappa h_{\mu\nu} \\ &= \eta_{\mu\nu} - \frac{\kappa^2}{2}\phi k_\mu k_\nu, \end{aligned} \tag{6.10}$$

with the scalar function

$$\phi = \frac{M(\tau)}{4\pi r}. \tag{6.11}$$

The null vector k_μ has the same geometric interpretation we described before. Then, in order to perform any computation for this metric, we use the identities

$$\begin{aligned} \partial_\alpha r &= k_\alpha \Delta + \lambda_\alpha, \qquad \Delta \equiv (-1 + rk\cdot\dot{\lambda}), \\ \partial_\alpha k_\beta &= r^{-1}[\eta_{\alpha\beta} - k_\beta\lambda_\alpha - k_\alpha\lambda_\beta - k_\alpha k_\beta \Delta], \\ \partial_\alpha \lambda_\beta &= k_\alpha \dot{\lambda}_\beta, \qquad \partial_\alpha \dot{\lambda}_\beta = k_\alpha \ddot{\lambda}_\beta, \end{aligned} \tag{6.12}$$

[1]Unlike the rest of the thesis, in this chapter the Minkowski metric is given by $\eta = \mathrm{diag}(1, -1, -1, -1)$, to make ease of comparison with ref. [2].

which we prove in appendix A. We can compute the energy-momentum tensor using the expression [2]

$$G^{\mu}{}_{\nu} \equiv R^{\mu}{}_{\nu} - \frac{R}{2}\delta^{\mu}{}_{\nu} = \frac{\kappa^2}{2}T_{\text{KS}}{}^{\mu}{}_{\nu} \tag{6.13}$$

We can use eqs. (6.12) to compute the energy momentum tensor. This yields

$$T^{\mu\nu}_{\text{KS}} = -k^{\mu}k^{\nu}\frac{M'(\tau) - 3M(\tau)k\cdot\dot{\lambda}}{4\pi r^2}. \tag{6.14}$$

We will only be concerned with solutions for constant mass, such that eq. (6.14) reduces to

$$T^{\mu\nu}_{\text{KS}} = k^{\mu}k^{\nu}\frac{3Mk\cdot\dot{\lambda}}{4\pi r^2}. \tag{6.15}$$

Thus, the use of Kerr-Schild coordinates for the accelerating particle leads to the presence of a non-trivial energy-momentum tensor on the RHS of the Einstein equations, in addition to the delta function source corresponding to the particle worldline itself. We can already see that this extra term vanishes in the stationary case ($\dot{\lambda} = 0$), consistent with the results of ref. [183]. More generally, this stress-energy tensor T_{KS} describes a pure radiation field present in the spacetime. The physical interpretation of this source is particularly clear in the electromagnetic "single copy" of this system, to which we now turn.

6.2 Single copy

Having examined the radiating particle in Kerr-Schild language, we may apply the classical single copy of eq. (3.27), in order to try and construct a gauge theory analogue. This procedure is not necessarily guaranteed to work, given that the single copy of refs. [1,183] only applies for stationary fields. However, we will see that we can indeed make sense of the single copy in the present context.

The Kerr-Schild approach to the double copy that we have studied so far consists of starting with a Kerr-Schild metric and constructing the gauge theory solution via the substitution $k_\mu k_\nu \to k_\mu$. Thus, the single copy of the graviton field

$$h^{\mu\nu} = \frac{\kappa}{2}\frac{M}{4\pi r}k^{\mu}k^{\nu}, \tag{6.16}$$

[2] In practice, the expression we used is

$$\kappa T^{\delta}{}_{\beta} = \frac{1}{2}\eta_{\gamma\beta}(\eta^{\alpha\epsilon}g^{\gamma\delta} - \eta^{\gamma\alpha}g^{\delta\epsilon} - \eta^{\delta\alpha}g^{\gamma\epsilon} + \eta^{\gamma\delta}g^{\alpha\epsilon})_{,\alpha\epsilon}\,.$$

This is equivalent to eq. (6.13) for Kerr-Schild solutions, and has the advantage of being much less time consuming when using software to evaluate it.

is the gauge vector field

$$A^{\mu} = \frac{g}{4\pi r} k^{\mu}, \tag{6.17}$$

where g is the coupling constant. One may then find that non-linear terms in the Yang-Mills equations vanish, leaving the Maxwell-like equations

$$\partial^{\mu} F_{\mu\nu} = j_{\mathrm{KS}\,\nu}, \tag{6.18}$$

where we have used the usual electromagnetic field strength tensor

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \tag{6.19}$$

The additional current density appearing on the RHS of eq. (6.18) turns out to be given by

$$j_{\mathrm{KS}\,\nu} = 2 \frac{g}{4\pi} \frac{k \cdot \dot{\lambda}}{r^2} k_{\nu} \bigg|_{\tau=\tau_{\mathrm{ret}}}. \tag{6.20}$$

This current corresponds to points $x \neq y(\tau)$, and we will ignore a delta function term corresponding to the world-line of the particle, since we have already seen in a previous chapter how the sources for point particles map to each other. It is interesting already to note that the current density in eq. (6.20) seems to be related to the energy-momentum tensor of eq. (6.15) in a double copy-like way in the sense that it involves a single factor of the Kerr-Schild vector k^{μ}, with similar prefactors, up to numerical constants. We will return to this in the following section.

The presence of the current density can be understood by interpreting the gauge field of eq. (6.17) in more detail. Computing the electromagnetic field strength tensor of this system yields

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} = \frac{g}{4\pi r^2} (k_{\mu} \lambda_{\nu} - \lambda_{\mu} k_{\nu}). \tag{6.21}$$

First of all, this field strength tensor goes as r^{-2}, and it is independent of the acceleration of the particle. Thus, it cannot describe the radiation of the accelerated point particle. Secondly, since this tensor is manifestly covariant under Lorentz transformation, we can choose any reference frame to explicitly evaluate it. We choose the instantaneous rest frame of the particle (this is, a frame where the particle velocity is zero in one given moment). Then, we have

$$\lambda^{\mu} = (1, 0, 0, 0), \qquad k^{\mu} = (1, \hat{\vec{r}}). \tag{6.22}$$

Substituting these values into eq. (6.21), we obtain the field strength tensor of the point charge, i.e. a Coulomb field. Therefore, if we take the inverse Lorentz transformation to get back to a general frame, we would simply obtain the boosted Coulomb field of a point charge. This absence of radiation field in the field-strength tensor makes obvious the interpretation of the current density eq. (6.20), as a source that contains all the information of the radiation field of the point particle.

To see this in an explicit way we can subtract the Kerr-Schild gauge field from the standard Liénard-Wiechert solution, which describes a point particle moving in an arbitrary manner in empty space (see e.g. [196]), and we'll show that the corresponding gauge field, indeed encodes the radiative effects. The Liénard-Wiechert potential takes the explicit form

$$A^\mu_{\text{LW}} = \frac{g}{4\pi r}\lambda^\mu, \tag{6.23}$$

so we construct the "radiative gauge field"

$$\begin{aligned} A^\mu_{\text{rad}} &= A^\mu_{\text{LW}} - A^\mu \\ &= \frac{g}{4\pi r}(\lambda^\mu - k^\mu). \end{aligned} \tag{6.24}$$

Using its explicit form, one may show that this vector field has the corresponding field-strength tensor

$$F^{\mu\nu}_{\text{rad}} \equiv \partial^\mu A^\nu_{\text{rad}} - \partial^\nu A^\mu_{\text{rad}} = \frac{g}{4\pi r}\left(k_\mu \beta_\nu - \beta_\mu k_\nu\right), \tag{6.25}$$

where

$$\beta_\mu = \dot{\lambda}_\mu - \lambda_\mu k \cdot \dot{\lambda}. \tag{6.26}$$

Thus, we may interpret $F^{\mu\nu}_{\text{rad}}$ as the radiative field strength tensor of the point particle. We may notice that, as expected, this field depends linearly on the acceleration $\dot{\lambda}$ and it goes as r^{-1} for large distances. One may then show that

$$\partial_\mu F^{\mu\nu}_{\text{rad}} = -j^\nu_{\text{KS}}, \tag{6.27}$$

and hence (from eq. (6.18)) that

$$\partial_\mu \left(F^{\mu\nu} + F^{\mu\nu}_{\text{rad}}\right) = 0, \tag{6.28}$$

as was expected from the fact that Liénard-Wiechert is a solution of the vacuum Maxwell equation. We can thus interpret the charge density as the divergence of the radiative field-strength tensor. This corresponds to having put the radiative part of the gauge field on the RHS on the Maxwell equations, rather than the left.

Let us now summarise what has happened. By choosing Kerr-Schild coordinates for the accelerating particle in gravity, radiation appears in the energy-momentum tensor, on the RHS of the Einstein equations, due to the fact that the LHS is forced to be linear. The single copy turns an energy density into a charge density (as in refs. [1, 156, 183]). Thus, the energy-momentum tensor in the gravity theory becomes a charge current in the gauge theory. We have seen that this current corresponds to the radiation effects of the accelerating charged particle. This allows us to interpret the gravitational energy momentum tensor as representing the same physical effect, namely that the gravitational radiation of an accelerating point mass.

Indeed, our use of Kerr-Schild coordinates forced the radiation to appear in this form. The vector k_μ which is so crucial for our approach is twist-free: $\partial_\mu k_\nu = \partial_\nu k_\mu$. It is known that Petrov type D metrics have no no gravitational radiation (see ref. [49] for a review). Then, since twist-free, vacuum, Kerr-Schild metrics are of Petrov type D, we see that there is no gravitational radiation in the metric. Correspondingly, the radiation is described by the Kerr-Schild sources.

One important question is how can we be sure that the Kerr-Schild double copy is indeed related to the BCJ procedure for scattering amplitudes? This is addressed in the following section, in which we interpret the radiative sources in terms of scattering amplitudes.

6.3 From Kerr-Schild sources to amplitudes

In the previous section, we saw that the Kerr-Schild double copy can indeed describe radiating particles, where the radiation appears as a source term on the RHS of the field equations. In this section, we consider a special case of this radiation, namely Bremsstrahlung associated with a sudden rapid change in direction. We will then show, by Fourier transforming the source terms in the gauge and gravity theory to momentum space, that they correspond to known scattering amplitudes, thus firmly establishing a relationship between the classical double copy and the BCJ amplitude result.

More specifically, we will consider a particle following the trajectory

$$y^\mu(\tau) = \tau[u^\mu(\tau) + f(\tau)(u'^\mu - u^\mu(\tau))], \tag{6.29}$$

where

$$f(\tau) = \begin{cases} 0, & \tau \leq -\epsilon \\ 1, & \tau \geq \epsilon \end{cases}, \tag{6.30}$$

and in the interval $(-\epsilon, \epsilon)$, $f(\tau)$ is smooth but is otherwise arbitrary . This corresponds to a particle with constant velocity $\lambda^\mu = u^\mu$ (u'^μ), for $\tau < -\epsilon$ $(> \epsilon)$ respectively i.e. with a rapid change of direction around $\tau = 0$, assuming ϵ to be small. The form of $f(\tau)$ acts as a regulator needed to avoid pathologies in the calculation that follows. However, dependence on this regulator cancels out, so that an explicit form for $f(\tau)$ will not be needed. Owing to the constant nature of u and u', the acceleration is given by

$$\dot{\lambda}^\mu = f'(\tau)\left(u'^\mu - u^\mu\right), \tag{6.31}$$

and according to eq. (6.30) is zero for $\tau < -\epsilon$ and $\tau > \epsilon$, but potentially large in the interval $(-\epsilon, \epsilon)$. Without loss of generality, we may choose the spatial origin to be the place at which the particle changes direction, so that $y^\mu(0) = 0$.

6.3.1 Gauge theory

We first consider the gauge theory case, and start by writing the current density of eq. (6.20) as

$$j_{\text{KS}\,\nu} = \frac{2g}{4\pi}\int d\tau \frac{k\cdot\dot{\lambda}}{r^2} k_\nu \delta(\tau - \tau_{\text{ret}}). \tag{6.32}$$

Now, recalling the expressions

$$k_\rho = \frac{x_\rho - y_\rho(\tau)}{r}, \qquad r = \left(x^\mu - y^\mu(\tau)\right)\lambda_\mu, \tag{6.33}$$

we can write eq. (6.32) in the form

$$j^\nu_{\text{KS}} = \frac{2g}{4\pi}\int d\tau \frac{\dot{\lambda}(\tau)\cdot(x - y(\tau))}{[\lambda(\tau)\cdot(x - y(\tau))]^4}(x - y(\tau))^\nu \delta(\tau - \tau_{\text{ret}}), \tag{6.34}$$

where we have introduced a delta function to impose the retarded time constraint. Using the identity (cf. eq. (A.10))

$$\frac{\delta(\tau - \tau_{\text{ret}})}{\lambda\cdot(x - y(\tau))} = 2\theta(x^0 - y^0(\tau))\delta\left((x - y(\tau))^2\right), \tag{6.35}$$

one may rewrite eq. (6.34) as

$$j^\nu_{\text{KS}} = \frac{4g}{4\pi}\int d\tau \frac{\dot{\lambda}(\tau)\cdot(x - y(\tau))}{[\lambda(\tau)\cdot(x - y(\tau))]^3}(x - y(\tau))^\nu \theta(x^0 - y^0(\tau))\delta\left((x - y(\tau))^2\right). \tag{6.36}$$

Any radiation field will be associated only with the non-zero acceleration for $|\tau| < \epsilon$, where $y^\mu(\tau)$ is small. We may thus neglect this with respect to x^μ in eq. (6.36), yielding

$$j^\nu_{\text{KS}} = \frac{4g}{4\pi}\int d\tau \frac{\dot{\lambda}(\tau)\cdot x}{(\lambda(\tau)\cdot x)^3} x^\nu \theta(x^0)\delta\left(x^2\right). \tag{6.37}$$

Substituting eq. (6.31) then gives

$$j^\nu_{\rm KS} = \frac{4g}{4\pi} x^\nu \theta(x^0)\delta(x^2) \int_{-\epsilon}^{\epsilon} d\tau \frac{bf'(\tau)}{(a+bf(\tau))^3}, \tag{6.38}$$

where

$$a = x\cdot u, \quad b = x\cdot u' - x\cdot u. \tag{6.39}$$

The integral is straightforwardly carried out, since

$$\frac{d}{d\tau}(a+bf(\tau))^{-2} = \frac{-2bf'(\tau)}{(a+bf(\tau))^3}, \tag{6.40}$$

so inserting this into eq. (6.40), we have

$$\begin{aligned} j^\nu_{\rm KS} &= -\frac{2g}{4\pi} x^\nu \theta(x^0)\delta(x^2) \left[\frac{1}{(a+bf(\tau))^2}\right]_{-\epsilon}^{\epsilon} \\ &= -\frac{2g}{4\pi} x^\nu \theta(x^0)\delta(x^2) \left[\frac{1}{(x\cdot u')^2} - \frac{1}{(x\cdot u)^2}\right], \end{aligned} \tag{6.41}$$

where we have used eqs. (6.30) and (6.39). This current can be expressed as a derivative in the form

$$j^\nu_{\rm KS} = \frac{2g}{4\pi} \theta(x^0)\delta(x^2) \left[\frac{\partial}{\partial u'_\nu}\left(\frac{1}{x\cdot u'}\right) - (u \leftrightarrow u')\right]. \tag{6.42}$$

This form is useful since we will now Fourier transform this expression to momentum space. We start by considering the transform of $(u\cdot x)^{-1}$, where we work explicitly in four spacetime dimensions:

$$\begin{aligned} \mathcal{F}\left\{\frac{1}{u\cdot x}\right\} &= \int d^4x \frac{e^{iq\cdot x}}{u\cdot x} \\ &= \frac{1}{u^0}\int d^3x e^{-i\vec{q}\cdot\vec{x}} \int dx^0 \frac{e^{iq^0x^0}}{x^0 - \frac{\vec{x}\cdot\vec{u}}{u^0}}. \end{aligned} \tag{6.43}$$

Closing the x^0 contour in the upper half plane to obtain a positive frequency solution $q^0 > 0$:

$$\begin{aligned} \mathcal{F}\left\{\frac{1}{u\cdot x}\right\} &= \frac{2\pi i}{u^0}\int d^3x e^{-i\vec{x}\cdot\left[\vec{q} - \frac{q^0}{u^0}\vec{u}\right]} \\ &= \frac{i(2\pi)^4}{u^0}\delta^{(3)}\left(\vec{q} - \frac{q^0}{u^0}\vec{u}\right). \end{aligned} \tag{6.44}$$

It is possible to regain a covariant form for this expression by introducing a mass variable m, such that

$$\begin{aligned}\mathcal{F}\left\{\frac{1}{u\cdot x}\right\} &= \frac{i(2\pi)^4}{u^0}\int_0^\infty dm\delta\left(m-\frac{q^0}{u^0}\right)\delta^{(3)}(\vec{q}-m\vec{u}) \\ &= i(2\pi)^4\int_0^\infty dm\delta^{(4)}(q-mu),\end{aligned} \tag{6.45}$$

where the integral is over non-negative values of m only, given that $q^0 > 0$. Given that $\theta(x^0)\delta(x^2)$ is a retarded propagator[3], one may also note the transform

$$\mathcal{F}\left\{\theta(x^0)\delta(x^2)\right\} = -\frac{2\pi}{q^2}. \tag{6.46}$$

We may now use the convolution theorem to perform the transform. This states that the Fourier transform of a product is equal to the convolution of the transforms of each term. This is

$$\mathcal{F}\{f\cdot g\} = \mathcal{F}\{f\} * \mathcal{F}\{g\}, \tag{6.47}$$

where the convolution operation is defined as[4]

$$(F*G)(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dqF(q)G(k-q). \tag{6.48}$$

Then we can compute the Fourier transform of the current

$$\begin{aligned}\tilde{j}^\nu(k) &= \mathcal{F}\{j^\nu_{\text{KS}}(x)\} \\ &= \frac{2g}{4\pi}\frac{\partial}{\partial u'_\nu}\left[\mathcal{F}\{\theta(x^0)\delta(x^2)\} * \mathcal{F}\left\{\frac{1}{x\cdot u'}\right\}\right] - (u'\to u),\end{aligned} \tag{6.49}$$

so inserting eqs. (6.46) and (6.45), and using the convolution definition of eq. (6.48)

$$\begin{aligned}\tilde{j}^\nu(k) &= \frac{2g}{4\pi}\frac{\partial}{\partial u'_\nu}\left[\frac{1}{(2\pi)^4}\int d^4q\left(-\frac{2\pi}{q^2}\right)\left(i(2\pi)^4\int_0^\infty dm\,\delta^{(4)}(k-q-mu')\right)\right] - (u'\to u) \\ &= -ig\int_0^\infty dm\left(\frac{\partial}{\partial u'_\nu}\left[\frac{1}{(k-mu')^2}\right] - (u\leftrightarrow u')\right).\end{aligned} \tag{6.50}$$

[3]The retarded nature of the propagator is implemented by the implicit Feynman prescription $\frac{1}{(p^0-i\varepsilon)^2-\vec{p}^2}$, where ε ensures convergence of the integrals in what follows.

[4]This normalisation differs from the literature by 2π. This is because of the way we have normalised our Fourier transform.

The derivative in the m integral can be carried out to give

$$\int_0^\infty dm \frac{2m(k-mu')^\nu}{(k-mu')^4} = -\int_0^\infty dm \frac{2m^2 u'^\nu}{(m^2 - 2mu'\cdot k)^2}. \tag{6.51}$$

On the RHS, we have used the on-shell condition $k^2 = 0$, and also neglected terms $\sim k^\mu$, which vanish upon contraction with a physical polarisation vector for the graviton. The remaining integral over m is easily carried out, and leads directly to the result

$$\tilde{j}^\nu_{\rm KS} = -ig\left(\frac{u'^\nu}{u'\cdot k} - \frac{u^\nu}{u\cdot k}\right). \tag{6.52}$$

We may now interpret this as follows. First, we note that the current results upon acting on the radiative gauge field with an inverse propagator. This is, we can write

$$\tilde{j}^\nu_{\rm KS}(k) = \mathcal{F}\{j^\nu_{\rm KS}(x)\} = \mathcal{F}\{\partial_\mu F^{\mu\nu}(x)\} = \Delta^{-1}{}_\mu{}^\nu A^\mu(k),$$

where $\Delta_{\mu\nu}$ is the gauge field propagator. We may now compare this with the LSZ procedure for amputating external legs, and interpret it as an amplitude

$$\tilde{j}^\nu_{\rm KS}(k) \sim \mathcal{A}^\nu.$$

Being precise, the contraction of $\tilde{j}^\nu_{\rm KS}$ with a polarisation vector gives the scattering amplitude for emission of a gluon. Upon doing this, one obtains the amplitude

$$\mathcal{A}_{\rm gauge} \equiv -i\epsilon_\nu(k)\tilde{j}^\nu_{\rm KS} = -g\left(\frac{\epsilon\cdot u'}{u'\cdot k} - \frac{\epsilon\cdot u}{u\cdot k}\right). \tag{6.53}$$

This corresponds to the well-known eikonal scattering amplitude for Bremsstrahlung (see e.g. [9]). We thus see once again that the additional current density in the Kerr-Schild approach corresponds to the radiative part of the gauge field.

6.3.2 Gravity

Following similar steps to the previous section, one may show that the energy-momentum tensor of eq. (6.15), for the present case of Bremsstrahlung

$$T^{\mu\nu}_{\rm KS} = k^\mu k^\nu \frac{3mk\cdot\dot{\lambda}}{4\pi r^2},$$

can be rewritten as

$$T^{\mu\nu}_{\text{KS}} = \frac{6m}{4\pi}\int d\tau \frac{\dot{\lambda}(\tau)\cdot(x-y(\tau)}{[\lambda(\tau)\cdot(x-y(\tau))]^4}(x-y(\tau))^\mu(x-y(\tau))^\nu\theta(x^0-y^0(\tau))\delta\left((x-y(\tau))^2\right). \tag{6.54}$$

Any radiation field will be associated only with the non-zero acceleration for $|\tau|<\epsilon$, where $y^\mu(\tau)$ is small. We may thus neglect this with respect to x^μ in eq. (6.54), yielding

$$T^{\mu\nu}_{\text{KS}} = \frac{6m}{4\pi}\int d\tau \frac{\dot{\lambda}(\tau)\cdot x}{(\lambda(\tau)\cdot x)^4}x^\mu x^\nu\theta(x^0)\delta\left(x^2\right). \tag{6.55}$$

Substituting eq. (6.31) then gives

$$T^{\mu\nu}_{\text{KS}} = \frac{6m}{4\pi}x^\mu x^\nu\theta(x^0)\delta(x^2)\int_{-\epsilon}^{\epsilon} d\tau \frac{bf'(\tau)}{(a+bf(\tau))^4}, \tag{6.56}$$

where

$$a = x\cdot u, \quad b = x\cdot u' - x\cdot u.$$

The integral is straightforwardly carried out, since

$$\frac{d}{d\tau}(a+bf(\tau))^{-3} = \frac{-3b\dot{f}(\tau)}{(a+bf(\tau))^4}, \tag{6.57}$$

so inserting this into eq. (6.40), we have

$$\begin{aligned} T^{\mu\nu}_{\text{KS}} &= -\frac{2m}{4\pi}x^\mu x^\nu\theta(x^0)\delta(x^2)\left[\frac{1}{(a+bf(\tau))^3}\right]_{-\epsilon}^{\epsilon} \\ &= -\frac{2m}{4\pi}x^\mu x^\nu\theta(x^0)\delta(x^2)\left[\frac{1}{(x\cdot u')^3}-\frac{1}{(x\cdot u)^3}\right], \end{aligned} \tag{6.58}$$

where we have used eqs. (6.30) and (6.39). This stress-energy tensor can be expressed as a derivative in the form

$$T^{\mu\nu}_{\text{KS}} = -\frac{m}{4\pi}\theta(x^0)\delta(x^2)\left[\frac{\partial}{\partial u'_\mu}\frac{\partial}{\partial u'_\nu}\left(\frac{1}{x\cdot u'}\right) - (u\leftrightarrow u')\right]. \tag{6.59}$$

Similar steps to those leading to eq. (6.50) can be used to rewrite eq. (6.59) in the form

$$T^{\mu\nu}_{\text{KS}} = \frac{im}{2}\int_0^\infty dn\left(\frac{\partial}{\partial u'_\mu}\frac{\partial}{\partial u'_\nu}\left[\frac{1}{(k-nu')^2}\right] - (u\leftrightarrow u')\right). \tag{6.60}$$

Carrying out the double derivative gives

$$\frac{\partial}{\partial u'_\mu}\frac{\partial}{\partial u'_\nu}\left[\frac{1}{(k-nu')^2}\right] = -\frac{2n^2\eta^{\mu\nu}}{(n^2-2nu'\cdot k)^4} + \frac{8n^2(k-nu')^\mu(k-nu')^\nu}{(n^2-2nu'\cdot k)^3}$$
$$\simeq \frac{8n^4u'^\mu u'^\nu}{(n^2-2nu'\cdot k)}, \tag{6.61}$$

where in the second line we have again used onshellness ($k^2 = 0$), and ignored terms which vanish when contracted with the graviton polarisation tensor. Substituting eq. (6.61) into eq. (6.60), the n integral is straightforward, and one obtains the result

$$\tilde{T}^{\mu\nu}_{\rm KS}(k) = -im\left(\frac{u'^\mu u'^\nu}{u'\cdot k} - \frac{u^\mu u^\nu}{u\cdot k}\right). \tag{6.62}$$

Again, one may interpret this as a scattering amplitude, after contraction with a polarisation tensor. Following the usual double copy procedure, the latter may be written as an outer product of two gauge theory polarisation vectors:

$$\epsilon^{\mu\nu}(k) = \epsilon^\mu(k)\epsilon^\nu(k). \tag{6.63}$$

The scattering amplitude is then given by

$$\mathcal{A}_{\rm grav.} \equiv -i\epsilon_\mu(k)\epsilon_\nu(k)\tilde{T}^{\mu\nu}_{\rm KS}(k) = -m\left(\frac{\epsilon\cdot u'\,\epsilon\cdot u'}{u'\cdot k} - \frac{\epsilon\cdot u\,\epsilon\cdot u}{u\cdot k}\right). \tag{6.64}$$

corresponding to the known eikonal amplitude for gravitational Bremsstrahlung [197]. Again we see that the additional source term in the Kerr-Schild approach corresponds to the radiative part of the field. Furthermore, it is in this form that the double copy is made manifest: numerical factors agree between eqs. (6.52) and (6.62), such that the mass in the gravity theory is replaced with the colour charge in the gauge theory, as expected from the usual operation of the classical single copy [1, 1, 183]. The more unusual numerical coefficients appearing in the position space sources of eqs. (6.15) and (6.20) turn out to be red herrings - it is in momentum space that the double copy looks more natural, consistent with the fact that the BCJ double copy for amplitudes is also set up in momentum space.

Let us summarise the results of this section. We have examined the particular case of a particle which undergoes a rapid change in direction, and confirmed that the additional source terms appearing in the Kerr-Schild description (in both gauge and gravity theory) are exactly given by known radiative scattering amplitudes. This strongly ties the classical double copy to the BCJ procedure for amplitudes. Furthermore, we see that the double copy for these sources is expressed more naturally in momentum rather than position space, as expected given that the BCJ procedure is also set up in momentum space.

Kerr-Schild coordinates are particularly useful for the classical double copy, in that they correspond to exact solutions of the resulting field equations. However, it is more conventional to calculate scattering amplitudes in e.g. the de Donder gauge in gravity. Given the above correspondence between the classical double copy and amplitudes, this begs the question as to whether one may set up a copy for the radiating particle directly in such gauges. The answer to this is yes, provided one pays the price of only being able to work order-by-order in perturbation theory. Nevertheless, it is instructive to do so, and we explore this further in the following chapters.

6.4 Gravitational energy conditions

In this section, we consider null, weak and strong energy conditions of General Relativity, and we will show that they are sensible for the case of the point particle. The null energy condition on a given energy-momentum tensor can be expressed by

$$T_{\mu\nu}\ell^\mu\ell^\nu \geq 0, \tag{6.65}$$

where ℓ^μ is any future-pointing null vector. The weak energy condition is similarly given by

$$T_{\mu\nu}t^\mu t^\nu \geq 0, \tag{6.66}$$

for any future-pointing timelike vector t^μ. The interpretation of this condition is that observers see a non-negative matter density. The null energy condition is implied by the weak energy condition (despite the names, the former is the weakest condition). One may also stipulate that the trace of the tidal tensor measured by such an observer is non-negative, which leads to the strong energy condition

$$T_{\mu\nu}t^\mu t^\nu \geq \frac{T}{2}g_{\mu\nu}t^\mu t^\nu, \quad T \equiv T^\alpha_\alpha. \tag{6.67}$$

Let us now examine whether these conditions are satisfied by the Kerr-Schild energy-momentum tensor of eq. (6.15). First, the null property of the vector k^μ implies that the trace vanishes, so that the weak and strong energy conditions are equivalent. We may further unify these with the null energy condition, by noting that eq. (6.15) implies

$$T^{\mu\nu}_{\rm KS}V_\mu V_\nu = -(k\cdot\dot{\lambda})\left[\frac{3M(k\cdot V)^2}{r^2}\right]. \tag{6.68}$$

for any vector V^μ. The quantity in the square brackets is positive definite, so that whether or not the energy conditions are satisfied is purely determined by the sign of $k\cdot\dot{\lambda}$. A physical interpretation of the latter has been given in e.g. ref. [49], namely that this represents

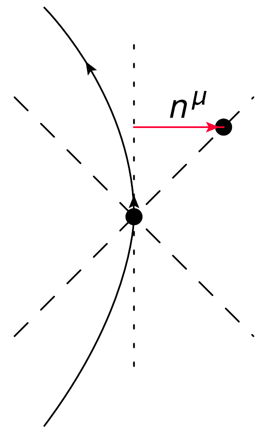

Figure 6.2: Physical interpretation of $(k \cdot \dot{\lambda})$, where this denotes the component of acceleration in the direction n^μ.

the component of acceleration in the direction given by the vector n^μ shown here in figure 6.2. If this quantity is positive (negative), the worldline curves to the right (left) at the retarded time associated with the given field point x^μ, and hence the particle is speeding up (slowing down). In the former case, the particle must absorb rather than emit radiation, and this is seen as a violation of the energy conditions.

The energy conditions were recently examined in the context of the Kerr-Schild double copy in ref. [156], where it was shown that extended charge distributions double copy to matter distributions that cannot simultaneously obey the weak and strong energy conditions, if there are no spacetime singularities or horizons. However, the result in the present section has no contradiction with ref. [156] due to the fact that there would be a horizon for an accelerating particle, so it is not one of the cases they discuss.

Closing remarks

In summary, in this chapter we studied Kerr-Schild solutions describing point-particles moving in arbitrary (timelike) worldlines, i.e. accelerated particles which, in general, include the stress energy tensor of a null fluid. We built the single copy as in the stationary case, and the Maxwell-like equations yield a null source current. This current and the null-fluid stress energy tensor of the gravity solution are related in a double copy-like way, in the sense that we can obtain the source term on the gravity side by taking a second copy of the vector appearing in the current, while we leave untouched the scalar part.

We noted that the current obtained from the single copy encodes the radiation from the particle, and this behaviour also occurs on the gravity side, so both sources contain the radiation effects and also satisfy a double copy relation. To further understand this, we studied the case of a particle that suffers a sudden acceleration during an infinitesimal time, and found that the Fourier transform of the current in gauge theory corresponds to the scattering amplitude for Bremsstrahlung. The same result is then obtained in gravity, as the Fourier transform of the null fluid energy-momentum tensor is also the scattering amplitude for the emission of gravitons in the soft limit. We interpret this as strong evidence of the classical double copy being the same as the BCJ double copy.

This point marks the end of the first part of the thesis, which dealt with Kerr-Schild solutions of the Einstein equations, and their corresponding single copy gauge fields built using a simple ansatz. We aimed to relate this process, that we sometimes refer as the Kerr-Schild double copy with the well-stablished BCJ double copy. Our philosophy in the second part of the thesis will no longer rely on the solutions themselves, but in establishing the double copy at a Lagrangian level, and then building tree-level (classical) solutions in a perturbative manner. Although the approach is very different to our previous work, we will shortly see that its motivation is closely related with the study of the radiation that we have performed in this chapter.

Part II

Perturbative spacetimes from the double copy

Preface

In the previous chapter we studied a double copy relation between the electromagnetic radiation emitted by an accelerated point particle and the gravitational radiation emitted by the corresponding point mass in general relativity, effectively an accelerated black hole[5].

Although we considered a very specific set up (instantaneous acceleration producing soft radiation), this result has pointed us towards the prospect of studying the important problem of describing the motion (scattering) and predicting the classical gravitational radiation generated by a system of merging black holes under their gravitational interactions, using the analogous solution in gauge theories coupled to (colour) sources via a double copy procedure.

This problem in general relativity, usually referred to as the binary black hole problem, persists as a key challenge of classical gravitational physics, being both of theoretical interest and pressingly relevant for astrophysical applications. Its study is crucial to the burgeoning field of gravitational wave astronomy, with the first three gravitational wave signals confidently detected by earth bound observatories each having originated from the final inspirals and mergers of black holes.

Then, in a similar manner to how the exacting program of collider physics required a revolution in the way that scattering amplitudes were computed, the recent experiment LIGO, signaling the dawn of gravitational waves measurements, will require novel and powerful methods of computation. We ultimately aim to explore the application of the double copy, to exploit the relative simplicity of gauge theory compared to perturbative gravity. This would provide an excellent avenue in the search for simplicity and efficiency in black hole scattering, and the efficient calculation of precision gravitational wave templates directly from Yang-Mills Feynman rules could play a role in comparisons between numerical and perturbative methods.

Several analytic approximation schemes have been used to study relativistic binary dynamics. One that has received much attention in recent years is the post-Newtonian approximation, which assumes weak fields and low speeds, and perturbs about the limit of Newtonian gravity (an expansion in $1/c$). This approximation is useful to describe bound

[5]Here, we do not refer to solutions like the Aichelburg-Sexl metric, which describes a boosted black hole, but to our treatment of metrics for point-particles moving in timelike, but otherwise arbitrary worldlines

systems of objects like black holes or neutron stars, which are the most likely sources for gravitational wave detection.

However, much closer to our purposes is the post-Minkowskian approximation, which does not assume low speeds, and perturbs about the limit of special relativity. Formally, this is equivalent to an expansion in G, which is the reason why it is suitable to be compatible with a double copy approach.

We had already encountered a construction of solutions to the equation of motion in a perturbative expansion in powers of the coupling constant in chapter 2, and we also briefly mentioned their relation to scattering amplitudes (they can be understood as the generating functional). In the next chapter, we will review a method developed by Duff to obtain spacetime metrics using Feynman graphs. This part of the thesis will rely heavily on similar ideas, to try and apply the double copy formalism to perturbative gravity solutions.

Chapter 7

Invitation: Schwarzschild from graphs (à la Duff)

A relation between a tree approximation to quantum field theory and classical solutions of the field equations was pointed out by Boulware and Brown in a paper [198] in 1968. This can be expressed in a few words as the classical field produced by an external source being the generating functional for the connected Green functions in the tree approximation.

Aiming to compute radiative corrections to solutions of the Einstein equation, Duff used this connection to first reproduce the classical result [199], by computing the vacuum expectation value (VEV) of the gravitational field in the presence of spherically symmetric sources and verified, to second order in perturbation theory, that the result corresponds to the classical Schwarzschild solution of the Einstein equations. In that same paper, Duff writes "No attempt is made to compute the four-point graph or higher order contributions because of the labor involved".[1]

We have seen, however, that we have come a long way since Duff's paper in terms of available techniques to compute scattering amplitudes in gravity. We are thus aiming to use double copy inspired methods to simplify computations in perturbative classical gravity. We will start by reviewing the example of a Schwarzschild black hole.

7.1 Tree graphs and classical fields relation

In chapter 2, we encountered the expression

$$\frac{\delta W[J, \phi_{\rm cl}]}{\delta J(x)} = \phi_{\rm cl}, \tag{7.1}$$

[1] Or, as put by Sheldon Cooper "Oh gravity, thou art a heartless bitch".

where the functional $W[J]$, defined by

$$e^{iW[J]} \equiv Z[J] = \int \mathcal{D}\phi \; e^{i\int d^4x \; [\mathcal{L}+J\phi]}, \tag{7.2}$$

is the generating functional for connected Green functions. The scattering amplitudes are then computed from these connected correlators by the LSZ procedure. In particular, the vacuum expectation value is computed by the means of

$$\langle 0 \,|\phi(x)|\, 0\rangle_J = \frac{\delta W[J, \phi_{\text{cl}}]}{\delta J(x)}. \tag{7.3}$$

Using eqns. (7.1) and (7.3), we have

$$\langle 0 \,|\phi(x)|\, 0\rangle_J = \phi_{\text{cl}}. \tag{7.4}$$

In conclusion, the vacuum expectation value of a field in presence of a classical source $J(x)$ corresponds to the solution of the classical field equation. In some sense, this is an inverse approach to the one we mentioned back in chapter 2, which uses ϕ_{cl} as a generating functional. We will now exploit this relation between vacuum expectation values and classical solutions to obtain the Schwarzschild spacetime metric using Feynman diagrams, but before diving into computations of greater difficulty, we may build some intuition looking at a simpler scalar example.

7.1.1 A scalar field example

We will now obtain the Yukawa potential in two ways. First, we will explicitly (and exactly) solve the equation of motion in the presence of a source. Then, we will show that we can obtain the same result by computing a VEV of tree graphs. As a starting point, we consider the action for a free Klein-Gordon field

$$S_{\text{KG}} = \int d^4x \; \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + J\phi. \tag{7.5}$$

We assume a source term corresponding to a point particle located at the origin. This is given by

$$J(x) = \delta(\vec{x}). \tag{7.6}$$

The classical field equation then takes the form

$$-\partial^2\phi(x) - m^2\phi(x) = J(x) = \delta(\vec{x}). \tag{7.7}$$

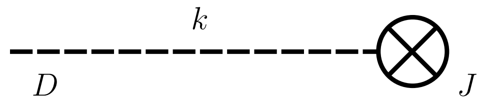

Figure 7.1: Diagram contributing to the Yukawa potential

We can find an exact solution for this equation by transforming into Fourier space. First, we note that the Fourier transform of the source is simply

$$\mathcal{F}[J(x)] = \mathcal{F}[\delta(\vec{x})] = \hat{\delta}(k^0). \tag{7.8}$$

Transforming eq. (7.7) to momentum space yields

$$k^2\phi(k) - m^2\phi(k) = \hat{\delta}(k^0), \tag{7.9}$$

and we can solve for $\phi(k)$ to get the solution

$$\phi(k) = \frac{\hat{\delta}(k^0)}{k^2 - m^2}. \tag{7.10}$$

Finally, Fourier transforming back into position space yields the result

$$\begin{aligned}\phi(x) &= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \frac{\hat{\delta}(k^0)}{k^2 - m^2} \\ &= \frac{e^{-mr}}{4\pi r}.\end{aligned} \tag{7.11}$$

This is the well known Yukawa potential [200], commonly associated with particle interaction through a massive mediator.

Although we were able to solve this exactly, this will not always be possible. In the examples we will encounter in this and later chapters, we will instead have non-linear theories, that we will need to approach using expansions in perturbation theory (encoded in Feynman diagrams). Let us now compute the VEV in presence of the source using such a method. Since this is a non-interacting theory, the only contributing diagram is shown in fig 7.1. The Feynman rules are then

$$D(k) = \frac{1}{k^2 - m^2} \tag{7.12}$$

for the propagator, and

$$J(k) = \hat{\delta}(k^0) \tag{7.13}$$

for every source. Applying these Feynman rules we compute the VEV in momentum space

$$\langle 0 |\phi(k)| 0\rangle_J = \frac{\delta\!\!\!{}^-(k^0)}{k^2 - m^2}. \tag{7.14}$$

Finally, we need to transform back into position space. This is the same Fourier transform that we encountered in the exact computation in eq. (7.11). Actually, we can note that the computation was pretty much identical both times. This lets us see that the Feynman diagram approach is to a certain extent just a handy way of bookkeeping the perturbative solution of the equation in momentum space.

After this warm up, we are ready to tackle the problem of computing the VEV of the gravitational field $h^{\mu\nu}$, in the presence of a stationary point source.

7.2 The Schwarzschild metric from tree graphs

We will now compute the VEV for a stationary point source to $\mathcal{O}(G^2M^2)$ in the tree-graph limit, following Duff [199]. We will see that this computation leads directly to a Schwarzschild black hole solution. The first step is to derive a set of Feynman rules for this theory. We deal with that in the following section.

7.2.1 Einstein-Hilbert Lagrangian (à la Goldberg)

Let us start by considering the Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\text{EH}} = \frac{1}{\kappa^2}\sqrt{-g}R, \tag{7.15}$$

as in eq. (1.74). We had seen that, since this theory is invariant under diffeomorphisms, we need to add a term that breaks such invariance in order to fix the gauge. We'll refer to that term as $\mathcal{L}_{\text{gf}}$. Finally, we need the term $\mathcal{L}_J$, which couples the field with the external source. We should then consider the action

$$S = \int d^4x \, [\mathcal{L}_{\text{EH}}(x) + \mathcal{L}_{\text{gf}}(x) + \mathcal{L}_J(x)] = S_{\text{EH}} + S_{\text{gf}} + S_J, \tag{7.16}$$

We can choose here to express this action in terms of the tensor densities

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}, \qquad \mathfrak{g}_{\mu\nu} = \frac{g_{\mu\nu}}{\sqrt{-g}}. \tag{7.17}$$

This form was used by Goldberg [201] (see also Capper [202]). In terms of these, it is possible to write the Einstein-Hilbert Lagrangian in the form

$$\mathcal{L}_{\text{EH}} = \frac{1}{8\kappa^2}\left[2\mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\lambda\mu}\mathfrak{g}_{\kappa\tau} - \mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\mu\kappa}\mathfrak{g}_{\lambda\tau} - \delta^{\sigma}{}_{\kappa}\delta^{\rho}{}_{\lambda}\mathfrak{g}_{\mu\tau}\right](\partial_\rho\mathfrak{g}^{\mu\kappa})(\partial_\sigma\mathfrak{g}^{\lambda\tau}). \tag{7.18}$$

Note that this Lagrangian contains only first order derivatives. To obtain it, we needed to perform integration by parts, and neglect the boundary terms. The reason to work with this form of the Lagrangian is that it has a relatively simple form, compared to the Lagrangian in terms of $g^{\mu\nu}$, and thus the Feynman rules will be simpler to work with. We choose to work in de Donder gauge. In terms of the tensor densities, this is defined by the condition

$$\partial_\nu \mathfrak{g}^{\mu\nu} = 0. \tag{7.19}$$

In practice, we fix the gauge by adding the term

$$\mathcal{L}_{\rm gf} = \frac{1}{2\kappa^2}\eta_{\mu\nu}(\partial_\alpha \mathfrak{g}^{\mu\alpha})(\partial_\beta \mathfrak{g}^{\nu\beta}), \tag{7.20}$$

in accordance with Fradkin and Tyutin [203]. In the same way that we usually consider the graviton to be a perturbation over a flat background, we can have an analogous definition of the graviton for the density, given by

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa \mathfrak{h}^{\mu\nu}. \tag{7.21}$$

We can now insert eq. (7.21) into the EH Lagrangian of eq. (7.18) to obtain an expansion in powers of the coupling constant κ

$$\mathcal{L}_{\rm EH} = \mathcal{L}^{(0)}_{\rm EH} + \kappa\mathcal{L}^{(1)}_{\rm EH} + \kappa^2\mathcal{L}^{(2)}_{\rm EH} + \dots, \tag{7.22}$$

where the ellipsis ... represents a finite number of higher orders in κ. The explicit expressions for the first terms are given by

$$\mathcal{L}^{(0)}_{\rm EH} = \frac{1}{8}[2\eta^{\rho\sigma}\eta_{\lambda\mu}\eta_{\kappa\tau} - \eta^{\rho\sigma}\eta_{\mu\kappa}\eta_{\lambda\tau} - 4\delta^\sigma{}_\kappa\delta^\rho{}_\lambda\eta_{\mu\tau}](\partial_\rho\mathfrak{h}^{\mu\kappa})(\partial_\sigma\mathfrak{h}^{\lambda\tau}), \tag{7.23}$$

$$\begin{aligned}\mathcal{L}^{(1)}_{\rm EH} = \frac{1}{8}\big[&-4\eta^{\rho\sigma}\eta_{\lambda\mu}\eta_{\kappa\alpha}\eta_{\tau\beta} + 2\eta^{\rho\sigma}\eta_{\mu\kappa}\eta_{\lambda\alpha}\eta_{\tau\beta} + \delta^\rho{}_\alpha\delta^\sigma{}_\beta\eta_{\kappa\tau}\eta_{\lambda\mu}\\ &-\delta^\rho{}_\alpha\delta^\sigma{}_\beta\eta_{\mu\kappa}\eta_{\lambda\tau} + 4\delta^\rho{}_\lambda\delta^\sigma{}_\kappa\eta_{\mu\alpha}\eta_{\tau\beta}\big]\,\mathfrak{h}^{\alpha\beta}(\partial_\rho\mathfrak{h}^{\mu\kappa})(\partial_\sigma\mathfrak{h}^{\lambda\tau}).\end{aligned} \tag{7.24}$$

Then, using eqs. (7.23) and (7.20), we can read off the propagator in de Donder gauge, which takes the form

$$D^{\alpha_1\beta_1\alpha_2\beta_2}(k) = -d^{\alpha_1\beta_1\alpha_2\beta_2}\frac{1}{k^2}, \tag{7.25}$$

where the tensor structure is given by

$$d^{\alpha_1\beta_1\alpha_2\beta_2} \equiv \eta^{\alpha_1\alpha_2}\eta^{\beta_1\beta_2} + \eta^{\alpha_1\beta_2}\eta^{\beta_1\alpha_2} - \eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2}. \tag{7.26}$$

To obtain Feynman rules for the three-graviton interaction vertex, we use DeWitt's [34] method of taking functional derivatives of the action with respect to the interacting fields. This is of course easier and more useful to express in momentum space, where it takes the explicit form

$$\begin{aligned}\Upsilon_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(k_1,k_2,k_3) = -\text{sym P}_6\frac{1}{8}\big[&-4\eta_{\alpha_3\alpha_2}\eta_{\beta_2\alpha_1}\eta_{\beta_3\beta_1}k_2\cdot k_3 + 2\eta_{\alpha_2\beta_2}\eta_{\alpha_3\alpha_1}\eta_{\beta_3\beta_1}k_2\cdot k_3 \\ &- \eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3}k_{2\alpha_1}k_{3\beta_1} + 2\eta_{\alpha_3\alpha_2}\eta_{\beta_2\beta_3}k_{2\alpha_1}k_{3\beta_1} \\ &+ 4\eta_{\alpha_2\alpha_1}\eta_{\beta_3\beta_1}k_{2\alpha_3}k_{3\beta_2}\big].\end{aligned} \tag{7.27}$$

In the last equation, "sym" stands for symmetrisation over every pair of indices, and "P_6" denotes summation over all six permutations of $\alpha_1\beta_1 k_1$, $\alpha_2\beta_2 k_2$ and $\alpha_3\beta_3 k_3$.

We mentioned before that the reason to work with a Lagrangian in terms of the densities from eq. (7.17) is that this set of Feynman rules turns out to be a lot simpler than the rules corresponding to the standard graviton field[2]

$$g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}. \tag{7.28}$$

However, it is the latter definition of the graviton that holds physical significance in our analysis. We could have used the last equation to perform an analogous expansions in terms of κ, and deduce the proper Feynman rules, but none of that will be necessary. Duff noticed that one can instead repeatedly use the relation

$$\frac{\delta g^{\alpha\beta}(x)}{\delta \mathfrak{g}^{\mu\nu}(x')} = \frac{1}{2\sqrt{-g}}\left(\delta^\alpha_{\ \mu}\delta^\beta_{\ \nu} + \delta^\alpha_{\ \nu}\delta^\beta_{\ \mu} - \mathfrak{g}_{\mu\nu}\mathfrak{g}^{\alpha\beta}\right)\delta(x-x'), \tag{7.29}$$

to prove the identity

$$\frac{1}{8}d^{\mu_1\nu_1\alpha_1\beta_1}d^{\mu_2\nu_2\alpha_2\beta_2}d^{\mu_3\nu_3\alpha_3\beta_3}\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} = \Upsilon^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} + \tilde{\Upsilon}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3},$$

where Γ is the three graviton vertex corresponding to the graviton as defined in eq. (7.28), and $\tilde{\Upsilon}$ is a correction to the new vertex, given by

$$\begin{aligned}\tilde{\Upsilon}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \equiv -\frac{1}{4}\text{P}_3[(&\delta^{\mu_1\nu_1\mu_2\nu_2}\eta^{\mu_3\nu_3} - \delta^{\mu_3\nu_3\mu_1\nu_1}\eta^{\mu_2\nu_2} \\ &- \delta^{\mu_2\nu_2\mu_3\nu_3}\eta^{\mu_1\nu_1} + \frac{1}{2}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3})k_3^2].\end{aligned} \tag{7.30}$$

Here, "P_3" means a sum over the three cyclic permutations of (μ_1,ν_1,k_1), (μ_2,ν_2,k_2) and

[2]Note that this convention is different from the usual $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, used throughout the thesis. We stick with this one to make ease of comparison with ref. [199]. Care is needed when interpreting the results.

(μ_3, ν_3, k_3). We have also used the tensor

$$\delta^{\mu\nu\alpha\beta} \equiv \frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}). \tag{7.31}$$

Thus, the set of Feynman rules in terms of densities will be sufficient for the computations we are about to perform.

7.2.2 VEV of the point source

In order to compute the VEV of the graviton field

$$\langle 0 \,| h^{\mu\nu}(x) |\, 0 \rangle_J \,, \tag{7.32}$$

we need to specify the definition of the source for the black hole. We choose a stationary point mass situated at the origin. This can be described by the stress-energy tensor

$$T_{\mu\nu}(x) = M u_\mu u_\nu \delta^{(3)}(\vec{x}), \tag{7.33}$$

where u_μ is the timelike unit vector,

$$u_\mu = (1, 0, 0, 0). \tag{7.34}$$

Then, the only non-vanishing component of the stress-energy tensor is T_{00}. We should then consider the source term in the action

$$S_J = \frac{1}{2}\int d^4x \mathfrak{g}^{\mu\nu} T_{\mu\nu} = \frac{1}{2}\int d^4x \sqrt{-g} g^{\mu\nu} T_{\mu\nu}. \tag{7.35}$$

It is interesting to take the Fourier transform of such a tensor. We find

$$T_{00}(k) = M \bar{\delta}(k_0). \tag{7.36}$$

Note that in terms of this stress-energy tensor, we can define the potential function

$$V(x) \equiv \frac{\kappa^2}{4}\int \bar{d}^4k \frac{e^{ikx}}{k^2} T_{00}(k). \tag{7.37}$$

Inserting the explicit value of eq. (7.36) we may write

$$V(x) = \frac{\kappa^2}{4}\int \bar{d}^4k \frac{e^{ikx}}{k^2} M \bar{\delta}(k_0) = \frac{\kappa^2 M}{16\pi r}. \tag{7.38}$$

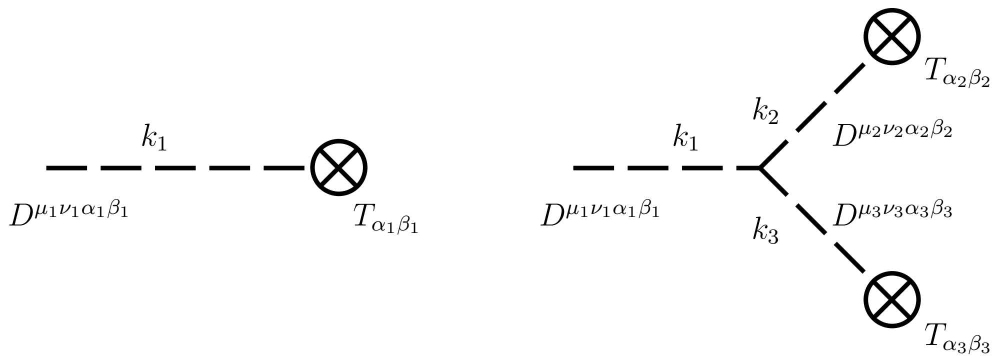

Figure 7.2: Diagrams contributing to $\mathcal{O}(G^2)$ in gravity

Using the relation $\kappa^2 = 16\pi G$, this reduces to

$$V(x) = \frac{MG}{r}, \tag{7.39}$$

so we can see that this function is indeed the Newtonian potential. This will play an important role shortly. We can now proceed to sum over the contributing diagrams, as seen in fig. 7.2. In momentum space, we have the expression

$$\begin{aligned}\kappa\langle h^{\mu_1\nu_1}(k_1)\rangle =& \frac{1}{2}\kappa^2 D^{\mu_1\nu_1\alpha_1\beta_1} T_{\alpha_1\beta_1}(k_1) \\ &+ \frac{1}{8}\kappa^4 \int đ^4 k_2 đ^4 k_3 D^{\mu_1\nu_1\alpha_1\beta_1}(k_1^2) D^{\mu_2\nu_2\alpha_2\beta_2}(k_2^2) D^{\mu_3\nu_3\alpha_3\beta_3}(k_3^2) \\ &\times \delta^{(4)}(k_1+k_2+k_3)\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} T_{\mu_2\nu_2}(k_2) T_{\mu_3\nu_3}(k_3).\end{aligned} \tag{7.40}$$

It is easy to show that since $T_{00}(k)$ is the only non-zero term in the source, at $\mathcal{O}(GM)$, the graviton VEV in position space is given by

$$\kappa\langle h^{\mu_1\nu_1}(\vec{x})\rangle = -2d^{\mu_1\nu_1 00} V(x) = -\frac{2GM}{r} d^{\mu_1\nu_1 00} + \mathcal{O}(G^2M^2). \tag{7.41}$$

The next order correction can be expressed in position space in the form

$$\kappa\langle h^{00}(\vec{x})\rangle = -2V - 4\eta_{kl}p^{kl}(\vec{x}) - 8\eta_{kl}f^{kl}(\vec{x}) + \mathcal{O}(G^3M^3), \tag{7.42}$$

$$\kappa\langle h^{ij}(\vec{x})\rangle = \left(-2V + 4\eta_{kl}p^{kl}(\vec{x})\right)\eta^{ij} + 4p^{ij}(\vec{x}) + \mathcal{O}(G^3M^3). \tag{7.43}$$

Here, the Latin letters $i, j, k, l, \ldots$ are purely spatial indices, unlike Greek letters $\mu, \nu \ldots$ which are spacetime indices. The appearing functions are defined as

$$p^{ij}(\vec{x}) \equiv \frac{\kappa^4}{16} \int đ^3 k_1 đ^3 k_2 đ^3 k_3 e^{i\vec{k}_1 \cdot \vec{x}_1} \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \frac{k_2^i k_3^j}{\vec{k}_1^2 \vec{k}_2^2 \vec{k}_3^2} T_{00}(\vec{k}_2) T_{00}(\vec{k}_3), \tag{7.44}$$

$$f^{ij}(\vec{x}) \equiv \frac{\kappa^4}{16} \int đ^3 k_1 đ^3 k_2 đ^3 k_3 e^{i\vec{k}_1 \cdot \vec{x}_1} \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \frac{k_2^i k_2^j}{\vec{k}_1^2 \vec{k}_2^2 \vec{k}_3^2} T_{00}(\vec{k}_2) T_{00}(\vec{k}_3). \tag{7.45}$$

In appendix B.1, we show a simple way to deal with these integrals. They yield the simple results

$$f^{ij} = -\frac{G^2 M^2}{4} \left(\frac{3x^i x^j - r^2 \eta^{ij}}{r^4} \right), \tag{7.46}$$

$$p^{ij} = -\frac{G^2 M^2}{4} \left(\frac{x^i x^j - r^2 \eta^{ij}}{r^4} \right). \tag{7.47}$$

Using the expressions from eqs. (7.46) and (7.47), it is easy to show the relations

$$\eta_{kl} f^{kl} = 0, \qquad 4\eta_{kl} p^{kl} = \frac{2G^2 M^2}{r^2}. \tag{7.48}$$

Inserting eq. (7.48) into our expressions for the VEV of the graviton, eq. (7.43), we obtain the result

$$\kappa \langle h^{00}(\vec{x}) \rangle = -\frac{2GM}{r} - \frac{2G^2 M^2}{r^2} + \mathcal{O}(G^3), \tag{7.49}$$

$$\kappa \langle h^{ij}(\vec{x}) \rangle = \left(-\frac{2GM}{r} + \frac{3G^2 M^2}{r^2} \right) \eta^{ij} - \frac{G^2 M^2}{r^2} \frac{x^i x^j}{r^2} + \mathcal{O}(G^3). \tag{7.50}$$

Finally, recalling the definition $g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}$ (and omitting the angle brackets), we can write the VEV for the gravitational field as

$$g^{00} = -1 - \frac{2GM}{r} - \frac{2G^2 M^2}{r^2} + \mathcal{O}(G^3), \tag{7.51}$$

$$g^{ij} = \left(1 - \frac{2GM}{r} + \frac{3G^2 M^2}{r^2} \right) \eta^{ij} - \frac{G^2 M^2}{r^2} \frac{x^i x^j}{r^2} + \mathcal{O}(G^3). \tag{7.52}$$

We finally have obtained an expression for the metric, up to $\mathcal{O}(G^3)$. However, in order to compare it with the already existing classical solution to the Einstein equation, we'll need to use a slight change of coordinates. To understand why that is, let us recall that, since we considered a localized, spherically symmetric, stationary source, Birkhoff's theorem implies we should expect it to correspond to the Schwarzschild metric, which is given in its most usual form as

$$ds^2 = -\left(1 - \frac{2GM}{r'}\right) dt^2 + \left(1 - \frac{2GM}{r'}\right)^{-1} dr'^2 + r'^2 d\Omega^2, \tag{7.53}$$

in the polar coordinate system (t, r', θ, φ). However, this metric fails to satisfy the de Donder condition, eq. (7.19). Actually, a form of the Schwarzschild metric that satisfies such a gauge condition has been used, for example, by Nakanishi in ref. [204]. This can be written in the form

$$ds^2 = -\frac{r - GM}{r + GM} dt^2 + \frac{r + GM}{r - GM} dr^2 - \left(1 + \frac{GM}{r}\right)^2 r^2 d\Omega^2, \tag{7.54}$$

in the polar coordinate system (t, r, θ, ϕ). It may be noted that the metric (7.54) is obtained from (7.53) by the simple transformation

$$r' = r + GM. \tag{7.55}$$

It is now not difficult to show that in the Cartesian coordinate system (t, x, y, z) we can express the (inverse) metric as

$$g^{00} = -\frac{r + GM}{r - GM}, \tag{7.56}$$

$$g^{ij} = \frac{r^2}{(r + GM)^2} \eta^{ij} - \frac{G^2 M^2}{r^2 (r + GM)^2} x^i x^j. \tag{7.57}$$

A Taylor expansion to $\mathcal{O}(G^2 M^2)$ of this last expression yields

$$g^{00} = -1 - \frac{2GM}{r} - \frac{2G^2 M^2}{r^2} + \mathcal{O}(G^3), \tag{7.58}$$

$$g^{ij} = \left(1 - \frac{2GM}{r} + \frac{3G^2 M^2}{r^2}\right) \eta^{ij} - \frac{G^2 M^2}{r^2} \frac{x^i x^j}{r^2} + \mathcal{O}(G^3), \tag{7.59}$$

which shows a perfect agreement with the expressions from eqs. (7.51) and (7.52).

Closing remarks

Let us summarise the results of this chapter. We have used the connection between classical solutions of the field equation and the tree level of a quantum field theory approximation to explicitly compute a classical solution for gravity as the vacuum expectation value of quantum tree graphs in the presence of a static spherically symmetric source.

Given the conditions imposed on the source, along with Birkhoff's theorem, we expected the solution obtained to correspond to the Schwarzschild metric. However, due to the fact that this was deduced using a quantum theory of gravity where the de Donder gauge condition was imposed, the form obtained for the metric is not immediately the usual one. The equivalence of both metrics is easily understood by explicitly showing the coordinate transformation connecting them.

The main difficulty with this example was dealing with the complexity of the Feynman rules obtained directly from the Einstein-Hilbert Lagrangian. This renders a next order computation rather difficult, as we would need to consider the four-graviton vertex (which in principle may contain thousands of individual terms). We won't deal with this issue, since in the next chapter, we repeat this procedure, using a gravity Lagrangian obtained as the double copy of Yang-Mills, which simplifies considerably these kinds of computations.

Chapter 8

Spacetimes from graphs through the double copy

In a paper published in 2010 (pretty much at the same time as the first double copy paper), Bern, Dennen, Huang and Kiermaier (BDHK) [56] proved that if it is possible to construct a set of numerators that satisfy BCJ duality, the double copy procedure would yield a gravity scattering amplitude. In general, the numerators obtained from a conventional Yang-Mills Lagrangian do not satisfy colour-kinematics duality. Nonetheless BDHK give an explicit expression for a Lagrangian that produces BCJ dual numerators to up to five-point scattering.

Such a BCJ-compliant Lagrangian can only differ from a regular Yang-Mills Lagrangian, by terms that don't affect amplitudes. This condition is satisfied, since the difference consists only of terms which vanish as a consequence of the Jacobi identity. However, the added terms cause the necessary rearrangements so that the colour-kinematics duality holds.

The construction of a Lagrangian whose numerators are BCJ dual has since been systematized by [130], to construct such a Lagrangian to arbitrary order in perturbation theory. This Lagrangian is non-local and contains Feynman vertices with an infinite number of fields. If desired, it is possible to obtain a local Lagrangian containing only three point vertices at the expense of introducing auxiliary fields.

To illustrate this ideas, let us start with the usual Yang-Mills Lagrangian

$$\mathcal{L} = \frac{1}{2}A^a_\mu \partial^2 A^{a\mu} - g f^{a_1 a_2 a_3} A^{a_1}_\nu A^{a_2\mu} A^{a_3\nu} - \frac{1}{4} g^2 f^{a_1 a_2 b} f^{b a_3 a_4} A^{a_1}_\mu A^{a_2}_\nu A^{a_3\mu} A^{a_4\nu}. \tag{8.1}$$

Now, at four points, the BCJ duality is satisfied in any gauge, so the Yang-Mills Lagrangian will generate diagrams whose numerators are colour-kinematics dual. This property will not hold if computing higher points amplitudes. In the case of five points, it is possible to keep the property of producing BCJ-dual numerators by adding the Lagrangian

$$\mathcal{L}_5 = -\frac{1}{2}g^3 f^{a_1a_2b}f^{ba_3c}f^{ca_4a_5} \times \left(\partial_{[\mu}A^{a_1}_{\nu]}A^{a_2}_\rho A^{a_3\mu} + \partial_{[\mu}A^{a_2}_{\nu]}A^{a_3}_\rho A^{a_1\mu} + \partial_{[\mu}A^{a_3}_{\nu]}A^{a_1}_\rho A^{a_2\mu}\right)\frac{1}{\partial^2}(A^{a_4\nu}A^{a_5\rho}). \tag{8.2}$$

Then, the contributions from the four and five-points interactions are assigned to the various diagrams with only three-point vertices according to their colour factors. It is easy to show that $\mathcal{L}_5$ is identically zero. Indeed, relabelling the colour indices yields

$$\mathcal{L}_5 = -\frac{1}{2}g^3(f^{a_1a_2b}f^{ba_3c} + f^{a_2a_3b}f^{ba_1c} + f^{a_3a_1b}f^{ba_2c})f^{ca_4a_5} \times \partial_{[\mu}A^{a_1}_{\nu]}A^{a_2}_\rho A^{a_3\mu}\frac{1}{\partial^2}(A^{a_4\nu}A^{a_5\rho}), \tag{8.3}$$

and the Jacobi identity ensures the terms identically vanish. We may also note that the terms have different colour factors, and thus appear in different diagrams, so they alter the numerators in such a way that BCJ duality is preserved.

Even though we now have a Lagrangian that will generate colour-kinematics duality satisfying numerators, another requirement to construct a tree level gravity Lagrangian is that the theory is strictly cubic. This can be done by introducing auxiliary fields.

For example, to four-point order, which is everything we'll need to consider in this and the following chapters, Bern, Dennen, Huang and Kiermaier (BDHK) introduced [56] an auxiliary field $B^a_{\mu\nu\rho}$ so as to write a cubic version of the Yang-Mills Lagrangian,

$$\mathcal{L}_{\text{BDHK}} = \frac{1}{2}A^{a\mu}\partial^2 A^a_\mu + B^{a\mu\nu\rho}\partial^2 B^a_{\mu\nu\rho} - gf^{abc}\left(\partial_\mu A^a_\nu - \partial^\rho B^a_{\rho\mu\nu}\right)A^{b\mu}A^{c\nu}, \tag{8.4}$$

where the equation of motion for the auxiliary field becomes

$$\partial^2 B^a_{\mu\nu\rho} = \frac{g}{2}f^{abc}\partial_\mu(A^b_\nu A^c_\rho). \tag{8.5}$$

It is not difficult to show that integrating out the auxiliary field, we recover the Yang-Mills Lagrangian in its usual form. This procedure to render the action cubic by introducing auxiliary fields can be repeated for arbitrary higher order terms in the Lagrangian, as explained in [56, 130]. Being a statement about scattering amplitudes, the BCJ double copy is better understood in momentum space. Thus it will be useful to Fourier transform the BDHK action. This takes the form

$$S_{\text{BDHK}} = \int d^4k_1 d^4k_2 \delta^4(k_1+k_2) k_2^2 [A_\mu(k_1)A^\mu(k_2) - 2B^{\mu\nu\rho}(k_1)B_{\mu\nu\rho}(k_2)] \\ + \int d^4k_1 d^4k_2 d^4k_3 \delta^4(k_1+k_2+k_3) P_6\Big([k_{1\mu}A_\nu(k_1) + k_1^\rho B_{\rho\mu\nu}(k_1)]A^\mu(k_2)A^\nu(k_3)\Big). \tag{8.6}$$

Now, with a cubic and BCJ duality satisfying action at hand, the process of squaring to obtain a gravity action is rather trivial. First, we note that, since the gravity Lagrangian has no colour structure, we need to encode the algebraic properties by introducing the operator P_6 to account for the antisymmetric behaviour[1]. The operator then sums over all permutations of $\{1, 2, 3\}$ with antisymmetrization signs included, i.e.

$$P_6\{k_{1\alpha_2}\eta_{\alpha_1\alpha_3}\} \equiv \eta_{\alpha_1\alpha_3}(k_{1\alpha_2} - k_{3\alpha_2}) + \eta_{\alpha_1\alpha_2}(k_{2\alpha_3} - k_{1\alpha_3}) + \eta_{\alpha_2\alpha_3}(k_{3\alpha_1} - k_{2\alpha_1}). \tag{8.7}$$

Using the BDHK Lagrangian as starting point, we can implement the identification

$$H_{\mu\nu}(k) = A_\mu(k)\tilde{A}_\nu(k). \tag{8.8}$$

Note that, since this is composed from two vector fields which contain two degrees of freedom each, this field should in principle contain four degrees of freedom. We call this field the “fat” graviton, as we expect it to contain the information of the usual (or “skinny") graviton $h_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$, and the dilaton ϕ.

We will now build a gravity Lagrangian in the following way

$$S_{\text{gravity}} = S_{\text{kin}} + S_{\text{int}}, \tag{8.9}$$

where the kinetic terms are given by

$$S_{\text{kin}} = \frac{1}{4}\int đ^4k_1 đ^4k_2 \delta\!\!\bar{}^4(k_1+k_2) k_2^2 [A_\mu(k_1)A^\mu(k_2) - 2B^{\mu\nu\rho}(k_1)B_{\mu\nu\rho}(k_2)] \\ \times [\tilde{A}_\sigma(k_1)\tilde{A}^\sigma(k_2) - 2\tilde{B}^{\sigma\tau\lambda}(k_1)\tilde{B}_{\sigma\tau\lambda}(k_2)], \tag{8.10}$$

[1] Not to be confused with the operator P$_6$ from section 7.2. The main difference being the antisymmetric behaviour of P_6.

while the interaction part of the Lagrangian is

$$S_{\text{int}} = \int d\!\!{}^-{}^4k_1 d\!\!{}^-{}^4k_2 d\!\!{}^-{}^4k_3 \delta\!\!{}^-{}^4(k_1+k_2+k_3) P_6\Big([k_{1\mu}A_\nu(k_1) + k_1^\rho B_{\rho\mu\nu}(k_1)]A^\mu(k_2)A^\nu(k_3)\Big) \times P_6\Big([k_{1\sigma}\tilde{A}_\tau(k_1) + k_1^\lambda \tilde{B}_{\lambda\sigma\tau}(k_1)]\tilde{A}^\sigma(k_2)\tilde{A}^\tau(k_3)\Big). \tag{8.11}$$

In a way similar to how we implemented the identification in eq. (8.8), we also need to consider a number of fields that are obtained as a product involving the auxiliary field $B^{\mu\nu\rho}$. Explicitly, we have the fields

$$\begin{aligned} g^{\mu\nu\rho\sigma} &\equiv A^\mu \tilde{B}^{\nu\rho\sigma}, \\ \tilde{g}^{\mu\rho\sigma\nu} &\equiv B^{\mu\rho\sigma}\tilde{A}^\nu, \\ f^{\mu\rho\sigma\nu\tau\lambda} &\equiv B^{\mu\rho\sigma}\tilde{B}^{\nu\tau\lambda}. \end{aligned} \tag{8.12}$$

However, as it will be shown later, they will not be significant for the purpose of this thesis. Using an explicit expression for the Lagrangian, it is possible to obtain a set of Feynman rules by taking functional derivatives of the Lagrangian with respect to the different fields. Instead of showing directly a set of Feynman rules, let us first study an example of how we want to use them. This is our subject in the next section, where we'll obtain a solution of the gauge theory, and then show how we can obtain a solution for gravity by the means of a double copy procedure.

8.1 Gauge theory solution

We will now use a procedure analogous to the one in the last chapter, to perturbatively obtain a solution to Yang-Mills equations by computing the VEV of the gauge field using Feynman diagrams. We start by considering the simplest possible configurations, which has static colour charges (note that this implies they are constant in time). Such a source has the Feynman rule in position space

$$J^a_\mu(x) = -iu_\mu c^a \delta(\vec{x}). \tag{8.13}$$

Of course, it is more useful to instead give the Feynman rule in momentum space. The Fourier transform of eq. (8.13) immediately yields

$$J^a_\mu(k) = -iu_\mu c^a \delta\!\!{}^-(k^0). \tag{8.14}$$

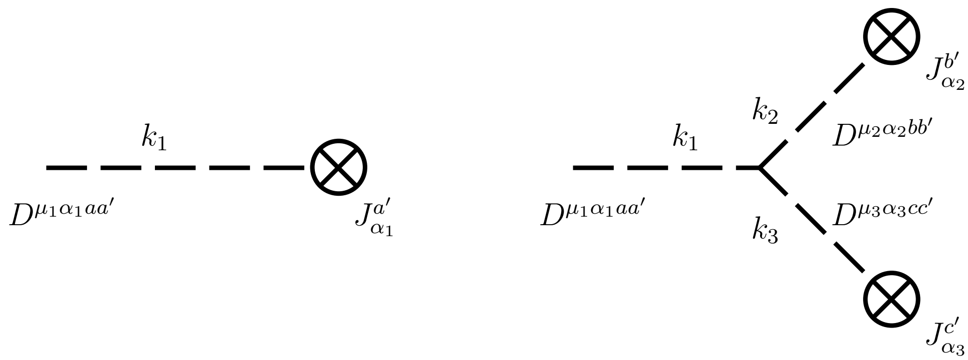

Figure 8.1: Diagrams contributing to $\mathcal{O}(g)$ in Yang-Mills

Now, since any second order (or higher) diagram will attach a structure constant (which is antisymmetric) to a couple of charges, this should vanish, i.e.

$$f^{abc}c^b c^c \sim 0 \tag{8.15}$$

Thus, the calculation terminates at $\mathcal{O}(g)$. We will ignore this for now, in order to explore some of the methods we will use. We now need to read Feynman rules directly from the momentum space Lagrangian eq. (8.6). The propagator for gluons is given by

$$D^{\mu_1\alpha_1 a_1 b_1}(k_1^2) = \frac{i\eta^{\mu_1\alpha_1}\delta^{a_1 b_1}}{k_1^2}, \tag{8.16}$$

while the three-gluon interaction vertex is given by

$$V^{a_1a_2a_3}_{\alpha_1\alpha_2\alpha_3}(k_1, k_2, k_3) = \frac{g}{\sqrt{2}}\tilde{f}^{a_1a_2a_3}P_6\{k_{1\alpha_2}\eta_{\alpha_1\alpha_3}\}. \tag{8.17}$$

Here, we have reinstated the colour factor by hand.

The first two terms contributing to the VEV of the vector field are those coming from the diagrams in fig. 8.1. Inserting our Feynman rules, we get

$$\begin{aligned} A^{a\mu_1}(k_1) = D^{\mu_1\alpha_1 aa'}(k_1)J^{a'}_{\alpha_1}(k_1) + i\int &đ^4k_2 đ^4k_3 D^{\mu_1\alpha_1 aa'}(k_1)D^{\mu_2\alpha_2 bb'}(k_2)D^{\mu_3\alpha_3 cc'}(k_3) \\ &\times \delta\!\!\!^{-}{}^4(k_1+k_2+k_3)V^{abc}_{\alpha_1\alpha_2\alpha_3}(k_1,k_2,k_3)J^{b'}_{\mu_2}(k_2)J^{c'}_{\mu_3}(k_3) \\ &+ \mathcal{O}(g^2), \end{aligned} \tag{8.18}$$

which, substituting the Feynman rules of eqs. (8.16) and (8.17) yields

$$\begin{aligned} A^{a\,\mu_1}(k_1) =& c^a \frac{\eta^{\mu_1\alpha_1}}{k_1^2}\delta(k_1^0)u_{\alpha_1} \\ &+ i\frac{g}{\sqrt{2}}\tilde{f}^{abc}c^b c^c \int đ^4 k_2 đ^4 k_3 \frac{\eta^{\mu_1\alpha_1}\eta^{\mu_2\alpha_2}\eta^{\mu_3\alpha_3}}{k_1^2 k_2^2 k_3^2}\delta^4(k_1+k_2+k_3) \\ &\times P_6\{k_{1\alpha_2}\eta_{\alpha_1\alpha_3}\}\delta(k_2^0)\delta(k_3^0)u_{\mu_2}u_{\mu_3} + \mathcal{O}(g^2). \end{aligned} \tag{8.19}$$

Now that we have used the Feynman rules to obtain an expression for the VEV of the gauge field, we use a Fourier transform to express this in position space, since it is there that we can interpret our results. It is straightforward to perform the integrals over k_i^0 using the delta functions. This then yields

$$\begin{aligned} A^{a\,\mu_1}(x) =& c^a \int đ^3 k_1 \frac{e^{-i\vec{k}_1\cdot\vec{x}}}{\vec{k}_1^2}u^{\mu_1} \\ &+ i\frac{g}{\sqrt{2}}\tilde{f}^{abc}c^b c^c \int đ^3 k_1 đ^3 k_2 đ^3 k_3 \frac{e^{-i\vec{k}_1\cdot\vec{x}}}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3) \\ &\times \eta^{\mu_1\alpha_1}P_6\{\vec{k}_{1\alpha_2}\eta_{\alpha_1\alpha_3}\}u^{\alpha_2}u^{\alpha_3} + \mathcal{O}(g^2). \end{aligned} \tag{8.20}$$

we may now note that since the spatial part of the u_μ vector vanishes, its product with any $\vec{k}^\mu$ will be null. Thus, we have the result

$$P_6\{\vec{k}_{1\alpha_2}\eta_{\alpha_1\alpha_3}\}u^{\alpha_2}u^{\alpha_3} = u^2(\vec{k}_{3\alpha_1} - \vec{k}_{2\alpha_1}). \tag{8.21}$$

Furthermore, we can note that this integration is symmetric under the interchange $\{2 \leftrightarrow 3\}$. Thus, the result of the integral is zero, and the VEV simply yields

$$A^{a\,\mu_1}(x) = \frac{c^a}{4\pi r}u^{\mu_1} + \mathcal{O}(g^2)$$

which is the Coulomb field for an Abelian theory. This should not be that surprising, since our choice of a constant colour charge leads to a linearised theory.

8.2 Gravity solution

Let us compute now the first two orders of a perturbative expansion for the fat graviton as the VEV of the field in the presence of the source. The two contributing Feynman diagrams can be seen in figure 8.2. In order to evaluate such diagrams, we need Feynman rules. We will use a set that comes directly from the double copy. The graviton propagator

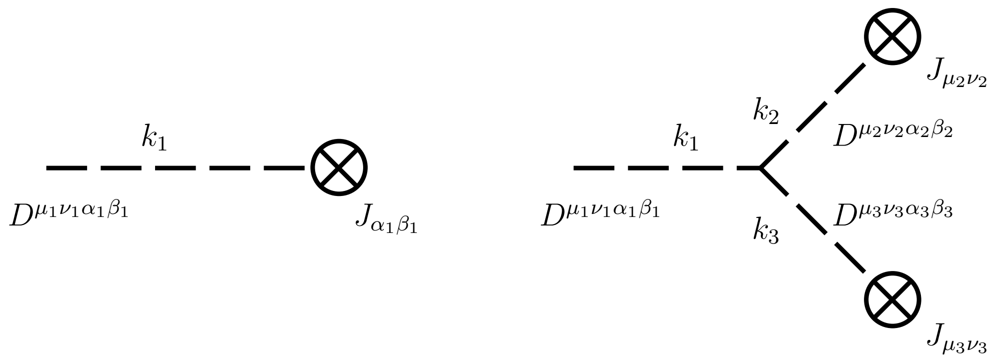

Figure 8.2: Diagrams contributing to $\mathcal{O}(G^2)$ in gravity

will be given by

$$D^{\mu_1\nu_1\alpha_1\beta_1}(k_1^2) = \frac{i\eta^{\mu_1\alpha_1}\eta^{\nu_1\beta_1}}{k_1^2}, \tag{8.22}$$

and the three-graviton interaction vertex is

$$\Gamma_{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}(k_1,k_2,k_3) = \frac{i\kappa}{2}V_{3\alpha_1,\alpha_2,\alpha_3}(k_1,k_2,k_3)V_{3\beta_1,\beta_2,\beta_3}(k_1,k_2,k_3), \tag{8.23}$$

where $\kappa^2 = 32\pi G$ defines the gravitational coupling constant[2], and V_3 is the colour-stripped three-gluon vertex of the last section. It takes the precise form

$$V_{3\alpha_1,\alpha_2,\alpha_3}(k_1,k_2,k_3) = \frac{i}{\sqrt{2}}P_6\{k_{1\alpha_1}\eta_{\alpha_2\alpha_3}\}, \tag{8.24}$$

where the P_6 operator is defined as in eq. (8.7). Finally, for every source we get the term

$$J_{\mu\nu} = -\frac{i\kappa}{2}T_{\mu\nu}, \tag{8.25}$$

where the stress-energy tensor is that of a point mass located at the origin, i.e.

$$T_{\mu\nu} = Mu_\mu u_\nu \delta^{(3)}(\vec{x}), \tag{8.26}$$

where $u_\mu = (1,0,0,0)$. As before, we will need this expression in momentum space. After taking the Fourier transform, the source takes the value

$$J_{\mu\nu}(k) = -\frac{i\kappa}{2}Mu_\mu u_\nu \delta(k^0). \tag{8.27}$$

We now have all the necessary Feynman rules to compute diagrams in this field theory.

[2]The convention here and in subsequent chapters is different from the one we found previously ($\kappa^2 = 16\pi G$). We maintain this to make ease of comparison with ref. [3]

8.2.1 Double copy solution to second order

Let us obtain the vacuum expectation value for the double copy solution. To do this, we have to compute the diagrams in fig. 8.2. They are evaluated as

$$\begin{aligned}\langle H^{\mu_1\nu_1}(k_1)\rangle =& D^{\mu_1\nu_1\alpha_1\beta_1} J_{\alpha_1\beta_1}(k_1)\\ &+ \frac{1}{2}\int đ^4k_2 đ^4k_3 D^{\mu_1\nu_1\alpha_1\beta_1}(k_1^2) D^{\mu_2\nu_2\alpha_2\beta_2}(k_2^2) D^{\mu_3\nu_3\alpha_3\beta_3}(k_3^2)\\ &\qquad \times \delta^{(4)}(k_1+k_2+k_3)\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} J_{\mu_2\nu_2}(k_2) J_{\mu_3\nu_3}(k_3).\end{aligned}$$

We now substitute the Feynman rules for the double copied Lagrangian (eqs. (8.22) and (8.23)). The expression takes the explicit form

$$\begin{aligned}\kappa\langle H^{\mu_1\nu_1}(k_1)\rangle =& \kappa\frac{i\eta^{\mu_1\alpha_1}\eta^{\nu_1\beta_1}}{k_1^2}\left(-\frac{i\kappa}{2}\right) đ(k_1^0) M u_{\alpha_1} u_{\beta_1}\\ &+ \frac{1}{2}\kappa\int đ^4k_2 đ^4k_3 \frac{i\eta^{\mu_1\alpha_1}\eta^{\nu_1\beta_1} i\eta^{\mu_2\alpha_2}\eta^{\nu_2\beta_2} i\eta^{\mu_3\alpha_3}\eta^{\nu_3\beta_3}}{k_1^2k_2^2k_3^2}\delta^{(4)}(k_1+k_2+k_3)\\ &\quad \times \frac{i\kappa}{2}\frac{i}{\sqrt{2}} P_6\{k_{1\alpha_2}\eta_{\alpha_1\alpha_3}\}\frac{i}{\sqrt{2}} P_6\{k_{1\beta_2}\eta_{\beta_1\beta_3}\}\\ &\quad\quad \times \left(-\frac{i\kappa}{2}\right) đ(k_2^0) M u_{\mu_2} u_{\nu_2}\left(-\frac{i\kappa}{2}\right) đ(k_3^0) M u_{\mu_3} u_{\nu_3}.\end{aligned} \tag{8.28}$$

Transforming to position space and performing the k_i^0 integrals we obtain the result

$$\begin{aligned}\kappa\langle H^{\mu_1\nu_1}(x)\rangle =& \frac{1}{2}\kappa^2\int đ^3k_1 \,\frac{e^{-i\vec{k}_1\cdot\vec{x}}}{\vec{k}_1^2} M u^{\mu_1} u^{\nu_1}\\ &+ \frac{1}{32}\kappa^4\int \eta^{\mu_1\alpha_1}\eta^{\nu_1\beta_1} đ^3k_1 đ^3k_2 đ^3k_3 \frac{e^{-i\vec{k}_1\cdot\vec{x}}}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\\ &\qquad \times P_6\{\vec{k}_{1\alpha_2}\eta_{\alpha_1\alpha_3}\} P_6\{\vec{k}_{1\beta_2}\eta_{\beta_1\beta_3}\} M u^{\alpha_2} u^{\beta_2} M u^{\alpha_3} u^{\beta_3}.\end{aligned} \tag{8.29}$$

The vertex structure is solved in a rather trivial way by noting the relation

$$P_6\{k_{1\alpha_2}\eta_{\alpha_1\alpha_3}\} u^{\alpha_2} u^{\alpha_3} = u^2(\vec{k}_{2\alpha_1} - \vec{k}_{3\alpha_1}), \tag{8.30}$$

thus yielding

$$P_6\{k_{1\alpha_2}\eta_{\alpha_1\alpha_3}\} u^{\alpha_2} u^{\alpha_3} P_6\{k_{1\beta_2}\eta_{\beta_1\beta_3}\} u^{\beta_2} u^{\beta_3} = u^4(\vec{k}_{2\alpha_1} - \vec{k}_{3\alpha_1})(\vec{k}_{2\beta_1} - \vec{k}_{3\beta_1}), \tag{8.31}$$

Moreover, we can also use the symmetry under the interchange $\{2 \leftrightarrow 3\}$ to obtain the expression

$$\kappa\langle H^{\mu_1\nu_1}(x)\rangle = 4u^{\mu_1}u^{\nu_1}\frac{1}{8}\kappa^2\int đ^3k_1\,\frac{e^{-i\vec{k}_1\cdot\vec{x}}}{\vec{k}_1^2}M + \delta_i^{\mu_1}\delta_j^{\nu_1}\frac{1}{32}\kappa^4\int đ^3k_1đ^3k_2đ^3k_3\frac{e^{-i\vec{k}_1\cdot\vec{x}}}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)2(k_2^ik_2^j - k_2^ik_3^j)M^2, \tag{8.32}$$

It is useful here to recall the definition of the function V from eq. (7.37). This is[3]

$$V = \frac{1}{8}\kappa^2\int đ^4k\,\frac{e^{-ik\cdot x}}{k^2}M\delta(k^0). \tag{8.33}$$

This was shown in chapter 7 to satisfy the relation $V = MG/r$. Also from that chapter, we recall the definition of the integrals

$$p^{ij}(\vec{x}) \equiv \frac{\kappa^4}{64}\int đ^3k_1đ^3k_2đ^3k_3e^{i\vec{k}_1\cdot\vec{x}_1}\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{k_2^ik_3^j}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}M^2, \tag{8.34}$$

$$f^{ij}(\vec{x}) \equiv \frac{\kappa^4}{64}\int đ^3k_1đ^3k_2đ^3k_3e^{i\vec{k}_1\cdot\vec{x}_1}\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{k_2^ik_2^j}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}M^2. \tag{8.35}$$

Using eqs (8.33-8.35), we can express the second order VEV of the gravity solution eq. (8.32) as

$$\kappa\langle H^{\mu_1\nu_1}(x)\rangle = 4u^{\mu_1}u^{\nu_1}V(x) + 4\delta_i^{\mu_1}\delta_j^{\nu_1}\big(f^{ij}(x) - p^{ij}(x)\big), \tag{8.36}$$

Finally inserting the values for f^{ij} and p^{ij} (given in eqs. (7.46) and (7.47) in the previous chapter) we get

$$\kappa\langle H^{\mu_1\nu_1}(x)\rangle = \frac{4MG}{r}u^{\mu_1}u^{\nu_1} - \frac{2M^2G^2}{r^2}k^{\mu_1}k^{\nu_1}. \tag{8.37}$$

where $k^\mu = (0, \vec{x}/r)$. We will explore the physical interpretation of this result, and how it compares to those of the previous chapter in a later section. However, now we turn our attention to extending our computation one order higher.

8.2.2 Double copy solution to third Order

It is possible to extend this formalism to go to the next order. To do this, it is useful to work in position space, since there we can take advantage of having computed the VEV

[3] The different coefficient with respect to eq. (7.37) comes from the different convention for κ.

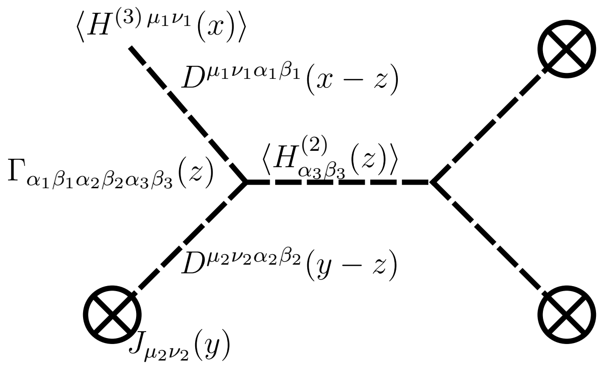

Figure 8.3: One diagram contributing to $\mathcal{O}(G^3)$

to second order previously. Hence, we need to consider the corresponding Feynman rules. The graviton propagator is given by

$$D^{\mu_1\nu_1\alpha_1\beta_1}(x-z) = \frac{i\eta^{\mu_1\alpha_1}\eta^{\nu_1\beta_1}}{4\pi|\vec{x}-\vec{z}|}, \tag{8.38}$$

while the three-graviton interaction vertex takes almost the same form as its position-space counterpart

$$\Gamma_{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}(z) = \frac{i\kappa}{2} V_{3\alpha_1\alpha_2\alpha_3}(z) V_{3\beta_1\beta_2\beta_3}(z), \tag{8.39}$$

where the three-vertex in position space takes the form

$$V_{3\alpha_1\alpha_2\alpha_3}(z) = \frac{i}{\sqrt{2}} P_6\{i\partial_{1\alpha_1}\eta_{\alpha_2\alpha_3}\}, \tag{8.40}$$

and the P_6 operator now reads

$$P_6\{i\partial_{1\alpha_2}\eta_{\alpha_1\alpha_3}\} \equiv \eta_{\alpha_1\alpha_3}(i\partial_{1\alpha_2} - i\partial_{3\alpha_2}) + \eta_{\alpha_1\alpha_2}(i\partial_{2\alpha_3} - i\partial_{1\alpha_3}) + \eta_{\alpha_2\alpha_3}(i\partial_{3\alpha_1} - i\partial_{2\alpha_1}). \tag{8.41}$$

Finally, for every source we get the term

$$J_{\mu\nu}(z) = -\frac{i\kappa}{2} M u_\mu u_\nu \delta^{(3)}(\vec{z}). \tag{8.42}$$

Now, in order to evaluate the third order term of the vacuum expectation value, we need to evaluate the diagram in fig. 8.3. This is

$$\begin{aligned}\langle H^{(3)\,\mu_1\nu_1}(x)\rangle = \int d^3y\, d^3z\, \Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(z) D^{\mu_1\nu_1\alpha_1\beta_1}(x-z) \\ \times D^{\mu_2\nu_2\alpha_2\beta_2}(y-z) J_{\mu_2\nu_2}(y) \langle H^{(2)\,\alpha^3\beta^3}(z)\rangle.\end{aligned} \tag{8.43}$$

recalling the second order expression from eq. (8.37)

$$\kappa\langle H^{(2)\,\mu_1\nu_1}(z)\rangle = -\frac{2M^2G^2}{r^2}k^{\mu_1}k^{\nu_1}, \tag{8.44}$$

where $k^\mu = (0, \vec{z}/r)$, and inserting our Feynman rules we get

$$\begin{aligned}\kappa\langle H^{(3)\,\mu_1\nu_1}(x)\rangle = \int d^3y\, d^3z \frac{i\kappa}{2}\frac{i}{\sqrt{2}}P_6\{i\partial_{1\alpha_1}\eta_{\alpha_2\alpha_3}\}\frac{i}{\sqrt{2}}P_6\{i\partial_{1\beta_1}\eta_{\beta_2\beta_3}\}\\ \times\frac{i\eta^{\mu_1\alpha_1}\eta^{\nu_1\beta_1}}{4\pi|\vec{x}-\vec{z}|}\frac{i\eta^{\mu_2\alpha_2}\eta^{\nu_2\beta_2}}{4\pi|\vec{y}-\vec{z}|}\left(-\frac{i\kappa}{2}\right)Mu_{\mu_2}u_{\nu_2}\delta^{(3)}(\vec{y})\left(-\frac{2M^2G^2}{|\vec{z}|^2}\right)k^{\alpha_3}k^{\beta_3}.\end{aligned} \tag{8.45}$$

Tidying up the last equation yields

$$\begin{aligned}\kappa\langle H^{(3)\,\mu_1\nu_1}(x)\rangle = \frac{\kappa^2G^2M^3}{16\pi}u_{\mu_2}u_{\nu_2}\int d^3y\, d^3z P_6\{\partial_{1\alpha_1}\eta_{\alpha_2\alpha_3}\}P_6\{\partial_{1\beta_1}\eta_{\beta_2\beta_3}\}\\ \times\frac{\eta^{\mu_1\alpha_1}\eta^{\nu_1\beta_1}}{4\pi|\vec{x}-\vec{z}|}\frac{\eta^{\mu_2\alpha_2}\eta^{\nu_2\beta_2}}{|\vec{y}-\vec{z}|}\delta^{(3)}(\vec{y})\frac{1}{|\vec{z}|^2}k^{\alpha_3}k^{\beta_3}.\end{aligned} \tag{8.46}$$

After performing the trivial integration over y, and substituting the relation $\kappa^2 = 32\pi G$, this yields the result

$$\kappa\langle H^{(3)\,\mu_1\nu_1}(x)\rangle = 2G^3M^3u^{\alpha_2}u^{\beta_2}\int d^3z P_6\{\partial_{1\alpha_1}\eta_{\alpha_2\alpha_3}\}P_6\{\partial_{1\beta_1}\eta_{\beta_2\beta_3}\}\frac{\eta^{\mu_1\alpha_1}\eta^{\nu_1\beta_1}}{4\pi|\vec{x}-\vec{z}|}\frac{1}{|\vec{z}|}\frac{1}{|\vec{z}|^2}k^{\alpha_3}k^{\beta_3}. \tag{8.47}$$

Now we can simplify the contraction $u^{\alpha_2}P_6\{\partial_{1\alpha_1}\eta_{\alpha_2\alpha_3}\}k^{\alpha_3}$, by noting that

$$u\cdot k = 0 = u\cdot\partial. \tag{8.48}$$

Using this, the product simplifies notably to yield

$$\begin{aligned}u^{\alpha_2}P_6\{\partial_{1\alpha_1}\eta_{\alpha_2\alpha_3}\}k^{\alpha_3} &= u^{\alpha_2}\eta_{\alpha_1\alpha_2}(\partial_{2\alpha_3}-\partial_{1\alpha_3})k^{\alpha_3},\\ &= u_{\alpha_1}(\partial_{2\alpha_3}-\partial_{1\alpha_3})k^{\alpha_3}.\end{aligned} \tag{8.49}$$

Inserting this into eq. (8.47), we get

$$\kappa\langle H^{(3)\,\mu_1\nu_1}(x)\rangle = 2G^3M^3u^{\mu_1}u^{\nu_1}\int d^3z(\partial_{2\alpha_3}-\partial_{1\alpha_3})(\partial_{2\beta_3}-\partial_{1\beta_3})\frac{1}{4\pi|\vec{x}-\vec{z}|}\frac{1}{|\vec{z}|^3}k^{\alpha_3}k^{\beta_3}. \tag{8.50}$$

We can write this in the compact form

$$\kappa\langle H^{(3)\,\mu_1\nu_1}(x)\rangle = 2G^3M^3u^{\mu_1}u^{\nu_1}g(x), \tag{8.51}$$

where we have defined the function

$$g(x) \equiv \int d^3z(\partial_{2\alpha_3} - \partial_{1\alpha_3})(\partial_{2\beta_3} - \partial_{1\beta_3})\frac{1}{4\pi|\vec{x}-\vec{z}|}\frac{1}{|\vec{z}|^3}k^{\alpha_3}k^{\beta_3}. \tag{8.52}$$

Recalling the explicit form for $k^\alpha = (0, \vec{z}/|\vec{z}|)$, the function $g(x)$ takes the form,

$$g(x) \equiv \int d^3z(\partial_{2i} - \partial_{1i})(\partial_{2j} - \partial_{1j})\frac{1}{4\pi|\vec{x}-\vec{z}|}\frac{1}{|\vec{z}|^5}z^iz^j. \tag{8.53}$$

This can be shown to yield the simple result

$$g(x) = -\frac{4}{3|\vec{x}|^3}, \tag{8.54}$$

which, being inserted in eq. (8.51) results in

$$\kappa\langle H^{(3)\,\mu_1\nu_1}(x)\rangle = -\frac{8}{3|\vec{x}|^3}G^3M^3u^{\mu_1}u^{\nu_1}. \tag{8.55}$$

8.2.3 Auxiliary fields contribution

Before going any further, we have to address the issue of the auxiliary fields appearing in the Yang-Mills Lagrangian. This is

$$\begin{aligned} S_{\text{YM}} = &\int d^4k_1d^4k_2\delta^4(k_1+k_2)k_2^2[A_\mu(k_1)A^\mu(k_2) - 2B^{\mu\nu\rho}(k_1)B_{\mu\nu\rho}(k_2)] \\ &+ \int d^4k_1d^4k_2d^4k_3\delta^4(k_1+k_2+k_3)P_6\Big([k_{1\mu}A_\nu(k_1) + k_1^\rho B_{\rho\mu\nu}(k_1)]A^\mu(k_2)A^\nu(k_3)\Big). \end{aligned}$$

From it, we can directly read the Feynman rules for a "BB" propagator, for example. This is

$$D_{BB}^{\mu_1\nu_1\rho_1\alpha_1\beta_1\gamma_1}(k_1) = -\frac{\eta^{\mu_1\alpha_1}\eta^{\nu_1\beta_1}\eta^{\rho_1\gamma_1}}{k_1^2}, \tag{8.56}$$

while the B-gluon-gluon vertex is given by

$$\Gamma_{AAB}^{\mu_1\nu_1\rho_1\alpha_2\alpha_3 ijk}(p_1,p_2,p_3) = f^{ijk}p_1^{\mu_1}(\delta^{\nu_1\alpha_2}\delta^{\rho_1\alpha_3} - \delta^{\nu_1\alpha_3}\delta^{\rho_1\alpha_2}). \tag{8.57}$$

The computation of the contribution of this auxiliary field is then trivial: to use a B-field at four points there must be a vertex connecting the B-field propagator with two sources.

This expression will always give zero due to the antisymmetry of the vertex

$$\Gamma_{AAB}^{\mu_1\nu_1\rho_1\alpha_2\alpha_3}(p_1,p_2,p_3)J_{\alpha_2}(p_2)J_{\alpha_3}(p_3) = g^2\bar{\delta}(p_2^0)\bar{\delta}(p_3^0)p_1^{\mu_1}(\delta^{\nu_1\alpha_2}\delta^{\rho_1\alpha_3} - \delta^{\nu_1\alpha_3}\delta^{\rho_1\alpha_2})u_{\alpha_2}u_{\alpha_3} = 0. \tag{8.58}$$

Therefore, there is no contribution at four points from the B-field in the single copy. In the gravity Lagrangian that we obtained as a double copy of the Yang-Mills one, there will be three additional propagators, for the fields $g^{\mu\nu\rho\sigma}$, $\tilde{g}^{\mu\rho\sigma\nu}$ and $f^{\mu\rho\sigma\nu\tau\lambda}$, along with several interaction vertices. However, the fact that to use an auxiliary field at four points there must be a vertex connecting its propagator with two sources guarantees that this expression will always give zero due to the antisymmetry of the vertex. This will apply to every different vertex, so we will also have no auxiliary field contributions in the gravity theory.

Closing remarks

Just for the sake of comparison, we note that the expression for the graviton obtained using the Einstein-Hilbert Lagrangian in the previous chapters (c.f. eqs. (7.51) and (7.52)) takes the form

$$\begin{aligned}\kappa h^{00} &= -\frac{2GM}{r} - \frac{2G^2M^2}{r^2} + \mathcal{O}(G^3),\\ \kappa h^{ij} &= \left(-\frac{2GM}{r} + \frac{3G^2M^2}{r^2}\right)\eta^{ij} - \frac{G^2M^2}{r^2}\frac{x^ix^j}{r^2} + \mathcal{O}(G^3),\end{aligned}$$

whereas collecting the results from the previous sections, we can see that the computation using the double copied Lagrangian can be written as

$$\kappa H^{\mu\nu}(x) = \frac{4MG}{r}u^\mu u^\nu - \frac{2M^2G^2}{r^2}k^\mu k^\nu - \frac{8M^3G^3}{3r^3}u^\mu u^\nu + \mathcal{O}(G^3). \tag{8.59}$$

While it is remarkable that we were able to compute the $\mathcal{O}(G^3)$ terms, meaning that as a computational tool this is much more efficient than a brute force approach, the interpretation of our result is not totally clear. Indeed, for the Einstein-Hilbert calculation, it is easy tu understand that it corresponds to the graviton for a Schwarzschild metric in de Donder gauge. However, for the double copy inspired Lagrangian, such an interpretation is not straightforward. We can understand the reason for this by recalling that the fat graviton $H^{\mu\nu}$ was built as a double copy of A^μ, so we expect it to contain, in principle, four degrees of freedom. We expect those degrees of freedom to correspond to the original (or skinny) graviton $h^{\mu\nu}$, the dilaton ϕ and the antisymmetric 2-form $B^{\mu\nu}$ (which in four dimensions is equivalent to a scalar axion field χ). The procedure to extract the information of the component (or skinny) fields from the fat graviton is rather involved. We devote the next chapter to such a task.

Chapter 9

Skinny fields from fat gravitons

We concluded the previous chapter by stating that the fat graviton $H_{\mu\nu}$ contains the degrees of freedom of the skinny fields, namely the graviton $h^{\mu\nu}$, the dilaton ϕ and the antisymmetric 2-form $B^{\mu\nu}$. We will devote this chapter to making such a statement more accurate, meaning that we want to give explicit relations between the fat graviton, and his component fields, i.e. we want a map

$$H_{\mu\nu} = H_{\mu\nu}(\mathfrak{h}_{\mu\nu}, B_{\mu\nu}, \phi), \tag{9.1}$$

but more importantly, we are interested in extracting the information for the component fields in terms of the fat graviton. That is, we wish to find relations of the form

$$\mathfrak{h}_{\mu\nu} = \mathfrak{h}_{\mu\nu}(H_{\alpha\beta}), \qquad B_{\mu\nu} = B_{\mu\nu}(H_{\alpha\beta}), \qquad \phi = \phi(H_{\alpha\beta}). \tag{9.2}$$

In order to gain some intuition, we will first work out the map for a linear approximation of the theory. Later, we will show how to consider higher order approximations by perturbatively constructing solutions to the equations of motion.

9.1 Linearised level

Let us start by considering the leading order of an expansion in powers of the coupling constant κ. We refer to this as the linearised level, since it will not include interactions between the fields.

To do this, we have to specify what theory of gravity we'll be working with. We note that the gravity theory associated with a double copy of non-supersymmetric Yang-Mills is the so-called $\mathcal{N} = 0$ supergravity. This theory is defined by the action

$$S = \int d^dx\sqrt{-g}\left[\frac{2}{\kappa^2}R - \frac{1}{2(d-2)}\partial^\mu\phi\partial_\mu\phi - \frac{1}{6}e^{-2\kappa\phi/d-2}H^{\lambda\mu\nu}H_{\lambda\mu\nu}\right], \tag{9.3}$$

where $H_{\lambda\mu\nu}$ is the field strength of $B_{\mu\nu}$. Performing the expansion in powers of the gravity coupling constant κ, the equations of motion for the linearised approximation are

$$\begin{aligned} &\partial^2 h_{\mu\nu} - \partial_\mu\partial^\rho h_{\rho\nu} - \partial_\nu\partial^\rho h_{\rho\mu} + \partial_\mu\partial_\nu h + \eta_{\mu\nu}\left[\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial^2 h\right] = 0, \\ &\partial^2 B_{\mu\nu} - \partial_\mu\partial^\rho B_{\rho\nu} + \partial_\nu\partial^\rho B_{\rho\mu} = 0, \\ &\partial^2\phi = 0. \end{aligned} \tag{9.4}$$

It will be useful again to work with the tensor density

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} = \eta^{\mu\nu} - \kappa\, \mathfrak{h}^{\mu\nu}, \tag{9.5}$$

as it is common in perturbation theory [205]. We will sometimes refer to the perturbation field $\mathfrak{h}^{\mu\nu}$ as the gothic graviton. In terms of it, the de Donder gauge condition is simply $\partial_\mu \mathfrak{h}^{\mu\nu} = 0$ to all orders, as we have seen previously. At the linear order, the two metric perturbations are related by the simple equation

$$\mathfrak{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \tag{9.6}$$

and the linear gauge transformation generated by $x^\mu \to x^\mu - \kappa\,\xi^\mu$ is

$$\mathfrak{h}_{\mu\nu} \;\to\; \mathfrak{h}'_{\mu\nu} = \mathfrak{h}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial\cdot\xi. \tag{9.7}$$

This transformation is more convenient in what follows than the standard gauge transformation for $h_{\mu\nu}$ (where the last term is missing), since it will also represent a transformation between skinny and fat gravitons. Finally, the linearised equation of motion is

$$\partial^2\mathfrak{h}_{\mu\nu} - \partial_\mu\partial^\rho\mathfrak{h}_{\rho\nu} - \partial_\nu\partial^\rho\mathfrak{h}_{\rho\mu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\mathfrak{h}_{\rho\sigma} = 0. \tag{9.8}$$

In de Donder gauge, we have simply

$$\partial^2\mathfrak{h}_{\mu\nu} = 0. \tag{9.9}$$

Finally, we also have the freedom to gauge transform the antisymmetric form $B_{\mu\nu}$ to satisfy a Lorentz-like condition $\partial^\mu B_{\mu\nu} = 0$, so that the equation of motion is given by

$$\partial^2 B_{\mu\nu} = 0. \tag{9.10}$$

In summary, we have the equations of motion

$$\partial^2\phi \;= 0, \qquad \partial^2\mathfrak{h}_{\mu\nu} = 0, \qquad \partial^2 B_{\mu\nu} = 0, \tag{9.11}$$

which means that our skinny fields obey wave equations at the linearised level, so we will be able to determine the relations between them beginning with the simplest case: linearised waves.

9.1.1 Linear waves

We have seen that to a first approximation, our fields should behave as linear waves. These are well-known to double copy between gauge and gravity theories (see e.g. [176]). This property is crucial for the double copy description of scattering amplitudes, whose incoming and outgoing states are plane waves. Here, we use linear waves to motivate a prescribed relationship between fat and skinny fields, which will be generalised in later sections.

Let us start by considering a gravitational plane wave in the de Donder gauge. The free equation of motion for the graviton is simply $\partial^2 \mathfrak{h}_{\mu\nu} = 0$. Plane wave solutions take the form

$$\mathfrak{h}_{\mu\nu} = a_{\mu\nu} e^{ip\cdot x}, \qquad p^\mu a_{\mu\nu} = 0, \qquad p^2 = 0, \tag{9.12}$$

where $a_{\mu\nu}$ is a constant tensor, and the last condition follows from the equation of motion. Symmetry of the graviton implies $a_{\mu\nu} = a_{\nu\mu}$, and one may also fix a residual gauge freedom by setting $a \equiv a^\mu_\mu = 0$, so that $\mathfrak{h}_{\mu\nu}$ becomes a traceless, symmetric matrix. It is useful to further characterise the matrix $a_{\mu\nu}$ by introducing a set of $(d-2)$ polarisation vectors ϵ^i_μ satisfying the orthogonality conditions

$$p \cdot \epsilon^i = 0, \qquad q \cdot \epsilon^i = 0, \tag{9.13}$$

where q^μ ($q^2 = 0$, $p \cdot q \neq 0$) is an auxiliary null vector used to project out physical degrees of freedom for an on-shell massless vector boson. These polarisation vectors are a complete set, so they satisfy a completeness relation

$$\epsilon^i_\mu \epsilon^i_\nu = \eta_{\mu\nu} - \frac{p_\mu q_\nu + p_\nu q_\mu}{p \cdot q}. \tag{9.14}$$

Then the equation of motion for $\mathfrak{h}_{\mu\nu}$,together with the symmetry and gauge conditions on $a_{\mu\nu}$, imply that one may write

$$a_{\mu\nu} = f^{\not{t}}_{ij} \epsilon^i_\mu \epsilon^j_\nu, \tag{9.15}$$

where $f^{\not{t}}_{ij}$ is a traceless symmetric matrix. Thus, the linearised gravitational waves have polarisation states which can be constructed from outer products of vector waves, times traceless symmetric matrices.

Similarly, one may consider linear plane wave solutions for a two-form and ϕ field. Imposing Lorenz gauge $\partial^\mu B_{\mu\nu} = 0$ for the antisymmetric tensor, its free equation of motion becomes simply $\partial^2 B_{\mu\nu} = 0$. Thus plane wave solutions are

$$B_{\mu\nu} = \tilde{f}_{ij}\epsilon^i_\mu \epsilon^j_\nu e^{ip\cdot x}, \tag{9.16}$$

where $\tilde{f}_{ij}$ is a constant antisymmetric matrix. Meanwhile the free equation of motion for the scalar field is $\partial^2\phi = 0$, with plane wave solution

$$\phi = f_\phi e^{ip\cdot x}. \tag{9.17}$$

The double copy associates these skinny waves with a single fat graviton field $H_{\mu\nu}$ satisfying the field equation

$$\partial^2 H_{\mu\nu} = 0. \tag{9.18}$$

Thus, we can express the fat graviton as

$$H_{\mu\nu} = f_{ij}\epsilon^i_\mu \epsilon^j_\nu e^{ip\cdot x}, \tag{9.19}$$

where now f_{ij} is a general $d-2$ matrix and we have chosen a Lorentz-like gauge condition

$$\partial^\mu H_{\mu\nu} = 0 = \partial^\mu H_{\nu\mu}. \tag{9.20}$$

One may write the decomposition of the f_{ij} matrix as

$$f_{ij} = f^{\mathfrak{t}}_{ij} + \tilde{f}_{ij} + \delta_{ij}\frac{f_\phi}{d-2}, \tag{9.21}$$

and comparing with eqs. (9.15-9.19), we have

$$H_{\mu\nu} = \mathfrak{h}_{\mu\nu} + B_{\mu\nu} + \left(\eta_{\mu\nu} - \frac{p_\mu q_\nu + p_\nu q_\mu}{p\cdot q}\right)\frac{\phi}{d-2}, \tag{9.22}$$

which explicitly constructs the fat graviton from skinny fields. Working in position space for constant q, this becomes

$$H_{\mu\nu}(x) = \mathfrak{h}_{\mu\nu}(x) + B_{\mu\nu}(x) + P^q_{\mu\nu}\phi, \tag{9.23}$$

where we have defined the projection operator

$$P^q_{\mu\nu} = \frac{1}{d-2}\left(\eta_{\mu\nu} - \frac{q_\mu\partial_\nu + q_\nu\partial_\mu}{q\cdot\partial}\right), \tag{9.24}$$

which will be important throughout this chapter. Notice that

$$\hat{P}^q_{\mu\nu} \equiv (d-2)P^q_{\mu\nu}, \tag{9.25}$$

satisfies the (properly normalised) projection equation

$$\hat{P}^{q\,\lambda}_{\mu} \hat{P}^{q\,\nu}_{\lambda} = \hat{P}^{q\,\nu}_{\mu}. \tag{9.26}$$

However, its trace is given by

$$\hat{P}^{q\,\mu}_{\mu} = d-2, \tag{9.27}$$

unlike the other version, whose trace is unity.

As stated before, our goal rather than to construct fat gravitons from skinny fields is to determine skinny fields using a perturbative expansion based on the double copy and the fat graviton, so we will decompose the matrix field $H_{\mu\nu}$ into its antisymmetric, traceless symmetric, and trace parts. To that end, recall that we have been able to choose a gauge so that the trace, $\mathfrak{h}$, of the metric perturbation vanishes. Therefore the trace of the fat graviton determines the dilaton, while we may use symmetry to determine the skinny graviton and antisymmetric tensor from the fat graviton:

$$\phi = H^{\mu}{}_{\mu} \equiv H, \tag{9.28}$$

$$B_{\mu\nu} = \frac{1}{2}\left(H_{\mu\nu} - H_{\nu\mu}\right), \tag{9.29}$$

$$\mathfrak{h}_{\mu\nu} = \frac{1}{2}\left(H_{\mu\nu} + H_{\nu\mu}\right) - P^q_{\mu\nu} H. \tag{9.30}$$

These relations are in some way inverse to eq. (9.23), and are the map that we were after, to linearised level. We will refer to them as the *guts equations*. We will eventually extend them to higher order in perturbation theory. However, before that, let us explore a generalisation to the process we just considered.

9.1.2 General linearised vacuum solutions

For plane waves, the fat graviton is given in terms of skinny fields in eq. (9.23), and at first glance this equation is not surprising: one may always choose to decompose an arbitrary rank two tensor into its symmetric traceless, antisymmetric and trace parts, but there is potentially a problem in that the relationship becomes ambiguous: the trace of the skinny graviton may be nonzero (as is indeed the case in general gauges), and one must then resolve how the trace degree of freedom in $H^{\mu\nu}$ enters the trace of the skinny graviton, and the scalar field ϕ. Here we will restrict ourselves to skinny gravitons that are in de

Donder gauge. However, we will relax the traceless condition on the skinny graviton which was natural in the previous section. To account for the trace, we postulate that eq. (9.23) should be replaced by

$$H_{\mu\nu}(x) = \mathfrak{h}_{\mu\nu}(x) + B_{\mu\nu}(x) + P^q_{\mu\nu}(\phi - \mathfrak{h}). \tag{9.31}$$

To be useful, this definition of the fat graviton must be invertible. First, note that since

$$P^{q\,\mu}_{\mu} = 1, \tag{9.32}$$

the trace of $H_{\mu\nu}$ determines ϕ as before, while the antisymmetric part of $H_{\mu\nu}$ determines $B_{\mu\nu}$, i.e.

$$\phi = H^\mu{}_\mu, \tag{9.33}$$

$$B_{\mu\nu} = \frac{1}{2}\left(H_{\mu\nu} - H_{\nu\mu}\right), \tag{9.34}$$

Finally, the traceless symmetric part of the fat graviton is

$$\frac{1}{2}\left(H_{\mu\nu} + H_{\nu\mu}\right) - P^q_{\mu\nu}H = \mathfrak{h}_{\mu\nu}(x) - P^q_{\mu\nu}\mathfrak{h}. \tag{9.35}$$

Recalling the nature of $P^q_{\mu\nu}$ in relation to eq. (9.7), we can understand

$$\mathfrak{h}'_{\mu\nu}(x) \equiv \mathfrak{h}_{\mu\nu}(x) - P^q_{\mu\nu}\mathfrak{h} \tag{9.36}$$

as a gauge transformation of $\mathfrak{h}_{\mu\nu}(x)$. In practice, though, we find it useful to work with $\mathfrak{h}_{\mu\nu}(x)$ rather than $\mathfrak{h}'_{\mu\nu}(x)$, because at higher orders the gauge transformation to $\mathfrak{h}'_{\mu\nu}(x)$ leads to more cumbersome formulae. We therefore construct the traceless skinny metric $\mathfrak{h}_{\mu\nu} - P^q_{\mu\nu}\mathfrak{h}$ directly from $H_{\mu\nu}$, and recover the trace $\mathfrak{h}$ by inspection of the coefficient of $P^q_{\mu\nu}$ in $H_{\mu\nu}$ when ϕ is known. To this end, it is very convenient to restrict the use of q so that it only appears in $P^q_{\mu\nu}$, and not, for example, in a gauge choice: this simple trick ensures that it is straightforward to identify the full metric at higher orders, as we will see. It is also worth noticing that both $\mathfrak{h}_{\mu\nu}$ and $\mathfrak{h}'_{\mu\nu}$ are in de Donder gauge, since

$$\partial^\mu P^q_{\mu\nu}\mathfrak{h} = \frac{1}{d-2}\left(\partial_\nu - \frac{q_\nu\partial^2 + q\cdot\partial\,\partial_\nu}{q\cdot\partial}\right)\mathfrak{h} \tag{9.37}$$

$$= -\frac{1}{d-2}\frac{q_\nu}{q\cdot\partial}\partial^2\mathfrak{h} = 0. \tag{9.38}$$

Our relationship between skinny and fat fields still holds only for linearised fields; we will explicitly find corrections to eq. (9.31) at higher orders in perturbation theory in the following section.

9.2 Perturbative Corrections

Now let us construct non-linear perturbative corrections to spacetime metrics and dilatons using the double copy. Thus, we will map the problem of finding perturbative corrections to a simple calculation in gauge theory.

9.2.1 Perturbative metrics from gauge theory

Since the basis of our calculations is the perturbative expansion of gauge theory, we begin with the the Yang-Mills Lagrangian

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}, \tag{9.39}$$

where the field strength tensor is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{9.40}$$

The vacuum Yang-Mills equation is then given by

$$\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{b\mu} F^c_{\mu\nu} = 0, \tag{9.41}$$

where g is the coupling constant. We are interested in a perturbative solution of these equations, so that the gauge field A^a_μ can be written as a power series in the coupling:

$$A^a_\mu = A^{(0)a}_\mu + g A^{(1)a}_\mu + g^2 A^{(2)a}_\mu + \cdots . \tag{9.42}$$

Again, the perturbative coefficients $A^{(i)a}_\mu$ have no dependence on g. To zeroth order in the coupling, the Yang-Mills equation in Lorenz gauge $\partial^\mu A^a_\mu = 0$ is simply

$$\partial^2 A^{(0)a}_\mu = 0. \tag{9.43}$$

For our present purposes, two basic solutions of this equation will be of interest: wave solutions, and Coulomb-like solutions with isolated singularities.

Given a solution $A^{(0)a}_\mu$ of the linearised Yang-Mills equation, it is easy to write down an expression for the first order correction $A^{(1)a}_\mu$ by expanding the Yang-Mills equation to first order in g:

$$\partial^2 A^{(1)a}_\nu = -2 f^{abc} A^{(0)b\mu} \partial_\mu A^{(0)c}_\nu + f^{abc} A^{(0)b\mu} \partial_\nu A^{(0)c}_\mu. \tag{9.44}$$

A Fourier transformation yields the solution for the first perturbative correction in Fourier space in the familiar form

$$A_\nu^{(1)\,a}(k) = \frac{igf^{abc}}{k^2}\int đ^d p_1 đ^d p_2 \bar{\delta}(p_1+p_2-k) \times \left(2k_\mu A^{(0)\,b\,\mu}(p_1)A_\nu^{(0)\,c}(p_2) - p_{2\,\nu}A^{(0)\,b\,\mu}(p_1)A_\mu^{(0)\,c}(p_2)\right). \quad (9.45)$$

We note that this may also be written in the form

$$A^{(1)a\mu}(-p_1) = \frac{i}{2p_1^2}f^{abc}\int đ^d p_2 đ^d p_3 \bar{\delta}^d(p_1+p_2+p_3) \times \left[(p_1-p_2)^\gamma\eta^{\mu\beta} + (p_2-p_3)^\mu\eta^{\beta\gamma} + (p_3-p_1)^\beta\eta^{\gamma\mu}\right] A_\beta^{(0)b}(p_2)A_\gamma^{(0)c}(p_3). \quad (9.46)$$

Notice that the factor in square brackets in this equation obeys the same antisymmetry properties as the colour factor, f^{abc}, appearing in the equation. This is a requirement of colour-kinematics duality. Before using the double copy, it is necessary to ensure that this duality holds. Also, we neglect here the effect of auxiliary fields, since they come from the four-gluon vertex, and it contributes to the next order. We will, however, deal with them when considering higher order contributions.

Perturbative solution to Gravity

Now that we have computed perturbative solutions to Yang-Mills, we will use them to construct an analogous solution for the fat graviton. To that end we use a similar notation for its perturbation series:

$$H^{\mu\nu} = H^{(0)\mu\nu} + \frac{\kappa}{2}H^{(1)\mu\nu} + \left(\frac{\kappa}{2}\right)^2 H^{(2)\mu\nu} + \cdots. \quad (9.47)$$

The power of the double copy is that it is now completely trivial to compute the perturbative correction $H_{\mu\nu}^{(1)}$ to a linearised fat graviton $H_{\mu\nu}^{(0)}$. All we need to do, following [39, 40, 56], is to square the numerator in eq. (9.46), ignore the colour structure, and assemble fat gravitons by the rule that $A_\mu^{(0)a}(p)A_\nu^{(0)b}(p) \to H_{\mu\nu}^{(0)}(p)$. This straightforward procedure leads to

$$H^{(1)\mu\mu'}(-p_1) = \frac{1}{4p_1^2}\int đ^d p_2 đ^d p_3 \bar{\delta}^d(p_1+p_2+p_3) \times \left[(p_1-p_2)^\gamma\eta^{\mu\beta} + (p_2-p_3)^\mu\eta^{\beta\gamma} + (p_3-p_1)^\beta\eta^{\gamma\mu}\right] H_{\beta\beta'}^{(0)}(p_2) \times \left[(p_1-p_2)^{\gamma'}\eta^{\mu'\beta'} + (p_2-p_3)^{\mu'}\eta^{\beta'\gamma'} + (p_3-p_1)^{\beta'}\eta^{\gamma'\mu'}\right] H_{\gamma\gamma'}^{(0)}(p_3). \quad (9.48)$$

Notice that the basic structure of the perturbative calculation is that of gauge theory. The double copy upgrades the gauge-theoretic perturbation into a calculation appropriate for gravity, coupled to a dilaton and an antisymmetric tensor.

Now, by analogy with eqs. (9.28-9.30), we could now straightforwardly extract the trace and the symmetric fields:

$$\tilde{\phi}^{(1)} \equiv H^{(1)}, \tag{9.49}$$

$$\tilde{\mathfrak{h}}^{(1)}_{\mu\nu} \equiv \frac{1}{2}\left(H^{(1)}_{\mu\nu} + H^{(1)}_{\nu\mu}\right), \tag{9.50}$$

but, we cannot directly state that this $\tilde{\phi}^{(1)}$ is the usual dilaton and that $\tilde{\mathfrak{h}}^{(1)}_{\mu\nu}$ is the first order correction to the metric in some well-known gauge, as there is freedom for field redefinitions and gauge transformations. We address in depth this topic in the next section.

9.2.2 Relating fat and skinny fields

In section 9.1, we argued that the relationship between the fat and skinny fields in linear theory is[1]

$$H^{(0)}_{\mu\nu}(x) = \mathfrak{h}^{(0)}_{\mu\nu}(x) + B^{(0)}_{\mu\nu}(x) + P^q_{\mu\nu}(\phi^{(0)}(x) - \mathfrak{h}^{(0)}(x)). \tag{9.51}$$

Beyond linear theory, we can expect perturbative corrections to this formula, so that

$$H_{\mu\nu}(x) = \mathfrak{h}_{\mu\nu}(x) + B_{\mu\nu}(x) + P^q_{\mu\nu}(\phi(x) - \mathfrak{h}(x)) + \mathcal{O}(\kappa). \tag{9.52}$$

We define a quantity $\mathcal{T}_{\mu\nu}$, which we call the *transformation function* to make this equation exact:

$$H^{(1)}_{\mu\nu}(x) = \mathfrak{h}^{(1)}_{\mu\nu}(x) + B^{(1)}_{\mu\nu}(x) + P^q_{\mu\nu}(\phi^{(1)}(x) - \mathfrak{h}^{(1)}(x)) + \mathcal{T}^{(1)}_{\mu\nu}. \tag{9.53}$$

Because of the way that $\mathcal{T}^{(1)}_{\mu\nu}$, appears in eq. (9.53) it can only be constructed from linearised fields, so that $\mathcal{T}^{(1)}_{\mu\nu} = \mathcal{T}^{(1)}_{\mu\nu}(\mathfrak{h}^{(0)}_{\alpha\beta}, B^{(0)}_{\alpha\beta}, \phi^{(0)})$. More generally, at the nth order of perturbation theory

$$H^{(n)}_{\mu\nu}(x) = \mathfrak{h}^{(n)}_{\mu\nu}(x) + B^{(n)}_{\mu\nu}(x) + P^q_{\mu\nu}(\phi^{(n)}(x) - \mathfrak{h}^{(n)}(x)) + \mathcal{T}^{(n)}_{\mu\nu}(\mathfrak{h}^{(m)}_{\alpha\beta}, B^{(m)}_{\alpha\beta}, \phi^{(m)}), \tag{9.54}$$

where $m < n$. We can therefore determine $\mathcal{T}^{(n)}_{\mu\nu}$ iteratively in perturbation theory.

Before going further, let us pause for a moment to discuss the physical significance of $\mathcal{T}^{(n)}_{\mu\nu}$. Our understanding of it rests on two facts. Firstly, the double copy is known to work to all orders in perturbation theory for tree amplitudes. Secondly, the classical background field which we have been discussing is a generating function for tree scat-

[1] This is eq. (9.31), but we have added the decoration $^{(0)}$ to emphasize that such result only holds to linearised level.

tering amplitudes. Therefore it must be the case that scattering amplitudes computed from the classical fat graviton background fields equal their known expressions. So consider computing $H^{(n)}_{\mu\nu}$ via the double copy, and computing $\mathfrak{h}^{(n)}_{\mu\nu}, B^{(n)}_{\mu\nu}$ and $\phi^{(n)}$ using a standard perturbative solution of their coupled equations of motion. Then the difference $H^{(n)}_{\mu\nu} - \mathfrak{h}^{(n)}_{\mu\nu} - B^{(n)}_{\mu\nu}(x) - P^q_{\mu\nu}(\phi^{(n)}(x) - \mathfrak{h}^{(n)}(x)) \equiv \mathcal{T}^{(n)}_{\mu\nu}$ must vanish upon use of the LSZ procedure. We conclude that $\mathcal{T}_{\mu\nu}$ parameterises redundancies of the physical fields which are irrelevant for computing scattering amplitudes: gauge transformations and field redefinitions. Indeed, the very definition of $\mathcal{T}_{\mu\nu}$ requires choices of gauge: for example, the choice of de Donder gauge for the skinny graviton.

While the information in the transformation function contains little content of physical interest, it may be of some interest from the point of view of the mathematics of colour-kinematics duality. Indeed, in the special case of the self-dual theory, it is known how to choose an explicit parameterisation of the metric perturbation so that the double copy is manifest [113]. Choosing these variables therefore sets $\mathcal{T}_{\mu\nu} = 0$ to all orders, for self-dual spacetimes. Once the relevant variables have been chosen, then the kinematic algebra in the self-dual case was manifest at the level of the equation of motion of self-dual gravity: the algebra is one of area-preserving diffeomorphisms. Perhaps it is the case that an understanding of the transformation function in the general case will open the way towards a simple understanding of the full kinematic algebra.

Since $\mathcal{T}_{\mu\nu}$ parameterises choices which can be made during a calculation, such as the choice of gauge, we do not expect a particularly simple form for it. Nevertheless, to compare explicit skinny gravitons computed via the double copy with standard metrics, it may be useful to have an explicit form of $\mathcal{T}^{(n)}_{\mu\nu}$. It is always possible to compute such function directly through its definition, at the expense of perturbatively solving the coupled Einstein, scalar and antisymmetric tensor equations of motion. To understand this, let us look directly to the first order example. Rearranging eq. (9.53), we obtain the explicit expression for the first order transformation function

$$\mathcal{T}^{(1)}_{\mu\nu} = H^{(1)}_{\mu\nu}(x) - \mathfrak{h}^{(1)}_{\mu\nu}(x) - B^{(1)}_{\mu\nu}(x) - P^q_{\mu\nu}(\phi^{(1)}(x) - \mathfrak{h}^{(1)}(x)). \tag{9.55}$$

Thus, we need explicit expressions not only for the first order fat graviton $H^{(1)}_{\mu\nu}$, but also for the skinny fields $\mathfrak{h}^{(1)}_{\mu\nu}$, $\phi^{(1)}$ and $B^{(1)}_{\mu\nu}$. We will obtain them now. First, for the graviton $\mathfrak{h}^{\mu\nu}$, we propose a perturbative expansion analogous to that of the fat graviton

$$\mathfrak{h}^{\mu\nu} = \mathfrak{h}^{(0)\mu\nu} + \frac{\kappa}{2}\mathfrak{h}^{(1)\mu\nu} + \left(\frac{\kappa}{2}\right)^2 \mathfrak{h}^{(2)\mu\nu} + \cdots , \tag{9.56}$$

and an analogous one for the dilaton field ϕ. In terms of them, the equations of motion

[205, 206] to first order are

$$\begin{aligned}\partial^2\mathfrak{h}^{\mu\nu} =& -\frac{\kappa}{2}\Big\{\partial^\mu\mathfrak{h}_{\alpha\beta}\,\partial^\nu\mathfrak{h}^{\alpha\beta} - \frac{1}{d-2}\partial^\mu\mathfrak{h}\,\partial^\nu\mathfrak{h} - 2\mathfrak{h}^{\alpha\beta}\partial_\alpha\partial_\beta\mathfrak{h}^{\mu\nu} \\ &+ 2\partial_\alpha\mathfrak{h}^{\mu\beta}\,\partial_\beta\mathfrak{h}^{\alpha\nu} + 2\partial_\alpha\mathfrak{h}^{\mu\beta}\,\partial^\alpha\mathfrak{h}^\nu{}_\beta - 2\partial_\alpha\mathfrak{h}^{\mu\beta}\,\partial^\nu\mathfrak{h}^\alpha{}_\beta - 2\partial_\alpha\mathfrak{h}^{\nu\beta}\,\partial^\mu\mathfrak{h}^\alpha{}_\beta \\ &+ \frac{1}{2}\eta^{\mu\nu}\Big(2\partial_\alpha\mathfrak{h}^{\beta\gamma}\,\partial_\beta\mathfrak{h}^\alpha{}_\gamma - \partial_\alpha\mathfrak{h}_{\beta\gamma}\,\partial^\alpha\mathfrak{h}^{\beta\gamma} + \frac{1}{d-2}\,\partial_\alpha\mathfrak{h}\,\partial^\alpha\mathfrak{h}\Big)\Big\} \\ &- \frac{\kappa}{2(d-2)}\Big(\partial^\mu\phi\,\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}\partial_\alpha\phi\,\partial^\alpha\phi\Big) + O(\kappa^2),\end{aligned} \tag{9.57}$$

while the dilaton satisfies the equation

$$\partial^2\phi = \kappa\mathfrak{h}^{\alpha\beta}\partial_\alpha\partial_\beta\phi. \tag{9.58}$$

Note that the fields are already coupled to this order, so we need to solve the equations simultaneously. As we have seen in previous chapters, in order to solve the equations it is convenient to perform a Fourier transform. The dilaton equation gives in momentum space

$$\phi^{(1)}(-p_1) = \frac{2}{p_1^2}\int đ^d p_2 đ^d p_3 \delta\!\!\!{}^-{}^{(d)}(p_1+p_2+p_3)\mathfrak{h}_2^{(0)\alpha\beta}p_{3\alpha}p_{3\beta}\phi_3^{(0)}, \tag{9.59}$$

while the equation for the graviton yields

$$\begin{aligned}\mathfrak{h}^{(1)\mu\nu}(-p_1) =& -\frac{1}{p_1^2}\int đ^d p_2 đ^d p_3 \delta\!\!\!{}^-{}^{(d)}(p_1+p_2+p_3) \\ &\times\Big\{ - 2\mathfrak{h}_2^{(0)\alpha\beta}p_{3\alpha}p_{3\beta}\mathfrak{h}_3^{(0)\mu\nu} + 2p_{3\alpha}\mathfrak{h}_2^{(0)\mu\beta}p_{3\beta}\mathfrak{h}_3^{(0)\alpha\nu} + 2(p_2\cdot p_3)\mathfrak{h}_3^{(0)\mu\alpha}\mathfrak{h}_3^{(0)\nu}{}_\alpha \\ &- 2p_{2\alpha}\mathfrak{h}_2^{(0)\mu\beta}p_3^\nu\mathfrak{h}_3^{(0)\beta}{}_\alpha - 2p_{2\alpha}\mathfrak{h}_2^{(0)\nu\beta}p_3^\mu\mathfrak{h}_3^{(0)\beta}{}_\alpha + \eta^{\mu\nu}p_{2\alpha}\mathfrak{h}_2^{(0)\beta\gamma}p_{3\beta}\mathfrak{h}_3^{(0)\alpha}{}_\gamma \\ &+ \Big(p_2^\mu p_3^\nu - \frac{1}{2}\eta^{\mu\nu}(p_2\cdot p_3)\Big)\,\mathfrak{h}_{2\alpha\beta}^{(0)}\mathfrak{h}_3^{(0)\alpha\beta} - \frac{\mathfrak{h}_2^{(0)}\mathfrak{h}_3^{(0)} - \phi_2^{(0)}\phi_3^{(0)}}{d-2}\Big\},\end{aligned} \tag{9.60}$$

where we have used a convenient short-hand notation

$$\mathfrak{h}_i^{\mu\nu} \equiv \mathfrak{h}^{\mu\nu}(p_i), \qquad \phi_i^{\mu\nu} \equiv \phi^{\mu\nu}(p_i). \tag{9.61}$$

The trace of the gothic graviton yields

$$\mathfrak{h}^{(1)}(-p_1) = -\frac{1}{2p_1^2}\int đ^d p_2 đ^d p_3 \bar{\delta}^{(d)}(p_1+p_2+p_3)\Big\{-4p_{3\alpha}\mathfrak{h}_2^{(0)\alpha\beta}p_{3\beta}\mathfrak{h}_3^{(0)} + 4p_{2\alpha}\mathfrak{h}_3^{(0)\alpha\beta}\mathfrak{h}_3^{(0)\gamma}{}_{\beta}p_{3\gamma}$$
$$+ 4(p_2\cdot p_3)\mathfrak{h}_{2\alpha\beta}^{(0)}\mathfrak{h}_3^{(0)\alpha\beta} - 8p_{2\alpha}\mathfrak{h}_3^{(0)\beta}{}_{\alpha}\mathfrak{h}_2^{(0)\gamma\beta}p_3^{\gamma} + 2dp_{3\alpha}\mathfrak{h}_2^{(0)\alpha\beta}\mathfrak{h}_3^{(0)\gamma}{}_{\beta}p_{2\gamma}$$
$$-(p_2\cdot p_3)\Big[(d-2)\mathfrak{h}_{2\alpha\beta}^{(0)}\mathfrak{h}_3^{(0)\alpha\beta} - \mathfrak{h}_2^{(0)}\mathfrak{h}_3^{(0)} + \phi_2^{(0)}\phi_3^{(0)}\Big]\Big\}, \tag{9.62}$$

which can be further simplified to

$$\mathfrak{h}^{(1)}(-p_1) = \frac{1}{2p_1^2}\int đ^d p_2 đ^d p_3 \bar{\delta}^{(d)}(p_1+p_2+p_3)\Big\{4p_{3\alpha}\mathfrak{h}_2^{(0)\alpha\beta}p_{3\beta}\mathfrak{h}_3^{(0)} - 2(d-2)p_3^{\alpha}\mathfrak{h}_{2\alpha\beta}^{(0)}\mathfrak{h}_3^{(0)\beta\gamma}p_{2\gamma}$$
$$+(p_2\cdot p_3)\Big[(d-6)\mathfrak{h}_{2\alpha\beta}^{(0)}\mathfrak{h}_3^{(0)\alpha\beta} - \mathfrak{h}_2^{(0)}\mathfrak{h}_3^{(0)} + \phi_2^{(0)}\phi_3^{(0)}\Big]\Big\}. \tag{9.63}$$

If we consider now, for example, a fat graviton $H_{\mu\nu}^{(1)}(x)$, where there is no antisymmetric tensor[2], we have all the terms entering eq. (9.55), so we can compute $\mathcal{T}_{\mu\nu}^{(1)}$ under the simplifying assumption that $B_{\mu\nu} = 0$ so that $H_{\mu\nu}$ is symmetric. We find that when $\partial_\mu\mathfrak{h}^{(0)\mu\nu} = \partial_\mu H^{(0)\mu\nu} = 0$, then the transformation function is

$$\mathcal{T}^{(1)\mu\nu}(-p_1) = \int đ^d p_2 đ^d p_3 \bar{\delta}^d(p_1+p_2+p_3)\frac{1}{4p_1^2}\Big\{H_{2\alpha\beta}^{(0)}H_3^{(0)\alpha\beta}p_1^{\mu}p_1^{\nu} + 8p_2^{\alpha}H_{3\alpha\beta}^{(0)}H_2^{(0)\beta(\mu}p^{\nu)}$$
$$+ 8p_2\cdot p_3\, H_2^{(0)\mu\alpha}H_3^{(0)\nu}{}_{\alpha} - 2\eta^{\mu\nu}p_2\cdot p_3\, H_{2\alpha\beta}^{(0)}H_3^{(0)\alpha\beta} + 4\eta^{\mu\nu}p_2^{\alpha}H_{3\alpha\beta}^{(0)}H_2^{(0)\beta\gamma}p_{3\gamma}$$
$$+ P_q^{\mu\nu}\Big[2(d-6)p_2\cdot p_3\, H_{2\alpha\beta}^{(0)}H_3^{(0)\alpha\beta} - 4(d-2)p_2^{\alpha}H_{3\alpha\beta}^{(0)}H_2^{(0)\beta\gamma}p_{3\gamma}\Big]\Big\}, \tag{9.64}$$

where we have again used the convenient short-hand notations

$$H_i^{\mu\nu} \equiv H^{\mu\nu}(p_i), \qquad p^{(\mu}q^{\nu)} \equiv \frac{1}{2}\left(p^{\mu}q^{\nu} + p^{\nu}q^{\mu}\right). \tag{9.65}$$

This expression is valid for any symmetric $H_{\mu\nu}^{(0)}$, and the extension to general $H_{\mu\nu}^{(0)}$ is straightforward. Now that we have obtained a somewhat general result for the transformation function, it will be enlightening to show a specific example of it. We do this now in the following section, considering the simplest solution possible.

[2] We will encounter an example of this form in the next section.

9.3 One dilaton gravity example

Since the fat graviton equation of motion is simply $\partial^2 H^{(0)}_{\mu\nu} = 0$, it is natural to consider the solution

$$H^{(0)}_{\mu\nu} = \frac{\kappa}{2}\frac{M}{4\pi r}u_\mu u_\nu, \qquad \text{with } \; u_\mu = (1,0,0,0), \tag{9.66}$$

which corresponds to inserting a delta function source at the origin. We can see that this solution has the physical interpretation of a point mass which is also a source for the scalar dilaton. Indeed, the dilaton contained in the fat graviton is given by its trace:

$$\phi^{(0)} = -\frac{\kappa}{2}\frac{M}{4\pi r}. \tag{9.67}$$

Since the fat graviton is symmetric, $B_{\mu\nu} = 0$. Meanwhile the skinny graviton is

$$\mathfrak{h}^{(0)}_{\mu\nu} - P^q_{\mu\nu}\mathfrak{h}^{(0)} = \left(u_\mu u_\nu + P^q_{\mu\nu}\right)\frac{\kappa}{2}\frac{M}{4\pi r}. \tag{9.68}$$

We can thus interpret the skinny graviton as having the same form as the fat graviton. We have then established a map between skinny and fat fields that is invertible and valid to linearised level. In order to extend it to non-linear fields, we need to develop some technology first.

As a simple example of this formalism at work, let us compute the first order correction to the simple fat graviton eq. (9.66) corresponding to a metric and scalar field. To begin, we need to write $H^{(0)}_{\mu\nu}(p)$ in momentum space. This is simply

$$H^{(0)\mu\nu}(p) = \frac{\kappa}{2}Mu^\mu u^\nu \frac{\hat{\delta}^1(p^0)}{p^2}. \tag{9.69}$$

Inserting this into our expression for $H^{(1)}$, eq. (9.48), we quickly find

$$H^{(1)\mu\mu'}(-p_1) = \left(\frac{\kappa}{2}\right)^2\frac{M^2}{4p_1^2}\int \hat{d}^3p_2\hat{d}^3p_3\hat{\delta}^4(p_1+p_2+p_3)\frac{(p_2-p_3)^\mu(p_2-p_3)^{\mu'}}{p_2^2\,p_3^2}, \tag{9.70}$$

where $p_2^0 = 0 = p_3^0$, and consequently $p_1^0 = 0$. For future use, we note that $p_{1\mu}H^{(1)\mu\mu'}(-p_1) = 0$. Since all of the components of $H^{(1)}$ in the time direction vanish, we need only calculate the spatial components $H^{(1)ij}$. To do so, it is convenient to Fourier transform back to position space and compute firstly the Laplacian of $\nabla^2 H^{(1)ij}(x)$; we find

$$\begin{aligned}\nabla^2 H^{(1)ij} &= -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4} \int đ^3 p_2 đ^3 p_3 \frac{e^{-i\mathbf{p}_2\cdot\mathbf{x}} e^{-i\mathbf{p}_3\cdot\mathbf{x}}}{\mathbf{p}_2^2\mathbf{p}_3^2} (\mathbf{p}_2 - \mathbf{p}_3)^i (\mathbf{p}_2 - \mathbf{p}_3)^j \\ &= \left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4} \int d^3 y\, \delta^{(3)}(\mathbf{x}-\mathbf{y}) (\nabla^i_{\mathbf{x}} - \nabla^i_{\mathbf{y}})(\nabla^j_{\mathbf{x}} - \nabla^j_{\mathbf{y}}) \frac{1}{4\pi|\mathbf{x}|} \frac{1}{4\pi|\mathbf{y}|} \\ &= -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4(4\pi)^2} \left(\frac{2\delta^{ij}}{r^4} - \frac{4x^i x^j}{r^6}\right). \end{aligned} \tag{9.71}$$

It is now straightforward to integrate this expression using spherical symmetry and the known boundary conditions to find

$$H^{(1)}_{\mu\nu}(x) = -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4(4\pi r)^2} k_\mu k_\nu, \tag{9.72}$$

where $k_\mu = (0, \mathbf{x}/r)$.

It is interesting to pause for a moment to contrast this calculation with its analogue in Yang-Mills theory. The simplest gauge counterpart of the linearised fat graviton eq. (9.66) is

$$A^{(0)a}_\mu(x) = g c^a u_\mu \frac{1}{4\pi r} \qquad \Rightarrow \qquad A^{(0)a}_\mu(p) = g c^a u_\mu \frac{\delta\!\!\!{}^-{}^1(p^0)}{p^2}. \tag{9.73}$$

To what extent is the first non-linear correction to the Yang-Mills equation similar to the equivalent in our double-copy theory? The answer to this question is clear: they are distinctly different. Indeed, the colour structure of $A^{(1)a}_\mu$ is $f^{abc}c^b c^c = 0$, so $A^{(1)a}_\mu = 0$. However, the kinematic numerator of $A^{(1)a}_\mu$ identified by colour-kinematics duality is non-zero, so there is no reason for $H^{(1)}_{\mu\nu}$ to vanish. How the double copy propagates physical information from one theory to the other is unclear, but as a mathematical statement there is no issue with using the double copy to simplify gravitational calculations.

Back to the task of obtaining skinny fields, given our expression, eq. (9.72), for the fat graviton, it is now straightforward to extract the trace and the symmetric fields. In view of the fact that there is no projector dependence in this fat graviton, these are

$$\tilde{\phi}^{(1)} \equiv H^{(1)} = -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4(4\pi r)^2}, \tag{9.74}$$

$$\tilde{\mathfrak{h}}^{(1)}_{\mu\nu} \equiv \frac{1}{2}\left(H^{(1)}_{\mu\nu} + H^{(1)}_{\nu\mu}\right) = -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4(4\pi r)^2} k_\mu k_\nu. \tag{9.75}$$

However, we cannot directly state that this $\tilde{\phi}^{(1)}$ is the usual dilaton and that $\tilde{\mathfrak{h}}^{(1)}_{\mu\nu}$ is the first order correction to the metric in some well-known gauge. Instead, as stated in the previous section, we need to consider the transformation function in order to account for the gauge and field redefinition freedom.

9.3.1 Transformation function

We are now in a position to convert our fat graviton $H^{(1)}_{\mu\nu}(x)$, eq. (9.72) into skinny fields. The simple form of the $H^{(0)}_{\mu\nu}(x)$ leads to a simplification in the gauge transformation/field redefinition, since $p \cdot u = 0$ for a stationary source. Thus $\mathcal{T}^{(1)\mu\nu}$ is simply

$$\begin{aligned} \mathcal{T}^{(1)\mu\nu}(-p_1) = -\left(\frac{\kappa}{2}\right)^2 M^2 \int & đ^4 p_2 đ^4 p_3 \delta\!\!\!^{-4}(p_1+p_2+p_3) \frac{1}{4p_1^2} \frac{\delta\!\!\!^{-1}(p_2^0)}{p_2^2} \frac{\delta\!\!\!^{-1}(p_3^0)}{p_3^2} \\ & \times \left\{ 8p_2 \cdot p_3 u^\mu u^\nu - p_1^\mu p_1^\nu + 2\eta^{\mu\nu} p_2 \cdot p_3 + P_q^{\mu\nu} \left[4p_2 \cdot p_3\right] \right\}, \end{aligned} \tag{9.76}$$

in $d = 4$. Performing the Fourier transform, we find

$$\mathcal{T}^{(1)\mu\nu}(x) = -\left(\frac{\kappa}{2}\right)^2 \left[3u^\mu u^\nu + 2k^\mu k^\nu + 2P_q^{\mu\nu}\right] \frac{M^2}{4(4\pi r)^2}. \tag{9.77}$$

Let us now extract the skinny fields in de Donder gauge from our fat graviton, eq. (9.72). The relation between the fat and skinny fields is now given by

$$\begin{aligned} \mathfrak{h}^{(1)}_{\mu\nu}(x) + P^q_{\mu\nu}\left[\phi^{(1)}(x) - \mathfrak{h}^{(1)}(x)\right] &= H^{(1)}(x) - \mathcal{T}^{(1)}_{\mu\nu}(x) \\ &= -\left(\frac{\kappa}{2}\right)^2 k_\mu k_\nu \frac{M^2}{4(4\pi r)^2} + \left(\frac{\kappa}{2}\right)^2 \left[3u^\mu u^\nu + 2k^\mu k^\nu + 2P_q^{\mu\nu}\right] \frac{M^2}{4(4\pi r)^2}. \end{aligned} \tag{9.78}$$

Thus, the dilaton vanishes, since

$$\phi^{(1)}(x) = H^{(1)}(x) - \mathcal{T}^{(1)}(x) = 0. \tag{9.79}$$

Consequently, the negative of the trace of the metric is the only term on which $P_q^{\mu\nu}$ acts, so we find

$$\mathfrak{h}^{(1)}(x) = -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{2(4\pi r)^2}. \tag{9.80}$$

The metric is easily seen to be

$$\mathfrak{h}^{(1)}_{\mu\nu}(x) = \left(\frac{\kappa}{2}\right)^2 \left(3u_\mu u_\nu + k_\mu k_\nu\right) \frac{M^2}{4(4\pi r)^2}, \tag{9.81}$$

consistent with the anticipated trace.

It is natural to ask what is the non-perturbative static spherically-symmetric solution for which we are finding the fields. Exact solutions of the Einstein equations minimally coupled to a scalar field of this form were discussed by Janis, Newman and Winicour (JNW) [207] and have been extensively studied in the literature [207–213]. The complete solution is, in fact, a naked singularity, consistent with the no-hair theorem. The general JNW metric and dilaton can be expressed as

$$ds^2 = -\left(1 - \frac{\rho_0}{\rho}\right)^{\gamma} dt^2 + \left(1 - \frac{\rho_0}{\rho}\right)^{-\gamma} d\rho^2 + \left(1 - \frac{\rho_0}{\rho}\right)^{1-\gamma} \rho^2 d\Omega^2, \tag{9.82}$$

$$\phi = \frac{\kappa}{2}\frac{Y}{4\pi\rho_0} \log\left(1 - \frac{\rho_0}{\rho}\right). \tag{9.83}$$

where the two parameters ρ_0 and γ can be given in terms of the mass M and the scalar coupling Y as

$$\rho_0 = 2G\sqrt{M^2 + Y^2} = \left(\frac{\kappa}{2}\right)^2 \frac{\sqrt{M^2 + Y^2}}{4\pi}, \qquad \gamma = \frac{M}{\sqrt{M^2 + Y^2}}. \tag{9.84}$$

For $Y = 0$ and $M > 0$, we recover the Schwarzschild black hole. For $|Y| > 0$ and $M > 0$, the solution also decays for large ρ, but there is a naked singularity at $\rho = \rho_0$. We can write the JNW solution in de Donder gauge by applying the coordinate transformation $\rho = r + \rho_0/2$, where r is the Cartesian radius in the de Donder coordinates. Expanding in κ, the result is

$$\mathfrak{h}_{\mu\nu} = \frac{\kappa}{2}\frac{M}{4\pi r} u_\mu u_\nu + \left(\frac{\kappa}{2}\right)^3 \frac{1}{8(4\pi r)^2}\big((7M^2 - Y^2)u_\mu u_\nu + (M^2 + Y^2)k_\mu k_\nu\big) + \mathcal{O}(\kappa^5), \tag{9.85}$$

$$\phi = -\frac{\kappa}{2}\frac{Y}{4\pi r} + \mathcal{O}(\kappa^5). \tag{9.86}$$

Note that for $Y = M$, the expansions become

$$\mathfrak{h}_{\mu\nu} = \frac{\kappa}{2}\frac{M}{4\pi r} u_\mu u_\nu + \left(\frac{\kappa}{2}\right)^3 \frac{M^2}{4(4\pi r)^2}\big(3u_\mu u_\nu + k_\mu k_\nu\big) + \mathcal{O}(\kappa^5), \tag{9.87}$$

$$\phi = -\frac{\kappa}{2}\frac{M}{4\pi r} + \mathcal{O}(\kappa^5), \tag{9.88}$$

reproducing the skinny fields obtained above. We conclude that the JNW solution with $Y = M$ is the exact solution associated to the linearised fat graviton (9.66). Thus, in some sense, we understand the JNW naked singularities as the (non-rotating) objects naturally living in $\mathcal{N} = 0$ supergravity, instead of black holes. It might be the case that if there is angular momentum in the system there would also be a non-vanishing axion ($B_{\mu\nu}$), but this question requires further investigation.

As a further check to this correspondence, we will compute the next order in perturbation theory in the following section.

9.3.2 Higher orders

In section 9.2.1, we saw how fat graviton fields can be obtained straightforwardly from perturbative solutions of the Yang-Mills equations. These can then be translated to skinny fields, if necessary, after obtaining the relevant transformation functions $\mathcal{T}^{\mu\nu}$. Now let us briefly describe how this procedure generalises to higher orders.

To illustrate the procedure in a non-trivial example, let us compute the second order correction to the JNW fat graviton, $H^{(2)}_{\mu\nu}(x)$. In fact, a number of simplifications make this calculation remarkably straightforward. Firstly, the momentum space equation of motion for the auxiliary field appearing in the BDHK Lagrangrian, eq. (8.4), is

$$p_1^2 B^{(1)a}_{\mu\nu\rho}(-p_1) = \frac{i}{4} f^{abc} \int \hat{d}^4 p_2 \hat{d}^4 p_3 \hat{\delta}^4(p_1+p_2+p_3) p_{1\mu} \left[\eta_{\nu\beta}\eta_{\rho\gamma} - \eta_{\nu\gamma}\eta_{\rho\beta}\right] A^{(0)b\beta}(p_2) A^{(0)c\gamma}(p_3). \tag{9.89}$$

Notice that the term in square brackets is antisymmetric under interchange of β and γ; imposing this symmetry is a requirement of colour-kinematics duality because the associated colour structure is antisymmetric under interchange of b and c. A consequence of this simple fact is that, in the double copy, the auxiliary field vanishes in the JNW case (to this order of perturbation theory). In fact, two auxiliary fields appear in the double copy: one can take two copies of the field B, or one copy of B times one copy of the gauge boson A. In either case, the expression for an auxiliary field in the double copy in momentum space will contain a factor

$$\begin{aligned} &p_{1\mu} \left[\eta_{\nu\beta}\eta_{\rho\gamma} - \eta_{\nu\gamma}\eta_{\rho\beta}\right] H^{(0)\beta\beta'}(p_2) H^{(0)\gamma\gamma'}(p_3) \\ &\qquad = p_{1\mu} \left[\eta_{\nu\beta}\eta_{\rho\gamma} - \eta_{\nu\gamma}\eta_{\rho\beta}\right] \frac{\delta(p_2^0)\delta(p_3^0)}{p_2^2 p_3^2} u^\beta u^{\beta'} u^\gamma u^{\gamma'} = 0, \end{aligned} \tag{9.90}$$

because of the antisymmetry of the vertex in square brackets, and the factorisability of the tensor structure of the zeroth order JNW expression.

Consequently, the Yang-Mills four-point vertex plays no role in the the double copy for JNW at second order. Thus the Yang-Mills equation to be solved is simply

$$\begin{aligned} p_1^2 A^{(2)a\mu}(-p_1) = i f^{abc} \int \hat{d}^4 p_2 \hat{d} p_3 \hat{\delta}^4(p_1+p_2+p_3) \\ \times \left[(p_1-p_2)^\gamma \eta^{\mu\beta} + (p_2-p_3)^\mu \eta^{\beta\gamma} + (p_3-p_1)^\beta \eta^{\gamma\mu}\right] A^{(0)b}_\beta(p_2) A^{(1)c}_\gamma(p_3), \end{aligned} \tag{9.91}$$

using the symmetry of the expression under interchange of p_2 and p_3. Thus, $H^{(2)}$ is the solution of

$$p_1^2 H^{(2)\mu\mu'}(-p_1) = \frac{1}{2}\int d\!\!{}^-{}^4p_2 d\!\!{}^-{}^4p_3 \delta\!\!{}^-{}^4(p_1+p_2+p_3) \times \left[(p_1-p_2)^\gamma \eta^{\mu\beta} + (p_2-p_3)^\mu \eta^{\beta\gamma} + (p_3-p_1)^\beta \eta^{\gamma\mu}\right] \times \left[(p_1-p_2)^{\gamma'} \eta^{\mu'\beta'} + (p_2-p_3)^{\mu'} \eta^{\beta'\gamma'} + (p_3-p_1)^{\beta'} \eta^{\gamma'\mu'}\right] H^{(0)}_{\beta\beta'}(p_2) H^{(1)}_{\gamma\gamma'}(p_3). \tag{9.92}$$

This expression simplifies dramatically when we recall that $H^{(0)}_{\beta\beta'}(p_2)$ and $H^{(1)}_{\gamma\gamma'}(p_3)$ both have vanishing components of momentum in the time direction, so that $p_2^0 = 0 = p_3^0 = p_1^0$. Meanwhile $H^{(0)}_{\beta\beta'}(p_2) \propto u_\beta u_{\beta'}$. Thus,

$$p_1^2 H^{(2)}_{\mu\mu'}(-p_1) = 2\int d\!\!{}^-{}^4p_2 d\!\!{}^-{}^4p_3 \delta\!\!{}^-{}^4(p_1+p_2+p_3) H^{(0)}_{\mu\mu'}(p_2)\, p_2^\alpha H^{(1)}_{\alpha\beta}(p_3) p_2^\beta. \tag{9.93}$$

We find it convenient to Fourier transform back to position space, where we must solve the simple differential equation

$$\partial^2 H^{(2)}_{\mu\mu'}(x) = 2H^{(1)}_{\alpha\alpha'} \partial^\alpha \partial^{\alpha'} H^{(0)}_{\mu\mu'}. \tag{9.94}$$

Inserting explicit expressions for $H^{(0)}$, eq. (9.66) and $H^{(1)}$, eq. (9.72), and bearing in mind that the situation is static, the differential equation simplifies to

$$\nabla^2 H^{(2)}_{\mu\mu'}(x) = -\left(\frac{\kappa}{2}\right)^3 \frac{M^3}{(4\pi r)^3} \frac{u_\mu u_{\mu'}}{r^2}, \tag{9.95}$$

with solution

$$H^{(2)}_{\mu\mu'}(x) = -\left(\frac{\kappa}{2}\right)^3 \frac{M^3}{6(4\pi r)^3} u_\mu u_{\mu'}. \tag{9.96}$$

We could now, if we wished, extract the metric perturbation and scalar field corresponding to this expression. Indeed, it is always possible to convert fat gravitons into ordinary metric perturbations in a specified gauge. This conversion may be cumbersome, but it may also be unnecessary since an alternative possibility exists, namely to calculate physical observables, which must be manifestly invariant under gauge transformations and field redefinitions, directly from fat graviton fields, without referring to skinny fields at all. However, work in this front is still in progress.

9.3.3 Comparing with the diagrams result

The astute reader will have noted by now that this computation is exactly the same as the one performed using Feynman diagrams in the previous chapter. Back there, we obtained results for the expansion

$$\langle H^{\mu_1\nu_1}(x)\rangle = \langle H^{(1)\,\mu_1\nu_1}(x)\rangle + \langle H^{(2)\,\mu_1\nu_1}(x)\rangle + \langle H^{(3)\,\mu_1\nu_1}(x)\rangle + \dots \tag{9.97}$$

with the values

$$\kappa\langle H^{(1)\,\mu_1\nu_1}(x)\rangle = \frac{4MG}{r} u^{\mu_1} u^{\nu_1}, \tag{9.98}$$

$$\kappa\langle H^{(2)\,\mu_1\nu_1}(x)\rangle = -\frac{2M^2G^2}{r^2} k^{\mu_1} k^{\nu_1}, \tag{9.99}$$

$$\kappa\langle H^{(3)\,\mu_1\nu_1}(x)\rangle = -\frac{8M^3G^3}{3r^3} u^{\mu_1} u^{\nu_1}. \tag{9.100}$$

However, in order to compare with the expansion from the present chapter

$$H^{\mu\nu} = H^{(0)\,\mu\nu} + \left(\frac{\kappa}{2}\right) H^{(1)\,\mu\nu} + \left(\frac{\kappa}{2}\right)^2 H^{(2)\,\mu\nu} + \dots \tag{9.101}$$

we need to insert back the relation $G = \frac{\kappa^2}{32\pi}$, so we obtain the expressions

$$H^{(0)\,\mu\nu} = \left(\frac{\kappa}{2}\right) \frac{M}{4\pi r} u^\mu u^\nu, \tag{9.102}$$

$$H^{(1)\,\mu\nu} = -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4} \frac{1}{(4\pi r)^2} k^\mu k^\nu, \tag{9.103}$$

$$H^{(2)\,\mu\nu} = -\left(\frac{\kappa}{2}\right)^3 \frac{M^3}{6} \frac{1}{(4\pi r)^3} u^\mu u^\nu, \tag{9.104}$$

which match the coefficients obtained for the expansion directly solving the equations of motion.

9.4 Towards Schwarzschild

The ultimate objective of our program is to be able to describe the scattering of black holes. However, the solution we found in the last section is stationary, and is not a black hole, but a naked singularity, due to the effect of the dilaton. Let us briefly comment on the second issue. An extension of our treatment to black holes is not difficult conceptually, but it would be technically cumbersome. Indeed, it is easy to construct a fat graviton for the linearised Schwarzschild metric: we begin by noticing that, in the case of Schwarzschild ($d = 4$), we have

$$\mathfrak{h}_{\mu\nu}(r) = \frac{\kappa}{2}\frac{M}{4\pi r} u_\mu u_\nu + \mathcal{O}(\kappa^2), \qquad B_{\mu\nu}(x) = 0, \qquad \phi(x) = 0, \qquad \text{with } \; u_\mu = (1,0,0,0). \tag{9.105}$$

Now, the equation that relates skinny and fat fields is

$$H_{\mu\nu}(x) = \mathfrak{h}_{\mu\nu}(x) + B_{\mu\nu}(x) + P^q_{\mu\nu}(\phi - \mathfrak{h}), \tag{9.106}$$

which, taking into account eq. (9.105), reduces to

$$\begin{aligned} H_{\mu\nu}(x) &= \mathfrak{h}_{\mu\nu}(x) - P^q_{\mu\nu}\mathfrak{h}, \\ &= \frac{\kappa}{2}\frac{M}{4\pi r}u_\mu u_\nu + P^q_{\mu\nu}\left(\frac{\kappa}{2}\frac{M}{4\pi r}\right) \end{aligned} \tag{9.107}$$

The fat graviton depends on an arbitrary constant null vector q^μ. In this section, for illustration, we will make an explicit choice of $q^\mu = (-1, 0, 0, 1)$, and evaluate the action of the projector (9.24) in position space. Let us compute now $P^q_{\mu\nu}(1/r)$. Recalling the explicit form of the projector (for $d = 4$)

$$P^q_{\mu\nu} = \frac{1}{2}\left(\eta_{\mu\nu} - \frac{q_\mu\partial_\nu + q_\nu\partial_\mu}{q\cdot\partial}\right), \tag{9.108}$$

we can see that we need to deal with the operator $1/q\cdot\partial$. This is where our choice of the vector q^μ comes handy. Working explicitly in the coordinates $(u, v, w, \bar{w})$, where $u = t - z$, the product $q\cdot\partial = -2\partial_u$, and so, we can interpret

$$\frac{1}{\partial\cdot q} = -\frac{1}{2}\partial_u^{-1} = -\frac{1}{2}\int du \tag{9.109}$$

Now, let us compute the action of this operator on $1/r$. This is

$$\begin{aligned} \frac{1}{\partial\cdot q}\left(\frac{1}{r}\right) = -\int du\frac{1}{2r(u,v,w,\bar{w})} &= -\int du\frac{1}{\sqrt{(v-u)^2 + 4w\bar{w}}} \\ &= \ln(2(r+z)) + f(v,w,\bar{w}), \end{aligned} \tag{9.110}$$

where $f(v, w, \bar{w})$ is an integration "constant" that we choose to be zero. Using this, it is easy to show that

$$P^q_{\mu\nu}\left(\frac{\kappa}{2}\frac{M}{4\pi r}\right) = \frac{\kappa}{2}\frac{M}{4\pi r}\left(\frac{1}{2}(\eta_{\mu\nu} - q_\mu l_\nu - q_\nu l_\mu)\right), \tag{9.111}$$

where $l_\mu = (0, x, y, r+z)/(r+z)$, such that $q\cdot l = 1$. Then, substituting this back into eq. (9.107), we obtain the fat graviton

$$H_{\mu\nu} = \frac{\kappa}{2}\frac{M}{4\pi r}\left(u_\mu u_\nu + \frac{1}{2}(\eta_{\mu\nu} - q_\mu l_\nu - q_\nu l_\mu)\right). \tag{9.112}$$

It is easy to check that this fat graviton satisfies the gauge requirement $\partial^\mu H_{\mu\nu} = 0$, and the equation of motion $\partial^2 H_{\mu\nu} = 0$.

Going in the other direction, it is easy to compute the skinny fields given this fat graviton. Since $H_{\mu\nu}$ is traceless, the dilaton vanishes. Similarly $H_{\mu\nu}$ is symmetric, and therefore $B_{\mu\nu} = 0$. The skinny graviton can therefore be taken to be equal to the fat graviton. While this result seems to be at odds with (9.105), recall that they differ only by a gauge transformation (which leaves ϕ and $B_{\mu\nu}$ unaffected at this order) and that the skinny graviton we recover is traceless, as we would expect from eq. (9.35). However, trying to repeat the higher order procedure would be very difficult, since there is an explicit dependence of the linearised fat graviton in q_μ, and thus the transformation function eq. (9.64) becomes cumbersome.

We can also ask what fat graviton would be associated to the general JNW family of solutions, with M and Y generic. Since we are dealing with linearised fields, we can superpose contributions, and so we arrive at

$$H_{\mu\nu} = \frac{\kappa}{2}\frac{1}{4\pi r}\left(M\, u_\mu u_\nu + (M-Y)\,\frac{1}{2}(\eta_{\mu\nu} - q_\mu l_\nu - q_\nu l_\mu)\right). \tag{9.113}$$

Note that both the fat gravitons for our "simplest" dilaton-gravity example eq. (9.66), and the Schwarzschild fat graviton eq. (9.112) are specific cases (for $Y = M$ and $Y = 0$, respectively) of this JNW fat graviton.

The gauge theory "single copy" associated to this field is simply the Coulomb solution, which presents an apparent puzzle: ref. [183] argued that the double copy of the Coulomb solution is a pure Schwarzschild black hole, with no dilaton field, as discussed also in chapter 3. In this chapter, however, the double copy produces a JNW solution. The latter was also found in ref. [157], which thus concluded that the Schwarzschild solution is not generally obtained by the double copy, but can only be true in certain limits (such as the limit of an infinite number of dimensions). The resolution of this apparent contradiction is simply that one can choose whether or not the dilaton is sourced upon taking the double copy. It is well-known in amplitude calculations, for example, that gluon amplitudes can double copy to arbitrary combinations of amplitudes for gravitons, dilatons and/or B-fields. A simple example are amplitudes for linearly polarised gauge bosons: the double copied "amplitude" involves mixed waves of gravitons and dilatons. Thus, the result in the gravity theory straightforwardly depends on the linear combinations of the pairs of gluon polarisations involved in the double copy. Here, we may say that the Schwarzschild solution is a double copy of the Coulomb potential, as given by the Kerr-Schild double copy [183], just as one may say that appropriate combinations of amplitudes of gluons lead to amplitudes of pure gravitons. The analogue of more general gravity amplitudes with both gravitons and dilatons, also obtained as a double copy, is the JNW solution. Therefore the double copy of the Coulomb solution is somewhat ambiguous: in fact, it is any member of the JNW family of singularities, including the Schwarzschild metric. Note that the Kerr-Schild double copy is applicable only in the Schwarzschild case since the other members of the JNW family of spacetimes do not admit Kerr-Schild coordinates.

For the vacuum Kerr-Schild solutions studied in [183], in particular for the Schwarzschild black hole, it was possible to give an exact map between the gauge theory solution and the exact graviton field, making use of Kerr-Schild coordinates (as opposed to the de Donder gauge used here). For the general JNW solution, the double copy correspondence was inferred above from the symmetries of the problem and from the perturbative results. A more general double copy map would also be able to deal with the exact JNW solution. This remains an important goal, but one which is not addressed in this thesis.

Closing remarks

To summarise this chapter, we have developed the means to extract the component (or skinny) fields from the fat graviton. To do this, we perturbatively solved the Yang-Mills equation for the gauge field $A^{\mu\, a}$, and the solution for the fat graviton was obtained as a double copy in a BCJ sense. That is, leave the propagators unchanged, substitute coupling constants, and replace colour factors with a second copy of the kinematic information (interaction vertices). The method to extract the skinny fields is based on the guts equation. This is exact in the linearised level, but to higher orders in perturbation theory, needs to be corrected by the transformation function $\mathcal{T}^{\mu\nu}$, which encodes the remaining freedom. To determine the transformation function we had to obtain perturbative solutions of the $\mathcal{N}=0$ equations, and we tested this approach with one example fat graviton that satisfies the equation of motion. We start with a stationary point source in Yang-Mills, and double copy it to the simplest possible fat graviton, which we interpreted to be the JNW metric.

Although we have developed a formalism to separate the contributions from the (unwanted) dilaton field, i.e. the transformation function, it is possible that this is not necessary. Indeed, it might be possible to calculate physical observables, which must be invariant under gauge transformations and field redefinitions directly from fat graviton fields, without referring to skinny fields at all. Research on this front is currently being undertaken.

Chapter 10

Conclusions

In this thesis, we study how a number of solutions to classical equations of motion in General Relativity and Yang-Mills theory are related by a procedure analogous to the double copy of Bern, Carrasco and Johansson. We refer to our procedure as the double copy of classical solutions or classical double copy.

We start by considering a certain type of General Relativity solutions known as Kerr-Schild solutions or metrics. These are such that a deviation from the background metric (that we call the graviton, even though we are working in a classical framework) may be factorised as the product of a scalar function ϕ and two copies of a vector k_μ which is null with respect to both the background and the full metric, and is also a geodetic congruence.

Drawing an analogy with the BCJ double copy, where the tensor structure in the gravity side consists of two copies of the structure for the gauge theory, with the denominator remaining untouched, we propose a Kerr-Schild like ansatz for the gauge theory vector field as the product of the same scalar function that entered the graviton, and (just one copy of) the vector k_μ. We refer to this vector field as the single copy of the graviton. It is worth noting that unlike the BCJ double copy that is expressed naturally in momentum space, the classical double copy is expressed in position space.

We consider first the set of Kerr-Schild solutions that are also stationary, this is, they have no explicit depedence on time. We have shown that the single copy of such metrics are also solutions to Abelian (or linearised) Yang-Mills equations, which can be effectively treated as Maxwell equations. This yields an infinite class of solutions that satisfy the classical double copy process.

We showed a number of examples, starting with the Schwarzschild black hole, which single copies to a Coulomb solution. We further showed that the point-like sources for these systems also obey a double copy relation, with the mass being substituted with an electric charge. Higher dimensional generalisations of these solutions also exhibit a classical double copy behaviour.

The next example we considered was the Kerr-Black hole, along with the higher dimensional generalisation known as the Myers-Perry black hole. Although the single copy was again a solution to Maxwell-like equations, the relation was a bit less transparent. This is a consequence of the extended sources, unlike the point-like sources of the Schwarzschild/Coulomb case, needing an extra pressure term in the stress-energy tensor to keep the system stationary.

It is remarkable that the single copy of the Kerr-Schild metrics we have considered turn out to be solutions of Abelian gauge theories. However, this does not exactly come as a surprise. The fact that the Kerr-Schild ansatz linearise the Einstein tensor suggests that a single copy of this solution may be unable to capture non-linear effects. One related result states that for a static solution (which would apply to the Schwarzchild/Coulomb case), it is possible to perform a gauge transformation whose effect gets rid of the non-Abelian character [214].

However, it is also possible that there exists a solution to a non-Abelian gauge theory that double copies to the same Kerr-Schild metric. In favour of this scenario, there exist a couple examples (for infrared singularities and for compton scattering), where the gravity case may be obtained as a double copy of either QED or QCD.

There is a number of ways we can consider extensions or generalisations of the classical double copy. The first one we explored was the application of this technology to the Taub-NUT black hole. This solution may be cast into a double Kerr-Schild form, where we have two different scalar functions and null vectors. Unlike the original Kerr-Schild ansatz, a double Kerr-Schild form does not, in general, linearise the Einstein tensor.

However, in the case of the Taub-NUT solution we considered, all the non-linear terms appearing in this tensor vanished, thus rendering the Einstein equation linear. We then found that the Taub-NUT black hole single copies to a gauge theory dyon, an object that bears both electric and magnetic monopole charges, with the mass and the NUT parameter being replaced with electric and magnetic charge, respectively. We also showed that higher dimensional generalisations of this metric also obey the classical double copy, with every extra NUT parameter mapping into another magnetic charge.

A second possible extension to the original Kerr-Schild ansatz is the substitution of the background metric with a non-flat space. We consider the specific examples of Schwarzschild, Kerr and Taub-NUT black holes over a (Anti)de Sitter background. The single copy of these solutions takes the exact same form as in the flat case, since we are not modifying the graviton.

In the three cases, we found that a single copy of these metrics are indeed solutions of Maxwell-like equations over the non-flat de Sitter background. The generalisation of the equations of motion is immediate, by considering covariant derivatives compatible with the background metric. However, a new feature appears for the zeroth copy. Unlike the flat background case, where the scalar function acted as a propagator by solving the equation of motion for the D'Alembertian operator, in the de Sitter background case a new term emerges. Due to the numeric factors accompanying it, this term suggests having its origin in a biadjoint theory that contains a conformal mass.

Thus far this is just a tentative interpretation, since we have not been able to study the double copy over less trivial non-flat backgrounds. It is also worth noting here that it is also possible to cast the de Sitter metric into a Kerr-Schild form, so the black holes over de Sitter space fit into a double Kerr-Schild scheme.

Further motivation for the study of the Kerr-Schild double copy over non-flat backgrounds comes from the recent paper by Adamo et.al. [143], which studies a double copy for amplitudes in curved space. Their work studies gauge fields which are perturbations around plane-wave solutions, whose amplitudes double copy corresponds to an amplitude for gravitons defined with respect to a gravitational plane wave background. This process belongs to a class of double copies that we call type A, where a gauge field over a non-trivial background is related to a graviton living on a non-Minkowski metric, with both metrics having a double copy-like relation. We have found that this is also possible in the context of the classical solutions of Kerr-Schild form, and this process also yields a consistent zeroth copy. We have also seen an alternative way to interpret the classical double copy, that we call type B, and it works by associating a graviton defined with respect to a non-flat background with a gauge field living on the same curved spacetime. One caveat here is that, unlike the previous case, we don't have a well defined interpretation for the zeroth copy, which suggests this is not a general feature, but a map that applies in certain cases. We have worked out a few examples that could have meaningful application, for example, performing this process over a de Sitter background is potentially relevant for cosmological purposes. Furthermore, we consider other examples where a type A double copy is not possible (due to not being a Kerr-Schild solution), but a type B double copy still works. A family of examples we have considered is that of conformally flat backgrounds (including, for example, the Kerr metric over an Einstein static universe background). In this context, we have found that for spherically symmetric gauge fields over such conformally flat metrics, a vacuum Maxwell equation over Minkowski implies a vacuum equation over the curved background, and this is a well defined type B double copy. Open questions include whether the type A double copy is valid when the background is not of Kerr-Schild form, as well as determining the extent of the type B double copy.

Although we started the treatment of the classical double copy by studying its stationary limit, it is most interesting to study time dependent situations. We first showed two examples with rather trivial explicit time dependence, namely plane-waves and shock-waves.

However, the study of time dependent solutions becomes more involved when we consider the case of solutions in General Relativity and gauge theory that describe accelerating particles, and thus include radiation. This study was motivated by the fact that there exist a class of GR solutions in Kerr-Schild form to describe accelerated particles. However, unlike the past examples where we had vacuum solutions, in this case there exists an additional stress energy tensor, corresponding to a null fluid. This source term has a nice physical interpretation that we describe below.

The procedure to construct the single copy is the same as for the stationary case, with the novelty that the Maxwell-like equations also yield a source term in the form of a null current. This null current and the null-fluid stress energy tensor in the gravity solution are related in a classical double copy-like fashion. Indeed, we can obtain the source term in the gravity side by taking a second copy of the vector appearing in the current, while we leave untouched the scalar part. There is a discrepancy in the numerical factors of the sources, but this fits a larger picture involving scattering amplitudes.

We noted that the current obtained from the single copy encodes the radiation from the particle. Indeed, we found that the Kerr-Schild-like vector field corresponds to a boosted Coulomb solution. Therefore, all the radiation effects need to be contained in the right hand side of the equation. This behaviour also occurs in the gravity side. Indeed, the linearisation of the Einstein tensor hints that it should be unable to encode the non-linear effects of radiation. We can go beyond this and note that the Kerr-Schild vector is twist-free. This implies the solution has Petrov type D, so it cannot contain radiation. Thus, all the radiation effects must be included in the energy momentum tensor.

It is an interesting fact that both sources contain the radiation effects and also satisfy a double copy relation. To further understand this, we studied the case of a particle that suffers a sudden acceleration during an infinitesimal time. We found that the Fourier transform of the current in gauge theory corresponds to the scattering amplitude for Bremsstrahlung. This led us to interpret the Fourier transform of the null fluid energy-momentum tensor on the gravity side as the scattering amplitude for the emission of gravitons.

Working in momentum space (after taking the Fourier transform), this is, with scattering amplitudes instead of solutions, the numerical factors between the two theories perfectly match, and the double copy becomes evident. We interpret this as strong evidence of the classical double copy being the same as the BCJ double copy.

We found, however, that the gravity solution we considered fails to satisfy the energy conditions (which are considered the test of physicality for a system). It should have, perhaps, been expected that this would happen, because we are not taking into account what accelerates the particle under study. In the gauge theory this poses no problem, since the agent responsible for this effect may be an external background field. However, because of the equivalence principle, any field that accelerates the particle contains energy and should then modify the metric.

The second part of the thesis focuses on applying the double copy formalism to perform computations in classical gravity by exploiting the relative simplicity of gauge theories. This is, in some sense, closer to the approach used by BCJ to compute integrands in supergravity. Unlike the results in the first part, where the Kerr-Schild condition linearises the Ricci tensor, and thus restricts the traetment to solutions of a linear (or linearised) gauge theory, in this case, we start with a (full) non-abelian theory. Of course, this comes at the cost of needing to treat the problem in a perturbative manner, since we are no longer working with exact solutions.

The idea of obtaining classical solutions order by order from a perturbation theory was first exploited by Duff, who showed that the Schwarzschild metric can be obtained by computing tree-graphs of a quantum gravity theory. In the original paper, he obtained results up to second order in the gravitational constant G. The results should be straightforwardly generalized to higher orders in G, but the complexity of the computation is prohibitive. (Actually, in a related front, he went on to compute quantum corrections to the metric).

We repeated this exercise using, instead of the Einstein-Hilbert Lagrangian, one obtained as a "double copy" of a Yang-Mills Lagrangian that was tailored to satisfy colour kinematics duality. Such a Lagrangian requires the introduction of auxiliary fields. We identified a double copy of the gluon field A^μ with the field $H^{\mu\nu}$, that we call the fat graviton, and using the double copy-inspired Lagrangian it is easy to compute its value up to third order in G by using the Duff approach . However, unlike Duff's computation, where the interpretation of the result is straightforward (the VEV of the graviton gives the classical perturbation of the Minkowski metric) we need further work to interpret our results.

This is because, in particular, we considered solutions that correspond to double copies of a non-supersymmetric Yang-Mills theory. Such a theory is (somewhat jokingly) referred to as $\mathcal{N} = 0$ supergravity, and in principle contains the degrees of freedom of a graviton $h_{\mu\nu}$ (although in practice we use $\mathfrak{h}_{\mu\nu}$), a dilaton ϕ and the antisymmetric 2-form $B_{\mu\nu}$ (in 4 dimensions, this is equivalent to the axion χ). All the information of our component fields is contained in the fat graviton. The means to extract the component (or skinny) fields from the fat graviton are central to our story. Before going into detail, let us comment on the method to obtain the solutions (we use this instead of Feynman diagrams).

For a distribution of charges, one perturbatively solves the Yang-Mills equation for the gauge field $A^{\mu\,a}$, and the solution for the fat graviton is given by double copying the gauge theory expression according to BCJ rules. That is, leave the propagators unchanged, substitute coupling constants, and replace colour factors with a second copy of the kinematic information (interaction vertices). Once we have a fat graviton, the method to extract the skinny information from it is based on the so-called guts equation. This is exact in the linearised level, and it comes simply from the fact that all the fields behave as plane waves. To higher order in perturbation theory, the guts equation needs a correction $\mathcal{T}^{\mu\nu}$, that we call a transformation function as, being supposed to vanish on shell we interpret it as encoding the possible gauge transformation and field redefinition freedom. In practice, to obtain it we have matched our fat graviton solution to a conventional perturbative solution of the $\mathcal{N} = 0$ equations. Once obtained, however, it can be used for arbitrary distributions. In order to test the approach, we developed the example of the simplest possible fat graviton that satisfies the linear (wave) equation of motion for the fat graviton. We start with a stationary point source in Yang-Mills, and double copy it to the simplest possible fat graviton. This results in the JNW metric, which has a naked singularity with a non-zero scalar field ϕ, where the mass and the dilaton charge are the same.

However, it was also noted that choosing carefully the fat graviton is equivalent to choosing whether or not to source the dilaton. This is similar to an aspect in amplitudes that it is possible to choose polarization states in the gauge theory such that the dilaton and the 2-form are sourced or not. As mentioned earlier, the ultimate goal of this program is to apply it to astrophysical problems, namely, to calculate gravitational observables for relevant physical sources. Related ideas are considered in ref. [157], where solutions of (radiating) dilaton gravity are obtained as a double copy of the solutions of classical Yang-Mills equations coupled to dynamical point particles carrying colour charge. This formalism has been extended to obtain radiating classical solutions to a biadjoint scalar theory from the Yang-Mills solution. This is similar to what we call the zeroth copy. As we have seen, it is possible to extend the transformation function formalism to higher orders, in order to extract the skinny fields from a fat graviton. However, it might be the case that we are able to extract physical information from the double copy without the need of explicitly computing the skinny fields, by calculating physical observables, which must be manifestly invariant under gauge transformations and field redefinitions, like the radiated energy-momentum as computed in [157].

These are very exciting times for physics, as the recent discoveries by the LIGO and VIRGO collaborations signal the dawn of gravitational wave astronomy. As the field moves towards a precision measurements era, we believe new methods, like those reviewed in this thesis, will be useful to help revolutionise science and achieve ground breaking discoveries.

Appendix A

Derivatives of retarded quantities

In chapter 6, in order to establish a geometric interpretation of the Kerr-Schild ansatz, we used the expressions.

$$\begin{aligned}\partial_\alpha r &= k_\alpha \Delta + \lambda_\alpha, \qquad \Delta \equiv (-1 + rk\cdot\dot{\lambda}),\\ \partial_\alpha k_\beta &= r^{-1}[\eta_{\alpha\beta} - k_\beta\lambda_\alpha - k_\alpha\lambda_\beta - k_\alpha k_\beta \Delta],\\ \partial_\alpha \lambda_\beta &= k_\alpha \dot{\lambda}_\beta, \qquad \partial_\alpha \dot{\lambda}_\beta = k_\alpha \ddot{\lambda}_\beta.\end{aligned} \tag{A.1}$$

These are derivatives with respect to x^ν. In this appendix, we show how to compute derivatives of these and other quantities. In order to do so we'll take a detour, and analyse the solution in classical electrodynamics that describes the fields generated by an accelerated particle. It is called the Liénard-Wiechert potential and we will now go through its derivation.

A.1 The Liénard-Wiechert potential

We want to solve the Maxwell equation (over a flat background),

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{A.2}$$

where the field-strength tensor $F^{\mu\nu}$ is defined as usual by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Then, considering Lorenz gauge $\partial_\mu A^\mu = 0$, the Maxwell equations reduce to the wave equation

$$\partial^2 A^\nu = j^\nu, \tag{A.3}$$

and we consider j^ν to be the current vector for a point particle,

$$j^\nu(x) = e\int d\tau \lambda^\nu \delta^4(x - y(\tau)), \tag{A.4}$$

where $y^\mu(\tau)$ describes the world line of the particle, and $\lambda(\tau)$ is its proper time derivative. The solution for eq. (A.3) takes the form

$$A_\mu = k \int d^4x' G(x-x') j_\mu(x'), \tag{A.5}$$

where $G(x-x')$ is called a Green function of the d'Alembertian operator, and is defined by

$$\partial^2 G(x-x') = \delta(x-x'). \tag{A.6}$$

One such Green function, known as "retarded" can be written in the manifestly covariant form

$$G_{\mathrm{R}}(x) = \frac{1}{2\pi}\Theta(x^0)\delta(x^2). \tag{A.7}$$

Note that this is truly covariant, since proper Lorentz transformations can't change the sign of x^0, thus keeping the Heaviside function invariant. With this retarded Green function in hand, we may now insert the tensor (A.4) into the perturbation (A.5), we get the expression,

$$A_\mu = k \int d^4x' d\tau \frac{1}{2\pi}\Theta(x^0 - x'^0)\delta((x-x')^2) e\lambda_\mu \delta^4(x' - y(\tau)). \tag{A.8}$$

We can now perform the integration over x' using $\delta^4(x' - y(\tau))$. This yields

$$A_\mu = \frac{ke}{2\pi}\int d\tau \Theta(x^0 - y(\tau)^0)\delta((x-y(\tau))^2)\lambda_\mu. \tag{A.9}$$

One relation that will be very important from now on is[1]

$$\Theta(x^0 - y(\tau)^0)\delta((x-y(\tau))^2) = \frac{\delta(\tau-\tau_*)}{2(x-y(\tau))\cdot\lambda}. \tag{A.10}$$

Inserting this result into eq. (A.9), we get

$$\begin{aligned} A_\mu &= \frac{ke}{2\pi}\int d\tau \frac{\delta(\tau-\tau_*)}{2(x-y(\tau))\cdot\lambda}\lambda_\mu \\ &= \frac{ke}{2\pi}\frac{\lambda_\mu}{2(x-y(\tau))\cdot\lambda}\bigg|_{\mathrm{ret}}. \end{aligned} \tag{A.11}$$

[1]This is derived, for example, in Jackson [196].

In the last equation, the label “ret" means we evaluate the expression in the retarded time $\tau = \tau_*$.

A.2 Derivatives

We will now apply the ideas of the last section to compute spacetime derivatives of the retarded distance r, and the null vector k^μ. The most important piece of information now is the relation eq. (A.10). Recalling the definition for the vector R^μ of eq. (6.8), that equation can be cast into the form

$$\Theta(R^0)\delta(R^2) = \frac{\delta(\tau - \tau_*)}{2R \cdot \lambda}. \tag{A.12}$$

Using eq. (A.12), we get

$$\int d\tau \Theta(R^0)\delta(R^2) f = \int d\tau \frac{\delta(\tau - \tau_*)}{2R \cdot \lambda} f. \tag{A.13}$$

Finally, we use the definition of the retarded distance eq. (6.6) to write

$$\left.\frac{f}{2r}\right|_{\text{ret.}} = \int d\tau \Theta(R^0)\delta(R^2) f. \tag{A.14}$$

We are now ready to obtain a spacetime derivative of the retarded distance r with the aid of eq. (A.14). Considering the case $f = 1$, we have

$$\left.\frac{1}{2r}\right|_{\text{ret.}} = \int d\tau \Theta(R^0)\delta(R^2), \tag{A.15}$$

and the derivative of this is[2]

$$\begin{aligned} \partial_\mu \left(\frac{1}{2r}\right) &= \partial_\mu \int d\tau \Theta(R^0)\delta(R^2) \\ -\frac{1}{2r^2}\partial_\mu r &= \int d\tau \Theta(R^0)\partial_\mu \delta(R^2). \end{aligned} \tag{A.16}$$

In principle, the spatial derivative should also act over $\Theta(R^0)$ and this yields a Dirac delta function, which results in t heo evaluation of the function at the point $y^\mu(\tau)$. However, since we are working with point charges, we are interested in computing things away from that point, so we ignore that part. The last important relation we need (but are not

[2]From now on we drop the "ret" label. Quantities inside integrals are non-retarded, though.

proving now) is

$$\partial_\mu \delta(R^2) = -\frac{R_\mu}{R \cdot \lambda} \frac{d}{d\tau} \delta(R^2). \tag{A.17}$$

Now, inserting eq. (A.17) in eq. (A.16) we obtain

$$\begin{aligned} -\frac{1}{2r^2}\partial_\mu r &= -\int d\tau \Theta(R^0) \frac{R_\mu}{R \cdot \lambda} \frac{d}{d\tau} \delta(R^2) \\ &= \int d\tau \Theta(R^0) \delta(R^2) \frac{d}{d\tau} \left(\frac{R_\mu}{r} \right). \end{aligned} \tag{A.18}$$

In the last line, we integrated by parts. In principle, the operator $\frac{d}{d\tau}$ should also act over $\Theta(R^0)$, but we'll ignore this, since it would yield a term evaluated on the particle world-line. Using again (although in an inverse way) eq. (A.14), we get

$$-\frac{1}{2r^2}\partial_\mu r = \frac{1}{2r} \frac{d}{d\tau} \left(\frac{R_\mu}{r} \right). \tag{A.19}$$

To evaluate the derivative appearing in eq (A.24),

$$\frac{d}{d\tau} \left(\frac{R_\mu}{r} \right) = \frac{\frac{d}{d\tau} R_\mu}{r} - \frac{R_\mu \frac{d}{d\tau} r}{r^2}, \tag{A.20}$$

we need an expression for the derivative in τ of the retarded distance. This is

$$\begin{aligned} \frac{d}{d\tau} r &= \frac{d}{d\tau}(R \cdot \lambda) \\ &= -\lambda \cdot \lambda + R \cdot \dot{\lambda} \\ &= 1 + rk \cdot \dot{\lambda}. \end{aligned} \tag{A.21}$$

We put everything together to obtain the expression

$$-\frac{1}{2r^2}\partial_\mu r = \frac{1}{2r^2}[-\lambda_\mu - k_\mu(1 + rk \cdot \dot{\lambda})], \tag{A.22}$$

which finally yields the result

$$\partial_\mu r = \lambda_\mu + k_\mu(1 + rk \cdot \dot{\lambda}). \tag{A.23}$$

We may now obtain an expression for the spacetime derivative of the vector k^μ. To do this, we start by noting that, from eqs. (6.9) and (A.14), we can write

$$\frac{k_\mu}{2} = \frac{R_\mu}{2r} = \int d\tau \Theta(R^0) \delta(R^2) R_\mu. \tag{A.24}$$

With this expression in hand, we can differentiate to get

$$\frac{1}{2}\partial_\nu k_\mu = \int d\tau \Theta(R^0)\delta(R^2)\partial_\nu R_\mu + \int d\tau \Theta(R^0)\partial_\nu \delta(R^2) R_\mu.$$

From the definition eq. (6.8) it is straightforward to find $\partial_\nu R_\mu = \eta_{\mu\nu}$. Using also eq. (A.17), we have

$$\begin{aligned}\frac{1}{2}\partial_\nu k_\mu &= \int d\tau \Theta(R^0)\delta(R^2)\eta_{\nu\mu} + \int d\tau \Theta(R^0)\delta(R^2)\frac{d}{d\tau}\left(\frac{R_\nu R_\mu}{r}\right), \\ &= \int d\tau \Theta(R^0)\delta(R^2)\left[\eta_{\nu\mu} + \frac{d}{d\tau}\left(\frac{R_\nu R_\mu}{r}\right)\right],\end{aligned}$$

and using eq. (A.14) we may write

$$\frac{1}{2}\partial_\nu k_\mu = \frac{1}{2r}\left[\eta_{\nu\mu} + \frac{d}{d\tau}\left(\frac{R_\nu R_\mu}{r}\right)\right]. \tag{A.25}$$

We need now to evaluate the proper time derivative

$$\begin{aligned}\frac{d}{d\tau}\left(\frac{R_\nu R_\mu}{r}\right) &= \frac{1}{r}\left(R_\mu \frac{d}{d\tau}R_\nu + R_\nu \frac{d}{d\tau}R_\mu\right) - \frac{1}{r^2}R_\mu R_\nu \frac{d}{d\tau}r \\ &= -k_\mu \lambda_\nu - k_\nu \lambda_\mu - k_\mu k_\nu (1 + rk\cdot\dot{\lambda}).\end{aligned} \tag{A.26}$$

In the last line we used the result from eq. (A.21). Inserting eq. (A.26) in eq. (A.25) we get our result:

$$\partial_\mu k_\nu = r^{-1}[\eta_{\mu\nu} - k_\mu\lambda_\nu - k_\nu\lambda_\mu - k_\mu k_\nu(1 + rk\cdot\dot{\lambda})]. \tag{A.27}$$

A.3 Direct techniques

Although we have obtained the identities eqs. (A.23) and (A.27) for the spacetime derivatives of k^μ and r, the method of the last section is rather slow. We will show that this can be done in a more immediate (though less thorough) way. We start with the derivative of r by differentiating its definition (eq. (6.6))

$$r = \lambda_\mu(x^\mu - y^\mu), \tag{A.28}$$

to get

$$\begin{aligned}\frac{\partial r}{\partial x^\nu} &= \lambda_\mu \frac{\partial}{\partial x^\nu}(x^\mu - y^\mu) + \frac{\partial \lambda_\mu}{\partial x^\nu}(x^\mu - y^\mu) \\ &= \lambda_\mu \delta^\mu_\nu - \lambda_\mu \frac{\partial y^\mu}{\partial x^\nu} + \frac{\partial \lambda_\mu}{\partial x^\nu} r k^\mu.\end{aligned}$$

In the last line, we've used eq. (6.9), to substitute $(x^\mu - y^\mu)$ with rk^μ. Now, using the chain rule

$$\begin{aligned}\frac{\partial r}{\partial x^\nu} &= \lambda_\nu - \frac{\partial \tau}{\partial x^\nu}\frac{\partial y^\mu}{\partial \tau}\lambda_\mu + \frac{\partial \tau}{\partial x^\nu}\frac{\partial \lambda_\mu}{\partial \tau} r k^\mu \\ &= \lambda_\nu - \frac{\partial \tau}{\partial x^\nu}\lambda^\mu \lambda_\mu + \frac{\partial \tau}{\partial x^\nu}\dot{\lambda}_\mu r k^\mu.\end{aligned}$$

Finally we can write

$$\begin{aligned}\frac{\partial r}{\partial x^\nu} &= \lambda_\nu + \frac{\partial \tau}{\partial x^\nu} + \frac{\partial \tau}{\partial x^\nu}\dot{\lambda}_\mu r k^\mu \\ &= \lambda_\nu + k_\nu(1 + r\dot{\lambda}_\mu k^\mu),\end{aligned}$$

using the relation (eq. (6.5))

$$\frac{\partial \tau}{\partial x^\nu} = k_\nu. \tag{A.29}$$

We turn now our attention to the derivative of k^μ. We start with its definition (eq. (6.7))

$$k^\mu = r^{-1}(x^\mu - y^\mu). \tag{A.30}$$

We then differentiate that expression and obtain

$$\begin{aligned}\partial_\nu k_\mu &= -\frac{1}{r^2}(x_\mu - y_\mu)\partial_\nu r + r^{-1}(\eta_{\mu\nu} - \partial_\nu y_\mu) \\ &= -r^{-1}k_\mu \partial_\nu r + r^{-1}(\eta_{\mu\nu} - k_\nu \lambda_\mu).\end{aligned} \tag{A.31}$$

In the last line, we again used eq. (6.9) to substitute $(x^\mu - y^\mu)$ with rk^μ as well as the relation

$$\partial_\nu y_\mu = \frac{\partial \tau}{\partial x^\nu}\frac{\partial y_\mu}{\partial \tau} = k_\nu \lambda_\mu. \tag{A.32}$$

Regrouping the terms in eq. (A.31), we have

$$\begin{aligned}\partial_\nu k_\mu &= r^{-1}(-k_\mu \partial_\nu r + \eta_{\mu\nu} - k_\nu \lambda_\mu) \\ &= r^{-1}(-k_\mu(k_\nu(1 + rk_\alpha \dot{\lambda}^\alpha) + \lambda_\nu) + \eta_{\mu\nu} - k_\nu \lambda_\mu),\end{aligned}$$

where, in the last line, we have inserted eq. (A.23). Finally, a rewrite of this gives

$$\begin{aligned}\partial_\nu k_\mu &= r^{-1}(-k_\mu k_\nu(1 + rk_\alpha \dot{\lambda}^\alpha) - k_\mu \lambda_\nu + \eta_{\mu\nu} - k_\nu \lambda_\mu) \\ &= r^{-1}(\eta_{\mu\nu} - k_\mu \lambda_\nu - k_\nu \lambda_\mu - k_\mu k_\nu(1 + rk_\alpha \dot{\lambda}^\alpha)),\end{aligned} \tag{A.33}$$

which is the same as eq. (A.27). Finally, the use of the chain rule, along with eq. (A.29), renders straightforward the proof of the identities

$$\partial_\alpha \lambda_\beta = k_\alpha \dot{\lambda}_\beta, \qquad \partial_\alpha \dot{\lambda}_\beta = k_\alpha \ddot{\lambda}_\beta. \tag{A.34}$$

Appendix B

Some Fourier computations

B.1 Computing integrals

In chapter 7 we encountered the Fourier transform integrals

$$p^{ij}(\vec{x}) \equiv \frac{\kappa^4}{16} \int đ^3 k_1 đ^3 k_2 đ^3 k_3 e^{i\vec{k}_1 \cdot \vec{x}_1} \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \frac{k_2^i k_3^j}{\vec{k}_1^2 \vec{k}_2^2 \vec{k}_3^2} T_{00}(\vec{k}_2) T_{00}(\vec{k}_3), \tag{B.1}$$

$$f^{ij}(\vec{x}) \equiv \frac{\kappa^4}{16} \int đ^3 k_1 đ^3 k_2 đ^3 k_3 e^{i\vec{k}_1 \cdot \vec{x}_1} \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \frac{k_2^i k_2^j}{\vec{k}_1^2 \vec{k}_2^2 \vec{k}_3^2} T_{00}(\vec{k}_2) T_{00}(\vec{k}_3), \tag{B.2}$$

where $T_{00}(k) = M\delta(k_0)$. In this appendix we show explicitly how to solve them. The first step is to show that they satisfy the simple relations

$$f^{ij}(\vec{x}) = \frac{1}{\nabla^2}(V \partial^i \partial^j V), \tag{B.3}$$

$$p^{ij}(\vec{x}) = \frac{1}{\nabla^2}(\partial^i V \partial^j V). \tag{B.4}$$

To do so, we can note that using the explicit form of T_{00} and the definition

$$V(x) \equiv \frac{\kappa^2}{4} \int đ^4 k \frac{e^{ikx}}{k^2} T_{00}(k), \tag{B.5}$$

we can perform the trivial integration in the k^0 component to obtain the expression

$$V(\vec{x}) = \frac{\kappa^2}{4} \int đ^3 k \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2} T_{00}(\vec{k}), \tag{B.6}$$

and thus

$$\partial^i V(\vec{x}) \partial^j V(\vec{x}) = -\frac{\kappa^4}{16} \int đ^3 k_2 đ^3 k_3 \frac{k_2^i k_3^j}{\vec{k}_2^2 \vec{k}_3^2} e^{i(\vec{k}_2 + \vec{k}_3)\cdot\vec{x}} T_{00}(\vec{k}_2) T_{00}(\vec{k}_3). \tag{B.7}$$

Now, defining $\rho(\vec{x}) \equiv \partial^i V(\vec{x})\partial^j V(\vec{x})$, we can take its Fourier transform

$$\rho(\vec{k}_1) = \int d\!\!{}^-{}^3 x e^{-i\vec{k}_1\cdot\vec{x}}\partial^i V(\vec{x})\partial^j V(\vec{x}), \tag{B.8}$$

and inserting here eq. (B.7), we have

$$\rho(\vec{k}_1) = \frac{\kappa^4}{16}\int d\!\!{}^-{}^3 x d\!\!{}^-{}^3 k_2 d\!\!{}^-{}^3 k_3 e^{i(\vec{k}_1+\vec{k}_2+\vec{k}_3)\cdot\vec{x}}\frac{k_2^i k_3^j}{\vec{k}_2^2\vec{k}_3^2}T_{00}(\vec{k}_2)T_{00}(\vec{k}_3). \tag{B.9}$$

Performing the integration over x yields the expression

$$\rho(\vec{k}_1) = \frac{\kappa^4}{16}\int d\!\!{}^-{}^3 k_2 d\!\!{}^-{}^3 k_3 \delta\!\!{}^-{}^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{k_2^i k_3^j}{\vec{k}_2^2\vec{k}_3^2}T_{00}(\vec{k}_2)T_{00}(\vec{k}_3). \tag{B.10}$$

Finally, transforming back to position space, we have

$$\frac{1}{\nabla^2}(\partial^i V\partial^j V)(\vec{x}) = \frac{\kappa^4}{16}\int d\!\!{}^-{}^3 k_1 d\!\!{}^-{}^3 k_2 d\!\!{}^-{}^3 k_3 e^{i\vec{k}_1\cdot\vec{x}}\delta\!\!{}^-{}^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{k_2^i k_3^j}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}T_{00}(\vec{k}_2)T_{00}(\vec{k}_3), \tag{B.11}$$

and comparing with the definition eq. (B.1), we have proven

$$p^{ij}(\vec{x}) = \frac{1}{\nabla^2}(\partial^i V\partial^j V). \tag{B.12}$$

Using an analogous process we can show the equality

$$f^{ij}(\vec{x}) = \frac{1}{\nabla^2}(V\partial^i\partial^j V). \tag{B.13}$$

A second step now is to look for particular solutions of

$$\nabla^2 p^{ij}(\vec{x}) = \partial^i V(\vec{x})\partial^j V(\vec{x}), \tag{B.14}$$
$$\nabla^2 f^{ij}(\vec{x}) = V(\vec{x})\partial^i\partial^j V(\vec{x}). \tag{B.15}$$

We will use the singularity as a boundary condition, which greatly simplifies the computation, since we circumvent the fact of the integrals being divergent[1]. This may be achieved by inspection. Indeed

$$\begin{aligned}\nabla^2 f^{xx}(\vec{x}) &= V(\vec{x})(\partial^x\partial^x V(\vec{x})),\\ &= \frac{G^2M^2}{r}\partial^x\partial^x\frac{1}{r},\\ &= -G^2M^2\left(\frac{-2x^2+y^2+z^2}{r^6}\right).\end{aligned} \tag{B.16}$$

[1]Credit is due to Niclas Westerberg, who developed such techniques in [215].

It is easy to show that this equation is solved by

$$f^{xx}(\vec{x}) = -\frac{G^2M^2}{2}\left(\frac{x^2 - \frac{1}{2}(y^2+z^2)}{r^4}\right). \tag{B.17}$$

Actually, we can extend this logic to obtain the relations

$$\nabla^2 f^{ij} = -G^2M^2\left(\frac{r^2\eta^{ij} - 3x^ix^j}{r^6}\right), \tag{B.18}$$

$$\nabla^2 p^{ij} = -G^2M^2\left(\frac{x^ix^j}{r^6}\right), \tag{B.19}$$

that will be equivalent to

$$f^{ij} = -\frac{G^2M^2}{4}\left(\frac{3x^ix^j - r^2\eta^{ij}}{r^4}\right), \tag{B.20}$$

$$p^{ij} = -\frac{G^2M^2}{4}\left(\frac{x^ix^j - r^2\eta^{ij}}{r^4}\right). \tag{B.21}$$

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