UNIVERSITY
of
GLASGOW

Pricing Derivatives with Stochastic Volatility

by

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Submitted in fulfilment of the requirements for
the degree of Doctor of Philosophy

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University of Glasgow

July 2016

Abstract

This Ph.D. thesis contains 4 essays in mathematical finance with a focus on pricing Asian option (Chapter 4), pricing futures and futures option (Chapter 5 and Chapter 6) and time dependent volatility in futures option (Chapter 7).

In Chapter 4, the applicability of the Albrecher et al.(2005)'s comonotonicity approach was investigated in the context of various benchmark models for equities and commodities. Instead of classical Lévy models as in Albrecher et al.(2005), the focus is the Heston stochastic volatility model, the constant elasticity of variance (CEV) model and the Schwartz (1997) two-factor model. It is shown that the method delivers rather tight upper bounds for the prices of Asian Options in these models and as a by-product delivers super-hedging strategies which can be easily implemented.

In Chapter 5, two types of three-factor models were studied to give the value of commodities futures contracts, which allow volatility to be stochastic. Both these two models have closed-form solutions for futures contracts price. However, it is shown that Model 2 is better than Model 1 theoretically and also performs very well empirically. Moreover, Model 2 can easily be implemented in practice. In comparison to the Schwartz (1997) two-factor model, it is shown that Model 2 has its unique advantages; hence, it is also a good choice to price the value of commodity futures contracts. Furthermore, if these two models are used at the same time, a more accurate price for commodity futures contracts can be obtained in most situations.

In Chapter 6, the applicability of the asymptotic approach developed in Fouque et al.(2000b) was investigated for pricing commodity futures options in a Schwartz (1997) multi-factor model, featuring both stochastic convenience yield and stochastic volatility. It is shown that the zero-order term in the expansion coincides with the Schwartz (1997) two-factor term, with averaged volatility, and an explicit expression for the first-order correction term is provided. With empirical data from the natural gas futures market, it is also demonstrated that a significantly better calibration can be achieved by using the correction term as compared to the standard Schwartz (1997) two-factor expression, at virtually no extra effort.

In Chapter 7, a new pricing formula is derived for futures options in the Schwartz (1997) two-factor model with time dependent spot volatility. The pricing formula can also be used to find the result of the time dependent spot volatility with futures options prices in the market. Furthermore, the limitations of the method that is used to find the time dependent spot volatility will be explained, and it is also shown how to make sure of its accuracy.

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Acknowledgements

First of all, I would like to express my deepest gratitude to my principal supervisor Professor Christian-Oliver Ewald for his invaluable wisdom and insight that deepen and enrich my knowledge of mathematical finance, and his innumerable help and patience on every aspect of my research in the past 3-4 years.

I would also like to take this opportunity to thanks to Dr Yuping Huang, who encouraged me at the beginning of my Ph.D. study and Zhe Zong, who gave me help in studying Kalman filter technique. I wish to thank all the follow friends, Huichou Huang, Yang Zhao, Xiao Zhang, Xuan Zhang and Suo Cao who give me chances to experience many interesting things beyond mathematical finance. I trust that all other people whom I have not specifically mentioned here are aware of my deep appreciation.

Finally, I am deeply obliged to my parents and my wife for giving me the opportunity, support and freedom to pursue my interests — any success I have achieved as a Ph.D. student is instantly related to them.

Dedication

To my parents, Dongyao Chen and Zhaomei Cai

and my wife, Liying Zhang.

Declaration

I declare that, the materials of this Ph.D. thesis is the result of my own work except where explicit reference is made to the contribution of others. These materials may also appear as published and/or working papers co-authored with my principal supervisor Professor Christian-Oliver Ewald (Chapter 4, Chapter 6 and Chapter 7), and my colleague Zhe Zong (Chapter 5). The bulk of materials (mathematical derivations and discussions) in Chapter 4, Chapter 6 and Chapter 7 are the results of my own ideas. Except the introduction of the Kalman filter technique and figures from the Kalman filter technique, the rest of Chapter 5 are undertaken by myself. Furthermore, part of materials from Chapter 6 are presented at 2015 Quantitative Methods in Finance conference.

Published papers resulting from this thesis.

[1] Chen, J. and Ewald, C.O., On the Performance of the Comonotonicity Approach for Pricing Asian Options in some Benchmark Models from Equities and Commodities, (to appear in) *Review of Pacific Basin Financial Markets and Policies*.

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July 2016

Chapter 1

Introduction

Options have been important financial instruments for hedging, speculation, arbitrage, and risk mitigation purposes in the financial markets over the past few years. They are fundamentally different from forward and futures contracts. For options' holders, they have the right to do something, but the holder does not have to exercise this right. However, in a forward or futures contract, the two parties have to do certain actions when the contract is expired. Furthermore, it costs nothing to enter into a forward or futures contracts, whereas a premium is necessary to buy an option contract.

Generally, there are two types of options. A call option gives the buyer (the owner or holder of the option) the right, but not the obligation, to buy an underlying asset or instrument at a specified price on a specified date. A put option gives the buyer (the owner or holder of the options) the right, but not the obligation, to sell an underlying asset or instrument at a specified price on a specified date. The price in the contract is known as the strike price; the date in the contract is known as the expiration date.

For a call option, it will normally be exercised when the strike price is below the market value of the underlying asset. The cost to have the underlying asset to the buyer is the strike price plus the premium. Compared with those who do not hold call option, the profit to the call option holder is the difference between the market price and strike price minus the premium. When the strike price is above the market value of the underlying asset, a call option will normally not be exercised. The loss to a call option holder is the premium, compared with those who do not hold call option.

For a put option, it will normally be exercised when the strike price is above the market value of the underlying asset. Compared with the non-put option holder, a put option holder can benefit from the profit of the difference between the market price and strike price minus the premium. When the strike price is below the market value of the underlying asset, a put option normally will not be exercised. The loss to a put option holder is the premium, compared with a non-put option holder.

Whether the call option or put option, the income to the option seller is the premium, and the loss to the option seller is the potential increment or decrement of the underlying price in the future market, depending on the form of option.

In terms of the underlying asset or the calculation of how or when the investor receives a certain payoff, options can be defined as vanilla options and exotic options.

Vanilla options contain European style options and American style options; the main difference being that American style options can be exercised before the expiration date, whereas the European style options can only be exercised on the expiration date. Therefore, generally American style options will be more expensive than European style options. They are often traded on exchange markets, and most options traded on exchange markets are American style. However, compared with American style options, in general, European style options are easier to analyse and frequently used as a benchmark for American style options.

An exotic option is an option that has more complex features than vanilla options. These features could reflect on the changing of the underlying, the payoff type, and the manner of settlement. For example, the payoff for a look-back option at maturity is not just on the value of the underlying instrument at maturity, but its maximum or minimum value during the option's life. Therefore, exotic options are generally traded on the over-the-counter (OTC) markets.

1.1 Asian Option

An Asian option is one of exotic options, in which the underlying is the average of a financial variable, such as prices of equities, commodities, interests or exchange rates. The pricing of such derivatives has been of utmost interest ever since trading started in the mid 1980's, initially mostly on OTC markets but for the last few years also on exchanges such as the London Metal Exchange. The most common claim of fixed strike asian call option is:

$$C(T) = max(A(0,T) - K, 0)$$

where A denotes the average price for the period $[0, T]$, and K is the strike price. The equivalent put option is given by:

$$P(T) = max(K - A(0,T), 0)$$

The average used in the calculation of Asian options can be defined in an arithmetic average or a geometric average. For example, in the case of discrete time, an arithmetic Asian option is:

$$A(0,T) = \frac{1}{N}\sum_{i}^{N} S_{t_i}$$

and a geometric Asian option is:

$$A(0,T) = \sqrt[N]{\prod_{i}^{N} S_{t_i}}$$

An Asian option has many obvious advantages. Firstly, because of the average feature, arithmetic Asian options can reduce the market risk of underlying assets over a certain time interval. Furthermore, arithmetic Asian options are typically less expensive than European or American options. In addition, they are also more appropriate than European or American options for meeting certain needs of the investors. Taking an investor who holds a large number of foreign currency exchanges as an example, the investor does not want to face the risk of currency exchanges, because the exchange rate may change everyday and will be highly volatile in the future. In this situation, an Asian put option can help the investor to reduce the exposure to the uncertain exchange rate, thereby guaranteeing that the average exchange rate is realized above a certain level during that time.

If S_{t_i} is assumed to follow a log normal distribution, a closed-form solution for the value of a geometric Asian option can be found because geometric average of log normal random variable is still log normal. These closed-form expressions are very similar to formulas for pricing vanilla options in the Black-Scholes model.

However, even in the Black-Scholes world, there is no simple closed-form solution for the value of arithmetic Asian options, since the arithmetic average of log normal random variables is no longer log normal. It is very difficult to price arithmetic Asian options since their payoff is determined by the value of arithmetic average of some underlying asset during a pre-set period of time. Generally, people can use the Monte Carlo simulation technique and partial differential equation method to get its price. Nevertheless, for the purpose of getting an accurate price for an arithmetic Asian option, the Monte Carlo simulation technique often requires a large amount of simulation trials. For example, one may use at least 100,000 simulation trials for giving the value of an arithmetic Asian option. Therefore, in general, the Monte Carlo simulation technique is very time-consuming in terms of getting an accurate result.

Geman and Yor (1993, 1996) derived analytic representations in the form of complex integrals for the price of a normalized Asian call option in the Black-Scholes model. This fundamental work was followed up and extended upon by a large number of authors, uncovering important relations to fundamental problems in probability theory and classical functions. The demands of financial practitioners however have long moved beyond the Black-Scholes model and the model error imposed by the Black-Scholes assumptions often outweighs any computational progress that some analytic formulas and techniques based on the Black-Scholes assumptions seem to offer.

1.2 Commodities Futures Option

Futures option, also known as option on futures, is similar to stock option, but the underlying is a single futures contract. It was first traded on an experimental basis in 1982 which was authorized by the Commodity Futures Trading Commission in the US. In 1987, permanent trading was approved and since then the popularity of futures options has grown very quickly with investors. Generally, they are American style options and traded on exchange markets. However, for some energy commodities, like crude oil and natural gas, the futures options are both European style and American style and are traded on exchange markets as well.

The buyer of a futures option has the right, but not the obligation, to enter into a futures contract at a certain futures contract price by a certain date. The price in the contract is known as the strike price; the date in the contract is known as the expiration date. Generally, the expiration date of futures options will be one day or two days in advance, compared with the expiration date of corresponding futures contracts.

A call futures option gives the holder the right to enter into a long futures contract at the strike price when the strike price of the futures option is lower than the price of a futures contract in the market. In this case, the holder will benefit from a cash amount, which equals the difference between the settlement futures contract price and the strike price. However, the holder must pay a premium to buy this right; thus, if the futures option is not exercised, the premium will be the capital loss to the holder.

A put futures option gives the holder the right to enter into a long futures contract at the strike price when the strike price of the futures option is higher than the price of a futures contract in the market. In this case, the holder will benefit from a cash amount, which equals the strike price minus the most recent settlement futures contract price. However, like the call futures option, a premium must be paid to have the long position in the futures option. If the strike price of the futures option is lower than the price of the futures contract in the market, the futures option will generally not be exercised; thus, the holder of the futures option will lose the premium.

The seller must be in the opposite futures position when the buyer exercises their right; however, no matter how the futures market changes in the future, the profit to a seller is the premium, which is paid by the buyer.

It is important to note that the underlying of a futures option is the futures contract, not the commodity, since an option on a commodity and an option on a futures contract is different. For example, a call option on crude oil will give the holder the right to buy crude oil at a price that equals the strike price; however, the holder of a call option on crude oil futures has the right to benefit from the difference between the futures contract price and strike price. If the crude oil futures option is exercised, the holder will also receive the corresponding futures contract. Therefore, the futures option price is related to the futures contract price, not the commodity price, even though the futures contract price tracks the corresponding commodity price closely. However, when the expiration date of a futures contract is the same as the expiration date of an option on a commodity, that is, the futures contract price equals the commodity price, then these two options are equivalent.

The popularity of trading options on futures contracts rather than options on the commodities is because of three main reasons. Firstly, a futures contract is more liquid than the commodity and the price of a futures contract can be known immediately from trading on the futures exchange. However, the commodity price can be known only by contacting one or more dealers. Secondly, a futures contract is easier to trade than a commodity. For instance, it is much easier to deliver a crude oil future contract than to deliver crude oil itself. Thirdly, in general, the delivery of a commodity will not happen since a futures contract will often be closed before the delivery date. Therefore, a futures option is normally settled in cash in the end, which is appealing to many investors who are interested in margin and leverage.

The futures option actually belongs to the vanilla option, thus if the futures contract price is assumed to follow a log normal process, a closed-form solution for the value of futures option can be obtained. The pricing formulas for European futures option were first presented by Fischer Black in 1976. The Black model is similar to the Black-Scholes model except that the underlying is the futures contract and the volatility is the futures contract volatility. Following Black's work, the pricing futures option has been further studied by many authors and some assumptions of the Black model have been eased. However, similar to the Black-Scholes world, some errors imposed by assumptions (e.g., constant volatility) were not solved by these extended models.

1.3 Structure of Thesis

It is natural to view volatility as a stochastic variable because it is clearly not constant. In this thesis, highly efficient methods to price derivatives with stochastic volatility are developed. Specifically, a tight upper bounds for the prices of Asian options in the Heston stochastic volatility model, the constant elasticity of variance (CEV) model and the Schwartz (1997) two-factor model are presented. In terms of stochastic spot volatility, also shown are closed-form solutions for the futures contract price and a very accurate approximated result for the futures option in the Schwartz (1997) two-factor model with stochastic spot volatility. Lastly, the behaviour of time dependent volatility in the Schwartz (1997) two-factor model is investigated.

In Chapter 2, firstly presented are some basic mathematical theorems that are related to this thesis, including fundamental theorem of asset pricing, stochastic calculus, It$\hat{o}$'s lemma and cholesky decomposition. After that, the Black-Scholes model and the stochastic volatility model for financial derivatives are introduced.

In Chapter 3, there will be a review of the literature of pricing Asian option, pricing futures contract and pricing futures option, which is not very long since a brief literature review is given for the corresponding topics in each chapter.

In Chapter 4, there will be a brief overview on the theoretical background of the Albrecher et al. (2005)'s approach to obtain the upper bound for an arithmetic Asian option and its corresponding static super-hedging strategy in form of European call options. Then, the application of Albrecher et al. (2005)'s approach in the Heston model, the CEV model and the Schwartz (1997) two-factor model is examined. After that, the performance of each model is compared and numerical illustrations for the corresponding hedging strategies are provided. This analysis includes Monte Carlo prices, the comonotonic-upper-bound prices for an arithmetic Asian call option and two further static super-hedging prices, including the trivial static super-hedging price using a single European call option only (with the same strike price and maturity) as well as a static super-hedging price where all call options in the portfolio share the same strike. Section 5 provides a comparison between the comonotonicity approach and an alternative optimization based method. Finally, there is a summary of the main conclusions of this chapter.

In Chapter 5, based on the Schwartz (1997) two-factor model, two types of three-factor models are developed, to give the value of a commodity futures contract, which allow volatility to be stochastic. It is shown that closed-form solutions for futures contracts price can be derived within these two models. After that, the Kalman filter technology and an extended Kalman filter algorithm are discussed to estimate the parameters in these two new developed models. Then, an empirical test with gold data for one of new developed models will be provided, and also its comparison to the results of the Schwartz (1997) two-factor model. Finally, there is a summary of the main conclusions of this chapter.

In Chapter 6, firstly, the theoretical background of the asymptotic approach is introduced and it is shown how to use this method to find the expression for the price of a futures option in the Schwartz (1997) two-factor model with stochastic spot volatility. After that, it is shown that the asymptotic formula has a better performance than the Schwartz (1997) two-factor model both in simulated data and real market data. Finally, there is a summary of the main conclusions of this chapter.

In Chapter 7, closed-form expressions for European style futures options with time dependent spot volatility are derived firstly. Secondly, it is shown how to use these expressions to find time dependent spot volatility for futures options with market data. After that, it is also shown how to examine the time dependent spot volatility is correct or not. Finally, there is a summary of the main conclusions of this chapter.

In Chapter 8, there is a summary of the main conclusions of this thesis.

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Chapter 2

Mathematical Background

2.1 Mathematical Theorem

One of the key problems in pricing derivatives is how to derive the fair value of derivatives (e.g., futures, options etc). The fair price will not give investors the opportunity to obtain extra profit without any risks. Hence, a no arbitrage world is very important, some basic background of arbitrage-free world will be introduced firstly.

2.1.1 Fundamental Theorem of Asset Pricing

In a multi-period market, investors can gather information over time; hence, it is needed to take care about the evolution over time of the information available to investors. This leads to the probabilistic concepts of a σ-algebra, a filtration and a probability measure.

Definition 2.1. *A collection $\mathcal{F}$ of subsets of the state space Ω is called a σ-algebra (or σ-field) whenever the following conditions hold:*

- $\Omega \in \mathcal{F}$,
- *If* $F \in \mathcal{F}$, *then* $F^c = \Omega \backslash F \in \mathcal{F}$,
- *If* $F_i \in \mathcal{F}$ *for* $i \in \mathbb{N}$ *then* $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$.

A state space Ω can be seemed as events or samples space, for example, the information that stock price changes over a year. σ-algebra is supposed to model a certain quantity of information, the larger the σ-algebra, the information conveyed by the σ-algebra is richer. In the example above, $\mathcal{F}$ could be the information that stock price goes up or goes down one day, one week or one month. In reality, a stock may in a state of suspension which is included in the state space Ω, however, this is not included in $\mathcal{F}$.

Definition 2.2. *A sequence $(\mathcal{F}_t)_{0\leq t\leq T}$ of σ-algebra on Ω is called a filtration if $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s < t$. For brevity, denote $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$.*

The main idea behind a filtration is to find a sequence $\mathcal{F}$ which records the information dynamically, that is, the information in $\mathcal{F}$ will be increased in time. If $\mathbb{F}$ is a filtration, then a stochastic process (X_t) (e.g., stock price) is called $\mathbb{F}$-adapted.

Lastly, the definition of probability measure is similar to that of σ-algebra, except that the probability is constrained between 0 and 1. Without probability measure, for example, investors can only say the stock price is very likely to rise, however, with probability measure, 'very likely' can be measured numerically.

Now the problem becomes which probability measure should be used for pricing since different investors have different attitudes on risk. This leads us into the world of risk-neutral probability measure.

Definition 2.3. *A measure $\mathbb{Q}$ on Ω is called a risk-neutral probability measure for a general multi period market model if*

- $\mathbb{Q}(w) > 0$ *for all* $w \in \Omega$*,*
- $\mathbb{E}_{\mathbb{Q}}(\frac{1}{1+r} S^i_{t+1} \mid \mathcal{F}_t) = S^i_t$ *for all* $0 \leq t \leq T-1$

In terms of mathematical finance, risk-neutral probability measure is a result of measure change, which is a mathematical tool for convenient calculation. In risk-neutral world, the present value of a derivatives claim is its discounted expected value by risk-free rate (r), which is the key for pricing derivatives by Monte Carlo method. The second condition also leads us to martingale.

Definition 2.4. *A $(\mathcal{F}_t)$-adapted process (S_t) on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale if and only if for all $s < t$*

$$\mathbb{E}_{\mathbb{P}}(S_t \mid \mathcal{F}_s) = S_s$$

Because the discounted stock price $(\hat{S}^i_t)$ for $i = 1, ..., n$ are martingales under risk measure $\mathbb{Q}$, risk measure $\mathbb{Q}$ are often referred to as martingale measures. In terms of finance, martingale has two meanings. Firstly, with current information, investors can only know the current stock price at most, that is, the stock price cannot be predicted with current information. Secondly, if market is completed or efficient, all information can be obtained before current time. It is also an implication of the efficient market hypothesis.

When it comes to completed market, it comes with no arbitrage opportunity. Therefore, martingale measure can be a useful criterion to check whether an arbitrage opportunity exists or not. The next two theorems are fundamental to the modern theory of mathematical finance. In general sense, they reveal the relationship between risk-neutral measure and no arbitrage opportunity. (see, e.g., Harrison and Kreps (1979), Harrison and Pliska (1981) and Delbaen and Schachermayer (1994))

Theorem 1 (**The First Fundamental Theorem of Asset Pricing**). *If there is a risk-neutral measure, then there are no arbitrage opportunity in market.*

The first theorem is important because it ensures a fundamental property of market models. Although it is not realistic, it is often assumed that the market is complete for market model (for instance the Black-Scholes model). With this assumption, all contingent claim in the market can be replicated. The next theorem implicates that there is only one replication strategy for derivatives securities. The replication strategy is typically achieved by assembling a portfolio which value is equal to the the value of derivatives.

Theorem 2 (**The Second Fundamental Theorem of Asset Pricing**). *If there is no arbitrage strategies in market, then there exist a risk-neutral measure.*

2.1.2 Stochastic Calculus

Now two important tools in mathematical finance for pricing derivatives will be introduced in this section.

A stochastic process S_t is called Itô process if

$$dS_t = \mu_t dt + \sigma_t dW_t \tag{2.1}$$

where μ_t is called drift, σ_t is the volatility or diffusion parameter, dW_t is the infinitesimal increment of Brownian motion.

This above equation is also referred to stochastic differential equation (SDE). To study the SDE world, the following rules of computation is fundamental, so called stochastic calculus.

$$dW_t \cdot dW_t = dt$$

also,

$$dW_t \cdot dt = 0$$

$$dt \cdot dW_t = 0$$

$$dt \cdot dt = 0$$

Itô process is an important tool for pricing derivatives, and Itô lemma play a key role in it.

Theorem 3 (**Itô Lemma**). *If stochastic process S_t is a Itô process and $F(S,t)$ is a 2-times continuously differential function on S and t, then $F(S,t)$ has a stochastic process given by*

$$dF = (F_t + \mu_t F_S + \frac{1}{2}\sigma_t^2 F_{SS})dt + \sigma_t F_S dW_t$$

Itô Lemma is the chain rule of stochastic calculus and it can also be applied into multi-variate stochastic processes,

$$dF = (F_t + \sum_{i=1}^{n} \mu_i F_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_j \sigma_j \rho_{ij} F_{ij})dt + \sum_{i=1}^{n} \sigma_i F_i dW_i(t) \tag{2.2}$$

where $F_i = \frac{\partial F}{\partial S_i}$, $F_{ij} = \frac{\partial^2 F}{\partial S_i S_j}$ and $\rho_{ij} dt = dW_i dW_j$.

2.1.3 Cholesky Decomposition

It is hard to find a closed-form solution for derivatives with multi-variate stochastic process; in practice, it is often given the value of derivatives by the Monte Carlo method and Cholesky decomposition is commonly used in this method for simulating systems with multiple correlated variables. In mathematic term, the Cholesky decomposition or Cholesky factorization is a decomposition of a Hermitian, positive-definite matrix ($\mathbf{A}$) into the product of a lower triangular matrix ($\mathbf{L}$) which is real and positive diagonal entries and its conjugate transpose ($\mathbf{L^T}$), that is $\mathbf{A} = \mathbf{L} \cdot \mathbf{L^T}$.

There are various methods for calculating the Cholesky decomposition, one of them is the Cholesky-Banachiewicz and Cholesky-Crout algorithms.

If the equation is written as:

$$\mathbf{A} = \mathbf{L} \cdot \mathbf{L^T} = \begin{pmatrix} L_{11} & 0 & \dots & 0 \\ L_{21} & L_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{i1} & L_{i2} & \dots & L_{ii} \end{pmatrix} \begin{pmatrix} L_{11} & L_{21} & \dots & L_{i1} \\ 0 & L_{22} & \dots & L_{i2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L_{ii} \end{pmatrix} \tag{2.3}$$

where

$$L_{i,i} = \sqrt{A_{i,i} - \sum_{k=1}^{i-1} L_{i,k}^2} \tag{2.4}$$

and

$$L_{i,j} = \frac{1}{L_{j,j}} \left(A_{i,j} - \sum_{k=1}^{j-1} L_{i,k} L_{j,k} \right) \tag{2.5}$$

for $i > j$.

For a simplified example, if two correlated Brownian motion x_1 and x_2 are needed to generate for the use of the Monte Carlo method. One just needs to generate two uncorrelated Gaussian random variables z_1 and z_2 and set $x_1 = z_1$ and $x_2 = \rho z_1 + \sqrt{1-\rho^2} z_2$.

2.2 Mathematical Models

The main question of pricing derivatives is how the value of derivatives depends on the underlying price and time, in mathematic term, that is, what is the exact expression for the value of derivatives. In 1973, Black, Scholes and Merton answered this question in their work on pricing options, that is, the Black-Scholes model. This model is the queen in option pricing world and has a significant influence on mathematical finance, changing the face of finance. It is also widely used in practice by people who works in derivatives, whether they are salesmen, traders or quants. However, there are some of flaws in the assumptions of the Black-Scholes model, which may lead the model's price far away from its real market price. For example, exotic options are frequently even more sensitive to the level of volatility than standard European style option, thus the price given by the Black-Scholes model can be widely inaccurate. Therefore, people are motivated to find models to take volatility into account when pricing options. To this extent, stochastic volatility models are particularly successful since they can capture, and potentially explain the smiles, skews and other structures in terms of volatility which have been observed in options market. In this section, there will be an overview on the Black-Scholes model and stochastic volatility models.

2.2.1 Black-Scholes Model

Even though all of the assumptions can be shown to be wrong to a greater or lesser extent, the Black-Scholes model is very popular because it is very simple and can provide an easy, quick result for the value of options. Therefore, it is often treated as a benchmark model that other models can be compared. However, it should be noticed that the formation mechanism of option price is not changed by the Black-Scholes model, but is always decided by the market demand and supply. The most important part in the Black-Scholes model is that they provide ideas about delta hedging and no arbitrage. This section reviews the delta hedging and no arbitrage in the Black-Scholes model theoretically.

Firstly, the Black-Scholes model assumes that the stock price is satisfied with an It$\hat{o}$ process:

$$dS_t = S_t\mu_t dt + S_t\sigma_t dW_t \tag{2.6}$$

where μ is the return of the stock price and σ is the volatility of stock price.

Now buying an option $V(S,t)$ and selling underlying S with some quantity Δ to construct a portfolio Π at time t, that is:

$$\Pi = V(S,t) - \Delta S \tag{2.7}$$

The change of the value of this portfolio from time t to $t + dt$ is:

$$d\Pi = dV(S,t) - \Delta dS \tag{2.8}$$

Note that Δ is constant during the time step.

From It$\hat{o}$ lemma, one can have:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}dt \tag{2.9}$$

Thus the portfolio becomes:

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}dt - \Delta dS \tag{2.10}$$

Now the right-hand side of the portfolio contains the deterministic term and random term, which are those with dt and dS respectively. The random term can be seemed as risk in the portfolio:

$$\left(\frac{\partial V}{\partial S} - \Delta\right) dS \tag{2.11}$$

To eliminate this risk, one could carefully choose a Δ:

$$\Delta = \frac{\partial V}{\partial S} \tag{2.12}$$

In this way, the randomness is reduced to zero, this perfect elimination of risk is generally called delta hedging.

Choosing the quantity Δ as suggested above, the portfolio changes by:

$$d\Pi = \frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt \tag{2.13}$$

Since the portfolio now is riskless, that means, there is no arbitrage opportunity, one can get:

$$d\Pi = r\Pi dt \tag{2.14}$$

Therefore, with some substitutions, dividing by dt and rearranging, one can obtained the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.15}$$

This equation was first written down in 1969, but the derivation of the equation was finally published in 1973. It is a linear parabolic partial differential equation. In fact, almost all partial differential equations in finance are of a similar form, meaning that if you have two solutions of the equation then the sum of these is itself also a solution.

Solving the Black-Scholes equation, one could get an analytical or closed-form solution for options price. In terms of European call option ($C(S,t)$):

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{2.16}$$

where

$$\begin{aligned} d_1 &= \frac{log(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

and $N(\cdot)$ is the standard normal cumulative distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt \tag{2.17}$$

With the Black-Scholes pricing formula, one could use the market value of the option to calculate the value of volatility for this option price. This volatility is called implied volatility. When the implied volatilities for market prices of options written on the same underlying price are plotted against a range of strikes and maturities, the resulting graph is typically like a smile, as shown in Figure 2.1. This observation shows the constant volatility assumption is not true.

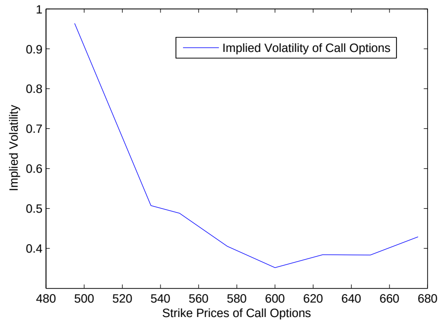

Figure 2.1: The implied volatility of Amazon call options [1]

2.2.2 Stochastic Volatility Models

It has been seen that volatility does not behave how the Black-Scholes equation would like it to behave; it is not constant, it is not predictable, it is not even directly observable. There is a plenty of evidence that the log returns on equities, currencies and commodities are not normally distributed. Actually they have higher peaks and fatter tails than predicted by a normal distribution. Volatility has a key role to play in the determination of risk and in the valuation of derivatives. In this section, there will be a review of models for options valuation with stochastic volatility.

Now, the value of derivatives (V) depends on underlying price S, time t and volatility σ, that is, $V = V(S, \sigma, t)$. One can assume the underlying price and volatility have following stochastic process:

$$\begin{aligned} dS_t &= S_t r d_t + \sigma S_t dW_1 \\ d\sigma_t &= a(S, \sigma, t)dt + b(S, \sigma, t)dW_2 \end{aligned} \tag{2.18}$$

and two increment Brownian motions have a relationship, $dW_1 \cdot dW_2 = \rho d_t$.

It can be found that the choice of functions $a(S, \sigma, t)$ and $b(S, \sigma, t)$ is the key to the evolution of the volatility. Note that the volatility is not a tradable asset in market, hence it is not easy to hedge the risk or randomness from stochastic volatility. Because there are two sources of randomness, the option must be hedged with two other contracts, one being the underlying asset as usual, but now it is also needed another option to hedge the volatility risk.

[1] The implied volatility is computed from Amazon equity options on Nasdaq with 3 weeks expiration time. Data are obtained from Yahoo Finance on April 28, 2016. The underlying price is 622.83 on that date

Considered a portfolio Π which contains one option with value $V(S,\sigma,t)$, a quantity $-\Delta$ of the asset and a quantity $-\Delta_1$ of another option with value denoted by $V_1(S,\sigma,t)$, then one can have:

$$\Pi = V - \Delta S - \Delta_1 V_1 \tag{2.19}$$

The jump of the value of this portfolio in one infinitesimal time step dt is:

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S b \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial \sigma^2}\right) dt \\ & -\Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma S b \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} b^2 \frac{\partial^2 V_1}{\partial \sigma^2}\right) dt \\ & + \left(\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta\right) dS + \left(\frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma}\right) d\sigma. \end{aligned} \tag{2.20}$$

where Itô lemma has been used on functions of S, σ and t.

Clearly one wish to eliminate all randomness by setting

$$\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta = 0 \tag{2.21}$$

and

$$\frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma} = 0 \tag{2.22}$$

so

$$\Delta_1 = \frac{\partial V}{\partial \sigma} \Big/ \frac{\partial V_1}{\partial \sigma} \tag{2.23}$$

and

$$\Delta = \frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} \tag{2.24}$$

Again, by using no arbitrage argument that the return on a risk-free portfolio must be equal to the risk-free rate, this riskless portfolio becomes:

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S b \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial \sigma^2}\right) dt \\ & -\Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma S b \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} b^2 \frac{\partial^2 V_1}{\partial \sigma^2}\right) dt \\ &= r\Pi dt = r(V - \Delta S - \Delta_1 V_1) dt. \end{aligned} \tag{2.25}$$

This equation can be rearranged by collecting all V terms on the left hand side and all V_1 terms on the right hand side, that is:

$$\begin{aligned} & \frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S b \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial \sigma^2} + rS\frac{\partial V}{\partial S} - rV}{\frac{\partial V}{\partial \sigma}} \\ = & \frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma S b \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} b^2 \frac{\partial^2 V_1}{\partial \sigma^2} + rS\frac{\partial V_1}{\partial S} - rV_1}{\frac{\partial V_1}{\partial \sigma}} \end{aligned} \tag{2.26}$$

Now the left hand side is a function of V only and the right hand side is a function of V_1 only. Because the two options will typically have different payoffs, strikes or time of expiration, the only way that this can be is for both sides to be equal to some functions, depending only on variable S, σ and t. Thus, it can be obtained at the general PDE for stochastic volatility:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma Sb\frac{\partial^2 V}{\partial S\partial\sigma}+\frac{1}{2}b^2\frac{\partial^2 V}{\partial\sigma^2}+rS\frac{\partial V}{\partial S}-rV=-(a-\lambda b)\frac{\partial V}{\partial\sigma} \quad (2.27)$$

Conventionally, the function $\lambda(S,\sigma,t)$ is called the market price of volatility risk since it tells us how much of the expected return of V is explained by the risk of volatility in market.

It is hard to solve above mentioned PDE, generally one can use numerical method to find the result. However, there are some popular models which can find analytical solutions for European options with stochastic volatility.

Hull & White

Hull & White considered both general and specific volatility modeling. The most important result of their analysis is that when the stock and the volatility are uncorrelated and the risk-neutral dynamics of the volatility are unaffected by the stock (i.e. $a-\lambda b$ and b are independent of S) then the fair value of an option is the average of the Black-Scholes values for the option, with the average taken over the distribution σ^2.

One of the risk-neutral stochastic volatility model considered by Hull & White is:

$$d(\sigma^2)=c(d-\sigma^2)dt+e\sigma^2 dW_2$$

Heston

However, empirical study shows the correlation for the stock and the volatility is not zero. Heston (1993) gives the following model which can give a closed-form solution for European options when the stock and volatility are correlated.

$$\begin{aligned} dS &= \mu S dt+\sqrt{v}SdW_1 \\ dv &= \lambda(\theta-v)dt+\eta\sqrt{v}dW_2 \\ \rho dt &= dW_1 dW_2 \end{aligned}$$

Ornstein-Uhlenbeck (OU) process

In addition, the following model can match the data well, in the long run, volatility is log normally distributed in this model.

$$d(y)=(c-dy)dt+edW_2$$

where $y=log(v)$, $v=\sigma^2$.

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Chapter 3

Literature Review

3.1 Pricing Asian Option

Accurate pricing for an arithmetic Asian option is really an important problem in practice. Different methods to this problem can be subdivided into three parts: the Monte Carlo method, the Partial Differential Equation (PDE) approach and the Bound Technique.

Boyle (1977) introduced Monte Carlo simulation for pricing option value to finance field. As he claimed, Monte Carlo simulation has many obvious advantages. Firstly, the Monte Carlo method is very flexible with distribution which describes the returns on the underlying stock. That means, in the Monte Carlo method, changing the underlying distribution merely involves generating the random variates by different process. Secondly, the Monte Carlo method does not need the distribution which generates the return on the underlying stock with an analytical expression. This advantage makes pricing option value based on the empirical distribution of stock returns become possible. Furthermore, the Monte Carlo method allows a distribution of stock return to solve any of the parameters of the problem rather than a point estimate. For instance, since the parameter is usually estimated from the empirical data, the Monte Carlo method can give a confidence interval to examine the accuracy of the estimators, which may be useful in some problem with regard to the variance.

Since then, with the popularity of computer, this approach is widely used by many authors. Kemna and Vorst (1990) pointed out that it is impossible to find an explicit formula for an arithmetic Asian option and explain it concretely. Also, they found the value of arithmetic Asian call option cannot readily be obtained by a finite difference method. Therefore, they applied the Monte Carlo method to price and hedge arithmetic Asian call options. As previously studied, the logarithm of stock price follows normal distribution with mean $(r - \frac{1}{2}\sigma^2)(T - t)$ and variance $\sigma^2\sqrt{(T-t)}\varepsilon$, thus the stock price can be expressed as:

$$S_T = S_t \cdot \exp\left((r - \frac{1}{2}\sigma^2)(T - t)) + \sigma^2(T - t)\right) \tag{3.1}$$

where S_T is the stock price at time T, S_t is the stock price at time t, r and σ are constant, representing the expected rate of return and the volatility of stock price respectively, and ε is a random number that followed by standardized normal distribution $N(0,1)$.

Assume:

$$A(T) = \frac{1}{n+1}\left(\sum_{i=0}^{n} S(T_i)\right) \tag{3.2}$$

Then, the price of arithmetic Asian call option was calculated as:

$$C = e^{-r(T-t)} max\left(A(T) - K, 0\right) \tag{3.3}$$

where K is the strike price.

In general, as Boyle et al. (1997) described, the Monte Carlo method follows three steps. Firstly, according to the risk-neutral measure, simulate sample paths of the underlying state variables over the relevant time horizon. For example, use Equation (3.1) to simulate the stock price under risk-neutral probability at each node of exercise opportunity. Then, evaluate the discounted cash flow of an underlying asset on each sample path, according to the structure of the underlying asset in the question. For example, use Equation (3.2) and Equation (3.3) to calculate the value of arithmetic Asian call options. At last, average the discounted cash flow among all sample paths.

Broadie and Glasserman (1996) pointed out that Monte Carlo simulation is a valuable approach for pricing options which do not have closed-form solutions and it is very suitable for pricing Asian options. However, Joy and Tan (1996) reported that the drawback of the standard Monte Carlo approach is that the use of pseudo number may yield an error bound which is probabilistic and that it can be computationally burdensome in order to get a high level of accuracy. Furthermore, the error term (e.g., ε) can encounter instabilities for the value of the certain underlying asset. Note that, the Monte Carlo method is very time-consuming without the enhancement of variance reduction techniques, and one must take the bias into account, which comes from the approximation of continuous time processes through discrete sampling (Broadie et al. (1999)). Therefore, a variety of variance reduction techniques have been developed to increase the accuracy and the speed of calculation.

Two classical variance reduction techniques are the antithetic variate approach and the control variate method. More recently, stratified sampling, important sampling and conditional Monte Carlo method have been applied in speeding up the calculation. These variance reduction techniques indeed improve the accuracy of valuation of option price as well as the computational speed. Moreover, the result come from these methods are still unbiased. Joy and Tan (1996) introduced another technique that is known as quasi-Monte Carlo method for improving the efficiency of the Monte Carlo simulation. The key idea of this method is to use a deterministic sequences (quasi-Monte Carlo sequences) to improve the convergence and give rise to the deterministic error bounds. In general, these sequences have a good convergence property even in the case of a large number of time steps. Furthermore, quasi-Monte Carlo sequences can be generated as quickly as the random numbers of normally distribution. However, among all these variance reduction techniques, Boyle et al. (1997) pointed out that control variates method is the most widely applicable, the easiest to use and the most effective variance reduction technique for pricing arithmetic Asian option. The core theory of this method is to use another similar option that the value of this option is easy to be found to price the arithmetic Asian option. Kemna and Vorst (1990) found that the characteristic of geometric Asian call option is similar with that of arithmetic Asian call option, and most importantly, the value of geometric Asian call option can be evaluated in the closed-form under the Black-Scholes framework. Therefore, choosing geometric Asian option as the control variate, they used control variates method for pricing arithmetic Asian call option. Concretely, let P_A be the price of an arithmetic Asian call option, P_G be the price of a geometric Asian call option, then the value of an arithmetic Asian call option can be expressed as follow:

$$P_A = P_G + E(\hat{P}_A - \hat{P}_G) \tag{3.4}$$

where $P_A = E(\hat{P}_A)$, $P_G = E(\hat{P}_G)$, $\hat{P}_A$ and $\hat{P}_G$ are the discounted value of options for a single simulated path of the underlying asset.

In other words, the value of an arithmetic Asian call option can be evaluated by the known value of a geometric Asian call option plus the expected difference between the discounted value of arithmetic Asian call option and that of a geometric Asian call option. The numerical results of Kemna and Vorst (1990) showed that this method is indeed effective. Different with Kemna and Vorst (1990), based on Laplace transform inversion methods, Fu et al. (1999), also investigated other control variate methods for pricing arithmetic Asian call option. For example, based on Geman-Yor transform, they developed a double Laplace transform of the arithmetic continuous Asian option in both its strike and maturity. And they found that, when a continuous Asian option price is sought, using suitably biased control vitiate has a great benefit for the correcting for the discretizated bias inherent in the simulation. However, Boyle et al. (1997) pointed out that these control variate methods are less strongly correlated with option price than the control variate method that Kemna and Vorst (1990) used.

It should be admitted that, although the variance reduction techniques have been enhanced, the fatal drawback of the Monte Carlo method is that in order to reach a fairly accurate level of option price, this method often need a large number of simulation trials, that is, using the entire path of the underlying asset as a sample greatly reduces the competitiveness of this approach.

The other main stream to price arithmetic Asian option is the Partial Differential Equation (PDE) approach. Ingersoll (1987) introduced a new variable state which represents the running sum of the stock process to help pricing arithmetic Asian option by PDE approach. Based on the new variable state, he pointed out that the price of an Asian option with floating strike can be found by solving a PDE in two space dimensions under the Black-Scholes model with constant volatility. Furthermore, he observed that, in some cases, the two-space dimensional PDE for a floating strike Asian option can be reduced to a one-space dimensional PDE, for instance, the case of no dividend payment on the stock. Geman and Yor (1993) computed the price of an out-of-the-money arithmetic Asian call option by Laplace transform. Moreover, by the use of only simple probabilistic method, they found that arithmetic Asian option may be more expensive than a standard European option (e.g., options on currencies or oil spread). They also gave a simple closed-form solution for the arithmetic Asian option which is in-the-money. However, Linetsky (2002) reported that this Laplace transform works well only when it is inverted numerically by applying a suitable numerical Laplace inversion algorithm and Fu et al. (1999) also found that this Laplace inversion is difficult to calculate, especially in the case of low volatility and/or short maturity. After Geman and Yor (1993), Rogers and Shi (1995) learned a similar scaling property for floating strike Asian option which is already observed by Ingersoll (1987). Using this similar scaling behavior, Rogers and Shi (1995) derived a one-dimensional PDE that can price both for floating and fixed strike Asian option. The reason is that they divided the $K - \bar{S}_t$ (K is the strike price and $\bar{S}_t$ is the average stock price during the period from 0 to t) by the stock price S_t. They claimed that the formula for pricing arithmetic Asian option can be easily computed once the function of distribution of stock process is known. However, Zvan et al. (1996) reported that this one-dimensional PDE is only suitable for the European style options and difficult to solve numerically since the diffusion term is very small for values of interest on the finite difference grid. Therefore, there are several authors who try to improve the numerical accuracy of this PDE style.

Barraquand and Pudet (1996) used the concept of symmetric multiplication for stochastic integral and the standard results both on discrete approximations of multiplication diffusion process and accessibility of deterministic control systems, and they found that the PDE of the arithmetic Asian option is non-holonomic. Then, they created a new numerical method called forward shooting grid method (FSG), which efficiently copes with arithmetic Asian options PDE style. Based on Rogers and Shis reduction technique, Andreasen (1998) noticed that a change of numeraire of the martingale method can make the two-dimensional PDE of arithmetic Asian option become one-dimensional PDE and they applied Crank-Nicolson scheme to the pricing of discretely sampled Asian option with both floating and fixed strike. Then, he proved that, comparing with Monte Carlo method, this approach is really competitive in terms of accuracy and speed. In order to obtain an accuracy result for arithmetic Asian option rapidly, Zvan et al. (1996) applied a high order non-linear flux limiter (van Leer flux limiter) for the convective term in the field of computational fluid dynamic techniques, thus the problem of spurious oscillations can be alleviated and the accuracy of result can be improved comparing with the approach that Rogers and Shi (1995) used. Moreover, they also showed that the application of van Leer flux limiter can rapidly obtain an accurate solution for both fixed strike and floating strike arithmetic Asian option of European style in a one-dimensional model. For instance, for general volatility or interest rate structure, maturity is up to one year, an accurate solution can be obtained within 10 seconds. Even for extreme volatility or interest rate structure, the average computational time is within 16 seconds.

Some other papers which intended to develop a unified pricing approach for different types of options including arithmetic Asian option are also developed during that time. Based on the concept of self-similarity, Lipton (1999) proved that the relationships among look back option, Asian option, passport option and imperfectly hedged European-style option have very similar properties. Then, based on the self-similarity reduction and Geman and Yor (1993)'s study, he gave a PDE based derivation of the valuation for Asian option. Different with Lipton (1999), Shreve and Večeř (2000) developed an alternative reduction method for pricing options on a traded account, which includes options that can be replicated by self-financing trade in the underlying asset, such as passport option, European option and arithmetic Asian option. Moreover, because the option holder can switch their position in an underlying asset during the life of the option through the traded account, the optimal strategy for buyer and seller can be obtained quickly by the use of a Mean Comparison Theorem. Vecer (2001) found that options (passport option, European option, American option, vacation option, Asian option) that on a traded account have a same type of one-dimensional PDE and applied aforementioned reduction technique to both continuous and discrete arithmetic Asian option. The result of the numerical implementation of this pricing method suggested that this method is fast, accurate and easy to implement as well. Even for the case of low volatility and/or short maturity, this method still has a stable performance. Similarly, by the use of scale invariance method, Hoogland and Neumann (2001) derived an alternative formulation for pricing various types of options. Moreover, they provided a more general semi-analytical solution for continuous sampled Asian options. Vecer (2002) presented an even simpler and unifying approach for pricing of continuous and discrete arithmetic Asian option. The result can be obtained extremely fast and accurately from the one-dimensional PDE for the price of arithmetic Asian option. This method is easy to implement and does not require implementing jump conditions for dividends. Fouque et al. (2003) proved that the method in Vecer (2002) is really an efficient, accurate and has stable performance method. Moreover, they found the assumption of constant volatility can be relaxed, so that through a singular perturbation technique and Vecers reduction, they approximated arithmetic Asian option under stochastic volatility model by the use of taking the observed implied volatility skew into account. However, it should be pointed out that they only consider the case of a short time scale volatility factor.

Quite different from above mentioned studies, using the spectral theory of singular Sturm-Liouville (Schrodinger) operators, Linetsky (2002) derived two alternative analytical formulas that allow exact pricing of the arithmetic Asian option, which not involved multiple integrals or Laplace transform inversion. In more details, the first analytical expression is an infinite series: the terms of series are explicitly characterized in terms of known special function. The second formula is single real integral of an expression, which is a limit serious formula and in the form of an integral transform. The exact pricing formula seems really good and works well; however, this approach is limited to diffusions, only used for continuously arithmetic Asian option and still needs to conduct more researches on the effectiveness of this area. For example, researches about continuity correction for arithmetic Asian option. Then, Vecer and Xu (2004) used special semi-martingale process models for pricing arithmetic Asian option, and they showed that, under this condition, the inherently path dependent problem of pricing arithmetic Asian option can be transformed into a problem without path dependence in the payoff function. They also derived a simple integro-differential pricing equation for arithmetic Asian option. Moreover, the pricing equation could be simpler in the case of a particular stock price model, such as geometric Brownian motion with Poisson jump model, the CGMY model, or the general hyperbolic model.

In addition, Albrecher et al. (2005) indicated that pricing arithmetic Asian option can be solved by the Bound techniques. Considering that the speed of the Monte Carlo method is relatively slow, Turnbull and Wakeman (1991) recognized the suitability of the log normal as a first-order approximation and described a quick algorithm for arithmetic Asian option. Based on Edgeworth series expansion techniques, the most difficult problem they mentioned that how to determine the probability of the distribution for arithmetic can be solved well. However, Levy (1992) claimed that Turnbull and Wakeman (1991) overlook the accuracy of the log normal assumption. Because only when the first two moments are taken into account in the approximation, the assumption is acceptably making redundant need to include additional terms in the expansion, which involves higher moments. Thus, he used a straight forward approach to approximate the arithmetic Asian options density function. This approach is similar with Edgeworth series expansion techniques, while the core key of this approach is that this method makes a closed-form analytical approximation for the valuation of arithmetic Asian options become possible, which has a great advantage on accuracy and implementation for typical ranges of volatility experienced. Curran (1992) also gave a fast method for the valuation of arithmetic Asian option by the use of lower bound, while this case is only for fixed-strike arithmetic Asian option. Bouaziz et al. (1994) agreed with Turnbull and Wakeman (1991) and Levy (1992)'s opinion that the closed-form analytical solution is not always available or it needs too many strict and unrealistic assumptions and they also noted it is possible to derive a simple formula for arithmetic Asian option which does not allow for early exercise in the case of a slight approximation. Thus, based on Turnbull and Wakeman and Lévys studies, they presented an alternative approximation method for pricing error by deriving an upper bound. However, it should be pointed out that the results are mutatis mutandis to the case of the price of fixed strike.

Although approximation methods are accurate for arithmetic Asian option, Curran (1994) claimed that these methods are not suitable for the case of portfolio option. When the number of assets rise above four or five, the distribution-approximation procedure for arithmetic Asian option are not always accurate because the computational time is exponential in the number of assets. Thus, he developed a new method so called Geometric Conditioning method which is based on conditioning on the geometric mean price. The numerical result implied that this method is simpler and more accurate than previous approach and it is also fast for any practical number of assets. Again, based on the conditioning approach, a very accurate lower bound for the price of arithmetic Asian option was obtained by Rogers and Shi (1995). For simplicity, the lower bound is expressed as an average of delayed payment European call option and it is efficient for both fixed strike and floating strike arithmetic Asian option. Considering the error from their lower bound, Rogers and Shi (1995) also obtained an upper bound. From the view of numerical result, this method is fast, taking less than 1 second. Since the expression is only a one-dimensional integral, it is also easy to compute. Jacques (1996) extended Turnbull and Wakeman and Levys approximation approach to the construction of hedging portfolio by giving two explicit formulas. One is based on usual log normal approximation as Turnbull and Wakeman and Levy used before, the other one is on an Inverse Gaussian approximation. Then, he proved that the result through Inverse Gaussian approximation is as good as log normal approximation. Simon et al. (2000) claimed that this approximation can be obtained by approximating the distribution of $\sum_{i=0}^{n-1} S(T-i)$, where $S()$ is the stock price, thus this method is more tractable than Turnbull and Wakeman and Levy used. Combining with the study from Curran (1992) and Rogers and Shi (1995), Thompson (1999) developed a simpler expression for the lower bound, and he also presented a new upper bound, which is accurate for both fixed strike and floating strike arithmetic Asian option in the case of typical parameter values.

Followed by Jacques (1996), Simon et al. (2000) found that the price of arithmetic Asian option can be bounded from above by a portfolio of European call options. Therefore, by the use of actuarial risk theory on stop-loss order and comonotone risk, they obtained an accurate upper bound for the price of arithmetic Asian option. The formula can be simply expressed as follow:

$$AA(t, S(t), n, K, T, r) = e^{-(T-t)r} E^Q \left(\left[\frac{1}{n} \sum_{i=0}^{n-1} S(T-i) - K \right]_+ \mid \mathcal{F}_t \right) \tag{3.5}$$

Compared to the simulated price, this method is rather accurate. Furthermore, it can optimize the portfolio of European call option by exercise price. However, in the case of options that are out-of-the-money, this method seems less accurate. Dhaene et al. (2002b) introduced the theoretical aspect of comonotonicity in actuarial science and finance. They claimed that it is interesting to know what the distribution function of a sum of random variable is. Thus, by the use of some simple powerful techniques, the distribution of a sum of random variable was determined, when the distributions of the terms are known, but the stochastic dependence structure between them is unknown or too cumbersome to work with. The main advantage as Dhaene et al. (2002b) described is that, this method not only easily computes the distribution functions and stop-loss premium, but also reduces the multidimensional problem to one-dimensional problem through comonotonic random vectors. Dhaene et al. (2002a) applied this method to derive a very tight upper bound as well as lower bound for the price of an arithmetic Asian option which are essentially a stop-loss premium of a sum of strongly dependent random variables. As can be seen from the numerical result, the upper bounds are especially sharp in case the random components of a sum are rather strongly positive-dependent. Furthermore, the lower bounds seem to work well even the dependencies are not very strongly positive.

Nielsen and Sandmann (2003) did a research in pricing formula for long term arithmetic Asian options written on the exchange rate in a two currency economy. They stressed that the total correlation structure is very important for the price in two specific country economies. Therefore, their model not only showed the exchange rate itself but the term structure of interest rates in these two countries. They extended the Rogers and Shi (1995)'s approach and developed a closed-form for fixed strike arithmetic Asian option. In addition, based on this bound, the pricing error also determined at the same time. Nielsen and Sandmann (2003) proved that the bound technique for pricing discrete arithmetic Asian option can readily get more information than the usually applied pricing approximation. For example, using the closed-form expression, the hedging position can be obtained from these bounds. The price of the underlying asset and the volatility can immediately be differentiated respectively through these bounds. Moreover, these bounds can also obtain the result in the position of Delta, Gamma and Vega as well.

More recently, based on the comonotonic approach again, Albrecher et al. (2005) introduced a simple static super-hedging strategy for the price of an arithmetic Asian option, which is consisted with a portfolio of European options. In particular, they replaced classical Black-Scholes model with Lévy market model, which can better describe the dynamics of stock price. Concretely, three popular Lévy processes (Variance Gamma process, Normal Inverse Gaussian process as well as Meixner process) have been used to describe the asset prices. Moreover, because this approach is also a static hedging strategy, it is less sensitive to the assumption of zero transaction costs as well as the hedging performance in the presence of large market movements. In addition, no dynamic rebalancing is required in this hedging strategy. The numerical result showed that these above mentioned advantages may sometimes compensate the gap of the hedging price and the option price even for at-the-money arithmetic Asian options.

However, Lévy processes are different with stochastic volatility model. In the Chapter 4, Albrecher et al. (2005)'s approach was followed, using results from comonotonic theory and some classical stochastic volatility models to obtain an accurate upper bound for the price of an arithmetic Asian option.

3.2 Pricing Futures and Futures Option

The market for commodity derivatives has been becoming an increasingly important part of the global derivatives market over the past few years. There is a lot of research to study in the development of an appropriate pricing model for futures and futures options.

European futures options can be valued by extending the results of the Black-Scholes model. Black (1976) presented this solution in 1976, as follows:

$$C = e^{-rT}\left[F_0 N(d_1) - KN(d_2)\right] \tag{3.6}$$

where

$$\begin{aligned} d_1 &= \frac{log(\frac{F_0}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

and σ is the volatility of the futures price, $N(\cdot)$ is the standard normal cumulative distribution function.

However, the relevant features of commodities require that the Black equation is adjusted so that they have a cost of carry. For example, the storage of commodities is not without cost. Brennan (1991) introduced the concept of convenience yield when pricing futures contracts; that is, the convenience yield of a commodity is the benefit obtained from holding the spot commodity not futures contracts. This benefit depends on the identity of storing the commodity. In Black's model, the only source of uncertainty is the spot price process. When the cost-of-carry formula is adapted to price commodity futures, the convenience yield is assumed to be constant. However, there is a lot of evidences show that the convenience yield should be a stochastic process. Based on the theory of storage, which was explained by Kaldor (1939), Working (1948), Brennan (1958), and Telser (1958), and by investigating agricultural products, wood products, animal products, and metal products, Fama and French (1987) found that the variation of convenience yield for most agricultural and animal products have seasonal behaviour. Moreover, by the use of futures data from the London Metal Exchange, Fama and French (1988) found there are sharp rises and declines around the peaks of 1973-1974 and 1979-1980, implying that the spot price and convenience yield of the metals has a mean reverting process. They also suggested futures prices are less variable than spot prices when inventory is low, but the spot price and futures price have similar variability when inventory is high. Furthermore, Brennan (1991) fitted a mean reverting process to the convenience yield of many sources, including plywood, lumber, heating oil, copper, platinum, silver and gold. The empirical study by Brennan (1991) revealed the relationship between inventory, spot price and convenience yield. When inventories area are low, spot prices are relatively high, and convenience yields are also high, which is consist with Fama and French (1988); that is, futures price will not increase as much as the spot price, and vice versa when inventories are high. Therefore, for some commodities, the correlation between commodity price and convenience yields is positive. This evidence can also be found in Working (1949) and Brennan (1958). Besides, Deaton and Laroque (1992) developed a simple model that is based on competitive storage with the study of thirteen commodities; the result is consistent with previous research, whereby the spot price and convenience yield have a positive correlation.

In addition to storage theory, there are many researchers who have studied the property of spot price and convenience yield with the mean reversion model. Gibson and Schwartz (1990) developed a Gaussian structure model for pricing financial and real assets contingent on the price of crude oil, taking spot price and convenience yield into account. In their model, the convenience yield is assumed to follow a mean reverting stochastic process. The pricing errors for short term futures contracts are quite comparable to previous studies. Gibson et al. (1991) found that a constant convenience did not work well when pricing oil-indexed bonds and suggested that a stochastic convenience yield should be considered in modeling any meaningful valuation model. Routledge et al. (2000) developed a one-factor model of forward prices for commodities. In this model, the convenience yield process is endogenous and captures the American option value of storage. The results also showed that the correlation between the spot price and the convenience yield is positive when there is shortage of the commodity. More recently, Casassus and Collin-Dufresne (2005) studied a three-factor model: spot price, convenience yield and interest rate. By using weekly data on crude oil, copper, gold, and silver futures contracts, they found that the relationship between spot price and convenience yield is positive. Moreover, they also showed evidence that the spot price process should have a mean reversion style.

This evidence and research confirms that the convenience yield should be time-varying and follows a mean reversion process, and the correlation between spot price and convenience yield is positive. These conclusions play a core role in pricing futures and futures options with convenience yield.

Many studies applied a mean reversion model for pricing commodity futures contract and futures options with a stochastic convenience yield. Gibson and Schwartz (1990) developed a two-factor model where the first factor is the spot price and the second factor is the instantaneous convenience yield. These two factors all followed stochastic processes. Schwartz (1997) extended this two-factor model to a three-factor model to price commodity futures contracts by introducing the instantaneous interest rate factor. The instantaneous interest rate in the model also followed a stochastic process. Parameters in models are estimated using historical data and the Kalman filter technique for two commercial commodities, copper and oil, and one precious metal, gold. This paper also develops hedging and capital budgeting application. Note that both the two-factor model and the three-factor model can give a analytical solution for futures contract price. Based on Schwartz (1997), Hilliard and Reis (1998) extended the two-factor model and the three-factor model by introducing jumps in the spot price of commodity and using the term structure of interest rates to eliminate the market price of interest risk in their fundamental pricing equation. Moreover, Hilliard and Reis (1998) also presented an analytical solution for the value of futures, forwards and futures options for both the Schwartz (1997) two-factor model and the Schwartz (1997) three-factor model. Furthermore, they claimed that a deterministic convenience yield can lead to a significant error for the price of futures options. However, they did not use any method to estimate parameters in their models; the value of parameters was chosen from Gibson and Schwartz (1990). Schwartz and Smith (2000) provided a short term and long term model to give the value for futures and futures options. Although this model does not consider a stochastic convenience yield, it can be proved that this model is actually equivalent to the two-factor model that is developed in Gibson and Schwartz (1990). In Schwartz and Smith's paper, movements in prices for long maturity futures contracts provide information about the equilibrium price level, and the differences between the prices for the short term and long term contracts provide information about short term variations in prices. More importantly, they also showed evidence that the spot price process should have a mean reversion style.

Another method for the valuation of futures and futures options is based on the Heath-Jarrow-Morton (HJM) methodology. Heath et al. (1992) developed an arbitrage-free model of the stochastic movements of the term structure of interest. This methodology does not need the estimation of the drift term since they can be expressed as functions of their volatilities and the correlations among themselves. For example, with the HJM framework, the risk premia does not need to be estimated since they are embedded in market prices. Some similar models were developed by Reisman (1991), Cortazar and Schwartz (1994), Amin et al. (1995) and Carr and Jarrow (1995). Note that whether it is a two-factor model with or without jump or a three-factor model with or without jump, the market premia of the convenience yield is treated as a parameter in their pricing formula. Miltersen and Schwartz (1998) combined the above mentioned multi-factor model and the HJM model to develop a new model that considers all of the information in the initial term structures of both interest rates and futures prices. By assuming that the interest rate and convenience yield follow normal distribution and the spot price of the underlying commodity follows log normal distribution, closed-form solutions for the pricing of options on futures prices as well as forward prices can be obtained, which are in the spirit of Black and Scholes (1973) and Black (1976). Furthermore, the forward and futures convenience yield is firstly distinguished in their paper.

However, the underlying still follows a normal distribution in these models, which is inconsistent with the fact. Trolle and Schwartz (2009) proposed a stochastic volatility model for pricing commodity derivatives by HJM methodology. In their paper, volatility can be defined as span volatility and unspanned volatility. One can only use futures contracts to hedge span volatility but must use the futures option to hedge unspanned volatility. They estimated their model on a panel set of derivatives data, including futures and futures options on Brent crude oil. They proved that it is necessary to consider stochastic volatility when pricing commodity futures options.

Therefore, it is necessary to take stochastic volatility into account when pricing futures and futures options. In Chapter 5 and Chapter 6, based on mean reversion models, futures and futures option will be priced with stochastic volatility.

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Chapter 4

On the Performance of the Comonotonicity Approach for Pricing Asian Options in some Benchmark Models from Equities and Commodities

4.1 Introduction

Asian options are derivatives, in which the underlying is the average of a financial variable, such as prices of equities, commodities, interest or exchange rates. The pricing of such derivatives has been of utmost interest ever since trading started in the mid 1980's, initially mostly on OTC markets but since a number of years also on exchanges such as the London Metal Exchange.

Even in the Black-Scholes framework, arguably the simplest of all derivatives pricing frameworks, there is no explicit solution for the price of an arithmetic Asian option. Geman and Yor (1993, 1996) derived analytic representations in form of complex integrals for the price of a normalized Asian call option in the Black-Scholes model. This fundamental work was followed up and extended upon by a large number of authors, uncovering important relations to fundamental problems in probability theory and classical functions. The demands of financial practitioners however have long moved beyond the Black-Scholes model and the model error imposed by the Black-Scholes assumptions often outweighs any computational progress that some analytic formulas and techniques based on the Black-Scholes assumptions seem to offer.

In this chapter, beyond the Black-Scholes model, the Heston stochastic volatility model, the constant elasticity of variance model (CEV) as well as the Schwartz (1997) two-factor model are included. All these models are frequently used by practitioners in the equity and commodity context and fix some of the most notorious faults of the Black-Scholes model. Obviously, as these models are mathematically more complex than the Black-Scholes model, it is not possible to derive explicit and exact formulas for arithmetic Asian options in these models either. A standard approach is to facilitate the Monte Carlo method, compare Broadie and Glasserman (1996). The pricing of Asian and Australian options under stochastic volatility by use of PDE techniques has been discussed in Ewald et al. (2013). Both methods are time intensive. Wong and Cheung (2004) had looked at the pricing of geometric Asian call options in stochastic volatility using asymptotic expansions, but do not discuss the more complex case of arithmetic Asian options, nor the Heston model. Wong and Lau (2008) discussed the pricing of some exotic options in the presence of mean reversion, but did not include Asian options in their analysis. Further the model discussed there corresponds to the Schwartz (1997) one-factor model, while this chapter relates to the Schwartz (1997) two-factor model, which is more general and contains the one-factor model as a special case.[1]

Here, following an approach that is based on Albrecher et al. (2005) and so far has mainly been applied to Lévy type jump models. This approach delivers an upper bound for the price of an arithmetic Asian option and is based on the theory of comonotonic functions.[2] It requires at some point the inversion of the distribution function of the sum of the comonotonic random vector with marginal distributions equal to the underlying at the times when the arithmetic average is sampled. This can be rather difficult for many models of practical relevance and often requires a case by case methodological approach. It is demonstrated how this can be done effectively in the cases considered here and shown that the Albrecher et al. (2005)'s approach delivers good results in this context. A nice by-product of the Albrecher et al. (2005)'s approach is that in addition to an upper price bound, it provides a super hedging strategy. This strategy is static and involves positions in the European call options used in the determination of the upper bound.

The remainder of this chapter is organized as follows: Section 2 briefly covers the theoretical background of the Albrecher et al. (2005)'s approach to obtain the upper bound for an arithmetic Asian option and its corresponding static super-hedging strategy in form of European call options. In Section 3 there is a briefly outline of the three benchmark models that underly the numerical analysis: the Heston model, the CEV model and the Schwartz (1997) two-factor model. In Section 4 the Albrecher et al. (2005)'s approach in the context of these models is examined, including the Black-Scholes model as a comparison, using market data for the VIX. The performance of each model is compared and numerical illustrations for the corresponding hedging strategies are provided. This analysis includes Monte Carlo prices, the comonotonic-upper-bound prices for an arithmetic Asian call option and two further static super-hedging prices, including the trivial static super-hedging price using a single European call option only (with the same strike price and maturity) as well as a static super-hedging price where all call options in the portfolio share the same strike. Section 5 provides a comparison between the comonotonicity approach and an alternative optimization based method. Section 6 summarizes the main conclusions.

[1]Other notable contributions from a numerical point of view include Costabile et al. (2006)

[2]Bounds for American option have been obtained by Chen and Yeh (2002).

4.2 Optimal Static Hedging for Arithmetic Asian Options with European Call Options

Under the assumption of constant interest rates and the existence of an equivalent martingale measure $\mathbb{Q}$ used for pricing contingent claims, the price of an arithmetic Asian call option written on a stock S_t sampled at times t_k for $k = 1, ..., n$ is given as:

$$\begin{aligned} AA_t &= \exp(-r(T-t))\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{\sum_{k=1}^{n} S_{t_k}}{n} - K\right)^+ \middle| \mathcal{F}_t\right], \\ &= \frac{\exp(-r(T-t))}{n}\mathbb{E}_{\mathbb{Q}}\left[\left(\sum_{k=1}^{n} S_{t_k} - nK\right)^+ \middle| \mathcal{F}_t\right]. \end{aligned}$$

Here K denotes the strike price, r denotes the risk-free interest rate, T denotes the time of maturity, S_t denotes the asset price at time t, n denotes the number of dates $0 \leq t_1 \leq t_2 \cdots \leq t_n \leq T$ over which the averaging takes place and $(x-K)^+ = \max(x-K, 0)$.

As is well known, the main difficulty in evaluating this expression is that the distribution of the average $\sum_{k=1}^{n} S_{t_k}/n$ is not known in explicit form. Here, one only attempt to find a sufficiently tight upper bound for the expression above. It follows from convexity of the function $(x-K)^+$ that for any $K_1, K_2, \cdots, K_n$ with $K = \sum_{k=1}^{n} K_k$

$$\begin{aligned} \left(\sum_{k=1}^{n} S_{t_k} - nK\right)^+ &= \Big((S_{t_1} - nK_1) + (S_{t_2} - nK_2) + \cdots + (S_{t_n} - nK_n)\Big)^+ \\ &\leq \sum_{k=1}^{n} (S_{t_k} - nK_k)^+. \end{aligned}$$

Therefore, one can have:

$$\begin{aligned} AA_0(K,T) &= \frac{\exp(-rT)}{n}\mathbb{E}_{\mathbb{Q}}\left[\left(\sum_{k=1}^{n} S_{t_k} - nK\right)^+ \middle| \mathcal{F}_0\right] \\ &\leq \frac{\exp(-rT)}{n}\sum_{k=1}^{n}\mathbb{E}_{\mathbb{Q}}\left[(S_{t_k} - nK_k)^+ \middle| \mathcal{F}_0\right] \\ &= \frac{\exp(-rT)}{n}\sum_{k=1}^{n}\exp(rt_k)EC_0(\kappa_k, t_k) \end{aligned} \tag{4.1}$$

where $EC_0(\kappa_k, t_k)$ denotes the price of a European call option at time 0 with maturity t_k and strike $\kappa_k = nK_k$.

From this expression, the static super-hedging strategy can easily be obtained. For each k, one can buy $\exp(-r(T-t_k))/n$ European call options at time 0 with strike κ_k and maturity t_k and hold these until expiry. The value of this position will at each time dominate the price of the Asian call option. The obtained hedge involves the trading of the corresponding European call options. If this is not aspired or if for liquidity reasons, such options are not available for trading, then it is possible to invest into the underlying stock only, according to the delta hedge ratios of the European options involved in Equation (4.1). The held amount of stock in the hedging portfolio at each time is then given by the cumulative sum of the relevant deltas, which then provides a bound for the actual hedge ratio.

Note that Equation (4.1) holds for all combination of $\kappa_k \geq 0$ that satisfy $\sum_{k=1}^{n} \kappa_k = nK$, thus one can have a variety of portfolios of European call options, which dominate the price of an arithmetic Asian option from above. From the perspective of super-hedging as well as pricing, it is natural to try to make this upper bound as tight as possible. This is equivalent to finding a combination of κ_k's which minimize the right hand side of Equation (4.1). Albrecher et al. (2005) showed that such combinations can be obtained by the use of stop-loss transforms and comonotonic risk theory. The solution can be obtained as follows:

$$\kappa_k = F^{-1}(F_{S^c}(nK); t_k), \quad k = 1, \cdots, n \tag{4.2}$$

where $F(x_k; t_k)$ is the conditional distribution of S_{t_k} with respect to the σ-algebra of initial information $\mathcal{F}_0$ under the risk-neutral measure $\mathbb{Q}$ and F_{S^c} is identified by the relationship:

$$F_{S^c}^{-1}(x) = \sum_{k=1}^{n} F_{x_k}^{-1}(x), \quad x \geq 0 \tag{4.3}$$

with $F_{x_k}^{-1}(x)$ denoting the inverse of $F(x_k; t_k)$ with respect to the argument x_k.

The solution is model independent and can be applied for any arbitrage-free model. Computational difficulties will prevail though whenever the distribution functions involved are not known in explicit form, or where it is difficult to carry out the inversion.

4.3 Some Benchmark Models: the Heston model, the CEV model and the Schwartz (1997) Two-factor Model

4.3.1 Heston Model

Heston (1993) proposed the following model:

$$\begin{aligned}
&dS_t = S_t r d_t + S_t \sqrt{V_t} dW_1 \\
&dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_2 \\
&dW_1 \cdot dW_2 = \rho d_t
\end{aligned}$$

where S_t denotes the stock price, V_t denotes the volatility, r denotes the risk-free rate of interest, κ denotes the rate of mean reversion of the volatility, θ denotes the long run mean variance, σ denotes the volatility of volatility, dW_1 and dW_2 denote the increments of two correlated Brownian motions with correlation ρ.

This model has become very popular because of the existence of semi-closed form solutions for European call options. Numerical code for pricing European call options under the Heston model are readily available, compare Moodley (2005). The numerical implementation of Moodly will be relied for the purpose of this chapter.

An explicit expression for the distribution function of the stock price S_t under the Heston's model is not available. The approach followed here is to accurately calibrate the Heston model to current market data and then to obtain the distribution function via Monte Carlo simulation. This is done once in a while and can be used for pricing a number of Asian options, until the market conditions (parameters κ, θ and σ) change so significantly that a recalibration becomes necessary.

4.3.2 CEV Model

Beckers (1980) studied forty-seven shares of companies using five years of daily data and concluded that a constant elasticity of variance (CEV) process describes actual share prices significantly better than the more traditional log-normal model.

Under the CEV assumption, the stock price features the following dynamics:

$$dS_t = S_t(rdt + \sigma {S_t}^{\beta} dW_t)$$

compare Davydov and Linetsky (2001). The instantaneous volatility in this model is hence σS_t^{β} and in order to make the initial calibration comparable with the classical Black-Scholes setup, one can set $\sigma = \sigma_0 {S_0}^{-\beta}$. In fact, the CEV model reduces to the Black-Scholes model if $\beta = 0$. For $\beta = -1/2$, one can obtain a type of square root process, as seen in the Heston model. The latter two cases are not considered in this chapter. In addition, assume that $r > 0$.

Let $\chi^2(\delta, \alpha; \cdot)$ denotes a non-central Chi-squared distribution with $\delta = 2 + \frac{1}{\beta}$ degrees of freedom and the non-centrality parameter α. Set $c(t) = \frac{\beta\sigma^2}{2r}(e^{2r\beta t} - 1)$, $\hat{y} = \frac{1}{c(t)} y^{-2\beta} e^{2r\beta t}$, $\alpha = \frac{S_0^{-2\beta}}{c(t)}$, then according to Jeanblanc-Picqué et al. (2009), the cumulative distribution function of the stock price conditional on S_0 can be expressed as follows:

For $\beta > 0$,

$$\begin{aligned} \mathbb{P}_{S_0}(S_t \leq y) &= 1 - \chi^2\left(\delta, \frac{S_0^{-2\beta}}{c(t)}; \hat{y}\right) \\ &= 1 - \sum_{n=1}^{\infty} g\left(n, \frac{S_0^{-2\beta}}{2c(t)}\right) G\left(n + \frac{1}{2\beta}, \frac{1}{2c(t)} y^{-2\beta} e^{2r\beta t}\right) \end{aligned}$$

where

$$g(\alpha, \mu) = \frac{\mu^{\alpha-1}}{\Gamma(\alpha)} e^{-\mu} \text{ and } G(\alpha, \mu) = \int_{\upsilon \geq \mu} g(\alpha, \upsilon) \mathbb{1}_{\upsilon \geq 0} d\upsilon.$$

For $\beta < -\frac{1}{2}$, one can have:

$$\mathbb{P}_{S_0}(S_t \leq y) = \chi^2\left(\delta, \frac{S_0^{-2\beta}}{c(t)}; \hat{y}\right)$$

and for $-\frac{1}{2} < \beta < 0$, one can have:

$$\mathbb{P}_{S_0}(S_t \le y) = 1 - \sum_{n=1}^{\infty} g\left(n - \frac{1}{2\beta}, \frac{S_0^{-2\beta}}{2c(t)}\right) G\left(n, \frac{1}{2c(t)} y^{-2\beta} e^{2r\beta t}\right)$$

The case of $\beta < 0$ was originally studied by Cox (1975) who also derived a formula for a European call option, and it was further developed by Schroder (1989). Based on the study of Schroder (1989) and Jeanblanc-Picqué et al. (2009), a simpler formula for this case can be derived as:

$$\mathbb{E}_{\mathbb{Q}}\big(e^{-rT}(S_T - K)^+\big) = S_0\big(1 - \chi^2(\hat{y}_1, \delta_1, \alpha_1)\big) - Ke^{-rT}\big(1 - \chi^2(\hat{y}_1, \delta_2, \alpha_1)\big)$$

where

$\hat{y}_1 = \frac{1}{c(T)} K^{-2\beta} e^{2r\beta T}, \quad \delta_1 = 2 - 1/\beta, \quad \delta_2 = 2 + 1/\beta, \quad \alpha_1 = \frac{1}{c(T)} S_0^{-2\beta}.$

Emanuel and MacBeth (1982) extended the result of Cox (1975) to the case $\beta > 0$, including the case of a European call option. Similarly, one can also derive a simpler formula for this case:

$$\mathbb{E}_{\mathbb{Q}}\big(e^{-rT}(S_T - K)^+\big) = S_0\chi^2(\hat{y}_1, \delta_1, \alpha_1) - Ke^{-rT}\chi^2(\hat{y}_1, \delta_2, \alpha_1)$$

The evaluation of the non-central Chi-squared distribution function however tends to be computational expensive, which is particularly relevant for the case considered in this article. Fortunately, approximations to the non-central Chi-squared distribution have been studied expensively and one particular good approximation has been derived by Sankaran (1959) and Sankaran (1963), as follows:

$$\chi^2(\delta, \alpha; \hat{y}) \sim 1 - N\left(\frac{1 - hp[1 - h + 0.5(2 - h)mp] - [\hat{y}/(\delta + \alpha)]^h}{h\sqrt{2p(1 + mp)}}\right) \tag{4.4}$$

where $N(\cdot)$ denotes the cumulative standard normal distribution function and

$$\begin{aligned} h &= 1 - \frac{2}{3}(\delta + \alpha)(\hat{y} + 3\alpha)(\delta + 2\alpha)^{-2}, \\ p &= 2(\delta + 2\alpha)(\delta + \alpha)^{-2}, \\ m &= (h - 1)(1 - 3h). \end{aligned}$$

4.3.3 Schwartz (1997) Two-factor Framework

Schwartz (1997) assumed that under the risk-neutral measure $\mathbb{Q}$, the spot price of the commodity and the instantaneous convenience yield follow the joint stochastic process:

$$\begin{aligned} dS_t &= (r - \delta_t)S_t dt + \sigma_s S_t dW_S, \\ d\delta_t &= (k(\alpha - \delta_t) - \lambda_c \sigma_c)dt + \sigma_c dW_c. \end{aligned}$$

with Brownian motions W_S andW_c and correlation $dW_S dW_c = \rho_{sc} dt$.

where $\lambda_c\sigma_c$ is constant, denoting the market price of convenience yield risk.

The distribution of the stock price under this model is log normal. Denote $X_t = \log(S_t)$, then an analytical expression for its distribution under $\mathbb{Q}$ can be found in Erb et al. (2011), that is:

$$\begin{aligned} E(X_t) &= X_0 + \left(r - \frac{1}{2}\sigma_s^2 - \hat{\alpha}\right)t + \left(\hat{\alpha} - \delta_0 \frac{1-e^{-kt}}{k}\right), \\ \sigma_{X_t}^2 &= \frac{\sigma_c^2}{k^2}\left(\frac{1}{2k}(1-e^{-2kt}) - \frac{2}{k}(1-e^{-kt}) + t\right) + 2\frac{\sigma_s\sigma_c\rho_{sc}}{k}\left(\frac{1-e^{-kt}}{k} - t\right) + \sigma_S^2 t, \end{aligned}$$

where $\hat{\alpha} = \alpha - \frac{\lambda_c\sigma_c}{k}$.

Hilliard and Reis (1998) presented a simple expression for the price of futures and forward contracts as well as European options on futures and forwards. Denote with $F(S_t, \delta_t, t, T)$ the futures price at time t with maturity at time T. Denote with $\tau = T - t$ the time to maturity, then the futures price can be obtained as follows:

$$F(S_t, \delta_t, t, T) = S_t A(\tau)\exp(-H_c(\tau)\delta_t)\frac{1}{P(t,T)}$$

with

$$\begin{aligned} A(\tau) &= \exp\left[\frac{(H_c - \tau)(k^2 - k\lambda_c\sigma_c - \sigma_c^2/2 + \rho_{sc}\sigma_s\sigma_c k)}{k^2} - \frac{\sigma_c^2 H_c^2}{4k}\right], \\ H_c(\tau) &= \frac{1-e^{-k\tau}}{k}, \end{aligned}$$

where $P(t,T)$ is the price of a zero-coupon bond with maturity at time T. In the following, it is assumed that interest rates are deterministic and constant, so that futures and forwards are equivalent. In this chapter:

$$P(t,T) = e^{-r\tau}$$

The formula for the valuation of a European call option on a future resembles the classical Black-Scholes formula with time dependent volatility:

$$\begin{aligned} C(t, T_1, T) &= P(t, T_1)[F(t,T)N(d_1) - KN(d_2)], \\ \textit{where} \quad d_1 &= \frac{\ln(F(t,T)/K) + 0.5v^2}{v}, \quad d_2 = d_1 - v, \\ v^2(t, T_1, T) &= \sigma_s^2(T_1 - t) \\ &\quad -\frac{2\rho_{sc}\sigma_s\sigma_c}{k}\left[(T_1 - t) - \frac{(e^{-k(T-T_1)} - e^{-k(T-t)})}{k}\right] \\ &\quad +\frac{\sigma_c^2}{k^2}\left[(T_1 - t) - \frac{2}{k}(e^{-k(T-T_1)} - e^{-k(T-t)}) \right.\\ &\quad \left. +\frac{1}{2k}(e^{-2k(T-T_1)} - e^{-2k(T-t)})\right]. \end{aligned}$$

Note that if $T_1 = T$ is chose, the European futures call is actually a classical

European call option on the spot, which allows us to apply the results above for pricing arithmetic Asian options written on the spot.[3]

4.4 Numerical Results

Ait-Sahalia and Kimmel (2007) has estimated parameters in a Heston model setting, using VIX daily data from January 2, 1990 to September 30, 2003.[4] The results are shown in Table 4.1.

Table 4.1: Parameters estimates for the Heston model

This table reports estimates of the Heston model on VIX daily data from January 2, 1990 to September 30, 2003. κ is the mean reversion rate, θ is the long run mean of volatility, σ is the volatility of volatility, ρ is the correlation between stock price and volatility.

Parameter	Estimate
κ	5.07
θ	0.0457
σ	0.48
ρ	-0.767

The mean value of implied volatility and the initial value of implied volatility during this period have been estimated as 0.5967 and 0.8596 respectively. The average risk-free rate has been close to 0.02 during this period. These results will be used in the context of the numerical investigation under the Heston model.

Parameters for the Schwartz (1997) two-factor model have been estimated by Gibson and Schwartz (1990), who used the spot price and convenience yield of crude oil during the period from January 1984 to November 1988: $k = 16.0747$, $\alpha = 0.1861$, $\sigma_s = 0.3534$, $\sigma_c = 1.1211$, $\rho_{sc} = 0.3200$, $\lambda_c = -1.796$. The initial convenience yield is chose $\delta_0 = 0.05$.

4.4.1 Black-Scholes Model

For the purpose of completeness and illustration, the Black-Scholes model is included in the analysis. An explicit expression for the optimal combination of κ_k for an arithmetic Asian option in this setup has been provided by Nielsen and Sandmann (2003):

$$\kappa_k = a_k n K = S_0 \left(\frac{a_1 n K}{S_0} \right)^{\sqrt{\frac{t_k}{t_1}}} \exp\left\{ \left(r - \frac{1}{2}\sigma^2 \right) (t_k - \sqrt{t_1 t_k}) \right\} \tag{4.5}$$

where $a_k, k = 1, \cdots, n$ can be interpreted as weights and a_1 is determined such that $\sum_{k=1}^{n} a_k = 1$.

[3] We could as well consider options written on the average futures price with the same methodology as developed here, but for consistency we choose the spot.

[4] Compare Psychoyios et al. (2010) who use a jump diffusion model to price VIX volatility options and futures.

An arithmetic Asian option is examined with a maturity of 1 year with monthly sampling (i.e. $n = 12$). In order to obtain the optimal hedging portfolio under the Black-Scholes assumption, one thus have to determine the strike price κ_k of the European call options at these 12 points. This can be obtained via Equation (4.5). Specifically, one can assume that the initial stock price S_0 is 100 and the strike price K is 80, 90, 100, 110, 120 respectively, reflecting the cases of in-the-money, at-the-money and out-of-the-money options. Assuming regular sampling one can have $t_k = \frac{k}{12}$ which can be substituted alongside S_0 and K into Equation (4.5) from which a_1 can then be obtained from:

$$\sum_{k=1}^{12} \kappa_k = \sum_{k=1}^{12} S_0 \left(\frac{a_1 12K}{S_0} \right)^{\sqrt{k}} \exp\left\{ (r - \frac{1}{2}\sigma^2) \left(\frac{k - \sqrt{k}}{12} \right) \right\} = 12K$$

and in consequence the strike prices κ_k can be calculated from Equation (4.5). Table 4.A.1 in the appendix contains the results of this exercise.

The price of the hedging strategy is then easily determined by the use of Equation (4.1). The prices of arithmetic Asian option computed by Monte Carlo simulation AA_{MC}, the comonotonic approach AA_C, the super-hedging price obtained from using one European call option with identical strike price and maturity EC and the super-hedging price AA_{tr} where $\kappa_k = K$ for all k are compared. Specifically, $100,000$ simulated trials are used for standard Monte Carlo prices. Table 4.2 shows the results of this exercise. The comonotonic approach works better than the two other super-hedging approaches, but the error margin is relatively high.

Table 4.2: Prices under the Black-Scholes model (monthly averaging)

This table presents the prices of arithmetic Asian option under the Black-Scholes model, which are computed by Monte Carlo simulation AA_{MC}, the comonotonic approach AA_C, the super-hedging price AA_{tr} where $\kappa_k = K$ for all k and European options EC respectively.

	AA_{MC}	AA_C	AA_{tr}	EC
K=80	24.827	26.2866	27.4783	33.37
K=90	18.877	20.6093	21.6303	28.436
K=100	13.965	16.0918	16.9091	24.227
K=110	10.2264	12.526	13.226	20.651
K=120	7.5555	9.7062	10.4042	17.62

4.4.2 Heston Model

Assuming general correlation between the volatility and stock in the Heston model, it is not possible to determine the distribution function of the stock price explicitly. Since this distribution function however is used as an input to the comonotonic method, an alternative way must be followed. Two possible pathways exist. First, one can compute the distribution function numerically via Monte Carlo method. Given a calibration of model parameters κ, θ and σ, this has only to be done once, and can then be used for pricing a range of options, until the calibration has changed significantly. Second, one can use an explicit approximation to the distribution, which has good analytical properties. This approach is followed for the CEV model.

Specifically, in the case of the Heston model, an arithmetic Asian option is considered under the assumption of monthly sampling and it is proceeded as follows:

1. Assume $S(0) = 100$ and simulate stock prices from the Heston model by Monte Carlo method for $100,000$ paths.
2. Sort these stock prices from small to large at t_1, then one can get the stock price distribution at t_1. Since 100,000 paths are taken, the probability for each stock price is 1/100000.
3. Compute the inverse of the stock price distribution function at t_1 by solving for the quantiles of these data with corresponding probability.
4. Fitting the data from the inversion of the stock price distribution by a polynomial of degree 10, then the coefficient vector $b_1 = (b_{11}, b_{12}, \cdots, b_{110})$ can be obtained.
5. Do the same thing as above for $t_2, t_3, \cdots, t_{12}$ to obtain the corresponding coefficient vectors $b_2, b_3, \cdots, b_{12}$.
6. Next, add these coefficient vectors to get a new coefficient vector $(c_1, c_2, \cdots, c_{10})$, note that $c_1 = b_{11} + b_{21} + \cdots + b_{121}$. This results in a new polynomial of degree 10, which according to Equation (4.3) can be used to describe the inverse of the distribution F_{S^c}.
7. After that, one can find the probability of the price nK in this new inverse distribution, that is, $F_{S^c}(nK)$[5]. Then the optimal strike prices can be provided by substituting $F_{S^c}(nK)$ as well as t_k according to Equation (4.2).
8. Given these strike price κ_k and their corresponding maturity t_k, the prices of European call options under the Heston model can be obtained by the use of the Matlab code in Moodley (2005).
9. Finally, according to Equation (4.1), the price of an arithmetic Asian option can be obtained. The corresponding super-hedging strategy comes as a by-product as discussed in Section 4.2.

[5]This probability has been corrected in the form $F_{S^c}(nK) = P + Q/100000$. Since there are a large number of repeated data in the new inverse distribution and many data are in fact really close to each other, we first remove the repeated data and then find the total number of stock prices centered around nK, denoted by Q. The range depends on the performance of the polynomial fitting process, we use $\pm 0.75\%$, $\pm 2.5\%$, $\pm 0.5\%$ for the Heston model, the CEV model and the Schwartz (1997) two-factor model respectively. After that, we find the probability P of the stock price which closest to the stock price nK.

Table 4.A.2 in the appendix presents the optimal strikes for different values of moneyness. The following Table 4.3 presents a number of arithmetic Asian option prices computed by the comonotonicity approach as compared to Monte Carlo prices AA_{MC}, the prices computed from the Black-Scholes optimal strikes AA_{C^*}, as well as AA_{tr} and EC computed in analogy to the previous subsection. Interestingly, the

Table 4.3: Prices under the Heston model (monthly averaging), $\theta = 0.0457$

This table presents the prices of arithmetic Asian option under the Heston model, which are computed by Monte Carlo simulation AA_{MC}, the comonotonic approach AA_C with optimal strike price, the comonotonic approach AA_{C*} with optimal strike price from the Black-Scholes model, the super-hedging price AA_{tr} where $\kappa_k = K$ for all k and European options EC respectively.

$S(0) = 100$	AA_{MC}	AA_C	AA_{C*}	AA_{tr}	EC
$K = 80$	25.882	25.9948	26.8813	26.8217	29.185
$K = 90$	19.542	19.9816	20.9128	20.7454	23.333
$K = 100$	14.288	15.0787	15.7291	15.7401	18.398
$K = 110$	10.102	11.1673	11.7627	11.7446	14.319
$K = 120$	6.8948	8.0985	8.5804	8.6396	11.011

results obtained from the comonotonicity approach are much better here than under the Black-Scholes assumption in the previous subsection, at least for the in-the-money cases.

The comonotonicity approach is also tested on $\theta = 0.5$ in case the volatility of underlying assets is extremely high. The optimal strikes for different values of moneyness are presented in Table 4.A.3 in the appendix. The result in Table 4.4 shows that the same result can be found as before.

Table 4.4: Prices under the Heston model (monthly averaging), $\theta = 0.5$

This table presents the prices of arithmetic Asian option under the Heston model, which are computed by Monte Carlo simulation AA_{MC}, the comonotonic approach AA_C with optimal strike price,the super-hedging price AA_{tr} where $\kappa_k = K$ for all k and European options EC respectively.

$S(0) = 100$	AA_{MC}	AA_C	AA_{tr}	EC
$K = 80$	28.5735	28.83502	31.5249	32.85754
$K = 90$	22.9539	23.98135	26.2347	27.76096
$K = 100$	18.2532	19.89097	21.7782	23.41524
$K = 110$	14.4003	16.37853	18.0798	19.74368
$K = 120$	11.2796	13.1391	15.0396	16.66156

4.4.3 CEV Model

Now focus is the CEV model. In contrast to the Heston model, the stock price distribution under the CEV model can be obtained explicitly, compare Section 4.3.2. However, the expression can not easily be inverted as required for the comonotonicity approach. For this reason, the approximation Equation (4.4) will be used in the following approach. Specifically, $\beta = 0.4, 0.3, 0.2, 0.1, -0.1, -0.2, -0.3, -0.4$ is chose in the CEV model and monthly averaging is considered.

1. Assume that the initial stock price is $S_0 = 100$ and consider stock prices in the range from 0 to 500 for the exact stock price distribution and 0 to 400 for the approximated stock price distribution respectively.

2. Use the stock price distribution function or its approximation to obtain the probability P of stock prices under the CEV model at t_1 to t_{12}.

3. However, the above data cannot provide a complete fit. Therefore, the probabilities P can be interpolated by the probability of $1/100000 : 1/100000 : 1$ and thus one can get $100,000$ corresponding stock prices at each time so that one can get a much better fit.

4. Do the same as it did in the case of the Heston model from step 3 to step 9, and from then get a very quick and accurate arithmetic Asian call option price and also its hedging strategy.

Table 4.5: Computational time for determining the option price using the analytical non-central Chi-squared distribution and its approximated distribution

Beta	Analytical(s)	Approximated(s)
0.4	4.5	3.87
0.3	4.95	3.86
0.2	5.62	3.84
0.1	7.3	3.89
-0.1	67	3.87
-0.2	38.5	3.89
-0.3	25.7	3.89
-0.4	19.4	3.89

Table 4.5 shows that for the analytical non-central Chi-squared distribution, the computational time depends on the beta that chose. It appears that it will take more time when β is close to zero, and the computational time for $\beta > 0$ is much less than for $\beta < 0$. In the case of the approximated non-central Chi-squared distribution, a stable performance of 3.8 seconds approximately for different β's can be observed. Furthermore, for almost all β's that selected, using the approximated non-central Chi-squared distribution is much faster than using the analytical exact alternative.

Table 4.A.4 in the appendix presents the optimal strike prices for different values of moneyness. Note that in the table the stock price distribution for the columns flagged

by K is calculated by an analytical non-central Chi-squared distribution, whereas the stock price distribution in the columns flagged by K^* is obtained by an approximated non-central Chi-squared distribution.

The price of arithmetic Asian call options under the CEV model for different β's can then be obtained once again by Equation (4.1). Let AA_{C**} be the arithmetic Asian option price using the approximated stock price distribution in connection with the comonotonicity approach. Tables 4.6 presents the corresponding results for different β's.

Table 4.6: Prices under the CEV model

	$S = 100$	AA_{MC}	AA_C	AA_{C**}	AA_{tr}	EC
	$K = 80$	24.409	24.0716	24.0829	26.9608	32.591
	$K = 90$	18.488	19.0382	19.0414	21.3438	28.041
$\beta = 0.4$	$K = 100$	13.946	15.1239	15.1463	16.9317	24.276
	$K = 110$	10.548	11.8113	12.0239	13.5664	21.156
	$K = 120$	8.0546	9.5907	9.6041	11.0234	18.557
	$K = 80$	24.529	24.245	24.2093	27.0855	32.777
	$K = 90$	18.583	19.0619	19.0849	21.4115	28.131
$\beta = 0.3$	$K = 100$	13.987	15.0937	15.0996	16.9217	24.255
	$K = 110$	10.522	11.6827	11.8973	13.476	21.018
	$K = 120$	7.955	9.3614	9.391	10.8616	18.309
	$K = 80$	24.665	24.4188	24.3368	27.2132	32.969
	$K = 90$	18.665	19.0882	19.1309	21.4818	28.227
$\beta = 0.2$	$K = 100$	14.015	15.0574	15.0567	16.9147	24.239
	$K = 110$	10.489	11.5702	11.7738	13.3892	20.888
	$K = 120$	7.8656	9.1361	9.1836	10.7047	18.07
	$K = 80$	24.739	24.592	24.4674	27.3441	33.166
	$K = 90$	18.697	19.1182	19.1796	21.5547	28.329
$\beta = 0.1$	$K = 100$	13.852	15.016	15.0155	16.9104	24.23
	$K = 110$	10.332	11.4732	11.6541	13.306	20.766
	$K = 120$	7.6264	8.916	8.9809	10.5524	17.841
	$K = 80$	25.059	25.1161	24.7382	27.6161	33.5806
	$K = 90$	18.926	19.0994	19.2858	21.7087	28.5492
$\beta = -0.1$	$K = 100$	14.028	14.9938	14.9414	16.9104	24.2299
	$K = 110$	10.241	11.4061	11.4235	13.1492	20.5425
	$K = 120$	7.3865	8.4059	8.591	10.2601	17.407
	$K = 80$	25.011	25.2472	24.8789	27.7577	33.7981
	$K = 90$	18.84	19.1401	19.3437	21.7901	28.6687
$\beta = -0.2$	$K = 100$	13.899	14.9299	14.9079	16.9147	24.2391
	$K = 110$	10.057	11.3527	11.312	13.0754	20.4406
	$K = 120$	7.172	8.2013	8.4033	10.1197	17.2013
	$K = 80$	25.167	25.3711	25.0256	27.9033	34.0234
	$K = 90$	18.948	19.2077	19.4068	21.8746	28.7947
$\beta = -0.3$	$K = 100$	13.922	14.8702	14.8803	16.9217	24.2546
	$K = 110$	10.007	11.2665	11.2034	13.0046	20.3451
	$K = 120$	7.0602	8.0223	8.2198	9.9829	17.0028
	$K = 80$	25.227	25.4959	25.1779	28.0533	34.2571
	$K = 90$	18.956	19.3066	19.4735	21.9623	28.9279
$\beta = -0.4$	$K = 100$	13.869	14.8146	14.8531	16.9317	24.2765

Table 4.6 (Continued)

$S = 100$	AA_{MC}	AA_C	AA_{C**}	AA_{tr}	EC
$K = 110$	9.8868	11.1619	11.0963	12.9368	20.2559
$K = 120$	6.8809	7.8689	8.041	9.8496	16.811

It can be observed that no matter what the value of β is, the analytical comonotonic prices (AA_c) and the approximated comonotonic prices (AA_{c**}) are all very close to each other. This means, the approximated Equation (4.4) can indeed provide a good description of the non-central Chi-squared distribution. Weighing up accuracy and computing time, it is more inclined to choose the approximated non-central Chi-squared distribution for pricing arithmetic Asian option with the comonotonic method under the CEV model.

In the context of the CEV model, it can be observed that the performance of the comonotonicity approach for $\beta < 0$ is better than it for $\beta > 0$. Fortunately, empirical evidence seems to support that β is largely negative, compare Beckers (1980) and Christie (1982). Comparing the results from the Heston model with those obtained for the CEV model, it can be found that the performance of the comonotonicity approach in the context of the CEV model is close to its performance in the context of the Heston model when β falls between -0.3 and -0.4.

4.4.4 Schwartz (1997) Two-factor Model

The case of the Schwartz (1997) two-factor model is similar to the case of the CEV model, in both cases, there will have explicit expressions for the distribution of the stock price. Therefore, in the case of the Schwartz (1997) two-factor model, the method that used in the case of the CEV model can be followed.

Table 4.A.5 in the appendix presents the optimal strike prices for different values of moneyness. The following Table 4.7 shows the results of the corresponding exercise.[6]

Table 4.7: Prices under the Schwartz (1997) two-factor model (monthly averaging)

This table presents the prices of arithmetic Asian option under the Schwartz (1997) two-factor model, which are computed by Monte Carlo simulation AA_{MC}, the comonotonic approach AA_C, the super-hedging price AA_{tr} where $\kappa_k = K$ for all k and European options EC respectively.

	AA_{MC}	AA_C	AA_{tr}	EC
K=80	10.5450	10.8753	12.3340	8.2096
K=90	5.3971	6.2467	7.1505	5.2299
K=100	2.4415	3.3445	3.7518	3.2682
K=110	1.0059	1.7145	1.8718	2.0154
K=120	0.3831	0.8382	0.9361	1.2322

[6]Note that since the model has been calibrated to a different data set, the results here are quantitatively different than in the previous sections.

It can be observed that the comonotonicity approach works well with the Schwartz (1997) two-factor model.

4.4.5 General

Tables 4.A.1, 4.A.2, 4.A.3, 4.A.4, 4.A.5 present the strike prices and maturities of options to be included in a portfolio of European call options for the purpose of static super-hedging in the case of monthly averaging for the Black-Scholes, the Heston model ($\theta = 0.0457$ and $\theta = 0.5$), the CEV model and the Schwartz (1997) two-factor model respectively. It can be observed that no matter whether the Black-Scholes model, the Heston model, the CEV model or the Schwartz (1997) two-factor model is applied, the static super-hedging portfolio contains only in-the-money call options, if the Asian option is in-the-money, and only out-of-the-money call options, if the Asian option is out-of-the-money. In the case of an at-the-money Asian option however the static super hedging portfolio consists of call options of which some are in-the-money and some are out-of-the-money.

Tables 4.2, 4.3, 4.4, 4.6 show that, for the same arithmetic Asian call option, prices derived from the Black-Scholes model, the Heston model ($\theta = 0.0457$ and $\theta = 0.5$) and the CEV model via the comonotonicity approach, i.e. (AA_c), are less than the prices obtained from the two trivial super-hedging strategies, (AA_{tr}) and (EC), which is as expected. Moreover, in the Heston model, the comonotonic hedging prices (AA_c) are less than the prices obtained from using the Black-Scholes comonotonic strikes (AA_{c*}), reflecting that the latter combination of strike prices are not optimal strikes for the Heston model. These results suggest that the comonotonicity approach leads to good results when pricing an arithmetic Asian call option under the Black-Scholes model, the Heston model as well as the CEV model.

From Table 4.7, it can be observed that the same conclusion can be obtained for the case of the Schwartz (1997) two-factor model, except that the prices obtained from using a single European call option with the same strike in the hedging portfolio now lie below the prices of the corresponding Asian option. This may at first appear surprising, however the convenience yield has a similar function as the dividend yield in equities and it is well known that at least for the case of constant dividend yield $q > r$, the price of the Asian option can be higher than the price of the corresponding European call option with the same strike.

Now the accuracy of the comonotonicity approach is investigated for each of the three models against the benchmark of Monte Carlo under variation of the strike price. As Table 4.2 and Table 4.7 show, for the cases of the Black-Scholes model and the Schwartz (1997) two-factor model, the accuracy is largely unaffected by the strike price. For the Heston model (compare Table 4.3) and the CEV model (compare Table 4.6), however, the difference between comonotonic hedging price and standard Monte Carlo price is increasing as the strike price increases. In conclusion, it is more appropriate to apply the comonotonicity approach to in-the-money arithmetic Asian options than to out-of-the-money arithmetic Asian options, when either the Heston or the CEV model is believed to provide an accurate model of the market.

It should be pointed out that for the cases where analytical solutions for the price of the European call options exist, the comonotonicity approach is quite fast. Even under the Heston model, where Monte Carlo method is required to obtain the stock price distribution by simulation initially, at least when a large number of options has to be priced, the comonotonicity approach is still relatively fast, as compared with the alternative of pricing each option individually by Monte Carlo method. Furthermore, from Table 4.2, Table 4.3, Table 4.6 and Table 4.7, it can be observed that while the difference between comonotonic hedging price and standard Monte Carlo price under the Black-Scholes model can be quite large, the difference under the Heston model, the CEV model and the Schwartz (1997) two-factor model are generally much lower, with the Schwartz (1997) two-factor model showing the best performance. In conclusion, while the comonotonicity approach shows mediocre performance under the Black-Scholes assumption, it works well under the Heston and CEV assumption with typical parameters, and works very well in the context of the Schwartz (1997) two-factor model.

4.5 Optimization Based Alternatives to the Comonotonicity Approach

As indicated earlier, the price of an arithmetic Asian option can be approximated from above by a combination of European call options $EC_0(\kappa_k, t_k)$. As a result, the pricing of an arithmetic Asian option can be treated as a non-linear optimization problem, where the objective is to minimize the expression $\frac{\exp(-rT)}{n}\sum_{k=1}^{n}\exp(rt_k)EC_0(\kappa_k, t_k)$ subject to the constraints $\sum_{k=1}^{n}\kappa_k = nK$ and $\kappa_k > 0$.

Since the optimal strike prices in the comonotonicity approach used for the Heston model, the CEV model and the Schwartz (1997) two-factor model feature correctional adjustments (compare footnote 5 on page 42)[7], their sums are no longer exactly equal to nK. In consequence it makes sense to weaken the constraints in order to obtain a fair comparison. For this reason, the above minimization problem will be investigated under a weaker constraint of $0.99 \cdot nK \leq \sum_{k=1}^{n}\kappa_k \leq 1.01 \cdot nK$. However, it will additionally be remained to consider the case of strong constraints.

Note that the application of optimization based methods for the pricing of arithmetic Asian options requires in the same way as the comonotonicity approach, that analytical solution for European call options under the corresponding models exist.

Table 4.8 shows the prices of arithmetic Asian call options under the Heston model calculated by different pricing methods with different constraints. In the case of strong constraints, the comonotonicity approach and the optimization based method lead to very similar results for a range of strike prices. In the case of weak constraints however, the comonotonicity approach shows a better performance than the optimization based method, if the corresponding Monte Carlo price is chose as a benchmark. However, the difference between the two are typically small (around 0.3) for different option

[7] Although we use $\pm 0.75\%$, $\pm 2.5\%$, $\pm 0.5\%$ to calculate Q for the Heston, CEV and Schwarz (1997) model respectively, the final effect on $\sum_{k=1}^{n}\kappa_k$ for these models are range from 1% to 2%. Hence the weak constraint for comonotonic method and optimization method are the same.

Table 4.8: Comparison under the Heston model

$S(0) = 100$	MC	Comonotonicity		Optimization	
		weak	strong	weak	strong
K=80	25.882	25.9948	26.7605	26.2403	26.762
K=90	19.542	19.9816	20.7274	20.2368	20.728
K=100	14.288	15.0787	15.7412	15.2974	15.7408
K=110	10.102	11.1673	11.7355	11.3486	11.7339
K=120	6.8948	8.0985	8.5978	8.2733	8.5974

positions. In conclusion, the optimization based method presents a good alternative to the comonotonicity approach in the context of the Heston model.

Tables 4.A.6 in the appendix present the analogue comparison for the CEV model with different β's. In the case of strong constraints, it can be concluded similar as in the case of the Heston model, that is, the results obtained from the comonotonicity approach and the optimization based approach are more or less the same. Further, in the case of weak constraints, the results obtained from the comonotonicity approach are now significantly better than those obtained from the optimization based method. Hence in the context of the CEV model, the comonotonicity approach has significant advantageous as compared to the optimization based method.

Table 4.9: Comparison under the Schwartz(1997) two-factor Model

$S(0) = 100$	MC	Comonotonicity		Optimization	
		weak	strong	weak	strong
K=80	10.5450	10.8753	11.3481	9.6804	4.5299
K=90	5.3971	6.2467	6.5744	0.0087	0.0102
K=100	2.4415	3.3445	3.5729	0.0105	0.0123
K=110	1.0059	1.7145	1.8468	0.0111	0.0129
K=120	0.3831	0.8382	0.9192	0.0106	0.0123

Finally, Table 4.9 shows an analogue comparison for the the Schwartz(1997) two-factor model. As can be seen, no matter whether the constraints are weak or strong, the optimization based method shows inferior performance as compared with the comonotonicity approach. The reason for this is not clear, but it might be the case that in the context of the Schwartz (1997) two-factor model multiple local minima exist, which cause the optimization based method to depend sensitively on the initial values.

4.6 Conclusion

In this chapter, the performance of the Albrecher et al. (2005)'s comonotonicity approach is investigated to pricing and hedging of Asian options in the context of the Heston model, the CEV model and the Schwartz (1997) two-factor model. These models admit (semi) closed-form solutions for plain vanilla European call options and are hence suitable for this approach. It is shown that the comonotonicity approach provides a simple, quick and effective method for the valuation of an arithmetic Asian option in these cases. In comparison to the Heston model, the CEV model and the Schwartz (1997) two-factor model have the advantage, that the distribution function of the stock price is available in explicit form, in the Heston model this distribution function needs to be computed via Monte Carlo method. Besides, for the Heston model, the CEV model and the Schwartz (1997) two-factor model, it seems that prices for in-the-money arithmetic Asian options are more accurate than prices for out-of-the-money arithmetic Asian option when following the comonotonicity approach. Furthermore, from the comparison results between the comonotonicity approach and the optimization based alternative, it can be drawn additional support for the comonotonicity approach in all three cases, the Heston model, the CEV model and the Schwartz (1997) two-factor model.

Appendix

4.A Optimal Strike Prices and Comparison

Table 4.A.1: Strike prices for the hedge portfolio under the Black-Scholes model (monthly averaging) with $S(0) = 100$ and $T = 1$

	a_1	t_k	κ_k
$K = 80$	0.097	0.083	93.12
		0.167	89.716
		0.25	86.922
		0.333	84.459
		0.417	82.219
		0.5	80.143
		0.583	78.198
		0.667	76.36
		0.75	74.613
		0.833	72.946
		0.917	71.349
		1	69.814
$K = 90$	0.0905	0.083	97.74
		0.167	96.075
		0.25	94.526
		0.333	93.048
		0.417	91.621
		0.5	90.235
		0.583	88.886
		0.667	87.568
		0.75	86.278
		0.833	85.016
		0.917	83.778
		1	82.564
$K = 100$	0.0852	0.083	102.24
		0.167	102.39
		0.25	102.19
		0.333	101.81
		0.417	101.32
		0.5	100.75
		0.583	100.13
		0.667	99.457
		0.75	98.752
		0.833	98.021

	a_1	t_k	κ_k
Table 4.A.1 (Continued)			
		0.917	97.267
		1	96.496
$K = 110$	0.0805	0.083	106.26
		0.167	108.13
		0.25	109.25
		0.333	109.98
		0.417	110.45
		0.5	110.74
		0.583	110.88
		0.667	110.92
		0.75	110.87
		0.833	110.73
		0.917	110.54
		1	110.29
$K = 120$	0.0765	0.083	110.16
		0.167	113.78
		0.25	116.29
		0.333	118.2
		0.417	119.72
		0.5	120.96
		0.583	121.98
		0.667	122.82
		0.75	123.53
		0.833	124.1
		0.917	124.58
		1	124.96

Table 4.A.2: Strike prices for the hedge portfolio under the Heston model (monthly averaging) with $S(0) = 100$, $\theta = 0.0457$

	t_k	κ_k
$K = 80$	0.083	88.787
	0.167	84.633
	0.25	82.428
	0.333	81.305
	0.417	80.61
	0.5	80.225
	0.583	79.927
	0.667	79.682
	0.75	79.435
	0.833	79.186
	0.917	79.053
	1	78.868
	0.083	95.922
	0.167	93.337
	0.25	91.973
	0.333	91.378
	0.417	91.006
$K = 90$		

Table 4.A.2 (Continued)		
	t_k	κ_k
	0.5	90.866
	0.583	90.639
	0.667	90.493
	0.75	90.398
	0.833	90.27
	0.917	90.179
	1	90.015
	0.083	102.8
	0.167	101.88
	0.25	101.39
	0.333	101.25
	0.417	101.22
$K = 100$	0.5	101.24
	0.583	101.28
	0.667	101.34
	0.75	101.41
	0.833	101.44
	0.917	101.41
	1	101.38
	0.083	109
	0.167	110.12
	0.25	110.75
	0.333	111.11
	0.417	111.52
$K = 110$	0.5	111.76
	0.583	112.11
	0.667	112.33
	0.75	112.49
	0.833	112.7
	0.917	112.76
	1	112.91
	0.083	115.29
	0.167	118.45
	0.25	120.17
	0.333	121.23
	0.417	122.04
$K = 120$	0.5	122.63
	0.583	123.06
	0.667	123.38
	0.75	123.61
	0.833	123.97
	0.917	124.18
	1	124.39

Table 4.A.3: Strike prices for the hedge portfolio under the Heston model (monthly averaging) with $S(0) = 100$, $\theta = 0.5$

	t_k	κ_k
	0.083	97.91

$K = 80$

Table 4.A.3 (Continued)

	t_k	κ_k
	0.167	94.79
	0.25	92.22
	0.333	90.11
	0.417	87.72
	0.5	85.41
	0.583	82.96
	0.667	80.76
	0.75	78.51
	0.833	76.59
	0.917	74.56
	1	72.49
$K = 90$	0.083	102.76
	0.167	101.42
	0.25	100.05
	0.333	98.8
	0.417	97.22
	0.5	95.65
	0.583	93.95
	0.667	92.28
	0.75	90.58
	0.833	89.24
	0.917	87.84
	1	86.19
$K = 100$	0.083	107.03
	0.167	107.65
	0.25	107.51
	0.333	107.21
	0.417	106.52
	0.5	105.81
	0.583	105
	0.667	103.94
	0.75	103.07
	0.833	102.24
	0.917	101.51
	1	100.51
$K = 110$	0.083	111.89
	0.167	114.75
	0.25	115.93
	0.333	116.7
	0.417	116.82
	0.5	116.81
	0.583	116.66
	0.667	115.98
	0.75	115.64
	0.833	114.8
	0.917	114.14
	1	113.63
$K = 120$	0.083	117.35
	0.167	122.38
	0.25	125.12

Table 4.A.3 (Continued)

	t_k	κ_k
	0.333	127.2
	0.417	128.37
	0.5	129.15
	0.583	129.71
	0.667	129.65
	0.75	129.81
	0.833	129.34
	0.917	128.89
	1	128.67

Table 4.A.4: Strike prices for the hedge portfolio under the CEV model with $S(0) = 100$

β	t_k	κ_k K=80	K*=80	K=90	K*=90	K=100	K*=100	K=110	K*=110	K=120	K*=120
	0.083	95.77	95.32	101.11	101.03	105.73	106.28	110.42	110.31	114.64	114.13
	0.167	92.93	92.46	100.52	100.64	107.01	108.42	113.61	113.99	120.13	119.42
	0.250	90.50	90.44	99.58	99.87	107.44	108.92	115.58	116.25	123.77	123.59
	0.333	88.38	88.65	98.38	98.75	107.38	108.55	116.95	117.55	126.27	126.68
	0.417	86.40	86.83	97.01	97.40	106.97	107.75	117.83	118.12	127.94	128.68
β=0.4	0.500	84.56	85.00	95.58	95.92	106.33	106.73	118.27	118.15	128.97	129.79
	0.583	82.79	83.18	94.12	94.38	105.54	105.56	118.37	117.80	129.54	130.28
	0.667	81.10	81.37	92.65	92.79	104.64	104.29	118.21	117.18	129.78	130.33
	0.750	79.50	79.64	91.23	91.23	103.69	102.98	117.86	116.37	129.77	130.05
	0.833	78.00	77.96	89.87	89.68	102.71	101.64	117.39	115.44	129.59	129.54
	0.917	76.58	76.33	88.55	88.15	101.72	100.28	116.82	114.40	129.29	128.86
	1.000	75.26	74.79	87.31	86.68	100.74	98.93	116.21	113.32	128.90	128.06
	0.083	95.55	95.35	100.84	100.86	105.37	105.95	109.95	110.01	114.13	113.82
	0.167	92.71	92.49	100.19	100.40	106.64	107.98	113.05	113.62	119.42	118.94
	0.250	90.29	90.38	99.27	99.65	107.12	108.63	114.98	115.85	123.02	122.91
	0.333	88.17	88.58	98.15	98.62	107.11	108.39	116.38	117.19	125.63	126.03
	0.417	86.20	86.80	96.88	97.36	106.76	107.70	117.37	117.87	127.49	128.22
β=0.3	0.500	84.38	85.00	95.54	95.96	106.19	106.77	117.96	118.02	128.75	129.56
	0.583	82.66	83.20	94.17	94.47	105.47	105.67	118.24	117.80	129.57	130.28
	0.667	81.01	81.40	92.80	92.92	104.66	104.46	118.28	117.28	130.06	130.53
	0.750	79.47	79.65	91.49	91.37	103.81	103.19	118.14	116.57	130.31	130.43
	0.833	78.03	77.94	90.22	89.82	102.93	101.87	117.86	115.71	130.38	130.07
	0.917	76.66	76.27	89.00	88.27	102.05	100.52	117.50	114.74	130.32	129.51
	1.000	75.40	74.68	87.86	86.77	101.18	99.18	117.07	113.70	130.17	128.82
	0.083	95.30	95.38	100.55	100.71	105.00	105.64	109.48	109.72	113.64	113.54
	0.167	92.47	92.53	99.83	100.17	106.24	107.53	112.48	113.24	118.76	118.54
	0.250	90.06	90.33	98.92	99.42	106.76	108.28	114.37	115.45	122.29	122.31
	0.333	87.93	88.50	97.86	98.46	106.81	108.19	115.78	116.83	124.96	125.40
	0.417	85.98	86.74	96.67	97.28	106.53	107.62	116.84	117.60	126.99	127.72
β=0.2	0.500	84.19	84.99	95.44	95.96	106.04	106.78	117.57	117.87	128.48	129.28
	0.583	82.52	83.21	94.20	94.54	105.42	105.77	118.03	117.77	129.55	130.22
	0.667	80.93	81.42	92.95	93.04	104.72	104.62	118.27	117.38	130.32	130.68
	0.750	79.46	79.67	91.76	91.51	103.99	103.40	118.34	116.77	130.84	130.77
	0.833	78.10	77.94	90.63	89.98	103.24	102.13	118.29	116.01	131.19	130.59
	0.917	76.81	76.24	89.54	88.43	102.49	100.80	118.14	115.11	131.41	130.17
	1.000	75.64	74.60	88.52	86.91	101.76	99.47	117.93	114.13	131.52	129.60
	0.083	95.03	95.39	100.24	100.57	104.63	105.36	109.00	109.45	113.18	113.29
	0.167	92.19	92.59	99.45	99.97	105.81	107.09	111.92	112.86	118.13	118.20
	0.250	89.80	90.32	98.52	99.20	106.35	107.90	113.78	115.06	121.60	121.81
	0.333	87.67	88.43	97.50	98.27	106.47	107.94	115.17	116.47	124.30	124.81
	0.417	85.74	86.66	96.41	97.17	106.27	107.50	116.28	117.31	126.46	127.21
β=0.1	0.500	83.99	84.94	95.29	95.93	105.89	106.77	117.13	117.69	128.15	128.94
	0.583	82.37	83.20	94.18	94.58	105.38	105.85	117.75	117.72	129.47	130.09
	0.667	80.86	81.43	93.09	93.14	104.81	104.78	118.19	117.45	130.53	130.78

Table 4.A.4 (Continued)

β	t_k	κ_k K=80	K*=80	K=90	K*=90	K=100	K*=100	K=110	K*=110	K=120	K*=120
	0.750	79.48	79.68	92.06	91.66	104.22	103.63	118.47	116.97	131.35	131.08
	0.833	78.21	77.95	91.08	90.15	103.63	102.41	118.65	116.31	132.00	131.08
	0.917	77.03	76.23	90.15	88.61	103.04	101.12	118.73	115.50	132.52	130.84
	1.000	75.96	74.56	89.29	87.10	102.47	99.82	118.75	114.60	132.93	130.42
β=-0.1	0.083	94.19	95.45	99.62	100.37	103.41	104.90	107.92	108.97	112.38	112.89
	0.167	91.22	92.72	98.72	99.68	104.22	106.34	110.63	112.17	117.10	117.69
	0.250	88.80	90.39	97.77	98.84	104.59	107.10	112.40	114.29	120.46	121.08
	0.333	86.69	88.36	96.78	97.92	104.75	107.36	113.73	115.77	123.14	123.87
	0.417	84.81	86.52	95.83	96.90	104.84	107.16	114.82	116.74	125.43	126.26
	0.500	83.17	84.80	94.96	95.79	104.91	106.66	115.74	117.31	127.44	128.19
	0.583	81.73	83.10	94.19	94.58	104.97	105.94	116.59	117.56	129.31	129.68
	0.667	80.46	81.38	93.53	93.27	105.06	105.06	117.40	117.55	131.11	130.78
	0.750	79.37	79.66	92.98	91.90	105.16	104.07	118.20	117.31	132.84	131.51
	0.833	78.42	77.94	92.53	90.48	105.28	102.99	119.01	116.89	134.53	131.93
	0.917	77.58	76.20	92.17	89.00	105.40	101.82	119.84	116.32	136.22	132.08
	1.000	76.86	74.49	91.88	87.51	105.53	100.61	120.68	115.62	137.86	132.03
β=-0.2	0.083	93.84	95.52	99.30	100.32	103.14	104.74	107.38	108.79	112.05	112.76
	0.167	90.82	92.83	98.35	99.61	103.90	106.07	109.96	111.89	116.71	117.52
	0.250	88.40	90.50	97.39	98.74	104.25	106.78	111.71	113.97	120.05	120.86
	0.333	86.33	88.41	96.44	97.81	104.44	107.07	113.08	115.45	122.76	123.57
	0.417	84.52	86.51	95.55	96.79	104.58	106.98	114.26	116.48	125.14	125.90
	0.500	83.00	84.75	94.79	95.71	104.73	106.58	115.31	117.13	127.27	127.86
	0.583	81.69	83.04	94.16	94.55	104.94	105.96	116.30	117.48	129.27	129.45
	0.667	80.58	81.32	93.64	93.28	105.18	105.17	117.29	117.57	131.23	130.70
	0.750	79.64	79.60	93.25	91.96	105.47	104.26	118.26	117.45	133.13	131.61
	0.833	78.86	77.88	92.96	90.58	105.80	103.24	119.25	117.15	135.00	132.22
	0.917	78.18	76.12	92.76	89.12	106.15	102.13	120.25	116.68	136.89	132.57
	1.000	77.62	74.39	92.65	87.65	106.51	100.97	121.25	116.08	138.74	132.70
β=-0.3	0.083	93.51	95.61	98.92	100.32	102.86	104.64	106.92	108.66	111.70	112.69
	0.167	90.43	92.96	97.90	99.60	103.56	105.88	109.40	111.68	116.27	117.42
	0.250	88.00	90.65	96.92	98.73	103.91	106.54	111.14	113.71	119.59	120.74
	0.333	85.97	88.53	96.01	97.77	104.13	106.83	112.55	115.19	122.33	123.39
	0.417	84.25	86.56	95.22	96.74	104.34	106.80	113.82	116.26	124.78	125.67
	0.500	82.83	84.74	94.58	95.66	104.59	106.49	115.00	116.97	127.02	127.60
	0.583	81.67	82.98	94.09	94.51	104.92	105.96	116.15	117.40	129.16	129.24
	0.667	80.71	81.23	93.73	93.27	105.32	105.24	117.31	117.58	131.27	130.59
	0.750	79.94	79.50	93.49	91.97	105.78	104.39	118.46	117.55	133.32	131.63
	0.833	79.33	77.75	93.36	90.60	106.29	103.43	119.62	117.34	135.34	132.39
	0.917	78.82	75.97	93.33	89.16	106.84	102.37	120.81	116.97	137.38	132.92
	1.000	78.43	74.20	93.37	87.69	107.43	101.24	121.99	116.46	139.38	133.21
β=-0.4	0.083	93.18	95.70	98.48	100.33	102.56	104.57	106.52	108.56	111.31	112.66
	0.167	90.03	93.08	97.37	99.61	103.22	105.75	108.93	111.52	115.79	117.37
	0.250	87.59	90.78	96.38	98.72	103.56	106.35	110.66	113.50	119.08	120.67
	0.333	85.61	88.63	95.51	97.74	103.81	106.61	112.13	114.95	121.85	123.28
	0.417	83.97	86.60	94.82	96.67	104.10	106.60	113.49	116.03	124.37	125.49
	0.500	82.68	84.70	94.31	95.57	104.47	106.34	114.78	116.78	126.72	127.38
	0.583	81.66	82.87	93.97	94.40	104.93	105.87	116.08	117.26	128.98	129.01
	0.667	80.86	81.08	93.77	93.17	105.48	105.23	117.39	117.52	131.22	130.41
	0.750	80.26	79.33	93.70	91.90	106.10	104.46	118.71	117.59	133.39	131.56
	0.833	79.80	77.61	93.73	90.59	106.77	103.60	120.04	117.51	135.54	132.48
	0.917	79.47	75.88	93.84	89.25	107.50	102.66	121.40	117.29	137.71	133.20
	1.000	79.24	74.20	94.04	87.91	108.26	101.69	122.76	116.97	139.84	133.74

Table 4.A.5: Strike prices for the hedge portfolio under the Schwartz (1997) two-factor model (monthly averaging) with $S(0) = 100$

t_k	κ_k
0.083	96.547

$K = 80$

Table 4.A.5 (Continued)

	t_k	κ_k
	0.167	93.337
	0.25	90.207
	0.333	87.199
	0.417	84.3
	0.5	81.544
	0.583	78.898
	0.667	76.323
	0.75	73.874
	0.833	71.513
	0.917	69.209
	1	67.011
	0.083	101.43
	0.167	100.06
	0.25	98.214
	0.333	96.203
	0.417	94.084
$K = 90$	0.5	91.992
	0.583	89.9
	0.667	87.797
	0.75	85.739
	0.833	83.708
	0.917	81.682
	1	79.713
	0.083	106.19
	0.167	106.7
	0.25	106.17
	0.333	105.2
	0.417	103.97
$K = 100$	0.5	102.58
	0.583	101.07
	0.667	99.467
	0.75	97.824
	0.833	96.141
	0.917	94.411
	1	92.683
	0.083	110.42
	0.167	112.74
	0.25	113.59
	0.333	113.74
	0.417	113.43
$K = 110$	0.5	112.87
	0.583	112.1
	0.667	111.15
	0.75	110.09
	0.833	108.93
	0.917	107.66
	1	106.35
$K = 120$	0.083	114.62
	0.167	118.82
	0.25	121.06

Table 4.A.5 (Continued)

	t_k	κ_k
	0.333	122.35
	0.417	123.13
	0.5	123.41
	0.583	123.36
	0.667	123.06
	0.75	122.56
	0.833	121.89
	0.917	121.07
	1	120.14

Table 4.A.6: Comparison under the CEV model

	$S(0) = 100$	MC	Comonotonicity		Optimization	
			weak	strong	weak	strong
$\beta = 0.4$	$K = 80$	24.529	24.245	26.6892	26.209	26.6903
	$K = 90$	18.488	19.0382	21.153	20.7265	21.1531
	$K = 100$	13.946	15.1239	16.8912	16.5182	16.89
	$K = 110$	10.548	11.8113	13.5591	13.2418	13.56
	$K = 120$	8.0546	9.5907	10.9675	10.6964	10.9663
$\beta = 0.3$	$K = 80$	24.529	24.245	26.6892	26.209	26.6903
	$K = 90$	18.583	19.0619	21.2313	20.7985	21.2318
	$K = 100$	13.987	15.0937	16.8872	16.5058	16.8863
	$K = 110$	10.522	11.6827	13.468	13.1413	13.4687
	$K = 120$	7.955	9.3614	10.7956	10.5164	10.7951
$\beta = 0.2$	$K = 80$	24.665	24.4188	26.8295	26.3453	26.83
	$K = 90$	18.665	19.0882	21.3125	20.872	21.3128
	$K = 100$	14.015	15.0574	16.8852	16.4958	16.885
	$K = 110$	10.489	11.5702	13.3798	13.0439	13.3805
	$K = 120$	7.8656	9.1361	10.6279	10.3408	10.6281
$\beta = 0.1$	$K = 80$	24.739	24.592	26.9739	26.4856	26.9738
	$K = 90$	18.697	19.1182	21.3965	20.9494	21.3963
	$K = 100$	13.952	15.016	16.8857	16.488	16.886
	$K = 110$	10.332	11.4732	13.2945	12.9496	13.2952
	$K = 120$	7.6264	8.916	10.4643	10.1693	10.465
$\beta = -0.1$	$K = 80$	25.059	25.1161	27.2797	26.777	27.2727
	$K = 90$	18.926	19.0994	21.5729	21.1093	21.5717
	$K = 100$	14.028	14.9938	16.8972	16.4795	16.8948
	$K = 110$	10.241	11.4061	13.1342	12.7683	13.1328
	$K = 120$	7.3856	8.4059	10.15	9.8376	10.1496
$\beta = -0.2$	$K = 80$	25.011	25.2472	27.4389	26.9285	27.4283
	$K = 90$	18.84	19.1401	21.6666	21.1945	21.6622
	$K = 100$	13.899	14.9299	16.9064	16.4795	16.9026
	$K = 110$	10.057	11.3527	13.0583	12.6818	13.0545
	$K = 120$	7.172	8.2013	9.9978	9.6771	9.997
$\beta = -0.3$	$K = 80$	25.167	25.3711	27.6025	27.0844	27.5883
	$K = 90$	18.948	19.2077	21.7641	21.2816	21.7573
	$K = 100$	13.922	14.8702	16.9189	16.4807	16.9129

Table 4.A.6 (Continued)

	$S(0)=100$	MC	Comonotonicity		Optimization	
	$K=110$	10.007	11.2665	12.9848	12.5978	12.9794
	$K=120$	7.0602	8.0223	9.8496	9.5199	9.8476
	$K=80$	25.227	25.4959	27.7719	27.2448	27.7531
	$K=90$	18.956	19.3066	21.8655	21.3719	21.8547
$\beta=-0.4$	$K=100$	13.869	14.8146	16.9337	16.4844	16.9264
	$K=110$	9.8868	11.1619	12.9138	12.5162	12.9066
	$K=120$	6.8809	7.8689	9.7047	9.366	9.7013

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Chapter 5

Pricing Gold Futures with Three-factor Models in Stochastic Volatility Case

5.1 Introduction

The price of gold is one of the world's most important global economic barometers. The trend of gold prices is carefully watched by policy-makers, consumers and financial market participants. In addition to traditional functions, gold nowadays also serves as the underlying asset for a large and growing part of the financial market. Its derivatives include futures, swaps and options and the derivatives market has been very active during the recent decade.

One has to admit that gold prices are highly volatile and sometimes suffer drastic shocks, and this has been true especially for the last four decades. Examples of significant and sudden large gold price movements include the market stress triggered by the Soviet war in Afghanistan in 1979, the following Iran hostage crisis in 1980, the invasion of Iraq in 2003, the subprime crisis in 2008 and the recent Eurozone crisis in 2011.

The concept of the convenience yield of commodities was proposed by Brennan (1991), which acts as a continuous dividend to the holder of the spot commodity. After that, many researchers have proved that the convenience yield of a commodity should indeed be considered when pricing futures based on the commodities. Gibson and Schwartz (1990) present a one-factor model to price futures, which assumes that the convenience yield of commodities is constant. However, a constant convenience yield assumption can only be held in a very strong assumption. In order to relax this strict assumption, Gibson et al. (1991), Schwartz (1997) and Schwartz and Smith (2000) take the stochastic convenience yield into account, presenting a two-factor model to price futures. Schroder (1989), Schwartz (1997), and Cortazar and Schwartz (2003) also developed a three-factor model wherein the price of futures also depends on the stochastic interest rate as well as the spot price of the commodities and the convenience yield.

In the above mentioned models, the volatility of the spot price is assumed to be constant so that the process of the spot price will follow a log normal distribution, which makes pricing futures and futures options easier. However, the truth is that the volatility time series shows that the volatility is very variable. Many empirical studies have shown that returns on equities, currencies and commodities have higher peaks and fatter tails than predicted by a normal distribution (Gatheral (2011)). Obviously, these models make the same mistake as that the Black-Scholes model did; that is, constant volatility. Consequently, it is very natural to make volatilities stochastic in the spot price of commodities.

In this chapter, two new developed three-factor models will be proposed, which consider stochastic volatility in both the spot price and the instantaneous convenience yield. These two models are developed from the Schwartz (1997) two-factor model. Both of them have closed-form solutions for the value of futures contracts. It can be seen that the new developed models have their advantages, compared with the Schwartz (1997) two-factor model. It is also demonstrated that an unexpected benefit comes from one of the new developed models and the Schwartz (1997) two-factor model when giving the value for futures contracts.

All parameters in the Schwartz (1997) two-factor model are estimated with daily gold spot price data from Bloomberg by the Kalman filter method. This method has successfully been applied to estimate the one-factor, two-factor, and three-factor models on crude oil data; see for example, Schwartz (1997) and Schwartz and Smith (2000). Since the new developed models take stochastic volatility into account, the basic Kalman filter method is no longer applicable; hence, extended Kalman filter method is used to estimate the value of the parameters in the new developed models with the same data.

The remainder of this chapter is organized as follows: Section 2 presents two three-factor models with stochastic volatility and a brief discussion about them is also included. In Section 3, there is a discussion of the Kalman filter technology and the extended Kalman filter algorithm. In Section 4, the data is described and the estimation of the parameters in the models is presented. Section 5 provides an empirical result for Model 2 and a comparison with the Schwartz (1997) two-factor model is also provided. Section 6 summarizes the main conclusions.

5.2 Three-factor Models

In this section, two types of three-factor models for pricing futures contract are presented. Both of these models are based on the Schwartz (1997) two-factor model and are extended to a stochastic volatility case. Closed-form solutions for futures contacts can be derived in both two new developed models. The Schwartz (1997) two-factor model will also be introduced in this section, and this model will be used as the benchmark model to measure against the two new developed models. In addition, a brief discussion about these models will also be included in this section.

5.2.1 Model 1

Here, the first model is presented, called Model 1. Under risk-neutral measure Q, assume the following model:

$$\begin{aligned}
dS_t &= (r - y_t)S_t d_t + S_t\sqrt{V_t}dW_1 \\
dy_t &= (\kappa_1(\alpha - y_t))dt + \sigma_1\sqrt{V_t}dW_2 \\
dV_t &= (\kappa_2(\theta - V_t))dt + \sigma_2\sqrt{V_t}dW_3 \\
dW_1 \cdot dW_2 &= \rho_{12}d_t \\
dW_1 \cdot dW_3 &= \rho_{13}d_t \\
dW_2 \cdot dW_3 &= \rho_{23}d_t
\end{aligned} \tag{5.1}$$

Since the convenience yield and spot volatility are non-traded in the market, a similar method to Hull and White (1987) and Scott (1987) can be used to construct a non-arbitrage portfolio, thereby finding the value of futures prices. This leads to the partial differential equation for futures prices in this model, as follows:

$$\frac{1}{2}S^2VF_{SS} + \frac{1}{2}\sigma_1^2VF_{yy} + \frac{1}{2}\sigma_2^2VF_{VV} + S\sigma_1V\rho_{12}F_{Sy} + S\sigma_2V\rho_{13}F_{SV} + V\sigma_1\sigma_2\rho_{23}F_{yV} + (r - y_t)SF_S + (\kappa_1(\alpha - y_t))F_y + (\kappa_2(\theta - V_t))F_V - F_\tau = 0 \tag{5.2}$$

subject to terminal boundary condition $F(S, y, V, 0) = 0$.

One can then use the Feyman-Kac theorem to find the solution for the futures price:

$$F(0, T) = \mathbb{E}(S_T) \tag{5.3}$$

To find the expression of the characteristic function explicitly, a method that is similar to Heston (1993) will be followed. Guessing $F = S\exp(A + By + CV)$, this function is substituted into Equation (5.2) to reduce Equation (5.1) to three ordinary differential equations (ODE):

$$\frac{1}{2}\sigma_1^2B^2 + \frac{1}{2}\sigma_2^2C^2 + \sigma_1\rho_{12}B + \sigma_2\rho_{13}C + \sigma_1\sigma_2\rho_{23}BC - \kappa_2C = \frac{\partial C}{\partial t} \tag{5.4}$$

$$-1 - \kappa_1B = \frac{\partial B}{\partial t} \tag{5.5}$$

$$r + \kappa_1\alpha B + \kappa_2\theta C = \frac{\partial A}{\partial t} \tag{5.6}$$

Firstly, Equation (5.5) can be easily solved with $B(0) = 0$, then the solution for B can be obtained:

$$B = \frac{1}{\kappa_1}(\exp(-\kappa_1t) - 1) \tag{5.7}$$

After that, $a = \frac{\sigma_2^2}{2}$, $b = \sigma_2\rho_{13} - \kappa_2$, $c = \frac{\sigma_1\sigma_2\rho_{23}}{\kappa_1}$, $d = \frac{\sigma_1^2}{2\kappa_1^2}$, $e = \frac{\sigma_1\rho_{12}}{\kappa_1}$ is assumed; thus, the Equation (5.4) can be expressed as:

$$aC^2 + (b + c(\exp(-\kappa_1 t) - 1)C + d(\exp(-\kappa_1 t) - 1)^2 + e(\exp(-\kappa_1 t) - 1) = \frac{\partial C}{\partial t} \quad (5.8)$$

The solution for Equation (5.8) can be solved with $C(0) = 0$ by Maple. The expression for C is very, very long (i.e., more than 20 pages) so it will not be shown in this thesis.

Finally, with solution B and solution C, solution A in Equation (5.6) can be solved. Since the solution A depends on the solution C, the expression for A is not shown in this thesis, either.

Note that, although the expression for C is very long, very complicated and contains particular functions, the result for C can still be solved analytically in Maple or Matlab in a very short time. Unfortunately, an explicit expression for A cannot be obtained since the solution for A depends on the expression of C and the particular function in expression C cannot be integrated analytically. Therefore, the result for A can only be obtained by a numerical method (e.g., the Runge-Kutta method) and then the futures price can be solved.

Since the expression for futures price is really complicated in Model 1, the Monte Carlo method will be used to test its accuracy. Assume at current time $t = 0$, the spot price is $S = 100$, the value of parameters are given with $y_0 = 0.05$, $V_0 = 0.8$, $r = 0.02$, $\sigma_1 = 1.1$, $\sigma_2 = 0.5$, $\kappa_1 = 0.8$, $\kappa_2 = 5$, $\alpha = 0.3$, $\theta = 0.05$, $\rho_{12} = 0.3$, $\rho_{13} = -0.7$, $\rho_{23} = 0.5$. The maturity time T is from 0 to 1, hence one can get futures contract prices by formula $F = S\exp(A + By + CV)$ in model 1 and Monte Carlo method separately.

Specifically, the relationship between three Brown motion processes in the Monte Carlo method can be solved by Cholesky decomposition:

$$\begin{bmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & 0 \\ \rho_{13} & \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} & \sqrt{1-\rho_{13}^2-(\frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}})^2} \end{bmatrix} \begin{bmatrix} dW1 \\ dW2 \\ dW3 \end{bmatrix}$$

The results for the new pricing formula and the Monte Carlo method are shown in Figure 5.1. As below, it can be seen that the prices that come from the new pricing formula are consistent with the Monte Carlo result. Therefore, a conclusion can be made that the new pricing formula for futures contracts is correct in Model 1.

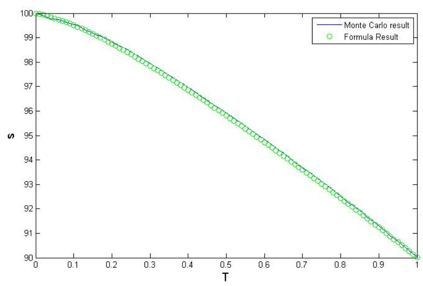

Figure 5.1: The prices of futures contract

5.2.2 Model 2

Now the second model is introduced, called Model 2. In this model, both convenience yield and spot price volatility are assumed to follow an Ornstein-Uhlenbeck stochastic process:

$$\begin{aligned}
dS_t &= (r - y_t)S_t d_t + S_t\sigma_t dW_1 \\
dy_t &= (\kappa_1(\alpha - y_t))dt + \sigma_1 dW_2 \\
d\sigma_t &= (\kappa_2(\theta - \sigma_t))dt + \sigma_2 dW_3 \\
dW_1 \cdot dW_2 &= \rho_{12} d_t \\
dW_1 \cdot dW_3 &= \rho_{13} d_t \\
dW_2 \cdot dW_3 &= \rho_{23} d_t
\end{aligned} \tag{5.9}$$

Similar to the discussion for Model 1, the futures contract price in Model 2 must satisfy the following partial differential equation:

$$\begin{aligned}
\frac{1}{2}S^2\sigma^2 F_{SS} + \frac{1}{2}\sigma_1^2 F_{yy} + \frac{1}{2}\sigma_2^2 F_{\sigma\sigma} + S\sigma_1\sigma\rho_{12}F_{Sy} + S\sigma_2\sigma\rho_{13}F_{S\sigma} + \sigma_1\sigma_2\rho_{23}F_{yV} + \\
(r - y_t)SF_S + (\kappa_1(\alpha - y_t))F_y + (\kappa_2(\theta - \sigma_t))F_\sigma - F_\tau = 0
\end{aligned} \tag{5.10}$$

subject to terminal boundary condition $F(S, y, \sigma, 0) = 0$.

The solution for futures price by Feyman-Kac theorem is:

$$F(0, T) = \mathbb{E}(S_T) \tag{5.11}$$

Again, to find the expression of the characteristic function explicitly, guessing $F = S\exp(A + By + C\sigma)$, then Equation (5.10) can be reduced to three ODEs by substituting this function:

$$\sigma_1\rho_{12}B + \sigma_2\rho_{13}C - \kappa_2 C = \frac{\partial C}{\partial t} \tag{5.12}$$

$$-1 - \kappa_1 B = \frac{\partial B}{\partial t} \tag{5.13}$$

$$\frac{1}{2}\sigma_1^2 B^2 + \frac{1}{2}\sigma_2^2 C^2 + \sigma_1\sigma_2\rho_{23}BC + r + \kappa_1\alpha B + \kappa_2\theta C = \frac{\partial A}{\partial t} \tag{5.14}$$

Firstly, Equation (5.13) can be easily solved with $B(0) = 0$. The solution for B is:

$$B = \frac{1}{\kappa_1}(\exp(-\kappa_1 t) - 1) \tag{5.15}$$

and the solution C can be given by solving Equation (5.12):

$$C = \frac{e}{bf}\exp(bt)(-b\exp(-ft) + f\exp(-bt) + b - f) \tag{5.16}$$

where $b = \sigma_2\rho_{13} - \kappa_2$, $e = \frac{\sigma_1\rho_{12}}{\kappa_1}$ and $f = b + \kappa_1$.

Note that, both solution B and solution C are not complicated so that Equation (5.14) is just a very simple differential function; hence, the expression for A can be easily integrated. The expression for A is a bit longer, but it is much better than in Model 1 (See the Appendix 5.A for detail).

5.2.3 Schwartz (1997) Two-factor Model

Schwartz (1997) assumed that under the risk-neutral measure $\mathbb{Q}$, the spot price of the commodity and the instantaneous convenience yield follow the joint stochastic process:

$$dS_t = (r - y_t)S_t dt + \sigma_s S_t dW_S, \tag{5.17}$$

$$dy_t = (k(\alpha - y_t) - \lambda_c \sigma_c)dt + \sigma_c dW_c. \tag{5.18}$$

with Brownian motions W_S andW_c and correlation $dW_S dW_c = \rho_{12} dt$.

where $\lambda_c \sigma_c$ is constant, denoting the market price of convenience yield risk.

This model has a good performance in pricing commodities and commodities derivatives; thus, this model will be used as the benchmark model.

A closed-form solution for the value of a futures contract can be derived from this model. Denote with $F(S_t, y_t, t, T)$ the futures price at time t with maturity at time T. Denote with $\tau = T - t$ the time to maturity, then the futures price is:

$$F(S_t, y_t, t, T) = S_t \exp(A(\tau) + B(\tau) y_t)$$

with

$$\begin{aligned} A(\tau) &= (r - \hat{\alpha} + \frac{1}{2}\frac{\sigma_c^2}{\kappa^2})\tau + \frac{1}{4}\sigma_c^2 \frac{1 - \exp^{-2\kappa\tau}}{\kappa^3} + (\kappa\hat{\alpha} + \sigma_s\sigma_c\rho_{12} - \frac{\sigma_c}{\kappa})\frac{1 - \exp^{\kappa\tau}}{\kappa^2}, \\ B(\tau) &= -\frac{1 - \exp^{-\kappa\tau}}{\kappa}, \end{aligned}$$

where $\hat{\alpha} = \alpha - \lambda_c \sigma_c / \kappa$.

5.2.4 Brief Discussion

Now, the performance of the above mentioned three models for futures price is investigated. For Model 1 and Model 2, the value of parameters was given in the previous section and σ_0 in Model 2 is 0.8. In addition to these parameters, $\kappa = 0.5$, $\sigma_s = 0.5$, $\hat{\alpha} = 0.3$ and $\sigma_c = 1.1$ are assumed in the Schwartz (1997) two-factor model.

Figure 5.2 shows the prices of futures contracts under the two new developed three-factor models and the Schwartz (1997) two-factor model given the data assumed above. As can be seen, in general, the performance of Model 2 is similar to the Schwartz (1997) two-factor model, whereas Model 1 is not.

In detail, as to Model 1, the performance is almost the same as the other two models in a short-time maturity. However, with the time maturity increasing, the

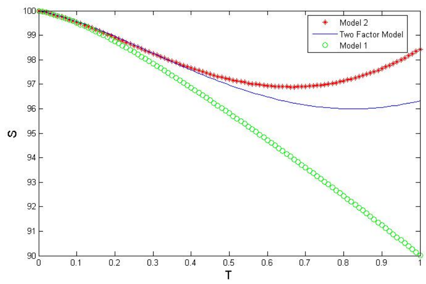

Figure 5.2: Price comparison between three models

price of the futures contract in this model goes down continually and has a much lower price when the time maturity is 1, compared with the other two models prices. Therefore, it may be said that this model is only good for pricing futures contract value in a short time maturity. Moreover, this model has three main drawbacks. Firstly, although the expression of parameter C can be found in a closed-form solution, this expression is difficult to apply with the Kalman filter technology or the least squares method since this expression has a particular function in it, the length of which is too long for these parameter estimation method, if used, would result in a very time-consuming process. Secondly, expression C has particular functions in it, which leads to no analytical expression for A, hence Kalman filter technology cannot be applied to estimate corresponding parameters in expression A. One can only obtain the value of expression A by a numerical method and finding the corresponding parameters by the least squares method, which is not convenient and also time-consuming. Thirdly, in the view of economics, the stochastic spot volatility that appears in the process of a convenience yield does not make sense.

As to Model 2, at the beginning, the prices of futures contracts are lower than those in the Schwartz (1997) two-factor model in a short time maturity. After that, the price of futures contracts continues decreasing, but the rate of reduction is lower than the Schwartz (1997) two-factor model. Consequently, with the time maturity increasing, the prices of futures contracts in Model 2 are gradually higher than those in the Schwartz (1997) two-factor model. Therefore, one may say that, in a short time maturity case, the values of futures contract given in Model 2 are lower than that in the Schwartz (1997) two-factor model, while the situation is opposite in a long time maturity case. Note that, except for Model 1, the prices of futures contract in both Model 2 and the Schwartz (1997) two-factor model do not continually decrease. When approaching time maturity equals 1, futures contracts prices have an upward growth trend but will not reach the original price.

In conclusion, although Model 1 has a closed-form solution, it seems that there is no reason to apply this model in pricing futures contract in practice except for a short time maturity contract. On the contrary, Model 2 has a more promising future, which is consistent with the Schwartz (1997) two-factor model in general and may fix the drawback of the Schwartz (1997) two-factor model both in short time maturity cases and long time maturity cases.

5.3 Kalman Filter Technique

5.3.1 Kalman Filter Algorithm

Nowadays, filtering technology has been a mature way in which a state-space model can be well analyzed. Indeed, many filter techniques have been used in many domains of communication technology, radar tracking, satellite navigation and applied physics. To be more specific, the filtering technology has been more and more introduced in many fields including signal processing, economics, econometrics, finance and so forth. Since the Kalman filter is the foundation of the extended Kalman filter, in this part, the Kalman Filter algorithm will be roughly explained before introducing the extended Kalman filter.

The basic principle behind filtering technology is not very complicated: using Bayes'theory, filters can use the information about current observation to predict the values of unobservable variables at next time point, and then update the information and forecast the situation at next time point (Pasricha (2006)).

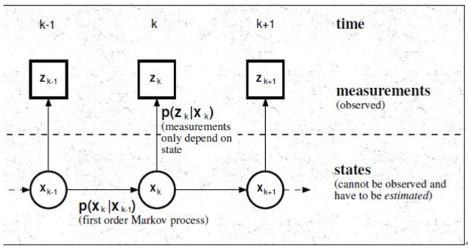

Figure 5.3: The process of filter technology

To be more specific, the process of filter technology can be described as Figure 5.3. In such a state space model, there are two parts that the state variable x_k for $k = 1, 2..., K$ and the observations y_k for $k = 1, 2..., K$ where K is normally the count of the observations of time variable. Normally, $x_k = f_k(x_{k-1}, v_{k-1})$, where x_k and x_{k-1} are the state at time point k and its previous time point, and assume the x_k is following a first-order Markov process as: $x_k \mid x_{k-1} \sim p_{x_k|x_{k-1}}(x_k \mid x_{k-1})$. As for the observations, the relationship between state variable(s) and the observations can be described as: $z_k = h_k(x_k, n_k)$, where x_k is the state variable (s) and n_k is the measurement noise at time point k.

Denote $z_{1:\,k}$ as the estimates of x_k from the start of time series to the updated time point k, and the observations are conditionally independent provided x_k as: $p = (z_k \mid x_k)$. Here, $p(x_0)$ can be either given or be obtained as an assumption. When $k \geq 1$, denote $p(x_k \mid x_{k-1})$ as the state transition probability.

Let f_k be any integrable function which depends on all the state and the whole trajectory in state space, then, the expectation of $f_k(x_{0:\,k})$ can be calculated as:

$$E(f_k(x_{0:\,k})) = \int f(x_{0:\,k}) p(x_{0:\,k} \mid z_{1:\,k}) dx_{0:\,k}$$

Essentially, the recursive filters consist of two steps: the first step is named as prediction step, which spreads state probability density function due to noise; the second step is update step, which combining the likelihood of the current measurement with the predicted state. Their mathematic expressions are $p(x_{k-1} \mid z_{1:\,k-1}) \to p(x_k \mid z_{1:\,k-1})$ and $p(x_k \mid z_{1:\,k-1}), z_k \to p(x_k \mid z_{1:\,k})$, respectively. Then, there are two probability density functions for the above steps. For the prediction step, assume the probability density function $p(x_{k-1} \mid z_{1:\,k-1})$ is available at time point $k?1$, using the Chapman-Kolmogoroff equation, the prior probability of the state at time point k can be expressed as:

$$p(x_k \mid z_{1:\,k-1}) = \int p(x_k \mid x_{k-1}) p(x_{k-1} \mid z_{1:\,k-1}) dx_{k-1}$$

As for the update step, the posterior probability density function is:

$$p(x_k \mid z_{1:\,k}) = \frac{p(z_k|x_k)p(x_k|z_{1:\,k-1})}{p(z_k|z_{1:\,k-1})}$$

Then,

$$p(z_k \mid z_{1:\,k-1}) = \int p(z_k \mid x_k) p(x_k \mid z_{1:\,k-1}) dx_k$$

can be obtained. (Muhlich (2003))

When this recursive system is considered in practice, in general, the recursive propagation of the posterior density is only a conceptual solution, but solutions definitely exist in some restrictive cases. For example, based on this recursive process, the Kalman Filter is developed by Kalman in 1960, and then rapidly become a widely used method in state-space models to calculate optimal estimates of unobservable state variables. In another word, the Kalman Filter can be seen as an optimal algorithm.

Recall of the measurement equation $z_k = h_k(x_k, n_k)$ and the state equation $x_k = f_k(x_{k-1}, v_{k-1})$, in a linear system, those two equations can be defined as:

$$\begin{cases} x_k = F_k x_{k-1} + v_{k-1} \\ z_k = H_k x_k + n_k \end{cases} \tag{5.19}$$

where the random variables v and n represent the noises which are assumed to be independent and with normal probability distributions $p(v) \sim N(0, Q)$ and $p(n) \sim N(0, R)$. If F_k and H_k are assumed to be constants, in order to simplify the system, the two equations can be then rewritten as:

$$\begin{cases} x_k = A x_{k-1} + v_{k-1} \\ z_k = H x_k + n_k \end{cases} \tag{5.20}$$

where A and H are known matrices. Denote $\widehat{x}_k^-$ and $\widehat{x}_k$ to be prior and posterior state estimates at time point k, respectively. Then the prior and posterior estimate errors can be defined as $e_k^- \equiv x_k - \widehat{x}_k^-$ and $e_k \equiv x_k - \widehat{x}_k$ at the time point k, respectively. In a similar way, the prior and posterior estimated error covariance can be obtained as $P_k^- = E[e_k^- e_k^{-T}]$ and $P_k = E[e_k e_k^T]$. Based on $z_k = H_k x_k + n_k$, $\widehat{x}_k = \widehat{x}_k^- + K(z_k - H)\widehat{x}_k^-$ can be obtained, where $K = P_k^- H^T (H P_k^- H^T + R)^{-1}$ (Jacobs (1993)).

The Kalman filter algorithm is an optimal algorithm to solve a system with state variable. However, there is an unignorable limitation. As described, the traditional

Kalman filter algorithm needs a strict Gaussian assumption for the posterior density at each time point. $p(x_k \mid z_{1:\,k})$ is proved to be Gaussian as $p(x_{k-1} \mid z_{1:\,k-1})$ is assumed to be Gaussian. Regarding the Kalman filter algorithm as a recursive process and connected with

$$p(x_k \mid z_{1:\,k-1}) = \int p(x_k \mid x_{k-1}) p(x_{k-1} \mid z_{1:\,k-1}) dx_{k-1}$$

and

$$p(x_k \mid z_{1:\,k}) = \frac{p(z_k|x_k)p(x_k|z_{1:\,k-1})}{p(z_k|z_{1:\,k-1})}$$

the prior and the posterior density probabilities can be written as:

$$p(x_{k-1} \mid z_{1:\,k-1}) = N(x_{k-1}; m_{k-1|k-1}, P_{k-1|k-1}) \; p(x_k \mid z_{1:\,k-1}) = N(x_k; m_{k|k-1}, P_{k|k-1})$$

and

$$p(x_k \mid z_{1:\,k}) = N(x_k; m_{k|k}, P_{k|k})$$

with

$$m_{k|k-1} = F_k m_{k-1|k-1},\; P_{k|k-1} = Q_{k-1} + F_k P_{k-1|k-1} F_k^T,$$
$$m_{k|k} = m_{k|k-1} + K_k(z_k - H_k m_{k|k-1})$$

and

$$P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1} = (I - K_k H_k) P_{k|k-1}$$

where $N(x; m, P)$ is a Gaussian density with argument x, mean m and covariance P.

Since

$$K = P_k^- H^T (H P_k^- H^T + R)^{-1}$$

is known,

$$K_k = P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1}$$

can be obtained.

In this case of implement of the Kalman Filter, there are unknown parameters needed to be estimated based on the initial set. According to Harvey (1989), the joint density can be written as: $L(z; \Psi) = \prod_{k=1}^{K} p(z_k)$, where $p(z_k)$ is the (joint) probability density function of t-th set of observations, and Ψ is the set of unknown parameters, when the T sets of observations $z_1, ... z_K$ are independently and identically distributed. However, the sets of observations are not independent, therefore, the above $L(z; \Psi)$ cannot be applied. The probability density need to be set conditionally as: $L(z; \Psi) = \prod_{k=1}^{K} p(z_k \mid Z_{k-1})$, where the capital Z_{k-1} is denoted as $Z_{k-1} = \{z_{k-1}, z_{k-2}, ..., z_1\}$. The distribution of z_k conditional on z_k is itself normal, if the initial state vector and the disturbances have multivariate normal distributions. Since the expectation of the z_k at time point $k-1$ only based on the information at $k-1$, the likelihood function can be finally written as:

$$\log L = -\frac{NK}{2} \log 2\pi - \frac{1}{2} \sum_{k=1}^{K} \log \mid D_k \mid - \frac{1}{2} \log \sum_{k=1}^{K} v_k' D_k^{-1} v_k$$

where $v_k = z_k - z_{k|k-1}$ and $D_k = H_k P_{k|k-1} H_k' + R$.

5.3.2 Extended Kalman Filter Algorithm

The extended Kalman filter algorithm is a useful development. By using the extended Kalman filter algorithm, the measurement function or/and the state function does not need to be linear anymore. Hence, the measurement equation $z_k = h_k(x_k, n_k)$ and the state equation $x_k = f_k(x_{k-1}, v_{k-1})$ cannot be expressed as below anymore:

$$\begin{cases} x_k = F_k x_{k-1} + v_{k-1} \\ z_k = H_k x_k + n_k \end{cases} \tag{5.21}$$

In order to run the filter algorithm, a local linearization of above equations might be a description of the nonlinear system. Then the $p(x_{k-1} \mid z_{1:\,k-1})$, $p(x_k \mid z_{1:\,k-1})$ and $p(x_k \mid z_{1:\,k})$ are approximated by a Gaussian distributions as:

$$p(x_{k-1} \mid z_{1:\,k-1}) \approx N(x_{k-1}; m_{k-1|k-1}, P_{k-1|k-1}) \; p(x_k \mid z_{1:\,k-1}) \approx N(x_k; m_{k|k-1}, P_{k|k-1})$$

and

$$p(x_k \mid z_{1:\,k}) \approx N(x_k; m_{k|k}, P_{k|k})$$

$$m_{k|k-1} = f_k(m_{k-1|k-1})$$

$$P_{k|k-1} = Q_{k-1} + \widehat{F_k} P_{k-1|k-1} \widehat{F_k}^T$$

$$m_{k|k} = m_{k|k-1} + K_k(z_k - h_k(m_{k|k-1}))$$

and

$$P_{k|k} = P_{k|k-1} - K_k \widehat{H_k} P_{k|k-1} = (I - K_k \widehat{H_k}) P_{k|k-1}$$

where

$$K_k = P_{k|k-1} \widehat{H_k}^T (\widehat{H_k} P_{k|k-1} \widehat{H_k}^T + R_k)^{-1}$$

is known as gain, and

$$\widehat{F_k} = \frac{df_k(x)}{dx} \big|_{x=m_{k-1|k-1}}$$

and

$$\widehat{H_k} = \frac{dh_k(x)}{dx} \big|_{x=m_{k-1|k-1}}$$

are Jacobian matrices. This process is known as the extended Kalman filter algorithm.

Based on the introduced model, the traditional Kalman filter does not work in this chapter. Hence, the model will be run with the extended Kalman filter. Except for the linearization, the principles of the Kalman filter algorithm and the extended Kalman filter algorithm are the same, hence, in this chapter, the likelihood function and the maximum likelihood value of the Kalman filter is seen reasonable approximations of the likelihood function and the maximum likelihood value of the extended Kalman filter algorithm.

5.4 Data and Estimation

In order to simplify the complex state space model, the weekends and other non-trading days can be ignored, which means that the trading days are considered to be continuous. The interest rate is assumed equal to 2% per year in this chapter. Furthermore, in order to simplify the calculation, the drift μ in the above model is replaced by the interest rate r.

The observable futures prices are the only needed data. Specifically, the futures of gold contracts are traded by the CME Group, and the data is collected from Bloomberg. Compared to other commodities, gold has a special feature of investment: against the inflation. Hence, the test period that is chosen should not be too long. In this chapter, the data is collected from the beginning of 2013 (4th Jan 2013) to the last trading day in 2013 (27th Dec 2013). In addition, the CME gold contracts are set to matured in every two months. Specifically, the gold futures contracts mature in February, April, June, August, October, and December each year. Therefore, six futures contracts will be used; they all start from the beginning of 2013, and their maturities are February, April, June, August, October, and December in 2014, respectively.

Table 5.1: Parameter estimates for three-factor model

Parameter	Estimate
κ_1	0.8035
κ_2	5.1814
σ_1	0.3041
σ_2	0.3058
ρ_{12}	0.5436
ρ_{13}	-0.9587
ρ_{23}	-0.2985
α	0.0976
θ	0.0416

The estimated parameters for Model 2 and the Schwartz (1997) two-factor model are shown in Table 5.1 and Table 5.2, respectively. Since the parameters between the two models might not be comparable, and the focus is on Model 2, Table 5.1 will be explained more comprehensively in this section. As to the value of parameter in Table

Table 5.2: Parameter estimates for the Schwartz (1997) two-factor model

Parameter	Estimate
κ	0.813
α	0.0144
σ_1	0.3339
σ_2	0.2703
ρ	0.4540

5.1, firstly, the coefficient of reverting of the volatility of the spot price is much higher than the coefficient of reverting the convenience yield (κ_2, which is much higher than κ_1). Secondly, the levels of the volatility of the volatility are similar to the volatility of the convenience yield, which is around 0.3. Thirdly, both correlation coefficients of the spot price and the volatility, and the convenience yield and the volatility are negative, while the correlation coefficient of the spot price and the convenience yield is positive.

5.5 Empirical Result

Figure 5.B.1 exhibits the estimated spot price of gold from Model 2. It is not hard to see that the spot price of gold keeps decreasing in the first half of the test period from over 1,700 dollars per ounce to about 1,200 dollars per ounce. After that, it fluctuates around 1,300 dollars per ounce. Figure 5.B.2 shows the estimated convenience yield of gold during the test period. To be more specific, the estimated convenience yield fluctuates around 0.01, and the interval of the fluctuation is really narrow. Furthermore, Figure 5.B.3 illustrates the estimated volatility of gold. Similar to the estimated convenience yield, the estimated volatility stably fluctuates around 0.1 during the entire test period.

The estimated spot price and the convenience yield from the Schwartz (1997) two-factor model are shown in Figure 5.B.4 and Figure 5.B.5. Firstly, the estimated spot price from the Schwartz (1997) two-factor model is extremely similar to the one estimated from Model 2. It also keeps decreasing in the first half of the test period from over 1,700 dollars per ounce to about 1,200 dollars per ounce; thereafter, it fluctuates around 1,300 dollars per ounce. Besides, the estimated convenience yield from the Schwartz (1997) two-factor model fluctuates around 0, and the interval of the fluctuation is wider than it is from Model 2.

Figure 5.B.6 to Figure 5.B.17 show the effectiveness of Model 2 and the Schwartz (1997) two-factor model, respectively. To be more specific, the first six figures show the usefulness of Model 2, while the last six figures show the usefulness of the Schwartz (1997) two-factor model. These figures correspond to six samples of futures for both Model 2 and the Schwartz (1997) two-factor model and these futures samples will mature in the Feb, Apr, Jun, Aug, Oct and Dec in 2014, respectively. For example, Figure 5.B.6 shows the prices of futures contract that will mature in Feb 2014, for both the observed futures prices and Model 2 prices. From the first six figures, it is easy to find that the prices from Model 2 fit the observed futures prices very well. Therefore, it may be concluded that Model 2 is useful for pricing a futures contract in reality. It should be noticed that the Schwartz (1997) two-factor model also works well, which can be seen from the last six figures.

The Model 2 of all six chosen futures contracts prices during the test period are compared in Figure 5.B.18, while the Schwartz (1997) two-factor model of all six chosen futures contracts prices during the test period are compared in Figure 5.B.19. Note that in order to show a simpler picture, only 26 trading days are chosen in these two figures. The Figure 5.B.20 exhibits the market observed prices for six futures contracts.

Lastly, the forward curves of Model 2 and the Schwartz (1997) two-factor model for the chosen 50th (14th Mar 2013), 100th (24th May 2013) and 200th (16th Oct 2013) trading days in the test period are shown in from Figure 5.4 to Figure 5.9, respectively. It can be found that the estimated price of a particular gold futures contract is lower than the real observed price of the same contract, when term of length to maturity is short in Model 2. With the length to maturity increasing , the difference between model price and market price decreases gradually, and the Model 2 price will eventually be higher than the real observed futures price. However, the above phenomenon did not happen in the Schwartz (1997) two-factor model, which is the same as expected.

More specifically, for the 50th trading day (14th Mar 2013), see Figure 5.4 and Figure 5.5. In general, both the Schwartz (1997) two-factor model and Model 2 can predict good results for futures contracts prices in market. Moreover, they both have one very accurate prediction price, compared with the real futures contracts prices. In addition, compared with the prices of Model 2, the market prices and the prices of the Schwartz (1997) two-factor model, it can be found that these six real futures contracts prices are between the prices of Model 2 and the prices of the Schwartz (1997) two-factor model, except the one that still has 21 months to maturity on the 50th trading day.

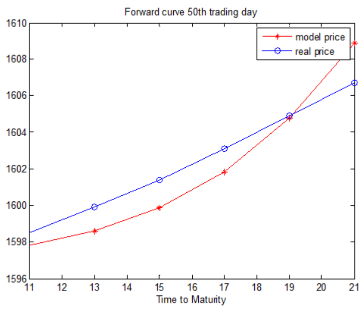

Figure 5.4: Model 2 forward curve, 50th

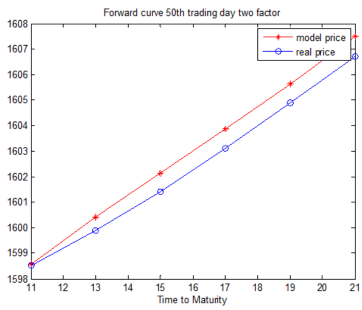

Figure 5.5: Schwartz (1997) two-factor model forward curve, 50th

For the 100th trading day (24th May 2013), see Figure 5.6 and Figure 5.7. The situation is similar to the situation of the 50th trading day. The difference is that for 100th trading day, the very accurate prediction prices from Model 2 and the Schwartz (1997) two-factor model are for the same futures contract, while for 50th trading day, the very accurate prediction prices are for different futures contracts.

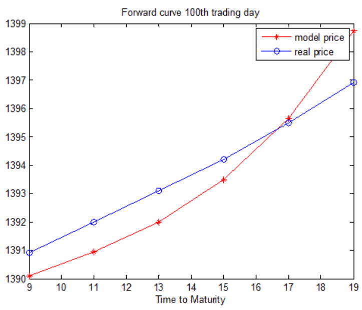

Figure 5.6: Model 2 forward curve, 100th

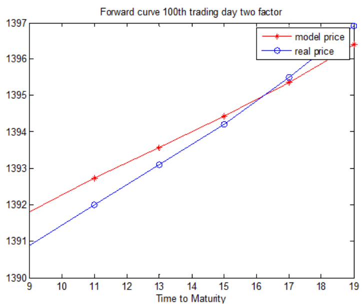

Figure 5.7: Schwartz (1997) two-factor model forward curve, 100th

For the 200th trading day (16th Oct 2013), see Figure 5.8 and Figure 5.9. A similar conclusion can be found as for 50th and 100th trading day. The difference is, for the

200th trading day, Model 2 has a price that hits the real futures contract price, while the Schwartz (1997) two-factor model does not.

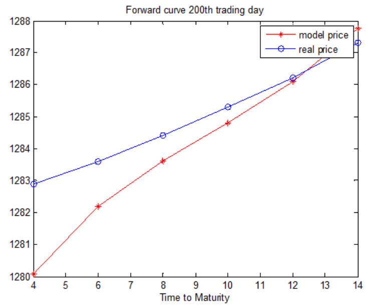

Figure 5.8: Model 2 forward curve, 200th

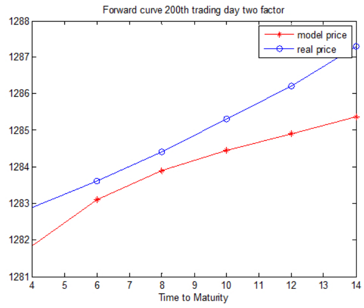

Figure 5.9: Schwartz (1997) two-factor model forward curve, 200th

In conclusion, both Model 2 and the Schwartz (1997) two-factor model can fit the real futures contracts prices quite well. Based on the above analysis and the exhibited figures, compared with the Schwartz (1997) two-factor model, Model 2 has three main advantages. Firstly, this three-factor model provides an estimable and assessable stochastic volatility of the spot price of the underlying commodity, which may be useful for traders who speculate and hedge with volatility. Secondly, in some specific lengths of time to maturity, this three-factor model performs better than the Schwartz (1997) two-factor model. Thirdly, one can combine Model 2 and the Schwartz (1997) two-factor model to predict a very accurate price for futures contracts, because in most situations, the real futures contracts prices are always between these two models prices. However, Model 2 also has its drawbacks. The main drawback is that the performance of this three-factor model is not as stable as the Schwartz (1997) two-factor model. For example, the errors from Model 2 are generally larger than those from the Schwartz (1997) two-factor model.

5.6 Conclusion

In this chapter, two types of three-factor models are investigated to give a value for futures contracts, which allow spot volatility to be stochastic. Both of these two three-factor models have closed-form solutions for futures contracts prices. However, the expression for the price of futures contract is very complicated in Model 1; hence, it is very time-consuming; while the futures price solution in Model 2 is very simple, so it is very quick and effective when giving a value for futures contracts in Model 2. Moreover, Model 1 seems to only work well in a short time maturity case. Therefore, it has to be admitted that Model 2 is better than Model 1. It it also shown that Model 2 can be applied in practice and can fit the prices of gold futures contracts very well. From the results, it can be concluded that both the Schwartz (1997) two-factor model and Model 2 are good models. However, in comparison to the Schwartz (1997) two-factor model, Model 2 can give an estimable and assessable stochastic volatility of the spot price of the underlying commodity. Besides, Model 2 can provide more information to the investor than the Schwartz (1997) two-factor model since price crosses happen more frequently in Model 2 than in the Schwartz (1997) two-factor model. Finally, in most situations in the sample that tested, it can be found that the market futures prices are bounded by prices from Model 2 and the Schwartz (1997) two-factor model. This unexpected result implies that one may use both of these two models to find the closest futures price to the real futures price.

Appendix

5.A Explicit Expression for Parameter A in Model 2

$$A = A1 + A2 + A3 + A4 + A5 + A6 - A7 \tag{5.22}$$

$$A1 = rt - \alpha t - \frac{\alpha e^{-\kappa 1_1 t}}{\kappa_1} \tag{5.23}$$

$$\begin{aligned} A2 &= \frac{\kappa_2\theta\sigma_1\rho_{12}t}{bf} - \frac{\kappa_2\theta\sigma_1\rho_{12}e^{bt}}{b^2f} + \frac{(\sigma_2\sigma_1\rho_{12}e^{bt})^2}{4b^3f^2} \\ &\quad + \frac{(\sigma_2\sigma_1\rho_{12})^2t}{2b^2f^2} - \frac{(\sigma_2\sigma_1\rho_{12})^2e^{bt}}{b^3f^2} \end{aligned} \tag{5.24}$$

$$\begin{aligned} A3 &= -\frac{\sigma_2^3\sigma_1^2\rho_{12}^2\rho_{13}\kappa_2 t}{b^2f^2\kappa_1^2} \\ &\quad + \frac{\sigma_2^4\sigma_1^2\rho_{12}^2\rho_{13}^2 t}{2b^2f^2\kappa_1^2} + \frac{\sigma_2^3\sigma_1^2\rho_{12}^2\rho_{13}t}{b^2f^2\kappa_1} + \frac{\sigma_2^2\sigma_1^2\rho_{12}^2\kappa_2 e^{bt}}{b^3f^2\kappa_1} \\ &\quad - \frac{\sigma_2^3\sigma_1^2\rho_{12}^2\rho_{13}e^{bt}}{b^3f^2\kappa_1} - \frac{\sigma_1^2\sigma_2\rho_{23}\rho_{12}t}{bf\kappa_1} + \frac{\sigma_1^2\sigma_2\rho_{23}\rho_{12}e^{bt}}{b^2f\kappa_1} \\ &\quad - \frac{\kappa_2^2\theta\sigma_1\rho_{12}t}{bf\kappa_1} + \frac{\kappa_2^2\sigma_2^2\sigma_1^2\rho_{12}^2 t}{2b^2f^2\kappa_1^2} \\ &\quad - \frac{\kappa_2\sigma_2^2\sigma_1^2\rho_{12}^2 t}{b^2f^2\kappa_1} - \frac{\sigma_1^2\sigma_2^2\rho_{23}\rho_{12}\rho_{13}t}{bf\kappa_1^2} + \frac{\sigma_1^2\sigma_2\rho_{23}\rho_{12}\kappa_2 t}{bf\kappa_1^2} \end{aligned} \tag{5.25}$$

$$
\begin{aligned}
A4 \;=\; & \frac{2\sigma_1^2\sigma_2\rho_{23}\rho_{12}\kappa_2}{bf\kappa_1^3e^{\kappa_1 t}} - \frac{\sigma_2^4\sigma_1^2\rho_{12}^2\rho_{13}^2}{4b^2f^2\kappa_1^3e^{2\kappa_1 t}} + \frac{\sigma_2^4\sigma_1^2\rho_{12}^2\rho_{13}^2}{b^2f^2\kappa_1^3e^{\kappa_1 t}} \\
& + \frac{\sigma_2^3\sigma_1^2\rho_{12}^2\rho_{13}}{b^2f^2\kappa_1^2e^{\kappa_1 t}} - \frac{\sigma_2^2\sigma_1^2\rho_{12}^2\kappa_2^2}{4b^2f^2\kappa_1^3e^{2\kappa_1 t}} + \frac{\sigma_2^2\sigma_1^2\rho_{12}^2\kappa_2^2}{b^2f^2\kappa_1^3e^{\kappa_1 t}} \\
& - \frac{\sigma_2^2\sigma_1^2\rho_{12}^2\kappa_2}{b^2f^2\kappa_1^2e^{\kappa_1 t}} - \frac{\sigma_1^2\sigma_2\rho_{12}\rho_{23}}{bf\kappa_1^2e^{\kappa_1 t}} + \frac{\sigma_2^3\sigma_1^2\rho_{12}^2\rho_{13}\kappa_2}{2b^2f^2\kappa_1^3e^{2\kappa_1 t}} \\
& - \frac{2\sigma_2^3\sigma_1^2\rho_{12}^2\rho_{13}\kappa_2}{b^2f^2\kappa_1^3e^{\kappa_1 t}} - \frac{\sigma_2\sigma_1^2\rho_{12}\rho_{23}\kappa_2}{2bf\kappa_1^3e^{2\kappa_1 t}} + \frac{\sigma_2^2\sigma_1^2\rho_{12}\rho_{13}\rho_{23}}{2bf\kappa_1^3e^{2\kappa_1 t}} \\
& - \frac{2\sigma_2^2\sigma_1^2\rho_{12}\rho_{13}\rho_{23}}{bf\kappa_1^3e^{\kappa_1 t}} - \frac{\kappa_2^2\theta\sigma_1\rho_{12}}{bf\kappa_1^2e^{\kappa_1 t}} + \frac{\kappa_2\theta\sigma_1\sigma_2\rho_{12}\rho_{13}}{bf\kappa_1^2e^{\kappa_1 t}}
\end{aligned} \tag{5.26}
$$

$$
A5 = \frac{\sigma_2^3\sigma_1^2\rho_{12}^2\rho_{13}e^{(b-\kappa_1)t}}{b^2f^2\kappa_1(b-\kappa_1)} - \frac{\sigma_2^2\sigma_1^2\rho_{12}^2\kappa_2e^{(b-\kappa_1)t}}{b^2f^2\kappa_1(b-\kappa_1)} - \frac{\sigma_2\sigma_1^2\rho_{12}\rho_{23}e^{(b-\kappa_1)t}}{bf\kappa_1(b-\kappa_1)} \tag{5.27}
$$

$$
A6 = \frac{\sigma_1^2e^{-\kappa_1 t}}{\kappa_1^3} - \frac{\sigma_1^2e^{-2\kappa_1 t}}{4\kappa_1^3} - \frac{\sigma_1^2\ln(e^{-\kappa_1 t})}{2\kappa_1^3} \tag{5.28}
$$

$$
\begin{aligned}
A7 \;=\; & \frac{2\sigma_1^2\sigma_2\rho_{23}\rho_{12}\kappa_2}{bf\kappa_1^3} - \frac{\sigma_2^4\sigma_1^2\rho_{12}^2\rho_{13}^2}{4b^2f^2\kappa_1^3} + \frac{\sigma_2^4\sigma_1^2\rho_{12}^2\rho_{13}^2}{b^2f^2\kappa_1^3} \\
& + \frac{\sigma_2^3\sigma_1^2\rho_{12}^2\rho_{13}}{b^2f^2\kappa_1^2} - \frac{\sigma_2^2\sigma_1^2\rho_{12}^2\kappa_2^2}{4b^2f^2\kappa_1^3} + \frac{\sigma_2^2\sigma_1^2\rho_{12}^2\kappa_2^2}{b^2f^2\kappa_1^3} \\
& - \frac{\sigma_2^2\sigma_1^2\rho_{12}^2\kappa_2}{b^2f^2\kappa_1^2} - \frac{\sigma_1^2\sigma_2\rho_{12}\rho_{23}}{bf\kappa_1^2} + \frac{\sigma_2^3\sigma_1^2\rho_{12}^2\rho_{13}\kappa_2}{2b^2f^2\kappa_1^3} \\
& - \frac{2\sigma_2^3\sigma_1^2\rho_{12}^2\rho_{13}\kappa_2}{b^2f^2\kappa_1^3} - \frac{\sigma_2\sigma_1^2\rho_{12}\rho_{23}\kappa_2}{2bf\kappa_1^3} + \frac{\sigma_2^2\sigma_1^2\rho_{12}\rho_{13}\rho_{23}}{2bf\kappa_1^3} \\
& - \frac{2\sigma_2^2\sigma_1^2\rho_{12}\rho_{13}\rho_{23}}{bf\kappa_1^3} - \frac{\kappa_2^2\theta\sigma_1\rho_{12}}{bf\kappa_1^2} + \frac{\kappa_2\theta\sigma_1\sigma_2\rho_{12}\rho_{13}}{bf\kappa_1^2} \\
& - \frac{\kappa_2\theta\sigma_1\rho_{12}}{b^2f} + \frac{(\sigma_2\sigma_1\rho_{12})^2}{4b^3f^2} - \frac{(\sigma_2\sigma_1\rho_{12})^2}{b^3f^2} \\
& + \frac{\sigma_2^3\sigma_1^2\rho_{12}^2\rho_{13}}{b^2f^2\kappa_1(b-\kappa_1)} - \frac{\sigma_2^2\sigma_1^2\rho_{12}^2\kappa_2}{b^2f^2\kappa_1(b-\kappa_1)} - \frac{\sigma_2\sigma_1^2\rho_{12}\rho_{23}}{bf\kappa_1(b-\kappa_1)} \\
& - \frac{\sigma_2^3\sigma_1^2\rho_{12}^2\rho_{13}}{b^3f^2\kappa_1} + \frac{\sigma_2^2\sigma_1^2\rho_{12}^2\kappa_2}{b^3f^2\kappa_1} + \frac{\sigma_1^2\sigma_2\rho_{23}\rho_{12}}{b^2f\kappa_1} \\
& - \frac{\alpha}{\kappa_1} + \frac{\sigma_1^2}{\kappa_1^3} - \frac{\sigma_1^2}{4\kappa_1^3}
\end{aligned} \tag{5.29}
$$

5.B Figures

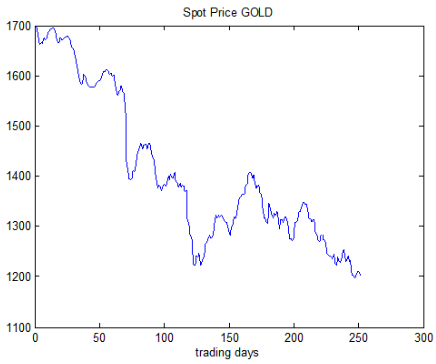

Figure 5.B.1: Estimated spot price from Model 2

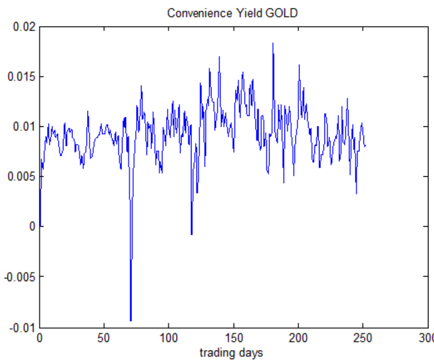

Figure 5.B.2: Estimated convenience yield from Model 2

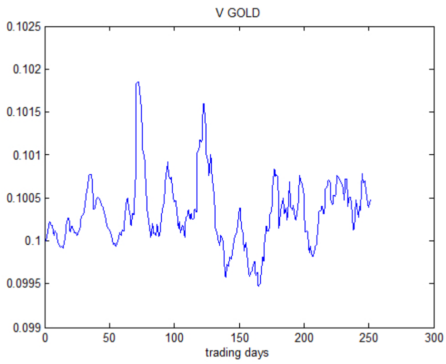

Figure 5.B.3: Estimated volatility of the gold from Model 2

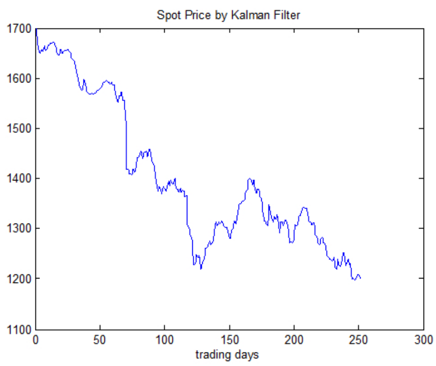

Figure 5.B.4: Estimated spot price from the Schwartz (1997) two-factor model

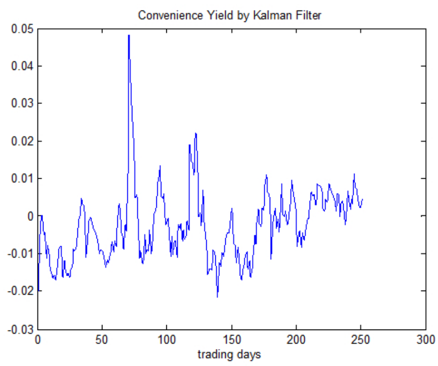

Figure 5.B.5: Estimated convenience yield from the Schwartz (1997) two-factor model

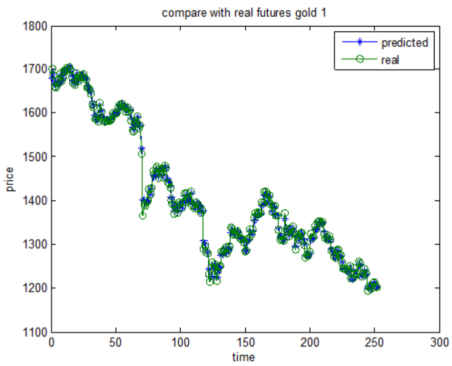

Figure 5.B.6: Effectiveness of Model 2, Feb

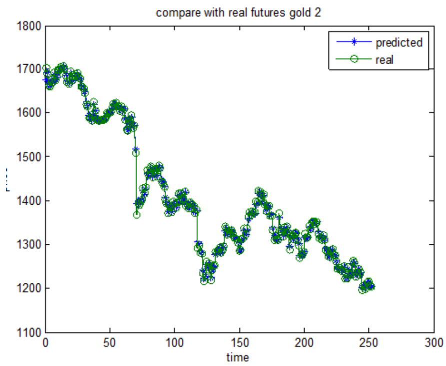

Figure 5.B.7: Effectiveness of Model 2, Apr

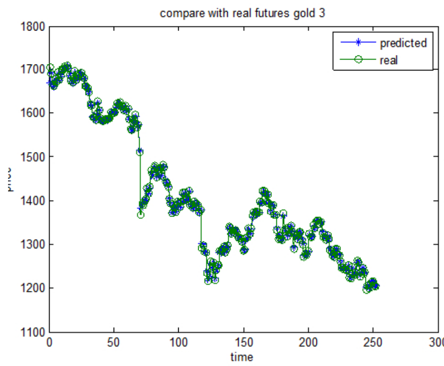

Figure 5.B.8: Effectiveness of Model 2, Jun

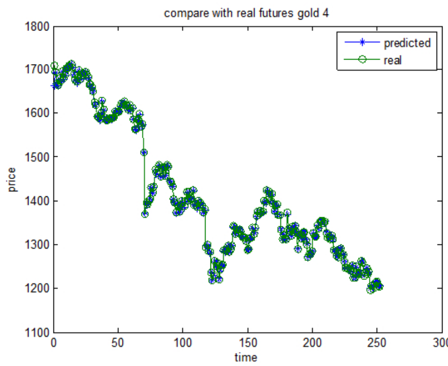

Figure 5.B.9: Effectiveness of Model 2, Aug

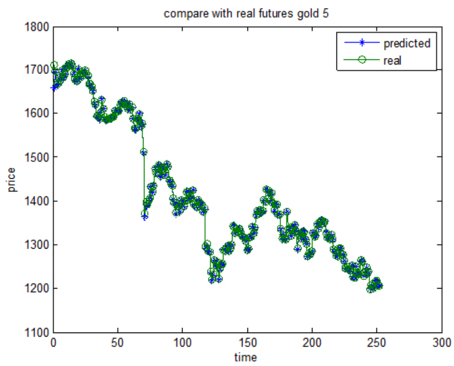

Figure 5.B.10: Effectiveness of Model 2, Oct

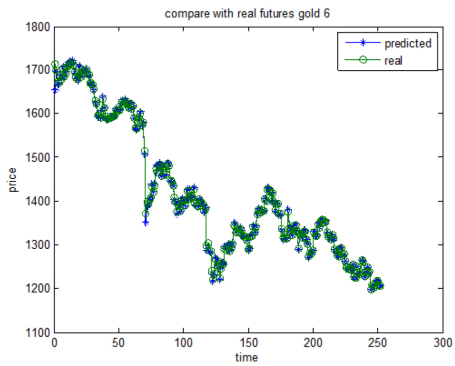

Figure 5.B.11: Effectiveness of Model 2, Dec

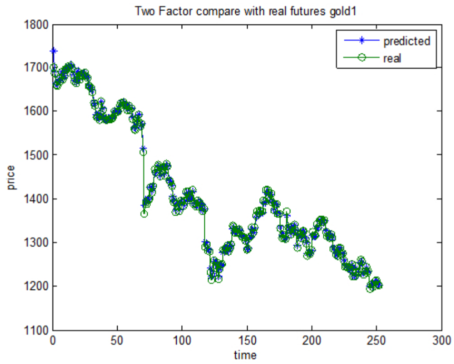

Figure 5.B.12: Effectiveness of the Schwartz (1997) two-factor model, Feb

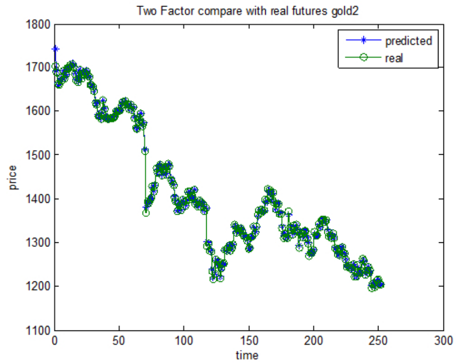

Figure 5.B.13: Effectiveness of the Schwartz (1997) two-factor model, Apr

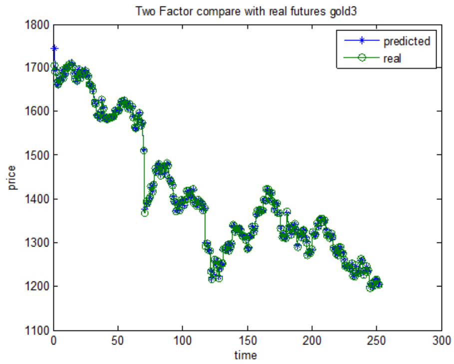

Figure 5.B.14: Effectiveness of the Schwartz (1997) two-factor model, Jun

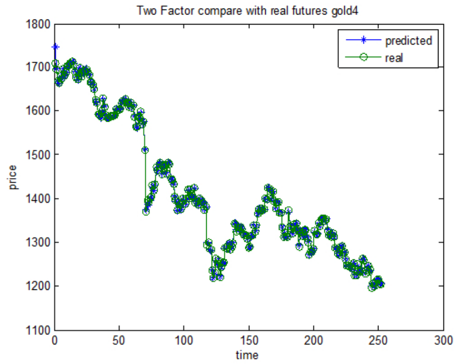

Figure 5.B.15: Effectiveness of the Schwartz (1997) two-factor model, Aug

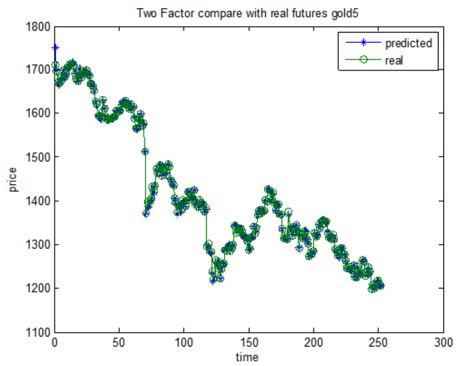

Figure 5.B.16: Effectiveness of the Schwartz (1997) two-factor model, Oct

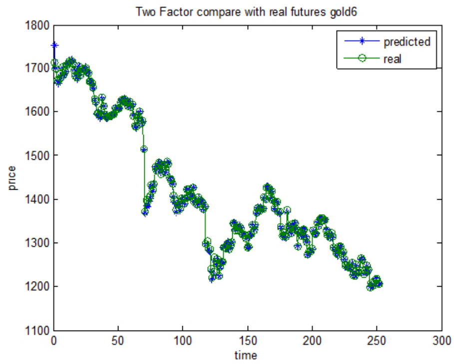

Figure 5.B.17: Effectiveness of the Schwartz (1997) two-factor model, Dec

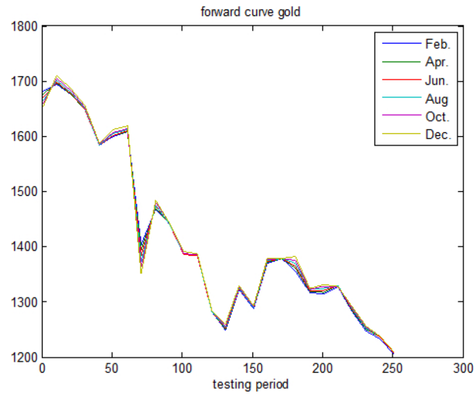

Figure 5.B.18: Prices of futures contracts from Model 2

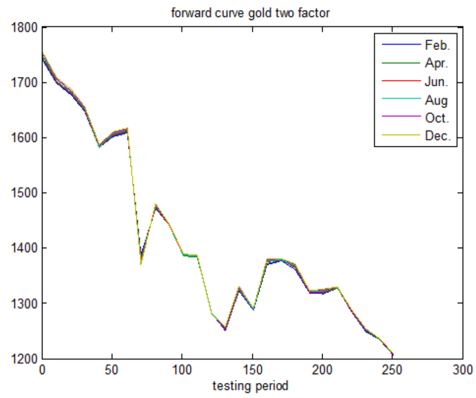

Figure 5.B.19: Prices of futures contracts from the Schwartz (1997) two-factor model

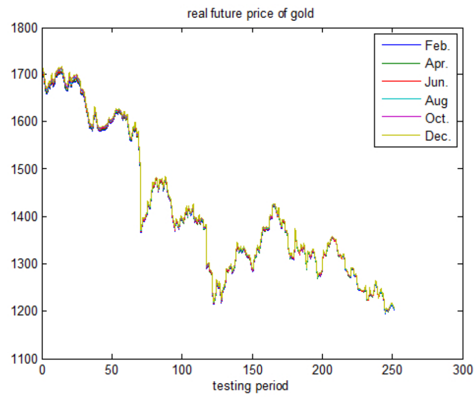

Figure 5.B.20: Real futures contracts prices

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Chapter 6

Pricing Commodity Futures Options with Stochastic Volatility by Asymptotic Method

6.1 Introduction

A futures option is an option contract in which the underlying is a single futures contract. The buyer of a futures option has the right, but not the obligation, to enter into a futures contract at a certain futures price at a certain date. The seller must take the opposite position in the futures contract when the buyer exercises this right. Specifically, a futures call option is the right to enter into a long futures contract at a certain futures price; a futures put option is the right to enter into a short futures contract at a certain futures price (Hull (2006)). Predominantly, futures options are American style options, which can be exercised at any time during the life of the contract. However, for a number of energy commodities, including crude oil and natural gas, futures options are also available as European style options. It is important to note that the underlying of a futures option is the futures price, not the commodity itself. Therefore, the futures option's price is tied to the futures price, not the commodity price, even though futures prices for contracts close to maturity track the corresponding commodity price closely.

The convenience yield of a commodity is the benefit obtained from holding the spot commodity instead of the futures contracts. This is important when pricing futures contracts (Brennan (1991)). Gibson and Schwartz (1990) found that a constant convenience yield did not work well for pricing futures contracts. Miltersen and Schwartz (1998) and Schwartz (1997) have developed one-, two- and three-factor models to price commodity futures contracts and futures options with stochastic convenience yields and interest rates. Their results confirmed that constant convenience yield is a rather ill-advised assumption and that stochastic convenience yield is far better able to fit the different observed shapes of the forward curves. Hilliard and Reis (1998) assumed additionally that the underlying spot price next to a stochastic convenience yield also features a Poisson jump term and showed that the relevant option pricing formula is a weighted sum over the corresponding the Schwartz (1997) two-factor expressions, similar as Merton (1976).

Stochastic volatility models have become more and more popular for derivatives pricing and hedging, especially since the 1987 crash (Fouque et al. (2000a)). In the Black-Scholes (1973) framework, the log returns of assets are assumed to follow a normal distribution and many other models also share this assumption, including the Schwartz (1997) two-factor model. While this has the advantage that it may lead to a closed-form solution for derivatives prices, empirical studies strongly contradict the normality assumption of log returns. Generally, empirical log returns of equities, currencies and commodities have higher peaks and fatter tails, which is indicative of a distribution with differing variances (Gatheral (2011)). Options, in terms of implied volatilities which are anything but flat, provide further evidence that underlyings are not log-normal. Recently, Trolle and Schwartz (2009) provided strong evidence that for both futures and options on Brent crude oil it is necessary to consider stochastic volatility when pricing derivatives . In difference to the classical Schwartz (1997) approach, they model the entire forward cost of carry curve in a fashion similar to the Heath-Jarrow-Morton approach for interest rates, with added stochastic volatility to both spot and cost of carry. However, they do not derive an explicit expression for the price of an option in their model.

In fact, it is very difficult to find closed-form solutions for option prices in models with stochastic volatility, especially when convenience yield is also assumed to be stochastic. Fouque et al. (2000b) assumed that the spot volatility follows a mean reverting Ornstein-Uhlenbeck (OU) process and presented an asymptotic expansion to calculate European derivatives prices. The zero-order term in this expansion corresponds to the classical Black-Scholes term, where the constant spot volatility is replaced by the long term average volatility. In the current chapter, the Schwartz (1997) multi-factor model with stochastic convenience yield is taken, which is a benchmark in the commodity literature, and add stochastic volatility of OU type. This model is a good candidate for the applicability of Fouque et al. (2000b)'s methodology, as in the constant volatility version, it admits closed-form expression for the price of European calls and puts which are of a modified Black-Scholes type. In fact, it can be shown that the zero-order term in the expansion coincides with the classical expression derived in Hilliard and Reis (1998) for the Schwartz (1997) two-factor model with constant volatility and in addition provide an explicit expression for the first-order correction term. This correction term is easy to evaluate and in fact the combined expression, consisting of zero-order term and correction term, is no harder to evaluate than the Hilliard and Reis (1998) formula. Then it is demonstrated that by taking account of the correction term, a significantly better fit can be obtained as compared to the Hilliard and Reis (1998) formula by looking at data for European call options on natural gas futures.

The remainder of this chapter is organized as follows: Section 2 briefly covers the three-factor model which includes spot price, convenience yield and stochastic spot volatility. In Section 3, the asymptotic expression for futures options is derived in terms of spot price. In Section 4, there is a briefly review of the pricing of European commodity options under the Schwartz (1997) two-factor model. In Section 5, it is shown how the asymptotic solution for futures options can also be expressed in terms of futures contract price. In Section 6 and Section 7, the asymptotic solution is examined for futures options through simulated and historical data and use the Schwartz (1997) two-factor model for comparison. Section 8 summarizes the main conclusions.

6.2 Three-factor Model

The model consists of the three factors: spot price S_t, convenience yield y_t and spot volatility V_t. For pricing futures and futures options their dynamics under a chosen risk-neutral measure $\mathbb{Q}$ is relevant. Assumed that under $\mathbb{Q}$, one can have that:

$$\begin{aligned} dS_t &= (r - y_t)S_t d_t + f(V)S_t dW_1 \\ dy_t &= (\kappa_1(\alpha - y_t) - \lambda)dt + \sigma_c dW_2 \\ dV_t &= (\kappa_2(m - V_t) - \Lambda(V_t))dt + \beta dW_3 \end{aligned} \tag{6.1}$$

with W_i correlated Brownian motions, s.t. $dW_i \cdot dW_j = \rho_{ij} d_t$. The parameter λ represents the market price of convenience yield risk and

$$\Lambda(V) = \rho_{13}\frac{\mu - r}{f(V)} + \gamma(V)\sqrt{1 - \rho_{13}^2} \tag{6.2}$$

where γ is the market price of volatility risk, which is a bounded function of V alone.

The key realization in the approach by Fouque et al. (2000b) is to consider the rate of mean reversion κ_2 as a large parameter and consequently the parameter $\varepsilon = 1/\kappa_2$ as a small parameter and then expand the pricing PDE and solution in terms of orders of ε.

Under the Ornstein-Uhlenbeck volatility assumption, the variance of the invariant distribution of V, denoted as v^2, can be expressed as $v^2 = \beta^2/(2\kappa_2)$; hence, $\beta = \frac{v\sqrt{2}}{\sqrt{\varepsilon}}$. Therefore, under the risk-neutral measure $\mathbb{Q}$, the stochastic differential Equation (6.1) can be rewritten:

$$\begin{aligned} dS_t &= (r - y_t)S_t d_t + f(V)S_t dW_1 \\ dy_t &= (\kappa_1(\alpha - y_t) - \lambda)dt + \sigma_c dW_2 \\ dV_t &= (\tfrac{1}{\varepsilon}(m - V_t) - \tfrac{v\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(V_t))dt + \tfrac{v\sqrt{2}}{\sqrt{\varepsilon}} dW_3 \end{aligned} \tag{6.3}$$

Under no-arbitrage, the value of a contingent claim $P(t, S, y, V)$ with payoff $h(S)$ must satisfy the following partial differential equation and boundary condition:

$$\frac{\partial P}{\partial t} + \frac{1}{2}f(V)^2 S^2 \frac{\partial^2 P}{\partial S^2} + \frac{1}{2}\sigma_c \frac{\partial^2 P}{\partial y^2} + \frac{v^2}{\varepsilon}\frac{\partial^2 P}{\partial V^2} + \rho_{12} f(V) S \sigma_c \frac{\partial^2 P}{\partial S \partial y} + \rho_{13} f(V) S \frac{v\sqrt{2}}{\sqrt{\varepsilon}}\frac{\partial^2 P}{\partial S \partial V} +$$
$$\rho_{23}\sigma_c \frac{v\sqrt{2}}{\sqrt{\varepsilon}}\frac{\partial^2 P}{\partial y \partial V} + (r - y)S\frac{\partial P}{\partial S} + (\kappa_1(\alpha - y) - \lambda)\frac{\partial P}{\partial y} + \left[\frac{1}{\varepsilon}(m - V) - \frac{v\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(V)\right]\frac{\partial P}{\partial V} - rP = 0 \tag{6.4}$$

and

$$P(T, S, y, V) = h(S) \tag{6.5}$$

6.2.1 The Operator Notation

In order to account for terms of order $1/\varepsilon$, $1/\sqrt{\varepsilon}$, 1 in the partial differential Equation (6.4), the following convenient notation is introduced:

$$\mathcal{L}_0 = v^2 \frac{\partial^2}{\partial V^2} + (m - V)\frac{\partial}{\partial V}, \tag{6.6}$$

$$\mathcal{L}_1 = \sqrt{2}v\rho_{13} f(V) S \frac{\partial^2}{\partial S \partial V} + \sqrt{2}\rho_{23} v \sigma_c \frac{\partial^2}{\partial y \partial V} - \sqrt{2} v \Lambda(V) \frac{\partial}{\partial V} \tag{6.7}$$

$$\begin{aligned} \mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2} f(V)^2 S^2 \frac{\partial^2}{\partial S^2} + \frac{1}{2}\sigma_c^2 \frac{\partial^2}{\partial y^2} + \rho_{12} f(V) S \sigma_c \frac{\partial^2}{\partial S \partial y} + (r - y) S \frac{\partial}{\partial S} \\ &+ (\kappa_1(\alpha - y) - \lambda)\frac{\partial}{\partial y} - r\cdot = \mathcal{L}_{TF}(f(V)). \end{aligned} \tag{6.8}$$

Note that $\mathcal{L}_2$ is the Schwartz (1997) two-factor model operator at the volatility level $f(V)$, which is also denoted as $\mathcal{L}_{TF}(f(V))$.

With this notation, the partial differential Equation (6.4) for the price of the contingent claim becomes:

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P = 0. \tag{6.9}$$

6.3 The Formal Expansion

The solution P of Equation (6.9) can be expanded in powers of $\sqrt{\varepsilon}$,

$$P = P_0 + \sqrt{\varepsilon} P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon} P_3 + \cdots . \tag{6.10}$$

Primary interest is in the the first two terms, $P_0 + \sqrt{\varepsilon}P_1$. The terminal conditions for the first term is $P_0(T, S, y, V) = h(S)$ and for the second term $P_1(T, S, y, V) = 0$ respectively.

Substituting Equation (6.10) into Equation (6.9), one can obtain:

$$\begin{aligned} \frac{1}{\varepsilon}\mathcal{L}_0 P_0 &+ \frac{1}{\sqrt{\varepsilon}}(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) \\ &+ (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) \\ &+ \sqrt{\varepsilon}(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) \\ &+ \cdots \\ &= 0 \end{aligned} \tag{6.11}$$

6.3.1 The Diverging Terms

To eliminate the terms of order $1/\varepsilon$, one must have:

$$\mathcal{L}_0 P_0 = 0$$

Note that the operator $\mathcal{L}_0$ takes derivatives with respect to V and that the equation above can only hold if P_0 is constant with respect to V, that is:

$$P_0 = P_0(t, S, y)$$

and independent of V. Similarly, in order to eliminate the terms of order $1/\sqrt{\varepsilon}$, one must have:

$$\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0$$

As the operator $\mathcal{L}_1$ factors through $\frac{\partial}{\partial V}$ one can have $\mathcal{L}_1 P_0 = 0$. Consequently, one can obtain $\mathcal{L}_0 P_1 = 0$.

Again, because $\mathcal{L}_0$ only acts on the V variable, one can get:

$$P_1 = P_1(t, S, y),$$

which also implies that $\mathcal{L}_1 P_1 = 0$. Therefore, to eliminate the terms of order 1 one must have:

$$\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0. \tag{6.12}$$

6.3.2 The Zero-order Term

Equation (6.12) is a Poisson equation and only has a solution if,

$$\langle \mathcal{L}_2 P_0 \rangle^{1} = 0,$$

where the brackets denote the average with respect to the invariant distribution of V.

Since P_0 does not depend on V, this means that $\langle \mathcal{L}_2 \rangle P_0 = 0$. From the definition of $\mathcal{L}_2$ in Equation (6.8), it can be derived that $\langle \mathcal{L}_2 \rangle = \mathcal{L}_{TF}(\bar{\sigma})$, where the volatility $\bar{\sigma}$ is defined by:

$$\bar{\sigma}^2 = \langle f^2 \rangle \tag{6.13}$$

where f is the invariant distribution of V.

Therefore, the zero-order term $P_0(t, S, y)$ is the solution of the Schwartz (1997) two-factor model, identified by:

$$\mathcal{L}_{TF}(\bar{\sigma}) P_0 = 0 \tag{6.14}$$

with the terminal condition $P_0(T, S, y) = h(S)$.

As the centering condition is satisfied, one can have:

$$\mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle = \frac{1}{2}(f(V)^2 - \bar{\sigma}^2) S^2 \frac{\partial^2 P_0}{\partial S^2} + \rho_{12}(f(V) - \bar{\sigma}) S \sigma_c \frac{\partial^2 P_0}{\partial S \partial y}.$$

[1] See centering condition for detailed in Fouque et al. (2000b), page 91.

Then the second-order correction P_2 can be given by:

$$\begin{aligned}
P_2(t,S,y,V) &= -\mathcal{L}_0^{-1}\left(\frac{1}{2}\big(f(V)^2-\bar{\sigma}^2\big)S^2\frac{\partial^2 P_0}{\partial S^2}+\rho_{12}\big(f(V)-\bar{\sigma}\big)S\sigma_c\frac{\partial^2 P_0}{\partial S\partial y}\right)\\
&= -\left(\frac{1}{2}\big(\phi_1(V)+c_1(t,S,y)\big)S^2\frac{\partial^2 P_0}{\partial S^2}\right.\\
&\quad\left.+\rho_{12}\big(\phi_2(V)+c_2(t,S,y)\big)S\sigma_c\frac{\partial^2 P_0}{\partial S\partial y}\right)
\end{aligned} \tag{6.15}$$

where $\phi_{1,2}(V)$ are solutions of the Poisson equations,

$$\mathcal{L}_0\phi_1 = f(V)^2-\langle f^2\rangle \tag{6.16}$$

$$\mathcal{L}_0\phi_2 = f(V)-\langle f\rangle \tag{6.17}$$

and $c_i(t,S,y), i=1,2$ does not depends on V variable, but may depend on (t,S,y), which is the same as presented in Fouque et al. (2000b).

6.3.3 The First Correction

Similarly, in order to eliminate the terms of order $\sqrt{\varepsilon}$, one can have:

$$\mathcal{L}_0P_3+\mathcal{L}_1P_2+\mathcal{L}_2P_1 = 0 \tag{6.18}$$

This is again a Poisson equation for P_3 with respect to $\mathcal{L}_0$, which requires the centering:

$$\langle\mathcal{L}_1P_2+\mathcal{L}_2P_1\rangle = 0 \tag{6.19}$$

The solution for P_2 is given by Equation (6.15), P_1 is independent of V and $\langle\mathcal{L}_2\rangle = \mathcal{L}_{TF}(\bar{\sigma})$, it can be derived that:

$$\begin{aligned}
\mathcal{L}_{TF}(\bar{\sigma})P_1 &= -\langle\mathcal{L}_1P_2\rangle\\
&= \frac{1}{2}\langle\mathcal{L}_1\phi_1(V)\rangle S^2\frac{\partial^2 P_0}{\partial S^2}+\rho_{12}\langle\mathcal{L}_1\phi_2(V)\rangle S\sigma_c\frac{\partial^2 P_0}{\partial S\partial y}
\end{aligned} \tag{6.20}$$

Note that $\mathcal{L}_1$ takes derivatives with respect to V and $c_i(t,S,y)$ does not depend on V, so that it can be derived that $\mathcal{L}_1c_i = 0$.

Then the problem now is to find out solutions for $\langle\mathcal{L}_1\phi_1(V)\rangle$ and $\langle\mathcal{L}_1\phi_2(V)\rangle$,

$$\begin{aligned}
\langle\mathcal{L}_1\phi_i(V)\cdot\rangle &= \sqrt{2}\rho_{13}\upsilon\langle f(V)\phi_i'(V)\rangle S\frac{\partial}{\partial S}\\
&+ \sqrt{2}\rho_{23}\upsilon\sigma_c\langle\phi_i'(V)\rangle\frac{\partial}{\partial y}-\sqrt{2}\upsilon\langle\Lambda(V)\phi_i'(V)\rangle, \quad i=1,2
\end{aligned} \tag{6.21}$$

Finally, one can deduce that:

$$
\begin{aligned}
\mathcal{L}_{TF}(\bar{\sigma})P_1 &= \frac{\sqrt{2}}{2}\rho_{13}v\langle f\phi_1'\rangle S^3\frac{\partial^3 P_0}{\partial S^3} \\
&+ \big(\frac{\sqrt{2}}{2}\rho_{23}v\langle\phi_1'\rangle + \sqrt{2}\rho_{12}\rho_{13}v\langle f\phi_2'\rangle\big)S^2\sigma_c\frac{\partial^3 P_0}{\partial S^2\partial y} \\
&+ \sqrt{2}\rho_{12}\rho_{23}v\langle\phi_2'\rangle S\sigma_c^2\frac{\partial^3 P_0}{\partial S\partial y^2} \\
&+ \big(\sqrt{2}\rho_{13}v\langle f\phi_1'\rangle - \frac{\sqrt{2}}{2}v\langle\Lambda\phi_1'\rangle\big)S^2\frac{\partial^2 P_0}{\partial S^2} \\
&+ \big(\sqrt{2}\rho_{12}\rho_{13}v\langle f\phi_2'\rangle - \sqrt{2}\rho_{12}v\langle\Lambda\phi_2'\rangle\big)S\sigma_c\frac{\partial^2 P_0}{\partial S\partial y} \qquad (6.22)
\end{aligned}
$$

Now the first correction, $\hat{P}_1(t,S,y) = \sqrt{\varepsilon}P_1(t,S,y)$, is introduced and a note is made that the right side of Equation (6.22) is G, that is:

$$\mathcal{L}_{TF}(\bar{\sigma})\hat{P}_1 = G(t,S,y) \qquad (6.23)$$

The solution for $\hat{P}_1(t,S,y)$ is $-(T-t)G$ with boundary condition.

It can be checked from the identity:

$$\mathcal{L}_{TF}(\bar{\sigma})(-(T-t)G) = G - (T-t)\mathcal{L}_{TF}(\bar{\sigma})G \qquad (6.24)$$

The second term on the right side of Equation (6.24) is equal to zero (see appendix 6.A for detail).

Therefore, the solution for the second term $\sqrt{\varepsilon}P_1$ in the expansion Equation (6.10) is:

$$
\begin{aligned}
\sqrt{\varepsilon}P_1 &= -(T-t)\bigg(A_2S^2\frac{\partial^2 P_0}{\partial S^2} + A_3S\frac{\partial^2 P_0}{\partial S\partial y} + A_4S^3\frac{\partial^3 P_0}{\partial S^3} \\
&\quad + A_5S^2\frac{\partial^3 P_0}{\partial S^2\partial y} + A_6S\frac{\partial^3 P_0}{\partial S\partial y^2}\bigg) \qquad (6.25)
\end{aligned}
$$

where A_2, A_3, A_4, A_5 and A_6 are five small coefficients, in terms of $\kappa_2 = 1/\varepsilon$, they can be given by:

$$A_2 = \frac{v}{\sqrt{2\kappa_2}}(2\rho_{13}\langle f\phi_1'\rangle - \langle\Lambda\phi_1'\rangle) \qquad (6.26)$$

$$A_3 = \bigg(\frac{\sqrt{2}\rho_{12}\rho_{13}}{\sqrt{\kappa_2}}\langle f\phi_2'\rangle - \frac{\sqrt{2}\rho_{12}}{\sqrt{\kappa_2}\langle\Lambda\phi_2'\rangle}\bigg)\sigma_c v \qquad (6.27)$$

$$A_4 = \frac{\rho_{13}v}{\sqrt{2\kappa_2}}\langle f\phi_1'\rangle \qquad (6.28)$$

$$A_5 = \big(\frac{\rho_{23}}{\sqrt{2\kappa_2}}\langle\phi_1'\rangle + \frac{\sqrt{2}\rho_{12}\rho_{13}}{\sqrt{\kappa_2}}\langle f\phi_2'\rangle\big)\sigma_c v \qquad (6.29)$$

$$A_6 = \frac{\sqrt{2}\rho_{12}\rho_{23}}{\sqrt{\kappa_2}}\langle\phi_2'\rangle\sigma_c^2 v \tag{6.30}$$

Therefore, the corrected price is given explicitly by:

$$\begin{aligned} P = P_0 - (T-t)\Big(A_2 S^2 \frac{\partial^2 P_0}{\partial S^2} &+ A_3 S \frac{\partial^2 P_0}{\partial S \partial y} + A_4 S^3 \frac{\partial^3 P_0}{\partial S^3} \\ &+ A_5 S^2 \frac{\partial^3 P_0}{\partial S^2 \partial y} + A_6 S \frac{\partial^3 P_0}{\partial S \partial y^2}\Big) \end{aligned} \tag{6.31}$$

where P_0 is the Schwartz (1997) two-factor model price with constant volatility $\bar{\sigma}$.

The solution for ϕ_1' has been done by Fouque et al. (2000b) from Equation (6.16),

$$\phi_1'(V) = \frac{1}{v^2\Phi(V)}\int_{-\infty}^{\cdot}(f^2 - \langle f^2\rangle)\Phi \tag{6.32}$$

where $\Phi(V)$ is the probability density of the $\mathcal{N}(m, v^2)$-invariant distribution.

Note that one have zero on both sides when interval of integration is infinity.

Therefore, with boundary condition, one can find:

$$\begin{aligned} \langle f\phi_1'\rangle &= \langle\frac{f}{v^2\Phi}\int_{-\infty}^{\cdot}(f^2 - \langle f^2\rangle)\Phi\rangle \\ &= \frac{1}{v^2}\int_{-\infty}^{+\infty} f \int_{-\infty}^{\cdot}(f^2 - \langle f^2\rangle)\Phi \\ &= -\frac{1}{v^2}\langle X(f^2 - \langle f^2\rangle)\rangle \end{aligned} \tag{6.33}$$

where X is the antiderivative of f, and

$$\langle\phi_1'\rangle = -\frac{1}{v^2}\langle V(f^2 - \langle f^2\rangle)\rangle \tag{6.34}$$

Similarly, the solution for ϕ_2' is:

$$\phi_2' = \frac{1}{v^2\Phi}\int_{-\infty}^{\cdot}(f - \langle f\rangle)\Phi \tag{6.35}$$

Therefore, one can find the solution for $\langle f\phi_2'\rangle$:

$$\langle f\phi_2'\rangle = -\frac{1}{v^2}\langle X(f - \langle f\rangle)\rangle \tag{6.36}$$

and

$$\langle\phi_2'\rangle = -\frac{1}{v^2}\langle V(f - \langle f\rangle)\rangle \tag{6.37}$$

Therefore, assuming the market price of volatility risk γ is 0 and $f(V) = V$, the quantities A_2, A_3, A_4, A_5 and A_6 can be obtained analytically by the remaining model parameters (see appendix 6.B for detail).

$$A_2 = -\frac{2\rho_{13}v}{\sqrt{2\kappa_2}}(2m^2 + v^2) \tag{6.38}$$

$$A_3 = -\frac{\sqrt{2}\rho_{12}\rho_{13}\sigma_c v m}{\sqrt{\kappa_2}} \tag{6.39}$$

$$A_4 = -\frac{\rho_{13}v}{\sqrt{2\kappa_2}}(2m^2 + v^2) \tag{6.40}$$

$$A_5 = -\sigma_c v\left(\frac{2m\rho_{23}}{\sqrt{2\kappa_2}} + \frac{\sqrt{2}\rho_{12}\rho_{13}m}{\sqrt{\kappa_2}}\right) \tag{6.41}$$

$$A_6 = -\frac{\sqrt{2}\rho_{12}\rho_{23}\sigma_c^2 v}{\sqrt{\kappa_2}} \tag{6.42}$$

6.4 European Commodity Call Options

The price of European options on commodity futures are presented by Miltersen and Schwartz (1998) and Hilliard and Reis (1998).

Here, it can be set that European call option C is at time zero with maturity t, exercise price K and commodity futures contract with maturity T. In this chapter, the constant spot price volatility is $\bar{\sigma}$, hence the value of a European call option is given by:

$$P_0(0, t, T) \;=\; e^{-rt}[F(0, T)N(d_1) - KN(d_2)] \tag{6.43}$$

with

$$d_1 \;=\; \frac{\ln(F(0, T)/K) + \frac{1}{2}\sigma^2}{\sigma}, \quad d_2 = d_1 - \sigma,$$

$$\begin{aligned}\sigma^2(0, t, T) \;=\; & \bar{\sigma}^2 t \\ & -\frac{2\rho_{12}\bar{\sigma}\sigma_c}{\kappa_1}\left[t - \frac{(e^{-\kappa_1(T-t)} - e^{-k_1 T})}{\kappa_1}\right] \\ & +\frac{\sigma_c^2}{\kappa_1^2}\left[t - \frac{2}{\kappa_1}(e^{-\kappa_1(T-t)} - e^{-k_1 T})\right. \\ & \left. +\frac{1}{2\kappa_1}(e^{-2\kappa_1(T-t)} - e^{-2\kappa_1 T})\right],\end{aligned}$$

and $N(\cdot)$ is the cumulative standard normal distribution function.

Hilliard and Reis (1998) also presented a simple expression for the price of futures contracts under the assumption of constant spot volatility, in this chapter, the value of a futures contract at time zero is given by:

$$F(S_0, y_0, 0, T) = S_0 e^{A(T)+H(T)y_0} \tag{6.44}$$

with

$$
\begin{aligned}
A(T) &= (r - \tilde{\alpha} + \frac{\sigma_c^2}{2\kappa_1^2} - \frac{\bar{\sigma}\sigma_c\rho_{12}}{\kappa_1})T + \frac{\sigma_c^2(1-e^{-2\kappa_1 T})}{4\kappa_1^3} \\
&\quad +(\kappa_1\tilde{\alpha} + \bar{\sigma}\sigma_c\rho_{12} - \frac{\sigma_c^2}{\kappa_1})\frac{1-e^{-\kappa_1 T}}{\kappa_1^2}, \\
H(T) &= -\frac{1-e^{-\kappa_1 T}}{\kappa_1},
\end{aligned}
$$

where $\tilde{\alpha} = \alpha - \lambda/\kappa_1$.

6.5 Asymptotic Two-factor Model Solution for Futures Options

Now one can try to rewrite Equation (6.31) in terms of futures contract. Because P_0 is an analytical expression, it can be computed that the terms of $\frac{\partial^2 P_0}{\partial S^2}$, $\frac{\partial^2 P_0}{\partial S \partial y}$, $\frac{\partial^3 P_0}{\partial S^3}$, $\frac{\partial^3 P_0}{\partial S^2 \partial y}$ and $\frac{\partial^3 P_0}{\partial S \partial y^2}$ in Equation (6.31).

The **Delta** can be computed on futures contracts as follows:

$$
\begin{aligned}
\frac{\partial P_0}{\partial S} &= \frac{\partial P_0}{\partial F} \cdot \frac{\partial F}{\partial S} \\
&= e^{-rt} N(d_1)(e^{A+Hy}) \\
&= \frac{\partial P_0}{\partial F} \cdot \frac{F}{S}
\end{aligned}
$$

The **Gamma**,

$$
\begin{aligned}
\frac{\partial^2 P_0}{\partial S^2} &= \frac{\partial^2 P_0}{\partial F^2} \cdot (\frac{\partial F}{\partial S})^2 + \frac{\partial P_0}{\partial F} \cdot \frac{\partial^2 F}{\partial S^2} \\
&= e^{-rt}\frac{N'(d_1)}{F\sigma}(e^{A+Hy})^2 \\
&= e^{-rt}\frac{N'(d_1)}{S\sigma}(e^{A+Hy}) \\
&= \frac{\partial^2 P_0}{\partial F^2} \cdot \left(\frac{F}{S}\right)^2
\end{aligned}
$$

Note $\frac{\partial^2 F}{\partial S^2} = 0$.

The **Speed**,

$$
\begin{aligned}
\frac{\partial^3 P_0}{\partial S^3} &= \frac{\partial^3 P_0}{\partial F^3}\cdot(\frac{\partial F}{\partial S})^3 + 3\frac{\partial^2 P_0}{\partial F^2}\cdot\frac{\partial F}{\partial S}\cdot\frac{\partial^2 F}{\partial S^2} + \frac{\partial P_0}{\partial F}\frac{\partial^3 F}{\partial S^3} \\
&= -e^{-rt}\frac{N'(d_1)}{F^2\sigma}(\frac{d_1}{\sigma}+1)(e^{A+Hy})^3 \\
&= -e^{-rt}\frac{N'(d_1)}{S^2\sigma}(\frac{d_1}{\sigma}+1)(e^{A+Hy}) \\
&= \frac{\partial^3 P_0}{\partial F^3}\cdot\left(\frac{F}{S}\right)^3
\end{aligned}
$$

Note $\frac{\partial^3 F}{\partial S^3} = 0$.

And one can find,

$$\frac{\partial^2 P_0}{\partial S \partial y} = \frac{\partial P_0}{\partial F}\frac{FH}{S} + \frac{\partial^2 P_0}{\partial F^2}\frac{F^2 H}{S}$$

also,

$$\frac{\partial^3 P_0}{\partial S^2 \partial y} = \frac{\partial^2 P_0}{\partial F^2}(1-\frac{d_1}{\sigma})H\left(\frac{F}{S}\right)^2\frac{\partial^2 P_0}{\partial F^2}$$

and,

$$\frac{\partial^3 P_0}{\partial S \partial y^2} = \frac{\partial P_0}{\partial F}\frac{FH}{S} + \frac{\partial^2 P_0}{\partial F^2}\left(\frac{F^2 H}{S} + \frac{F^2 H^2}{S\sigma}(1-d_1)\right)$$

Therefore, the corrected price Equation (6.31) can be expressed as:

$$
\begin{aligned}
P &= P_0 - (T-t)\Bigg(A_2 F^2\frac{\partial^2 P_0}{F^2} + A_3\big(FH\frac{\partial P_0}{\partial F} + F^2 H\frac{\partial^2 P_0}{\partial F^2}\big) \\
&+ A_4 F^3\frac{\partial^3 P_0}{\partial F^3} + A_5 F^2 H(1-\frac{d_1}{\sigma})\frac{\partial^2 P_0}{\partial F^2} \\
&+ A_6\big(FH\frac{\partial P_0}{\partial F} + F^2 H\frac{\partial^2 P_0}{\partial F^2} + \frac{F^2 H^2}{\sigma}(1-d_1)\frac{\partial^2 P_0}{\partial F^2}\big)\Bigg) \qquad (6.45)
\end{aligned}
$$

where P_0 is the Schwartz (1997) two-factor model price with constant volatility $\bar{\sigma}$.

6.6 Asymptotic Results on Simulated Data

For the purpose of demonstrating the accuracy of the asymptotic formula, the numerical example is presented in this section for different option position. The results are compared that obtained from the asymptotic formula to those from the Schwartz (1997) two-factor model, and those from the Monte Carlo simulation.

The Euler Maruyama discretization is employed for the spot, convenience yield and stochastic volatility dynamic in Monte Carlo simulation:

$$\begin{aligned}
V_t &= V_{t-\Delta t} + (\kappa_2(m - V_{t-\Delta t}))\Delta t + \beta dW_{3_{t-\Delta t}} \\
y_t &= y_{t-\Delta t} + (\kappa_1(\alpha - y_{t-\Delta t}) - \lambda)\Delta t + \sigma_c dW_{2_{t-\Delta t}} \\
S_t &= S_{t-\Delta t}(1 + (r - y_{t-\Delta t})\Delta t + V_{t-\Delta t} dW_{1_{t-\Delta t}})
\end{aligned} \tag{6.46}$$

and the relationship between three Brown motion processes in Monte Carlo method can be solved by Cholesky decomposition:

$$\begin{bmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & 0 \\ \rho_{13} & \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} & \sqrt{1-\rho_{13}^2 - (\frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}})^2} \end{bmatrix} \begin{bmatrix} dW_1 \\ dW_2 \\ dW_3 \end{bmatrix}$$

Note that the futures options underlying is futures price not commodity price, however, the futures price is equal to commodity price when it comes to maturity.

The initial spot price is assumed 100, initial convenience yield is assumed 0.05, the initial spot volatility is assumed 0.8944 and both futures contracts and futures options contracts' maturity is assumed 1. All parameters showed in three-factor model are listed in Table 6.1.

Table 6.1: Parameter choices for three-factor model

Parameter	Value	Parameter	Value
r	0.02	κ_1	0.8
$\tilde{\alpha}$	0.3	σ_c	1.1
κ_2	5	m	0.05
β	0.5	ρ_{12}	0.3
ρ_{13}	-0.7	ρ_{23}	-0.5

Then it can be found that the average spot volatility $\bar{\sigma}$ from Monte Carlo simulation is 0.2162, the standard deviation of volatility v is 0.1581. From Equation (6.38) to Equation (6.42), it can be found that A_2 is 0.0021, A_3 is 0.0012, A_4 is 0.001, A_5 is 0.0039 and A_6 is 0.0182.

With these parameters, a number of futures call option prices can be computed by the asymptotic method (Asymptotic) as compared by Monte Carlo (MC) and the prices computed from the Schwartz (1997) two-factor model (TF). The Monte Carlo simulation results are chose as the benchmark price. The gap1 is the error(%) between the prices of MC and Asymptotic and the gap2 is the error(%) between the prices of MC and TF.

Table 6.2 shows that the results of the asymptotic method here are better than those of the standard Schwartz (1997) two-factor model, except for the in-the-money options, whose strike price is 80. Specifically, the error for asymptotic results seems to depend on the position of options, while the error for the Schwartz (1997) two-factor model increased as the strike price increased, from 1.86% to 8.9%. Hence, it can be concluded that the asymptotic method can provide more stable results than the standard Schwartz (1997) two-factor model. Particularly, when it comes to at-the-money options, the asymptotic method result is much better than the standard Schwartz (1997) two-factor model result; the errors are 1.26% and 5.13% respectively. Therefore, the new formula in this thesis can be a very useful tool to price futures options in the financial market.

Table 6.2: Futures options prices

This table shows the numerical result on asymptotic formula with 1 year maturity. The initial spot price is 100, and strike prices are 80, 90, 100, 110 and 120. The gap1 is the error between Monte Carlo result (MC) and asymptotic result (Asymptotic), the gap2 is the error between Monte Carlo (MC) and the standard Schwartz (1997) two-factor model (TF).

S=100	MC	Asymptotic	gap1	TF	gap2
K=80	27.0215	27.8974	3.24%	26.5188	1.86%
K=90	22.1095	22.4095	1.36%	21.3547	3.41%
K=100	18.0338	17.8067	1.26%	17.1092	5.13%
K=110	14.684	14.0133	4.57%	13.6617	6.96%
K=120	11.95	10.9333	8.51%	10.8869	8.90%

6.7 Asymptotic Results on Market Data

6.7.1 Data

Now this section will look at European natural gas futures (GKJ4) whose underlying is the natural gas (Henry Hub) physical futures, which will be used to test the asymptotic two-factor model formula. The data is chosen from 27/12/2012 to 26/09/2013, which has a 2-year, 1-year, 0.75-year and 0.5-year time maturity point; that is, the dates 27/12/2012, 26/03/2013, 26/06/2013, and 26/09/2013, respectively. The interest free rate (r) during this time period is 0.5%. The natural gas futures prices at these times are in the following table:

Table 6.3: Natural gas futures prices (GKJ4)

This table presents the natural gas futures prices on 27/12/2012(T=2), 26/03/2013(T=1), 26/06/2013(T=0.75) and 26/09/2013(T=0.5).

	Natural Gas Future Prices
T=2	3.82
T=1	4.101
T=0.75	3.934
T=0.5	3.793

Since the natural gas futures prices are around 4, the natural gas futures options contracts is chose with strike price 2.5, 3, 3.5, 4, 4.5 and 5 to see the performance of different option position. The corresponding natural gas futures options prices are shown in Table 6.4.

Table 6.4: Natural gas futures call options prices

This table presents the prices of natural gas futures options (GKJ4) on 227/12/2012(T=2), 26/03/2013(T=1), 26/06/2013(T=0.75) and 26/09/2013(T=0.5), with strike price (K), 2.5, 3, 3.5, 4, 4.5, 5.

	K=2.5	K=3	K=3.5	K=4	K=4.5	K=5
T=2	1.3874	0.9976	0.6871	0.4623	0.3106	0.2068
T=1	1.6226	1.1648	0.7767	0.4821	0.2816	0.1583
T=0.75	1.4377	0.9719	0.5798	0.3086	0.1534	0.0745
T=0.5	1.2978	0.8325	0.45	0.2065	0.0837	0.0314

6.7.2 Calibration

Calibration is performed using both the standard Schwartz (1997) two-factor model and the asymptotic two-factor model. Parameters in these two models are estimated by the minimization of the least squares method through the Matlab routine lsqnonlin. Since the natural gas futures contract price is already known, $\tilde{\alpha}$ does not need to be estimated.

Table 6.5: Parameter estimation from market data for asymptotic two-factor model and standard two-factor model

Parameter	Asymptotic formula	Two-factor model
$\bar{\sigma}$	0.461	0.498
σ_c	3.1124	3.5793
κ_1	6.7193	6.4295
κ_2	2.3604	N/A
ρ_{12}	0.8817	0.8919
ρ_{13}	-0.8912	N/A
ρ_{23}	-0.1827	N/A
m	0.2638	N/A
v	0.0188	N/A

Table 6.5 shows the estimation of necessary parameters to compute the natural gas futures options price in both the asymptotic two-factor model and the standard Schwartz (1997) two-factor model.

Tables 6.6 and 6.7 present prices of natural gas futures call options from the asymptotic two-factor formula and the standard Schwartz (1997) two-factor model, respectively. The values of the sum of squared of the residual (RSS) at natural gas futures call options for the asymptotic solution and the standard Schwartz (1997) two-factor model are 0.0018 and 0.0026, respectively. The difference between these two numbers is quite small (0.0008); however, the natural gas futures contract is quoted in U.S. dollars, the contract size is 10,000 mmBtu and the daily exchange volume is around 100,000, so, actually, the difference between these two models is quite large. On the

Table 6.6: Natural gas futures call option prices from asymptotic two-factor solution

This table presents the prices of natural gas futures call options from asymptotic two-factor model on 27/12/2012(T=2), 26/03/2013(T=1), 26/06/2013(T=0.75) and 26/09/2013(T=0.5), with strike price (K), 2.5, 3, 3.5, 4, 4.5, 5.

	K=2.5	K=3	K=3.5	K=4	K=4.5	K=5
T=2	1.3885	1.0009	0.6950	0.4691	0.3103	0.2022
T=1	1.6149	1.1526	0.7609	0.4675	0.2706	0.1495
T=0.75	1.4441	0.9791	0.5950	0.3261	0.1640	0.0772
T=0.5	1.2989	0.8277	0.4477	0.2060	0.0827	0.0298

Table 6.7: Natural gas futures call option prices from the Schwartz (1997) two-factor model

This table presents the prices of natural gas future call options from Schwartz (1997) two-factor model on 27/12/2012(T=2), 26/03/2013(T=1), 26/06/2013(T=0.75) and 26/09/2013(T=0.5), with strike price(K), 2.5, 3, 3.5, 4, 4.5, 5.

	K=2.5	K=3	K=3.5	K=4	K=4.5	K=5
T=2	1.3723	0.9938	0.6945	0.4721	0.3144	0.2064
T=1	1.6022	1.1464	0.7611	0.4703	0.2731	0.1508
T=0.75	1.4350	0.9756	0.5962	0.3279	0.1644	0.0764
T=0.5	1.2929	0.8259	0.4489	0.2060	0.0812	0.0283

other hand, the asymptotic method can improve 30% accuracy, compared with the standard Schwartz (1997) two-factor model. Therefore, it can be concluded that the benefit from the asymptotic method is significant.

Besides, from Table 6.8, it can be seen that the value of RSS for the asymptotic two-factor model solution is lower than the value of RSS for the Schwartz (1997) two-factor model in all maturity cases. Especially for long time maturity cases (T=2), the performance of the asymptotic method is much better than the performance of the Schwartz (1997) two-factor model. Therefore, it is worth applying asymptotic formula in pricing the value of futures options.

Table 6.8: The value of RSS for asymptotic two-factor model and standard two-factor model in terms of time maturity

Time maturity	Asymptotic formula	Two-factor model
$T = 2$	0.0001	0.0004
$T = 1$	0.0009	0.0013
$T = 0.75$	0.0007	0.0008
$T = 0.5$	0.0001	0.0001
Sum	0.0018	0.0026

6.8 Conclusion

In this chapter, the performance of the asymptotic two-factor model in futures options pricing is investigated. It is shown that the asymptotic formula can be expressed both in the form of spot price and futures price. The asymptotic two-factor model solution is found to be a good approximation for both simulated data and real market data. Compared with the results from the standard Schwartz (1997) two-factor model, the results coming from asymptotic two-factor model solution improve by 30% accuracy in the illustrated samples, and especially perform much better in a long maturity case and for at-the-money options. Besides, since the asymptotic formula can be solved analytically, the asymptotic solution will not increase the computing time, compared with that from the standard Schwartz (1997) two-factor model. Therefore, clearly, the asymptotic two-factor model formula can take the place of the standard Schwartz (1997) two-factor model in pricing futures options in the financial market. In addition, the asymptotic formula can also be expressed in terms of futures prices, which can further improve the efficiency of the asymptotic method for futures options prices. Furthermore, it is also believed that the asymptotic formula will give a better performance in hedging.

Appendix

6.A First Correction Proof

Here, it can be proved that $\mathcal{L}_{TF}(\bar{\sigma})G$ in Equation (6.24) is zero.

In Equation (6.23), the operator $\mathcal{L}_{TF}(\bar{\sigma})$ is expressed as Equation (6.8), which is related to S and y. Actually it can be transferred into operator which is only related with F, that is:

$$\mathcal{L}_{TF}(\bar{\sigma}) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2}{\partial F^2} - r\cdot \tag{6.47}$$

where $F = Se^{A+Hy}$, $\sigma = \sqrt{\bar{\sigma}^2 + B^2\sigma_c^2 + 2\rho_{12}\bar{\sigma}\sigma_c B}$.

Similarly, G can also be rewritten as a function of F, that is:

$$\begin{aligned} &(A_2 F^2 \frac{\partial^2 P_0}{F^2} + A_3\big(FH\frac{\partial P_0}{\partial F} + F^2 H \frac{\partial^2 P_0}{\partial F^2}\big) + A_4 F^3 \frac{\partial^3 P_0}{\partial F^3} + A_5 F^2 H (1 - \frac{d_1}{\sigma})\frac{\partial^2 P_0}{\partial F^2} \\ + \quad & A_6\big(FH\frac{\partial P_0}{\partial F} + F^2 H \frac{\partial^2 P_0}{\partial F^2} + \frac{F^2 H^2}{\sigma}(1-d_1)\frac{\partial^2 P_0}{\partial F^2}\big) \end{aligned} \tag{6.48}$$

Therefore, one can prove that second term $\mathcal{L}_{TF}(\bar{\sigma})G$ in Equation (6.24) is zero because,

$$\mathcal{L}_{TF}(\bar{\sigma})\left(F^n \frac{\partial^n P_0}{\partial F^n}\right) = F^n \frac{\partial^n}{\partial F^n}\mathcal{L}_{TF}(\bar{\sigma})P_0 = 0 \tag{6.49}$$

Equation (6.49) can be proved by following the method provided in Ting (2012).

6.B Solutions for A_2, A_3, A_4, A_5 and A_6

In order to compute A_2, A_3, A_4, A_5 and A_6 analytically, one must find the results for $\langle f\phi_1'\rangle$, $\langle \phi_1'\rangle$, $\langle f\phi_2'\rangle$ and $\langle \phi_2'\rangle$.

Note that if $f(V) = V$ and V is followed a normal distribution $N(m, v^2)$, with the property of normal distribution moments, one can find:

$$\begin{aligned}
\langle f\phi_1'\rangle &= -\frac{1}{v^2}\langle X(f^2-\langle f^2\rangle)\rangle \\
&= -\frac{1}{v^2}\int_{-\infty}^{+\infty}\frac{1}{2}V^2(V^2-\langle V^2\rangle)\Phi(V)dV \\
&= -\frac{1}{v^2}\left(\frac{1}{2}\big(\mathbb{E}(V^4)-\mathbb{E}(V^2)\mathbb{E}(V^2)\big)\right) \\
&= -\frac{1}{v^2}\big(\frac{1}{2}(m^4+6m^2v^2+3v^4-m^4-2m^2v^2-v^4)\big) \\
&= -(2m^2+v^2) \qquad (6.50)
\end{aligned}$$

Similarly, the solution for $\langle\phi_1'\rangle$, $\langle f\phi_2'\rangle$ and $\langle\phi_2'\rangle$ are:

$$\langle\phi_1'\rangle = -2m \qquad (6.51)$$

$$\langle f\phi_2'\rangle = -m \qquad (6.52)$$

and

$$\langle\phi_2'\rangle = -1 \qquad (6.53)$$

Therefore, the solutions for A_2, A_3, A_4, A_5 and A_6 can be obtained by substituting these above results.

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Chapter 7

Time Dependent Volatility in Futures Options

7.1 Introduction

In the Black-Scholes (1973) framework, the log returns of assets are assumed to follow a normal distribution since it may lead to a closed-form solution for derivatives pricing, which would make pricing problem easier. However, the empirical studies show that this is not the case. Generally, empirical log returns on equities, currencies and commodities have higher peaks and fatter tails, which is indicative of a distribution with differing variances (Gatheral (2011)). A large volume of literature has studied volatility, which present the fact that volatility is crucial for option pricing.

However, volatility cannot be directly observed in the market. Moreover, it is not a good idea to use historical volatility to predict volatility since volatility is not predictable. One way is to use stochastic volatility models to produce stochastic volatility instead of constant volatility when pricing options. There are a large number of research studies in this area; however, most stochastic volatility models are quite complicated and not easy to implement.

The other way is to make volatility time dependent in a simple model; hence, one can still get an easy and quick result for the options price. Wilmott (2013) showed that the Black-Scholes formula is still valid when the volatility is time dependent and the time dependent volatility can be measured by options market prices. However, the Black-Scholes formula is not suitable for futures options since the commodities futures market is different from the equity market. For example, futures contracts have expired time and most futures contract prices have periodicity. More important, the convenience yield must be considered when it comes to the futures market.

The convenience yield of a commodity is the benefit obtained from holding the spot commodity rather than the futures contracts, which is important when pricing futures contracts (Brennan (1991)). Gibson and Schwartz (1990) found that a constant convenience yield did not work well for pricing futures contracts, presenting a two-factor model for futures contracts on the price of oil. Miltersen and Schwartz (1998) and Schwartz (1997) have developed a two-factor model and a three-factor model to price commodities futures contracts and futures options contracts with stochastic convenience yields and interest rates. The results confirmed that a constant convenience yield is a bad assumption and stochastic convenience yield assumption has a better performance. Therefore, the stochastic convenience yield should be considered when pricing futures and futures derivatives, obviously, the time dependent volatility derived by the Black-Scholes formula cannot be applied into the commodities futures market.

Fortunately, an analytical solution for the value of futures options for the Schwartz (1997) two-factor model was presented by Hilliard and Reis (1998), which is very similar to the Black-Scholes formula. Hence, in this chapter, it is shown that the spot volatility in the Schwartz (1997) two-factor model can also be a function of time, and the corresponding partial differential equation is still valid. It will also be demonstrated how to measure time dependent volatilities in the commodities futures market empirically with natural gas futures and its corresponding futures options. Furthermore, the limitations of the method that is used to find the time dependent spot volatility will be explained, and it will be shown how to make sure of its accuracy.

The remainder of this chapter is organized as follows: Section 2 briefly covers the Schwartz (1997) two-factor model and the pricing formula of European futures options in this model. In Section 3, it is shown how to derive a formula for futures options when spot volatility is time dependent. In Section 4 and Section 5, the result of time dependent spot volatility with natural gas futures options historical data is demonstrated, and the correctness of the result of time dependent spot volatility is tested. Section 6 summarizes the main conclusions.

7.2 Schwartz (1997) Two-factor Model

Schwartz (1997) assumed that the spot price of the commodity and the instantaneous convenience yield follow the joint stochastic process under measure $\mathbb{P}$:

$$dS_t = (r - \delta_t)S_t d_t + S_t\sigma_s dW_1 \tag{7.1}$$
$$d\delta_t = \kappa(\alpha - \delta_t)dt + \sigma_c dW_2 \tag{7.2}$$

with Brownian motions W_1 andW_2 and correlation $dW_1 dW_2 = \rho_{12}dt$.

Under the risk-neutral measure $\mathbb{Q}$, the model becomes:

$$dS_t = (r - \delta_t)S_t d_t + S_t\sigma_s dW_1 \tag{7.3}$$
$$d\delta_t = (\kappa(\alpha - \delta_t) - \lambda)dt + \sigma_c dW_2 \tag{7.4}$$

where λ is constant, denoting the market price of convenience yield risk.

The dynamic of convenience yield can be simplified, setting

$$\hat{\alpha} = \alpha - \lambda/\kappa \tag{7.5}$$

which leads to:

$$dS_t = (r - \delta_t)S_t d_t + S_t \sigma_s dW_1 \tag{7.6}$$
$$d\delta_t = \kappa(\hat{\alpha} - \delta_t)dt + \sigma_c dW_2 \tag{7.7}$$

An analytical expression for the price of futures contract was presented by Schwartz (1997) and Hilliard and Reis (1998). Hilliard and Reis (1998) also derived an analytical solution for the price of European futures options. Erb et al. (2011) reviewed their work and presented a simple expression for both the value of futures and the value of European futures options.

Denote with $F(S_t, \delta_t, t, T)$ the futures price at time t with maturity at time T, $\tau = T - t$ the time to maturity, then

$$F(S_t, \delta_t, t, T) = S_t \exp(A(\tau) + B(\tau)\delta_t) \tag{7.8}$$

with

$$\begin{aligned} A(\tau) &= (r - \hat{\alpha} + \frac{1}{2}\frac{\sigma_c^2}{\kappa^2})\tau + \frac{1}{4}\sigma_c^2 \frac{1 - e^{-2\kappa\tau}}{\kappa^3} + (\kappa\hat{\alpha} + \sigma_s\sigma_c\rho_{12} - \frac{\sigma_c}{\kappa})\frac{1 - e^{\kappa\tau}}{\kappa^2}, \\ B(\tau) &= -\frac{1 - e^{-\kappa\tau}}{\kappa}. \end{aligned}$$

Denote with $C(t, T_1, T)$ the futures options price at time t with maturity at time T_1, K the futures options strike price. Assume interest rates r are deterministic and constant then the formula for the valuation of a European call option on a futures contract resembles the classical Black-Scholes formula:

$$C(t, T_1, T) = e^{-r(T_1 - t)}[F(t, T)N(d_1) - KN(d_2)] \tag{7.9}$$

where $N(\cdot)$ is cumulative standard normal distribution, and

$$\begin{aligned} d_1 &= \frac{\ln(F(t, T)/K) + 0.5v^2}{v}, \quad d_2 = d_1 - v, \\ v^2(t, T_1, T) &= \sigma_s^2(T_1 - t) \\ &\quad - \frac{2\rho_{12}\sigma_s\sigma_c}{\kappa}\left[(T_1 - t) - \frac{(e^{-\kappa(T - T_1)} - e^{-\kappa(T - t)})}{\kappa}\right] \\ &\quad + \frac{\sigma_c^2}{\kappa^2}\Bigg[(T_1 - t) - \frac{2}{\kappa}(e^{-\kappa(T - T_1)} - e^{-\kappa(T - t)}) \\ &\quad + \frac{1}{2\kappa}(e^{-2\kappa(T - T_1)} - e^{-2\kappa(T - t)})\Bigg]. \end{aligned}$$

7.3 Time Dependent Spot Volatility in the Schwartz (1997) Two-factor Model

The time dependent spot volatility pricing formula for futures options will be derived in this section. The method that Wilmott (2013) used for parameter generalization in the Black-Scholes model will be followed, to make the spot volatility be time dependent in the Schwartz (1997) two-factor model, since these two models are quite similar to each other.

Recall the Schwartz (1997) two-factor model in the risk-neutral world is:

$$\begin{aligned} dS_t &= (r-\delta_t)S_t d_t + S_t\sigma_s dW_1 \\ d\delta_t &= (\kappa(\hat{\alpha}-\delta_t))dt + \sigma_c dW_2 \\ dW_1 \cdot dW_2 &= \rho_{12} dt \end{aligned} \tag{7.10}$$

Assume $F(S,\delta,t)$ is the futures price, according to Itô lemma, the stochastic differential equation for futures price F is:

$$\begin{aligned} dF &= (F_t + \frac{1}{2}F_{SS}\sigma_s^2 S^2 + \frac{1}{2}F_{\delta\delta}\sigma_c^2 + F_{S\delta}\sigma_s\sigma_c\rho_{12})dt \\ &\quad +((F_S(r-\delta)) + F_\delta(\kappa(\hat{\alpha}-\delta)))dt + F_S S\sigma_s dW_1 + F_\delta\sigma_c dW_2 \end{aligned} \tag{7.11}$$

and futures price must satisfy the following partial differential equation in the Schwartz (1997) two-factor model:

$$\frac{1}{2}F_{SS}\sigma_s^2 S^2 + \frac{1}{2}F_{\delta\delta}\sigma_c^2 + F_{S\delta}\sigma_s\sigma_c\rho_{12} + (F_S(r-\delta)) + F_\delta(\kappa(\hat{\alpha}-\delta)) - F_\tau = 0 \tag{7.12}$$

with the boundary condition $F(S,\delta,0) = S$.

Therefore, Equation (7.11) can be rewritten as:

$$dF = F_S S\sigma_s dW_1 + F_\delta\sigma_c dW_2 \tag{7.13}$$

Remember an analytical solution for futures price can be found in the Schwartz (1997) two-factor model:

$$F = S\exp(A + B\delta) \tag{7.14}$$

Thus, substitute this analytical expression into Equation (7.13), Equation (7.13) can be expressed as:

$$dF = F\sigma_s dW_1 + FB\sigma_c dW_2 \tag{7.15}$$

Now one can define a new standard Wiener process dW_3 and a new parameter σ as:

$$\sigma dW_3 = \sigma_s dW_1 + B\sigma_c dW_2 \tag{7.16}$$

Then, Equation (7.15) can be simplified to:

$$dF = F\sigma dW_3 \tag{7.17}$$

where

$$\sigma = \sqrt{\sigma_s^2 + B^2\sigma_c^2 + 2\rho_{12}\sigma_s\sigma_c B} \tag{7.18}$$

Denote with $V(F,t)$ the futures options price, note that $V(F,t)$ is based on futures price and time, with Itô lemma, one can get:

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial F^2}F^2\sigma^2\right)dt + \frac{\partial V}{\partial F}F\sigma dW_3 \tag{7.19}$$

Now consider a portfolio Π, which is consisting of derivatives (V) and futures contracts ΔF. Because it costs nothing to enter into a futures contract,

$$\Pi = V \tag{7.20}$$

Then the total change of this portfolio dZ can be expressed as:

$$dZ = dV - \Delta dF \tag{7.21}$$

Substitute Equation (7.19) into differential Equation (7.21), Equation (7.21) becomes:

$$dZ = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial F^2}F^2\sigma^2\right)dt + (\frac{\partial V}{\partial F} - \Delta)F\sigma dW_3 \tag{7.22}$$

Note that the portfolio will be risk free, if one set $\Delta = \frac{\partial V}{\partial F}$,

$$dZ = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial F^2}F^2\sigma^2\right)dt \tag{7.23}$$

Because now the portfolio is risk free, one can get:

$$dZ = \Pi r dt \tag{7.24}$$

so that,

$$\Pi r = \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial F^2}F^2\sigma^2 \tag{7.25}$$

and then,

$$Vr = \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial F^2}F^2\sigma^2 \tag{7.26}$$

Therefore, the following formula can be obtained:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial F^2}F^2\sigma^2 = rV \tag{7.27}$$

This formula is similar to the Black-Scholes formula, but now the σ is expressed as Equation (7.18), which can be seemed as the term structure of volatility of futures price.

Note that σ is already time dependent, but in this chapter, σ_s is also made time dependent. Hence it can be rewritten Equation (7.27) as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial F^2} F^2 \sigma^2(t) - rV = 0 \tag{7.28}$$

Now three new variables can be defined, $\bar{V} = Ve^{\beta(t)}$, $\bar{F} = Fe^{\beta(t)}$ and $\bar{t} = \gamma(t)$. Note that β and γ can be choose as any functions so that one can eliminate all time dependent coefficients from Equation (7.28). After changing, Equation (7.28) can be given by:

$$\frac{\partial \bar{V}}{\partial \bar{t}} \dot{\gamma}(t) + \frac{1}{2}\frac{\partial^2 \bar{V}}{\partial \bar{F}^2} \bar{F}^2 \sigma^2(t) - (r + \dot{\beta}(t))\bar{V} = 0 \tag{7.29}$$

where $\dot{} = \frac{d}{dt}$.

As above mentioned, function $\beta(t)$ and $\gamma(t)$ can be any functions, thus assume $\beta(t) = r(T-t)$, then Equation (7.29) can be eliminated to

$$\frac{\partial \bar{V}}{\partial \bar{t}} \dot{\gamma}(t) + \frac{1}{2}\frac{\partial^2 \bar{V}}{\partial \bar{F}^2} \bar{F}^2 \sigma^2(t) = 0 \tag{7.30}$$

and then it can be assumed that $\gamma(t) = \int_t^T \sigma^2(\tau)d\tau$, finally, the Equation (7.30) can be further eliminated to

$$\frac{\partial \bar{V}}{\partial \bar{t}} = \frac{1}{2}\frac{\partial^2 \bar{V}}{\partial \bar{F}^2} \bar{F}^2 \tag{7.31}$$

Now the equation only depends on the variable of time. Assume $\bar{V}(\bar{F}, \bar{t})$ is the solution of Equation (7.31), then the solution of Equation (7.28) is:

$$V = e^{-\beta(t)}\bar{V}(\bar{F}, \bar{t}) = e^{-\beta(t)}\bar{V}(Fe^{\beta(t)}, \gamma(t)) \tag{7.32}$$

Make a note that V_{TF} is the solution of the Schwartz (1997) two-factor model for constant spot volatility, then this solution can be written in the form:

$$V_{TF} = e^{-\beta(t)}\bar{V_{TF}}(Fe^{\beta(t)}, v) \tag{7.33}$$

where

$$v = \int \sigma_1^2(t)dt \tag{7.34}$$

$$\sigma_1^2(t) = \sigma_s^2 + B^2(t)\sigma_c^2 + 2\rho_{sc}\sigma_s\sigma_c B(t) \tag{7.35}$$

v can be seemed as the variance of futures price in the Schwartz (1997) two-factor model and σ_1 is the term structure of volatility of futures price in the Schwartz (1997) two-factor model.

Note that σ_s is the constant spot volatility in the Schwartz (1997) two-factor model and $\sigma_s(t)$ is the time dependent volatility of spot volatility in the Schwartz (1997) two-factor model. To make spot volatility in the Schwartz (1997) two-factor model time dependent, it should be made sure that $v = \gamma(t)$. Recall that there is the assumption that

$$\gamma(t) = \int \sigma^2(t)dt \tag{7.36}$$

$$\sigma^2(t) = \sigma_s^2(t) + B^2(t)\sigma_c^2 + 2\rho_{sc}\sigma_s(t)\sigma_c B(t) \tag{7.37}$$

thus one should make sure,

$$v = \int \sigma^2(t)dt \tag{7.38}$$

that is, σ_1^2 should equal to σ^2.

To make this established, the following two equations can be obtained:

$$\sigma_s^2(T-t) = \int_t^T \sigma_s^2(\tau)d\tau \tag{7.39}$$

$$\int_t^T \sigma_s B(\tau)d\tau = \int_t^T \sigma_s(\tau)B(\tau)d\tau \tag{7.40}$$

where

$$B(t) = \frac{e^{-\kappa(T-t)} - 1}{\kappa} \tag{7.41}$$

The result is similar to the result of time dependent volatility in the Black-Scholes model, except that there is one more constraint for the Schwartz (1997) two-factor model.

The formula for a European call futures option with constant spot volatility in the Schwartz (1997) two-factor model was presented by Hilliard and Reis (1998); thus, in case of time dependent spot volatility, the explicit expression for a European call option $C(t, T_1, T)$ should be:

$$C(t, T_1, T) \;\; = \;\; e^{-r(T_1-t)}[F(t,T)N(d_1) - KN(d_2)] \tag{7.42}$$

with

$$d_1 \;\; = \;\; \frac{\ln(F(t,T)/K) + \frac{1}{2}v^2}{v}, \quad d_2 = d_1 - v,$$

$$
\begin{aligned}
v^2(t,T_1,T) &= \int_t^{T_1} \sigma_s^2(\tau)d\tau + 2\rho_{12}\sigma_c\Big(\int_t^{T_1} \sigma_s(\tau)B(\tau)d\tau\Big) \\
&\quad + \frac{\sigma_c^2}{\kappa^2}\Big[(T_1-t) - \frac{2}{\kappa}(e^{-\kappa(T-T_1)} - e^{-\kappa(T-t)}) \\
&\quad + \frac{1}{2\kappa}(e^{-2\kappa(T-T_1)} - e^{-2\kappa(T-t)})\Big].
\end{aligned}
$$

Now one needs to verify that the formula of a European call futures option with time dependent spot volatility satisfies the Equation (7.27); thus, one can have:

$$
\frac{\partial C}{\partial t} = rV + e^{-r(T_1-t)}FN'(d_1)\frac{\partial v}{\partial t} \tag{7.43}
$$

$$
\frac{1}{2}\frac{\partial^2 C}{\partial F^2}F^2\sigma^2 = \frac{1}{2}e^{-r(T_1-t)}N'(d_1)\frac{F\sigma^2}{v} \tag{7.44}
$$

By inserting Equation (7.43) and Equation (7.44) into partial differential Equation (7.27), one can have:

$$
\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\sigma^2}{v} = 0 \tag{7.45}
$$

This equation can be further transferred to:

$$
\frac{1}{2v}\frac{\partial v^2}{\partial t} + \frac{1}{2}\frac{\sigma^2}{v} = 0 \tag{7.46}
$$

that is:

$$
\frac{\partial v^2}{\partial t} + \sigma^2 = 0 \tag{7.47}
$$

Since v^2 is actually the integral of σ^2, it can then be verified that $C(t,T_1,T)$ satisfies the partial differential Equation (7.27). Therefore, this formula can be used to price futures options in the Schwartz (1997) two-factor model with time dependent spot volatility.

7.4 Empirical Study

The pricing formula for futures options in the Schwartz (1997) two-factor model with time dependent spot volatility is tested with market data in this section. The expired time of futures options (T_1) are only one day later than the expired time of futures (T). Hence, for notational convenience, it is assumed that $T_1 = T$ in what follows.

Natural gas futures are chosen as the underlying; specifically, GKZ5, GKG6, GKK6, GKQ6, GKX6 are chosen for the testing, whose expired times are 24/11/2015, 26/01/2016, 26/04/2016, 26/07/2016 and 26/10/2016 respectively. For simplicity, the date 26/10/2015 is assumed as the current date; that is, $t = 0$ in what follows. Therefore, the pricing formula is tested by natural gas futures with maturity times T=1/12, T=1/4, T=1/2, T=3/4, and T=1. The detailed natural gas futures data is presented in the following Table 7.1.

Table 7.1: Natural gas futures prices

This table presents the natural gas futures prices on 26/10/2015.

Symbol	Maturity time	Natural Gas Futures prices
GKZ5	T=1/12	2.353
GKG6	T=1/4	2.534
GKK6	T=1/2	2.483
GKQ6	T=3/4	2.591
GKX6	T=1	2.715

It is known that time dependent spot volatility cannot be directly found in the market, but the market data can be used to find it by the new derived pricing formula. However, although the futures options price (C), futures price (F), interest rate (r) and strike price (K) can be easily obtained from market, the mean reversion of convenience yield (κ), the correlation between instantaneous increment of spot price and instantaneous increment of convenience yield (ρ_{12}), and the volatility of convenience yield (σ_c) still need to be known in order to calculate the time dependent spot volatility ($\sigma_s(t)$) by using the new derived pricing formula. The European call futures options prices with different strike prices for corresponding natural gas futures are showed in Table 7.2 as follows:

Table 7.2: European call natural gas futures options prices with different strike prices

This table presents the European call natural gas futures options prices with different strike prices on at $t = 0$.

	K	1.5	2	2.5	3	3.5
GKZ5C	T=1/12	0.8551	0.3886	0.0957	0.0167	0.0037
GKG6C	T=1/4	1.0469	0.6009	0.2813	0.1172	0.0504
GKK6C	T=1/2	0.9961	0.571	0.2591	0.0972	0.0327
GKQ6C	T=3/4	1.1046	0.6657	0.3302	0.1479	0.0543
GKX6C	T=1	1.2214	0.775	0.4345	0.2207	0.1078

The interest-free rate during this period is 0.7%. As mentioned above, the mean reversion of convenience yield (κ), the correlation between spot price and convenience yield (ρ_{12}), and the volatility of convenience yield (σ_c) need to be estimated firstly. This can be done easily. Specifically, for a certain maturity time, these parameters are calibrated within the standard Schwartz (1997) two-factor model by futures prices and futures options prices with varied strike prices. The strike prices are chosen from 1.5, 2, 2.5, 3, 3.5 so that lsqnonlin can be run in Matlab since lsqnonlin requires that the amount of valid data is not less than the number of parameters estimated. The

Table 7.3: Estimated results of parameters

This table presents the calibrated results of parameters for pricing futures options in the Schwartz (1997) two-factor model.

	σ_s	σ_c	ρ_{12}	κ
T=1/12	0.6135	3.2468	0.3865	2.6223
T=1/4	0.5147	3.0578	0.2912	2.9119
T=1/2	0.374	1.8702	0.4389	3.1299
T=3/4	0.3069	1.5344	0.4629	3.4655
T=1	0.2667	1.6889	0.4311	4.6671

calibrated results of the parameters for pricing futures options in the Schwartz (1997) two-factor model can be seen in Table 7.3.

The value of parameters (σ_c, ρ_{12} and κ) will be used in the pricing formula for futures options. The σ_s shown in Table 7.3 can be seemed as the implied spot volatility in the Schwartz (1997) two-factor model. Theoretically, the implied spot volatility can be used to find the time dependent spot volatility according to Equation (7.39) and Equation (7.40); however, different outcomes will be obtained in general since a continuous set of data points is unavailable. Therefore, the value of implied spot volatility σ_s will not be used to find the value of time dependent spot volatility $\sigma_s(t)$ in this chapter but this value will be used to test the result of time dependent spot volatility in the next section.

Finally, the time dependent spot volatility can be found by the new derived pricing formula with the value of futures prices, futures options prices on market and parameters estimated in Table 7.3. Note that the results of time dependent spot volatility would be different if different strike prices were applied in the pricing formula even at the same maturity time. However, for time dependent spot volatility in this chapter, it should only be dependent on the variable of time; that is, for a certain maturity time, the result of time dependent spot volatility is unique. Therefore, one futures options strike price needs to be chosen that has the most representative significance. In this case, the at-the-money option is the most suitable for this role. Since the value of natural gas futures prices are all around 2.5 for different maturity times, obviously the strike price with 2.5 is chosen to compute the time dependent spot volatility.

Note that since there is only a discrete set of points (5 points in this chapter), it is assumed that time dependent spot volatility $\sigma_s(t)$ is piecewise constant. Moreover, a discrete futures options pricing Equation (7.48) should be used to find the time dependent spot volatility instead of the continuous futures options pricing Equation (7.42).

$$C(t,T) = e^{-r(T-t)}[F(t,T)N(d_1) - KN(d_2)] \tag{7.48}$$

with

$$d_1 = \frac{\ln(F(t,T)/K) + \frac{1}{2}v^2}{v}, \quad d_2 = d_1 - v,$$

$$
\begin{aligned}
v^2(t,T) \quad = \quad & \sum \sigma_s^2(\tau_i)\tau_i + 2\rho_{12}\sigma_c\left(\sum \sigma_s(\tau_i)B(\tau_i)\tau_i\right) \\
& + \frac{\sigma_c^2}{\kappa^2}\left[(T-t) - \frac{2}{\kappa}(1-e^{-\kappa(T-t)})\right. \\
& \left. + \frac{1}{2\kappa}(1-e^{-2\kappa(T-t)})\right].
\end{aligned}
$$

The results of time dependent spot volatility for each maturity time are presented in the following Table 7.4:

Table 7.4: Time dependent spot volatility $\sigma_s(t)$ for futures options

This table shows the value of time dependent spot volatility in the Schwartz (1997) two-factor model for different maturity time.

Maturity time	$\sigma_s(t)$
T=1/12	0.6003
T=1/4	0.4588
T=1/2	0.0804
T=3/4	0.0678
T=1	0.0639

As can be seen, in Table 7.4, the value of time dependent spot volatility is almost equal to the value of implied spot volatility in Table 7.2 for maturity time $T = 1/12$. This makes sense because, theoretically, the value of time dependent spot volatility should equal the value of implied spot volatility at the initial data point under the given assumption. The relationship between implied spot volatility and time dependent spot volatility is shown in Figure 7.1:

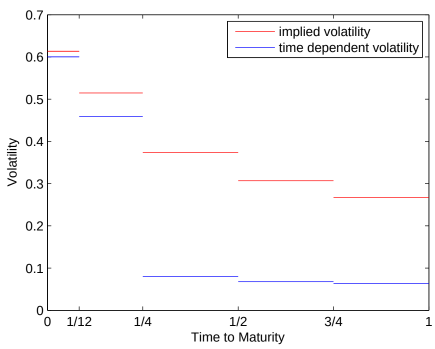

Figure 7.1: Implied spot volatility and time dependent spot volatility

It can be easily found that implied spot volatility and piecewise time dependent spot volatility is consistent in Figure 7.1.

7.5 Test the Result of Time Dependent Spot Volatility

In this section, it will be explained why the result of time dependent spot volatility needs to be tested, and it will be shown how one can make sure of the correctness of the results of time dependent spot volatility in the Schwartz (1997) two-factor model.

There are mainly two reasons that one needs to verify the time dependent spot volatility results. The first reason is that the stability of the value of parameters that are estimated in Table 7.3 cannot be guaranteed absolutely. Firstly, lsqnonlin is used in Matlab to the estimate parameters since it is very quick and only requires a small amount of data. However, the estimated results depend on the initial value given in this program and it is not always easy to find which value is the best initial value in the program. Secondly, there are only a few data points that may decrease the precision of the estimated results. For example, with more data points, it means that there are more conditions to constrain the program; therefore, the estimated results will be more accurate. However, even though the accuracy of the estimated results is at a very high level, the results from these data points may not be enough to describe the situation for the whole period of time.

The second reason is the uncertainty of the method to find the results of time dependent spot volatility when using the new derived pricing formula. It comes from two cases: one is that there could be more than one result obtained, and the other is that no result may be found. In the first case, it can be easily found out which result is correct when one result is positive and the other is negative. However, it is needed to consider carefully before choosing the result of time dependent spot volatility when the two results are both positive or negative. In the second case, there may be no result for time dependent spot volatility that can be found to make the price of futures options from the pricing formula equal to the market price of futures options. If that happens, one has to choose the result of time dependent spot volatility that leads to the value of futures options from the pricing formula that is the closest to the market price. However, this may result in the decrease of accuracy for time dependent spot volatility, since except for the initial data point, the value of time dependent spot volatility for other data points will be affected by the front.

The above mentioned two reasons will have a negative impact on the confidence of the results of time dependent spot volatility. To reduce the fear from the limitations of the method that was used to find the time dependent spot volatility, as to the first reason, one may make the data set much bigger than before and apply the Kalman filter technique or a more advanced econometric method to estimate parameters that will come with their confidence interval. As for the second reason, one may change the criterion to find a closer value that equals the market data.

However, all of these methods will make things complicated and obviously increase the computing time for time dependent spot volatility in the Schwartz (1997) two-factor model. If there could be a way to verify whether the results of time dependent spot volatility by the simple and quick method that we used is correct, then the aforementioned two reasons no longer need to be worried about.

Fortunately, remember that there are two constraints between implied spot volatility and time dependent spot volatility; see Equation (7.39) and Equation (7.40). These two equations can be used as the criterion to analyse whether the results of time dependent spot volatility are correct or not. In terms of discrete time, these two equations can be transferred into Equation (7.49) and Equation (7.50), respectively:

$$\sigma_s^2 T = \sum \sigma_s^2(\tau_i)(\tau_i) \tag{7.49}$$

$$\sigma_s \frac{T - (1 - e^{-\kappa T})}{\kappa} = \sum \sigma_s(\tau_i) \frac{\tau_i - (1 - e^{-\kappa_i \tau_i})}{\kappa_i} \tag{7.50}$$

If the value of implied spot volatility and time dependent spot volatility is substituted into Equations (7.49) and Equation (7.50), generally, one could find that the sum of the implied spot volatility on the left side of the equation is larger than the sum of time dependent spot volatility on the right side of equation, since there will have small errors in time dependent spot volatility. Otherwise, the result of time dependent spot volatility cannot be trusted.

The results of first constraint can be found in the following Table 7.5:

Table 7.5: Test result for first constraint

This table shows the results for both sides of Equation (7.49).

Maturity time	Sum of time dependent volatility	Sum of implied volatility
T=1/12	0.030030	0.031365
T=1/4	0.065113	0.066229
T=1/2	0.066729	0.069938
T=3/4	0.067878	0.070641
T=1	0.068899	0.071129

It can be clearly seen that the sum of implied spot volatilities is larger than the sum of time dependent spot volatilities for futures options at different time maturities. It can also be found that the sum of implied spot volatility and the sum of time dependent spot volatility that is consistent.

A similar conclusion can also be found for the second constraint; the calculating result for both sides of Equation (7.50) is shown in Table 7.6.

Combined with conclusions from both the first constraint and the second constraint, one can deduce that the results of time dependent spot volatility that come from the new derived pricing formula with market data are correct.

Table 7.6: Test result for second constraint

This table shows the results for both sides of Equation (7.50).

Maturity time	Sum of time dependent volatility	Sum of implied volatility
T=1/12	0.005089	0.0052
T=1/4	0.020974	0.037271
T=1/2	0.027132	0.092493
T=3/4	0.032744	0.1482
T=1	0.039291	0.210092

7.6 Conclusion

Pricing options with constant volatility are really outdated nowadays. In this chapter, it has been shown that the Schwartz (1997) two-factor model is still valid when the spot price volatility is time dependent. The results of time dependent spot volatility can be calculated by a simple and quick method with the new derived pricing formula. Although one cannot tell how reliable the result of time dependent spot volatility is because of the limitations of the method used, those criteria can be used to make sure whether the result of time dependent spot volatility is correct or not. Although the results of time dependent spot volatility cannot be used for prediction, they can still be applied in pricing and hedging exotic contracts in futures and commodities markets. Furthermore, the time dependent spot volatility that is found in exchange can also be used as a benchmark for other consistent futures and commodities that are traded on the OTC market.

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Chapter 8

Conclusion

This Ph.D. thesis is composed of four essays that address pricing derivatives with stochastic volatility. In particular, accurate approximated results that can be solved analytically for arithmetic Asian options, futures and futures options are presented.

First, the pricing of arithmetic Asian options with the Heston model, the CEV model, and the Schwartz (1997) two-factor model was investigated. These models can all provide closed-form solutions for the plain vanilla European call options, hence, are suitable for Albrecher et al. (2005)'s comonotonicity approach. This approach has, so far, mainly been applied to Lévy type jump models. It was shown that it can also provide a simple, quick, and accurate result for the valuation of an arithmetic Asian option in the above mentioned models. The distribution function is also needed for applying the comonotonicity approach. In this case, the CEV model and the Schwartz (1997) two-factor model have the advantage that the distribution function of the stock price is available in explicit form, while the distribution function in the Heston model needs to be computed via the Monte Carlo method. For the Heston model, the CEV model, and the Schwartz (1997) two-factor model, the results showed that prices for in-the-money arithmetic Asian options are more accurate than prices for out-of-the-money arithmetic Asian option when following the comonotonicity approach. Furthermore, an optimization method for pricing arithmetic Asian options in these three models was also provided. The results provide a further support for the comonotonicity approach in all three cases: the Heston model, the CEV model, and the Schwartz (1997) two-factor model.

Following the study of pricing arithmetic Asian options in the Schwartz (1997) two-factor model, there was a focus on the pricing of futures with stochastic spot volatility. Two three-factor models for pricing futures contracts with closed-form solutions were derived, which include the spot price factor, instantaneous convenience yield factor, and stochastic spot volatility factor. These factors are all assumed to follow mean reversion processes. In comparison to Model 1, Model 2 is simpler and can give an effective result for the futures contract price in practice. Also tested were Model 2 and the Schwartz (1997) two-factor model with market gold futures data. The result shows that both of these models can provide a good result. Moreover, comparing the result of the Schwartz (1997) two-factor model, it can be found that for most cases, when the market price is overestimated in the Schwartz (1997) two-factor model, it is underestimated in Model 2, and vice versa. Hence, one can find a very accurate price by using the Schwartz (1997) two-factor model and Model 2.

After the pricing of futures contracts, the pricing of futures options was discussed. A closed-form solution with the asymptotic method for pricing futures options in the context of the Schwartz (1997) two-factor model was presented. The closed-form solution can give a very accurate approximation for the price of futures options in both simulated data and market natural gas futures options data. It was shown that the computational accuracy of futures options is improved by using this closed-form solution, especially for long time maturity contract, compared with the results of the standard Schwartz (1997) two-factor model.

Finally, time dependent spot volatility in futures options was studied. A closed-form solution for the value of European style futures options in the Schwartz (1997) two-factor model with time dependent spot price volatility was derived. These analytical expressions can give a result for time dependent spot volatility with the market options price by a simple and quick method. Through theoretical analysis and numerical examples, it was shown how to make sure that the time dependent spot volatility derived from this new pricing formula is correct.

Several directions may be worthy of further pursuit. Firstly, the value of parameters estimated in this thesis's model were either chosen from previous studies or by a mathematic algorithm with a small amount of data, which may not appropriately suit for the extended models or may not consider the real economic situation. Even in those parameters estimated by the Kalman filter or extended Kalman filter with large data, the variance of the value of the parameters was not given. Hence, future study on finding the most reliable way to estimate the parameters is an important issue. Secondly, since the pricing formulas used in this thesis for futures and futures options have closed-form solutions, one could extend the pricing purpose to a hedging purpose, which may be more meaningful in practice. Thirdly, volatility derivatives (e.g., VIX and VIX derivatives) have developed quickly in the past few years and will certainly be popular in the future. Therefore, the study of time dependent volatility on futures options could also be developed to local volatility by considering various strike prices, which can more truly reflect the market view of future's volatility.

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