Essays in Monetary Economics

by

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Abstract

This dissertation can be thematically grouped into two categories: monetary theory in the so called New Monetarist search models where money and credit are essential in terms of improving social welfare, and optimal time-consistent monetary and fiscal policy in New Keynesian dynamic stochastic general equilibrium (DSGE) models when the government cannot commit. Arguably, the methodology and conceptual frameworks adopted in these two lines of work are quite different. However, they share a common goal in helping us understand how and why monetary factors can affect the real economy, and how monetary and fiscal policy should respond to developments in the economy to improve social welfare. There are two chapters in each part. In the first chapter, recent advances based on the pre-eminent Lagos-Wright (LW) monetary search model are reviewed. Against this background, chapter two introduces collateralized credit inspired by a communal responsibility system into the creditless LW model, in order to study the role of money and credit as alternative means of payment. In contrast, the third chapter revisits the classic inflation bias problem associated with optimal time-consistent monetary policy in the cashless New Keynesian framework. In this chapter, fiscal policy is trivial, due to the assumption of lump-sum tax. As a follow-up work, chapter four studies optimal time-consistent monetary and fiscal policy mix as well as debt maturity choice in an environment with only distortionary taxes, endogenous government spending and government debt of various maturities.

Chapter 1 introduces the tractable and influential Lagos-Wright (LW) search-theoretic framework and reviews the latest developments in extending it to study issues concerning the role of money, credit, asset pricing, monetary policy and economic growth. In addition, potential research topics are discussed. Our main message from this review is that the LW monetary model is flexible enough to deal with numerous issues where fiat money plays an essential role as a medium of exchange.

Chapter 2, based on the LW framework, develops a search model of money and credit motivated by a historical medieval institution - the community responsibility system. The aim is to examine the role of credit collateralized by the community responsibility system as a supplementary medium of exchange in long-distance trade, assuming that

entry cost and the cost of using credit are proportional to distance, due to factors like direct verification and settlement cost and indirect transportation cost. We find that both money and credit are useful in the sense of improving welfare. In addition, the Friedman rule can be sub-optimal in this economy, due to the interaction between the extensive margin (that is, the range of outside villages which the representative household has trade with) and the intensive margin (that is, the scope of villages where credit is used as a supplementary medium of exchange). Finally, higher entry cost narrows down the extensive margin, and similarly, higher cost of using credit, ceteris paribus, reduces the usage of credit and hence lowers social welfare.

Chapter 3 reconsiders the inflation bias problem associated with the renowned rules versus discretion debate in a fully nonlinear version of the benchmark New Keynesian DSGE model. We ask whether the inflation bias problem related to discretionary monetary policy differs quantitatively under two dominant forms of nominal rigidities - Calvo pricing and Rotemberg pricing, if the inherent nonlinearities are taken seriously. We find that the inflation bias problem under Calvo contracts is significantly greater than under Rotemberg pricing, despite the fact that the former typically exhibits far greater welfare costs of inflation. In addition, the rates of inflation observed under the discretionary policy are non-trivial and suggest that the model can comfortably generate the rates of inflation at which the problematic issues highlighted in the trend inflation literature. Finally, we consider the response to cost push shocks across both models and find these can also be significantly different. Thus, we conclude that the nonlinearities inherent in the New Keynesian DSGE model are empirically relevant and the form of nominal inertia adopted is not innocuous.

Chapter 4 studies the optimal time-consistent monetary and fiscal policy when surprise inflation (or deflation) is costly, taxation is distortionary, and non-state-contingent nominal debt of various maturities exists. In particular, we study whether and how the change in nominal government debt maturity affects optimal policy mix and equilibrium outcomes, in the presence of distortionary taxes and sticky prices. We solve the fully nonlinear model using global solution techniques, and find that debt maturity has drastic effects on optimal time-consistent policies in New Keynesian models. In particular, some interesting nonlinear effects are uncovered. Firstly, the equilibrium value for debt is negative and close to zero, which implies a slight undershooting of the inflation target in steady state. Secondly, starting from high level of debt-GDP ratio, the optimal policy will gradually reduce the level of debt, but with radical changes in the policy mix along the transition path. At high debt levels, there is a reliance on a relaxation of monetary policy to reduce debt through an expansion in the tax base and reduced debt service costs, while tax rates are used to moderate the increases in inflation. However, as debt levels fall, the use of monetary policy in this way is diminished and the policy maker turns to fiscal policy to continue the reduction in debt. This is akin to a switch from an active to passive fiscal policy in rule based descriptions of policy, which occurs endogenously under the optimal policy as debt levels fall. It can also be accompanied by a switch from passive to active monetary policy. This switch in the policy mix occurs at higher debt levels, the longer the average maturity of government debt. This is largely because high debt levels induce an inflationary bias problem, as policy makers face the temptation to use surprise inflation to erode the real value of that debt. This temptation is then more acute when debt is of shorter maturity, since the inflationary effects of raising taxes to reduce debt become increasingly costly as debt levels rise. Finally, in contrast to the Ramsey literature with real bonds, in the current setting we find no extreme portfolios of short and long-term debt. In addition, optimal debt maturity, implicitly, lengthens with the level of debt.

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"There is only one difference between a bad economist and a good one: the bad economist confines himself to the visible effect; the good economist takes into account both the effect that can be seen and those effects that must be foreseen."

Frederic Bastiat, Selected Essays on Political Economy

"The fact that economics is not physics does not mean that we should not aim to apply the same fundamental standards for what constitutes legitimate argument; we can insist that the ultimate criterion for judging economic ideas is the degree to which they help us order and summarize data, that it is not legitimate to try to protect attractive theories from the data."

Christopher A. Sims, 1996. "Macroeconomics and Methodology", Journal of Economic Perspectives, 10(1), p. 111.

"The era of closed-form solutions for their own sake should be over. Newer generations get similar intuitions from computer-generated examples than from functional expressions."

Jose-Victor Rios-Rull, JME (2008)

Declaration

I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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Preface

The four chapters in this thesis employ two distinct theoretical frameworks to study a pure theory of money and credit, and optimal monetary and fiscal policy, respectively. The first two chapters - more qualitative and with strong micro-foundations for money and credit- delve into the so-called New Monetarist economics, while the remaining two chapters - closer to policy practices - contribute to the optimal policy literature in the mainstream New Keynesian dynamic stochastic general equilibrium (DSGE) models. Admittedly, these two lines of work advocate quite different methodologies and conceptual frameworks. However, both are helpful for us to understand the role of money and credit in the real economy, and how monetary and fiscal policy should stabilize business cycles in order to improve social welfare.

There has been a quest to understand aggregate economic phenomena in terms of the behaviour of individual economic entities and their interactions, since the 1970s onwards, and in particular, following the publication of the "Lucas critique" (Lucas, 1976). As a response to this criticism, building macroeconomic models with solid micro-foundations has become the dominant research program, which involves formulating, solving and estimating models with parameters that are independent of the policy regime, so that they can be used for evaluating alternative policies. Against this background, there also has been a continuing effort to seek sound micro-foundations for money and credit, for instance, see the prominent conference volume in Kareken and Wallace (1980). Why would intrinsically worthless fiat money have value? How can fiat money and credit improve the efficiency of resource allocations? Why is money dominated in the rate of return by other assets and, in particular, by government issued nominal bonds? Classic questions like these can not be answered via ad hoc monetary models like putting money in the utility function or arbitrarily assuming a "cash in advance" constraint, since money in these models is not essential. Essentiality of money means that it improves the efficiency of resource allocations relative to an economy without money, see Kocherlakota (1998) and Wallace (2001). Search models of money, initiated by Kiyotaki and Wright (1989), provide such micro-foundations for monetary economics that endogenizes the value and essentiality of fiat money by explicitly specifying the frictions that impede the functioning of markets. Recently, this distinctive and extensive literature on monetary theory and policy, on banking, financial intermediation, payments, and on asset markets has been termed New Monetarist Economics, see Williamson and Wright (2010a) and Williamson and Wright (2010b) for surveys, and Nosal and Rocheteau (2011) for a textbook exposition.

Chapter 1, as a complement to the above-mentioned surveys, highlights the influential Lagos and Wright (2005) (LW) framework and reviews the latest developments in extending it to study issues concerning the role of money, credit, asset pricing, monetary policy and economic growth. Two useful assumptions - an alternating frictional decentralized market and a frictionless Walrasian centralized market within each period, and quasi-linear preferences - make the LW model tractable. This desirable feature renders it amenable to policy analysis. Along with an overview of the literature, we also provide detailed discussions about possible future research topics. Our main message from this comprehensive review is that the LW search-theoretic model is flexible enough to deal with numerous issues where fiat money plays an essential role as a medium of exchange.

The hallmark of New Monetarist models is to explicitly deal with frictions in the exchange process. Random search and bilateral matching is a natural way to generate a double coincidence problem, and to motivate incomplete record keeping, limited commitment and other frictions that make monetary exchange socially useful. Unfortunately, it is these frictions which render money essential that make credit arrangements impossible in standard search-theoretic models. As a result, a growing number of studies aim to solve this dilemma, and in general to clarify the relationship among money, credit and banking.

Chapter 2 makes a theoretical contribution in this direction. Inspired by the historical narrative of an interesting medieval institution - the community responsibility system in Greif (2006), we develop a search model of money and credit based on the LW framework. Under such a scheme, a local, community court held all members of a different commune legally liable for default by any one involved in contracts with a member of the local community. If the defaulter's communal court refused to compensate the injured party, the local court confiscated the property of any member of the defaulter's commune present in its jurisdiction as compensation. This institutional innovation is a credible commitment technology, if trade links between two communes are sufficiently strong. A commune could avoid compensating for the default of one of its members only by ceasing to trade with the other commune. When this cost was too high, a commune court's best response was to dispense impartial justice to nonmembers who had been cheated by a member of the commune. With this historical story in mind, we aim to examine the role of credit collateralized by the community responsibility system as a supplementary medium of exchange in long-distance trade, assuming that entry cost and the cost of using credit are proportional to distance, due to factors like direct verification and settlement cost and indirect transportation cost. We find that both money and credit are useful in the sense of improving welfare, and that the Friedman rule can be sub-optimal. In addition, higher entry cost narrows down the range of villages which have trade with the representative household. Similarly, higher cost of using credit, ceteris paribus, reduces the usage of credit and hence lowers social welfare.

Chapter 3 revisits the inflation bias problem associated with the renowned rules versus discretion debate initiated by Kydland and Prescott (1977) and Barro and Gordon (1983) in a fully nonlinear New Keynesian DSGE model. We ask whether the inflation bias problem related to discretionary monetary policy differs quantitatively under two workhorse models of sticky prices due to Calvo (1983) and Rotemberg (1982), respectively, if the inherent nonlinearities are taken seriously. This is an important consideration, since these two forms of nominal inertia are commonly used to give monetary policy a meaningful role in New Keynesian DSGE models. Moreover, recent empirical work suggests the discretion offers more data coherent description of policymaking than commitment. Given the literature on trend inflation shows that there are potentially significant nonlinearities in New Keynesian DSGE models, especially when steady state inflation rate is nonzero, it is necessary to assess the extent to which the inflationary bias problem affects the equilibrium and properties of a New Keynesian economy that is not subject to a linearized approximation.

The underlying reason for the inflation bias and time consistency problem in general is that policymakers are unable to make credible commitments regarding future policies. As an example, assume that the objective of the monetary authority is low inflation and that it announces such a policy. If households and firms believe this policy announcement, then inflationary expectations are low and therefore small wage increases will be demanded. In retrospect, however, the monetary authority may be tempted to conduct a more inflationary monetary policy via setting low interest rates, as this would reduce unemployment in the short run. If workers understand the policymakers' motives, the announcement of low inflation loses its credibility, and rational employees will ask a positive growth rate of wages to avoid losses from inflation. In equilibrium the monetary authority is not able to affect unemployment, but there is a positive rate of inflation. This outcome is inefficient since by convincingly committing not to inflate in advance the monetary authority could achieve the same level of unemployment but with zero inflation. Therefore, the lack of commitment by the monetary authority will lead to an inefficiently high level of inflation.

In this chapter, we assume the monetary authority makes discretionary decisions sequentially, and hence can not commit to a plan in the hope of influencing economic agents' expectations. The steady state output is inefficient, due to monopolistic distortion. Hence, the discretionary government has incentive to engineer some (ex ante) unexpected inflation to make output closer to the efficient level, even though (ex post) realized inflation is costly. Unexpected inflation raises output because of sticky prices, and it reduces the monopoly distortion. In a fully micro-founded model, we then numerically solve for the resulting fully nonlinear time consistent optimal policy using powerful projection methods, as in Anderson et al. (2010). This is the point of departure from the extant linear quadratic (LQ) literature where nonlinearities are either not adequately captured, or even ignored. We find that the inflation bias problem under Calvo contracts is significantly greater than under Rotemberg pricing, despite the fact that the former typically exhibits far greater welfare costs of inflation. The rates of inflation observed under the discretionary policy are non-trivial and suggest that the model can comfortably generate the rates of inflation at which the problematic issues highlighted in the trend inflation literature emerge, as well as the movements in trend inflation emphasized in empirical studies of the evolution of inflation, see Ascari and Sbordone (2014) for a survey. Finally, we consider the response to cost push shocks across both models and find these can also be significantly different. Thus, we conclude that the nonlinearities inherent in the New Keynesian DSGE model are empirically relevant and the form of nominal inertia adopted is not innocuous.

Chapter 4 makes a contribution to the literature which combines the New Keynesian paradigm of optimal monetary policy with the Neoclassical paradigm of optimal fiscal policy. It studies the optimal time-consistent policy problem when surprise inflation (or deflation) is costly, taxation is distortionary, and non-state-contingent nominal debt of various maturities exists. Schmitt-Grohe and Uribe (2004) show that in a New Keynesian model with one-period government debt, even a mild degree of price stickiness implies nearly constant inflation and near random walk behaviour in government debt and tax rates, in response to government spending disturbances. In other words, monetary policy should not be used to stabilize debt. However, Sims (2013) questions the robustness of this result when government can issue long-term nominal bonds. When government debt is short term, inflation or deflation is the only way to change its market value in cushioning fiscal shocks. In contrast, if debt is long term, large changes in the value of debt can be produced by sustained changes in the nominal interest rate (or bond price), with much smaller changes in current inflation. Based on these considerations, Sims sketches out a theoretical argument for using nominal debt - of which the real value can be altered with surprise changes in inflation and interest rates - as a cushion against fiscal disturbances to substitute for large movements in distorting taxes. Both papers assume commitment, that is, the social planner's promises are credible.

We develop a New Keynesian DSGE model augmented with fiscal policy and a portfolio of mixed maturity bonds and solve the optimal time-consistent policy problem using global non-linear solution techniques. In particular, we study how the change in nominal government debt maturity affects optimal monetary and fiscal policy decisions in stabilizing business cycles and equilibrium outcomes in the presence of distortionary taxes and sticky prices. Leeper and Zhou (2013) ask some similar questions and they solve the optimal monetary and fiscal policies in a LQ model from the timeless perspective. In addition, Bhattarai et al. (2014), and Burgert and Schmidt (2014) study optimal time consistent monetary and fiscal policies as well, but the former employ the LQ method and the latter examine only one-period debt. In contrast, we consider government debt of various durations and apply global solution methods to accurately deal with the inherent nonlinearities in New Keynesian models.

In the model, both fiscal and monetary policy are useful stabilization tools, thanks to the combination of incomplete insurance and price stickiness. Consider a negative productivity shock. Lack of complete markets (and lump-sum tax) implies that lowering the current tax rate necessarily results in a primary deficit that has to be financed by an increase in public debt and thus an increase in future taxes which is distortionary. The government thus finds it optimal to decrease the tax rate by less than what it would have done under complete markets to offset the negative shock. That is, fiscal policy alone can not fully deal with the shock. As a result, it becomes optimal to also use monetary policy for the purpose of output stabilization. In fact, an unexpected increase in inflation not only stimulates aggregate demand but also lowers the real value of nominal debt and thus minimizes the need to vary distortionary income taxes over the business cycle. However, monetary policy cannot do it all either, since inflation surprises are costly, due to nominal rigidities. That is, it is optimal for the government to raise inflation by an amount less than what would be necessary to fully stabilize output and cover the primary deficit as with flexible prices. In addition, we argue that the average maturity of government debt affects how the government optimally makes this tradeoff of choosing the inflation path.

We find the following key results. Firstly, the steady-state balances an inflation and debt stabilization bias to generate a small negative long-run optimal value for debt, which implies a slight undershooting of the inflation target in steady state. Secondly, starting from levels of debt consistent with currently observed debt-GDP ratios, the optimal policy will gradually reduce that debt, but with radical changes in the policy mix along the transition path. At high debt levels, there is a reliance on a relaxation of monetary policy to reduce debt through an expansion in the tax base and reduced debt service costs, while tax rates are used to moderate the increases in inflation. However, as debt levels fall, the use of monetary policy in this way is diminished and the policy maker turns to fiscal policy to continue the reduction in debt. This is akin to a switch from an active to passive fiscal policy in rule based descriptions of policy, which occurs endogenously under the optimal policy as debt levels fall. It can also be accompanied by a switch from passive to active monetary policy. This switch in the policy mix occurs at higher debt levels, the longer the average maturity of government debt. This is largely because high debt levels induce an inflationary bias problem as policy makers face the temptation to use surprise inflation to erode the real value of that debt. This temptation is then more acute when debt is of shorter maturity since the inflationary effects of raising taxes to reduce debt become increasingly costly as debt levels rise. Finally, in contrast to the Ramsey literature with real bonds, in current setting we find no extreme portfolios of short and long-term debt. In addition, optimal debt maturity, implicitly, lengthens with the level of debt.

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1 Search Models of Money: Recent Advances

In this chapter, we focus on the Lagos and Wright (2005) monetary search model - a workhorse in the so-called New Monetarist Economics. After describing this microfounded yet tractable framework, we review recent developments in extending it to study a variety of issues from monetary theory to policy analysis. The topics include the role of money, credit and financial intermediation in facilitating the exchange process in decentralized economies which are impeded by explicitly specified frictions, the implication of liquidity for asset pricing in monetary environments, monetary policy analysis when money is essential, the welfare costs of inflation, and the interaction between money and capital accumulation in economic growth models. Besides mapping out the literature, we also provide some suggestions on how to apply the benchmark model in possible ways to generate new insights about classic topics or deal with new issues in the light of the recent financial crisis.

1.1 Introduction

As a microfoundation of money, the search theory of money gained its momentum in 1980s (e.g., Kiyotaki and Wright, 1989), and has been moving forward rapidly since then. Recently, this distinctive and extensive literature on monetary theory and policy, on banking, financial intermediation, payments, and on asset markets has been termed New Monetarist Economics (Lagos et al., 2014; Williamson and Wright, 2010a,b). Here we do not attempt to review this voluminous literature. Instead, we will focus on the work based on the Lagos and Wright (2005) model. Hence, this chapter serves as a complement to the above-mentioned surveys which are more about methodology and the evolution of the whole literature. The influential Lagos-Wright framework highlights two defining features, that is, alternating frictional decentralized market and frictionless Walrasian centralized market within each period, and quasi-linear preferences. The competitive market in the second sub-period allows agents to readjust their money holdings, while quasi-linear utility function eliminates wealth effects so that the choice of money holdings of an agent is independent of his idiosyncratic trading history. This ingenious modelling strategy makes the distribution of money holdings at the beginning of each period degenerate, and hence keeps the model tractable and user-friendly for policy analysis.

Hence, the Lagos-Wright framework has emerged as a workhorse in modern monetary economics, given its ability to address the divisibility of money and goods simultaneously. In the first generation search models of money, represented by Kiyotaki and Wright (1993, 1989), both money and goods are indivisible for tractability. In addition, it is typically assumed that agents can only hold one object at a time, and only agents without money are willing to produce. The terms of trade are exogenously determined, that is, one unit of money buys one unit of commodity. Hence monetary policy cannot be meaningfully discussed under these restrictive assumptions. Shi (1995) and Trejos and Wright (1995) in the second generation models endogenize prices by allowing divisible goods, though indivisible money is still necessary for analytic results. It is the endogenous distribution of money holdings complicates the analysis in previous generations of monetary search models. Since there is random matching in these models, this generates idiosyncratic uncertainty concerning consumption and production opportunities, and therefore the equilibrium distribution of money holdings across agents is typically non-degenerate. Shi (1997) and Lagos and Wright (2005), as two seminal third generation models, directly tackle the troublesome non-degenerate distribution of money holdings.[1] As pointed above, the latter use quasi-linear preferences and periodic access to a centralized market. In contrast, the former employs a large household structure. More specifically, each household consists of a continuum of members who follow the family's instruction to trade at decentralized markets, share consumption, and aim to maximize household utility rather than individual utility. At the end of each period, the members of the same family pool their money holdings. By the law of large numbers, this eliminates match-specific risks within each household. As a result, the distribution of money holdings is degenerate across households in the symmetric monetary equilibrium.[2]

After describing the benchmark setup in section 2, we then arrange this chapter by various topics and issues. Hence, we divide the papers reviewed into six categories, with reference to the review work by Shi (2006), Williamson and Wright (2010a,b), Lagos et al. (2014), and the recent book by Nosal and Rocheteau (2011). However,

[1] Alternatively, we can also use advanced numerical methods to deal with the non-degenerate distribution of money, as Molico (2006), Chiua and Molico (2014) and Rocheteau et al. (2015) do.

[2] Berentsen and Rocheteau (2002) and Lagos and Wright (2004) have detailed discussions about these two distinct methods.

the distinctions among these categories are not always so clear. After all, they are all intellectual fruits growing from the same tree, that is, the Lagos-Wright model. For simplicity, hereafter LW refers to Lagos-Wright when appropriate. In section 3, we review papers studying the existence and robustness of monetary equilibrium in variants of the LW model. In Section 4, we survey models where fiat money is valued even though other real or nominal assets are available as well. A key theme is the so-called rate-of-return dominance puzzle. Section 5 evaluates work aiming to introduce various forms of credit into modified LW monetary models. The common goal is build models with both money and credit as observed in real economies so that we can deal with issues where monetary and financial frictions matter. The recent financial crisis reminds us that this is a worthwhile intellectual investment. Section 6 scrutinizes contributions which investigate the role of liquidity in determining asset prices within the context of monetary economies and how monetary policy affects asset prices. Extended LW models are used to study business cycle issues like monetary transmission mechanism in section 7. The shared objective is to take monetary search models to data and conduct quantitative analysis like the typical practice in real business cycle (RBC) models or New Keynesian dynamic stochastic general equilibrium (DSGE) models. In section 8, we report progress in integrating the LW monetary search model with neoclassical or endogenous growth models. This line of work is still quite thin. We conclude in section 9.

1.2 The Underlying Model

The following environment description is in large part based on Lagos and Wright (2005). In particular, we follow the convention of notation implemented in this paper, given the fact that most papers more or less keep the same symbols.

1.2.1 The Basic Environment

Time is discrete and indexed by $t \geq 1$. There is a $[0, 1]$ continuum of ex ante identical and infinitely lived agents. Each period is divided into two sub-periods, called day and night, during which different activities happen. Agents discount between periods with the discount factor $\beta \in (0, 1)$, but not between the two sub-periods within a period. During the day, agents search and trade in a decentralized market or DM for short according to time-consuming and anonymous bilateral matching. The probability of a meeting is α. Each agent specializes in production and can turn labour one-for-one into goods, which are produced in many varieties and henceforth are called DM goods or special goods. To motivate trade, we require agents do not consume goods made by themselves. Clearly, for any two randomly drawn agents i and j, there are four possible events. With probability δ, a double-coincidence of wants happens, that is, both consume what the other can produce. With probability σ, i desires what j produces but not vice versa, which corresponds to the case of a single coincidence of wants. Symmetrically, with probability σ agent j wants what i produces but not vice versa. The remaining case that neither desires what the other produces takes place with probability$1-2\sigma-\delta$. For convenience, we will label i the buyer and j the seller in a single coincidence meeting, if the former wants the DM good the latter produces.

At night agents interact in a Walrasian centralized market or CM, where the problem of double-coincidence of wants does not arise. As a result, we can reasonably assume that at night all agents produce and consume a general good or CM good. Similarly, the technology of producing the CM good is linear. By implication, the real wage in the CM is $w = 1$. Both the CM goods and the DM goods are perfectly divisible and non-storable.[3] The non-storability of both types of goods precludes the emergence of commodity money. There is another intrinsically useless, perfectly divisible and storable object, called money, which is supplied by the government. The two main frictions in the DM, double-coincidence problem and anonymity, make money as a medium of exchange essential, since credit is infeasible.[4] Here money is essential in the sense that it can support desirable allocations which are unattainable in its absence (see Kocherlakota, 1998; Wallace, 2001, for detailed discussions).

Agents make decisions in consuming and supplying labour in both sub-periods. Let $(x,\, h)$ and $(X,\, H)$ represent consumption and labour pairs during the day and night, respectively. Then the period utility function is

$$\mathcal{U}(x, h, X, H) = u(x) - c(h) + U(X) - H, \tag{1.1}$$

where u, c and U are twice continuously differentiable with $u' > 0$, $c' > 0$, $U' > 0$, $u'' < 0$, $c'' > 0$, and $U'' \leq 0$. In addition, $u(0) = c(0) = 0$, $u'(q^*) = c'(q^*)$ for some $q^* \in (0, \infty)$, and for some X^* such that $U'(X^*) = 1$ with $U'(X^*) > X^*$. Note that $\mathcal{U}$ is linear in H.[5] Also note that q^* is the optimal amount of output per trade chosen by a central planner. This modelling strategy renders the distribution of money holdings degenerate and allows us to characterize equilibrium tractably. The government controls the money

[3] Non-storable means that the goods cannot be stored from one sub-period to the next.

[4] The presence of search friction $\alpha \in [0, 1]$ has nothing to do with the essentiality of fiat money. Limited commitment and anonymity or imperfect record keeping are the key frictions to make money have exchange value.

[5] Wong (2014) shows that degenerate distribution of money can be obtained for a larger class of preference specifications, with quasi-linear utility as a special case.

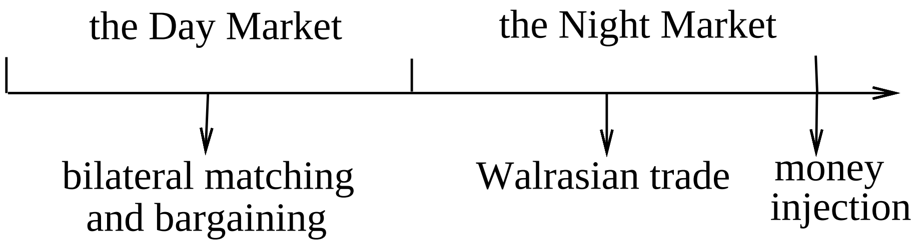

Figure 1.1: The timing of events during period t in the Lagos-Wright model.

supply so that $M_{t+1} = (1+\tau)M_t$, with τ constant, via lump-sum monetary transfers at the end of the CM sub-period. The timing of events is illustrated in Figure 1.1.

1.2.2 Decisions and Equilibrium

Agents in the CM and DM maximize their expected discounted utility net of production costs. Let $F_t(m)$ denote the distribution of money holdings across agents, then we have $\int m_t \,\mathrm{d}F_t(m) = M_t$, which is the total amount of money at time t. In addition, let $V_t(m)$ be the value function or discounted life time utility for an agent with m dollars when entering the DM, and $W_t(m)$ the value function entering the CM. Note that m is an individual state variable, while the distribution F is an aggregate state variable. In a single-coincidence meeting, let $q_t(m, \tilde{m})$ denote the amount of goods and $d_t(m, \tilde{m})$ be the amount of money exchanged, where m and $\tilde{m}$ are the money holdings of the buyer and the seller, respectively. In a double-coincidence meeting, $B_t(m, \tilde{m})$ denotes the payoff for an agent holding m who meets another agent with $\tilde{m}$. Then the Bellman equation for an ex ante identical agent satisfies

$$\begin{aligned} V_t(m) = \alpha\sigma &\int \{u[q_t(m,\widetilde{m})] + W_t[m - d_t(m,\widetilde{m})]\} dF_t(\widetilde{m}) \\ &+ \alpha\sigma \int \{-c[q_t(\widetilde{m},m)] + W_t[m + d_t(\widetilde{m},m)]\} dF_t(\widetilde{m}) \qquad (1.2) \\ &+ \alpha\delta \int \ B_t(m,\widetilde{m}) dF_t(\widetilde{m}) + (1 - 2\alpha\sigma - \alpha\delta) W_t(m), \end{aligned}$$

where the first term is the expected payoff from buying in a single-coincidence match; the second is the expected gain from selling in a single-coincidence meeting; the third is the expected payoff from a double-coincidence match; and the last term is the expected value of not trading in the day market and going to the CM with m. In the CM, an agent solves

$$W_t(m) \ = \ \max_{X,H,m'} \{U(X) - H + \beta V_{t+1}(m' + \tau M)\} \qquad (1.3)$$

subject to

$$X = H + \phi_t m - \phi_t m',$$

$$X \geq 0, \quad 0 \leq H \leq \bar{H}, \quad \text{and } m' \geq 0,$$

where ϕ_t is the price of money in the CM, $\bar{H}$ is an upper bound on labour hours, and m' is money left over after trading. Note that $m_{t+1} = m'_t + \tau M_t$ dollars are taken into the next period by the agent.

In bilateral trading, the generalized Nash bargaining is a natural solution concept to characterize the terms of trade in the decentralized market. For double-coincidence transactions, it can be shown that no money is swapped for goods (see Lagos and Wright, 2004), and matched pairs give each other q^* with $u'(q^*) = c'(q^*)$. Therefore, $B_t(m, \tilde{m}) = u(q^*) - c(q^*) + W_t(m)$. In single-coincidence meetings, the terms of trade (q, d) solves

$$\max_{q,d} [u(q) + W_t(m - d) - W_t(m)]^{\theta} [-c(q) + W_t(\tilde{m} - d) - W_t(\tilde{m})]^{1-\theta} \tag{1.4}$$

subject to

$$d \leq m \text{ and } q \geq 0,$$

where $\theta \in (0, 1]$ is the bargaining power of the buyer, m and $\tilde{m}$ the buyer's and the seller's money holdings.

In the following, we outline the four key steps to solve for the monetary equilibrium (see Nosal and Rocheteau, 2011, for a textbook treatment). First, characterize some desirable properties of the optimization problem in the CM. Note that (1.3) can be written as

$$W_t(m) = \phi_t m + \max_{X, m'} \{U(X) - X - \phi_t m' + \beta V_{t+1}(m' + \tau M)\}, \tag{1.5}$$

Hence, $X_t(m) = X^*$ with $U'(X^*) = 1$, the decision variable $m'(m)$ is independent of m, and the continuation value $W_t(m)$ is linear in m with slope ϕ_t. This implies that the distribution of money $F_t(m)$ is degenerate. In addition, the lump-sum monetary transfer is evenly distributed to every agent. Hence, each agent takes the same amount of money out of the CM and enters with that money into the next round of transactions.

The second step is to determine the terms of trade in the DM. Given the linearity of W_t in m, the Nash bargaining problem (1.4) simplifies to

$$\max_{q,d} [u(q) - \phi_t d]^{\theta} [-c(q) + \phi_t d]^{1-\theta} \tag{1.6}$$

subject to

$$d \leq m \text{ and } q \geq 0.$$

It is easy to show that the solution to (1.6) is

$$q_t(m,\tilde{m}) = \begin{cases} \hat{q}_t(m) & \text{if } m < m_t^* \\ q^* & \text{if } m \geq m_t^*, \end{cases} \quad d_t(m,\tilde{m}) = \begin{cases} m & \text{if } m < m_t^* \\ m^* & \text{if } m \geq m_t^*, \end{cases} \tag{1.7}$$

where $\hat{q}_t(m)$ is the solution to $\phi_t m = z(q_t, \theta)$, with

$$z(q,\theta) \equiv \frac{\theta c(q)u'(q) + (1-\theta)u(q)c'(q)}{\theta u'(q) + (1-\theta)c'(q)}, \tag{1.8}$$

and $m_t^* = z(q^*, \theta)/\phi_t$. Notice that the terms of trade is independent of the seller's money balances. Hence, we can simply denote $q_t(m,\tilde{m}) = q_t(m)$, and $d_t(m,\tilde{m}) = d_t(m)$ in the following. It should point out that in equilibrium $d = m$ is the only rational choice, since, intuitively, no one is willing to hold more money than is needed for transactions, given the access to the CM where all agents can adjust their money holdings by producing the general good. Technical details on this point can be found in the original paper. In the following, we will use this result to derive a difference equation which summarizes the monetary equilibrium.

Now the third step characterizes the DM value function. Given the desirable properties associated with the CM value function, the value function (1.2) can be simplified to

$$\begin{aligned} V_t(m) &= v_t(m) + \phi_t m + \max_{m'} \{-\phi_t m' + \beta V_{t+1}(m' + \tau M)\} \\ &= v_t(m) + \phi_t m \\ &\quad + \sum_{j=t}^{\infty} \beta^{j-t} \max_{m_{j+1}} \{-\phi_j m_{j+1} + \beta[v_{j+1}(m_{j+1}) + \phi_{j+1} m_{j+1}]\}, \end{aligned}$$

where

$$\begin{aligned} v_t(m) &= \alpha\sigma\{u[q_t(m)] - \phi_t[d_t(m)]\} + \alpha\sigma \int \{-c[q_t(\widetilde{m})] + \phi_t d_t(\widetilde{m})\} dF_t(\widetilde{m}) \\ &\quad + \alpha\delta[u(q^*) - c(q^*)] + U(X^*) - X^*. \end{aligned}$$

In the final step, we decide the optimal choice of money holdings in the CM. That is, the choice of the sequence m_{t+1} is determined by solving the following optimization problem

$$\max_{m_{t+1}} \{-\phi_t m_{t+1} + \beta[v_{t+1}(m_{t+1}) + \phi_{t+1} m_{t+1}]\}.$$

Also note that a necessary condition for the existence of monetary equilibrium is $\phi_t \geq \beta\phi_{t+1}$. Otherwise, the buyer would demand an infinite amount of money balances, since the rate of return on money is less than the discount rate. Now we are ready to derive the aforementioned difference equation. For m_{t+1}, the first order condition is $\phi_t = \beta[\phi_{t+1} + v'_{t+1}(m_{t+1})]$. In addition, from the definition of $v_t(m)$ we have $v'_{t+1}(m_{t+1}) = \alpha\sigma[u'(q_{t+1})q'(m_{t+1}) - \phi_{t+1}]$, and remember $z(q_t, \theta) = \phi_t m_t = \phi_t M_t$ in stationary equilibrium. Thus, we have a difference equation in q

$$\frac{z(q_t, \theta)}{M_t} = \beta \frac{z(q_{t+1}, \theta)}{M_{t+1}}[\alpha\sigma \frac{u'(q_{t+1})}{z_q(q_{t+1}, \theta)} + 1 - \alpha\sigma], \tag{1.9}$$

which defines a monetary equilibrium if $q_t \geq 0$ for all t. Again, rigorous discussions about the existence and uniqueness of equilibrium are presented in Lagos and Wright (2004) and Lagos and Wright (2005)[6]. Since $M_{t+1} = (1+\tau)M_t$, the equilibrium condition (1.9) for stationary monetary equilibrium with $\phi_t M_t = \phi_{t+1}M_{t+1}$ is

$$\begin{aligned} \frac{u'(q)}{z_q(q, \theta)} &= 1 + \frac{1+\tau-\beta}{\alpha\sigma\beta} \\ &= 1 + \frac{i}{\alpha\sigma}, \end{aligned} \tag{1.10}$$

where $1 + i = (1 + r)(1 + \pi)$, $\pi = \tau$, and $r = (1 - \beta)/\beta$.

Up to now, we have reviewed the key elements of the LW model. Some basic results are highlighted. First, the Friedman rule is always optimal in this model, though not always efficient. The reason is that there is a holdup problem on money holdings when buyers do not have full bargaining power. Second, this novel source of inefficiency makes the estimated welfare cost of inflation considerably higher than conventional models. In the following sections, there will be more discussion of these points in reviewing recent contributions, which build on the basic framework just outlined.

1.3 Equilibrium with Money Being Essential

It is a classic and foundational question to ask why intrinsically useless fiat money is valued, and why money is essential in modern economic systems. Money is essential when it expands the set of incentive-feasible allocations or improves the efficiency of resource allocations relative to an economy without money (see Kocherlakota (1998) and Wallace (2001)). Modelling money in an essential rather than ad hoc way is one of

[6] Wright (2010) shows that the steady state is always unique in the LW model, while Hiraguchi and Kobayashi (2015) prove that multiple monetary steady states can exist in modified model where the centralized market opens once, but the decentralized markets open twice in each period.

the guiding principles to build microfounded monetary economics. Shi (2006) explains the need for a microfoundation of monetary economics and discusses the limitations of earlier efforts to integrate money into general equilibrium theory, while Williamson and Wright (2010b) emphasizes the methodological distinctions of search-based monetary theory from Monetarist and Keynesian approaches. Previous generations of New Monetarist models show that trading frictions such as a lack of double-coincidence of wants, an environment with anonymous agents and imperfect record-keeping are responsible for the emergence of a medium of exchange, which overcomes the double-coincidence problem (e.g.,Kiyotaki and Wright (1989)), plays the role of record-keeping (e.g.,Ostroy (1973), Townsend (1987) and Dong and Jiang (2010)), and serves as a public memory device (Kocherlakota, 1998). Unlike its predecessors, the LW framework is tractable with both perfectly divisible money and goods. These long desired qualities have stimulated numerous papers to explain why fiat money has value in equilibrium and how this intrinsically worthless token improves the efficiency of resource allocations, and to study the properties of monetary equilibria under different trading protocols.

1.3.1 Essentiality of Money

Anonymous trading and random pairwise matching are used to guarantee the essentiality of money in Lagos and Wright (2005). Aliprantis et al. (2007a,b) challenge this assertion and show that anonymity and random pairings are not per se sufficient to ensure money being essential, since informal enforcement schemes like trigger strategies or social norms can be used to support an efficient nonmonetary equilibrium if individual actions are observable in the centralized market and trading partners are patient enough. Lagos and Wright reply to this allegation that centralized trading is neither necessary nor sufficient for individual actions being observable, and the original LW environment never assumes the observability of actions except prices (Lagos and Wright, 2008). In addition, the disturbing results in Aliprantis et al. (2007a,b) are not robust, in the sense that a small amount of noise in the observation of individual behaviour can make money essential. However, Araujo et al. (2012) strike back, and argue that the specification of exchange process in the CM is nontrivial. Using a modified LW setup where the CM is modelled as a strategic market game along the lines of Shapley and Shubik (1977), and finite agents can only observe prices in this market, they show that if agents are patient enough, an efficient non-monetary equilibrium can be supported, even when prices are noisy. Intuitively, as long as individual behaviour has a measurable impact on prices, then market prices can convey relevant information about individual actions and this mechanism can be used to sustain cooperative behaviour. Indeed, when there are infinite agents in the economy, hence agents have no market power, the essentiality of money in LW can be restored as the case in Aliprantis et al. (2007a). Recently, there is a new strand of literature using experiments to test the predictions from the LW model. For example, Duffy and Puzzello (2014) test the LW model using laboratory experiments. They find that money - as an coordination device - improves social welfare, relative to the gift exchange equilibria.[7]

Other studies aim to clarify how fiat money enhances economic efficiency. Reed and Waller (2006) explicate money as a self-insurance device to share idiosyncratic consumption risk, when intertemporal credit contracts are not feasible due to private information over identity, endowment and trading history and the lack of record-keeping. More specifically, in their LW type model with only one perishable good, all markets are competitive rather than bilateral bargaining as in Berentsen et al. (2005), and in place of random matching, ex ante identical agents are partitioned into buyers or sellers by randomly receiving an endowment of the good at the beginning of the first sub-period, then they trade and consume, while in the second sub-period, anonymous agents also can produce the common good. To motivate aggregate risk, the amount of endowments received by agents is also stochastic with two states, high and low. Under the Friedman rule, when holding money is costless, agents will take as much cash as is needed to efficiently smooth consumption across both time and aggregate states. In a series of papers, Andolfatto and Martin (2013) and Andolfatto et al. (2014) show that fiat money is an "informationally insensitive" means of payment relative to risky durable assets represented as Lucas trees, whose payoffs are responsive to bad or good news. In contrast, the return on fiat money is not news-driven. By implication, there is a trade-off in disclosing payoff-related information which can be privately valuable but socially undesirable, when these risky assets serve as a payment instrument or as collateral for credit.

In general, both theoretical and empirical papers articulating the essentiality of money belong to the body of literature that aims to answer whether money per se is essential for modern mainstream macroeconomics. We will return to this matter later, but at first it is helpful to look into papers on the properties of equilibrium in search-theoretic monetary models.

1.3.2 Properties of Monetary Equilibrium

In Lagos and Wright (2005), the steady-state equilibrium is characterized under random bilateral matching and Nash bargaining. It is natural to ask whether existence and

[7] Anbarci et al. (2015) provide an experimental implementation of the competitive search monetary equilibrium in Rocheteau and Wright (2005). See Duffy (2014) for a survey on related topics.

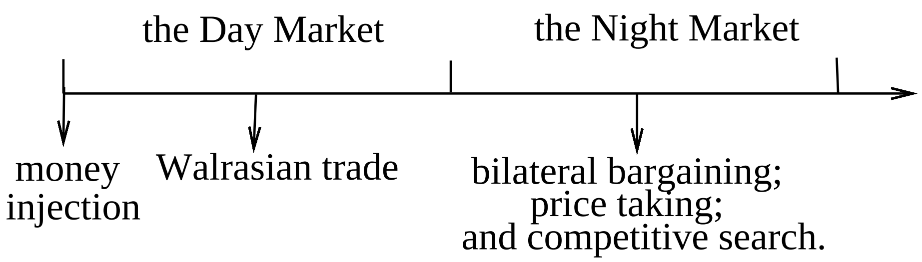

Figure 1.2: The timing of events during period t in the Rochetau-Wright model.

uniqueness of equilibrium, the optimal monetary policy and other results depend on specific characteristics of the environment such as preferences, information structure, and alternative trading protocols.

Rocheteau and Wright (2005) extend the standard model by introducing a general matching technology and endogenizing sellers' entry decision into the decentralized market to compare how three market structures affect equilibrium and efficiency. We would like to sketch the details of this model, since it is modified from the LW model and is itself an often-cited benchmark. A frictionless Walrasian market operates in the day, and at night there is a more decentralized and frictional market, in which three alternative pricing mechanisms will be compared. A continuum of agents is ex ante divided into buyers and sellers to motivate sellers' entry decision and hence to capture the extensive margin effect (the number of trades). More precisely, during the day all agents produce and consume, while at night buyers want to consume but cannot produce, and sellers are able to produce but do not want to consume. There is only one perishable good. Let $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{S}}$ denote the sets of buyers and sellers, respectively. To focus on sellers' entry, the measure of $\boldsymbol{\mathcal{B}}$ is normalized to 1 and all buyers have costless access to the night market, whereas only a subset $\overline{\boldsymbol{\mathcal{S}}}_t \subseteq \boldsymbol{\mathcal{S}}$ with measure n_t of sellers enter the night market at cost χ at time t. New money is injected, or withdrawn by lump-sum transfers at the beginning of each period, such that $M_{t+1} = (1+\tau)M_t \equiv \gamma M_t$. Without loss of generality, we also assume only buyers receive these monetary transfers. $\boldsymbol{F}_t^b$ and $\boldsymbol{F}_t^s$ denote the distributions of money holdings across buyers and sellers at t, respectively. The timeline is illustrated in Figure 1.2. The period utility of a buyer at t is

$$\boldsymbol{U}_t^b = U(X_t) - H_t + \beta_d u(x_t), \tag{1.11}$$

where X_t the quantity consumed and H_t is labour (also the quantity produced due to a linear production technology) during the day, x_t is consumption at night, and β_d is a discount factor between the two sub-periods. The aforementioned discount factor $\beta = \beta_d\beta_n$, where β_n is discount factor between night and the next day. Similarly, the instantaneous utility of a seller is

$$\boldsymbol{U}_t^s = U(X_t) - H_t - \beta_d c(x_t). \tag{1.12}$$

The functions U and c have the same properties as in the canonical LW model, though here we normalize $U(X^*) = X^*$. In the frictional night market, only a subset $\widetilde{\boldsymbol{\mathcal{B}}}_t \subseteq \boldsymbol{\mathcal{B}}$ of buyers and a subset $\widetilde{\boldsymbol{\mathcal{S}}}_t \subseteq \overline{\boldsymbol{\mathcal{S}}}_t \subseteq \boldsymbol{\mathcal{S}}$ of sellers participate and trade in this market in each period t. The measure of $\widetilde{\boldsymbol{\mathcal{B}}}_t$ is $\alpha_b(n_t)$ and the measure of $\widetilde{\boldsymbol{\mathcal{S}}}_t$ is $n_t\alpha_s(n_t)$. Agents in $\widetilde{\boldsymbol{\mathcal{B}}}_t \cup \widetilde{\boldsymbol{\mathcal{S}}}_t$ are chosen randomly. Therefore, the probabilities of trading for buyers and sellers at night are $\alpha_b(n_t)$ and $\alpha_s(n_t)$, respectively. Note that the trading probabilities are supposed to depend on the ratio of sellers to buyers, n_t. Typically, $\alpha_b(n) = \alpha(n)$ and $\alpha_s(n) = \alpha(n)/n$ are assumed. Intuitively, the number of buyers and that of sellers trade in the night market is equal. In addition, the matching function $\alpha(n)$ is required to satisfy $\alpha'(n) > 0$, $\alpha''(n) < 0$, $\alpha(n) \leq \min\{1, n\}$, $\alpha(0) = 0$, $\alpha'(0) = 1$ and $\alpha(\infty) = 1$.

We will characterize the equilibrium under alternative pricing protocols in the following. Under Nash bargaining, let $V^b(z_b)$ and $W^b(z_b)$ denote the value functions for a buyer with real balance z_b in the night market and day market, respectively, and similarly, $V^s(z_s)$ and $W^s(z_s)$ for a seller. Recall that real balance z is defined as $\phi_t m$. In a meeting involving a buyer with real balances z_b and a seller with z_s, let $d = d(z_b, z_s)$ and $q = q(z_b, z_s)$ represent the real money and units of the good exchanged, respectively. Then, we have

$$\begin{aligned} V^b(z_b) \quad &= \quad \alpha(n) \int \{u[q(z_b, z_s)] + \beta_n W^b[\frac{z_b - d(z_b, z_s)}{\gamma}]\} d\boldsymbol{F}^s(z_s) \\ &\quad +[1 - \alpha(n)]\beta_n W^b(\frac{z_b}{\gamma}). \end{aligned} \tag{1.13}$$

That is, with probability $\alpha(n)$ a buyer meets a seller who holds z_s, in which case he consumes $q(z_b, z_s)$, and enter into the next day with real balances $[z_b - d(z_b, z_s)]/\gamma$; and with probability $1 - \alpha(n)$ he has no trade opportunity and enter into the next day with z_b/γ. Note that for a buyer holds $z_t = m_t\phi_t$ at the end of period t, his real balances at the beginning of period $t+1$ are $z_{t+1} = m_t\phi_{t+1} = z_t\phi_{t+1}/\phi_t = z_b/\gamma$. Similarly, for sellers we have

$$\begin{aligned} V^s(z_s) \quad &= \quad \frac{\alpha(n)}{n} \int \{-c[q(z_b, z_s)] + \beta_n W^s[\frac{z_s + d(z_b, z_s)}{\gamma}]\} d\boldsymbol{F}^b(z_b) \\ &\quad +[1 - \frac{\alpha(n)}{n}]\beta_n W^s(\frac{z_s}{\gamma}) - \chi. \end{aligned} \tag{1.14}$$

Intuitively, a seller first pays χ for participation, then with probability $\alpha(n)/n$ he trades with a buyer who holds z_b, in which case he produces $q(z_b, z_s)$, and enters into the next day with real balances $[z_s + d(z_b, z_s)]/\gamma$; and with probability $1 - \alpha(n)/n$ he does not trade and enters into the next day with z_s/γ. In the day market, a buyer solves

$$W^b(z_b) = \max_{\hat{z},X,H} U(X) - H + \beta_d V^b(\hat{z}) \tag{1.15}$$

subject to

$$\hat{z} + X = z_b + T + H,$$

$$X \geq 0, 0 \leq H \leq \bar{H}, \text{and } \hat{z} \geq 0,$$

where $T_t = \phi_t(\gamma - 1)M_{t-1}$ is a monetary transfer in real terms, and $\hat{z}$ is money taken into the second sub-period. Likewise, for sellers,

$$W^s(z_s) = \max_{\hat{z},X,H} U(X) - H + \beta_d \max[V^s(\hat{z}), \beta_d W^s(\frac{z_s}{\gamma})] \tag{1.16}$$

subject to

$$\hat{z} + X = z_s + H,$$

$$X \geq 0, 0 \leq H \leq \bar{H}, \text{and } \hat{z} \geq 0.$$

As in the benchmark case, it is quite straightforward to show that $\hat{z}$ does not depend on z_b or z_s, and that $W^b(z_b)$ and $W^s(z_s)$ are linear in z_b and z_s, respectively. Now $d(z_b, z_s)$ and $q(z_b, z_s)$ are determined by the following generalized Nash bargaining problem

$$\max_{q,d} [u(q) - \frac{\beta_n}{\gamma} d]^\theta [-c(q) + \frac{\beta_n}{\gamma} d], \tag{1.17}$$

where $q \geq 0$ and $d \leq z_b$. We can see that (1.17) is basically the same problem as (1.6). Without proof, we simply write out the stationary equilibrium conditions

$$\frac{\beta_n}{\gamma} z_b = z(q, \theta),$$

and

$$\frac{\gamma - \beta}{\beta \alpha(n)} + 1 = \frac{u'(q)}{z_q(q, \theta)},$$

where $z(q, \theta)$ is defined in (1.8). The equilibrium condition for seller's entry is

$$\frac{\alpha(n)}{n} [-c(q) + \frac{\beta_n}{\gamma} d] = \chi. \tag{1.18}$$

Intuitively, the participation cost is equal to a seller's expected surplus which is the probability of trading multiplied by the seller's surplus per trade in the night market. As in the LW, the key results with bargaining are that the intensive margin(q) and extensive margin(n) are both socially inefficient, that is, $q < q^*$ and $n < n^*$, where q^* and n^* satisfy $u'(q^*) = c'(q^*)$ and $\alpha'(n^*)[u(q^*) - c(q^*)] = \chi$, and that the Friedman rule(i.e.,$\gamma = \beta$) is optimal but it cannot generally correct the inefficiencies on the intensive or the extensive margins. However, an efficient intensive margin is obtained at the Friedman rule if buyers have full bargaining power, since the holdup problem on money is eliminated in this case.

Under price taking, the night market is a Walrasian market with some search frictions. More precisely, not all agents can enter and trade in this competitive market. For successful entrants, after observing the price p, each seller and each buyer supplies q^s and demands q^b units of the good, respectively. The double coincidence problem and anonymity make money essential, as in the case of bargaining. The previously defined $\alpha_b(n)$ and $\alpha_s(n)$ can be interpreted as the probability of getting into the night market. For a buyer, the value function is

$$\begin{aligned} V^b(z_b) &= \alpha_b(n)\max_{q^b}\{u(q^b) + \beta_n W^b(\frac{z_b - pq^b}{\gamma})\} \\ &\quad +[1-\alpha_b(n)]\beta_n W^b(\frac{z_b}{\gamma}), \end{aligned} \tag{1.19}$$

subject to

$$pq^b \leq z_b, \text{and } W^b(z_b) \text{ satisfies (1.15).}$$

Similarly, for a seller, we have

$$\begin{aligned} V^s(z_s) &= \alpha_s(n)\max_{q_s}\{-c(q^s) + \beta_n W^s(\frac{z_s + pq^s}{\gamma})\} \\ &\quad +[1-\alpha_s(n)]\beta_n W^s(\frac{z_s}{\gamma}) - \chi, \end{aligned} \tag{1.20}$$

where $W^s(z_s)$ is defined by (1.16). The equilibrium conditions in steady state are

$$\begin{aligned} n\alpha_s(n)q^s &= \alpha_b(n)q^b, \text{ (market clearing)} \\ \frac{\beta_n}{\gamma}z_b &= q^b c'(q^s), \\ \frac{\gamma-\beta}{\beta\alpha_b(n)} + 1 &= \frac{u'(q^b)}{c'(q^s)}, \end{aligned}$$

and the free entry condition analogous to (1.18) is

$$\alpha_s(n)[q^s c'(q^s) - c(q^s)] = \chi. \tag{1.21}$$

In this case, the Friedman rule restores efficiency along the intensive margin but not the extensive margin. Consequently, inflation has ambiguous effects, and in particular positive inflation may be welfare-improving.

In competitive search (price posting with directed search), there are agents acting as market makers,[8] who set up sub-markets and post the terms of trade (q, d).Within any sub-market, random bilateral matching prevails. Search is directed in the sense that buyers and sellers have full knowledge of (q, d) across sub-markets, and they can go to any sub-market freely. Clearly, it is rational for an agent to consider other agents' choice of sub-markets. In equilibrium, the set of sub-markets is complete in the sense that extra sub-markets are redundant in improving buyers and sellers' welfare. The timing of events in a period goes as follows: at the beginning of each day, market makers announce the set of sub-markets Ω to open at night; agents then trade and hence readjust their real balances in the day market, and travel to their selected sub-markets represented by triplet $\omega = (q, d, n) \in \Omega$, where n is the ratio of the expected number of sellers to the anticipated number of buyers at each sub-markets; in the sub-markets at night agents trade goods and money in pairs, simply following the posted terms of trade (q, d). For a buyer, the value function is

$$\begin{aligned} V^b(z_b) &= \max_{\omega}\{\alpha(n)\mathbb{1}(z_b \geq d)[u(q) + \beta_n W^b(\frac{z_b - d}{\gamma})]\} \\ &\quad +[1-\alpha(n)]\mathbb{1}(z_b \geq d)\beta_n W^b(\frac{z_b}{\gamma}), \end{aligned} \tag{1.22}$$

where $\mathbb{1}(z_b \geq d)$ is the indicating function. Intuitively, a buyer chooses $\omega \in \Omega$, and then he decides to trade if he meets a seller and has enough money to meet the posted price $z_d \geq d$. As in the benchmark, the buyer rationally carries just enough money to meet the posted price in his chosen sub-market, that is, $z_b = d$ in equilibrium. Similarly, for a seller we have

$$\begin{aligned} V^s(z_s) &= \max_{\omega}\{\frac{\alpha(n)}{n}[-c(q) + \beta_n W^s(\frac{z_s + d}{\gamma})]\} \\ &\quad +[1-\frac{\alpha(n)}{n}]\beta_n W^s(\frac{z_s}{\gamma}) - \chi. \end{aligned} \tag{1.23}$$

$W^b(z_b)$ and $W^s(z_s)$ satisfy (1.15) and (1.16), respectively. Again, for comparison we just give the equilibrium conditions without technical details. In steady state,

$$\frac{\gamma - \beta}{\beta\alpha(n)} = \frac{u'(q)}{c'(q)}, \tag{1.24}$$

[8]One can imagine that the market makers can charge buyers and sellers fees for entering their sub-market and seek to maximize profits, but competition among market makers drives their profits to zero and hence entry costs are zero in equilibrium.

$$\frac{\beta_n}{\gamma} z_b = z[q, 1-\eta(n)], \tag{1.25}$$

$$\frac{\alpha(n)}{n} z[q, 1-\eta(n)] - c(q) = \chi, \tag{1.26}$$

where z is given in (1.8), and $\eta(n) = n\alpha'(n)/\alpha(n)$ measures sellers' contribution to buyers' probability of trading. Note that (1.25) is the first-order condition for a generalized Nash problem where the seller's bargaining power is $\eta(n)$. (1.24) indicates that the Friedman rule achieves the first best along the intensive margin. In addition, we have $z[q^*, 1-\eta(n)] - c(q^*) = \eta(n)[u(q^*) - c(q^*)]$, hence the free entry condition (1.26) at the Friedman rule is $\alpha'(n)[u(q^*) - c(q^*)] = \chi$, which delivers an efficient extensive margin. Summing up, market structure specifications indeed matter for equilibrium and efficiency, whereas they per se do not alter the essentiality of money. In this sense, the properties of monetary equilibrium in the LW framework are robust.

As a caveat, Faig and Huangfu (2007) elaborate the role of market makers in Rocheteau and Wright (2005). More specifically, market makers play an active role in transferring payments from buyers to sellers, and they can charge buyers a positive entry fee and use the collected revenue to remunerate sellers in their sub-markets. They show that the introduction of market makers is nontrivial and should be used with caution, since it could induce counterfactual results like perfect predictability of payments. In fact, there is a general tendency in competitive search models which make payments predictable and allow buyers to avoid idle money balances, since the search is not purely random, but directed. Two related studies extend the Rocheteau and Wright (2005) model by introducing private information over preferences into the common LW environment. One is Faig and Jerez (2006), which uses preference shocks to capture precautionary money demand under uncertain expenditure needs and the use of non-linear price schedules in screening out buyers with heterogeneous valuations for the same products in retail markets, find that inflation impedes the role of prices in achieving efficient allocations. Mathematically, $\boldsymbol{U}_t^b = U(X_t) - H_t + \varepsilon_t u(x_t)$, and $\boldsymbol{U}_t^s = U(X_t) - H_t - c(x_t)$, where ε_t is the idiosyncratic match-specific preference shock and is uniformly distributed in $[1, \bar{\varepsilon}]$. Quantitatively, the proposed competitive search monetary model with private information fits historical US data on velocity and interest rates well. The other is Dong and Jiang (2014) with price posting and undirected search[9]. Private information about ε_t prevents the seller from extracting all the trading surplus and ensure a unique monetary equilibrium, though the Friedman rule is optimal but inefficient in this model. Similarly, Ennis (2008) introduces imperfect information into the LW environment. Buyers have private information over tastes,

[9] Prior to a bilateral meeting, the seller posts the terms of trade (q, d), and the buyer does not observe the posted price-quantity pairs until they are matched.

that is, $\boldsymbol{U}_t^b = U(X_t) - H_t + \varepsilon u(x_t)$, where $\varepsilon \in \{\varepsilon_H, \varepsilon_L\}$. Sellers make take-it-or-leave-it (i.e.,$\theta = 0$) price–quantity pairs $\{(q_H, d_H), (q_L, d_L)\}$ to buyers. The steady-state monetary equilibrium in their model is robust in the sense that agents with heterogeneous unobservable tastes are more likely to use money for transactions. In addition, only a proportion of the agents hold money in equilibrium, that is, an endogenous extensive margin arises, even though there are no fixed costs imposed on the participation decision into the decentralized market. Note that sellers have all the bargaining power, which worsens the hold-up problem[10] on money holdings. However, the presence of private information limits the ability of sellers to exercise their hold-up power. In comparison, Shao (2009) investigates how private information about the quality of goods affects the role of money, efficiency and optimality of monetary policy in a modified version of the LW model. He finds that in alternative environments with adverse selection and moral hazard respectively, the Friedman rule is always optimal, and efficient if sellers have some mechanism to signal their products' quality. However, the effectiveness of money differs considerably in ameliorating the two types of informational frictions.[11]

Recently, Aruoba et al. (2007) compare Nash bargaining and egalitarian bargaining and find that they have a quite different impact on the efficiency of the monetary equilibrium, the optimality of the Friedman rule and the welfare cost of inflation, especially in models with endogenous participation. In particular, they point out that the inefficiency identified by Lagos and Wright (2005) at the Friedman rule does not result from a holdup problem on real balances. Rather, it is a consequence of non-monotonicity[12] of the Nash bargaining solution. Notably, proportional bargaining meets the monotonic property, which makes the Friedman rule efficient. Using the notation in the benchmark model, the buyer in a bilateral match solves the following problem under egalitarian bargaining,

$$\max_{q,d}[u(q) - \phi_t d] \tag{1.27}$$

such that $u(q) - \phi_t d = \phi_t d - c(q), d \leq m,$ and $q \geq 0$. Intuitively, the buyer choose an offer to maximize his own surplus from trading subject to the constraint that the match surplus is shared equally between the buyer and the seller. In contrast, Hu et al. (2009) use a mechanism design approach to determine the terms of trade in

[10] A holdup problem occurs when an agent makes an ex-ante, costly and irreversible investment decision, and the ex-post return can be appropriated by other agents. In a bilateral trade, two types of holdup problem can arise. One is that a seller can holdup buyers' money holdings for trade surplus. The other is that a buyer can holdup sellers' capital investment.

[11] As a related study, Shao (2014) considers the LW environment with competitive search, and shows that under sufficiently low counterfeiting cost, there is no monetary equilibrium. The reason is that the possibility of counterfeiting can destroy sellers' confidence in using money, and hence discourage them from entering the DM market so that this market may shut down.

[12] Monotonicity means agents' shares of match surplus are monotonically increasing as the bargaining set expands. Note that money is essential when it enlarges the set of incentive-feasible allocations.

the LW economy, and find that the first-best allocation is implementable if agents are sufficiently patient (i.e., β is close to 1.), even though lump-sum tax schemes are absent to finance interest payments on money. This result is important, since all the previous studies which assumed trading protocols and other LW features conclude that the first-best allocation is not achievable without the use of lump-sum taxes.

Quasi-linear preferences are emphasized by Lagos and Wright (2005) as being necessary to keep the model tractable. However, Faig (2008) argues that the quasi-linearity of preferences is not necessary to generate degenerate distributions of money holdings when ex ante identical agents make endogenous decisions to participate in the decentralized market as buyers or as sellers. This binary buyer–seller choice is motivated as follows. At the beginning of each period, agents can gamble their money holdings via atemporal fair lotteries; if they win the gamble, they go to the decentralized market as buyers; if they lose, they go to the search market as penniless sellers. As in Rocheteau and Wright (2005), the probabilities of trading for buyers and sellers dependent on the ratio of buyers over sellers. Thanks to the nonconvex participation choice, under general preferences the value function $W_t(m)$ has a linear segment where agents behave as if their preferences were quasi-linear. Rocheteau et al. (2008) provide another way to relax the somewhat strong quasi-linear preference assumption. In a nonconvex version of the LW economy with indivisible labour and general preferences, they show that agents act as if they have quasi-linear preferences in the case of interior solutions. Interestingly, these studies suggest that the setup in Lagos and Wright (2005) is quite robust.

All the papers we have reviewed focus on steady-state equilibria. There is another strand of work which goes beyond stationary analysis and examines dynamics in monetary equilibrium in search-based models. Lagos and Wright (2003) characterize dynamic equilibria including cycles and chaos, and sunspot equilibria in their famous framework, which abstracts from capital accumulation. Ferraris and Watanabe (2011) move a step forward. Based on the collateralized credit model of Kiyotaki and Moore (1997) and the LW model, they provide sufficient conditions for endogenous fluctuations in both consumption and capital in an economy where capital serves both as productive input and collateral for monetary loans. Collateral constraints or borrowing limits play a crucial role in generating interesting cycles, chaotic trajectories and sunspot equilibria.

1.3.3 Discussion

In this section, we reviewed the work which is concerned with the robustness of the LW framework in particular, and the search theory of money in general. Basically, the existence of efficient monetary equilibrium or essentiality of money has not been fundamentally challenged, though the choice of trading protocols has substantial implications for the effects of monetary policy, efficiency and the optimality or otherwise of the Friedman rule. Still, the paper by Araujo et al. (2012) reminds us that more work is required to settle the issue of whether money is essential in the LW model. This is worth pursuing for at least two reasons. First of all, a unified monetary framework with solid microfoundations and essential money is highly desired for both theoretical and policy analysis. Second, the alternating centralized market and decentralized market structure stimulates our imagination to introduce labour, capital and other markets, as well as Neoclassical ingredients like asymmetric information, and New Keynesian elements like nominal rigidities into the model. Ultimately, money is brought into general equilibrium theory in an essential way rather than via ad hoc assumptions, and this microfounded theory of money is fully integrated with modern macroeconomics.

An interesting topic for future research is to study the robustness of monetary equilibrium in the presence of heterogeneous expectations and learning. There are always, at least, two steady states in monetary search models: one is monetary and the other is non-monetary. The monetary equilibrium under rational expectation is tenuous in the LW model, since even a small perturbation can push the model economy away from the monetary stationary equilibrium and towards the non-monetary steady state. Baranowski (2015) shows that when agents use adaptive learning to form expectations about the value of money, the monetary equilibrium is locally stable and agents never learn the non-monetary equilibrium. In this sense, the monetary steady state is robust.[13]

Before considering policy analysis, we want to explore papers concerned with the coexistence of money and other media of exchange.

1.4 Models with Competing Media of Exchange

In this section, we review models which aim to rationalize the coexistence of fiat money and other real or nominal assets. In the presence of competing media of exchange or rather interest-bearing assets, why is fiat money valued? This has been a central issue in pure monetary theory. In general, we are interested in the interaction between

[13] Also see Branch (2014), Branch and McGough (2014) and Branch et al. (2014).

money and other assets in facilitating exchange either as a payment instrument or as collateral.

1.4.1 Money and Real Assets

Lagos and Rocheteau (2008) assume that the general goods in the LW environment are storable and in this way examine the essentiality of money when physical capital can be used directly as an alternative medium of exchange. They show that money is essential if there is a lack of capital to be used as means of payment. In fact, agents are induced to overaccumulate capital in the case of liquidity shortage, and the introduction of fiat money helps to eliminate this inefficiency. The Friedman Rule is optimal and efficient. The model is briefly outlined below. For an ex ante identical agent, let q_t^b, q_t^s, and y_t denote the quantity of special goods consumed, the quantity of special goods produced and the net consumption of general goods in period t, respectively. Then, the instantaneous utility is $u(q_t^b) - c(q_t^s) + y_t$. Note that $y = X - H$ by assuming $U(X) = X$ in our basic model. Agents have two storage technologies. By storing x_{lt} units of general goods at time t, an agent reaps $k_{lt+1} = f_l(x_{lt})$ units of general goods before entering the DM of the following period. Alternatively, by storing x_{it} at time t, an agent gets $k_{it+1} = f_i(x_{it})$ units of general goods after leaving the DM of the following period. This timing of storage implies that the goods stored using the f_i technology cannot be brought into the DM as a medium of exchange. f_l and f_i are assumed to be strictly concave, with $f_l'(0) = f_i'(0) = +\infty$, $f_l(0) = f_i(0) = 0$, $\lim_{x_l \to \infty} f_l'(x_l) < \beta^{-1}$, and $\lim_{x_i \to \infty} f_i'(x_i) < \beta^{-1}$. Clearly, (z, k_l, k_i) is the individual state vector, where $z = \phi m$ are the real money balances. The value functions $V_t(z, k_l, k_i)$ and $W_t(z, k_l, k_i)$ are quite similar to the $V_t(m)$ and $W_t(m)$ in our benchmark case. In addition, the terms of trade is determined by Nash bargaining. Aruoba and Schorfheide (2010) extends Lagos and Rocheteau (2008) by allowing capital also as a factor of production in the decentralized market(see also Aruoba and Schorfheide (2011)). They find that money and capital claims can coexist as competing media of exchange in the decentralized market if liquid capital is only a small fraction of the overall capital stock.

The so-called rate-of-return dominance puzzle—why is money held when higher-return assets exist—is not solved in Lagos and Rocheteau (2008), since in their model liquid capital (k_l) and real balances (z) earn an equal rate of return in monetary equilibrium. Hu and Rocheteau (2013) adopts a mechanism design approach to attack this puzzle in a LW style economy where fiat money and risk-free capital compete as means of payment. He shows that a positive equilibrium return differential between money and capital is both socially optimal and individually rational even if capital goods are perfectly divisible and recognizable. More specifically, the nondegenerate core in pairwise meetings accommodates a pricing mechanism which can align agents' incentives with the social optimal, since an agent who deviates from collectively agreed allocations can be punished by triggering his least-preferred trade in the core. Andolfatto and Martin (2009) introduce uncertainty into the Lagos and Rocheteau (2008) model where the short-term return to capital is subject to aggregate risk. Fiat money, on the contrary, is "informationally insensitive", or rather its expected return is relatively less news-driven. Hence, money improves welfare by mitigating the excess volatility associated with liquid asset as alternative medium of exchange. Naturally, capital dominates money in long-run expected return, since capital or claims against capital are more responsive to news shocks.[14]

1.4.2 Money and Nominal Assets

Lagos (2013) builds on Lagos and Wright (2005) to study the coexistence of money and nominal risk-free interest-bearing bonds (i.e., each bond represents a certain claim to fiat money). Based on a realistic observation that fiat money is heterogeneous in some extraneous attribute like a serial number (dubbed as a moneyspot by analogy with sunspots), he shows that, without the typical exogenous imposition that agents can only use money as means of payment to buy goods or securities, there exist equilibria in which money coexists with interest-bearing bonds. The basic logic is that moneyspots can be treated differently by agents' beliefs, and hence money may be valued asymmetrically in equilibrium. These asymmetric and expectation-drive valuations of extraneous attributes then can be used to construct monetary equilibria with the declared property. In addition, the model also can rationalize the liquidity effects of open-market operations, and even the possibility of negative nominal interest rates. Intuitively, agents' self-fulfilling expectations can affect the proportion of the money supply that agents are willing to use for bond purchases. In some cases, there may be too much, and in other cases too little cash to be traded for bonds. When the size of open-market operation is relatively small, agents are willing to buy bonds at a premium, since some type of money is expected to be inferiorly valued in particular moneyspot equilibria. Note that the coexistence puzzle is rationalized purely from the self-fulfilling beliefs of agents, without assuming any intrinsic difference between money and bonds. In contrast, Ferraris and Mattesini (2014) approach the rate of return puzzle by assuming that bonds are subject to a minimum purchase requirement and untrustworthy intermediaries. These intermediaries issue private notes to pool the cash of the agents who face a binding minimum purchase constraint, and then use the

[14] Also see Andolfatto et al. (2015) for an application to the practice of rehypothecation.

revenue to acquire the interest-bearing bonds. In equilibrium, a positive interest rate motivates the intermediaries to redeem their private notes. In both models, money and bonds are equally liquid, that is, can be freely chosen as means of payment.

A related literature is about the social benefits of illiquid nominal bonds. Illiquidity here means that bonds cannot be directly used to purchase goods at no cost. In order to reconcile the coexistence of positive nominal interest rates and illiquid government bonds, Boel and Camera (2006) introduce heterogeneous discounting and idiosyncratic consumption risk into the LW model, where spatial separation in spirit of Townsend (1980), anonymity, limited commitment and enforcement make money essential. Heterogeneity in discounting and idiosyncratic shocks render the Friedman rule sub-optimal, and the constrained-efficient allocation has to be supported by illiquid bonds and hence positive nominal interest rates. In a version of the LW setup with price taking, Marchesiani and Senesi (2009) study the welfare-enhancing role of illiquid one-period nominal bonds as a mechanism to reallocate idle money balances toward liquidity-constrained agents, as the banking sector does in Berentsen et al. (2007). For tractability and robustness, Andolfatto (2011) recasts the Kocherlakota (2003) model in the LW environment with competitive markets, and reconsiders the idea that illiquid bonds smooth marginal utility across heterogeneous agents with different intertemporal marginal rates of substitution. He shows that this welfare-improving role of illiquid bonds holds in the steady state rather than just in one period, when household types are private information (that is, liquidity needs or desired consumption needs are private information).

Berentsen and Waller (2011a) compare the allocative effects of nominal outside (government bonds) and inside bonds. They adopt a version of the LW model with three perfectly competitive markets, the first being a bond market. Money is essential in the second market, where agents trade money for goods. The main result is that allocations with inside bonds which relax liquidity constraints can be replicated by those with outside bonds, while the converse is not true, when neither private households and nor the government have any enforcement power. Interestingly, the allocations in the two cases are equivalent if participation in bond markets is costly. As above, illiquid bonds are socially beneficial whenever the inflation rate is positive.

As for formal modelling issues, the key point is that (m, B) now acts as the state variables, where B denotes agents' bond holdings. Then, $V_t(m, B)$ and $W_t(m, B)$ can be formulated in a way similar to the benchmark case. Rather than going into details, we recommend interested readers refer to Nosal and Rocheteau (2011) for a textbook exposition.

1.4.3 Discussion

It is still an ongoing project to explain why money is dominated in the rate of return by other assets like government issued nominal bonds. For example, Li et al. (2012) introduce the threat of fraud to endogenize asset-specific liquidity properties. This study sheds new light on the distribution and time-variation of liquidity premia across assets, and in turn offers fresh insight on rate-of-return differentials. In particular, a satisfactory rationalization of money remaining essential even when capital goods can be used as medium of exchange provides us a solid microfoundation to examine the relationship between inflation and capital accumulation, like the famous Tobin (1965) effect. We will discuss money and growth further in the following.

The determination of the exchange rate between two fiat currencies in an economy can be framed as a special case of rate-of-return dominance question. For example, Gomis-Porqueras et al. (2013b) use a two-country version of the LW model to account for exchange rate determination and its dynamics, while Kannan (2009) quantifies welfare benefits of an international currency in an open-economy variant of the LW setup.[15] Zhang (2014) extends the LW model into open-economy context and aims to answer three classic questions in international monetary economics: (i) the conditions under which a currency can emerge as an international money, (ii) how monetary authorities should set inflation when there is no restrictions on which currencies can be used as medium of exchange, and (iii) how big the welfare gains from currency internationalization are. In the two-country, two-currency model, She finds that strategic complementarities between currency portfolio choices and information acquisition decisions lead to multiple equilibria where zero, one, or two international monies can emerge. The Friedman rule is generally suboptimal, due to the trade-off between the seigniorage benefit from inflation and the threat of losing international status. In addition, the welfare gains from internationalization can be quantitatively sizeable. Similarly, issues associated with monetary unions or currency areas need more attention, though Li and Matsui (2009) have shed some light on issues like the welfare costs and benefits of a monetary union in a simple two-country two-currency search-theoretic model with indivisible goods.

We next move to the closely related section which deal with the coexistence of money and credit, and as we will see, this issue deserves extra attention.

[15] Also see Gomis-Porqueras et al. (2014b).

1.5 Models with Money and Credit

The hallmark of New Monetarist models is to explicitly deal with frictions in the exchange process. Random bilateral matching in the search theory of money is a natural way to generate a double coincidence problem, and to motivate incomplete record keeping, limited commitment and other frictions that make monetary exchange socially useful. Unfortunately, it is these frictions which render money essential that make credit arrangements impossible in standard search-theoretic models. As a result, an increasing number of studies are aiming to solve this dilemma, and in general to clarify the relationship among money and credit arrangements.

1.5.1 Money and Credit as Means of Payment

There is no role for credit in the original LW model. Credit is impossible in the decentralized market because of the assumption that agents are anonymous. Meanwhile, credit is not necessary in the centralized market due to the assumption that all agents can work and have linear utility in hours. Telyukova and Wright (2008) introduce a third sub-period where a centralized market with no anonymity assumption operates. In addition, some agents want to consume but cannot produce, hence bilateral credit is feasible and useful in this market. They show that agents choose to use interest-bearing credit even when they have money at hand, due to idiosyncratic uncertainty about liquidity need. This prediction provides an explanation to the "credit card debt puzzle" in their paper. By construction, there is a dichotomy in the sense that the output traded through money is independent of those via credit, since money and credit are not used simultaneously within the same market. Hence, monetary policy does not affect the composition of monetary and credit transactions. Another simplification is that enforcement problems with credit arrangements are assumed away. See Telyukova (2013) for a quantitative analysis of the role of liquidity demand in accounting for this puzzle. Gomis-Porqueras et al. (2014a) quantify the size of the shadow economy in a slightly modified LW model. Costless record-keeping technology is available and hence credit exchange can be supported. However, money is still useful since monetary transactions are not reported to the government, which allows agents to evade income taxes. The trade-off of paying the inflation tax or the income tax determines the scale of the informal sector.

In order to introduce credit into the LW environment with imperfect record-keeping and lack of commitment, Sanches and Williamson (2010) resort to the threat of theft associated with money holding. They show that money and credit can coexist as competing media of exchange under different record-keeping technologies and ways of money injections, as long as the cost of theft is sufficiently small. In general, following the Friedman rule is not optimal. It is useful to note that in a bilateral meeting, the buyer can only use credit in transactions when the seller monitors the transaction with some exogenous probability, hence the choice of using credit is essentially exogenous in this model. In addition, other types of costs of operating a monetary system, such as counterfeiting, the costs of deterring counterfeiting, or the costs of replacing worn currency can be considered as well.[16] In fact, Fung and Shao (2011) extend Nosal and Wallace (2007) to study counterfeiting in the LW framework with divisible money and competitive search. They show counterfeiting can exist in a monetary equilibrium if the cost of producing counterfeits is sufficiently low. Also assuming imperfect record-keeping, Bethune et al. (2015) integrate the LW with a frictional Mortensen-Pissarides labour market in order to study the relationship between household unsecured debt, liquid assets, and aggregate unemployment.[17] Households can not commit and are heterogeneous in terms of their access to unsecured credit. Due to imperfect record-keeping technology, some households can be monitored and hence firms are willing to extend credit to them. Defaulted households are excluded from credit arrangements permanently. They show that when the rate of return of liquid assets are neither too high nor too low, liquid assets including money and unsecured pairwise credit - as alternative means of payment - can coexist in equilibrium. Meanwhile, there is an virtuous cycle between credit markets and labour markets: higher aggregate employment leads to looser endogenous debt limits for households, and vice versa. However, a bad equilibrium with high unemployment and credit crunch is possible as well, due to the households' choice of liquid assets and firms' participation decisions.

Full enforcement is a strong assumption. Boerner and Ritschl (2011) adopt a village economy version of the LW model and analyze communal responsibility as a collateralizing device to overcome enforcement problems associated with bilateral exchange.[18] Communal responsibility is a medieval institution of merchants, by which merchants from a given city would be held collectively liable for each other's debts when trading abroad. They show that fiat money and interest-bearing bills of exchange issued by

[16] In a similar environment without money, Carapella and Williamson (2015) study the role of government debt as collateral in private credit contracts and hence facilitating decentralized exchange. In the model, limited memory softens the punishment for default. Government debt can be easily recognized and seized, which helps to discourage default on private or public debts. This improves social welfare. Notably, default can occur in equilibrium.

[17] See Dong and Xiao (2015) and Rocheteau and Rodriguez-Lopez (2014) for related models.

[18] In the DM, agents from different villages trade village-specific goods bilaterally. Agents are anonymous but with verifiable citizenship. So agents may use money and (or) credit as a medium of payment in the inter-village market during the day. The CM in this economy is an Walrasian intra-village market, where only the inhabitants of each village participate, and all claims can be enforced costlessly. See also Faig (2004) and Liu (2015).

agents may coexist in equilibrium if there is a fixed but sufficiently low cost of using credit. In contrast, Dong (2011b) develops a three-sub-period version of the LW model where competitive financial intermediaries exist and have access to a costly record-keeping technology to identify agents. Bilateral credit is feasible in this environment when the buyer pays a fixed cost in order to make the seller and herself identified to a financial intermediary which enforce the settlement of IOUs via money. She shows that inflation and credit exhibit an inverse U-shaped relationship, while the use of credit has inflation-dependent effect on money demand. A innovative feature of this model is that money and credit can simultaneously serve as means of payment. In addition, the choice of using means of payment, money or credit, is endogenous.

So far, we have exclusively focused on the coexistence of fiat money and bilateral credit (i.e., personal IOUs), and their implications. Again, here we would like to omit the technical details, since instructive textbook exposition can be found in Nosal and Rocheteau (2011). In the next subsection, we are going to discuss papers concerning credit issued by financial intermediaries, like bank loans.

1.5.2 Credit, Banking and Liquidity Reallocation

There are models where money and banking are substitutes, since currency and bank liabilities are alternative payment instruments. He et al. (2005) examine the safekeeping function of banking in the LW model with indivisible money. Money in their model faces the risk of endogenous or exogenous theft, while checks on deposit accounts in banks are safe. In this way, money and banking arise endogenously, and interestingly the presence of banks enhances the essential role of money in exchange process, though money and checks can compete as means of payment in equilibrium. As a natural extension, He et al. (2008) introduce divisible money and study the relationship between money and banking further. Similarly, they show that money and bank liabilities (check on deposit accounts) can circulate as media of exchange simultaneously. Notably, negative nominal interest rates are feasible and optimal under certain conditions. This unusual result is due to the assumption that holding money is risky. Similarly, Li (2011) adds a preference shock into the LW model with both currency and checking deposits as means of payment. He shows that in equilibrium checks are used only in sizeable transactions while cash is used in all transactions, since there is a record keeping cost to use checks.

Money and banking can be complements as well, since a bank is where one goes to get cash. Berentsen et al. (2007) use a competitive pricing version of the LW framework to explore the liquidity reallocation of banks, which can record financial transaction

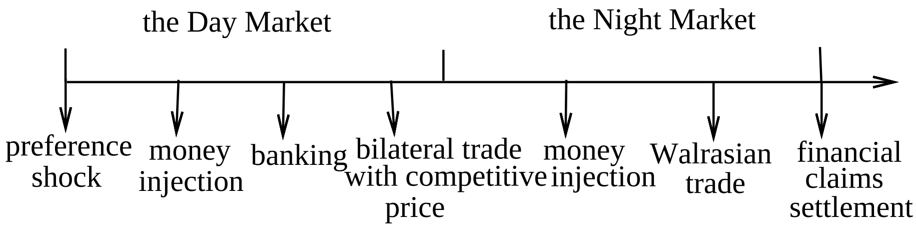

Figure 1.3: The timing of events during period t in a modified Lagos-Wright model with banking.

histories at no cost, but they cannot record goods trade histories.[19] In this case, uncollateralized credit is available as bank loans in the form of money, when buyers are cash constrained.[20] Briefly, the model has the following elements. There is one perishable good and the DM in the LW is replaced by a perfectly competitive market, as in the price taking case in Rocheteau and Wright (2005). At the beginning of the day market, there is a preference shock such that with probability $1-N$ an agent can consume but cannot produce while with probability N the agent can produce but cannot consume. Following the convention, consumers and produces are called buyers and sellers, respectively. In contrast, all agents consume and produce in the night market. We still have $M_{t+1}=(1+\tau)M_t$, but with $\tau=\tau_1+\tau_2$, where $\tau_1 M_t$ denotes the lump-sum monetary transfers in the day market and $\tau_2 M_t$ the transfers in the night market. In addition, $\tau_1=(1-N)\tau_b+N\tau_s$, where τ_b and τ_s represent the shares of DM transfer going to buyers and sellers, respectively. The timing of events is depicted in Figure 1.3. Let $V_t(m)$ be the value function for an agent with m dollars when entering the day market, and $W_t(m,L,D)$ the value function for an agent entering the night market with m dollars,L loans, and D deposits at time t. Using the notation above, then we have

$$W_t(m,L,D) \;=\; \max_{X,H,m'}[U(X)-H+\beta V_{t+1}(m')] \tag{1.28}$$

such that

$$X+\phi_t m' \;=\; H+\phi_t(m+\tau_2 M_{t-1})+\phi_t(1+i_d)D-\phi_t(1+i_l)L,$$

where i_l is the nominal loan rate, and i_d the nominal deposit rate. Notably, $W_t(m,L,D)$

[19]In addition, banks can commit. In a model where banks suffer from limited commitment, Monnet and Sanches (2014), show that banking regulation is necessary for the optimal provision of private money.

[20]Bank loans are one-period contracts. The quasi-linear preference (due to linear disutility in labour hours in the night market) implies one-period debt contracts are optimal in the LW framework.

is linear in m, L and D. Likewise,

$$\begin{aligned} V_t(m) &= (1-N)[u(q^b) + W_t(m + \tau_b M_{t-1} + L - pq^b, L, 0)] \\ &\quad + N[-c(q^s) + W_t(m + \tau_s M_{t-1} - D + pq^s, 0, D)], \end{aligned} \tag{1.29}$$

where p is nominal price of goods in the day market, q^b and q^s the corresponding quantities consumed by a buyer and produced by a seller. Note that buyers will never deposit money in the bank and sellers will never take out loans, and that sellers cannot deposit receipts of cash pq^s, since the bank closes before the onset of goods trading in the day market. Buyers solve

$$\max_{q^b, L}[u(q^b) + W_t(m + \tau_b M_{t-1} + L - pq^b, L, 0)]$$

$$\text{s.t. } pq^b \leq m + \tau_b M_{t-1} + L - pq^b, L \leq \overline{L},$$

where $\overline{L}$ is buyers' borrowing constraint. A seller faces the problem

$$\max_{q^s, D}[-c(q^s) + W_t(m + \tau_s M_{t-1} - D + pq^s, 0, D)]$$

$$\text{s.t. } D \leq m + \tau_s M_{t-1}.$$

The banking sector is perfectly competitive with free entry and there is no operating costs or reserve requirements, so a representative bank solves

$$\max_{L}(i_l - i_d)L$$

$$\text{s.t. } L \leq \overline{L}, u(q^b) - (1 + i_l)L\phi_t \geq \Gamma,$$

where Γ is a borrower's surplus by accepting a loan from another bank. The main result is that consumption loans extended by banks are welfare-improving since they can reallocate money across agents who have heterogeneous preferences for consumption and production.

Similarly, Bencivenga and Camera (2011) assume that ex ante identical agents are hit by an idiosyncratic preference shock and divided into sellers and buyers in the beginning of the CM. Each buyer has either high ($\theta^H u(x)$) or low ($\theta^L u(x)$) marginal utility of consuming the specialized goods, with $\theta^H > \theta^L$. In this way, there is an ex post inefficiency since some agents are holding idle balances while others are cash constrained. Hence, they introduce costly banking into a model of money and capital based on Lagos and Wright (2005) and Aruoba and Wright (2003), and they find that banks can reallocate liquidity, eliminate idle balances and hence improve welfare by providing agents with demand deposit contracts. Note that banks in their model cannot make loans or create private money due to enforcement problems. That is, money is the only means of payment as in Berentsen et al. (2007). Chiu and Meh (2011) study how money and banking interact to affect allocation and welfare in an environment based on Lagos and Wright (2005) and Silveira and Wright (2010). The competitive banking system operates like the one in Berentsen et al. (2007), though with two modifications, enforcement at a finite fixed cost and endogenous fractions of borrowers and lenders. They find that banking makes inflation less welfare-damaging, and in turn inflation has a nonlinear impact on the welfare effects of banking. In terms of welfare effects of banking, Rojas Breu (2013) analyzes the welfare implications of limited access to costless bank credit with full enforcement. In her model, the intra-period credit can be used simultaneously with money, but access to credit is determined by an exogenous technology, so the use of credit is exogenous like the case in Sanches and Williamson (2010). She finds that broadening the use of credit has an ambiguous effect on welfare, but when access to credit is broadened sufficiently, the cost of inflation is more likely to be effectively reduced. Similarly, Chiu et al. (2012) differentiate nominal credit arrangement (supported by record-keeping technology for financial history and facilitating inter-temporal trade of money balances) from real credit arrangement (supported by record-keeping technology for goods transaction history and facilitating inter-temporal trade of goods) and then compare welfare among real loan, nominal loan and monetary economy. They find that increasing credit usage may not necessarily increase welfare. Note that nominal bank credit complements the use of money, while real bank credit substitutes the use of money.

In an environment similar to Berentsen et al. (2007) but with some modifications, Ferraris and Watanabe (2008) consider the collateralizing role of productive capital in a competitive pricing version of the LW model with banks which do not have record-keeping or enforcement technologies. That is, agents are anonymous in both the goods and credit markets. Nevertheless, agents can obtain one-period consumption loans in the form of money from the private competitive banks, by pledging their capital asset as collateral. To allow capital accumulation, the night good is durable, while the day good is perishable. In addition, capital can be used to produce the night good, whereas capital per se or promises backed by capital cannot be used as media of exchange, hence fiat money is the only means of payment. They show that money and bank credit can coexist in a steady state monetary equilibrium, where the efficiency of capital accumulation and the effect of inflation on capital investment decision depend on whether the borrowing constraint is binding. As a follow-up work, Ferraris and Watanabe (2011) explicitly explore how fluctuations in the value of collateralizing assets generate cyclical movement in consumption and capital accumulation. Similarly, in a model with money loans collateralized by assets, Li and Li (2013) show that inflation may reduce asset prices when enforcement is sufficiently efficient. The reason is that higher inflation raises the borrowing cost, which dampens the demand for the asset as collateral.

1.5.3 Discussion

In this section, we have reviewed the papers concerned with the coexistence of money and credit, and its implications in different settings. As indicated above, we have to modify the standard LW framework in order to develop a microfounded model of money and credit. Record-keeping technology, collateral, specialized banking sector are useful elements to introduce credit into monetary models. Credit trades are intertemporal and involve a delayed settlement, so enforcement is an inherent issue for models with credit. Basically, the credit contracts in the models we mentioned are static, in the sense that they do not roll over across periods. This is due to the quasi-linear preference assumption[21]. As an example of dynamic credit model, Sanches (2011) use dynamic contracting to study the terms of long-term unsecured credit arrangements in the LW environment.

Money and credit are typically substitutes in making transactions in reality. Hence, it is natural and interesting to ask what determines the composition of payment system and how to price the payment service. Monnet and Roberds (2008) extend the LW framework to analyze the existence of relatively costly card-based payment and its pricing when fiat money is available, while Koeppl et al. (2008) introduce private information into the same underlying framework and find that settlement, the discharge of past obligations through the transfer of an asset, is an essential part of an optimal payment system.[22] Models with endogenous choice of the use of money and credit are promising tools with which to analyze the structure and evolution of payments systems.

An interesting direction for further research is to explore the price of durable assets which serve as collateral in explaining the persistence and amplification of monetary shocks (see Ferraris and Watanabe (2011) for an example). In addition, investment loans rather than consumption loans deserve more attention in microfounded monetary models. Ferraris and Watanabe (2012) allow an entrepreneur to pledge a part of the returns of a project as collateral to buy investment goods from an investor, and then study liquidity constraints and under-investment in a three-sub-period competitive LW

[21] The quasi-linear preferences also make the insurance function of banks disappear. See Williamson and Wright (2010b) for an example which incorporates the Diamond-Dybvig risk-sharing role of banks.

[22] Additionally, see Lee (2014) on the effect of a uniform pricing constraint for cash and debit card as alternative payment methods.

model.[23] For a related paper, Venkateswaran and Wright (2013) reinterpret the DM as a Kiyotaki and Moore (1997) market where money competes with collateralized credit as medium of exchanges. Due to limited commitment, unsecured credit is not feasible. Different assets have different pledgability - the extent to which they are useful to secure loans - which in turn implies various degrees of liquidity. This macroeconomic model is flexible enough to encompass money, capital, and other assets, and hence can be used to study the effects of monetary policy on asset prices and financial innovation like home-equity loans. For example, in He et al. (2015), housing bears a liquidity premium because it collateralizes consumption loans. Due to the self-fulfilling nature of liquidity, house prices can display boom-bust cycles even in a stationary environment without any shocks. Mechanisms like this would be helpful in understanding what happened in the housing market during the recent financial crisis.

Another potentially fruitful direction is to study international banking issues. For instance, Bignon et al. (2015) develop a two-country version of the banking model in Berentsen et al. (2007) and evaluate the welfare gains from a currency union when credit markets are imperfectly integrated in the sense that the cost for banks to grant credit is higher for cross-border purchases. They show that a currency union without a banking union is welfare decreasing, since monetary integration in this case may strengthen default incentives and worsen credit rationing.

1.6 Liquidity, Asset Prices and Monetary Policy

In this section, liquidity refers to an asset's usefulness to directly facilitate exchange as means of payment or indirectly as collateral to secure credit. Many factors affect the liquidity of assets. For example, the intrinsic properties such as portability, storability, divisibility and recognizability, and the extrinsic determinants like informational frictions and even subjective beliefs. In a frictional economy, liquidity is a valuable property. As a special case, fiat money can have positive value, though it is intrinsically valueless, i.e., the discounted sum of its dividends is zero. Clearly, money and other intrinsically valued assets of varying liquidity are supposed to compete as media of exchange. In the following, we will investigate the role of liquidity in determining asset prices within the context of monetary economies, and how monetary policy and asset prices interact via the liquidity channel. From the modelling perspective, we

[23]In this paper, money is the medium used to transfer resources on the spot, while liquidity refers to the availability of a medium to transfer resources over time. Apparently, money can be used as a medium of spot trade and a medium of trade over time. With this fact in mind, the authors develop a liquidity constraint model with the notion of limited pledgeability of returns of a project.(e.g., Kiyotaki and Moore (2002)).

generalize the search theory of money to become a search theory of liquidity.

1.6.1 Asset Prices with Liquidity Premia

Many asset pricing anomalies in finance can be satisfactorily explained when we take into account the transaction role of assets. Lagos (2010a) develops a liquidity-based asset-pricing model by recasting the Lucas (1978) consumption-based asset pricing model into the LW environment. In the model, there are three divisible and non-storable consumption goods: general goods, apples as dividends from Lucas trees and endowed coconuts. There is no fiat money, but two perfectly divisible real financial assets exist, durable equity shares against the trees and one-period risk-free government-issued real bonds. Each tree backs up one equity share which proves the holder's ownership of a tree and gives her the right to collect the stochastic apple dividends, while each of the bonds pays off an apple at maturity. In the DM, agents trade coconuts and financial assets and the terms of trade are determined by bargaining, while in the CM, agents trade apples, labour services, general goods and financial assets. He then shows that this model is well-suited to rationalize the equity-premium puzzle and the risk-free rate puzzle both qualitatively and quantitatively. Nosal and Rocheteau (2013) extend the trading mechanism in Zhu and Wallace (2007) and show a LW style monetary model can generate rate-of-return-differences between fiat money and a real asset in fixed supply, without imposing any exogenous trading restrictions and without violating Pareto-efficiency in pairwise trades. In order to characterize the generation and bursting of bubbles, Rocheteau and Wright (2013) modify the LW framework by using liquid real assets rather than unbacked fiat money as a means of payment. They show that strategic complementarity between buyers' asset holdings and sellers' participation decisions is a mechanism to generate multiple steady-state equilibria.

In addition, the decentralized market in the highlighted monetary model is ideal to model OTC asset markets. Li et al. (2012) develop a search-theoretic model of over-the-counter markets by combining useful elements in Lagos and Wright (2005), and Duffie et al. (2005). They feature the vulnerability of the asset to fraud as a fundamental determinant of assets' liquidity and show that this friction can contribute to the endogenous cross-sectional differences in liquidity premia.[24]

[24] Relatedly, Li and Lin (2014) study how the threat of payments fraud affects the liquidity of deposits and banks' capacity in providing credit in the form of overdraft.

1.6.2 Monetary Policy and Asset Prices

The effects of money on asset returns is another issue that recently has attracted much attention in the search-theoretic models of money. Geromichalos et al. (2007) exploit a multiple-asset version of the LW framework with real assets (Lucas trees) and money as media of exchange to study the relationship between asset prices and monetary policy. The durable real assets exogenously fixed in supply deliver the general good as dividends to their shareholders, who can trade the shares in the CM. They find a negative relationship between inflation and asset returns, given money and the real asset have similar liquidity properties. Intuitively, inflation increases and the return on money decreases when the money supply grows. Since there are no prior differences in the liquidity properties between fiat money and real assets, the rate of return on both assets has to be equal in equilibrium.

Lagos (2011) moves a step further by assuming exogenous differences in the acceptability of assets and explores interaction between monetary policy and asset prices. More specifically, a set of infinitely lived Lucas trees measured at the number of agents is introduced into the standard model. Each tree yields an equal and random amount of fruit e_t in the CM of every period t, and each tree backs up one durable and divisible equity share which proves the holder's ownership of a tree and gives her the right to collect the stochastic fruits. Therefore, e_t can be considered as an aggregate shock and exogenously determined. The underlying e_t process is assumed to be Markovian with appropriate properties. Agents know the actual realization of e_t at the onset of the DM. As usual, the three consumption goods, general goods, special goods and fruit, are non-storable and perfectly divisible. Note that fruit and general goods are homogeneous goods, while special goods come in many varieties. Let $\mathbf{s}_t = (e_t, M_t)$ be the aggregate state vector at time t. $\mathbf{a} = (a^s, a^m)$ denotes the portfolio of an agent who holds a^s equity shares and a^m dollars. Now $V(\mathbf{a}, \mathbf{s})$ and $W(\mathbf{a}, \mathbf{s})$ are the value function for an agent entering the DM and CM, respectively. Then, we have

$$\begin{aligned} W(\mathbf{a}_t, \mathbf{s}_t) &= \max_{y_t, X_t, H_t, \mathbf{a}_{t+1}} \{U(y_t) + X_t - H_t + \beta \mathbb{E}_t V(\mathbf{a}_{t+1}, \mathbf{s}_{t+1})\} \\ \text{s.t. } y_t + \zeta_t X_t + \boldsymbol{\phi}_t \mathbf{a}_{t+1} &= (\phi_t^s + e_t) a_t^s + \phi_t^m (a_t^m + \tau M_t) + \zeta_t H_t \end{aligned}$$

and $0 \le y_t, 0 \le H_t \le \overline{H}, 0 \le \mathbf{a}_{t+1}$, where $\boldsymbol{\phi}_t \equiv (\phi_t^s, \phi_t^m)$. In words, an representative agent consumes y_t units of fruit and X_t general goods, works H_t hours, and chooses to hold an end-of-period asset portfolio $\mathbf{a}_{t+1}$. Note that here the numeraire is fruit, ζ_t is the relative price of the general good, ϕ_t^s is the ex-dividend price of an equity share, and ϕ_t^m is the price of fiat money. We just point out that $W(\mathbf{a}_t, \mathbf{s}_t)$ satisfies the desired properties as $W_t(m)$ in the benchmark model. In a bilateral meeting between a buyer with portfolio $\mathbf{a}_t$ and a seller with $\tilde{\mathbf{a}}_t$, the terms of trade are determined by Nash bargaining where the buyer has all the bargaining power ($\theta = 1$). Let $(q_t, \mathbf{d}_t)$ with $\mathbf{d}_t = (d_t^s, d_t^m)$ denote the quantity of special good, equity share and money traded. Then we have

$$\max_{q_t, \mathbf{d}_t \leq \mathbf{a}_t} [u(q_t) + W(\mathbf{a}_t - \mathbf{d}_t, \mathbf{s}_t) - W(\mathbf{a}_t, \mathbf{s}_t)]$$
$$\text{s.t. } W(\tilde{\mathbf{a}}_t + \mathbf{d}_t, \mathbf{s}_t) - q_t \geq W(\tilde{\mathbf{a}}_t, \mathbf{s}_t).$$

Note that $c(q) = q$ here. Then using the procedure applied in LW, the value function in the DM is

$$V(\mathbf{a}_t, \mathbf{s}_t) \;=\; \sigma u[q(\boldsymbol{\lambda}_t \mathbf{a}_t)] - q(\boldsymbol{\lambda}_t \mathbf{a}_t) + W(\mathbf{a}_t, \mathbf{s}_t),$$

where $\boldsymbol{\lambda}_t = (\lambda_t^s, \lambda_t^m)$ with $\lambda_t^s \equiv \frac{1}{\zeta_t}(\phi_t^s + e_t)$ and $\lambda_t^m \equiv \frac{\phi_t^m}{\zeta_t}$. Note that $\alpha = 1$ and $\delta = 0$, in comparison with the basic model. Also note that money and equity shares coexist as means of payment in the model, while the latter are subject to price fluctuations, since they are claims to the Lucas trees with risky returns. After characterizing the equilibrium, Lagos shows that equity prices do not depend on shares' liquidity property at the Friedman rule, while there are liquidity premia in assets prices under a set of monetary policies that target a constant and positive nominal interest rate. In particular, this liquidity channel can support persistent deviations between the real prices of assets and their fundamental values, when monetary policy persistently deviates from the Friedman rule.

Even further, Lester et al. (2012) adopt a multiple-asset version of the LW model with proportional bargaining to study how asymmetric information associated with recognizability generates endogenous liquidity differentials, and the implications for monetary policy. They show that multiple equilibria can exist, with different transaction patterns, which are not invariant to monetary policy. In particular, small changes in information structure may generate large responses in asset prices, allocations and welfare. In comparison, the liquidity in Jacquet and Tan (2012) is endogenous as well, that is, different hedging properties lead to different liquidity properties across assets. But they focus on how monetary policy influences the market liquidity[25] of assets and ultimately asset price dynamics. Another notable difference is that the model is an overlapping generations version of the LW framework with infinitely-lived households and finitely-lived entrepreneurs. Geromichalos and Herrenbrueck (2013) extend the LW framework with three assets: a real asset, a risky financial asset and fiat money, and correspondingly with three markets to study the role of money in over-the-counter markets. An interesting innovation is that they explicitly consider the securitization

[25]See Brunnermeier and Pedersen (2009) for the distinction between market liquidity and funding liquidity. Basically, market liquidity refers to the re-saleability of assets, while funding liquidity concerns borrowing constraints of agents.

process whereby agents transform a safe real asset into a risky financial asset.

Turning to international issues, Rose and Spiegel (2012a,b) make the assumption that unrecognized assets are not accepted in a bilateral match, and then develop an international version of the LW model to account for the surprising dollar appreciation during the recent global financial crisis. For comparison, Geromichalos and Simonovska (2014) develop a two-country version of the LW setup with real financial assets and without fiat money to study how liquidity properties of domestic and foreign assets can account for the "home bias" phenomenon in international portfolio choice.

Furthermore, the liquidity properties of assets also have crucial implications for unconventional monetary policy. Williamson (2012a) integrates financial intermediation theory a la Diamond and Dybvig (1983) into monetary search framework a la Lagos and Wright (2005) to model explicitly public and private liquidity, and study unconventional monetary policy like quantitative easing.[26] He finds that money can have persistent nonneutral effect, and a liquidity trap can exist in equilibrium away from the Friedman rule.

1.6.3 Discussion

The literature surveyed in this section demonstrates that the unified framework is quite useful to examine questions like how frictions and policy affect the liquidity of assets, their prices, and the trading volume in these markets. As a result, finance theory and monetary theory bond even more closely.[27] Of course, this integration is an ongoing agenda. Unresolved questions and new issues are analyzed in variants of the workhorse model. The recent global financial turmoil provides us a good opportunity to scrutinize issues related to liquidity and credit with the help of search-based monetary models. Berentsen et al. (2014a) provide a new example, which shows that restricting access to financial markets - in order to reduce the frequency of trading - can be welfare improving, due to a pecuniary externality of portfolio choices between money and nominal government bonds.[28] In the following section, we will deal with a traditional but central topic in monetary economics.

[26] Also see Williamson (2012b),Williamson (2014a), Williamson (2014b), and Herrenbrueck (2014).

[27] For some recent examples, see Lagos and Rocheteau (2009), Mattesini and Nosal (2013), Lagos and Zhang (2014), and Lester et al. (2014).

[28] In a similar environment, Geromichalos et al. (2014) shows that a liquidity premium exists in the primary bonds market and rationalize the term premium by the idea that short maturity bonds are inherently more liquid.

1.7 Monetary Propagation and Business Cycles

How are policy-induced changes in the nominal money stock or the short-term nominal interest rate propagated to affect nominal variables like the price level and real variables such as aggregate output and employment over business cycles? This question about the monetary transmission mechanism is one of our focuses in this section. Specific channels of monetary propagation including interest rates, exchange rates, equity and real estate prices, bank lending, and firm balance sheets have been identified in traditional monetary business cycle models. As we will see, the search theory of money provides us a new way with microfoundations to construct general equilibrium models in order to analyze how monetary shocks affect output, prices, and interest rates. On the normative side, we will review papers concerning optimal monetary policy and the welfare costs of inflation.

1.7.1 Monetary Transmission Mechanism

The search theory of money has been used to examine the relationship between inflation and output. Rocheteau et al. (2007) adopt the LW framework with nonseparable preferences and indivisible labour (i.e., $h, H \in \{0, 1\}$) to derive the inflation-unemployment trade-off, without positing any permanent nominal rigidities, any departure from rational expectations or any form of money illusion. The implied Phillips curve can be upward or downward sloping, depending on cross-derivatives of the utility function. Intuitively, since inflation is a tax on economic activity in the DM, inflation reduces unemployment when either CM goods are substitutes for DM goods or leisure is complementary with DM activity, and vice versa. In addition, the optimal policy is the Friedman rule, even though the Phillips curve provides a long-run, stable, trade-off for monetary policy. We want to mention that agents act as if they have quasi-linear utility, due to the assumption of indivisible labour. Dong (2011a) uses competitive search in place of Nash bargaining, and incorporates free entry decisions by sellers into Rocheteau et al. (2007) model. However, the results are similar.

In contrast, Berentsen et al. (2011) integrate the Mortensen-Pissarides (MP) model of unemployment rather than the indivisible labour model into the LW model with separable and quasi-linear utility, and obtain an unambiguous upward-sloping long-run Phillips curve. In their model, each period is divided into three sub-periods, during which a MP labour market, the DM and the CM operate sequentially. A $[0, 1]$ continuum of households, indexed by h, and a arbitrarily large number of firms, indexed by f, interact in these three markets, where the former work, consume and the latter maximize profits and pay dividends. Let e represents employment status, with $e = 1$ if h and f is matched, and $e = 0$ otherwise. $U_e^j(z)$, $V_e^j(z)$ and $W_e^j(z)$ denote the value function for the MP, DM and CM markets, where $j \in \{h, f\}$, $e \in \{0, 1\}$ and $z \in [0, \infty)$ are real balances. A household h entering the CM with m dollars solves

$$W_e^h(z) \;=\; \max_{X,\hat{z}} \{X + (1-e)\mathcal{L} + \beta U_e^h(\hat{z})\}$$

subject to

$$X \;=\; ew + (1-e)b + \Delta - T + z - \hat{z},$$

where X is consumption of a general good, $\mathcal{L}$ is the utility of leisure, w the wage, b unemployment insurance benefits, Δ dividend income, T a lump-sum tax. $z = m/p$, $\hat{z} = \hat{m}/p$, where p is the current price level, and $\hat{m}$ is the money taken out of the CM and into the next period. Note that the utility in CM is linear, and that w is received in AD, even though matching occurs in MP. As in LW, $W_e^h(z)$ is linear in z, and $\hat{z}$ is independent of z. In the DM, households buy search goods q from firms, yielding utility $v(q)$ with $v(0) = 0$, $v' > 0$, and $v'' < 0$, and the value function

$$V_e^h(z) \;=\; \alpha_h v(q) + \alpha_h W_e^h[\rho(z-d)] + (1-\alpha_h)W_e^h(\rho z),$$

where is α_h the probability of trade, (q, d) the terms of trade, $\rho = p_t/p_{t+1}$. α_h is endogenously determined by constant-return-to-scale matching function $\mathcal{M}(B, S)$, that is, $\alpha_h = \mathcal{M}(B, S)/B = \mathcal{M}(Q, 1)/Q$, where $Q = B/S$, B and S are the measures of buyers and sellers in the DM, respectively. Standardly, $\mathcal{M}(Q, 1)$ is strictly increasing in Q, with $\mathcal{M}(0, 1) = 0$ and $\mathcal{M}(\infty, 1) = 1$, and $\mathcal{M}(Q, 1)/Q$ is strictly decreasing with $\mathcal{M}(0, 1)/0 = 1$ and $\mathcal{M}(0, 1)/\infty = 0$. Note that $B = 1$ in equilibrium, since every h participates in the DM. In contrast, only matched firms can participate in the DM, since unmatched firm has nothing to sell, given inventories are liquidated in the CM. Let u denote unemployment entering the DM, then $\alpha_h = \mathcal{M}(1, 1-u)$. For households in the MP market,

$$U_1^h(z) \;=\; V_1^h(z) + \pi[V_0^h(z) - V_1^h(z)]$$

$$U_0^h(z) \;=\; V_0^h(z) + \lambda_h[V_1^h(z) - V_0^h(z)],$$

where π is the job destruction rate and λ_h the job creation rate. Job destruction is exogenous, but job creation is given by a matching function $\mathcal{N}(u, \nu)$, that is, $\lambda_h = \mathcal{N}(u, \nu)/u = \mathcal{N}(1, \xi)$, where $\xi = \nu/u$ is labour market tightness, u unemployment and ν vacancies. $\mathcal{N}$ has similar properties to those of $\mathcal{M}$. Firms, like sellers in LW, do not carry money out of the CM. In the MP market, we have

$$U_1^f = \pi V_0^f + (1-\pi)V_1^f$$

$$U_0^f = \lambda_f V_1^f + (1-\lambda_f)V_0^f,$$

where $\lambda_f = \mathcal{N}(u,\nu)/\nu = \mathcal{N}(1,\xi)/\xi$. Let y denote output in a match and be measured in units of the CM good. Firms takes it to the DM and search trading opportunity with households. q units traded incur a transformation cost $c(q)$, with $c' > 0$ and $c'' > 0$. Hence, $y - c(q)$ is left over to bring to the CM. In the DM, the value function is

$$V_1^f = \alpha_f W_1^f(y - c(q), \rho d) + (1-\alpha_f)W_1^f(y, 0),$$

where $\alpha_f = \mathcal{M}(B,S)/S$. In the CM,

$$W_1^f(x,z) = x + z - w + \beta U_1^f,$$

where x is inventory, z real balances, and w wage commitment. As is standard, unmatched firms can pay χ in units of the general good in the CM to enter the next MP market with a vacancy. Hence, the value function for free entry is

$$W_0^f = \max\{0, -\chi + \beta\lambda_f V_1^f + \beta(1-\lambda_f)V_0^f\},$$

with $V_0^f = W_0^f = 0$. To close the model, the government faces the budget constraint $G + bu = T + \tau M/p$, where G is government consumption, b the unemployment insurance benefit, T the lump-sum tax, and τ money growth rate. As in LW, agents take the price as given in the CM, and bargain over the terms of trade in MP and the DM. For completeness, We list the key conditions for steady state equilibrium below, but skip all the technical details. The equilibrium in the DM is characterized by

$$\frac{i}{\mathcal{M}(1, 1-u)} = \frac{v'(q)}{g'(q,\theta)} - 1, \tag{1.30}$$

where $i = (1+\tau)/\beta - 1$ is nominal interest rate, $g(q,\theta)$ equal to $z(q,\theta)$ in (1.8) with u and u' replaced by v and v', respectively. The (u,q) relationship in (1.30) is called the LW curve by the authors. The labour market equilibrium is given by

$$\chi = \frac{\eta\frac{\mathcal{N}(u,\nu(u))}{\nu(u)}\{y - b - \mathcal{L} + \frac{\mathcal{M}(1,1-u)}{1-u}[g(q,\theta) - c(q)]\}}{r + \pi + (1-\eta)\frac{\mathcal{N}(u,\nu(u))}{u}}, \tag{1.31}$$

where η is the bargaining power of firms, r real interest rate, and $\nu(u)$ implicitly determined by steady-state condition $(1-u)\pi = \mathcal{N}(u,\nu)$. The (u,q) relationship in (1.31) is termed the MP curve in the paper. The intersection of the LW curve and the MP curve delivers a general equilibrium. It can be shown that $\partial u/\partial i > 0$, that is, there is an upward-sloping long-run Phillips curve. Intuitively, a higher i increases the opportunity cost of holding money, leading households to economize on real balances, which worsens firms' profitability, and ultimately reduces employment.

However, in a similar environment, Liu (2009) shows that the long-run Phillips curve relationship can be either positive or negative, if the observation that inflation affects unemployed agents more heavily than employed ones is considered. The basic logic is like this. On the one hand, increasing inflation reduces the value of outside options for the employed, which lowers the reservation wage, thereby holding back unemployment. On the other hand, inflation as a tax on the cash-intensive activities, reduces firms' return to job creation and hence raises unemployment. The equilibrium inflation-unemployment relationship can be either positive or negative, depending the relative strength of these two opposing effects. About the short-run relationship between unanticipated inflation and output, Huangfu (2009) considers the effect of asymmetric information over monetary shocks in a competitive search environment based on Rocheteau and Wright (2005). She finds that uninformed sellers have to produce more in order to extract information from buyers when positive monetary shocks hit the economy. Taking stock, the nature of Phillips curve, particularly in the long-run, is a debatable issue once again. A potentially productive direction for further research is to combine features of monetary search theory with reasonable elements of the mainstream New Keynesian macroeconomics. In this spirit, Aruoba and Schorfheide (2011) provides an illuminating example. They introduce price rigidities into the centralized market and construct an estimable search-based DSGE model for business cycle analysis.

There are other researches using search-theoretic models in order to shed new light on other issues about monetary transmission mechanism. Guerrieri and Lorenzoni (2009) employ a three-sub-period version of the LW model to examine how liquidity constraints resulting from limited credit access and a low value of real money balances affects the response of an economy to aggregate shocks. More specifically, the economy is populated by a $[0,1]$ continuum of infinitely lived households composed of a consumer and a producer. In the first two sub-periods, consumers and producers travel to spatially separated markets, or islands, where competitive goods markets exist. In addition, the consumer and the producer from the same household do not communicate or share resources while travelling during these two sub-periods, but convene at the end of each sub-period. In the last sub-period, all consumers and producers trade in a single centralized market. In this way, they find a coordination motive[29] which can amplify the effect of aggregate shocks on output and induce greater comovement across different sectors of the economy. Focusing on the quantitative implications of a precautionary demand for money in explaining the business cycle behaviour of nominal aggregates including velocity, Telyukova and Visschers (2013) develop a combined model with the features of the Lucas and Stokey (1987) cash-credit good models and the Lagos and Wright (2005) model. In standard New Keynesian models, price stickiness plays a prominent role in analyzing the transmission of nominal and real shocks. Head et al. (2012) embed the Burdett and Judd (1983) pricing mechanism into a version of the LW model, where sticky prices are an endogenous outcome and can be viewed as a simple corollary of price dispersion. In particular, they find money is neutral, though not superneutral in the model. Hence, the logical basis that nominal rigidities imply nonneutrality is called into question. Aruoba et al. (2014) examine whether household production is a potentially important channel of monetary transmission in an extended Aruoba et al. (2011) model, which is also based on the LW framework. They find inflation unambiguously reduces market consumption and employment.

1.7.2 Optimal Monetary Policy

Several models with microfoundation show that the zero nominal interest rate policy is optimal. Lagos and Rocheteau (2005) find that under the benchmark LW model with bargaining, the Friedman rule is always optimal but in general inefficient, while efficiency is restored with competitive search, by which they also formalize the idea that agents try to avoid the inflation tax through socially wasteful efforts. That is, they endogenize the search intensity of buyers. Dutu et al. (2009) embed the competing auctions framework a la Peters and Severinov (1997) into the LW model with directed search, and they discover that the Friedman rule is optimal, whether buyers bid prices or quantities. Using a mechanism design approach in a version of the LW model, Andolfatto (2013) shows that the Friedman rule, implementable via extracting money from agents with positive balances in the day market, is desirable but not incentive-feasible, when agents are sufficiently impatient. More specifically, all agents in the day can consume or produce, but at night an agent is hit by a private idiosyncratic shock, which determines whether he has an opportunity to produce or a desire to consume with equal probability. If agents are sufficiently impatient, then ex-consumers in the day submarket will find it undesirable to consume less to hold money. In this case, the monetary authority has to slow down the speed of money withdrawal, since in this way agents are induced to accumulate money in order to insure themselves against

[29] That is, agents are less willing to trade (buy and sell) when they expect others to trade less.

future desire to consume. In the quest for a class of monetary policies that make the Friedman Rule optimal and robust, Lagos (2010b) augments the LW model to allow for both risky equity shares on Lucas trees and money as means of payment, just as in Lagos (2011). In this way, aggregate liquidity shocks are introduced, and he shows that this kind of monetary policies have two defining characteristics about the level and the growth rate of money supply. That is, the money supply must be arbitrarily close to zero for an infinite number of dates, and asymptotically, the averaging growth rate of the money supply over the dates when money being essential, must be no less than the rate of time preference. Masters (2013) adds additional realism to study the welfare effects of inflation in retail markets. Specifically, he develops a model with prior production and imperfectly directed search[30], based on Rocheteau and Wright (2005). Given free-entry of sellers, the Friedman rule is optimal.

On the other hand, there are models where the Friedman rule is suboptimal. Bhattacharya et al. (2005) provide examples including the LW model with heterogeneous preferences, a turnpike model of Townsend (1980), and an OLG model with stochastic relocation in tradition of Schreft and Smith (1997), where the Friedman rule is not the ex post welfare maximizing monetary policy. They show that a necessary condition for the Friedman rule to be suboptimal is that changes in the rate of growth of the money supply have redistributive effects. A type-specific lump-sum transfer is then required to restore the optimality. Craig and Rocheteau (2008b) introduce menu costs into a continuous-time version of the LW model. Provided menu costs are small, they show that the optimal monetary policy requires a deflation, which does not necessarily correspond to the the Friedman rule. Hoerova et al. (2012) argue that deviation from the zero nominal interest rate policy is a credible way for central banks to transmit their information to private investors in a model based on Berentsen and Monnet (2008)[31].Under the restriction that all trade is voluntary in the sense that the government cannot use coercive lump-sum taxation as a policy instrument, Andolfatto (2010) demonstrates that the Friedman rule is not incentive-feasible in a variant of the LW model with two centralized markets. In particular, he shows that a non-negative inflation rate and a strictly positive nominal interest rate are equilibrium properties of an array of welfare-improving incentive-feasible policies. However, a nonlinear mone-

[30]Imperfectly directed search here means that buyers may not know everything about a good before they go purchasing it. In contrast, the notion of partially directed search refers to the case that sellers cannot commit to their posted prices but use them to signal the quality of their goods. Due to imperfect knowledge of goods, buyers' instantaneous utility of consuming the DM goods is $\epsilon u(q)$, where ϵ is a random variable and represents match-specific shocks.

[31]The model is a variant of the LW model with three perfectly competitive markets. The authors aim to study the three stylized facts of channel systems, also see Berentsen et al. (2014b) on the implementation of monetary policy via a channel system or a floor system. In an extension of this paper, Kahn (2013) examine competition between private and public payment systems.

tary mechanism involving the payment of interest on money holdings can implement the unconstrained efficient allocation in a pure monetary economy even if lump-sum taxes cannot be used as an instrument. Gomis-Porqueras and Sanches (2013) impose an additional restriction that interest payments can be made only to agents whose trade histories are registered to further examine the same monetary mechanism in an extended LW model with private credit and fiat money. A positive inflation rate is necessary to make credit arrangement incentive-feasible.

Another two related papers study optimal monetary policy in models with fiat money and inside money. Berentsen and Waller (2011b) adopt an extension of the LW environment in which anonymous agents have access to a credit market.[32] Monetary policy consists of a short-run and a long-run component, the former focused on stabilizing real activity in the presence of short-run shocks and the latter on the long-run inflation trend. They show that stabilization policy is useful in controlling inflation expectations under price-level targeting, and the optimal policy involves smoothing nominal interest rates. Head and Qiu (2011) extend the analysis in Berentsen and Waller (2011b) by replacing the credit market with a large number of identical private banks. In their model, aggregate shocks affect households asymmetrically due to exogenous bank credit constraints. They demonstrate that positive short-term nominal interest rates are desirable in some aggregate states, since this interest rate policy can affect inside money creation and the distribution of wealth.

Furthermore, interactions between fiscal and monetary policies have been studied in models with micro-foundations for money. Aruoba and Chugh (2010) study the properties of optimal monetary and fiscal policy in business cycles by reframing the dynamic Ramsey problem(with commitment) in the LW framework with government bonds and capital assets. Fiscal and monetary instruments are restricted to production and capital taxes in the centralized market, and to open market operations. Equilibrium in their model is not efficient, and the Friedman rule is generically not optimal. In addition, a subsidy on capital income is desirable to offset capital underaccumulation due to the holdup problem associated with capital investment and non-optimality of the Friedman rule. Hagedorn (2010) discusses tax cycles or tax smoothing in Ramsey taxation problems in several frictional economies. In the Lagos and Wright (2005) monetary economy, he finds that tax smoothing is not always the optimal policy even if nondistortionary taxes are infeasible. Without considering government debt, Gomis-Porqueras and Peralta-Alva (2010) consider the role of optimal monetary and fiscal policies in restoring efficiency of monetary equilibria in the LW model with general-

[32] Berentsen and Waller (2015) introduce congestion externality associated with buyers' entry decision into Berentsen and Waller (2011b), and show that the optimal policy deviates from the Friedman rule and optimal stabilization policy is nontrivial.

ized Nash bargaining. The Friedman rule is always an optimal policy regardless of the bargaining power of the buyer, while a monetary production subsidy can be used to increases the return of holding money and hence increase output in the decentralized market. In contrast, when costless coercive lump-sum monetary transfers are not available, introducing sales taxes into the decentralized market is welfare-improving, and the optimality of the Friedman rule depends largely on the bargaining power of the buyer. Similarly, Gomis-Porqueras et al. (2013a) examine the same question in a model with search frictions in both labour and goods markets as in Berentsen et al. (2011). Given governments' limited ability to make intertemporal commitment, Martin (2011) studies the joint determination of fiscal and monetary policies in a variant of the LW model with a benevolent government. When net nominal government liabilities are positive, the tradeoff between the objective of smoothing distortions intertemporally and the time-consistency problem created by the interaction between debt and monetary policy determines a unique equilibrium with positive taxes and inflation above the Friedman rule. As an extension, Martin (2013a) introduces aggregate uncertainty and limited commitment into the model in Martin (2011) and then studies optimal time-consistent policy in three variants of the underlying monetary economy featuring different environmental frictions. The long-run response of policy variables to permanent changes in fundamentals is quantitatively different across environments, mainly due to the idiosyncratic behaviour of the money demand.

The distributional effects of monetary policy cannot be discussed in Lagos and Wright (2005), due to the degenerate distribution of money holdings in equilibrium. In fact, random monetary injections in the LW model are neutral regardless of when they occur. Berentsen et al. (2005) develop a three-Walrasian-market version of the LW model to study the effects of random money injections by the central bank that occur in periods when the distribution of money is nondegenerate.[33] More specifically, agents need to trade in two competitive markets (i.e., the price taking case in Rocheteau and Wright (2005).) before they can readjust their money holdings in the centralized market. They show that in all equilibria monetary shocks have temporarily nonneutral effects, which are direct consequences of the nondegenerate distribution of money holdings. The monetary shocks are introduced through unexpected changes in the money supply, that is, $M_{t+1} = \gamma_t M_t$, where γ_t is a random variable with

$$\gamma_t = \begin{cases} \gamma^H = \mu(1+\epsilon^H) \text{ with probability } \mathcal{P}, \\ \gamma^L = \mu(1-\epsilon^L) \text{ with probability } 1-\mathcal{P}, \end{cases}$$

[33]See Jiang and Shao (2014) for a similar setup, but with different focus. They study how credit expansion affects cash velocity, allocation and money demand.

and μ, ϵ^L, $\epsilon^H > 0$, and $\mathcal{P} = \epsilon^L/(\epsilon^L + \epsilon^H)$ such that $\mathbb{E}(\gamma_t) = \mu$. Williamson (2006) studies the role of limited participation in financial markets in generating distributional effect of monetary injections or withdrawals. He adapts the LW framework by letting centralized and decentralized markets open concurrently. Changes in the money supply have persistent real effects regardless of the shocks being anticipated or unanticipated. As expected, the Friedman rule is not optimal since redistribution is desirable. In the presence of private information over productivity, Galenianos and Kircher (2008) examine the distributional effects of inflation among heterogeneous agents in a multilateral matching version of the LW model with divisible money and indivisible goods. In particular, a second-price auction is used to determine the terms of trade. The Friedman rule is desirable since inflation acts as a regressive tax in their model. Xiang (2013) investigates the trade-off between distribution and production effects of inflation using a model similar to the one in Andolfatto (2011). The Friedman rule is likely to be suboptimal when both distribution and production efficiencies matter. Sanches and Williamson (2011) develop a variant of the LW model with segmented centralized markets and asymmetrically informed agents. Price dispersion exists in equilibrium, and this nondegeneracy lead to private-information inefficiency in decentralized trade. The Friedman rule is welfare-improving, since it can correct both the informational inefficiency and the intertemporal distortion. More recently, Sun (2012) investigates the distributional effects of monetary and fiscal policies simultaneously in a modified LW environment with competitive search and non-degenerate money distributions in equilibrium. The model features a block recursive structure and hence tractably delivers endogenous dispersions of prices, income and wealth. When heterogeneous agents suffer from uninsurable idiosyncratic risks, long-run inflation and income taxation have distinct effects on real activities and welfare. However, an optimal policy mix, that is, simultaneous determination of monetary and fiscal policies, is required to maximize social welfare.

Besides all the papers mentioned above, the LW framework has also been used as a basis to study other issues in monetary policy. Based on a competitive pricing version of the LW framework with heterogeneous agents, Berentsen and Strub (2009) discuss the determination of monetary policy under alternative institutional rules including simple-majority voting, super-majority voting, and bargaining in the central bank's decision-making. The efficiency of equilibrium and optimality of the Friedman rule depend on specific institutional arrangements. Jacquet and Tan (2011) investigate the closely-related dual role of money, that is, as self-insurance device (store of value) and means of payment, in an overlapping-generations version of the LW model. They find atypical results about the effects of inflation when agents face a risk in the centralized market. More specifically, if idiosyncratic shocks are correlated across agents, that is, when there are aggregate shocks, a state-contingent inflation rate(featuring a pro-cyclical monetary policy) is desirable for agents to better insure themselves against the shocks. Another very interesting paper is Burdett et al. (2015). They view search and bargaining frictions, taxation and inflation as transactions costs, and study household formation by analogy with the Coasian theory of firm formation. A quite straightforward result is that inflation stimulates household production and hence household formation, since inflation is a tax on market activity, which is assumed to be relatively cash-intensive for single agents. Lastly, Waknis (2014) reformulates the LW economy into an infinitely repeated game between a utility maximizing central bank and agents. In this way, the money supply is endogenous and the optimal inflation rate is positive.

So far, we have surveyed the papers which study optimal monetary policy basically in an analytical and qualitative manner. Logically, quantitative analysis about the welfare cost of inflation will be discussed in the following subsection.

1.7.3 Welfare Costs of Inflation

Lagos and Wright (2005) quantify the welfare cost of expected inflation under Nashing bargaining using annual U.S. data. They find that the welfare cost of 10 percent inflation is worth 3%-5% of total consumption in both the centralized and decentralized markets. Subsequent researches extend this benchmark numerical analysis along various lines. Craig and Rocheteau (2008a) generalize the LW framework by considering alternative pricing mechanisms, participation decisions, and capital accumulation decisions to estimate and compare the welfare cost of inflation. Overall, the choice of pricing mechanisms matters for the estimated results, holdup problems on capital investment tend to exacerbate the inefficiency of inflation, and endogenous participation choices and search externalities possibly render the Friedman rule suboptimal. Instructively, the authors provide a search-based microfoundation for the Bailey-Lucas "welfare triangle" methodology. Small menu costs in Craig and Rocheteau (2008b) reduce the welfare cost of inflation, since in this case inflation erodes sellers' market power and hence induces them to internalize the congestion externality (too many sellers). In addition, buyers are more willing to raise real balances, given favourable bargaining power. Earlier, Reed and Waller (2006) focused on money as a device to share consumption risk when credit arrangements are not feasible. In this context, they find that the welfare cost of inefficient risk sharing associated with 10% inflation is approximately 1%-1.5% of steady-state consumption. Faig and Jerez (2006) estimate the welfare implications of buyers' private information over preferences in a competitive search version of the LW model. Based on historical US data on velocity and interest rates, their estimation of the welfare cost of inflation is about 0.5% of GDP, which matches the area below the predicted money demand curve. More recently, Faig and Li (2009) simultaneously calculate the welfare loss of expected and unexpected inflation in the LW model with imperfect information about monetary shocks. They find that the former (around 0.25% of GDP) is far greater than the latter (below 0.0003% of GDP) for U.S. postwar data. The main reason is that monetary shocks during $1947 - 2007$ have been small. It is worth pointing out that monetary shocks may be welfare enhancing, since they increase price dispersion. Consequently, buyers tend to hold more precautionary balances and, in doing so, partly offset the inefficiently low money demand due to the inflation tax. Dong (2010) endogenizes product variety in the LW framework with Nash bargaining and competitive search as alternative trading protocols. The total welfare cost of 10% inflation ranges from 4.77% to 8.4% under bargaining and is 1.52% under competitive search. In particular, the novel welfare cost of inflation associated with product variety can be substantial (more than half of the total estimation) in a bargaining equilibrium, while it is very small in price posting equilibrium. Rojas Breu (2013) finds that broadening credit access is likely to reduce the costs of inflation. Similarly, Chiu and Meh (2011) consider the impact of banking on the welfare loss of inflation. Inflation tends to be less damaging when banking is introduced to facilitate decentralized trading of production projects in an LW economy. Taking the redistributive effects of inflation into consideration, Chiu and Molico (2011) estimate the welfare cost of inflation in the LW environment without the simplifying quasi-linear preferences restriction. Under buyer-take-it-all bargaining, the gains of decreasing inflation from 10% to 0% are about 0.59% of stationary consumption, significantly lower than the Lagos and Wright estimate of 1.3%. Quite intuitively, when the distribution of money holdings across agents is nondegenerate, relatively rich agents (with above average money holdings) view inflation as a tax while relatively poor agents view inflation as a subsidy. Hence, the redistributive effect of increasing inflation has the potential to be welfare improving and offset some of the detrimental real balance effects. Indeed, the welfare costs of inflation are 40% to 55% smaller than previous estimates which abstract from the redistributive effect. Closely related, Chiu and Molico (2010) consider uninsurable idiosyncratic uncertainty about trading opportunities and a fixed cost of entering the centralized liquidity market. This combination generates a liquidity management problem and in equilibrium a non-degenerate distribution of money across agents. They find that the distributional effects of inflation lower welfare costs of inflation substantially. In contrast, Berentsen et al. (2015) assume limited participation in a version of Berentsen et al. (2007), calibrate the model to U.S. economy and show that improved access to money markets makes the allocation of liquidity more efficient, which decreases the welfare cost of inflation considerably.

Moreover, there are several related papers which focus on the “hot potato” effect of inflation. Lagos and Rocheteau (2005) are able to generate the "hot potato" effect, but this result is not robust, since it depends critically on how prices are determined in decentralized trade. Hence, Ennis (2009) modifies the LW model by asymmetrically restricting buyers and sellers' access to the centralized market in order to rationalize the classic idea that agents tend to spend their money holdings speedily in response to inflation. Liu et al. (2011) offer another approach to formalize the "hot potato" effect of Inflation. More specifically, they highlight the extensive margin rather than the intensive margin by endogenizing the participation decisions of buyers based on Rocheteau and Wright (2005). Notably, the model can unambiguously and robustly predict that a rise in inflation leads to an increase in the speed with which agents spend their money. Nosal (2011) also builds on Lagos and Rocheteau (2005), and Rocheteau and Wright (2005), but features a reservation strategy, that is, there is an opportunity cost associated with the buyer accepting a trade. When inflation rises, the reservation value falls, and hence the buyer increases the speed at which he trade. Similarly, the model presented in Dong and Jiang (2014), which is based on Rocheteau and Wright (2005) with private information, is able to capture the "hot potato" effect of inflation along both the intensive margin and the extensive margin. Unfortunately, none of the mentioned studies provide a systematically numerical analysis. In general, however, positive inflation can be desirable and hence qualitatively the welfare cost of inflation is possibly smaller when the the "hot potato" effect is considered.

Lastly, several papers deal with the welfare implication of price dispersion. Dutu et al. (2012) extend the LW model with competitive search and free entry by buyers to isolate the welfare effect of price dispersion through comparing one economy with price dispersion and the other without at the same inflation rate. They find a nonlinear welfare impact of price dispersion. That is, price dispersion improves welfare at low inflation, while decreases welfare for high inflation. The Friedman rule is efficient. See also Dutu (2013). In contrast, Wang (2014) integrates Burdett and Judd (1983) pricing into the LW framework. In this way, price dispersion is endogenous, which then amplifies the negative real balance effect of inflation.[34]

1.7.4 Discussion

About business cycle analysis, an integration between micro-founded models of money and estimatable DSGE models is an emerging direction for future research. Aruoba and Schorfheide (2011) show us a nice example. In terms of optimal monetary policy, more studies are needed to clarify the interactions between monetary policy and fiscal policy

[34]See Liu et al. (2014) for another example.

when nondistortionary policy is unavailable. Besides Ramsey policy, it is also interesting to study optimal time-consistent monetary policy (Markov-perfect equilibrium), as demonstrated in Martin (2013b) and Martin (2012). In addition, the limited commitment associated with government's intertemporal decisions, heterogeneous agents, asymmetric information and other frictions are being examined intensively. Lastly, mechanism design, as suggested by Wallace (2001) and Wallace (2010), is a promising approach to advance monetary theory. For instance, Rocheteau (2012) applies mechanism design in the LW environment to quantify the welfare cost of inflation. Using the annual data for the US over 1900-2006, the estimated welfare cost of 10 percent inflation is 0. The reason for this "surprising" result is that the trading mechanism in the decentralized market is constructed to be optimal so that the only frictions in the environment are the ones that make money essential. Hence, the larger welfare costs of inflation in previous monetary search models are probably due to the adoption of inefficient trading mechanisms like the Nash bargaining solution.[35]

1.8 Money in Economic Growth Models

Compared with monetary policy analysis, integrating monetary search with neoclassical or endogenous growth models is still largely undone, though a few papers have moved forward in this potentially fruitful direction. Aruoba and Wright (2003) introduce a labour market, capital accumulation, and a neoclassical production function into the centralized market in the LW model. However, the undesirable dichotomy indicates that this initiation is not very successful. Basically, the limitation is due to the model structure that money is essential in the decentralized sector but inessential in the centralized market, and capital is essential in the centralized market for production but plays no role in the decentralized sector. In order to remedy this problem, Aruoba et al. (2011) extend Aruoba and Wright (2003) by allowing the sellers in the decentralized market to have access to capital for production purposes. In this way, monetary policy affects centralized market investment, because inflation is a tax on decentralized trade. They show that monetary frictions matter quantitatively for the long-run effects of anticipated inflation on capital formation.

In order to get a flavour of this line of work, we outline the model below. As in the LW, the DM and CM markets open sequentially in each period. The general good in the CM is storable and produced by a $[0,1]$ continuum of firms with technology

[35] For more examples, see Araujo and Hu (2014) on unconventional monetary policy, Chiu and Wong (2014) on payment systems, Gu et al. (2013) on credit cycles, Gu et al. (2014) on the essentiality of money and credit, and Hu and Rocheteau (2015) on the effects of monetary policy on asset prices.

$F(K, H)$, where H is labour hired and K capital rented from agents. In the DM, these neoclassical firms stop operating, while a seller[36] can use his effort e and capital k with technology $g(k, e)$ to produce the non-storable specialized goods. With probability $1 - \varpi$, a bilateral meeting is monitored, and hence the buyer can trade with credit due in the next CM, whereas a meeting is anonymous with probability ϖ. The utility function is $u(x) - e + U(X) - AH$ with typical monotonicity and curvature properties. The government controls the money supply so that $M_{t+1} = (1 + \tau)M_t$, consumes G, collects a lump-sum tax T, imposes labour income tax t_h, capital income tax t_k, and sales tax t_x in the CM. Let κ denote the depreciation rate of capital, which is routinely tax deductible, and $p = 1/\phi$ the price level in the CM. Then, the government budget constraint is $G = T + t_h wH + (\nu - \kappa)t_k K + t_x X + \tau M/p$, where $w = F_H(K, H)$ is the wage rate and $\nu = F_K(K, H)$ the rental rate of capital. Let $W(m_t, k_t, l)$ be the value function for an agent entering the CM with m dollars and k units of capital and owing l from the previous DM in period t. Similarly, $V(m_t, k_t)$ is the value function for an agent entering the DM in period t. In the CM, an agent solves

$$W(m_t, k_t, l) = \max_{X,H,m_{t+1},k_{t+1}} \{U(X) - AH + \beta V(m_{t+1}, k_{t+1})\} \tag{1.32}$$

subject to

$$(1 + t_x)X = w(1 - t_h)H + [1 + (\nu - \kappa)(1 - t_k)]k_t - k_{t+1} - T + \frac{m_t - m_{t+1} - l}{p}.$$

Again, as in the baseline model, $W(m_t, k_t, l)$ is linear in m,k and l, and agents exiting the CM will choose the same (m_{t+1}, k_{t+1}). The neoclassical firms use the constant-returns-to-scale technology $F(K, H)$ to produce the general good, yielding $w = F_H(K, H)$ and $\nu = F_K(K, H)$. In the DM, we have

$$V(m_t, k_t) = \sigma V^b(m_t, k_t) + \sigma V^s(m_t, k_t) - (1 - 2\sigma)W(m_t, k_t, 0), \tag{1.33}$$

where we assume $\alpha = 1$, δ=0, while $V^b(m_t, k_t)$ and $V^s(m_t, k_t)$, respectively, denote the value function for a buyer and a seller, with

$$\begin{aligned} V^b(m_t, k_t) &= \varpi[u(q^b) + W(m_t - d^b, k_t, 0)] + (1 - \varpi)[u(\hat{q}^b) + W(m_t, k_t, l^b)], \\ V^s(m_t, k_t) &= \varpi[-c(q^s, k_t) + W(m_t + d^s, k_t, 0)] + (1 - \varpi)[u(\hat{q}^s) + W(m_t, k_t, -l^s)]. \end{aligned}$$

In the above expressions, q^b and d^b(q^s and d^s) represent the quantity of specialized

[36] To preclude capital used as medium of exchange, the capital of all agents is importable during the DM and hence buyers have to visit sellers' locations for trading. In addition, the claims against capital are counterfeitable. Therefore, money is necessary in the DM.

goods and money exchanged for being a buyer(a seller), while $\hat{q}^b$ and l^b($\hat{q}^s$ and $-l^s$) denote the quantity of DM goods traded and the value of a loan for the buyer(seller) in a monitored meeting. In addition, $c(q, k) \equiv e = g^{-1}(q, k)$ with $q = g(k, e)$. The terms of trade(q, d) and $(\hat{q}, l)$ can be determined by Nash bargaining like Lagos and Wright (2005) or price taking like Rocheteau and Wright (2005). This completes the sketch of the model without replicating the details from Aruoba et al. (2011). Recently, Aruoba (2011) studies the business cycle properties of the model in Aruoba et al. (2011) and brings it to US data. The calibrated results demonstrate that search-theoretic models are able to make better predictions about nominal variables than RBC models with flexible prices. Waller (2011) introduces exogenous labour-enhancing technological change into Aruoba et al. (2011), that is, $F(K_t, Z_tH_t) = Z_tF(K_t/Z_t, H_t)$ and $g(k_t, Z_te_t) = Z_tg(k_t/Z_t, e_t)$ with $Z_{t+1} = (1 + \rho)Z_t$, and then characterizes steady-states and transitional dynamics under proportional bargaining and competitive price taking respectively. Inflation lowers the capital intensity along the balanced growth path for both bargaining and price taking.

The modelling choice of exogenous or endogenous growth models can have distinct implications. Chu et al. (2014) develop a monetary growth model based on Aruoba et al. (2011) to analyze the effects of inflation on economic growth and social welfare. They find that inflation affects the welfare effect nonlinearly in the endogenous growth model whereas linearly in the exogenous counterpart. Quantitatively, the welfare cost of inflation under the former is up to four times as large as the welfare cost of inflation under the latter. In addition, the channel through which inflation affects economic growth in the search-based model is different from the traditional cash-in-advance model.

In comparison, two illuminating papers are worthy of close examination, though they are not directly based on the LW framework. Berentsen et al. (2012) integrate the monetary model in Shi (1997) into an endogenous growth model in order to study the effects of inflation and financial development on economic growth. Money is essential for the decentralized innovation goods market, and hence financial intermediaries arise endogenously to provide liquid funds to the innovation sector. They find that inflation and the efficiency of financial sector matter qualitatively and quantitatively for welfare and growth. Similarly, Chiu et al. (2013) develop a endogenous growth model with a market for ideas (see Silveira and Wright (2010)) to examine the roles of liquidity and financial institutions in overcoming trading and financial frictions in the knowledge market. Both papers provide a good starting point for further studies along this line.

1.8.1 Discussion

One interesting research direction is to study the role of monetary expansion in the development process, as depicted in Gurley and Shaw (1960) and McKinnon (1973). Is the increasing trend of long-run M2-to-GDP ratio (or the decreasing trend of long-run velocity) an essential aspect of the growth process, given similar situations across countries and over time? If so, how can we rationalize it in a monetary growth model? In particular, can we get a U-shaped income velocity in this model? Intuitively, at first, increasing monetization causes velocity to fall, while recently increasing financial sophistication and the growth of money substitutes cause velocity to rise. This is a pervasive pattern across countries (see Bordo and Jonung (1987)). The search-based monetary growth model could shed some new lights on these old questions.

1.9 Concluding Remarks

Lagos and Wright (2005) offer a flexible and productive framework to understand the process of exchange in the presence of various frictions, and how this process might be facilitated by institutions, including money, but also various forms of credit, financial intermediation, and the use of different assets as payment instruments or as collateral. The number of papers we have surveyed forcefully supports this claim. The topics identified in each section are being actively researched. It is increasingly clear that micro-founded models of money are more than elaborated money-in-utility or cash-in-advance monetary models.[37] As we have seen, monetary search models based on the LW framework improve our understanding in questions about the usefulness of money, the relationship among money, credit and banking, the mechanisms by which policy can affect allocations and welfare, liquidity and asset pricing, and about economic growth in monetary economies. However, much work still needs to be done, such as the issues discussed in previous sections.

[37] However, Camera and Chien (2013) argue that monetary search models are mathematically equivalent to cash-in-advance models. See Lagos et al. (2014) for a counterargument.

Appendix

1.A Shi (1997) in Detail

We will closely follow the original paper to describe the environment and characterize the equilibrium, but with necessary modifications based on Rauch (2000), Shi (1999), Berentsen and Rocheteau (2003) and Zhu (2008). Time is discrete and β is the rate of time preference. The economy is populated with a $[0, 1]$ continuum of infinitely lived households that specialize in consumption and production. Money and goods are perfectly divisible. Each household has a continuum of members with measure one, and each member acts like a robot in carrying out different tasks, regarding the household's utility as the common objective. This simplifying specification rules out potential incentive problems. At the beginning of each period, each household divides money balances evenly among its money holders. Let N_{jt}, exogenously valued at constant $N \in [0, 1]$ or endogenously chosen by household j, denotes the fraction of money holders from household j in period t. A member without money is a producer endowed with the capacity to produce his family-specific good, which is perishable and randomly chosen from a continuum of different types of goods with measure one. Production takes no time but incurs a utility cost $\phi(q)$ to produce the quantity q, with $\phi(0) = 0, \phi' > 0$ and $\phi'' > 0$. Each member in one household consumes the same subset of goods with measure $z \in (0, 1)$, which is randomly assigned and does not include the good produced by his family. Goods in the subset are equally preferred, and consuming q units of them yields utility $u(q) = aq$ with $a > 0$. After the division of money balance, each member of a household is randomly matched to one agent from other families. The two matched agents decide whether to trade. Two types of trade are possible: barter and monetary trade. In monetary trade, the money holder is subject to a cash-in-advance constraint, that is, he cannot spend more money than he is carrying. The terms of trade-the quantities of goods in barter and the quantity of the good and the amount of money in a monetary exchange-are determined by a Nash bargaining problem. Here it is assumed that the buyer and seller have the same bargaining power in every match. After goods are exchanged, members of each family bring their receipts back to the household.

Then the household allocates pooled goods evenly to its members for consumption. After consumption, the household receives a lump-sum monetary transfer τ such that the money stock $M_{t+1} = \gamma M_t$ with $\gamma \geq \beta$. This implies that $\tau_{t+1} = (\gamma - 1)M_t$. After the transfer, time proceeds to the next period and the sequence of events repeats.

Note that the specified large household construct smoothes the matching-specific risks within a household and hence eliminates aggregate uncertainty for households. This modeling device makes the distribution of money holdings across households degenerate and allows us to focus on a representative household. In the following, we will derive the binding symmetric monetary equilibria, which are defined by the three conditions below:

1. Every family holds the same amount of money $M_{t+1} = \gamma M_t$;
2. In every monetary trade, the same amount of money $L_t = M_t/N$ is traded for the constant quantity q^m of a consumption good;
3. If N_{jt} is endogenously determined, then $N_{jt} = N$ for all j and t.

During period t, in barter the efficient quantities $\bar{q}$ is given by $\phi'(\bar{q}) = a$. In fact, consider two producers from household i and $-i$ in a barter match, and they bargain over the quantities of goods to be produced, (q^b_{it}, q^b_{-it}). The utility of household i is increased by $[aq^b_{-it} - \phi(q^b_{it})]$, and the utility of household $-i$ by $[aq^b_{it} - \phi(q^b_{-it})]$. The terms of trade satisfies

$$\max_{(q^b_{it}, q^b_{-it})} [aq^b_{-it} - \phi(q^b_{it})]^{\frac{1}{2}} [aq^b_{it} - \phi(q^b_{-it})]^{\frac{1}{2}},$$

which yields $q^b_{it} = q^b_{-it} = \bar{q}$.

In a monetary trade between a household i money holder with M_{it}/N_{it} units of money, and a producer from household $-i$, L_{it} units of money are traded for q_{-it} units of consumption goods. Let λ_{it} denote the value of a marginal relaxation of the cash-in-advance constraint for the money holder, and $\omega_{jt}(j = i, -i)$ the value of a marginal unit of money for household j. By trade, the household i increase its utility by $[aq_{-it} - \omega_{it}L_{it}] \geq 0$, while the household j by $[\omega_{-it}L_{it} - \phi(q_{-it})] \geq 0$. The Nash bargaining problem is

$$\max_{(L_{it}, q_{-it})} [aq_{-it} - \omega_{it}L_{it}]^{\frac{1}{2}} [\omega_{-it}L_{it} - \phi(q_{-it})]^{\frac{1}{2}}$$

subject to the cash-in-advance constraint $\frac{M_{it}}{N_{it}} - L_{it} \geq 0$. Using Lagrange multiplier method, we can solve this problem. Denote the Lagrange multiplier by $\mu_{it,-it}$, then the first order conditions are

$$\begin{aligned}\omega_{-it}[aq_{-it} - \omega_{it}L_{it}] - \omega_{it}[\omega_{-it}L_{it} - \phi(q_{-it})] &= \mu_{it,-it},\\ L_{it}[a\omega_{-it} + \phi'(q_{-it})\omega_{it}] &= a[\phi(q_{-it}) + q_{-it}\phi'(q_{-it})].\end{aligned}$$

In symmetric equilibrium, $\omega_{it} = \omega_{-it} = \omega_t$, $L_{it} = L_t$, $q_{it} = q^m$, and $\mu_{it,-it} = \mu_t$, then we have

$$\begin{aligned}L_t\omega_t &= \frac{aq^m + \phi - \mu_t/\omega_t}{2},\\ L_t\omega_t &= \frac{a(\phi + q^m\phi')}{a + \phi'}.\end{aligned}$$

Therefore, $q^m = \bar{q}$ if the cash-in-advance constraint is not binding($\mu = 0$), and $q^m < \bar{q}$ if it is binding. In a binding monetary equilibrium, $L_t = M_t/N, M_{t+1} = \gamma M_t$, N and M are constant, hence

$$\begin{aligned}M_t\omega_t &= \frac{Na(\phi + q^m\phi')}{a + \phi'},\\ \omega_t &= \frac{1}{\gamma}\omega_{t-1}.\end{aligned}$$

Now we consider the decision of a household j with initial money holding M_{j0}. Taking other households' N, M_t as given, household j chooses N_{jt} to maximize its utility from consuming $C_{jt}(N_{jt}, N)$ net of production $\phi_{jt}(N_{jt}, N)$ and holding money M_{jt}. The optimization problem is

$$\max_{N_{jt}} \sum_{t=0}^{\infty} \beta^t (aC_{jt} - \phi_{jt})$$

subject to

$$\begin{aligned}C_{jt}(N_{jt}, N) &= z^2(1-N)(1-N_{jt})\bar{q} + z(1-N)N_{jt}q_{-jt},\\ \phi_{jt}(N_{jt}, N) &= z^2(1-N)(1-N_{jt})\phi(\bar{q}) + z(1-N_{jt})N\phi(q_{jt}),\\ M_{jt+1} - M_{jt} &= \tau_{t+1} + z(1-N_{jt})NL_{-jt} - z(1-N)N_{jt}L_{jt},\\ L_{jt} \leq \frac{M_{jt}}{N_{jt}},&\\ M_{j0} \text{ is given.}&\end{aligned}$$

Technical details and more discussions can be found in Rauch (2000). We skip this and make some remarks below. Faig (2004, 2008) argue that the LW model and the large household model in Shi (1997) can be encompassed in a more general setup, while

Berentsen and Rocheteau (2003) and Ritter (2010) point out the subtle differences in the implications of formulating bargaining solution between the two alternative frameworks. See Liu and Shi (2010) on inflation coordination, Head et al. (2010) on price adjustment and Shi (2015) on financial frictions in business cycles for examples using the Shi model.

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2 A Spatial Search Model with Money and Credit

This chapter incorporates credit into a monetary search model with explicit consideration of spatial frictions. When agents face a liquidity shortage problem, credit supported by a communal responsibility system supplements fiat money as a medium of exchange at a distance-dependent cost. In addition, the Friedman rule can be sub-optimal. Money and credit are essential in the sense of improving welfare. Finally, numerical analysis of a calibrated example supports these main findings, and also provides some comparative statics results.

2.1 Introduction

One of the main results from New Monetarist Economics (Williamson and Wright, 2010a,b) is that money or other forms of media of exchange are essential to realize gains from decentralized trade or centralized trade when agents cannot commit. In this chapter, we develop a New Monetarist search model with money and costly credit in order to study the role of these two alternative means of payment, their interactions and the effect of inflation on social welfare. The model features spatial frictions and endogenous market entry decision. The motivation comes from Greif (2006a,b) and Boerner and Ritschl (2011).

Greif (2006a,b) describes an interesting medieval institution, the community responsibility system, by which credit is supported in long-distance trade. Specifically, a local, community court held all members of a different commune legally liable for default by any one involved in contracts with a member of the local community. If the defaulter's communal court refused to compensate the injured party, the local court confiscated the property of any member of the defaulter's commune present in its jurisdiction as compensation. This institutional innovation is a credible commitment technology, if trade links between two communes is very strong. A commune could avoid compensating for the default of one of its members only by ceasing to trade with the other commune. When this cost was too high, a commune court's best response was to dispense impartial justice to non-members who had been cheated by a member of the commune. With this historical background in mind, we aim to examine the role of credit collateralized by the community responsibility system in inter-community exchange by considering a monetary search model with spatial elements.

Having a similar objective, Boerner and Ritschl (2011) use the Lagos and Wright (2005) model to examine the coexistence of money and credit in the presence of communal responsibility. It is an interesting idea to study this historical issue in monetary search models. In their model, buyers can use interest-bearing-bond-like credit to purchase goods at a fixed cost, when experiencing favourable preference shocks. In this way, credit makes precautionary balances unnecessary and hence reduces the opportunity cost of holding money. They do not consider the effects of inflation on allocations and social welfare.

In comparison, there are two additional things notably different from their model. First, we view the credit supported by the communal responsibility, of which bill of exchange is the historical counterpart, as a supplementary medium of exchange in long-distance trade, rather than a substitute for money. This is a reasonable perspective, given relevant facts such as currency shortage, safety concerns in carrying a large quantity of money during inter-community trade and actual operation of bills of exchange[1]. Second, participation in decentralized markets and credit is costly. In addition, we assume entry costs and the cost of using credit are proportional to distance, due to factors like direct verification and settlement cost, and indirect transportation cost. Of course, new things could be discussed in our model, such as the intensive and extensive margins of trade, the implications of market expansion and the evolution of monetary and credit trade.

With these new elements, we aim to make a contribution to the New Monetarist literature by building a search model with money and credit supported by the multi-tiered collective responsibility arrangement. As usual, lack of double coincidence of wants and anonymity keep money essential. Credit can supplement money as medium of exchange when agents have insufficient money balances. Indeed, this is consistent with the general lesson that if real assets or outside money are in limited supply, then adding privately issued, transferable debt to the set of alternatives extends the economy's payment capacity, and hence improves welfare by fully exploiting trading opportunities. In particular, we combine the features from the two influential third-generation monetary search models, that is, Shi (1997) and Lagos and Wright (2005). The main result of this chapter is that money and credit can meaningfully coexist as

[1] In his monograph about mediaeval institutions, Greif (2006b) uses a whole chapter to describe the communal responsibility system and build a simple repeated game model to endogenize it.

means of payment under the communal responsibility system with spatial frictions.

2.1.1 Related Literature

On the theoretical side, it is nontrivial to build a microfounded model with both money and credit being essential in the sense of Kocherlakota (1998) that these media of exchange support desirable allocations which are unattainable in their absence, since these two alternative means of payment require different even opposite frictions. For example, the useful existence of money needs imperfect monitoring or record-keeping, while perfect monitoring ensures a meaningful role for credit. In the burgeoning monetary search literature, the coexistence of money and different forms of credit are modelled via various ways. Indivisible money models include Shi (1996), Kocherlakota and Wallace (1998), Cavalcanti and Wallace (1999a,b), Aiyagari and Williamson (2000), Corbae and Ritter (2004), Jin and Temzelides (2004), Mills (2007) and among others, while Berentsen et al. (2007), Telyukova and Wright (2008), Sanches and Williamson (2010) and Li (2011) are representative models with divisible money.

More specifically, Shi (1996) introduces a decentralized credit arrangement by assuming that a consumption tool like a spoon can be used as collateral. To ensure future consumption, the debtor must redeem his collateral from the creditor. As long as the continuation value of future consumption is positive to the debtor, it is in his interest to repay the debt as soon as possible, since delaying repayments only reduces utility due to positive discounting. Kocherlakota and Wallace (1998) assume an imperfect public record-keeping technology with updating lag to model credit when agents can not commit. The main finding is that the set of implementable allocations is weakly larger the shorter the lag. Similarly, Cavalcanti and Wallace (1999a,b) also use imperfect monitoring to accommodate inside money in search models. An exogenous fraction of agents are perfectly monitored and hence are the potential issuers of private money, whereas the rest are not monitored at all and therefore needs fiat money for exchange. Mills (2007) combines this idea with the lagged record-keeping technology in Kocherlakota and Wallace (1998), and shows that both outside and inside money are essential for lags that are neither too long nor too short. Relatedly, Jin and Temzelides (2004) suggest a search model with local and faraway trades. Record-keeping technology is available at the local level so that local trade can be facilitated with credit. In contrast, agents from different communities need to trade with money. Aiyagari and Williamson (2000) construct a dynamic risk sharing model where agents can write long-term contracts with financial intermediaries. Random limited participation in the financial market gives a transaction role for money. In each period, agents can defect from their long-term credit arrangements and trade in a competitive money market thereafter. They show that the value of the outside option depends on monetary policy. Without a public record keeping device, Corbae and Ritter (2004) consider an environment with pairwise meetings where agents can form long-term partnerships to sustain credit arrangements. Relative to these papers, our contribution is to construct an environment with divisible money which is essential along with credit.

Microfounded monetary models with divisible money and credit are more desirable, since they enable us to more realistically study issues like the relationship between inside and outside money and the effect of monetary policy on their interaction. Some progresses has been made in this direction. Berentsen et al. (2007) use a competitive pricing version of the Lagos and Wright (2005) framework to explore the liquidity reallocation of banks, which can record financial transaction history at no cost, but they cannot record goods trade history. In this way, uncollateralized credit is available as nominal bank loans, when buyers are cash constrained. Similarly, Li (2011) study means of payment decisions between money and checking deposits by introducing preference shocks and record-keeping cost into Lagos and Wright (2005) framework. Banks have a technology for record keeping on financial histories but not the trading histories in the goods markets. In this way, agents can write checks against their deposits at some fixed cost to make payments in the decentralized goods market. There is an equilibrium where checks are used only in big transactions while cash is used in all transactions. In a different way, Telyukova and Wright (2008) introduce an additional centralized market into Lagos and Wright (2005) model to study the card debt puzzle. In this market, some agents want to consume but cannot produce, hence bilateral credit is feasible and useful. They show that agents choose to use interest-bearing credit even when they have money at hand, due to idiosyncratic uncertainty about liquidity need. Within the same framework, Sanches and Williamson (2010) explore the interaction among imperfect record-keeping, limited commitment and theft. They show that money and credit can coexist as competing media of exchange under certain assumptions of record-keeping technology and punishment of defection, as long as the cost of theft is sufficiently small.

The rest of this chapter is organized as follows. In section 2, the environment is given. Then in section 3, we describe the representative household's optimization problems. Next, stationary equilibrium is characterized in section 4. Numerical results are presented in section 5. The model's limitations are discussed in section 6. Section 7 concludes.

2.2 The Environment

The underlying environment is a mixture of the large household model of Shi (1997), and the alternating decentralized and centralized markets model of Lagos and Wright (2005). For exposition, we use the parable of a village economy.[2]

2.2.1 Agents

There is a continuum of spatially separated villages with unit mass. We can imagine that these villages are evenly and clockwise distributed[3] on a circle of circumference 1. In each village, there lives a continuum of measure one identical infinitely lived households. However, households in different villages have different preferences and production capabilities. Every household consists of a large number of sellers and buyers (normalized to one) who share consumption and regard the family's utility as the common objective. The demographic composition of a representative family is determined endogenously, which will be discussed later. Within each village, individuals know each other well, that is, they share common information about their trading opportunities and their credit histories. However, individuals are anonymous outside their village, but their village of origin is verifiable. Time is discrete and the horizon is infinite. Each period is divided into two sub-periods, say day and night, that differ in terms of economic activity. During the day, buyers travel to another village and match in pair with local sellers in a decentralized market, where sellers do not want to consume but they can produce village-specific goods, while buyers want to consume but they are unable to produce. This generates a temporal double coincidence of wants problem. Buyers do not consume the goods produced in their own villages, but desire goods from all the other villages. That is, all households have preference for variety. These village-specific consumption goods are non-storable and perfectly divisible. In the night, intra-village centralized markets become active, in which all agents want to consume and are able to produce one unit of perishable general goods with one unit of labour. Without loss of generality, we assume both the utility function of consuming general goods and the disutility function of producing general goods are linear.

[2]This is inspired by Faig and Jerez (2007).

[3]Without loss of generality, we will assume buyers travel clockwise as well. In principle, we can assume buyers search in both directions.

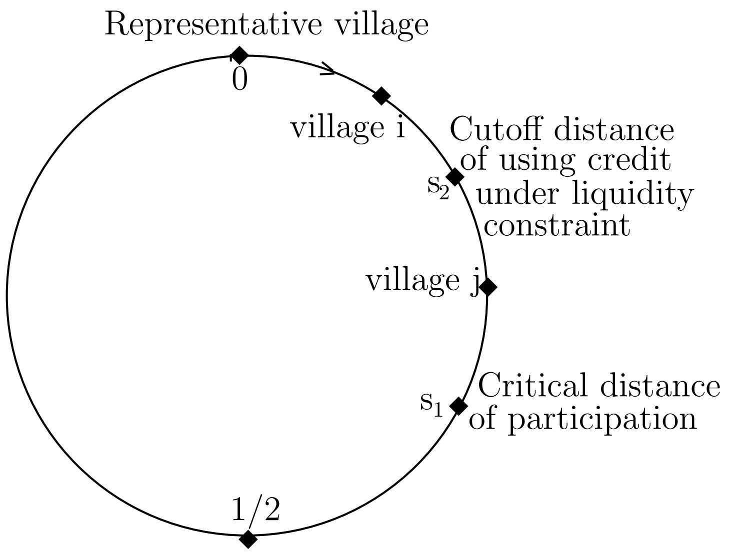

Figure 2.1: The village economy of a spatial search model

2.2.2 Preferences and technology

Without loss of generality, we can arbitrarily select one of the villages as representative village 0, and then each village is indexed by $j \in [0, 1)$ corresponding to its distance, moving clockwise around the circle from village 0, as illustrated in Figure 2.1. Since all the households within a village j are identical, we can also denote households by their origin. Similarly, we select an arbitrary household as the representative household. Following the convention, henceforth lower-case variables denote this representative household's choices, while upper-case variables denote the corresponding choices of other households, that is, aggregate variables, which are taken as given by the representative household. In a symmetric equilibrium, lower-case variables are equal to the corresponding capital-case variables. During the day, the representative household divides its members into two groups: sellers who stay at home, produce and sell goods to visiting buyers, and buyers who travel to other villages to purchase goods. Each buyer visits only one village.

Imagine a buyer from the representative household travelling to village j during the day sub-period. The village j goods can be of two varieties, of high and low quality. To avoid confusion, we would like to say the village j goods are produced in different colors, red or white[4], since quality differentials usually make us think of

[4]We borrow this from Shi (2008), who studies the social benefit of illiquid bonds in the large household framework.

asymmetric information. However, there is no private information here. Once the buyer is successfully matched with a seller, the seller receives a shock that determines he can produce the red good with probability α, and the white good with $1-\alpha$. This product-variety shock is identically and independently distributed across matches and over time. Since the representative household sends only one buyer to the village j, the trading outcome for this buyer is uncertain, due to matching and product-variety risk. In order to collapse this kind of uncertainty, we assume that all the buyers from the same village will pool their purchases and commit to divide the village j goods equally on the way back to their home village[5]. This assumption is consistent with the model, as agents know their fellow villagers well and contracts among them are enforceable at no cost. As a result, the law of large numbers is applicable, which implies that the proportion of village-specific goods of each color is deterministic for each household.

In each trade, the buyer makes a take-it-or-leave-it offer. Denote $(q^r(j), d^r(j))$ and $(q^w(j), a^w(j))$ as the offer for red goods and white goods, respectively, where q is the quantity of goods, and d is the payment. The cost of producing village j goods of two colours is given by the same disutility function, $\psi(q)$, with $\psi(0) = 0, \psi'(q) > 0$, and $\psi''(q) \geq 0$ for $q > 0$. The consumption of red goods yields utility $u(q^r)$, and the consumption of white goods gives utility $\varepsilon u(q^w)$, where $0 < \varepsilon < 1$. The function u is strictly concave, increasing and continuously differentiable, with $u(0) = 0$ and $u'(0) = +\infty$. All households have the same discount factor across periods, $\beta \in (0, 1)$.

2.2.3 Money Supply

Money is an intrinsically useless, perfectly divisible, and storable object. Suppose that each household in each village is initially endowed with M units of money. Subsequently, the government makes equal lump-sum transfers at the beginning of the night to all the households in such a way that money holdings per household grow at a constant gross rate μ. When $\mu < 1$, the transfers are negative, that is, money is withdrawn in the centralized market. Before visiting outside villages, each household decides how to allocate money among the buyers in the family. However, this decision has to be made before the realization of matching shocks. As a result, the household allocates the money holdings evenly among the buyers. In fact, it cannot be optimal to allocate money balances randomly among buyers in the household, as this would reduce household expected utility.

[5]As Greif (2006a, p.227) describes, "In general, merchants of the same community travelled together, lodged together (often in their own special residences), and witnessed each other's contracts. Communal identification was facilitated by the fact that members of distinct communities had different dialects and customs. Indeed, contracts and court cases reflect the great extent to which medieval merchants knew of one another's communal affiliations."

2.2.4 Market Structure

As in Lagos and Wright (2005), agents trade in two sequential markets in every period, as demonstrated in Figure 2.2. That is, in the day decentralized market, agents are randomly and bilaterally matched. In the night centralized market, agents within each village trade under competitive price.

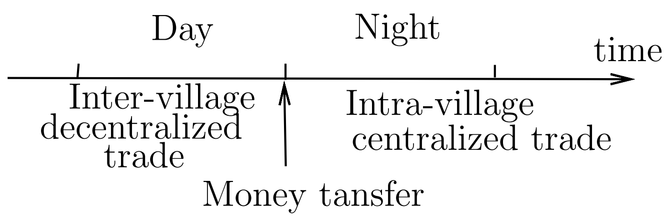

Figure 2.2: The timing of events during period t in the village economy

The fundamental difference is that we assume there is a fixed entry cost for buyers in order to visit outside villages. This transaction cost is independent of the quantity of goods traded and is increasing in the distance. For a buyer from the representative household, he will have to pay $\chi(j)$ in terms of the general goods (or utility) to enter into the village j's decentralized market, with $\chi'(j) > 0$.[6] As a result, the representative household will balance the costs of more distant transactions against the gains from increased variety. To minimize the heterogeneity, we assume that the representative household pays this type of entry cost instead, just like buying permits to trade. Let us denote that the fraction of buyers in the typical household is s_1, and the remaining $1 - s_1$ fraction are sellers. s_1 will be endogenously determined. Note that village s_1 is also the farthest village to which the household will send a buyer.

In the day decentralized market hosted in village j, $j \leq s_1$, the measure of buyers is S_1, and the measure of sellers is $1 - S_1$. Recall that we use upper-case variables to denote choices of other households. Let θ_j be the ratio of buyers to sellers in the

[6] Chiu and Molico (2010) assume that there is a fixed cost to enter the centralized financial markets at night, in order to estimate the long-run welfare costs of inflation in Lagos and Wright (2005) model with uninsurable idiosyncratic trading risk and costly liquidity management.

village:

$$\theta_j = \frac{S_1}{1-S_1}. \tag{2.1}$$

We assume that buyers and sellers perform at most one transaction during the trading period. Then, the probability that a seller meets a buyer is $\pi^s(\theta_j)$, where $\pi^s : [0,\infty] \to [0,1]$ is continuously differentiable, increasing and concave. Similarly, the probability that a buyer meets a seller is $\pi^b(\theta_j)$, where $\pi^b : [0,\infty] \to [0,1]$ is continuously differentiable, decreasing and convex. Intuitively, if θ_j is high, many buyers seek a few sellers, then it is easy for a seller to find a buyer and hard for a buyer to find a seller. Implicitly, we assume a constant returns matching function $\mathcal{M}$ such that

$$\mathcal{M}(S_1, 1-S_1) = S_1\pi^b(\theta_j) = (1-S_1)\pi^s(\theta_j), \tag{2.2}$$

where $\mathcal{M}$ maps the measure of sellers and buyers onto the measure of trading matches in village j decentralized market. General discussions about matching functions can be found in Pissarides (2000).

To better understanding matching functions, three interesting cases are briefly exposited. The first case is

$$\pi^s(\theta_j) = 1 - \exp(-\theta_j). \tag{2.3}$$

This case is the so called urn-ball matching function in frictional assignment models and arises when buyers use identical mixed strategies to select a seller among those who post equivalent price offers, and when each seller can serve at most one customer at one period[7]. Another special case is

$$\pi^s(\theta_j) = \frac{\theta_j}{1+\theta_j}. \tag{2.4}$$

This case emerges in directed search where buyers and sellers have full knowledge of terms of trade across sub-markets, and they can go to any sub-market freely. Intuitively, $\pi^s(\theta_j)$ is equal to the fraction of buyers over all traders in submarket j. The third case is

$$\pi^s(\theta_j) = \min\{\theta_j, 1\}. \tag{2.5}$$

That is, the short side of the market is always matched, hence there is no search friction. Since θ_j is independent of j, we henceforth simply use π^b and π^s rather than $\pi^b(\theta_j)$ and $\pi^s(\theta_j)$, respectively. This last case was used in Faig and Jerez (2007).

[7]Suppose there are u unemployed workers (balls) and v vacancies (urns). If each ball is independently placed into each urn with equal probability, then the number of balls per urn is a Poisson random variable with mean u/v, so a fraction $e^{-\frac{u}{v}}$ of the urns do not get any balls. Let u and v go to infinity with $q = u/v$ fixed, then a fraction $1 - e^{-q}$ of firms get a worker. See Burdett et al. (2001).

For the representative household, it will visit a range $(0, s_1]$ of villages during the day, due to love for variety and entry cost. The household maximizes total utility, which will be given shortly. Thanks to the assumption of risk sharing among fellow villagers, each buyer brings home deterministic proportions of red and white goods from chosen villages. At night, all the buyers return to their home villages, and intra-village centralized markets open, where all agents consume and produce one unit of general goods with one unit of effort. For simplicity, one unit of general goods yields one unit of utility. Let x denote net consumption of each member in the representative household. $x < 0$ means net production.

The timing of events is as follows. In the day, after the allocation of money balances, buyers in each households travel to another village and trade bilaterally with local sellers in decentralized markets with the terms of trade determined by bargaining. While buyers shop for goods, sellers remain at home and sell goods to visiting buyers. When night falls, agents return home, trade the general goods in intra-village markets, settle debts and adjust money balances.

2.2.5 Communal Responsibility System

We quote a story from Greif (2006a, pp.222-3) to facilitate understanding of this institutional innovation.

In 1323 the goods of a merchant from London, John de Grantham, were transported through the important English port of Dover. A local man, William Virgil, captured John's goods with authorization from Dover's court, although no legal claim was advanced against John. John complained to the court in London about the confiscation and the mayor, who presided over the court, wrote to Dover requesting restitution. Otherwise, he threatened, Dover's merchants' goods in London will be confiscated in retaliation. Dover's mayor, who headed the local court, responded by explaining the reason for the confiscation. John's goods were impounded because the mayor of London didn't act upon his earlier request to collect a debt that a Londoner, Henry Nasard, had failed to pay to William Virgil of Dover. London's mayor replied that he had no record of this earlier request and asked that it be resubmitted. The evidence suggests that the dispute ended at this point. London verified the default on the debt and collected the amount due. Dover released John's goods. (Reference within omitted.)

Motivated by this episode, we assume that in each village there is a community court which keeps the records of its citizens' personal identity, knows their trading histories[8], and can verify the communal identity of agents from other villages. Under

[8] Hence, agents will not pretend to be victims, since the following assumptions imply that the local

the communal responsibility system, the community court confiscates defector or his fellow villagers' assets for compensation when one of its citizens suffers from default. Similarly, when one of its villagers defaults, the local court seizes the defector's assets and the defaulter is banned from trading inside or outside her village forever[9]. Note that this implies that the whole household will be deprived of its asset holdings and of the opportunity to trade, if any one of its buyers reneges. In order to motivate the local court to deliver impartial justice, we simply assume that villages have strong trading relationships such that it is not in their interest to cease partnership. In addition, it is so costly and lengthy to gain affiliation with a new village that agents have no incentive to give up their original citizenship. To simplify analysis, we abstract from strategic behaviour between villages.

With such a intra-community contract enforcement institution, credit in inter-community exchange could be supported in this village economy, although on the individual level, neither buyers or sellers can commit to their promises. The use of credit, however, costs $\kappa(j) > 0$ units of general goods for the seller in village i during the day sub-period. This fixed cost could capture the loss in issues like verifying identity of origin, resolving potential conflicts or in settlement. Like entry cost, this cost of using credit is an increasing function of distance, that is, $\kappa'(j) > 0$.[10] Indeed, Greif (2006a, p.229) explains the decline of communal responsibility system like this, "the system became less effective because intercommunity interactions and the growth in the number and size of communities reduced the cost of falsifying community affiliation and increased the cost of verification". Again, we assume that the cost of using credit is paid at the household level for tractability.

2.2.6 First Best

The first best allocations for the village-specific goods, denoted by q_*^r and q_*^w, respectively, are characterized by the first order conditions,

$$u'\left(q_*^r\right) = \psi'(q_*^r)$$

$$\varepsilon u'\left(q_*^w\right) = \psi'(q_*^w)$$

which say that for each pair of matched buyer and seller, at the optimum, the marginal benefit of consumption of both red and white goods should equal the marginal cost

court cares more about collective reputation.

[9]This individual autarky punishment is added to simplify analysis.

[10]Li (2011) studies money and checking deposits as alternative means of payment, with the assumption that there is a fixed cost to use checks in decentralized markets.

of their production, respectively. The same quantity is assigned to consumption and production, which implies feasibility. The first best allocation for the general good in the night only involves the feasibility condition, due to the linearity of the preference for the general good.

2.3 The Representative Household's Problems

In this section, we characterize the behaviour of a representative household whose buyers travels around and sellers stay at home. Recall that lower-case letters denote the decision variables of the household, while upper-case letters denote the decisions of other households and hence aggregate quantities, which are taken as given by our representative household. In a symmetric equilibrium,lower-case letters are equal to the corresponding upper-case letters. In the following, the subscript ± 1 denotes for $t \pm 1$, where t is an arbitrary period.

2.3.1 The Problem in Day Markets

Let m denote the household's money holdings at the beginning of the day, and let ϕ denote the value of money in terms of general goods in a centralized intra-village market at date t. Recall that all villages are identical, so for the whole economy the value of money in the night sub-period is also ϕ at period t.

As Boerner and Ritschl (2011), we are interested in the case where only money is used for white goods, and credit is used as supplementary means of payment for red goods, if the cost of using credit is not too high[11]. In fact, a similar argument is made in Li (2011). Here, we want to clarify the role of credit in our model. When meeting a seller with high-marginal-utility goods, the buyer first makes whatever money purchases of the seller's goods he can and then attempts to secure a credit to obtain further quantities. Expecting ex post dispensation of impartial justice, the seller is motivated to extend credit. In addition, the initial money exchanges reduced the desired loan and hence the buyer's incentive to renege. This view has been adopted by Bernhardt (1989).

To determine the terms of trade in a decentralized exchange, we assume that the buyer makes a take-it-or-leave-it offer to the seller. This particular bargaining protocol

[11] It seems arbitrary to focus on the case where only money is used for white goods, and credit is used as supplementary means of payment for red goods. However, this objection is not valid, since we can always adjust the tastes for the two types of goods to ensure this simplifying division, see the argument in Shi (2008).

simplifies the analysis without loss of generality. It is pointed out in Lagos and Wright (2005) that a holdup problem on money balances arises from the generalized Nash bargaining. Consequently, the monetary equilibrium is not efficient. We would rather not tussle with this unnecessary issue.

Now imagine a buyer from the representative household meet a domestic seller in village j. The buyer makes a take-it-or-leave-it offer. Let us consider white goods first. In this case, the buyer only use money as means of payment. For another household's seller in village j to be indifferent between accepting and rejecting the buyer's offer $(q^w(j), d^w(j))$ in the random match, the offer must satisfy the seller's participation constraint:

$$\phi d^w(j) - \psi(q^w(j)) = 0, \tag{2.6}$$

where $d^w(j) \leq \frac{m}{s_1}$. That is, the buyer cannot spend more money than what he has.

Under our assumptions, the representative household are willing to consume more red goods, hence the buyer will be instructed to use costly credit. Since the cost of using credit is increasing with respect to distance, we can conjecture that there is a threshold value $s_2 \leq s_1$ such that buyers from the representative household will use costly credit only in villages within $(0, s_2]$, and they will only use money in villages $(s_2, s_1]$, even though encountering red goods. The threshold s_2 can be considered as a measure of the intensive margin in inter-village trade, while s_1 as a measure of the extensive margin. Let $(q^r(j), d^r(j))$ denote the buyer's offer, and $l(j), 0 < j \leq s_2$ denote the quantity of monetary loan that the buyer requests from the seller, when his holdings of money is not enough to make the payment. Then, the seller's participation constraint is,

$$\phi d^r(j) - \psi(q^r(j)) = 0, \tag{2.7}$$

where $d^r(j) \leq \frac{m}{s_1} + l(j)$, and $j \leq s_2$. Note that the buyer ends up paying the fixed cost of using credit, since (2.7) is binding at the optimum. The benefit for the buyer is that credit relaxes his cash constraint.[12]

For the decentralized market j located within $(s_2, s_1]$, the participant constraint for sellers who produce red goods is similar to that of sellers who produce white goods,

$$\phi d^r(j) - \psi(q^r(j)) = 0, \tag{2.8}$$

[12] Boerner and Ritschl (2011) view credit supported by communal responsibility as a substitutes for money as a means of payment. In particular, credit makes precautionary balances unnecessary and hence reduces the opportunity cost of holding money. See also Faig and Jerez (2007), who examine precautionary money demand in a search model with continuous idiosyncratic preference shock. In contrast, we argue here credit supplements money as a medium of exchange. In fact, currency shortages are recorded during the relevant historical period.

where $d^r(j) \le \frac{m}{s_1}$ and $j \in (s_2, s_1]$.

Let $V(m)$ denotes the value function entering the day market with money holdings m, and similarly $W(m)$ denotes the value function entering the night market with money holdings m. Taking other households' decisions as given, the representative household then instruct its buyers at the day market j to choose $d^w(j)$, $d^r(j)$, $q^w(j)$, $q^r(j)$, $l(j)$, s_1, s_2 such that

$$V(m) = \max \left\{ \begin{array}{c} \pi^b \alpha \int_0^{s_1} u(q^r(j))\, dj - \pi^s \alpha (1 - S_1) \int_0^{S_1} \psi(Q^r(j))\, dj \\ +\pi^b (1-\alpha) \int_0^{s_1} \varepsilon u(q^w(j))\, dj - \pi^s (1-\alpha)(1-S_1) \int_0^{S_1} \psi(Q^w(j))\, dj \\ - \int_0^{s_1} \chi(j)\, dj - \pi^b \alpha \int_0^{s_2} \kappa(j)\, dj + W(\widetilde{m}) \end{array} \right\} \tag{2.9}$$

subject to the constraints (2.6), (2.7), (2.8), and where

$$\begin{aligned} \widetilde{m} = m - \pi^b \alpha r \int_0^{s_2} l(j) dj - \pi^b \alpha \int_0^{s_1} d^r(j) dj - \pi^b (1-\alpha) \int_0^{s_1} d^w(j) dj \\ +\pi^s \alpha (1 - S_1) \int_0^{S_1} D^r(j) dj + \pi^s (1-\alpha)(1-S_1) \int_0^{S_1} D^w(j) dj \\ + \pi^s \alpha (1 - S_1) r \int_0^{S_2} L(j) dj \end{aligned} \tag{2.10}$$

denotes the representative household's overall balance of money taken to the night market. That is, (2.10) describes the evolution of the representative household's money holdings: the first item is its money holdings before entering the day market, the second item is the interest rate payment, the third and fourth items represent expenditure in purchasing village-specific goods, the fifth and sixth items are revenues via selling red goods and white goods, and the final integral is the interest earnings from lending. Note that the period utility function is the sum of utility from consuming goods, disutility from production, and total cost of entering decentralized markets and using credit.

2.3.2 The Problem in Night Markets

At the beginning of the night, household makes choice over net consumption x, money holdings for the future, m_{+1}, to solve the following problem

$$W(\widetilde{m}) = \max \{x + \beta V(m_{+1})\} \tag{2.11}$$

subject to the budget constraint,

$$-x + \phi \widetilde{m} + T - \phi m_{+1} = 0$$

which says that net production, the real value of current money holdings and government lump-sum transfer can be used to acquire money for the future.

The problem for the household problem at night can be rewritten as,

$$W\left(\widetilde{m}\right)=\max\left\{\phi\widetilde{m}+T-\phi m_{+1}+\beta V\left(m_{+1}\right)\right\}$$

and the first-order condition is,

$$-\phi+\beta V'\left(m_{+1}\right)=0 \tag{2.12}$$

Intuitively, (2.12) says that the cost of an extra unit of cash at night must equal the benefit generated in the following period. As typically in the Lagos and Wright (2005) framework, these decisions over future money are the same for all the agents, and hence there are no distributional issues. The quasi-linearity of preference is responsible for the separation between the decisions of future money holdings and current money holdings, as it can be seen from the first term in (2.12), which depend on the current price, but not on the current individual holdings of money.

2.3.3 The Envelope Conditions

The envelope condition for money between day and night are, respectively,

$$W'\left(\widetilde{m}\right)=\frac{\partial W\left(\widetilde{m}\right)}{\partial\widetilde{m}}=\phi \tag{2.13}$$

which says that a unit of cash is worth ϕ units of the general good at night.

The envelope conditions between night and day are,

$$V'\left(m\right)=\frac{\partial\mathcal{L}}{\partial m}=\frac{\partial W\left(\widetilde{m}\right)}{\partial\widetilde{m}}\frac{\partial\widetilde{m}}{\partial m}+\frac{\phi\pi^b}{2}\int_0^{s_1}\lambda_2^w(j)dj+\frac{\phi\pi^b}{2}\int_0^{s_1}\lambda_2^r(j)dj$$

simplifying,

$$\frac{\partial V\left(m\right)}{\partial m}=\phi+\frac{\phi\pi^b}{2}\int_0^{s_1}\lambda_2^w(j)dj+\frac{\phi\pi^b}{2}\int_0^{s_1}\lambda_2^r(j)dj \tag{2.14}$$

which says that an extra unit of cash will be brought into the night market if not used, and relaxes the liquidity constraints for buyers of both red goods and white goods if the buyers have successful matches.

Along with (2.12), we have

$$\phi=\beta V'\left(m_{+1}\right)=\beta\phi_{+1}\left[\ 1+\tfrac{\pi^b}{2}\int_0^{s_1}\lambda_{2,+1}^w(j)dj+\tfrac{\pi^b}{2}\int_0^{s_1}\lambda_{2,+1}^r(j)dj\ \right] \tag{2.15}$$

which says that the current value of cash (nominal asset) at the household level should be equal to its future value, which reflects the expected benefit of using money to purchase red goods and white goods.

2.3.4 Market Clearing Conditions

Besides the village-specific goods, there are three items exchanged and their corresponding market clearing conditions are given below:

1. For monetary loans in each market j during the day, market clearing requires
$$\int_0^{s_2} l(j)dj = \int_0^{S_2} L(j)dj$$
2. For money at night, market clearing requires
$$m = M$$
3. The night market for the general good clears whenever the markets for money does by Walras Law.

2.3.5 The Value of Defection

As we have assumed, when the representative household reneges on private liabilities, its assets will be seized and the family will be deprived of access to both inside- and outside-village markets. This harsh punishment imposed by local courts aims to sustain trading relationship between communities and support credit arrangements involved. Therefore, the value function or expected discounted utility upon default should be 0.

2.4 Symmetric Stationary Equilibrium

An symmetric stationary equilibrium consists of the representative household's choices, d^w, d^r, q^w, q^r, l, s_1, s_2, m, x, price ϕ, the value functions, V, W, and other households' choices satisfying the following properties:

1. Optimality: Given other households' choices, the representative household's choices solve the problems (2.9) and (2.11), and satisfy the market clearing conditions;

2. Symmetry: The optimal choices and value functions are the same across households. That is,

   $d^w = D^w$, $d^r = D^r$, $q^w = Q^w$, $q^r = Q^r$, $s_1 = S_1$, $s_2 = S_2$, and $l = L$;

3. Stationarity: All real variables and real money balances are constant.

Symmetry implies that $m = M$, while stationarity suggests that $\phi M = \phi_{+1} M_{+1}$. Recall that $M_{+1} = \mu M$. Hence, $\phi/\phi_{+1} = \mu$.

Without causing any confusion, the market index j, also the distance measure, is omitted when appropriate. From the first-order conditions, we have,

$$\lambda_1^w = \frac{\varepsilon u'(q^w)}{\psi'(q^w)},\ 0 < j \leq s_1 \tag{2.16}$$

which says that the shadow value of the white goods is given by the ratio between its marginal utility and its marginal cost;

$$\lambda_1^r = \frac{u'(q^r)}{\psi'(q^r)},\ 0 < j \leq s_1 \tag{2.17}$$

which says that the shadow value of the red goods is given by the ratio between its marginal utility and its marginal cost;

$$\int_0^{s_1} \lambda_2^w(j)dj = \frac{1}{s_1}\left[\int_0^{s_1}\left(\frac{\varepsilon u'(q^w)}{\psi'(q^w)} - 1\right)dj\right] \tag{2.18}$$

which says that the shadow value of liquidity at the household level encountering white goods is equal to the sum of the shadow value of liquidity (adjusted by extensive margin) for buyers of white goods, which is given by the marginal utility of consumption of white goods net of its marginal cost;

$$\int_0^{s_1} \lambda_2^r(j)dj = \frac{1}{s_1}\left[\int_0^{s_1}\left(\frac{u'(q^r)}{\psi'(q^r)} - 1\right)dj\right] \tag{2.19}$$

which says that the shadow value of liquidity at the household level encountering red goods is equal to the sum of the shadow value of liquidity (adjusted by extensive margin) for buyers of red goods, which is given by the marginal utility of consumption of red goods net of its marginal cost;

$$\int_0^{s_2} \lambda_2^r(j)dj = \frac{s_2 r}{s_1} \tag{2.20}$$

which says that the sum of shadow value of liquidity (adjusted by extensive margin and intensive margin) for buyers of red goods in the credit region is equal to the cost of monetary loans, which, in turn, is simply the nominal interest rate;

$$u\left(q^{r}(s_1)\right)+\varepsilon u\left(q^{w}(s_1)\right)=\left(1+\lambda_2^{r}(s_1)\right)\psi(q^{r}(s_1))+\left(1+\lambda_2^{w}(s_1)\right)\psi(q^{w}(s_1))+\frac{2}{\pi^{b}}\chi\left(s_1\right) \tag{2.21}$$

which says that the range of decentralized markets participated by the representative household is determined by the expected trade surplus net of the entry cost, and by the fact that the further the marginal market (higher s_1), less amount of money (lower m/s_1) distributed to each buyer of the household, which tightens the liquidity constraints of both buyers of red and white goods;

$$u\left(\psi^{-1}\left(\phi l(s_2)+\frac{\phi m}{s_1}\right)\right)-u\left(\psi^{-1}\left(\frac{\phi m}{s_1}\right)\right)=\phi r l(s_2)+\kappa(s_2) \tag{2.22}$$

which says that credit is used as medium of exchange up to the point when its benefit of relaxing liquidity constraint equals to the cost of using credit.

Finally, the Euler equation gives

$$\frac{\mu}{\beta}=\ 1+\tfrac{\pi^{b}}{2}\int_0^{s_1}\lambda_2^{w}(j)dj+\tfrac{\pi^{b}}{2}\int_0^{s_1}\lambda_2^{r}(j)dj \tag{2.23}$$

since the Lagrange multipliers λ_2^w and λ_2^r are non-negative. In other words, it is necessary to have $\mu \geq \beta$, in order to ensure the existence of monetary equilibrium. Otherwise, households will demand an infinite amount of money balances, since the rate of return on money is less than the discount rate. This is one of the standard results in monetary search literature, see Lagos and Wright (2005) and Wright (2010).

About the optimality of the Friedman rule in our model, we have the following proposition.

Proposition 1 *The Friedman rule is optimal, and it is feasible when there is no liquidity shortage problem. When there is liquidity shortage problem, however, the Friedman rule can be sub-optimal.*

Proof. Combining (2.19) and (2.20), we have

$$s_1\int_{s_2}^{s_1}\lambda_2^{r}(j)dj+s_2 r=\int_0^{s_1}\left(\frac{u'\left(q^{r}\right)}{\psi'(q^{r})}-1\right)dj \tag{2.24}$$

which says that the shadow cost of liquidity for red goods at the household level is equal to the the shadow value of liquidity for red goods at the household level. We already know that $\frac{u'(q^r)}{\psi'(q^r)}$ is equal to 1 when the first best allocations of red goods are obtained. In other words, the smaller value of the left hand of (2.24), the closer the allocations of red goods to the first best. Therefore, the Friedman rule, that is, $r = 0$, is optimal in our economy, when the liquidity constraints for red goods in the no-credit region are unbinding ($\lambda_2^r(j) = 0,\ s_2 < j \leq s_1$).

Imagine the government approximately withdraw money at the rate of discount factor β as typically done in the monetary literature to implement the Friedman rule, then the Euler equation (3.8) requires that

$$\lambda_2^w = \lambda_2^r = 0,\ \ 0 < j \leq s_1 \tag{2.25}$$

which, by the complementarity slackness conditions, implies that the liquidity constraints for both red and white goods are not binding in all of the villages where buyers shop. That is, the Friedman rule is also feasible when there is no liquidity shortage problem at all.

However, if the condition (2.25) does not meet, that is, some of the Lagrange multipliers λ_2^w and λ_2^r are non-zero, which implies that some of the liquidity constraints binds, and hence $\mu > \beta$. As a result, the Friedman rule is not implementable. In addition, the Friedman rule can be sub-optimal, due to the fact that the growth rate of money affects the Lagrange multipliers as well. We will see this result in the numerical analysis. ■

As indicated before, we are only interested in the case where only money is used for white goods, and credit is used as supplementary means of payment for red goods, if the cost of using credit is not too high. We construct an equilibrium with both money and credit used as media of exchanges as follows: for $j \in (0, s_2]$, $\lambda_2^r(j)$ satisfies (2.20); for $j \in (s_2, s_1]$, $\lambda_2^r(j)$ satisfies (2.19); while for $j \in (0, s_1]$, $\lambda_2^w(j) = 0$, which says the liquidity constraint for white goods is always not binding, and hence credit is not useful. Note that this type of equilibrium is closely motivated by the historical communal responsibility arrangement, where buyers borrow from the sellers on some occasions. Without this background, one could argue that there is some credit arrangement where buyers of red goods borrow from buyers of white goods and unmatched buyers (ex post), instead of the sellers. We deal with this potential problem by assuming that only one buyer (ex ante) from each household is sent to each village, as stated above, which implies that all of the buyers in a specific village are anonymous and hence there is enforcement problem in implementing the alternative credit arrangement.

Recall that the entry cost and the cost of using credit are paid at the household level, and as a result, each buyer from the representative household has the same amount of money when travelling to outside villages. This implies that at the buyer level, both $\lambda_2^r(j)$ at the credit region ($0 < j \leq s_2$), and $\lambda_2^r(j)$ at the no-credit region ($s_2 < j \leq s_1$)

are independent of j. Intuitively, in near villages the buyers will use the same amount of credit when red goods are traded, while in farway villages the buyers have the same amount of money to execute trades. In this way, we get rid of the heterogeneities caused by village-specific bilateral trades at the buyer level. Note that the Lagrange multipliers associated with the liquidity constraints for red goods at the two regions have different values.

With the above intuition in mind, we can solve for the allocations affiliated with the equilibrium of interest from the optimality conditions and constraints, listed as below:

$$q^w = q_*^w, \text{ for } j \in (0, s_1]$$

$$\frac{u'\left(q^r\left(s_2\right)\right)}{\psi'(q^r\left(s_2\right))} = r + 1 \tag{2.26}$$

$$\frac{u'\left(q^r\left(s_1\right)\right)}{\psi'(q^r\left(s_1\right))} = s_1 \lambda_2^r\left(s_1\right) + 1 \tag{2.27}$$

$$\phi l\left(s_2\right) = \psi(q^r\left(s_2\right)) - \psi(q^r\left(s_1\right)) \tag{2.28}$$

$$\phi M = s_1 \psi(q^r\left(s_1\right)) \tag{2.29}$$

$$\frac{\mu}{\beta} - 1 = \frac{\pi^b}{2}\left[\frac{s_2 r}{s_1} + \left(s_1 - s_2\right)\lambda_2^r\left(s_1\right)\right] \tag{2.30}$$

$$u\left(q^r(s_1)\right) + \varepsilon u\left(q_*^w\right) = \psi(q_*^w) + \left(1 + \lambda_2^r(s_1)\right)\psi(q^r(s_1)) + \frac{2}{\pi^b}\chi\left(s_1\right) \tag{2.31}$$

$$u\left(q^r(s_2)\right) - u\left(q^r(s_1)\right) = \phi r l(s_2) + \kappa(s_2) \tag{2.32}$$

$$r + 1 = \frac{\mu}{\beta} \tag{2.33}$$

where (2.33) is the Fisher equation. Note that there are eight variables

$$\left\{q^r_{j\in(0,s_2]},\, q^r_{j\in(s_2,s_1]},\, r,\, s_1,\, \lambda_2^r,\, \phi l,\, s_2,\, \phi M\right\}$$

and eight equations (2.26) - (2.33). Given the well-defined properties of functions u and ψ, the solution exists. To get further qualitative or quantitative results, we need to specify the functional forms and parameters. This will be done in the numerical analysis section.

2.4.1 Steady State Welfare

Since we focus on a symmetric equilibrium, the standard measure of social welfare is the steady-state lifetime utility $\overline{V}$:

$$\overline{V} = \frac{1}{1-\beta}\left\{\begin{array}{c} \frac{\pi^b}{2}\int_0^{s_1} u\left(q^r(j)\right)dj + \frac{\pi^b}{2}\int_0^{s_1} \varepsilon u\left(q^w(j)\right)dj \\ -\frac{\pi^s(1-S_1)}{2}\int_0^{S_1}\psi\left(Q^r(j)\right)dj - \frac{\pi^s(1-S_1)}{2}\int_0^{S_1}\psi\left(Q^w(j)\right)dj \\ -\int_0^{s_1}\chi(j)\,dj - \frac{\pi^b}{2}\int_0^{s_2}\kappa(j)\,dj \end{array}\right\}$$

$$= \frac{1}{1-\beta}\left\{\begin{array}{c} \frac{\pi^b}{2}\int_0^{s_1} u\left(q^r(j)\right)dj + \frac{\pi^b}{2}\int_0^{s_1} \varepsilon u\left(q^w(j)\right)dj \\ -\frac{\pi^s(1-s_1)}{2}\int_0^{s_1}\psi\left(q^r(j)\right)dj - \frac{\pi^s(1-s_1)}{2}\int_0^{s_1}\psi\left(q^w(j)\right)dj \\ -\int_0^{s_1}\chi(j)\,dj - \frac{\pi^b}{2}\int_0^{s_2}\kappa(j)\,dj \end{array}\right\} \tag{2.34}$$

Intuitively, (2.34) says that the steady-state lifetime utility is the discounted expected trade surplus minus costs of market entry and of using credit in decentralized markets. Note that the welfare in the night market is not included, due to the assumption of linear preference for the general good.

2.4.2 Welfare Without Credit

For comparison, we can consider an otherwise identical monetary village economy without credit, derive the value function of the representative household, and then construct the steady state welfare as we do in the case with credit. This value function can be considered as a lower bound such that the representative household keeps its promises. If the resulting social welfare decreases due to lack of credit, then we can claim that credit is essential in our environment. The unavailability of credit can be thought due to prohibitively high cost of using credit. Mathematically, we need to impose $l(j) = 0$, for any $j \in (0, s_1)$.

We have the following proposition about the role of money and credit played in this economy.

Proposition 2 *Both money and credit are essential in the sense that they improve social welfare. Furthermore, social welfare is higher when credit is available and desired by the buyers of red goods.*

Proof. In the day markets, buyers and sellers are anonymous, so bilateral trade between them is impossible if there is no money or credit. In other words, money and credit matter as media of exchange. Money is essential, since it is valued in buying both white and red goods across all of villages visited by the representative household.

In addition, credit supported by the communal responsibility system only supplements money as medium of exchange, since it is only desirable occasionally in a limited region. By construction, the social welfare associated with the equilibrium of interest is higher when credit is available. This result will be checked in the numerical analysis as well. ■

2.5 Numerical Analysis

In the numerical example, following Faig and Jerez (2007), the functional forms for the utility of consuming village-specific goods and the disutility of producing them are assumed to be isoelastic and linear, respectively. That is,

$$u(q) = \frac{q^{1-\sigma}}{1-\sigma}, \quad \sigma \in (0,1), \tag{2.35}$$

$$\psi(q) = zq, \quad z \geq 1, \tag{2.36}$$

where z is productivity parameter for village-specific goods production. That is, higher z implies lower productivity.

For simplicity, we choose the matching function so that

$$\mathcal{M}(S_1, 1-S_1) = S_1\,(1-S_1)$$

which then implies that

$$\pi^b = 1 - s_1 \tag{2.37}$$

and hence π^s is given by

$$\pi^s = s_1$$

The cost functions $\chi(j)$ and $\kappa(j)$ are assumed to be linear function of distance j. That is,

$$\kappa(j) = \delta j, \quad \chi(j) = \gamma j,$$

where δ and γ are positive.

The benchmark parameter values are specified in the Table 2.1. Note that we choose the values quite freely as long as they make economic sense, since the aim of this numerical analysis is for illustrative purpose only. For example, the relative risk aversion parameter (or the inverse of elasticity of intertemporal substitution) is chosen for computational convenience. Faig and Jerez (2007) set $\sigma = 0.435$. Another reason we choose $\sigma < 1$ is that we need $u(0) = 0$, that is, the utility of autarky.

Table 2.1: Benchmark Parameterization

Parameter	Value	Definition
β	0.96	discount factor
σ	0.5	relative risk aversion coefficient
z	1	productivity of village-specific goods
ε	0.5	scaling paramter of the marginal utility from white goods
μ	1	the growth rate of money
γ	1	scaling paramter associated with entry cost
δ	1	scaling paramter associated with cost of using credit

Given the functional forms, it is quite straightforward to get

$$\varepsilon\left(q_*^w\right)^{-\sigma}=z$$

which yields

$$q_*^w=\left(\frac{z}{\varepsilon}\right)^{-\frac{1}{\sigma}}$$

Similarly, the equilibrium conditions (2.26) - (2.33) give the following results:

$$r=\frac{\mu}{\beta}-1 \tag{2.38}$$

$$q^r\left(s_2\right)=\left(\left(r+1\right)z\right)^{-\frac{1}{\sigma}} \tag{2.39}$$

$$\lambda_2^r\left(s_1\right)=\frac{\left(q^r\left(s_1\right)\right)^{-\sigma}}{zs_1}-\frac{1}{s_1} \tag{2.40}$$

$$\phi l\left(s_2\right)=zq^r\left(s_2\right)-zq^r\left(s_1\right) \tag{2.41}$$

$$\phi M=zq^r\left(s_1\right)s_1 \tag{2.42}$$

$$2\left(\frac{\mu}{\beta}-1\right)=\frac{1-s_1}{s_1}\left[s_2r+\left(s_1-s_2\right)\left(\frac{\left(q^r\left(s_1\right)\right)^{-\sigma}}{z}-1\right)\right] \tag{2.43}$$

$$\frac{q^r(s_1)^{1-\sigma}}{1-\sigma}+\frac{\varepsilon\left(\frac{z}{\varepsilon}\right)^{-\frac{1-\sigma}{\sigma}}}{1-\sigma}=z\left(\frac{z}{\varepsilon}\right)^{-\frac{1}{\sigma}}+\frac{\left(q^r\left(s_1\right)\right)^{1-\sigma}}{s_1}-\frac{z\left(1-s_1\right)q^r(s_1)}{s_1}+\frac{2\gamma s_1}{1-s_1} \tag{2.44}$$

$$\frac{q^r(s_2)^{1-\sigma}}{1-\sigma}-\frac{q^r(s_1)^{1-\sigma}}{1-\sigma}=zr\left(q^r\left(s_2\right)-q^r\left(s_1\right)\right)+\delta s_2 \tag{2.45}$$

With respect to the key parameters, we have the following comparative statics results.

Proposition 3 *Higher money growth (or higher inflation), μ, increases the nominal* *interest rate paid on monetary loans, which, in turn, lowers the demand for credit and, as a result, decreases the consumption of red goods in the credit region.*

Proof. (2.38) says that credit requires higher interest rate as compensation when the government increases the growth rate of money supply. As a result, buyers of red goods demand less amount of credit, other things being equal. This can be seen from (2.41). Finally, (2.39) supports the claim that higher borrowing cost lowers the consumption of red goods in the credit region. ■

Conjecture 4 *Higher marginal entry cost, γ, reduces the extensive margin, s_1. In addition, consumption of red goods in the no-credit region is higher, ceteris paribus, since the amount of money carried by each buyer is higher.*

Conjecture 5 *Higher marginal cost of using credit, δ, reduces the intensive margin, s_2. In addition, social welfare is smaller.*

It is intuitive to understand the two conjectures, but not straightforward to prove them, since we do not have explicit expressions for the relevant variables. However, given the benchmark parameter values, we can verify them numerically, as shown in Table 2.2.

Table 2.2: Comparative static analysis

Parameter	Values			Steady state results					
	μ	γ	δ	s_1	s_2	$q^r(s_1)$	$q^r(s_2)$	$\phi l\,(s_2)$	welfare $\overline{V}$
μ	0.961	1	1	0.2842	0.2812	0.7318	0.9980	0.2662	0.1630
	1	1	1	0.3075	0.1569	0.7716	0.9216	0.1500	0.1617
	1.005	1	1	0.3036	0.1552	0.7641	0.9125	0.1483	0.1612
	1.015	1	1	0.2955	0.1517	0.7497	0.8946	0.1449	0.1601
	1.025	1	1	0.2870	0.1478	0.7360	0.8772	0.1412	0.1588
γ	1	1	1	0.3075	0.1569	0.7716	0.9216	0.1500	0.1617
	1	1.01	1	0.3052	0.1556	0.7728	0.9216	0.1488	0.1614
	1	1.02	1	0.3029	0.1543	0.7740	0.9216	0.1476	0.1611
δ	1	1	1	0.3075	0.1569	0.7716	0.9216	0.1500	0.1617
	1	1	1.01	0.3069	0.1565	0.7705	0.9216	0.1511	0.1616
	1	1	1.02	0.3063	0.1562	0.7695	0.9216	0.1521	0.1615

When the growth rate of money supply approximates the Friedman rule, the nominal interest rate is close to zero, which makes credit more desirable, and as a result, the range of using credit or intensive margin (s_2) follows the extensive margin (s_1) quite closely. Notably, when the monetary policy is slightly away from the Friedman rule ($\mu = 1$), the range of market participation (s_1) is higher and so is the consumption of red goods in the no-credit region, since a lower degree of deflation relaxes the liquidity constraints in regions where credit is not used. This suggests that the Friedman rule may be not optimal under some circumstances. However, in the case of higher inflation, the consumption of red goods in the no-credit region decreases, since inflation erodes agents' real money balances. In addition, a higher nominal interest rate is required to clear the credit market, which lowers both the demand for credit and the consumption of red goods in the credit region.

Higher cost of market entry (γ) lowers the extensive margin (s_1), but reduces the consumption of red goods in the no-credit region. The reason is that, ceteris paribus, a smaller range of market participation implies that more money holdings can be assigned to each buyer, which relaxes their liquidity constraints, and consequently lowers the intensive margin (s_2) as well as the total amount of credit usage. In comparison, with a higher cost of using credit, the intensive margin (s_2) and the the extensive margin (s_1) decrease slightly, while the total usage of credit even increases marginally. Intuitively, the representative household desires to sustain the level of the consumption of red goods in the credit-region, even though the consumption of red goods in the no-credit region is reduced. Overall, these choices yield relatively higher welfare. Finally, the household will pay back their debt, since social welfare is always positive, implying repayment is a better option than default. In other words, there is no default in equilibrium.

As mentioned above, the Friedman rule can be sub-optimal in the presence of distance-dependent costs. Indeed, Figure 2.3 supports this result. It says that when the entry cost is not equal to the cost of using credit, a mild deviation from the Friedman rule can be welfare improving. There are two reasons for this result. One is that relatively higher money supply can relax the buyers' liquidity constraints even if the value of money is relatively lower. The other is that a lesser degree of deflation affects the extensive and intensive margins and their interactions. Figure 2.4 shows that a mild deviation from the Friedman rule lowers the extensive margins in the cases with unequal costs of entry and using credit, which, in turn, reduces the utility loss of participating in faraway inter-village markets. This effect is somewhat similar to the search externality effect in monetary search models when market entry is endogenous (e.g., Shi, 2008). That is, there are too many market entries relative to the socially efficient level, since the representative household does not internalize the search externality, which widens the difference between the consumption of red goods in the credit-region and the consumption of red goods in the no-credit region. However, efficiency requires that the consumption of red goods in both regions should be as close as possible. A minor deviation from the Friedman rule, thus, shifts consumption from the red goods in the credit region to those in the no-credit region. This smoothes marginal utilities across regions, which is welfare improving.

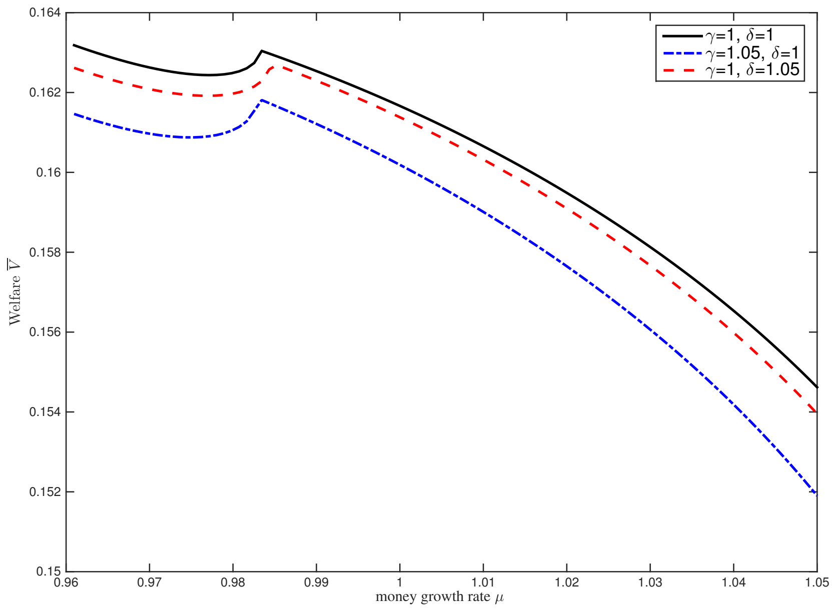

Figure 2.3: The effect of inflation on social welfare under alternative combinations of costs of entry and using credit

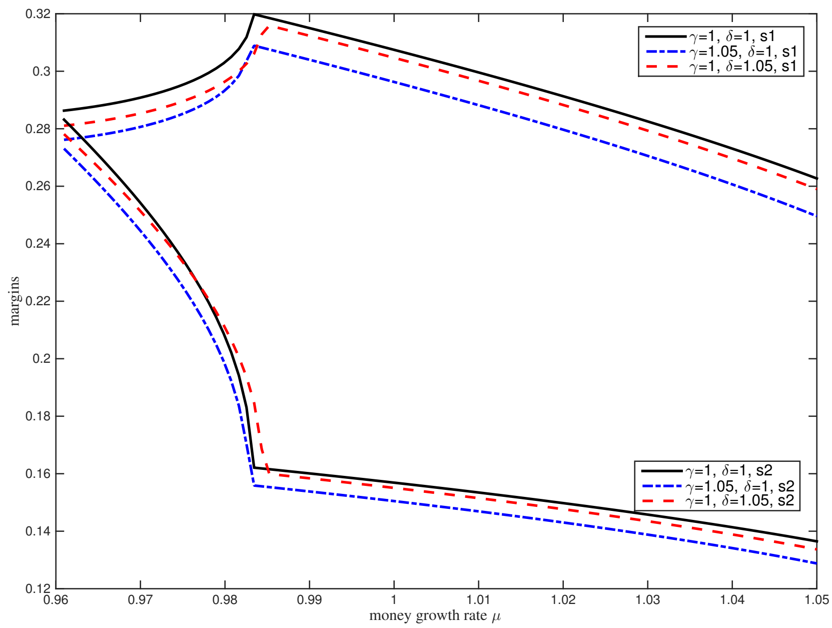

Figure 2.4: The effect of inflation on the extensive and intensive margins under alternative combinations of costs of entry and using credit

2.6 Discussion

The relationship between money and credit has been examined from different perspectives. Gomis-Porqueras and Sanches (2013) take the view that fiat money complements the use of credit in exchange. With the introduction of fiat money, agents have an alternative mechanism to trade in the decentralized market. This implies that it is more difficult to induce buyers to cooperate when the value of money is constant over time. In order to strengthen the punishment of default, the authors argue that the government can engineer a positive inflation rate. Jin and Temzelides (2004), which uses the Kiyotaki and Wright (1989) model introduce spatial heterogeneity with explicitly modelling distance, takes the view that monetary trade is a outside option to credit trade. In nearer villages, agents meet with higher probability. The main finding is that when there is no direct cross-location informational flow, money is an imperfect substitute for credit or record-keeping. In fact, Kocherlakota (1998) shows that money is dominated by memory at least when agents are patient in the second-generation search models. Intuitively, monetary and credit transactions differ in their scope. If an agent fails to perform a credit transaction, he loses the opportunity to trade again with the village of the agent that experienced the deviation. On the other hand, if an agent fails to produce in exchange for money, he loses the opportunity to trade with agents from a large number of villages in the near future. Thus, monetary exchange allows for the possibility of a punishment that is less severe but broader in scope than denial of future credit.

We view this chapter complementary to existing work, which studies the relationship between money and credit in monetary search models as well. Given the historical background, it is arguable that commodity money should be more relevant, and that the model here is helpful to explain the use of credit during currency shortage. In fact, this is one of the considerations that shape our view that costly credit complements fiat money as means of payment in decentralized exchange. Nevertheless, the Lagos and Wright (2005) framework is also suitable to model commodity money. As Wright (2010) does, we can think commodity money as the "Lucas tree" bearing fruit (general goods), and fiat money as fruitless trees. Hence, the main results in this chapter are valid for both fiat money and commodity money, as long as agents sometimes face a liquidity shortage problem.

There are several qualifications to our treatment. One limitation of our analysis is that we simply assume local communities have the incentive to sustain their partnership and hence punish defectors. Taking strategic interaction between villages into consideration, we maybe construct a much richer model with additional economic implications, but this kind of model is probably more analytically complicated. Hence in this chapter we do not attempt to explain the communal responsibility system as an equilibrium phenomenon[13]. Instead we exclusively ask whether credit can be supported with this interesting historical institution in place and what the implications are. Another extension we can pursue is to compare different pricing protocol, as in Rocheteau and Wright (2005).

2.7 Conclusion

We have taken the view that credit complements fiat money as means of payment in decentralized exchange, and have considered whether the communal responsibility system can support credit with anonymous agents. In our analysis, lack of double coincidence of wants and anonymity keep money essential, while credit can relax agents' liquidity constraints when shocks are favourable. The distance-dependent cost of using credit sustains money as means of payment in the village economy. In this way, we have shown that money and credit can meaningfully coexist under the communal responsibility system, in the sense of improving welfare. In addition, the Friedman rule can be sub-optimal, due to the interaction between the extensive and intensive margins.

[13]We think that combining the work of Aliprantis et al. (2007) and Deb (2012) could be a good starting point.

Appendix

2.A Technical Appendix

2.A.1 The Problem in Day Markets

Without loss of generality, let $\alpha = \frac{1}{2}$. During the day, the household's problem can be summarized by the following Lagrangian function,

$$\mathcal{L} = \frac{\pi^b}{2}\int_0^{s_1} u\left(q^r(j)\right)dj - \frac{\pi^s}{2}\left(1-S_1\right)\int_0^{S_1}\psi\left(Q^r(j)\right)dj$$

$$+\frac{\pi^b}{2}\int_0^{s_1}\varepsilon u\left(q^w(j)\right)dj - \frac{\pi^s}{2}\left(1-S_1\right)\int_0^{S_1}\psi\left(Q^w(j)\right)dj - \int_0^{s_1}\chi\left(j\right)dj - \frac{\pi^b}{2}\int_0^{s_2}\kappa\left(j\right)dj + W\left(\widetilde{m}\right)$$

$$+\frac{\pi^b}{2}\lambda_1^w(j)\left[\phi d^w(j) - \psi(q^w(j))\right] + \frac{\pi^b}{2}\lambda_2^w(j)\left[\phi m - \phi s_1 d^w(j)\right]$$

$$+\frac{\pi^b}{2}\lambda_1^r(j)\mathbf{I}_{\{0<j\leq s_2\}}\left[\phi d^r(j) - \psi(q^r(j))\right]$$

$$+\frac{\pi^b}{2}\lambda_2^r(j)\mathbf{I}_{\{0<j\leq s_2\}}\left[\phi m + \phi s_1 l\left(j\right) - \phi s_1 d^r(j)\right]$$

$$+\frac{\pi^b}{2}\lambda_1^r(j)\mathbf{I}_{\{s_2<j\leq s_1\}}\left[\phi d^r(j) - \psi(q^r(j))\right]$$

$$+\frac{\pi^b}{2}\lambda_2^r(j)\mathbf{I}_{\{s_2<j\leq s_1\}}\left[\phi m - \phi s_1 d^r(j)\right]$$

The first-order conditions are given as follows:

$$\frac{\partial\mathcal{L}}{\partial q^w(j)} = \varepsilon u'\left(q^w(j)\right) - \lambda_1^w(j)\psi'(q^w(j)) = 0$$

which says that an extra consumption of the white goods increases utility of buyers, but tightens the participation constraint of sellers;

$$\frac{\partial\mathcal{L}}{\partial q^r(j)} = u'\left(q^r(j)\right) - \lambda_1^r(j)\psi'(q^r(j)) = 0$$

which says that an extra consumption of the red goods increases utility of buyers, but tightens the participation constraint of sellers;

$$\frac{\partial \mathcal{L}}{\partial d^w(j)} = \int_0^{s_1} \lambda_1^w(j)dj - s_1 \int_0^{s_1} \lambda_2^w(j)dj - \int_0^{s_1} \frac{W'(\widetilde{m})}{\phi} dj = 0$$

which says that an extra payment of cash for the white goods relaxes the participation constraint of sellers, tightens the liquidity constraint, and reduces liquidity for the following sub-period;

$$\frac{\partial \mathcal{L}}{\partial d^r(j)} = \int_0^{s_1} \lambda_1^r(j)dj - s_1 \int_0^{s_1} \lambda_2^r(j)dj - \int_0^{s_1} \frac{W'(\widetilde{m})}{\phi} dj = 0$$

which says that an extra payment of cash for the red goods relaxes the participation constraint of sellers, tightens the liquidity constraint, and reduces liquidity for the following sub-period;

$$\frac{\partial \mathcal{L}}{\partial l(j)} = \phi s_1 \int_0^{s_2} \lambda_2^r(j)dj - \int_0^{s_2} rW'(\widetilde{m})\, dj = 0$$

which says that an extra unit of cash borrowed for buyers of red goods in the credit region increases the cash available for consumption purposes, and increases the net repayment in the following sub-period;

$$\frac{\partial \mathcal{L}}{\partial s_1} = \frac{\pi^b}{2} u\left(q^r(s_1)\right) + \frac{\pi^b}{2} \varepsilon u\left(q^w(s_1)\right) - \chi\left(s_1\right)$$

$$-W'(\widetilde{m}) \left(\frac{\pi^b}{2} d^r(s_1) + \frac{\pi^b}{2} d^w(s_1) \right)$$

$$-\frac{\pi^b}{2} \lambda_2^w(s_1) \phi d^w(s_1) - \frac{\pi^b}{2} \lambda_2^r(s_1) \phi d^r(s_1) = 0$$

which says that the range of decentralized markets participated by the representative household is determined by the expected trade surplus net of the entry cost, and by the fact that the further the marginal market, less amount of money distributed to each buyers of the household, which which tightens the liquidity constraints of both buyers of red and white goods;

$$\frac{\partial \mathcal{L}}{\partial s_2} = u\left(\psi^{-1}\left(\phi l(s_2) + \frac{\phi m}{s_1} \right) \right) - u\left(\psi^{-1}\left(\frac{\phi m}{s_1} \right) \right)$$

$$-\kappa\left(s_2\right) - W'(\widetilde{m})\, rl(s_2) = 0$$

which says that credit is used as medium of exchange up to the point when its benefit of relaxing liquidity constraint equals to the cost of using credit.

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3 The Inflation Bias under Calvo and Rotemberg Pricing*

New Keynesian analysis relies heavily on two workhorse models of nominal inertia - due to (Calvo, 1983) and (Rotemberg, 1982), respectively - to generate a meaningful role for monetary policy. These are often used interchangeably, since they imply an isomorphic linearized Phillips curve and, if the steady-state is efficient, the same policy conclusions. In this chapter we compute time-consistent optimal monetary policy in benchmark New Keynesian models containing each form of price stickiness, using global solution techniques. We find that the inflation bias problem under Calvo contracts is significantly greater than under Rotemberg pricing, despite the fact that the former typically exhibits far greater welfare costs of inflation. The rates of inflation observed under this policy are non-trivial and suggest that the model can comfortably generate the rates of inflation at which the problematic issues highlighted in the trend inflation literature emerge, as well as the movements in trend inflation emphasized in empirical studies of the evolution of inflation. Finally, we consider the response to cost push shocks across both models and find these can also be quite different. The non-linearities inherent in the New Keynesian model are significant and the form of nominal inertia adopted is not innocuous.

3.1 Introduction

Mainstream macroeconomic analysis of both monetary and fiscal policy relies heavily on the New Keynesian model. The distinguishing feature of this model, relative to a more classical approach, is that it contains some form of nominal inertia. This allows

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monetary policy to have real effects, and widens the degree of interaction between monetary and fiscal policies, since monetary policy affects both the size of the tax base and real debt service costs in such models. Typically, one of two workhorse forms of nominal inertia are adopted in the literature - Calvo (1983) price contracts, and Rotemberg (1982) price adjustment costs. In the former, firms are only able to adjust their prices after random intervals of time, such that, outside of a zero inflation steady-state there will be a costly dispersion of prices across firms. While the latter implies that all firms behave symmetrically in setting the same price, but that they face quadratic adjustment costs in doing so. Despite this fundamental difference, researchers have typically treated the two approaches as being equivalent since the New Keynesian Phillips Curve (NKPC) they imply are, to a first order of approximation, isomorphic when linearized around a zero inflation steady state. Moreover, when that zero inflation steady-state is also efficient (that is, it matches the output level that would be chosen by a benevolent social planner) it can be shown that the second order approximation to welfare rewritten in terms of inflation and the output gap is also the same across the two approaches (see Nistico, 2007). Under these conditions, to a first order of approximation, the two approaches would yield the same policy implications. For these reasons the two approaches have largely been treated as synonymous within the New Keynesian literature.

However, despite this broad consensus, there are examples within the literature where the two approaches do differ. The first is where the steady-state around which we approximate the New Keynesian economy is not efficient. For example, Lombardo and Vestin (2008) relax the assumption of Nistico (2007) and consider the second order approximation to welfare when the steady state is not efficient. They find that the costs of such inefficiencies are typically larger in the Calvo economy. This mirrors the results in Damjanovic and Nolan (2011) .

The second assumption underpinning the equivalence result, is that the economy is approximated around a zero inflation steady state (or that any steady-state inflation is perfectly indexed and therefore costless, see Yun (1996)). A literature considering the importance of trend inflation argues that this is not the case, and that the implications of failing to account for trend inflation can be dramatic, see Ascari and Sbordone (2014) for a survey. The presence of even a modest degree of (unindexed) steady-state inflation can radically overturn determinacy results, undermine the learnability of rational expectations equilibria, affect the monetary policy transmission mechanism and change the nature of optimal policy. Moreover, these effects can differ across the two forms of nominal inertia (Ascari and Rossi, 2012) with the larger impact of trend inflation being felt under Calvo. The large costs of trend inflation under Calvo is also reflected in the analysis of Damjanovic and Nolan (2010a) where the seigniorage maximizing rate of inflation is at double digit levels under Rotemberg pricing, but only single digits under Calvo. In short, there appears to be significant non-linearities in the New Keynesian model which are affected by the size of the steady-state distortion, the degree of unindexed inflation and the type of nominal inertia adopted. However, this evidence largely comes from studies which linearize such economies, either to a first or second order approximation, after allowing for such factors.

In this chapter, we solve the benchmark New Keynesian model non-linearly, using the two standard approaches to modelling price stickiness. Since we are not imposing any kind of approximation around a steady-state, we can fully explore the non-linearities inherent in the New Keynesian model and see clearly the extent to which the two approaches differ. Moreover, rather than consider the Ramsey problem or commitment to a simple monetary rule, we shall consider time-consistent optimal policy (commonly known as discretion). This in turn, given that we are not using any artificial devices to ensure the model's steady-state is efficient, implies that we can measure the extent of the inflationary bias problem under the two forms of nominal inertia. This identifies the extent to which a policy maker who is constrained to be time-consistent would be unable to prevent a costly rise in steady-state or trend inflation. This is an important measure of the non-linearities across the two descriptions of pricing behavior, but also serves as a plausibility check on the relevance of the effects highlighted in the literature on trend inflation. The inflation bias thus measures the maximum level of unindexed inflation that a policy maker would be forced to tolerate - the policy maker which allowed inflation to rise above this level is behaving sub-optimally even given the constraint that they cannot commit. Therefore, if the level of inflation bias is significantly below that required to generate the perverse results found in the trend-inflation literature, then we would need to find a reason why policy makers are not only failing to commit, but are generating inflation levels well beyond the maximum inflation bias, before we need worry about these properties of the New Keynesian model. While if the model implies a sizeable inflation bias, then the issues raised by the trend inflation literature and, more generally, the non-linearities inherent in the New Keynesian model need to be taken more seriously.

There is also an empirical literature which focusses on these two distortions in helping to explain inflation dynamics. Ireland (2007) allows for time variation in the Fed's inflation target to explain the evolution of US inflation. Cogley and Sargent (2002) argue that much of the movement in US inflation reflects movements in an underlying trend, rather than in fluctuations relative to that trend. Meanwhile, several authors have sought to identify the level of trend inflation using generalizations of the new Keynesian Phillips curve which allow for time varying (unindexed) trend inflation. As an example of the findings of this literature, Cogley and Sbordone (2008) argue that trend inflation rather than any kind of backward-looking indexation behavior is a major component of observed movements in inflation. Again we can ask - can the benchmark model, using either Rotemberg or Calvo pricing plausibly deliver the size of unindexed steady state or trend inflation these papers infer to explain the data?

Moving away from the Ramsey description of policy is important, as such a policy implies that the optimal rate of inflation the policy maker would commit to would be zero in the benchmark model employing either Calvo or Rotemberg pricing (Woodford, 2003), and in the case of Calvo contracts very close to zero in models with other distortions due to, for example, fiscal policy (Schmitt-Grohe and Uribe, 2004) or a desire to generate seigniorage revenues (Damjanovic and Nolan, 2011). Under Rotemberg, the example of Damjanovic and Nolan suggests that this may not be a general result across the two descriptions of nominal inertia, since the welfare costs of nominal inertia do not rise as sharply as the rate of inflation rises under Calvo. Nevertheless, the fact remains that Ramsey policy would typically imply that inflation was far lower and stable than appears to be found in the data.[14]

There are some recent papers using global solution techniques which also consider optimal discretionary policy in the New Keynesian model under Calvo contracts - see Van Zandweghe and Wolman (2011) and Anderson et al. (2010), which is then extended in Ngo (2014) to allow for discount factor shocks which imply that policy must account for the zero lower bound (ZLB).[15] Other authors also consider issues relating to the ZLB in models which use Rotemberg pricing, but also introduce extensions such as capital (see Gavin et al. (2013), Braun and Korber (2011), Johannsen (2014)), consumption habits (Gust et al. (2012) and Aruoba and Schorfheide (2013)), labor market frictions (Roulleau-Pasdeloup (2013)) or fiscal policy (Nakata (2013), Niemann et al. (2013) and Johannsen (2014)).[16] Solving non-linear representations of an enriched New Keynesian model is typically far more computationally intensive than conventional perturbation methods, and these latter authors have all adopted the Rotemberg description of price stickiness since this reduces the number of state variables one must consider. Further-

[14] Chen et al. (2014) assess the relative empirical performance of a New Keynesian model with habits and inflation inertia with policy described by not only by simple rules, but also optimal policy under discretion, commitment and quasi-commitment. They find that discretion fits the data far better than any other description of policy, especially commitment which is simply too effective in stabilising the economy to be consistent with the data.

[15] Fernandez-Villaverde et al. (2012), Wieland (2013) and Richter et al. (2013) also explore equilibrium dynamics around the ZLB in variants of the New Keynesian model which adopt Calvo price contracts, but which adopt a rule-based description of policy.

[16] Within this group, Shibayama and Sunakawa (2012), Nakata (2013), and Niemann et al. (2013) explore optimal policy in various New Keynesian models using Rotemberg pricing. The others utilise a rule-based description of policy.

more, in calibrating the Rotemberg price adjustment cost parameter almost all these authors use a conventional parameterization which matches the slope of the linearized NKPC across the Rotemberg and Calvo variants of the New Keynesian model after assuming a zero inflation steady-state. In other words, the literature is typically implicitly assuming that the equivalence of the two forms of nominal inertia is retained in non-linear solutions of the New Keynesian model where the steady-state is distorted and the rate of inflation will typically not be zero. To our knowledge, the current chapter is the first to formally compare and contrast time-consistent optimal policy under the two forms of price-setting using global solution algorithms, and therefore to assess how innocuous the choice of one form of price-setting over the other actually is.

We find that the inflationary bias problem is non-trivial under both descriptions of nominal inertia, but is much greater under Calvo. This is despite earlier results implying that the costs of inflation are much higher under Calvo than Rotemberg. This essentially arises because of the different average mark-up behavior under the two models. Under Calvo higher inflation causes those firms who are able to adjust prices in a particular period to raise that price in anticipation of not being able to readjust that price for a prolonged period, despite the general rise in the price level. This leads to an increase in the average mark-up as inflation rises. In contrast, under Rotemberg all firms set the same price, period by period, but face adjustment costs in doing so. In discounting future profits, they also discount future price adjustment costs. As a result, in the face of higher inflation the firms postpone some of the required price adjustment due to this discounting effect, which serves to reduce the average markup. Accordingly, for a given degree of monopolistic competition which induces an inflation bias, this further raises (lowers) the markup under Calvo (Rotemberg) and thereby worsens (improves) the inflationary bias problem. This effect also tends to imply that the inflationary impact of a given cost-push shock is greater under Calvo pricing, ceteris paribus. Meanwhile, the presence of an additional state variable under Calvo price-setting, namely price dispersion, can also result in a hump-shaped response in output to cost-push shocks which would not be the case under either Rotemberg pricing or the benchmark linearized model. The fact that steady-state inflation would ceteris paribus, and using standard calibration approaches, be significantly higher under Calvo also has implications for the probability of hitting the ZLB such that studies adopting Rotemberg pricing are more likely to experience such episodes.

The rest of the chapter is organized as follows. In section 2, we describe the basic model under both Calvo and Rotemberg pricing. In section 3, we formulate the optimal discretionary policy problem with Rotemberg and Calvo pricing, respectively. In section 4, we present numerical results. In section 5, we extend the analysis to allow for a tax-driven cost-push shock to assess policy trade-offs. We conclude in section 6.

3.2 The Model

This section describes the basic economic structure in our model.

3.2.1 Households

There are a continuum of households of size one. We shall assume full asset markets, such that, through risk sharing, they will face the same budget constraint and make the same consumption plans. As a result, at period 0 the typical household will seek to maximize the following objective function,

$$E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \tag{3.1}$$

where $0 < \beta < 1$ denotes the discount factor, C_t and N_t are a consumption aggregate, and labour supply at period t, respectively.

The household purchases differentiated goods in a retail market and combines them into composite goods using a CES aggregator:

$$C_t = \left(\int_0^1 C_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}, \epsilon > 1 \tag{3.2}$$

where $C_t(j)$ is the demand for differentiated goods of type j.

The budget constraint at time t is given by

$$\int_0^1 P_t(j) C_t(j) dj + E_t \left\{ Q_{t,t+1} D_{t+1} \right\} = \Xi_t + D_t + W_t N_t - T_t \tag{3.3}$$

where $P_t(j)$ is the nominal price of type j goods, D_{t+1} is the nominal payoff of the nominal bonds portfolio held at the end of period t, Ξ is the representative household's share of profits in the imperfectly competitive firms, W are wages, and T are lump-sum taxes/transfers.[17] $Q_{t,t+1}$ is the stochastic discount factor for one period ahead payoffs. The labor market is perfectly competitive and wages are fully flexible.

Households must first decide how to allocate a given level of expenditure across the various goods that are available. They do so by adjusting the share of a particular good

[17] In Section 5 we shall analyse cost push shocks driven by fluctuations in a revenue tax which shall be rebated to households in a lump-sum form.

in their consumption bundle to exploit any relative price differences—this minimizes the costs of consumption. The demand curve for each good j is,

$$C_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} C_t \tag{3.4}$$

where the aggregate price level P_t is defined to be

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}. \tag{3.5}$$

The dynamic budget constraint at period t can therefore be rewritten as

$$P_t C_t + E_t \{Q_{t,t+1} D_{t+1}\} = \Xi_t + D_t + W_t N_t - T_t. \tag{3.6}$$

Households' problem

The household's decision problem can be dealt with in two stages. First, regardless of the level of C_t the household purchases the combination of individual goods that minimizes the cost of achieving this level of the composite good. Second, given the cost of achieving any given level of C_t, the household chooses C_t, D_{t+1} and N_t optimally. We have solved the first stage problem above. For tractability, we assume that (3.1) takes the specific form

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}\right). \tag{3.7}$$

where $\sigma > 0$ is a risk aversion parameter and $\varphi > 0$ is the inverse of the Frisch elasticity of labor supply.

We can then maximize utility subject to the budget constraint (3.6) to obtain, after taking expectations, the optimal allocation of consumption across time,

$$\beta R_t E_t \left\{\left(\frac{C_t}{C_{t+1}}\right)^{\sigma} \left(\frac{P_t}{P_{t+1}}\right)\right\} = 1, \tag{3.8}$$

where $R_t \equiv \frac{1}{E_t(Q_{t,t+1})}$ is the gross nominal return on a riskless one period bond paying off a unit of currency in $t+1$. This is the familiar consumption Euler equation which implies that consumers are attempting to smooth consumption over time such that the marginal utility of consumption is equal across periods (after allowing for tilting due to interest rates differing from the households' rate of time preference).

The second first order condition concerning labour supply decision is given by

$$\left(\frac{W_t}{P_t}\right) = N_t^{\varphi} C_t^{\sigma}. \tag{3.9}$$

3.2.2 Firms

Each firm produces a differentiated good j using a constant returns to scale production function:

$$Y_t(j) = A_t N_t(j) \tag{3.10}$$

where $Y_t(j)$ is the output of firm j, and $N_t(j)$ denotes the hours hired by the firm, A_t is an exogenous aggregate productivity shock at period t, and $a_t = \log(A_t)$ is time varying and stochastic.[18]

Similar to the household's problem, we first consider the cost minimization problem of firm j, which implies that the real marginal costs of production are given by

$$mc_t = \frac{W_t}{P_t A_t}, \tag{3.11}$$

Note that the real marginal cost described in (3.11) does not depend on the output level of an individual firm, since its production function exhibits constant returns to scale and prices of inputs (here labor) are fully flexible.

The demand curve the firm j faces is given by

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t,$$

where $Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}}$.

The intermediate-good sector is monopolistically competitive and the intermediate good producer therefore has market power. In the following, we consider two alternative forms of price stickiness - firstly that due to Rotemberg (1982) and then that of Calvo (1983).

Rotemberg Pricing

The Rotemberg model assumes that a monopolistic firm faces a quadratic cost of adjusting nominal prices, which can be measured in terms of the final good and given

[18] Typically, the logarithm of A_t is assumed to follow an $AR(1)$ process: $a_t = \rho_a a_{t-1} + e_{at}$, $0 \leq \rho_a < 1$ where technology shock e_{at} is an $i.i.d.$ random variable, which has a zero mean and a finite standard deviation σ_a.

by

$$\frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)}-1\right)^2 Y_t \tag{3.12}$$

where $\phi \geq 0$ measures the degree of nominal price rigidity. The adjustment cost, which accounts for the negative effects of price changes on the customer–firm relationship, increases in magnitude with the size of the price change and with the overall scale of economic activity Y_t.

The problem for firm j is then to maximize the discounted value of nominal profits,

$$\max_{\{P_t(j)\}_{t=0}^{\infty}} E_t \sum_{s=0}^{\infty} Q_{t,t+s}\Xi_{t+s}$$

where nominal profits are defined as

$$\Xi_t = P_t(j)Y_t(j) - mc_t Y_t(j) P_t - \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)}-1\right)^2 Y_t P_t \tag{3.13}$$

Firms can change their price in each period, subject to their demand curve and payment of the adjustment cost. Hence, all the firms face the same problem, and thus will choose the same price, and produce the same quantity such that, $P_t(j) = P_t$ and $Y_t(j) = Y_t$ for any j. Hence, the first-order condition for a symmetric equilibrium is

$$(1-\epsilon) + \epsilon mc_t - \phi\Pi_t\left(\Pi_t - 1\right) + \phi\beta E_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\frac{Y_{t+1}}{Y_t}\Pi_{t+1}\left(\Pi_{t+1}-1\right)\right] = 0. \tag{3.14}$$

This is the Rotemberg version of the non-linear Phillips curve that relates current inflation to future expected inflation and to the level of output.

Calvo Pricing

Each period, the firms that adjust their price are randomly selected, and a fraction $1-\theta$ of all firms adjust while the remaining θ fraction do not adjust. Those firms that do adjust their price at time t do so to maximize the expected discounted value of current and future profits. Profits at some future date $t+s$ are affected by the choice of price at time t only if the firm has not received another opportunity to adjust between t and $t+s$. The probability of this is θ^s.

The firm's pricing decision problem then involves picking $P_t(j)$ to maximize discounted nominal profits Using the demand curve for the firm's product, this objective function can be written as

$$E_t \sum_{s=0}^{\infty} \theta^s Q_{t,t+s} \left[P_t(j) \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} - mc_{t+s} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} P_{t+s} \right].$$

where the discount factor $Q_{t,t+s}$ is given by $\beta^s \left(\frac{C_t}{C_{t+s}}\right)^{\sigma} \frac{P_t}{P_{t+s}}$, and mc_{t+s} is the marginal cost of production.

Let P_t^* be the optimal price chosen by all firms able to reset their price at time t. The first order condition for the optimal choice of P_t^* is,

$$\frac{P_t^*}{P_t} = \left(\frac{\epsilon}{\epsilon - 1} \right) \frac{K_t^p}{F_t^p} \tag{3.15}$$

where

$$\begin{aligned} K_t^p &= C_t^{-\sigma} mc_t Y_t + \theta\beta E_t \left[\left(\frac{P_{t+1}}{P_t} \right)^{\epsilon} K_{t+1}^p \right] \\ F_t^p &= C_t^{-\sigma} Y_t + \theta\beta E_t \left[\left(\frac{P_{t+1}}{P_t} \right)^{\epsilon - 1} F_{t+1}^p \right]. \end{aligned}$$

The price index evolves according to

$$1 = (1-\theta) \left(\frac{P_t^*}{P_t} \right)^{1-\epsilon} + \theta (\Pi_t)^{\epsilon - 1} \text{ with } \Pi_t \equiv \frac{P_t}{P_{t-1}} . \tag{3.16}$$

and price dispersion is described by

$$\Delta_t \equiv \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} dj = (1-\theta) \left(\frac{P_t^*}{P_t} \right)^{-\epsilon} + \theta \left(\frac{P_t}{P_{t-1}} \right)^{\epsilon} \Delta_{t-1}. \tag{3.17}$$

3.2.3 Aggregate Conditions

Under Rotemberg pricing, as all the firms will employ the same amount of labour, the aggregate production function is simply given by

$$Y_t = A_t N_t.$$

and the aggregate resource constraint is given by

$$Y_t = C_t + \frac{\phi}{2} (\Pi_t - 1)^2 Y_t.$$

Note that the Rotemberg adjustment cost creates an inefficiency wedge ψ_t^R between output and consumption

$$C_t = \left(1 - \psi_t^R\right) Y_t = \left(1 - \psi_t^R\right) A_t N_t \tag{3.18}$$

where $\psi_t^R = \frac{\phi}{2}\left(\Pi_t - 1\right)^2$.

In the case of Calvo pricing, firms changing prices in different periods will generally have different prices. Thus, the model features price dispersion. When firms have different relative prices, there are distortions that create a wedge between the aggregate output measured in terms of production factor inputs and aggregate demand measured in terms of the composite goods. Specifically,

$$N_t(j) = \frac{Y_t(j)}{A_t} = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} \frac{Y_t}{A_t}$$

which yields,

$$N_t = \int_0^1 N_t(j)dj = \frac{Y_t}{A_t}\int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj = \frac{Y_t \Delta_t}{A_t}$$

after integrating across firms. $\Delta_t \geq 1$ implies that price dispersion is always costly in terms of aggregate output: the higher Δ_t, the more labour is needed to produce a given level of output. Moreover, under Calvo different firms with different prices will employ different amounts of labor. This explains why higher price dispersion acts as a negative productivity shift in the aggregate production function: $Y_t = (A_t/\Delta_t)N_t$. In addition, price dispersion is a backward-looking variable, and introduces an inertial component into the model.

Under Calvo, the aggregate resource constraint is simply given by

$$Y_t = C_t.$$

Hence, defining $\psi_t^c = \Delta_t - 1$ as an inefficiency wedge under Calvo, then

$$C_t = Y_t = \frac{A_t N_t}{(1 + \psi_t^c)} \tag{3.19}$$

Comparing (3.18) and (3.19), it is illuminating to note that the Rotemberg adjustment cost creates a wedge ψ_t^R between aggregate consumption and aggregate output, while the Calvo price dispersion creates a wedge ψ_t^c between aggregate hours and aggregate output. In addition, both wedges are non-linear functions of inflation, and they are minimized at one when steady-state net inflation equals zero ($\Pi = 1$), and both wedges increase as trend inflation moves away from zero. See Ascari and Rossi (2012) for a discussion.

Appendix 3.A.1 summarizes the models under Rotemberg and Calvo pricing.

3.3 Optimal Policy Problem Under Discretion

Under discretion, the monetary authority solves a sequential or period-by-period optimization problem, which maximizes the representative household's expected discounted utility subject to the optimality conditions from market participants, the aggregate conditions, and the law of motion for the state variables. Therefore, under optimal discretion, the policymaker cannot commit to a plan in the hope of influencing economic agents' expectations.

3.3.1 Rotemberg Pricing

Let $V(A_t)$ represents the value function at period t in the Bellman equation for the optimal policy problem. The optimal monetary policy then solves the following optimization problem:

$$V(A_t) = \max_{\{C_t, Y_t, \Pi_t\}} \left\{ \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi} + \beta E_t \left[V(A_{t+1}) \right] \right\} \tag{3.20}$$

subject to,

$$C_t = \left[1 - \frac{\phi}{2} \left(\Pi_t - 1 \right)^2 \right] Y_t \tag{3.21}$$

and,

$$(1-\epsilon) + \epsilon Y_t^{\varphi} C_t^{\sigma} A_t^{-\varphi-1} - \phi \Pi_t \left(\Pi_t - 1 \right) + \phi \beta E_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \frac{Y_{t+1}}{Y_t} \Pi_{t+1} \left(\Pi_{t+1} - 1 \right) \right] = 0 \tag{3.22}$$

Defining an auxilliary function,

$$M(A_{t+1}) \equiv C_{t+1}^{-\sigma} Y_{t+1} \Pi_{t+1} \left(\Pi_{t+1} - 1 \right)$$

we can rewrite the Phillips curve (4.12) as,

$$(1-\epsilon) + \epsilon Y_t^{\varphi} C_t^{\sigma} A_t^{-\varphi-1} - \phi \Pi_t \left(\Pi_t - 1 \right) + \phi \beta C_t^{\sigma} Y_t^{-1} E_t \left[M(A_{t+1}) \right] = 0$$

which captures the fact that the policy maker recognizes that any change in the state variable will affect expectations, but cannot promise to behave in a particular way tomorrow in order to influence expectations today. The optimal policy problem can then be formulated as the following Lagrangian,

$$\mathcal{L} = \frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi} + \beta E_t\left[V(A_{t+1})\right] + \lambda_{1t}\left\{\left[1-\frac{\phi}{2}\left(\Pi_t-1\right)^2\right]Y_t - C_t\right\}$$
$$+\lambda_{2t}\left\{(1-\epsilon)+\epsilon Y_t^{\varphi}C_t^{\sigma}A_t^{-\varphi-1} - \phi\Pi_t\left(\Pi_t-1\right)+\phi\beta C_t^{\sigma}Y_t^{-1}E_t\left[M(A_{t+1})\right]\right\}$$

where λ_{1t} and λ_{2t} are the Lagrange multipliers. The first order conditions and complementary slackness conditions are given as follows,

$$C_t^{-\sigma} = \lambda_{1t} - \lambda_{2t}\left\{\sigma\epsilon Y_t^{\varphi}C_t^{\sigma-1}A_t^{-\varphi-1} + \sigma\phi\beta C_t^{\sigma-1}Y_t^{-1}E_t\left[M(A_{t+1})\right]\right\},$$
$$Y_t^{\varphi}A_t^{-1-\varphi} = \lambda_{1t}\left[1-\frac{\phi}{2}\left(\Pi_t-1\right)^2\right] + \lambda_{2t}\left\{\epsilon\varphi Y_t^{\varphi-1}C_t^{\sigma}A_t^{-\varphi-1} - \phi\beta C_t^{\sigma}Y_t^{-2}E_t\left[M(A_{t+1})\right]\right\},$$
$$\lambda_{1t}\phi\left(1-\Pi_t\right)Y_t = \lambda_{2t}\phi\left(2\Pi_t-1\right),$$
$$C_t = \left[1-\frac{\phi}{2}\left(\Pi_t-1\right)^2\right]Y_t,$$
$$0 = (1-\epsilon)+\epsilon Y_t^{\varphi}C_t^{\sigma}A_t^{-\varphi-1} - \phi\Pi_t\left(\Pi_t-1\right)+\phi\beta C_t^{\sigma}Y_t^{-1}E_t\left[M(A_{t+1})\right].$$

Note that consumption Euler equation is non-binding from the point of view of maximizing utility, because R_t (a variable of no direct interest in utility) can effectively be chosen to achieve the desired level of consumption.

The fully nonlinear problem is then to find five policy functions which relate the three choice variables $\{Y_t, C_t, \Pi_t\}$ and two Lagrange multipliers $\{\lambda_{1t}, \lambda_{2t}\}$ to the state variable A_t, that is, $Y_t = Y(A_t)$, $C_t = C(A_t)$, $\Pi_t = \Pi(A_t)$, $\lambda_{1t} = \lambda_1(A_t)$, and $\lambda_{2t} = \lambda_2(A_t)$. We will use the Chebyshev collocation method to approximate these five time invariant rules.

3.3.2 Calvo Pricing

Let $V(\Delta_{t-1}, A_t)$ denote the value function at period t in the Bellman equation for the optimal policy problem. The optimal monetary policy under discretion then can be described as a set of decision rules for $\{C_t, Y_t, \Pi_t, \frac{P_t^*}{P_t}, K_t^p, F_t^p, \Delta_t\}$ which maximize,

$$V(\Delta_{t-1}, A_t) = \max\left\{\frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{(\Delta_t Y_t/A_t)^{1+\varphi}}{1+\varphi} + \beta E_t\left[V(\Delta_t, A_{t+1})\right]\right\}$$

subject to the following constraints,

Resource constraint:

$$Y_t = C_t$$

Phillips curve:

$$\frac{P_t^*}{P_t} = \left(\frac{\epsilon}{\epsilon - 1}\right) \frac{K_t^p}{F_t^p}$$

with

$$\begin{aligned} K_t^p &= (\Delta_t Y_t)^{\varphi} A_t{}^{-\varphi-1} Y_t + \theta\beta E_t \left[M(\Delta_t, A_{t+1})\right] \\ F_t^p &= Y_t C_t^{-\sigma} + \theta\beta E_t \left[L(\Delta_t, A_{t+1})\right], \end{aligned}$$

where we have utilized two auxilliary functions,

$$M(\Delta_t, A_{t+1}) = (\Pi_{t+1})^{\epsilon} K_{t+1}^p$$

and

$$L(\Delta_t, A_{t+1}) = (\Pi_{t+1})^{\epsilon-1} F_{t+1}^p,$$

which again captures the highlights the fact that the policy maker recognizes that any change in the state variable will affect expectations, but cannot make credible promises about their future behavior. Inflation:

$$1 = (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{1-\epsilon} + \theta(\Pi_t)^{\epsilon-1}$$

Price dispersion:

$$\Delta_t = (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\epsilon} + \theta\left(\Pi_t\right)^{\epsilon}\Delta_{t-1}.$$

As before, the policy problem can be written in Lagrangian form as follows:

$$\begin{aligned} \mathcal{L} = & \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{(\Delta_t Y_t / A_t)^{1+\varphi}}{1+\varphi} + \beta E_t \left[V(\Delta_t, A_{t+1})\right] \\ & + \lambda_{1t}[Y_t - C_t] \\ & + \lambda_{2t}\left[\frac{P_t^*}{P_t} - \left(\frac{\epsilon}{\epsilon-1}\right)\frac{K_t^p}{F_t^p}\right] \\ & + \lambda_{3t}\left\{K_t^p - (\Delta_t Y_t)^{\varphi} A_t{}^{-\varphi-1} Y_t - \theta\beta E_t\left[M(\Delta_t, A_{t+1})\right]\right\} \\ & + \lambda_{4t}\left\{F_t^p - Y_t C_t^{-\sigma} - \theta\beta E_t\left[L(\Delta_t, A_{t+1})\right]\right\} \\ & + \lambda_{5t}\left[1 - (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{1-\epsilon} - \theta(\Pi_t)^{\epsilon-1}\right] \\ & + \lambda_{6t}\left[\Delta_t - (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\epsilon} - \theta\left(\Pi_t\right)^{\epsilon}\Delta_{t-1}\right] \end{aligned}$$

where λ_{jt} $(j = 1, .., 6)$ are the Lagrange multipliers. The first order conditions are given as follows: for consumption,

$$C_t^{-\sigma} - \lambda_{1t} + \sigma Y_t C_t^{-\sigma-1}\lambda_{4t} = 0$$

output,

$$-(\Delta_t/A_t)^{1+\varphi} Y_t^{\varphi} + \lambda_{1t} - (1+\varphi)(\Delta_t Y_t)^{\varphi} A_t{}^{-\varphi-1}\lambda_{3t} - C_t^{-\sigma}\lambda_{4t} = 0$$

optimal price,

$$\lambda_{2t} + (1-\theta)(\epsilon-1)\left(\frac{P_t^*}{P_t}\right)^{-\epsilon}\lambda_{5t} + \epsilon(1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\epsilon-1}\lambda_{6t} = 0$$

inflation,

$$-(\epsilon-1)\theta\lambda_{5t} - \epsilon\theta\Delta_{t-1}\Pi_t\lambda_{6t} = 0$$

numerator of optimal price K_t^p,

$$-\left(\frac{\epsilon}{\epsilon-1}\right)\frac{1}{F_t^p}\lambda_{2t} + \lambda_{3t} = 0$$

denominator of optimal price F_t^p,

$$\left(\frac{\epsilon}{\epsilon-1}\right)\frac{K_t^p}{(F_t^p)^2}\lambda_{2t} + \lambda_{4t} = 0$$

and price dispersion,

$$\begin{aligned} 0 = &-(Y_t/A_t)^{1+\varphi}\Delta_t^{\varphi} + \beta\frac{\partial E_t\left[V(\Delta_t, A_{t+1})\right]}{\partial\Delta_t} - \varphi(\Delta_t)^{\varphi-1}A_t{}^{-\varphi-1}Y_t^{\varphi+1}\lambda_{3t} \\ &- \theta\beta\frac{\partial E_t\left[M(\Delta_t, A_{t+1})\right]}{\partial\Delta_t}\lambda_{3t} - \theta\beta\frac{\partial E_t\left[L(\Delta_t, A_{t+1})\right]}{\partial\Delta_t}\lambda_{4t} + \lambda_{6t} \end{aligned}$$

Note that the envelope theorem yields

$$\frac{\partial V(\Delta_{t-1}, A_t)}{\partial\Delta_{t-1}} = -\theta\left(\Pi_t\right)^{\epsilon}\lambda_{6t}$$

which allows us to rewrite the first order condition for price dispersion as,

$$\begin{aligned} 0 = &-(Y_t/A_t)^{1+\varphi}(\Delta_t)^{\varphi} - \theta\beta E_t\left[\left(\Pi_{t+1}\right)^{\epsilon}\lambda_{6t+1}\right] - \varphi(\Delta_t)^{\varphi-1}A_t{}^{-\varphi-1}Y_t^{\varphi+1}\lambda_{3t} \\ &- \theta\beta\frac{\partial E_t\left[M(\Delta_t, A_{t+1})\right]}{\partial\Delta_t}\lambda_{3t} - \theta\beta\frac{\partial E_t\left[L(\Delta_t, A_{t+1})\right]}{\partial\Delta_t}\lambda_{4t} + \lambda_{6t} \end{aligned}$$

We can solve the nonlinear system consisting of these seven first order conditions and the six constraints to yield the time-consistent optimal policy under Calvo pricing. Specifically, without commitment, we need to find these thirteen time-invariant policy rules which are functions of the two state variables $\{\Delta_{t-1},\ A_t\}$. That is, we need to find policy functions such as $F_t^P = F^P(\Delta_{t-1}, A_t)$, $K_t^P = K^P(\Delta_{t-1}, A_t)$, and $\Pi_t = \Pi(\Delta_{t-1}, A_t)$. Similar to the Rotemberg case, the Chebyshev collocation method will be used to approximate these policy functions.

3.4 Numerical Analysis

3.4.1 Solution Method

We use the Chebyshev collocation method to globally approximate the policy functions.[19] In contrast to the linear-quadratic approximation method, this projection method can capture the extent to which the two approaches to modelling price stickiness differ, due to the non-linearities inherent in the New Keynesian models. First, we discretize the state space into a set of collocation nodes. In the Rotemberg model, there is one state variable (A_t), while in the Calvo model there are two state variables (Δ_{t-1}, A_t). Accordingly, the space of the approximating functions for the Rotemberg pricing consists of one-dimensional Chebyshev polynomials. In comparison, the space of approximating functions for the Calvo pricing is two-dimensional, and is, given by the tensor products of two sets of Chebyshev polynomials. Then we define the residual functions based on the equilibrium conditions. Gaussian-Hermite quadrature is used to approximate expectation terms. Under Calvo pricing, the partial derivatives with respect to price dispersion, are approximated by differentiating the Chebyshev polynomials. Finally, we solve the resultant system of nonlinear equations consisting of the residual functions evaluated over all the collocation nodes.[20] See appendix 3.A.2 for details.

3.4.2 Numerical Results

Benchmark Parameters and Solution Accuracy

The benchmark parameters for Calvo pricing are taken from Anderson et al. (2010) and are standard. We conduct a sensitivity analysis below. To make the results from

[19] Judd (1992) and Judd (1998) are good references.

[20] We also tried the time iteration method. That is, a smaller system of nonlinear equations, composed of the residual functions evaluated at each collocation node, is solved repeatedly. For the benchmark case in this chapter, both methods find identical solutions. However, the time iteration method will be used for other cases since it is generally faster and more robust.

Rotemberg pricing comparable, the value of price adjustment cost is calibrated so that the linear quadratic approximation for both cases are equivalent[21]. This implies an equivalence between the two forms of pricing provided the steady-state is undistorted with a rate of inflation of zero. Such an approach is typically adopted in the literature even where authors are considering models where these conditions are not met. Table 3.1 summarizes the relevant parameter values.

Table 3.1: Parameterization

Parameter	Value	Definition
β	0.99	Quarterly discount factor
σ	1	Relative risk aversion coefficient
φ	1	Inverse Frisch elasticity of labor supply
ϵ	11	Elasticity of substitution between varieties
θ	0.75	Probability of fixing prices in each quarter
ρ_a	0.95	AR-coefficient of technology shock
σ_a	0.01	Standard deviation of technology shock
ϕ	116	Rotemberg adjustment cost

With this benchmark parameterization, we solve the fully nonlinear models via the Chebyshev collocation method. Following Anderson et al. (2010) , the relative price dispersion Δ_t is bounded by $[1, 1.02]$, and the logged productivity a_t takes values from $[-2\sigma_a/(1-\rho_a), 2\sigma_a/(1-\rho_a)] = [-0.4, 0.4]$. For the Rotemberg case, the order of approximation n_a is chosen to be 6, and the number of nodes for Gauss-Hermite quadrature $q = 12$. This combination is quite accurate, since the maximum Euler equation error is of the order of 10^{-8}. For the Calvo case, the order of approximation n_a and n_Δ are both assigned to be 6, and $q = 12$ for Gauss-Hermite quadrature. The maximum Euler equation error over the full range is of the order of 10^{-7}. As suggested by Judd (1998) , this order of accuracy is reasonable.

Steady State Inflation Bias

Figure 3.1 illustrates the solution of the discretionary equilibria for the Calvo case. Similar to the results in Anderson et al. (2010) , the red dotted line plots the value of Δ_t as a function of Δ_{t-1} in a narrow interval of $[1, 1.02]$. The steady state relative price dispersion is about 1.0026 which is the intersection point between the red line and

[21] That is, $\phi = \frac{(\epsilon-1)\theta}{(1-\theta)(1-\beta\theta)}$.

the solid 45-degree line. At this fixed point, the value of optimal gross inflation $\overline{\Pi}$ (the dashed line) is about 1.0054, implying an annualized inflation rate of 2.2%. In contrast, the discretionary inflation rate for the Rotemberg case is 1.0047 or 1.89% per year. It is well known that the optimal rate of inflation under commitment is zero, hence the inflation bias is equal to the optimal rate of inflation under discretion. Therefore, the inflation bias problem under Calvo pricing is more severe than that under Rotemberg pricing for the benchmark parameters. We now turn to discuss this result, as well as undertaking a sensitivity analysis.

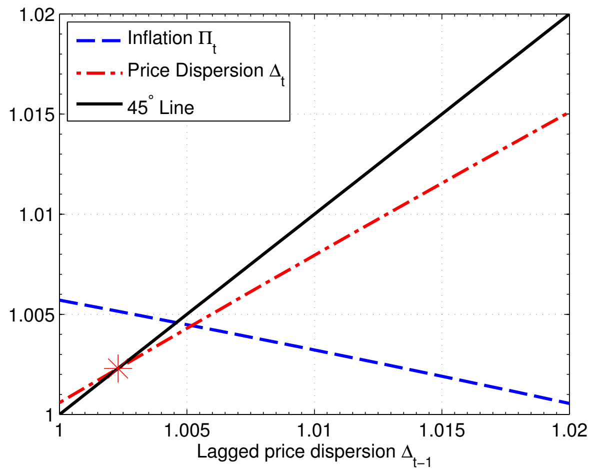

Figure 3.1: Relative price dispersion and inflation as functions of lagged relative price dispersion. This figure shows how the relative price dispersion converges to its steady state.

To explore this difference further, we change the value of the monopolistic competition distortion defined by $\epsilon/(\epsilon-1)$ by varying ϵ and assessing its effect on the equilibrium inflation bias. We interchangeably describe this measure of the monopolistic competition distortion as the flexible-price markup since it measures the markup that would be observed under flexible prices. This approach is based on the fact that the size of the inflation bias depends on the degree of monopolistic distortion, which makes steady state (even flexible-price) output inefficient and generates the temptation on the part of the policy maker to inflate the economy. Figures 3.2 and 3.3 show how the size of inflation bias changes as the markup is varied for the Calvo and Rotemberg pricing, respectively. The benchmark $\epsilon = 11$ yields a gross flexible-price markup of 1.1. When ϵ decreases, the corresponding monopolistic competition distortion and inflation bias increases. To illustrate the impact of the monopoly distortion on the non-linearity, the inflation bias for both cases under the linear-quadratic approximation (LQ) are also presented. The traditional linear-quadratic method becomes increasingly inaccurate for larger distortions.

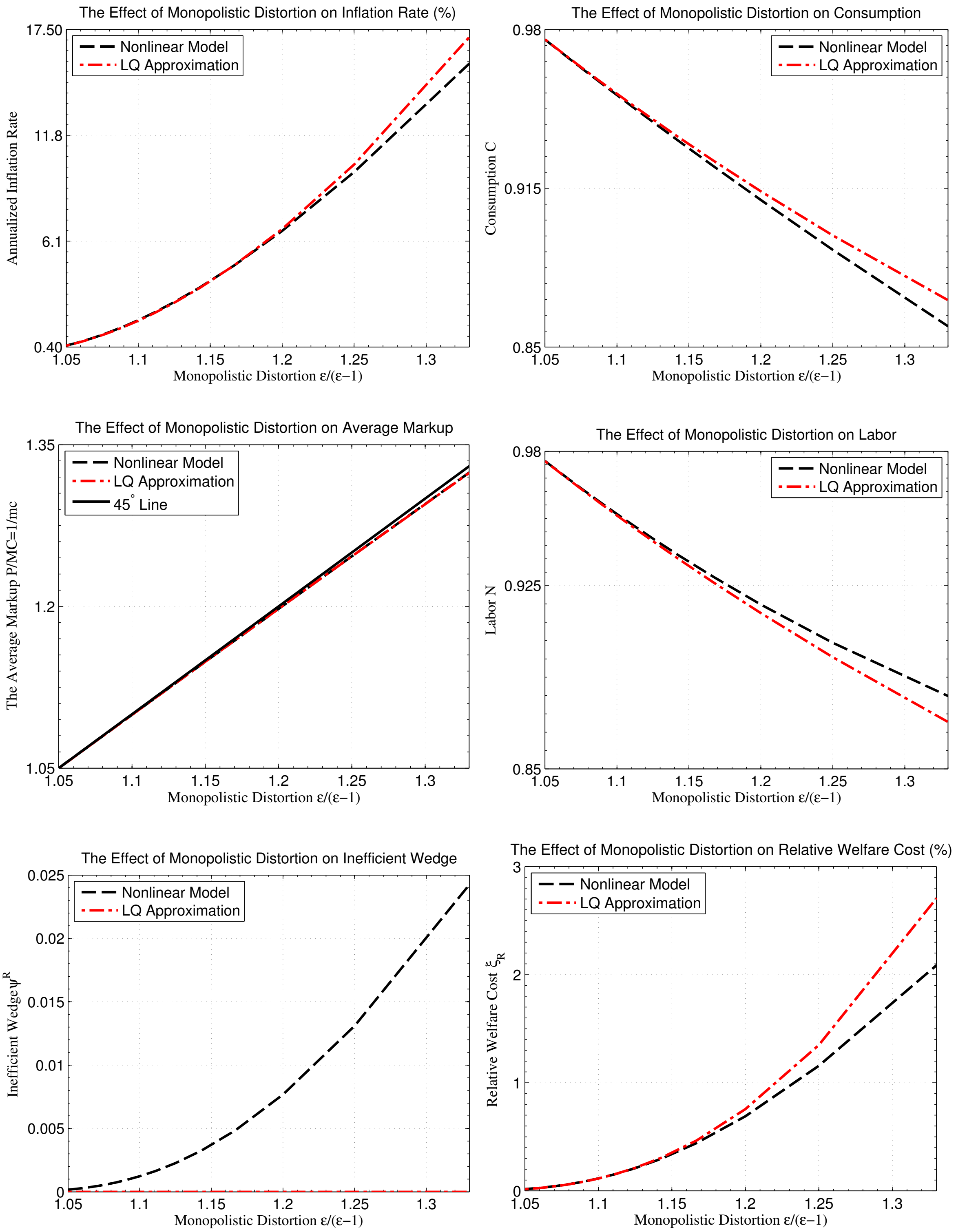

Figure 3.2: This figure shows the effect of monopolistic distortion under Rotemberg pricing. The monopolistic distortion is measured by markup at the deterministic steady state with zero inflation rate. The results from LQ and projection method are compared.

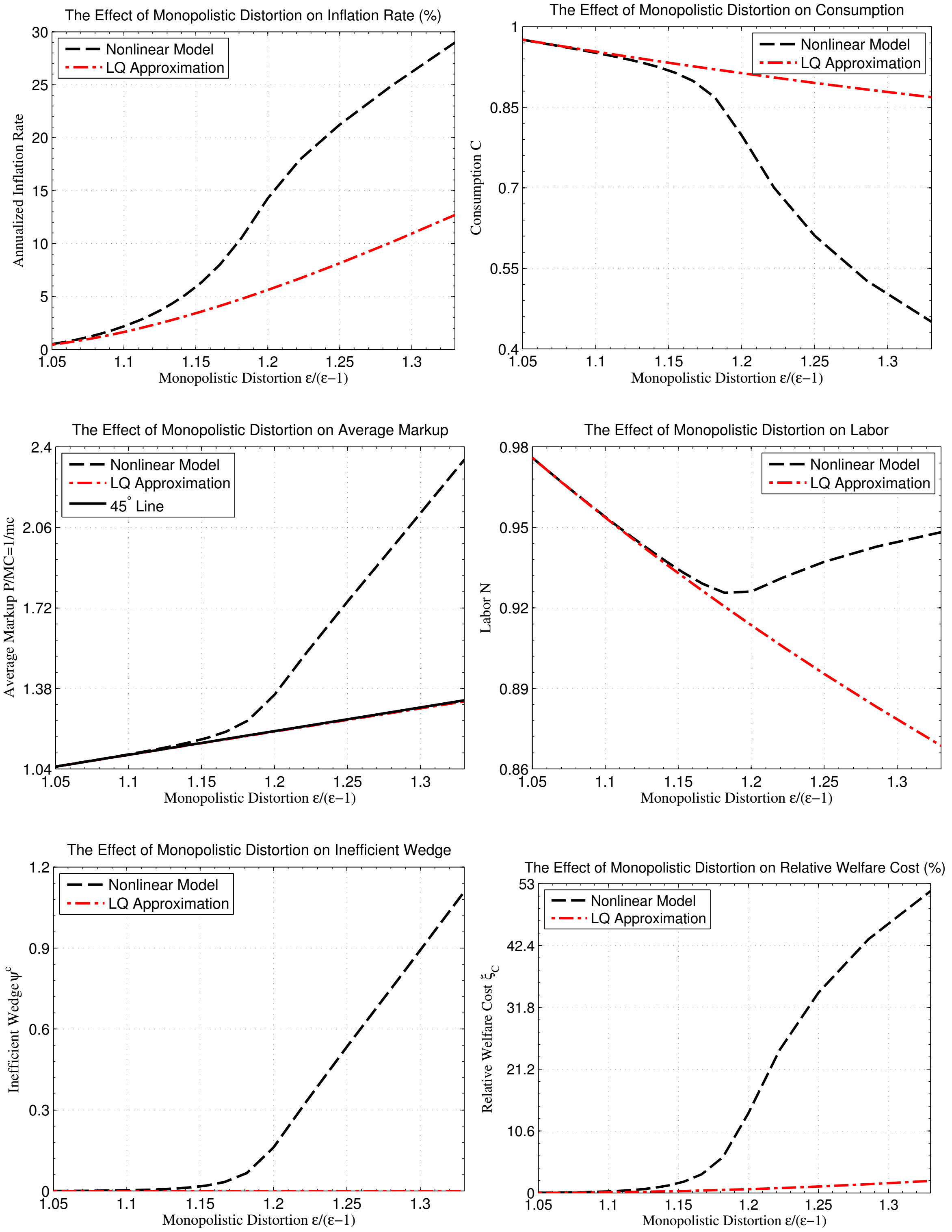

Figure 3.3: This figure shows the effect of monopolistic distortion under Calvo pricing. The monopolistic distortion is measured by markup at the deterministic steady state with zero inflation rate. The results from LQ and projection method are compared.

Table 3.2: Sensitivity analysis for Calvo pricing

Parameter	Values				Numerical results		
	θ	σ	φ	ϵ	Price dispersion	Nonlinear solution	LQ solution
θ	0.05	1	1	11	1.0000	1.71	1.66
	0.3	1	1	11	1.0001	1.78	1.66
	0.5	1	1	11	1.0003	1.84	1.66
	0.75	1	1	11	1.0026	2.18	1.65
	0.85	1	1	11	1.0275	3.01	1.64
	0.90	1	1	11	1.0726	2.55	1.60
σ	0.75	0.3	1	11	1.0308	5.64	2.54
	0.75	1	1	11	1.0026	2.18	1.65
	0.75	5	1	11	1.0002	0.60	0.56
φ	0.75	1	0.36	11	1.0139	4.31	2.42
	0.75	1	1	11	1.0026	2.18	1.65
	0.75	1	4.75	11	1.0002	0.60	0.56

Note: the last two columns contain the annualized inflation rate in percentage solved by the projection method and the LQ method, respectively. The numbers are rounded up.

Table 3.3: Sensitivity analysis for Rotemberg pricing

Parameter	Values				Annualized Inflation rate (%)	
	θ	σ	φ	ϵ	Nonlinear solution	LQ solution
θ	0.05	1	1	1	1.94	1.83
	0.3	1	1	1	1.94	1.83
	0.5	1	1	11	1.93	1.83
	0.75	1	1	11	1.90	1.82
	0.85	1	1	11	1.83	1.80
	0.90	1	1	11	1.72	1.76
σ	0.75	0.3	1	11	2.95	2.54
	0.75	1	1	11	1.90	1.82
	0.75	5	1	11	0.64	0.56
φ	0.75	1	0.36	11	2.83	2.42
	0.75	1	1	11	1.90	1.82
	0.75	1	4.75	11	0.64	0.56

Note: for comparison, θ, which affects ϕ, is included. The numbers are rounded up.

Finally, we do some comparative statics with the model under both pricing approaches, in order to explore how other parameters affect the severity of the inflation bias problem and the sensitivity of the results obtained from the linear-quadratic approach. Table 3.2 and Table 3.3 summarize the robustness outcomes for the Calvo and Rotemberg pricing, respectively. In general, the inflation bias problem is much worse under Calvo pricing.

3.4.3 Discussion

The Average Markup and Inflation Bias

We find that the inflationary bias problem is significantly greater under Calvo, especially as the monopolistic competition distortion is increased. At the same time, consumption falls by more, and hours worked by less under Calvo as we increase this distortion, and the average markup rises above the flexible price markup under Calvo, while decreases under Rotemberg, as a result of the non-linear effects of the inflation bias. See Figures 3.2 and 3.3.

In understanding the results, it is helpful to consider the effects of inflation on the two models. Ascari and Rossi (2012) discuss how inflation affects both models through a 'wedge' effect as well as an average markup effect. We shall consider the wedge effect first, before turning to the average markup effects, which will turn out to be key. Under both forms of nominal inertia, the 'wedge' implies that the representative household's aggregate consumption will be lower for a given level of labour input as inflation rises. Under Calvo, this is because the dispersion of prices means that they need to consume relatively more of the cheaper goods to compensate for the expensive goods, given diminishing marginal utility in the consumption of each good. As Damjanovic and Nolan (2010b) note, this is akin to a negative productivity shock, where we can combine the resource and aggregate production function to yield,

$$C_t = \frac{A_t}{(1+\psi_t^c)} N_t$$

where the inefficient wedge under Calvo, $\psi_t^c = \Delta_t - 1$, captures the extent to which price dispersion has been raised above one.

Under Rotemberg, the micro-foundation of the wedge is different - adjusting prices uses up consumption goods directly. However, we can similarly combine the aggregate production function and resource constraint to obtain a similar expression under Rotemberg,

$$C_t = A_t(1-\psi_t^R)N_t$$

where the Rotemberg wedge, $\psi_t^R = \frac{\phi}{2}(\Pi_t - 1)^2$ reflects the costs per unit of output of changing prices. Therefore, in both cases the labour costs of attaining a particular level of aggregate consumption are higher, ceteris paribus, as inflation rises.

In order to assess how this affects the inflation bias problem facing the policy maker, it is helpful to imagine how a social planner would respond to the existence of such wedges, were he to imagine them to be exogenously given in the manner of a technology shock. Given the form of household utility, the social planner would choose an optimal level of labour input of

$$N_t^{\sigma+\varphi} = \left(\frac{A_t}{(1+\psi_t^c)}\right)^{1-\sigma}$$

under Calvo, and

$$N_t^{\sigma+\varphi} = \left(A_t(1-\psi_t^R)\right)^{1-\sigma}$$

under Rotemberg. Therefore, for our benchmark calibration of $\sigma = 1$ the social planner would not seek to adjust the labour input into the production process as a result of increases in either of the wedges, but would simply allow consumption to fall. In other words, for our benchmark calibration the efficiencies implied by these wedges do not give the policy maker a further desire to generate a surprise inflation, ceteris paribus. While if $\sigma > 1$ the social planner would seek to reduce the labour input as either of these inefficiency wedges increased. That is, in this case the wedges would reduce the desire to encourage firms to employ more workers, ceteris paribus. We can see this from Tables 3.2 and 3.3 where raising the inverse of the intertemporal elasticity of substitution, σ, reduces the inflation bias under both pricing models. Therefore, the different inefficiency wedges under Calvo and Rotemberg are not responsible for the observed inflation biases.

Instead the differences in inflation bias across the two models are generated by their average mark-up behavior, which is fundamentally different. Consider the steady-state of the average markup (equal to the inverse of real marginal cost) under Rotemberg, which is obtained by rearranging the deterministic steady state of the new Keynesian Phillips curve (NKPC) as,

$$mc^{-1} = \left[\frac{\epsilon-1}{\epsilon} + \frac{(1-\beta)}{\epsilon}\phi(\Pi-1)\Pi\right]^{-1}$$

The second term within the square brackets exists as a combination of steady-state inflation and discounting on the part of firms (on behalf of their owners, the representative household). Essentially, as the firm discounts future profits, they also discount future price adjustment costs. As a result, in the face of ongoing inflation, they will opt to partially delay the required price adjustment, such that the average mark-up is decreasing in inflation.

The effect of inflation on the average mark-up under Calvo is,

$$mc^{-1} = \frac{\epsilon}{\epsilon - 1} \left(\frac{1 - \theta\beta\Pi^{\epsilon-1}}{1 - \theta\beta\Pi^{\epsilon}} \right) \left(\frac{1 - \theta\Pi^{\epsilon-1}}{1 - \theta} \right)^{\frac{1}{\epsilon-1}}$$

In this case the effects of inflation on the average markup are ambiguous. However, following King and Wolman (1996), this can be decomposed into two elements - the marginal markup,

$$\frac{P^*}{MC} = \frac{\epsilon}{\epsilon - 1} \left(\frac{1 - \theta\beta\Pi^{\epsilon-1}}{1 - \theta\beta\Pi^{\epsilon}} \right)$$

and the price adjustment gap,

$$\frac{P}{P^*} = \left(\frac{1 - \theta\Pi^{\epsilon-1}}{1 - \theta} \right)^{\frac{1}{\epsilon-1}}.$$

Here we can see that higher inflation raises the marginal markup. Firms facing the possibility of being stuck with the current price for a prolonged period will tend to raise their reset price, when that price is likely to be eroded by inflation throughout the life of that contract. The effect of inflation on the price adjustment gap will tend to reduce this element of the average markup. However, except at very low rates of inflation, the effects of inflation on the average markup through the marginal mark-up effect are positive.

Therefore, we would expect to see average markups rise with inflation under Calvo, but fall under Rotemberg. This, in turn, implies that the inflationary bias problem is worsened under Calvo as the rising markups increase the policy makers incentives to introduce a surprise inflation, ceteris paribus, at the same time, as it is mitigated under Rotemberg. As a result, the inflation bias problem is significantly higher under Calvo where consumption falls by more and hours by less than it does under Rotemberg.

Sensitivity Analysis

Tables 3.2 and 3.3 consider the robustness of our results across various parameters for Calvo and Rotemberg pricing, respectively. The first three rows of each Table increase the degree of nominal inertia (where the Rotemberg price adjustment parameter is adjusted in line with the changes in the Calvo parameter such that the linearized NKPC is equivalent across both forms of nominal inertia). As we increase the degree of nominal inertia, we find that the inflation bias rises under Calvo, but falls under Rotemberg. This is for the reasons discussed above. Under Calvo greater price stickiness means that firms are likely to be stuck with their current price for longer, meaning that they aggressively raise prices when given the opportunity to do so. This will tend to raise average markups and worsen the inflationary bias problem.[22] In contrast, under Rotemberg, higher price adjustment costs result in firms wishing to delay price adjustment which reduces average markups and reduces the inflation bias problem.

The next piece of sensitivity analysis looks at various parameterization of the inverse of the intertemporal elasticity of substitution, σ. As noted above, at the benchmark value of $\sigma = 1$, the social planner would not wish to expand employment as either of the efficiency wedges due to the two forms of nominal inertia increase. While if $\sigma < (>)$ 1 then they would wish to increase (decrease) the labour input as either efficient wedge increased. Therefore, we see the inflationary bias falling as σ increases across both forms of nominal inertia. Finally, we consider an increase in the inverse of the Frisch elasticity of labour supply, φ, which serves to reduce the inflationary bias problem across both types of price stickiness. As labour supply becomes less elastic, there is less desire to use costly inflation surprises to achieve only marginal increases in the level of output, and hence the inflation bias falls.

Relevance of Results

In order to assess the implications of our calculated levels of inflation bias under the Rotemberg and Calvo forms of nominal inertia, we contrast our inflation rates with both the empirical estimates of trend inflation and the critical values of trend inflation at which the standard model develops non-standard properties.

Empirical Estimates of Trend Inflation Cogley and Sargent (2002) 's estimates of trend inflation in a Bayesian VAR with time varying coefficient suggests that a large part of the movements in inflation in the post-war period (its rise in the 1970s to its fall in the 1980s) was due to the evolution of trend inflation rather than fluctuations around that trend. Similarly, Cogley and Sbordone (2008) find that there is no inertia in price-setting behavior due to indexation-type behavior, but that the inertia in the data can be described by the evolution of trend-inflation in a generalized NKPC. Meanwhile, Ireland (2007) finds that changes in the Fed's inflation target can help explain inflation dynamics, where that target rose from 1.25% in 1959 to over 8% in the late 1970s, before falling back to 2.5% in 2004. Therefore, to the extent that observed inflation reflects movements in an underlying trend, it would suggest that the empirical measures

[22] At extremely high levels of price stickiness ($\theta = 0.9$, or an average price contract duration of two and a half years) the steady state rate of inflation begins to fall as the costs of price dispersion begin to overturn the average markup effect.

of trend inflation could easily be consistent with our measures of the inflationary bias without having to resort to implausibly high monopolistic competition distortions. Moreover, when we augment the model with an estimated process for mark-up shocks (see below), the magnitude of the resultant inflation volatility can easily account for the observed volatility of inflation around its time-varying trend.

Trend Inflation and Determinacy In order to further assess the implications of our calculated levels of inflation bias under the Rotemberg and Calvo forms of nominal inertia, we contrast our inflation rates with the key values of trend inflation at which the standard model develops issues with interest rate determinacy. We could, of course, have looked at other features highlighted in the trend inflation literature such as the learnability of the model as trend inflation rises or its impulse responses to monetary policy shocks and so on, but since the bifurcation in determinacy conditions reflects a common underlying non-linearity which drives all the phenomena in the trend inflation literature, we choose to focus on this as a straight-forward way of assessing whether or not our calculations suggest the concerns raised by the trend inflation literature are significant or not. We find that our model with the common parameterizations adopted in the literature can easily imply trend inflation rates which generate the issues highlighted by the trend-inflation literature (see Ascari and Sbordone (2014) for a comprehensive survey of this literature).

Accordingly, we follow Ascari and Rossi (2012) and linearise our two economies around a deterministic steady-state with an arbitrary rate of steady-state inflation (details of the linearised models are provided in the Appendix 3.A.4). We then assume a standard parameterisation of a Taylor rule for monetary policy, $R_t = 1.5\pi_t + 0.5y_t$, and for a range of values of the mark-up, $\epsilon/(\epsilon - 1)$, we compute the steady-state rate of inflation at which the standard Taylor rule flips from being determinate to being indeterminate. We then plot this determinacy frontier in markup-inflation space along side our inflation bias estimates, see Figure 3.4 and Figure 3.5. We find that at low levels of the mark-up the inflationary bias number lies below the determinacy frontier - in other words the standard Taylor rule would remain determinate at the rates of inflation implied by our inflationary bias calculations. However, as the markup is increased the inflation bias estimates cross the determinacy frontier implying that at the rates of inflation implied by the inflation bias estimates a standard Taylor rule would be indeterminate in a log-linearised representation of the model. This is particularly true in the case of Calvo where a markup of just over 11%, which is well within the range of standard parameterisations in the literature. In contrast, under Rotemberg the markup needs to be double that push us beyond the determinacy frontier.

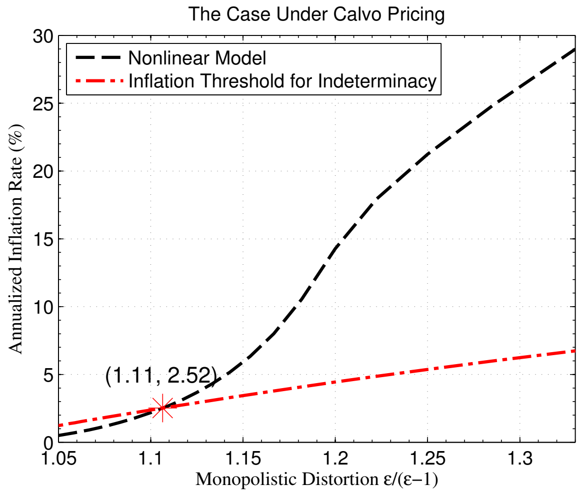

Figure 3.4: The threshhold of inflation rate for indeterminacy under Calvo pricing

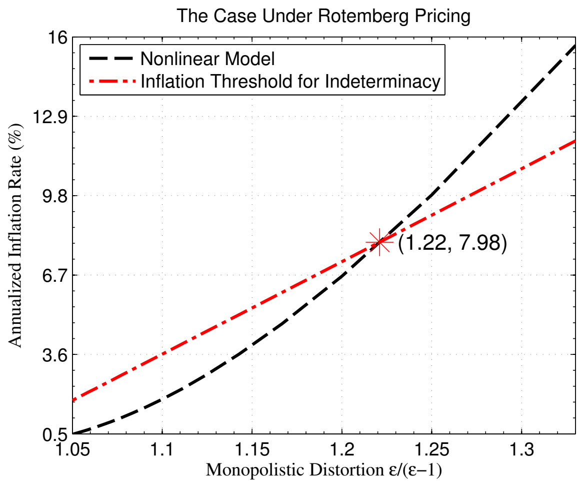

Figure 3.5: The threshhold of inflation rate for indeterminacy under Rotemberg pricing

3.5 The Effects of Cost-push Shocks

In the analysis above we have focussed on the stochastic steady state of the non-linear policy problem to reveal the extent of the inflation bias. However, the response to shocks can also be markedly different across the two forms of nominal inertia. In order to explore the effect of cost-push shocks[23] on policy trade-offs under discretion in our fully nonlinear model, we, adopt the estimated shock process from Chen et al. (2014)

[23]The technology shocks already present in our model do not create meaningful policy trade-offs under our benchmark calibration largely resulting in offsetting interest rate movements regardless of the form of nominal inertia.

which is modelled as a revenue tax rate fluctuating around a steady state value of zero,

$$\ln(1-\tau_{pt}) = (1-\rho^{\tau_p})\ln(1-\tau_p) + \rho^{\tau_p}\ln(1-\tau_{pt-1}) - e_{\tau t}$$

where $e_{\tau t} \sim N(0, 0.00486^2)$ and $\rho^{\tau_p} = 0.939$. In a log-linearized model this is equivalent to allowing for fluctuations in a desired mark-up through variations in ϵ. However, in our non-linear model allowing ϵ to be time varying has a direct impact on the measure of price dispersion in a way which would not normally be considered to be an inherent part of a cost push shock. Therefore, we focus on variations in a revenue tax a means of generating an autocorrelated cost push shock which is consistent with the data. The complete model with the time-varying revenue tax rate is presented in appendix 3.A.5.

We present two sets of results. In the first we consider the impact of an inflationary cost push shock with our benchmark parameterization, but with $\theta = 0.625$, and $\phi = 57.3684$. These respective measure of price stickiness imply an identical steady-state rate of inflation of 1.95%. Figure 3.6 reveals that even at this relatively modest degree of inflation bias, there are non trivial differences in the impulse responses to an identical cost push shock. These are driven by the same economic mechanisms observed in the steady state analysis above, as average markups rise under Calvo exacerbating the effects of the cost push shock.

It should be noted that the conventional way of parameterizing the Rotemberg price adjustment cost parameter such that the slopes of the linearized Phillips curves are identical would have implied a far lower value of $\phi = 43.7$. In fact, given the significant differences in the inflation bias across the two forms of price-stickiness, it is generally not possible to calibrate the Rotemberg parameter by seeking to mimic the steady-state rate of inflation observed under Calvo, ceteris paribus. Therefore, in a second exercise we ensure a common steady-state rate of inflation of 2.54% by adopting the following set of parameters, $\epsilon = 11$, $\theta = 0.8$ under Calvo, and $\epsilon = 9.7076$, $\phi = 57.3684$ under Rotemberg. This calibration ensures that both forms of nominal inertia generate identical steady-state rates of inflation and levels of output. Despite sharing a steady state in these dimensions, the response to the identical shock is markedly different across Calvo and Rotemberg. In Figure 3.7, we can see that inflation is 0.2% higher on impact from an identical cost push shock under Calvo, while other variables, particularly output and consumption, exhibit a hump-shaped response to the shock due to the gradual evolution of price dispersion, which is not a feature of the response to the shock under Rotemberg pricing.

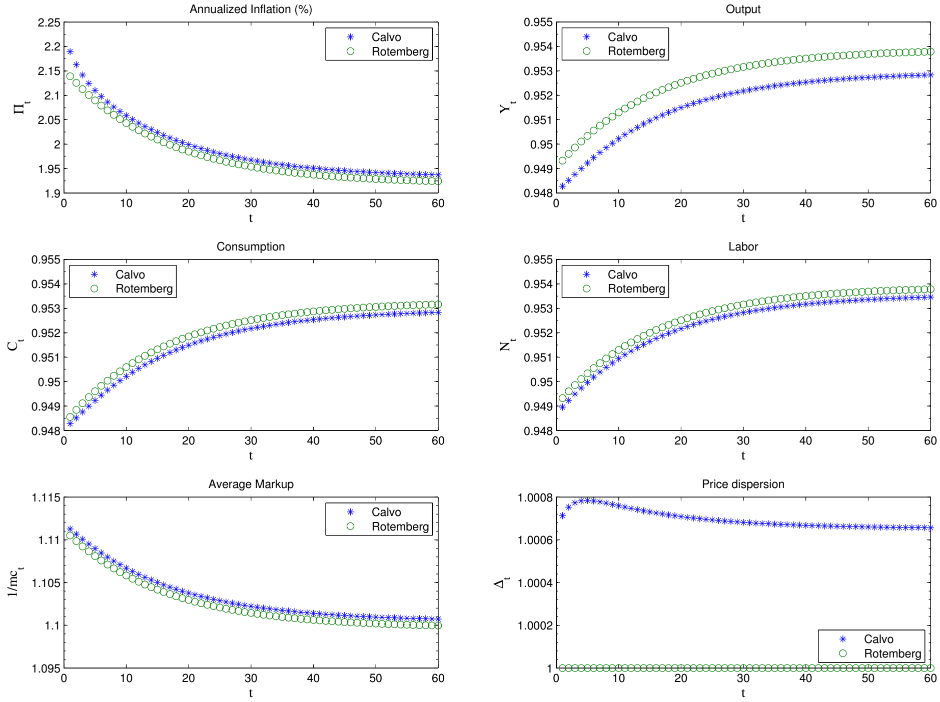

Figure 3.6: The impulse response functions to one percent positive tax-driven cost-push shock under Calvo and Rotemberg pricing. Note that the two cases are calibrated so that the steady state inflation rate is equal.

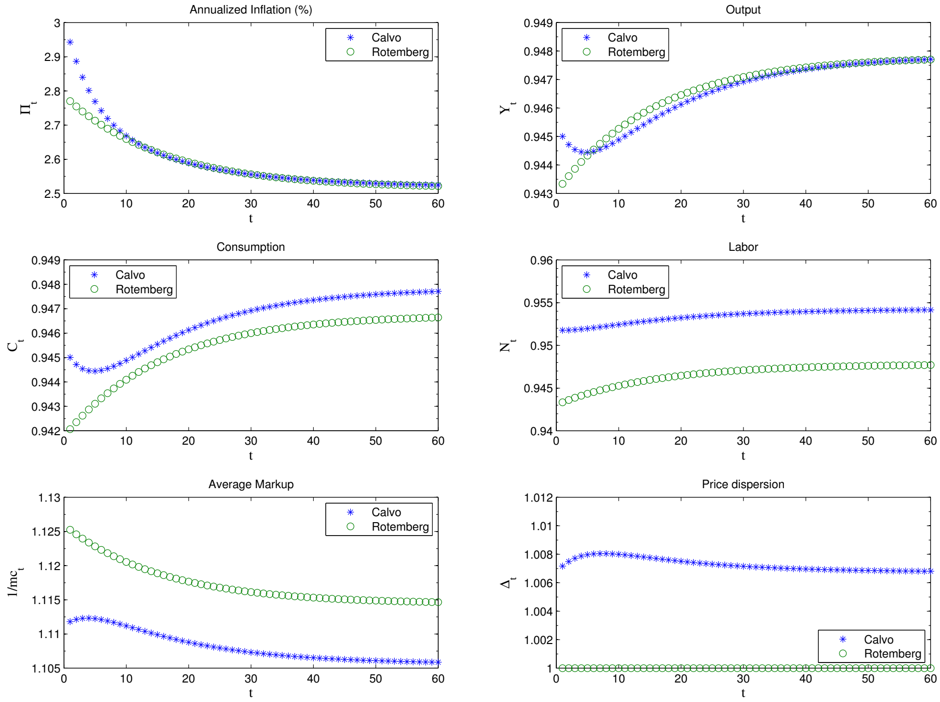

Figure 3.7: The impulse response functions to one percent positive tax-driven cost-push shock under Calvo and Rotemberg pricing. Note that each model is calibrated so that the steady state output and inflation rates are equal across both cases.

The above results focus on the response to a cost push shock under both forms of nominal inertia, given that the economy is at its stochastic steady state. We can also explore how the state of the economy affects its response to shocks. We find that conditioning on the state of the economy in terms of levels of productivity lying significantly above or below their steady state value, the marginal impact of technology or cost push shocks does not give rise to significant asymmetries. However, conditioning on the level of price dispersion either being higher or lower than its steady state value, can give rise to significant differences in the response to a given shock. We explore this by plotting, in Figure 3.8 , the difference in the impulse response functions to a positive cost-push shock when the economy begins at the 75th and 25th percentile of the grid for price dispersion, respectively. In the presence of a high level of price dispersion there is an ongoing reduction in inflation to reduce the distortion associated with price dispersion, which also depresses output. When the cost push shock hits, this raises inflation, the average markup and reduces output. These effects of a given cost push shock are larger when the economy begins from a high rather than low level of price dispersion as the policy maker faces a relatively stronger inflationary bias problem as a result of the distortion caused by price dispersion. Therefore, even the simplest New Keynesian model may imply significant state-dependent optimal policy responses to shocks when the non-linearities inherent in the model are properly accounted for. However, such effects depend on which state variable one conditions on, (in our model they are not present when conditioning on the level of productivity, for example) which may explain the mixed evidence on non-linearities in policy and responses to policy found in the empirical literature[24]. Such effects would be lost in the standard linearized version of the model.

[24]See, for example, Barnichon and Matthes (2014) for a discussion.

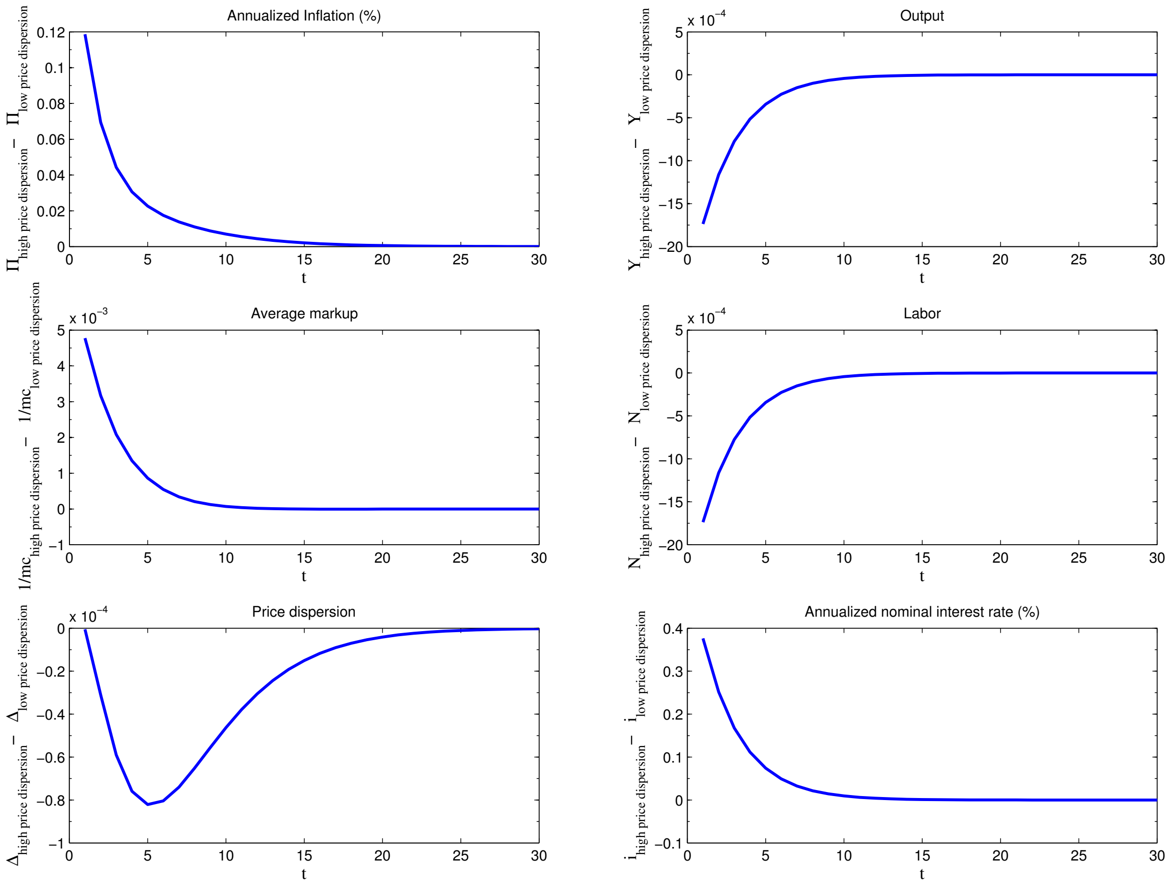

Figure 3.8: Difference between the impulse response functions to five percent positive tax-driven cost-push shock under high and low price dispersion. Specifically, we first calculate the impulse response functions to the cost-push shock, given initial price dispersion being high or low. Then, we calculate and plot the differences of the impulse response functions between these two cases, as the labels on the vertical axis indicate. The reason for choosing a big shock is to highlight the asymmetries.

To further highlight the differences in the stochastic properties of the model under the two forms of pricing behavior, Table 3.4 computes the mean, variance and persistence of inflation under the various calibrations used in our sensitivity analysis above. It is important to note, that to a first order of approximation these statistics would be identical across Calvo and Rotemberg pricing given the standard approach to calibration. However, the fact that they are significantly different across all these dimensions highlights the fact that the choice of model of nominal inertia not only affects the average inflation bias, but also the variance and (in the case of Calvo) persistence of inflation in the face of an identical cost push shock process when the non-linearities inherent in the model are properly accounted for.

Table 3.4: The inflation volatility and persistence under Calvo and Rotemberg pricing

Parameter	Values				Calvo			Rotermberg		
	θ	σ	φ	ϵ	mean (%)	volatility (%)	persistence	mean (%)	volatility (%)	persistence
θ	0.05	1	1	11	1.75	0.25	0.93	1.94	0.31	0.93
	0.3	1	1	11	1.79	0.27	0.93	1.94	0.31	0.93
	0.5	1	1	11	1.85	0.29	0.93	1.94	0.31	0.93
	0.75	1	1	11	2.21	0.40	0.93	1.91	0.29	0.93
	0.85	1	1	11	3.02	0.27	0.86	1.83	0.25	0.93
	0.90	1	1	11	2.55	0.09	0.87	1.72	0.21	0.93
σ	0.75	0.3	1	11	4.51	0.71	0.86	2.95	0.45	0.93
	0.75	1	1	11	2.21	0.40	0.90	1.91	0.29	0.93
	0.75	5	1	11	0.63	0.10	0.93	0.65	0.10	0.93
φ	0.75	1	0.36	11	4.38	0.71	0.86	2.83	0.43	0.93
	0.75	1	1	11	2.21	0.40	0.90	1.91	0.29	0.93
	0.75	1	4.75	11	0.62	0.09	0.93	0.65	0.10	0.93

Note: This table reports the mean, volatility (standard deviation) and persistence (first order autocorrelation) of annualized inflation rate in the model with cost-push shock driven by time-varying revenue tax rate. The numbers are rounded up. The coefficient of Rotermberg price adjustment cost is calibrated such that $\phi = \frac{(\epsilon-1)\theta}{(1-\theta)(1-\beta\theta)}$.

3.6 Conclusion

In this chapter we have contrasted the properties of the Calvo and Rotemberg forms of nominal inertia which are commonly used in New Keynesian analyses of macroeconomic policy. They are often treated as being interchangeable, largely because they generate equivalent NKPC and policy implications when linearized around an efficient zero-inflation steady state. However, our non-linear solution of the discretionary policy problem reveals some striking differences across the two models of price stickiness, which have significant implications for the importance of non-linearities in New Keynesian policy analyses more generally.

Firstly, the inflation bias problem is far greater under Calvo pricing than Rotemberg pricing, despite the fact that the costs of inflation are significantly higher under the former. The reason for this is that inflation raises the average markup under Calvo pricing as firms seek to raise their prices more aggressively whenever they can to avoid the erosion of their relative price due to inflation. This increase in average markups worsens the inflationary bias problem. In contrast, under Rotemberg pricing firms can adjust prices in every period, and will moderate their average markups as inflation rises as they attempt to delay some of the costs of price adjustment due to the discounting inherent in their objective function.

Secondly, for empirically reasonable levels of monopoly power the inflation bias that emerges from both models implies that the rates of inflation identified as being 'trend' inflation in empirical studies are reasonable. Moreover, the rates of inflation implied by the model are sufficient for the non-linearities inherent in the model to place the economy in the zone where the effects of trend inflation are found, in studies which approximate the economy around a non-zero rate of steady-state inflation, to have profound implications for the determinacy properties of rules, the learnability of the rational expectations equilibrium and the transmission of monetary policy. That is, the degree of inflation bias generated by the model implies that the non-linearities inherent in the model and the choice of form of nominal inertia matter.

Thirdly, we extended the model to consider the stabilization of the economy in the face of mark-up shocks. Here we find that the non-linearities that generate radically different degrees of inflation bias in the steady state also imply profound differences in the monetary policy response to the same shock both across models, with the inflation response to a cost-push shock being significantly greater under Calvo, while possibly also being associated with a hump-shaped output/consumption response as a result of the evolution of price dispersion which is absent from the Rotemberg model and typically ignored in the linearized New Keynesian model. As a result, the implied stochastic properties for inflation can differ substantially across the two forms of nominal inertia, ceteris paribus, even when they would be identical under a linear approximation.

Appendix

3.A Technical Appendix

3.A.1 Summary of Models

Households' Utility Maximization Problem

The Lagrangian function for the utility maximization problem is

$$\begin{aligned}\mathcal{L} = {} & E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) \\ & + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\Xi_t + D_t + W_t N_t - T_t - P_t C_t - E_t \left\{ Q_{t,t+1} D_{t+1} \right\} \right]\end{aligned}$$

where λ_t is the Lagrangian multiplier. The FOCs are given as follows:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial C_t} &= C_t^{-\sigma} - P_t \lambda_t = 0 \\ \frac{\partial \mathcal{L}}{\partial N_t} &= -N_t^{\varphi} + \lambda_t W_t = 0 \\ \frac{\partial \mathcal{L}}{\partial D_{t+1}} &= -\lambda_t E_t \left\{ Q_{t,t+1} \right\} + \beta E_t \{ \lambda_{t+1} \} = 0\end{aligned}$$

These conditions can be simplified further into

$$\begin{aligned}-\frac{C_t^{-\sigma}}{P_t} E_t \left\{ Q_{t,t+1} \right\} + \beta E_t \left\{ \frac{C_{t+1}^{-\sigma}}{P_{t+1}} \right\} = 0 &\Rightarrow E_t \left\{ Q_{t,t+1} \right\} = \beta E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \right\} \\ \frac{N_t^{\varphi}}{W_t} = \frac{C_t^{-\sigma}}{P_t} &\Rightarrow \left(\frac{W_t}{P_t} \right) = N_t^{\varphi} C_t^{\sigma}.\end{aligned}$$

Firms' Profit Maximization Under Calvo Pricing

$$\begin{aligned}
&E_t \sum_{s=0}^{\infty} \theta^s Q_{t,t+s} \left[P_t(j) Y_{t+s}(j) - mc_{t+s} Y_{t+s}(j) P_{t+s} \right] \\
&= E_t \sum_{s=0}^{\infty} \theta^s Q_{t,t+s} \left[P_t(j) \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} - mc_{t+s} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} P_{t+s} \right] \\
&= E_t \sum_{s=0}^{\infty} \theta^s Q_{t,t+s} \left[\left(\frac{1}{P_{t+s}} \right)^{-\epsilon} P_t(j)^{1-\epsilon} - mc_{t+s} \left(\frac{1}{P_{t+s}} \right)^{-\epsilon-1} P_t(j)^{-\epsilon} \right] Y_{t+s}
\end{aligned}$$

Denoting the optimal price $P_t(j) = P_t^*$, the FOC is

$$E_t \sum_{s=0}^{\infty} \theta^s Q_{t,t+s} \left[(1-\epsilon) \left(\frac{1}{P_{t+s}} \right)^{-\epsilon} (P_t^*)^{-\epsilon} + \epsilon mc_{t+s} \left(\frac{1}{P_{t+s}} \right)^{-\epsilon-1} (P_t^*)^{-\epsilon-1} \right] Y_{t+s} = 0$$

that is,

$$E_t \sum_{s=0}^{\infty} \theta^s Q_{t,t+s} \left[(1-\epsilon) \left(\frac{P_t^*}{P_{t+s}} \right)^{-\epsilon} + \epsilon mc_{t+s} \left(\frac{P_t^*}{P_{t+s}} \right)^{-\epsilon-1} \right] Y_{t+s} = 0.$$

Using $Q_{t,t+s} = \beta^s \left(\frac{C_t}{C_{t+s}} \right)^{\sigma} \frac{P_t}{P_{t+s}}$ and rearranging gives:

$$E_t \sum_{s=0}^{\infty} \theta^s \beta^s \left(\frac{C_t}{C_{t+s}} \right)^{\sigma} \frac{P_t}{P_{t+s}} \left[(1-\epsilon) \left(\frac{P_t^*}{P_{t+s}} \right)^{-\epsilon} + \epsilon mc_{t+s} \left(\frac{P_t^*}{P_{t+s}} \right)^{-\epsilon-1} \right] Y_{t+s} = 0$$

$$\begin{aligned}
&\Rightarrow E_t \sum_{s=0}^{\infty} \theta^s \beta^s \left(\frac{C_t}{C_{t+s}} \right)^{\sigma} \frac{P_t}{P_{t+s}} Y_{t+s} (1-\epsilon) \left(\frac{P_t^*}{P_{t+s}} \right)^{-\epsilon} \\
&= -E_t \sum_{s=0}^{\infty} \theta^s \beta^s \left(\frac{C_t}{C_{t+s}} \right)^{\sigma} \frac{P_t}{P_{t+s}} Y_{t+s} \epsilon mc_{t+s} \left(\frac{P_t^*}{P_{t+s}} \right)^{-\epsilon-1}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow (1-\epsilon) E_t \sum_{s=0}^{\infty} \theta^s \beta^s \left(\frac{C_t}{C_{t+s}} \right)^{\sigma} \left(\frac{1}{P_{t+s}} \right)^{1-\epsilon} P_t Y_{t+s} (P_t^*)^{-\epsilon} \\
&= -\epsilon E_t \sum_{s=0}^{\infty} \theta^s \beta^s \left(\frac{C_t}{C_{t+s}} \right)^{\sigma} \left(\frac{1}{P_{t+s}} \right)^{-\epsilon} P_t Y_{t+s} mc_{t+s} (P_t^*)^{-\epsilon-1}
\end{aligned}$$

$$\Rightarrow (1-\epsilon)E_t\sum_{s=0}^{\infty}\theta^s\beta^s\left(\frac{C_t}{C_{t+s}}\right)^{\sigma}\left(\frac{1}{P_{t+s}}\right)^{1-\epsilon}P_tY_{t+s}$$
$$= -\epsilon E_t\sum_{s=0}^{\infty}\theta^s\beta^s\left(\frac{C_t}{C_{t+s}}\right)^{\sigma}\left(\frac{1}{P_{t+s}}\right)^{-\epsilon}P_tY_{t+s}mc_{t+s}(P_t^*)^{-1}$$

$$\Rightarrow P_t^* = \frac{\epsilon}{\epsilon-1}\frac{E_t\sum_{s=0}^{\infty}\theta^s\beta^s\left(\frac{C_t}{C_{t+s}}\right)^{\sigma}\left(\frac{1}{P_{t+s}}\right)^{-\epsilon}P_tY_{t+s}mc_{t+s}}{E_t\sum_{s=0}^{\infty}\theta^s\beta^s\left(\frac{C_t}{C_{t+s}}\right)^{\sigma}\left(\frac{1}{P_{t+s}}\right)^{1-\epsilon}P_tY_{t+s}}$$
$$\Rightarrow \frac{P_t^*}{P_t} = \frac{\epsilon}{\epsilon-1}\frac{E_t\sum_{s=0}^{\infty}\theta^s\beta^s\left(\frac{C_t}{C_{t+s}}\right)^{\sigma}\left(\frac{P_t}{P_{t+s}}\right)^{-\epsilon}Y_{t+s}mc_{t+s}}{E_t\sum_{s=0}^{\infty}\theta^s\beta^s\left(\frac{C_t}{C_{t+s}}\right)^{\sigma}\left(\frac{P_t}{P_{t+s}}\right)^{1-\epsilon}Y_{t+s}}$$

Note that if there is no price-stickiness, so that $\theta = 0$, then $P_t^* = \frac{\epsilon}{\epsilon-1}P_tmc_t = \frac{\epsilon}{\epsilon-1}MC_t$. That is, each firm sets its price P_t^* equal to a markup $\eta \equiv \frac{\epsilon}{\epsilon-1} > 1$ over its nominal marginal cost. This is the standard result in a model of monopolistic competition. Because price exceeds marginal cost, output will be inefficiently low.

Note that we can have
$$\frac{P_t^*}{P_t} = \frac{\epsilon}{\epsilon-1}\frac{K_t^p}{F_t^p}$$
via defining two recursive equations

$$K_t^p = E_t\sum_{s=0}^{\infty}\theta^s\beta^sC_{t+s}^{-\sigma}\left(\frac{P_t}{P_{t+s}}\right)^{-\epsilon}Y_{t+s}mc_{t+s}$$

$$F_t^p = E_t\sum_{s=0}^{\infty}\theta^s\beta^sC_{t+s}^{-\sigma}\left(\frac{P_t}{P_{t+s}}\right)^{1-\epsilon}Y_{t+s}$$

We can prove this relationship as follows: using Law of Iterated Expectations,

$$K_t^p = E_t\sum_{s=0}^{\infty}\theta^s\beta^sC_{t+s}^{-\sigma}\left(\frac{P_t}{P_{t+s}}\right)^{-\epsilon}Y_{t+s}mc_{t+s}$$

$$= C_t^{-\sigma}Y_tmc_t + E_t\sum_{s=1}^{\infty}\theta^s\beta^sC_{t+s}^{-\sigma}\left(\frac{P_t}{P_{t+s}}\right)^{-\epsilon}Y_{t+s}mc_{t+s}$$

and
$$E_t\sum_{s=1}^{\infty}\theta^s\beta^sC_{t+s}^{-\sigma}\left(\frac{P_t}{P_{t+s}}\right)^{-\epsilon}Y_{t+s}mc_{t+s}$$
$$= \theta\beta E_t\sum_{s=1}^{\infty}\theta^{s-1}\beta^{s-1s}C_{t+s}^{-\sigma}\left(\frac{P_t}{P_{t+s}}\right)^{-\epsilon}Y_{t+s}mc_{t+s}$$

$$= \theta\beta E_t \sum_{s^*=0}^{\infty} \theta^{s^*} \beta^{s^*} C_{t+s^*+1}^{-\sigma} \left(\frac{P_t}{P_{t+s^*+1}} \right)^{-\epsilon} Y_{t+s^*+1} mc_{t+s^*+1}$$

$$= \theta\beta E_t \left[\left(\frac{P_t}{P_{t+1}} \right)^{-\epsilon} \sum_{s^*=0}^{\infty} \theta^{s^*} \beta^{s^*} C_{t+s^*+1}^{-\sigma} \left(\frac{P_{t+1}}{P_{t+s^*+1}} \right)^{-\epsilon} Y_{t+s^*+1} mc_{t+s^*+1} \right]$$

$$= \theta\beta E_t \left[\left(\frac{P_t}{P_{t+1}} \right)^{-\epsilon} K_{t+1}^p \right],$$

since

$$K_{t+1}^p = E_{t+1} \sum_{s=0}^{\infty} \theta^s \beta^s C_{t+1+s}^{-\sigma} \left(\frac{P_{t+1}}{P_{t+1+s}} \right)^{-\epsilon} Y_{t+1+s} mc_{t+1+s}$$

Hence,

$$K_t^p = C_t^{-\sigma} Y_t mc_t + \theta\beta E_t \left[\left(\frac{P_t}{P_{t+1}} \right)^{-\epsilon} K_{t+1}^p \right]$$

Similarly,

$$F_t^p = C_t^{-\sigma} Y_t + E_t \sum_{s=1}^{\infty} \theta^s \beta^s C_{t+s}^{-\sigma} \left(\frac{P_t}{P_{t+s}} \right)^{1-\epsilon} Y_{t+s}$$

$$= C_t^{-\sigma} Y_t + \theta\beta E_t \sum_{s=1}^{\infty} \theta^{s-1} \beta^{s-1} C_{t+s}^{-\sigma} \left(\frac{P_t}{P_{t+s}} \right)^{1-\epsilon} Y_{t+s}$$

$$= C_t^{-\sigma} Y_t + \theta\beta E_t \sum_{s^*=0}^{\infty} \theta^{s^*} \beta^{s^*} C_{t+s^*+1}^{-\sigma} \left(\frac{P_t}{P_{t+s^*+1}} \right)^{1-\epsilon} Y_{t+s^*+1}$$

$$= C_t^{-\sigma} Y_t + \theta\beta E_t \left[\left(\frac{P_t}{P_{t+1}} \right)^{1-\epsilon} \sum_{s^*=0}^{\infty} \theta^{s^*} \beta^{s^*} C_{t+s^*+1}^{-\sigma} \left(\frac{P_{t+1}}{P_{t+s^*+1}} \right)^{1-\epsilon} Y_{t+s^*+1} \right]$$

$$= C_t^{-\sigma} Y_t + \theta\beta E_t \left[\left(\frac{P_t}{P_{t+1}} \right)^{1-\epsilon} F_{t+1}^p \right]$$

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$$\max_{\{P_t(j)\}_{t=0}^{\infty}} E_t \sum_{s=0}^{\infty} Q_{t,t+s} \left\{ \begin{array}{c} P_{t+s}(j)^{1-\epsilon} P_{t+s}^{\epsilon} Y_{t+s} - mc_{t+s} P_{t+s}(j)^{-\epsilon} P_{t+s}^{1+\epsilon} Y_{t+s} \\ -\frac{\phi}{2} \left(\frac{P_{t+s}(j)}{P_{t+s-1}(j)} - 1 \right)^2 Y_{t+s} P_{t+s} \end{array} \right\}$$

$$= \max_{\{P_t(j)\}_{t=0}^{\infty}} E_t \sum_{s=0}^{\infty} \beta^s \left(\frac{C_t}{C_{t+s}} \right)^{\sigma} \frac{P_t}{P_{t+s}} \left\{ \begin{array}{c} P_{t+s}(j)^{1-\epsilon} P_{t+s}^{\epsilon} Y_{t+s} - mc_{t+s} P_{t+s}(j)^{-\epsilon} P_{t+s}^{1+\epsilon} Y_{t+s} \\ -\frac{\phi}{2} \left(\frac{P_{t+s}(j)}{P_{t+s-1}(j)} - 1 \right)^2 Y_{t+s} P_{t+s} \end{array} \right\}$$

The FOC is

$$\beta^t \left\{(1-\epsilon)P_t(j)^{-\epsilon}P_t^{\epsilon}Y_t + \epsilon mc_t P_t(j)^{-\epsilon-1}P_t^{1+\epsilon}Y_t - \frac{\phi}{P_{t-1}(j)}\left(\frac{P_t(j)}{P_{t-1}(j)}-1\right)Y_tP_t\right\}$$
$$+ E_t\beta^{t+1}\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\frac{P_t}{P_{t+1}}\left\{\phi\frac{P_{t+1}(j)}{P_t(j)^2}\left(\frac{P_{t+1}(j)}{P_t(j)}-1\right)Y_{t+1}P_{t+1}\right\}$$
$$= 0$$

For the symmetric equilibrium, we have

$$(1-\epsilon)Y_t + \epsilon mc_tY_t - \frac{\phi}{P_{t-1}}\left(\frac{P_t}{P_{t-1}}-1\right)Y_tP_t$$
$$+ \beta E_t\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\frac{P_t}{P_{t+1}}\left\{\phi\frac{P_{t+1}}{P_t^2}\left(\frac{P_{t+1}}{P_t}-1\right)Y_{t+1}P_{t+1}\right\}$$
$$= 0$$

that is,

$$(1-\epsilon) + \epsilon mc_t - \phi\frac{P_t}{P_{t-1}}\left(\frac{P_t}{P_{t-1}}-1\right) + \beta E_t\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\phi\left(\frac{P_{t+1}}{P_t}-1\right)\frac{Y_{t+1}P_{t+1}}{Y_tP_t} = 0.$$

Define $\Pi_t \equiv \frac{P_t}{P_{t-1}}$, then we get

$$(1-\epsilon) + \epsilon mc_t - \phi\Pi_t\left(\Pi_t - 1\right) + \phi\beta E_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\frac{Y_{t+1}}{Y_t}\Pi_{t+1}\left(\Pi_{t+1}-1\right)\right] = 0.$$

This is the Phillips curve under Rotemberg pricing.

The equilibrium conditions of the cases are summarized below.

Rotemberg Pricing

The equilibrium conditions are given as follows:

Consumption Euler equation:

$$\beta R_tE_t\left\{\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{P_t}{P_{t+1}}\right)\right\} = 1$$

Labor supply:

$$\left(\frac{W_t}{P_t}\right) = N_t^{\varphi}C_t^{\sigma}$$

Resource constraint:

$$\left[1-\frac{\phi}{2}\left(\Pi_t-1\right)^2\right]Y_t=C_t$$

Phillips curve:

$$(1-\epsilon)+\epsilon mc_t-\phi\Pi_t\left(\Pi_t-1\right)+\phi\beta E_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\frac{Y_{t+1}}{Y_t}\Pi_{t+1}\left(\Pi_{t+1}-1\right)\right]=0$$

Technology:

$$Y_t=A_tN_t$$

Marginal costs:

$$mc_t=\frac{W_t}{P_tA_t}=\frac{N_t^{\varphi}C_t^{\sigma}}{A_t}=\frac{(Y_t/A_t)^{\varphi}C_t^{\sigma}}{A_t}=Y_t{}^{\varphi}C_t^{\sigma}A_t{}^{-\varphi-1}$$

We can simplify these equilibrium conditions by eliminating the interest rate and labour supply from the constraints, so that consumption can be considered as the monetary policy instrument. Specifically,

Resource constraint:

$$\left[1-\frac{\phi}{2}\left(\Pi_t-1\right)^2\right]Y_t=C_t$$

Phillips curve:

$$(1-\epsilon)+\epsilon Y_t{}^{\varphi}C_t^{\sigma}A_t{}^{-\varphi-1}-\phi\Pi_t\left(\Pi_t-1\right)+\phi\beta E_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\frac{Y_{t+1}}{Y_t}\Pi_{t+1}\left(\Pi_{t+1}-1\right)\right]=0$$

while the objective function is given by

$$E_0\sum_{t=0}^{\infty}\beta^t\left(\frac{C_t^{1-\sigma}-1}{1-\sigma}-\frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi}\right)$$

Note that the state variables are productivity (and any other exogenous shock processes we choose to add).

Calvo Pricing

The equilibrium conditions are given below:

Consumption Euler equation:

$$\beta R_tE_t\left\{\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{P_t}{P_{t+1}}\right)\right\}=1$$

Labor supply:

$$\left(\frac{W_t}{P_t}\right) = N_t^{\varphi} C_t^{\sigma}$$

Resource constraint:

$$Y_t = C_t = \frac{A_t N_t}{\Delta_t}$$

Phillips curve:

$$\frac{P_t^*}{P_t} = \left(\frac{\epsilon}{\epsilon - 1}\right) \frac{K_t^p}{F_t^p}$$

Inflation:

$$1 = (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{1-\epsilon} + \theta(\Pi_t)^{\epsilon-1}$$

Price dispersion:

$$\begin{aligned}
\Delta_t &= (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\epsilon} + \theta\left(\frac{P_t}{P_{t-1}}\right)^{\epsilon} \Delta_{t-1} \\
&= (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\epsilon} + \theta\left(\Pi_t\right)^{\epsilon} \Delta_{t-1}
\end{aligned}$$

Marginal costs:

$$mc_t = \frac{W_t}{P_t A_t} = \frac{N_t^{\varphi} C_t^{\sigma}}{A_t} = (Y_t \Delta_t)^{\varphi} C_t^{\sigma} A_t^{-\varphi-1}$$

Note that the state variables are not just productivity, but also price dispersion.

3.A.2 The Chebyshev Collocation Method

Algorithm for Rotemberg Pricing

In the following, let s_t denote the state of the economy at time t. There are five functional equations associated with five endogenous variables $\{C_t, Y_t, \Pi_t, \lambda_{1t}, \lambda_{2t}\}$.

The state is $s_t = a_t \equiv \ln A_t$, which evolves according to the following motion equation:

$$a_{t+1} = \rho_a a_t + e_{at}$$

where $0 \leq \rho_a < 1$ and technology innovation e_{at} is an *i.i.d.* normal random variable, which has a zero mean and a finite standard deviation σ_a.

Let's define a new function $X : \mathbb{R} \to \mathbb{R}^5$, in order to collect the policy functions of endogenous variables as follows:

$$X(s_t) = (C_t(s_t), Y_t(s_t), \Pi_t(s_t), \lambda_{1t}(s_t), \lambda_{2t}(s_t))$$

Given the specification of the function X, the equilibrium conditions can be written more compactly as,

$$\Gamma(s_t, X(s_t), E_t\left[Z\left(X(s_{t+1})\right)\right]) = 0$$

where $\Gamma : \mathbb{R}^{1+5+1} \to \mathbb{R}^5$ summarizes the full set of dynamic equilibrium relationship, and $Z\left(X(s_{t+1})\right) = M(A_{t+1})$. Then the problem is to find a vector-valued function X that Γ maps to the zero function. Projection methods, hence, can be used.

The Chebyshev collocation method which we use to approximate policy functions under Rotemberg pricing can be described as follows:

1. Choose an order of approximation n_a, compute the $n_a + 1$ roots of the Chebychev polynomial of order $n_a + 1$ as

$$z_a^i = \cos\left(\frac{(2i-1)\pi}{2(n_a+1)}\right)$$

for $i = 1, 2, ..., n_a + 1$, and formulate initial guesses for θ_y and θ_Π.

2. Compute collocation points

$$a_i = \frac{\overline{a} + \underline{a}}{2} + \frac{\overline{a} - \underline{a}}{2} z_a^i = \frac{\overline{a} - \underline{a}}{2}\left(z_a^i + 1\right) + \underline{a}$$

for $i = 1, 2, ..., n_a + 1$, where $a = \log(A)$ is logged technology shock. Note that the number of collocation nodes is $n_a + 1$.

3. Formulate the approximating policy functions. Let $T_i(z) = \cos(i \cos^{-1}(z))$ denote the Chebyshev polynomial of order i, $z \in [-1, 1]$, and let ξ denote a linear function mapping the domain of $x \in [\underline{x}, \overline{x}]$ into $[-1, 1]$. In this way, $T_i(\xi(x))$ are Chebyshev polynomials adapted to $x \in [\underline{x}, \overline{x}]$ for $i = 0, 1, ...$ Apparently, $\xi(x) = 2\left(x - \underline{x}\right) / \left(\overline{x} - \underline{x}\right) - 1$. Then, a degree n_a Chebyshev polynomial approximation for the five decision rules at each nodes a_i can be written as in vector form

$$X(a_i) = T(\xi(a_i))\Theta_X$$

where $\Theta_X = [\theta_y, \theta_c, \theta_\pi, \theta_{\lambda_1}, \theta_{\lambda_2}]$ is a $(1 + n_a) \times 5$ matrix comprised of the Chebyshev collocation coefficients, and $T(\xi(a_i))$ is a $1 \times (1 + n_a)$ matrix of the Chebyshev polynomials evaluated at node a_i.

4. At each collocation point a_i, calculate the values of the five residual functions defined by the five equilibrium conditions as follows: assuming a Gaussian distribution for the shock $e_{at} \sim N(0, \sigma_a^2)$, To compute the integral part, we make the following change of variables: $z = e_a/\sqrt{2\sigma_a^2} \sim N(0, 1/2)$, then

$$\begin{aligned} & E_t\left[M\left(s_{t+1}\right)\right] \\ & = \frac{1}{\sigma_a\sqrt{2\pi}} \int_{-\infty}^{+\infty} C_{t+1}^{-\sigma} Y_{t+1} \Pi_{t+1}\left(\Pi_{t+1} - 1\right) \exp\left(-\frac{e_{at+1}^2}{2\sigma_a^2}\right) de_{at+1} \\ & = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} C_{t+1}^{-\sigma} Y_{t+1} \Pi_{t+1}\left(\Pi_{t+1} - 1\right) \exp\left(-z^2\right) dz \end{aligned}$$

where we employ a Gauss-Hermite quadrature to approximate the integral. We compute the nodes z_j and weights ω_j for the quadrature such that

$$\begin{aligned} & \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} C_{t+1}^{-\sigma} Y_{t+1} \Pi_{t+1}\left(\Pi_{t+1} - 1\right) \exp\left(-z^2\right) dz \\ & \approx \frac{1}{\sqrt{\pi}} \sum_{j=1}^{q} \left[\begin{array}{c} \omega_j \widehat{C}\left(\rho_a a_i + z_j\sqrt{2\sigma_a^2}; \theta_y, \theta_\pi\right)^{\sigma} \widehat{Y}\left(\rho_a a_i + z_j\sqrt{2\sigma_a^2}; \theta_y\right) \times \\ \widehat{\Pi}\left(\rho_a a_i + z_j\sqrt{2\sigma_a^2}; \theta_\pi\right)\left(\widehat{\Pi}\left(\rho_a a_i + z_j\sqrt{2\sigma_a^2}; \theta_\pi\right) - 1\right) \end{array} \right] \\ & \equiv \Psi\left(a_i; \theta_y, \theta_\pi, q\right) \end{aligned}$$

for $i = 1, 2, ..., n_a + 1$.

Then, the residual functions can be written as

$$R_1 = \widehat{C}(a_i;\theta_c)^{-\sigma} - \widehat{\lambda}_1(a_i;\theta_{\lambda_1}) + \widehat{\lambda}_2(a_i;\theta_{\lambda_2}) \left[\begin{array}{c} \sigma\epsilon\widehat{Y}(a_i;\theta_y)^{\varphi}\widehat{C}(a_i;\theta_c)^{\sigma-1}\exp\left(a_i\right)^{-\varphi-1} \\ +\sigma\phi\beta\widehat{C}(a_i;\theta_c)^{\sigma-1}\widehat{Y}(a_i;\theta_y)^{-1}\Psi\left(a_i;\theta_y,\theta_\pi,q\right) \end{array} \right]$$

$$\begin{aligned} R_2 = {} & \widehat{Y}(a_i;\theta_y)^{\varphi}\exp\left(a_i\right)^{-\varphi-1} - \left[1 - \frac{\phi}{2}\left(\widehat{\Pi}(a_i;\theta_\pi) - 1\right)^2\right]\widehat{\lambda}_1(a_i;\theta_{\lambda_1}) \\ & - \widehat{\lambda}_2(a_i;\theta_{\lambda_2}) \left[\begin{array}{c} \epsilon\varphi\widehat{Y}(a_i;\theta_y)^{\varphi-1}\widehat{C}(a_i;\theta_c)^{\sigma}\exp\left(a_i\right)^{-\varphi-1} \\ -\phi\beta\widehat{C}(a_i;\theta_c)^{\sigma}\widehat{Y}(a_i;\theta_y)^{-2}\Psi\left(a_i;\theta_y,\theta_\pi,q\right) \end{array} \right] \end{aligned}$$

$$R_3 = \widehat{\lambda}_1(a_i;\theta_{\lambda_1})\left(1 - \widehat{\Pi}(a_i;\theta_\pi)\right)\widehat{Y}(a_i;\theta_y) - \widehat{\lambda}_2(a_i;\theta_{\lambda_2})\left(2\widehat{\Pi}(a_i;\theta_\pi) - 1\right)$$

$$R_4 = \widehat{C}(a_i;\theta_c) - \left[1 - \frac{\phi}{2}\left(\widehat{\Pi}(a_i;\theta_\pi) - 1\right)^2\right]\widehat{Y}(a_i;\theta_y)$$

$$
\begin{aligned}
R_5 = {} & (1-\epsilon) + \epsilon \widehat{Y}(a_i;\theta_y)^{\varphi} \widehat{C}(a_i;\theta_c)^{\sigma} \exp{(a_i)}^{-\varphi-1} \\
& - \phi \widehat{\Pi}(a_i;\theta_\pi) \left(\widehat{\Pi}(a_i) - 1 \right) + \phi\beta \widehat{C}(a_i;\theta_c)^{\sigma} \widehat{Y}(a_i;\theta_y)^{-1} \Psi\left(a_i;\theta_y,\theta_\pi,q\right)
\end{aligned}
$$

where the hat symbol indicates the corresponding approximate policy functions.

5. If all residuals are close enough to zero then stop, else update $\{\theta_y,\ \theta_c,\ \theta_\pi,\ \theta_{\lambda_1},\ \theta_{\lambda_2}\}$, and go back to step 3.

The last step uses Christopher A. Sims' *csolve*[25] to solve the system of nonlinear equations, $R_j = 0$ for $j = 1, ..., 5$. When implementing the above algorithm, we first use lower order Chebyshev polynomials where steady states can be good initial guesses. Then, we increase the order of approximation and take as starting value the solution from the previous lower order approximation. This informal homotopy continuation method follows the advice from Anderson et al. (2010).

Algorithm for Calvo Pricing

Now the state space is $s_t = (\Delta_{t-1}, A_t)$, where price dispersion Δ_{t-1} is endogenous and technology A_t is exogenous and respectively, with the following law of motion:

$$
\begin{aligned}
\Delta_t &= (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\epsilon} + \theta\left(\Pi_t\right)^{\epsilon}\Delta_{t-1} \\
a_t &= \rho_a a_{t-1} + e_{at}
\end{aligned}
$$

There are 7 endogenous variables and 6 Lagrangian multipliers, hence 13 functional equations. Similar to Rotemberg pricing, we can rewrite this nonlinear system a more compact form,

$$
\Gamma\left(s_t, X(s_t), E_t\left[Z\left(X(s_{t+1})\right)\right], E_t\left[Z_\Delta\left(X(s_{t+1})\right)\right]\right) = 0
$$

where $\Gamma : \mathcal{R}^{2+13+3+3} \to \mathcal{R}^{13}$ summarizing the equilibrium relationship,

$$
X(s_t) = \left(C_t(s_t), Y_t(s_t), \Pi_t(s_t), \frac{P_t^*(s_t)}{P_t(s_t)}, K_t^p(s_t), F_t^p(s_t), \Delta_t(s_t), \lambda_{1t}(s_t), \lambda_{2t}(s_t), \lambda_{3t}(s_t), \lambda_{4t}(s_t), \lambda_{5t}(s_t), \lambda_{6t}(s_t) \right)
$$

collecting the policy functions we need to solve, with $X : \mathcal{R}^2 \to \mathcal{R}^{13}$, and

[25] The solver can be found at http://dge.repec.org/codes/sims/optimize/csolve.m.

$$Z\left(X(s_{t+1})\right)=\begin{bmatrix} Z_1\left(X(s_{t+1})\right) \\ Z_2\left(X(s_{t+1})\right) \\ Z_3\left(X(s_{t+1})\right) \end{bmatrix}=\begin{bmatrix} M(\Delta_t,a_{t+1}) \\ L(\Delta_t,a_{t+1}) \\ \left(\Pi_{t+1}\right)^{\epsilon}\lambda_{6t+1} \end{bmatrix}$$

and

$$\begin{aligned} Z_{\Delta}\left(X(s_{t+1})\right) &= \begin{bmatrix} \frac{\partial Z_1(X(s_{t+1}))}{\partial\Delta_t} \\ \frac{\partial Z_2(X(s_{t+1}))}{\partial\Delta_t} \\ \frac{\partial Z_3(X(s_{t+1}))}{\partial\Delta_t} \end{bmatrix}=\begin{bmatrix} \frac{\partial M(\Delta_t,a_{t+1})}{\partial\Delta_t} \\ \frac{\partial L(\Delta_t,a_{t+1})}{\partial\Delta_t} \\ \frac{\partial[(\Pi_{t+1})^{\epsilon}\lambda_{6t+1}]}{\partial\Delta_t} \end{bmatrix} \\ &= \begin{bmatrix} \epsilon\left(\Pi_{t+1}\right)^{\epsilon-1}K^p_{t+1}\frac{\partial\Pi_{t+1}}{\partial\Delta_t}+\left(\Pi_{t+1}\right)^{\epsilon}\frac{\partial K^p_{t+1}}{\partial\Delta_t} \\ (\epsilon-1)\left(\Pi_{t+1}\right)^{\epsilon-2}F^p_{t+1}\frac{\partial\Pi_{t+1}}{\partial\Delta_t}+\left(\Pi_{t+1}\right)^{\epsilon-1}\frac{\partial F^p_{t+1}}{\partial\Delta_t} \\ \epsilon\left(\Pi_{t+1}\right)^{\epsilon-1}\lambda_{6t+1}\frac{\partial\Pi_{t+1}}{\partial\Delta_t}+\left(\Pi_{t+1}\right)^{\epsilon}\frac{\partial\lambda_{6t+1}}{\partial\Delta_t} \end{bmatrix} \end{aligned}$$

Note we are assuming $E_t\left[Z_{\Delta}\left(X(s_{t+1})\right)\right]=\partial E_t\left[Z\left(X(s_{t+1})\right)\right]/\Delta_t$, which is normally valid using the Interchange of Integration and Differentiation Theorem.

The Chebyshev collocation method which we use to solve this nonlinear system can be described as follows:

1. Choose an order of approximation n_Δ and n_a for each dimension of the state space $s_t=(\Delta_{t-1},a_t)$, then there are $N_s=(n_\Delta+1)\times(n_a+1)$ nodes in the state space.

2. Compute the $n_\Delta+1$ and n_a+1 roots of the Chebychev polynomial of order $n_\Delta+1$ and n_a+1 as

$$\begin{aligned} z^i_\Delta &= \cos\left(\frac{(2i-1)\pi}{2(n_\Delta+1)}\right), \text{ for } i=1,2,...,n_\Delta+1. \\ z^i_a &= \cos\left(\frac{(2i-1)\pi}{2(n_a+1)}\right), \text{ for } i=1,2,...,n_a+1. \end{aligned}$$

   and formulate initial guesses for the approximating coefficients.

3. Compute collocation points a_i as

$$a_i=\frac{\overline{a}+\underline{a}}{2}+\frac{\overline{a}-\underline{a}}{2}z^i_a=\frac{\overline{a}-\underline{a}}{2}\left(z^i_a+1\right)+\underline{a}$$

   for $i=1,2,...,n_a+1$, where $a=\log(A)$ is logged technology shock. Note that the number of collocation nodes is n_a+1. Similarly, compute collocation points Δ_i as

$$\Delta_i=\frac{\overline{\Delta}+\underline{\Delta}}{2}+\frac{\overline{\Delta}-\underline{\Delta}}{2}z^i_\Delta=\frac{\overline{\Delta}-\underline{\Delta}}{2}\left(z^i_\Delta+1\right)+\underline{\Delta}$$

for $i = 1, 2, ..., n_\Delta + 1$, which map $[-1, 1]$ into $[\underline{\Delta}, \overline{\Delta}]$.

4. At each node (Δ_i, a_j), for $i = 1, 2, ..., n_\Delta + 1$ and $j = 1, 2, ..., n_a + 1$, compute $X(s_t)$, that is,

$$X(\Delta_i, a_j) = \Omega(\Delta_i, a_j)\Theta_X$$

where $\Omega(\Delta_i, a_j) \equiv [T_{j_\Delta}(\xi(\Delta_i)T_{j_a}(\xi\,(a_j))]$, $j_\Delta = 0, ..., n_\Delta$, and $j_a = 0, ..., n_a$, is a $1 \times N_s$ matrix of two-dimensional Chebyshev polynomials evaluated at node (Δ_i, a_j), and

$$\Theta_X = \left[\theta^c, \theta^y, \theta^\pi, \theta^p, \theta^k, \theta^f, \theta^\Delta, \theta^{\lambda_1}, \theta^{\lambda_2}, \theta^{\lambda_3}, \theta^{\lambda_4}, \theta^{\lambda_5}, \theta^{\lambda_6}\right]$$

is a $N_s \times 13$ matrix of the collocation coefficients.

5. At each node (Δ_i, a_j), for $i = 1, 2, ..., n_\Delta + 1$ and $j = 1, 2, ..., n_a + 1$, compute the possible values of future policy functions $X(s_{t+1})$ for $k = 1, ..., q$. That is,

$$X_{t+1}(\Delta_i, a_j) = \Omega_{t+1}(\Delta_i, a_j)\Theta_X$$

where q is the number of quadrature nodes, and the subscript $t+1$ indicates next period values. Note that

$$\Omega_{t+1}(\Delta_i, a_j) \equiv \left[T_{j_\Delta}(\xi(\widehat{\Delta}(\Delta_i, a_j; \theta^\Delta)))T_{j_a}(\xi(\rho_a a_j + z_k\sqrt{2\sigma_a^2}))\right]$$

with $j_\Delta = 0, ..., n_\Delta$, and $j_a = 0, ..., n_a$, is a $q \times N_s$ matrix of Chebyshev polynomials evaluated at $t+1$ nodes (Δ_t, a_{t+1}), and the hat symbol indicates the corresponding approximate policy functions.

The two auxilliary functions can be calculated as follows:

$$M(s_{t+1}) \approx \left(\widehat{\Pi}(s_{t+1}; \theta^\pi)\right)^\epsilon \widehat{K}(s_{t+1}; \theta^k)$$

and,

$$L(s_{t+1}) \approx \left(\widehat{\Pi}(s_{t+1}; \theta^\pi)\right)^{\epsilon-1} \widehat{F}(s_{t+1}; \theta^f).$$

6. Calculate the expectation terms at each node (Δ_i, a_j). Let $z = e_a/\sqrt{2\sigma_a^2}$, and we

have

$$\begin{aligned}
E_t\left[M(s_{t+1})\right] &= \frac{1}{\sigma_a\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\Pi_{t+1}\right)^{\epsilon}K_{t+1}^{p}\exp\left(-\frac{e_{at+1}^{2}}{2\sigma_a^{2}}\right)de_{at+1}\\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\left(\Pi_{t+1}\right)^{\epsilon}K_{t+1}^{p}\exp\left(-z^{2}\right)dz\\
&\approx \frac{1}{\sqrt{\pi}}\sum_{k=1}^{q}\omega_k\left(\widehat{\Pi}(s_{t+1};\theta^{\pi})\right)^{\epsilon}\widehat{K}(s_{t+1};\theta^{k})\\
&\equiv \Psi\left(\Delta_i, a_j, q\right),
\end{aligned}$$

$$\begin{aligned}
E_t\left[L(s_{t+1})\right] &= \frac{1}{\sigma_a\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\Pi_{t+1}\right)^{\epsilon-1}F_{t+1}^{p}\exp\left(-\frac{e_{at+1}^{2}}{2\sigma_a^{2}}\right)de_{at+1}\\
&\approx \frac{1}{\sqrt{\pi}}\sum_{k=1}^{q}\omega_k\left(\widehat{\Pi}(s_{t+1};\theta^{\pi})\right)^{\epsilon-1}\widehat{F}(s_{t+1};\theta^{f})\\
&\equiv \Theta\left(\Delta_i, a_j, q\right),
\end{aligned}$$

and

$$\begin{aligned}
E_t\left[\left(\Pi_{t+1}\right)^{\epsilon}\lambda_{6t+1}\right] &= \frac{1}{\sigma_a\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\Pi_{t+1}\right)^{\epsilon}\lambda_{6t+1}\exp\left(-\frac{e_{at+1}^{2}}{2\sigma_a^{2}}\right)de_{at+1}\\
&\approx \frac{1}{\sqrt{\pi}}\sum_{k=1}^{q}\omega_k\left(\widehat{\Pi}(s_{t+1};\theta^{\pi})\right)^{\epsilon-1}\widehat{\lambda}_6(s_{t+1};\theta^{\lambda_6})\\
&\equiv \Lambda\left(\Delta_i, a_j, q\right).
\end{aligned}$$

7. Calculate the two partial derivatives under expectation, that is,

$$\begin{aligned}
\frac{\partial E_t\left[M(s_{t+1})\right]}{\partial\Delta_t} &= E_t\left[\frac{\partial M(s_{t+1})}{\partial\Delta_t}\right]\\
&= E_t\left[\epsilon\left(\Pi_{t+1}\right)^{\epsilon-1}K_{t+1}^{p}\frac{\partial\Pi_{t+1}}{\partial\Delta_t}+\left(\Pi_{t+1}\right)^{\epsilon}\frac{\partial K_{t+1}^{p}}{\partial\Delta_t}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial E_t\left[L(s_{t+1})\right]}{\partial\Delta_t} &= E_t\left[\frac{\partial L(s_{t+1})}{\partial\Delta_t}\right]\\
&= E_t\left[(\epsilon-1)\left(\Pi_{t+1}\right)^{\epsilon-2}F_{t+1}^{p}\frac{\partial\Pi_{t+1}}{\partial\Delta_t}+\left(\Pi_{t+1}\right)^{\epsilon-1}\frac{\partial F_{t+1}^{p}}{\partial\Delta_t}\right].
\end{aligned}$$

Hence, we only need to compute $\partial\Pi_{t+1}/\partial\Delta_t$, $\partial K_{t+1}^{p}/\partial\Delta_t$ and $\partial F_{t+1}^{p}/\partial\Delta_t$. Note that

$$\frac{\partial \Pi_{t+1}}{\partial \Delta_t} \approx \sum_{j_\Delta=0}^{n_\Delta} \sum_{j_a=0}^{n_a} \frac{2\theta^{\pi}_{j_\Delta j_a}}{\overline{\Delta} - \underline{\Delta}} T'_{j_\Delta}(\xi(\Delta_i)) T_{j_a}(\xi(a_j)) \equiv \widehat{\Pi}_\Delta$$

$$\frac{\partial K^p_{t+1}}{\partial \Delta_t} \approx \sum_{j_\Delta=0}^{n_\Delta} \sum_{j_a=0}^{n_a} \frac{2\theta^{k}_{j_\Delta j_a}}{\overline{\Delta} - \underline{\Delta}} T'_{j_\Delta}(\xi(\Delta_t)) T_{j_a}(\xi(a_{t+1})) \equiv \widehat{K}_\Delta$$

$$\frac{\partial F^p_{t+1}}{\partial \Delta_t} \approx \sum_{j_\Delta=0}^{n_\Delta} \sum_{j_a=0}^{n_a} \frac{2\theta^{f}_{j_\Delta j_a}}{\overline{\Delta} - \underline{\Delta}} T'_{j_\Delta}(\xi(\Delta_t)) T_{j_a}(\xi(a_{t+1})) \equiv \widehat{F}_\Delta$$

Now, we can calculate

$$\frac{\partial E_t\left[M(s_{t+1})\right]}{\partial \Delta_t} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \left[\begin{array}{c} \epsilon\left(\Pi_{t+1}\right)^{\epsilon-1} K^p_{t+1} \frac{\partial \Pi_{t+1}}{\partial \Delta_t} \\ +\left(\Pi_{t+1}\right)^{\epsilon} \frac{\partial K^p_{t+1}}{\partial \Delta_t} \end{array} \right] \exp\left(-z^2\right) dz$$

$$\begin{aligned} &\approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{q} \omega_k \left[\epsilon\left(\widehat{\Pi}(s_{t+1};\theta^\pi)\right)^{\epsilon-1} \widehat{K}(s_{t+1};\theta^k)\widehat{\Pi}_\Delta + \left(\widehat{\Pi}(s_{t+1};\theta^\pi)\right)^{\epsilon} \widehat{K}_\Delta \right] \\ &\equiv \widehat{M}\left(\Delta_i, a_j, q\right), \end{aligned}$$

$$\frac{\partial E_t\left[L(s_{t+1})\right]}{\partial \Delta_t} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \left[\begin{array}{c} (\epsilon-1)\left(\Pi_{t+1}\right)^{\epsilon-2} F^p_{t+1} \frac{\partial \Pi_{t+1}}{\partial \Delta_t} \\ +\left(\Pi_{t+1}\right)^{\epsilon-1} \frac{\partial F^p_{t+1}}{\partial \Delta_t} \end{array} \right] \exp\left(-z^2\right) dz$$

$$\begin{aligned} &\approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{q} \omega_k \left[(\epsilon-1)\left(\widehat{\Pi}(s_{t+1};\theta^\pi)\right)^{\epsilon-2} \widehat{F}(s_{t+1};\theta^f)\widehat{\Pi}_\Delta + \left(\widehat{\Pi}(s_{t+1};\theta^\pi)\right)^{\epsilon-1} \widehat{F}_\Delta \right] \\ &\equiv \widehat{L}\left(\Delta_i, a_j, q\right). \end{aligned}$$

8. At each collocation point (Δ_i, a_j), calculate the values of the thirteen residual functions defined by the equilibrium conditions as follows:

$$R_1 = \widehat{Y}(\Delta_i, a_j; \theta^y) - \widehat{C}(\Delta_i, a_j; \theta^c)$$

$$R_2 = \widehat{p}(\Delta_i, a_j; \theta^p) - \left(\frac{\epsilon}{\epsilon-1}\right) \frac{\widehat{K}(\Delta_i, a_j; \theta^k)}{\widehat{F}(\Delta_i, a_j; \theta^f)}$$

$$\begin{aligned} R_3 = &\widehat{K}(\Delta_i, a_j; \theta^k) - (\widehat{\Delta}(\Delta_i, a_j; \theta^\Delta)\widehat{Y}(\Delta_i, a_j; \theta^y))^\varphi \exp(a_j)^{-\varphi-1}\widehat{Y}(\Delta_i, a_j; \theta^y) \\ &- \theta\beta\Psi\left(\Delta_i, a_j, q\right) \end{aligned}$$

$$R_4 = \widehat{F}(\Delta_i, a_j; \theta^f) - \widehat{Y}(\Delta_i, a_j; \theta^y)\widehat{C}(\Delta_i, a_j; \theta^c)^{-\sigma} + \theta\beta\Theta\left(\Delta_i, a_j, q\right)$$

$$R_5 = 1 - (1-\theta)\widehat{p}(\Delta_i, a_j; \theta^p)^{1-\epsilon} - \theta\widehat{\Pi}(\Delta_i, a_j; \theta^\pi)^{\epsilon-1}$$

$$R_6 = \widehat{\Delta}(\Delta_i, a_j; \theta^\Delta) - (1-\theta)\widehat{p}(\Delta_i, a_j; \theta^p)^{-\epsilon} - \theta\widehat{\Pi}(\Delta_i, a_j; \theta^\pi)^{\epsilon}\Delta_i$$

$$R_7 = \widehat{C}(\Delta_i, a_j; \theta^c)^{-\sigma} - \widehat{\lambda}_1(\Delta_i, a_j; \theta^{\lambda_1}) + \sigma\widehat{Y}(\Delta_i, a_j; \theta^y)\widehat{C}(\Delta_i, a_j; \theta^c)^{-\sigma-1}\widehat{\lambda}_4(\Delta_i, a_j; \theta^{\lambda_4})$$

$$\begin{aligned} R_8 = & -\widehat{\Delta}(\Delta_i, a_j; \theta^\Delta)^{1+\varphi}\widehat{Y}(\Delta_i, a_j; \theta^y)^{\varphi}\exp(a_j)^{-1-\varphi} + \widehat{\lambda}_1(\Delta_i, a_j; \theta^{\lambda_1}) \\ & - (1+\varphi)(\widehat{\Delta}(\Delta_i, a_j; \theta^\Delta)\widehat{Y}(\Delta_i, a_j; \theta^y))^{\varphi}\exp(a_j)^{-1-\varphi}\widehat{\lambda}_3(\Delta_i, a_j; \theta^{\lambda_3}) \\ & - \widehat{C}(\Delta_i, a_j; \theta^c)^{-\sigma}\widehat{\lambda}_4(\Delta_i, a_j; \theta^{\lambda_4}) \end{aligned}$$

$$\begin{aligned} R_9 = & \widehat{\lambda}_2(\Delta_i, a_j; \theta^{\lambda_2}) + (1-\theta)(\epsilon-1)\widehat{p}(\Delta_i, a_j; \theta^p)^{-\epsilon}\widehat{\lambda}_5(\Delta_i, a_j; \theta^{\lambda_5}) \\ & + \epsilon(1-\theta)\widehat{p}(\Delta_i, a_j; \theta^p)^{-\epsilon-1}\widehat{\lambda}_6(\Delta_i, a_j; \theta^{\lambda_6}) \end{aligned}$$

$$R_{10} = -(\epsilon-1)\theta\widehat{\lambda}_5(\Delta_i, a_j; \theta^{\lambda_5}) - \epsilon\theta\widehat{\Pi}(\Delta_i, a_j; \theta^\pi)\widehat{\lambda}_6(\Delta_i, a_j; \theta^{\lambda_6})\Delta_i$$

$$R_{11} = -\left(\frac{\epsilon}{\epsilon-1}\right)\widehat{F}(\Delta_i, a_j; \theta^f)^{-1}\widehat{\lambda}_2(\Delta_i, a_j; \theta^{\lambda_2}) + \widehat{\lambda}_3(\Delta_i, a_j; \theta^{\lambda_3})$$

$$R_{12} = \left(\frac{\epsilon}{\epsilon-1}\right)\frac{\widehat{K}(\Delta_i, a_j; \theta^k)}{\widehat{F}(\Delta_i, a_j; \theta^f)^2}\widehat{\lambda}_2(\Delta_i, a_j; \theta^{\lambda_2}) + \widehat{\lambda}_4(\Delta_i, a_j; \theta^{\lambda_4})$$

$$\begin{aligned} R_{13} = & -\widehat{Y}(\Delta_i, a_j; \theta^y)^{1+\varphi}\widehat{\Delta}(\Delta_i, a_j; \theta^\Delta)^{\varphi}\exp(a_j)^{-1-\varphi} - \theta\beta\Lambda\left(\Delta_i, a_j, q\right) \\ & - \varphi\widehat{\Delta}(\Delta_i, a_j; \theta^\Delta)^{\varphi-1}\widehat{Y}(\Delta_i, a_j; \theta^y)^{1+\varphi}\widehat{\lambda}_3(\Delta_i, a_j; \theta^{\lambda_3})\exp(a_j)^{-1-\varphi} \\ & - \theta\beta\widehat{M}\left(\Delta_i, a_j, q\right)\widehat{\lambda}_3(\Delta_i, a_j; \theta^{\lambda_3}) - \theta\beta\widehat{L}\left(\Delta_i, a_j, q\right)\widehat{\lambda}_4(\Delta_i, a_j; \theta^{\lambda_4}) + \widehat{\lambda}_6(\Delta_i, a_j; \theta^{\lambda_6}) \end{aligned}$$

9. Finally, check the stopping rules. If all residuals are close enough to zero then stop, else update the approximation coefficients and go back to step 4.

The equation solver *csolve* written by Christopher A. Sims is employed to solve the resulted system of nonlinear equations. When implementing the above algorithm, we start from lower order Chebyshev polynomials and formulate the initial guesses based on the results in Anderson et al. (2010). Then, we increase the order of approximation and take as starting value the solution from the previous lower order approximation. This informal homotopy continuation idea ensures us to find the solution.

3.A.3 Welfare Comparison

In order to compare the social welfare under Calvo and Rotemberg pricing in a fully nonlinear model, we first describe the second-order approximation to welfare. Then we transform the welfare as the fraction of the consumption path under the Ramsey allocation that must be given up in order to equalize welfare under the Ramsey policy and discretionary policy.

The Quadratic Approximation to Welfare

Individual utility in period t is

$$U_t \equiv U(C_t, N_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}$$

Let $\widehat{X}_t \equiv \log\left(X_t/\overline{X}\right)$ denote the log-deviation of variable X_t from its steady state $\overline{X}$. In addition, let $\widetilde{X}_t = X_t - \overline{X}$ denote the linear deviation of X_t around its steady state value. Then using a second-order Taylor approximation,

$$\frac{X_t - \overline{X}}{\overline{X}} = \frac{\widetilde{X}_t}{\overline{X}} = \widehat{X}_t + \frac{1}{2}\widehat{X}_t^2 + o(2) \tag{3.23}$$

where $o(2)$ represents terms that are of order higher than 2 in the bound on the amplitude of the relevant shocks. We will repeatedly use (3.23) to derive a second-order approximation to the social welfare.

Now consider the second-order approximation to per period utility,

$$U_t = \overline{U} + \overline{C}^{1-\sigma}\left[\widehat{C}_t + \frac{1-\sigma}{2}\widehat{C}_t^2\right] - \overline{N}^{1+\varphi}\left[\widehat{N}_t + \frac{1+\varphi}{2}\widehat{N}_t^2\right] + o(2)$$

where

$$\overline{U} = \frac{\overline{C}^{1-\sigma} - 1}{1-\sigma} - \frac{\overline{N}^{1+\varphi}}{1+\varphi}$$

Rotemberg Pricing The second-order approximation to market clearing condition, $C_t = \left[1 - \frac{\phi}{2}\left(\Pi_t - 1\right)^2\right] Y_t$, is

$$\widehat{C}_t + \frac{1}{2}\widehat{C}_t^2 = \widehat{Y}_t + \frac{1}{2}\widehat{Y}_t^2 - \frac{\phi}{2}\widehat{\Pi}_t^2 + o(2)$$

such that,

$$\begin{aligned} U_t = \overline{U} - \frac{(\sigma+\varphi)\,\overline{C}^{1-\sigma}}{2}\left[(x_t - x^*)^2 + \frac{\phi}{\varphi+\sigma}\widehat{\Pi}_t^2\right] \\ + \overline{C}^{1-\sigma}\left[\begin{array}{c} \frac{\Phi^2}{2(\varphi+\sigma)} - \frac{(1-\sigma)(1-\Phi)-(1+\varphi)}{1+\varphi}\widehat{Y}_t^f \\ + \frac{(1-\sigma)(\sigma+\varphi)}{2(1+\varphi)}\left(\widehat{Y}_t^f\right)^2 \end{array}\right] + o(2) \end{aligned} \tag{3.24}$$

where $\widehat{Y}_t^f = \log\left(Y_t^f/\overline{Y}^f\right)$ denote the log-deviation of output from its steady state under flexible price, $x_t \equiv \widehat{Y}_t - \widehat{Y}_t^f$ is the output gap, $x^* \equiv \ln\overline{Y} - \ln\overline{Y}^f = -\ln(1-\Phi)/(\sigma+\varphi) \approx \Phi/(\sigma+\varphi)$ is a measure of the distortion created by the presence of monopolistic competition alone, *t.i.p.* are terms independent of policy, and terms like $\Phi\left(\widehat{Y}_t^f\right)^2$ and $\Phi\widehat{Y}_t\widehat{Y}_t^f$ are omitted[26]. In addition, the fact that $\overline{N}^{1+\varphi} = (1-\epsilon^{-1})\overline{C}^{1-\sigma} \equiv (1-\Phi)\overline{C}^{1-\sigma}$, and $\widehat{A}_t = (\varphi+\sigma)/(1+\varphi)\widehat{Y}_t^f$ is used in deriving (3.24).

Hence,

$$
\begin{aligned}
W_R \equiv E_0\sum_{t=0}^{\infty}\beta^t U_t &= \frac{\overline{U}}{1-\beta} - \frac{(\sigma+\varphi)\overline{C}^{1-\sigma}}{2}E_0\sum_{t=0}^{\infty}\beta^t\left[(x_t-x^*)^2 + \frac{\phi}{\sigma+\varphi}\widehat{\Pi}_t^2\right] \\
&\quad + \left[\frac{\Phi^2\overline{C}^{1-\sigma}}{2(\varphi+\sigma)(1-\beta)} - \frac{(1-\sigma)(1+\varphi)\overline{C}^{1-\sigma}\sigma_a^2}{2(\varphi+\sigma)(1-\beta)(1-\rho_a)}\right] + o(2) \\
&= \frac{\overline{U}}{1-\beta} - \Omega_R E_0\sum_{t=0}^{\infty}\beta^t\left[\lambda_R(x_t-x^*)^2 + \widehat{\Pi}_t^2\right] \\
&\quad + \left[\frac{\Phi^2\overline{C}^{1-\sigma}}{2(\varphi+\sigma)(1-\beta)} - \frac{(1-\sigma)(1+\varphi)\overline{C}^{1-\sigma}\sigma_a^2}{2(\varphi+\sigma)(1-\beta)(1-\rho_a)}\right] + o(2)
\end{aligned}
\tag{3.25}
$$

where

$$\Omega_R \equiv \frac{\phi\overline{C}^{1-\sigma}}{2}$$
$$\lambda_R \equiv \frac{\sigma+\varphi}{\phi}$$

Note that we derive the LQ welfare function explicitly retaining the relevant *t.i.p* in order to make a legitimate comparison with the social welfare obtained from the fully nonlinear model.

In order to calculate the inflation bias under LQ method, we write down the log-linearized IS equation and NKPC below. The IS curve is,

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}\left(\widehat{R}_t - E_t\widehat{\Pi}_{t+1}\right) + \frac{1+\varphi}{\varphi+\sigma}(\rho_a-1)\widehat{A}_t$$

and the NKPC is,

$$\widehat{\Pi}_t = \beta E_t\widehat{\Pi}_{t+1} + \frac{(\epsilon-1)(\varphi+\sigma)}{\phi}x_t$$

[26] When $\Phi = 1/\epsilon$ is so small that the product of Φ with a second-order term can be ignored as negligible.

Calvo Pricing The second-order approximation to market clearing condition is

$$\widehat{C}_t + \frac{1}{2}\widehat{C}_t^2 = \widehat{Y}_t + \frac{1}{2}\widehat{Y}_t^2 + o(2)$$

and it can be shown (see Woodford, 2003, chap 6) that,

$$\widehat{N}_t = \left(\widehat{Y}_t - \widehat{A}_t\right) + \frac{\epsilon}{2} var_j \left(\widehat{P}_t(j)\right) + o(2)$$

Hence, similar to the Rotemberg case,

$$U_t = \overline{U} - \frac{(\varphi + \sigma)\,\overline{C}^{1-\sigma}}{2}\left[(x_t - x^*)^2 + \frac{\epsilon}{\varphi + \sigma} var_j \left(\widehat{P}_t(j)\right)\right]$$

$$+\overline{C}^{1-\sigma}\left[\frac{\Phi^2}{2(\varphi + \sigma)} - \frac{(1-\sigma)(1-\Phi) - (1+\varphi)}{(1+\varphi)}\widehat{Y}_t^f + \frac{(1-\sigma)(\sigma+\varphi)}{2(1+\varphi)}\left(\widehat{Y}_t^f\right)^2\right] + o(2)$$

The next step is to relate price dispersion $\Delta_t \equiv var_j \left(\widehat{P}_t(j)\right)$ to the average inflation rate across all firms. Walsh (2003, p.554) shows that

$$\Delta_t \approx \theta \Delta_{t-1} + \left(\frac{\theta}{1-\theta}\right)\pi_t^2$$

which implies

$$\sum_{t=0}^{\infty} \beta^t \Delta_t = \frac{\theta}{(1-\theta)(1-\theta\beta)} \sum_{t=0}^{\infty} \beta_t \pi_t^2$$

Therefore,

$$W_C = \frac{\overline{U}}{1-\beta} - \Omega_C E_0 \sum_{t=0}^{\infty} \beta^t \left[\lambda_C (x_t - x^*)^2 + \pi_t^2\right]$$

$$+\left[\frac{\Phi^2 \overline{C}^{1-\sigma}}{2(\varphi+\sigma)(1-\beta)} - \frac{(1-\sigma)(1+\varphi)\overline{C}^{1-\sigma}\sigma_a^2}{2(\varphi+\sigma)(1-\beta)(1-\rho_a)}\backslash\right] + o(2)$$

where

$$\Omega_C \equiv \frac{(\sigma+\varphi)\,\overline{C}^{1-\sigma}}{2}\frac{\epsilon}{\kappa}$$

$$\lambda_C \equiv \kappa/\epsilon$$

$$\kappa \equiv \frac{(1-\theta)(1-\theta\beta)(\sigma+\varphi)}{\theta}$$

The log-linearized IS equation and NKPC are given, respectively, as follows:

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}\left(\widehat{R}_t - E_t \widehat{\Pi}_{t+1}\right) + \frac{1+\varphi}{\varphi+\sigma}(\rho_a - 1)\,\widehat{A}_t$$

$$\widehat{\Pi}_t = \beta E_t \widehat{\Pi}_{t+1} + \kappa x_t$$

Note that when

$$\phi = \frac{(\epsilon - 1)\,\theta}{(1-\theta)\,(1-\theta\beta)}$$

the NKPC under both Rotemberg pricing and Calvo pricing are the same. Also note that $\lambda_R = \left(\frac{\epsilon}{\epsilon-1}\right)\lambda_C$, and $\Omega_R = \left(\frac{\epsilon-1}{\epsilon}\right)\Omega_C$. The inflation weights λ_R and λ_C differ only marginally, since ϵ usually takes values between 7 and 10 in the applied literature.

Inflation Bias Under LQ Method

We can rewrite the above LQ model as follows, using $\pi_t = \Pi_t - 1 \approx \ln(\Pi_t) - \ln\left(\overline{\Pi}\right) = \widehat{\Pi}_t$ and $i_t = R_t - 1 \approx \ln(R_t) - \ln\left(\overline{R}\right) = \widehat{R}_t$:

$$\max_{\{x_t,\pi_t\}} -\Omega_j E_0 \sum_{t=0}^{\infty} \beta^t \left[\lambda_j \left(x_t - x^*\right)^2 + \pi_t^2\right]$$

subject to

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t \tag{3.26}$$

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1}\right) + \frac{1+\varphi}{\varphi+\sigma}\left(\rho_a - 1\right)\widehat{A}_t$$

where $j = R, C$. Woodford (2003, p.471) shows that the equilibrium inflation under optimal discretion is

$$\pi_t = \frac{\lambda_j}{\lambda_j + \kappa^2}\left(\beta E_t \pi_{t+1} + \kappa x^*\right)$$

hence the steady state $\overline{\pi}$ under rational expectation satisfies

$$\overline{\pi} = \frac{\lambda_j}{\lambda_j + \kappa^2}\left(\beta\overline{\pi} + \kappa x^*\right)$$

that is,

$$\overline{\pi} = \frac{\lambda_j \kappa}{(1-\beta)\,\lambda_j + \kappa^2} x^* = \frac{\lambda_j \kappa}{(1-\beta)\,\lambda_j + \kappa^2}\frac{\Phi}{(\sigma+\varphi)}$$

with $j = R, C$. $\overline{\pi}$ is the so-called inflation bias, relative to the targeted zero rate of inflation which is optimal under perfect commitment.

Relative Welfare Cost

The welfare under discretion from the LQ method is calculated as follows. Unless stated otherwise, the superscript d denotes the discretion case, and subscripts R and C represent the Rotemberg and Calvo pricing, respectively. From (4.18), $\overline{x} = (1-\beta)\,\overline{\pi}/\kappa$, then using the log-linearized model we can solve for steady state values for deviations $\widehat{C}_t$ and $\widehat{N}_t$, denoted as $\widehat{C}$ and $\widehat{N}$, respectively. It is straightforward to show that $\widehat{C} = \widehat{N} = \widehat{Y} = \overline{x}$. Finally, the steady state values for levels C_t and N_t, are

$$\overline{C}_j^d = \overline{C}^r e^{\widehat{C}} \approx \overline{C}^r \left(1 + \widehat{C}\right) = \overline{C}^r \left(1 + \overline{x}\right)$$

$$\overline{N}_j^d = \overline{N}^r e^{\widehat{N}} \approx \overline{N}^r \left(1 + \widehat{N}\right) = \overline{N}^r \left(1 + \overline{x}\right)$$

where $j = R, C$, and

$$\overline{C}^r = \overline{N}^r = \left(\frac{\epsilon - 1}{\epsilon}\right)^{1/(\sigma+\varphi)}$$

are the Ramsey steady states around which we log-linearize the model. Therefore,

$$W_j = \frac{1}{1-\beta}\left[\frac{\left(\overline{C}_j^d\right)^{1-\sigma} - 1}{1-\sigma} - \frac{\left(\overline{N}_j^d\right)^{1+\varphi}}{1+\varphi}\right] - \frac{\Omega_j}{1-\beta}\left[\lambda_j\left(\frac{(1-\beta)\overline{\pi}}{\kappa} - \frac{\Phi}{(\varphi+\sigma)}\right)^2 + \overline{\pi}^2\right]$$

$$+\frac{\left(\overline{C}_j^d\right)^{1-\sigma}}{2(\varphi+\sigma)(1-\beta)}\left[\Phi^2 - \frac{(1-\sigma)(1+\varphi)\sigma_a^2}{(1-\rho_a)}\right]$$

where $j = R, C$.

For the fully nonlinear method, the welfare under discretion is calculated by adding corresponding policy functions into optimal policy problem and then approximated by the Chebyshev collocation method. That is,

$$W_{R,t}^d = W_R^d(A_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi} + \beta E_t\left[W_R^d(A_{t+1})\right]$$

$$W_{C,t}^d = W_C^d(\Delta_{t-1}, A_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{(\Delta_t Y_t/A_t)^{1+\varphi}}{1+\varphi} + \beta E_t\left[W_C^d(\Delta_t, A_{t+1})\right]$$

and the steady state welfare, denoted as W_R^d and W_C^d for ease of notation, can be correspondingly found.

Note that W_R, W_R^d and W_C, W_C^d which represent the conditional expectation of life-time utility, are absolute welfare measures under Rotemberg pricing and Calvo pricing, respectively. However, the utility function is ordinal, so a welfare measure based on the value function is not very revealing. Hence, we calculate the relative welfare cost in terms of the consumption equivalent units under the Ramsey allocation. Specifically, we want to find ξ such that

$$E_0 \sum_{t=0}^{\infty} \beta^t U(C_t^d, N_t^d) = E_0 \sum_{t=0}^{\infty} \beta^t U((1-\xi) C_t^r, N_t^r)$$

where the r superscript denotes the Ramsey allocation (under full commitment), and the d superscript stands for the allocation under discretion. Given the utility function adopted, the expression for ξ in percentage terms is

$$\xi = \left\{1 - \exp\left[(1-\beta)\left(W^d - W^r\right)\right]\right\} \times 100 \tag{3.27}$$

where

$$W^d \equiv E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t^d - \frac{\left(N_t^d\right)^{1+\varphi}}{1+\varphi} \right)$$

represents the unconditional expectation of lifetime utility in the economy under discretion, and

$$W^r \equiv E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t^r - \frac{(N_t^r)^{1+\varphi}}{1+\varphi} \right) = \frac{1}{1-\beta} \left[\ln \overline{C}^r - \frac{\left(\overline{N}^r\right)^{1+\varphi}}{1+\varphi} \right]$$

is the unconditional expectation of lifetime utility associated with the economy under full commitment. Recall that $\sigma = 1$ is the benchmark case here.

Hence, under the Rotemberg case,

$$\xi_R = \begin{cases} \{1 - \exp\left[(1-\beta)\left(W_R - W^r\right)\right]\} \times 100 & \text{, using LQ method} \\ \left\{1 - \exp\left[(1-\beta)\left(W_R^d - W^r\right)\right]\right\} \times 100 & \text{, using projection method} \end{cases}$$

and under the Calvo case,

$$\xi_C = \begin{cases} \{1 - \exp\left[(1-\beta)\left(W_C - W^r\right)\right]\} \times 100 & \text{, using LQ method} \\ \left\{1 - \exp\left[(1-\beta)\left(W_C^d - W^r\right)\right]\right\} \times 100 & \text{, using projection method} \end{cases}$$

3.A.4 Trend Inflation

In this section we explore the determinacy properties of our simple New Keynesian models at the levels of steady-state inflation implied by our non-linear optimal policy exercise.

The Rotemberg Case

Following Ascari and Rossi (2012) the linearised version of our New Keyensian model under Rotemberg pricing can be shown to be,

$$\pi_t = \gamma_f \beta E_t \pi_{t+1} + \gamma_y \beta(1-\sigma)\Delta E_t y_{t+1} + \gamma_{mc} mc_t$$

$$\begin{aligned} mc_t &= (\sigma+\varphi)y_t - \zeta_c \sigma \pi_t - (1+\varphi)a_t \\ y_t &= E_t y_{t+1} - \zeta_c \Delta E_t \pi_{t+1} - \frac{1}{\sigma} E_t(R_t - \pi_{t+1}) \end{aligned}$$

where

$$\begin{aligned} \zeta_c &= \frac{\phi(\pi-1)\pi}{1-\frac{\phi}{2}(\pi-1)^2} \\ \frac{C}{Y} &= 1 - \frac{\phi}{2}(\pi-1)^2 \\ \rho &= (2\pi^2-\pi)C/Y + \beta[(\pi-1)\pi]^2\sigma\phi \\ \gamma_f &= \frac{(2\pi^2-\pi)C/Y + [(\pi-1)\pi]^2\sigma\phi}{\rho} \\ \gamma_y &= \frac{(\pi^2-\pi)C/Y}{\rho} \\ \gamma_{mc} &= \frac{(\varepsilon - 1 + \phi(\pi^2-\pi)(1-\beta))C/Y}{\phi\rho} \end{aligned}$$

This can be written in matrix algebra form as,

$$A0 \begin{bmatrix} \pi_t \\ y_t \\ i_t \\ E_t\pi_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A1 \begin{bmatrix} \pi_{t-1} \\ y_{t-1} \\ i_{t-1} \\ \pi_t \\ y_t \end{bmatrix}$$

where

$$A0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\alpha_\pi & -\alpha_y & 1 & 0 & 0 \\ -\gamma_{mc}\zeta_c\sigma & -\gamma_y\beta(1-\sigma)+\gamma_{mc}(\sigma+\varphi) & 0 & \gamma_f\beta & \gamma_y\beta(1-\sigma) \\ \zeta_c & 0 & -\frac{1}{\sigma} & -\zeta_c+\frac{1}{\sigma} & 1 \end{bmatrix}$$

$$A1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Calvo Case

$$\pi_t = [\beta\pi + \eta(\theta-1)]E_t\pi_{t+1} + \kappa y_t + \lambda\varphi s_t + \eta E_t\psi_{t+1}$$

$$\psi_t = (1-\sigma)(1-\theta\beta\pi^{\varepsilon-1})y_t + \theta\beta\pi^{\varepsilon-1}[(\varepsilon-1)E_t\pi_{t+1} + E_t\psi_{t+1}]$$

$$s_t = \xi\pi_t + \theta\pi^\varepsilon s_{t-1}$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}E_t(R_t - \pi_{t+1})$$

where

$$\lambda = \frac{(1-\theta\pi^{\varepsilon-1})(1-\theta\beta\pi^\varepsilon)}{\theta\pi^{\varepsilon-1}}$$

$$\eta = \beta(\pi-1)(1-\theta\pi^{\varepsilon-1})$$

$$\kappa = \lambda(\sigma+\varphi) + \eta(1-\sigma)$$

$$\xi = \frac{\varepsilon\theta\pi^{\varepsilon-1}(\pi-1)}{1-\theta\pi^{\varepsilon-1}}$$

$$B0\begin{bmatrix} \pi_t \\ y_t \\ i_t \\ s_t \\ E_t\pi_{t+1} \\ E_t y_{t+1} \\ E_t\psi_{t+1} \end{bmatrix} = B1\begin{bmatrix} \pi_{t-1} \\ y_{t-1} \\ i_{t-1} \\ s_{t-1} \\ \pi_t \\ y_t \\ \psi_t \end{bmatrix}$$

where

$$B0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_\pi & -\alpha_y & 1 & 0 & 0 & 0 & 0 \\ -\xi & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & \lambda\varphi & \beta\pi + \eta(\theta-1) & 0 & \eta \\ 0 & 0 & -\frac{1}{\sigma} & 0 & \frac{1}{\sigma} & 1 & 0 \\ 0 & (1-\sigma)(1-\theta\beta\pi^{\varepsilon-1}) & 0 & 0 & \theta\beta\pi^{\varepsilon-1}(\varepsilon-1) & 0 & \theta\beta\pi^{\varepsilon-1} \end{bmatrix}$$

$$B1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta\pi^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This enables us to assess the determinacy properties of the underlying dynamic systems by considering the eigenvalues of the transition matrices, $A0^{-1}A1$ and $B0^{-1}B1$, in the cases of Rotemberg and Calvo, respectively, We require two roots with modulus in excess of one to ensure determinacy in the case of Rotemberg, and three under Calvo.

Notice that when $\pi = 1$, the linearised systems reduce to.

$$\pi_t = \beta E_t \pi_{t+1} + \frac{\varepsilon - 1}{\phi}[(\sigma + \varphi)y_t - (1+\varphi)a_t]$$

under Rotemberg, and,

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta}[(\sigma + \varphi)y_t - (1+\varphi)a_t]$$

under Calvo, with both representations sharing the same Euler equation,

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}E_t(R_t - \pi_{t+1})$$

Therefore, linearised around a zero inflation steady-state the two systems are identical provided,

$$\frac{\varepsilon - 1}{\phi} = \frac{(1-\theta)(1-\beta\theta)}{\theta}$$

3.A.5 The Model With Time-Varying Tax Rate

To indirectly introduce cost push shock, we consider the revenue tax τ_{pt} which is assumed to follow the following autoregressive process,

$$\ln(1-\tau_{pt}) = (1-\rho^{\tau_p})\ln(1-\tau_p) + \rho^{\tau_p}\ln(1-\tau_{pt-1}) - e_{\tau t}$$

$$e_{\tau t} \overset{i.i.d}{\sim} N\left(0, \sigma_\tau^2\right)$$

With revenue tax τ_{pt}, the expected discounted sum of nominal profits under Rotemberg pricing is given by

$$E_t \sum_{s=0}^{\infty} Q_{t,t+s}\left[(1-\tau_{pt})P_t(j)Y_t(j) - mc_t Y_t(j)P_t - \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)^2 Y_t P_t\right]$$

and under Calvo it can be written as

$$E_t \sum_{s=0}^{\infty} \theta^s Q_{t,t+s}\left[(1-\tau_{pt})P_t(j)Y_{t+s}(j) - mc_{t+s}Y_{t+s}(j)P_{t+s}\right]$$

Based on the derivation of the benchmark model, it is quite straightforward to write down the complete system of non-linear equations describing the discretionary equilibrium. We will use Chebyshev collocation with time iteration method to solve the models with time-varying tax for optimal policy functions.

The Rotemberg Pricing

Since we want to focus on the effect of tax rate, then the technology shock can be shut down by setting $A_t \equiv 1$. This, in fact, can simplify numerical computation.

The Lagrangian is

$$\mathcal{L} = \frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} + \beta E_t\left[V(\tau_{pt+1})\right] + \lambda_{1t}\left\{\left[1-\frac{\phi}{2}(\Pi_t-1)^2\right]N_t - C_t\right\}$$
$$+ \lambda_{2t}\left\{(1-\epsilon)(1-\tau_{pt}) + \epsilon C_t^\sigma N_t^\varphi - \phi\Pi_t(\Pi_t-1) + \phi\beta C_t^\sigma Y_t^{-1}E_t\left[M(\tau_{pt+1})\right]\right\}$$

where λ_{jt} $(j=1,2)$ are the Lagrange multipliers, and

$$M(\tau_{pt+1}) \equiv C_{t+1}^{-\sigma}N_{t+1}\Pi_{t+1}(\Pi_{t+1}-1)$$

The equilibrium conditions for time-consistent policy are,

$$C_t^{-\sigma} = \lambda_{1t} - \lambda_{2t}\left\{\epsilon\sigma C_t^{\sigma-1}N_t^{\varphi} + \sigma\phi\beta C_t^{\sigma-1}N_t^{-1}E_t\left[M(\tau_{pt+1})\right]\right\}$$

$$N_t{}^{\varphi} = \lambda_{1t}\left[1 - \frac{\phi}{2}\left(\Pi_t - 1\right)^2\right] + \lambda_{2t}\left\{\begin{array}{c}\epsilon\varphi N_t{}^{\varphi-1}C_t^{\sigma}\\ -\phi\beta C_t^{\sigma}N_t^{-2}E_t\left[M(\tau_{pt+1})\right]\end{array}\right\}$$

$$\lambda_{1t}\phi\left(1 - \Pi_t\right)N_t = \lambda_{2t}\phi\left(2\Pi_t - 1\right)$$

$$C_t = \left[1 - \frac{\phi}{2}\left(\Pi_t - 1\right)^2\right]N_t$$

$$0 = (1-\epsilon)(1-\tau_{pt}) + \epsilon C_t^{\sigma}N_t^{\varphi} - \phi\Pi_t\left(\Pi_t - 1\right) + \phi\beta\frac{C_t^{\sigma}}{N_t}E_t\left[M(\tau_{pt+1})\right].$$

The Calvo Pricing

Similar to the Rotemberg case, we solve a simpler question by shutting down the technology shock. Then, there are two state variables, τ_{pt} and Δ_{t-1}. The Lagrangian is given as follow:

$$\begin{aligned}\mathcal{L} = &\frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{N_t{}^{1+\varphi}}{1+\varphi} + \beta E_t\left[V(\Delta_t, \tau_{pt+1})\right]\\ &+ \lambda_{1t}[N_t/\Delta_t - C_t]\\ &+ \lambda_{2t}\left[(1-\tau_{pt})\frac{N_t}{\Delta_t C_t^{\sigma}} + \theta\beta E_t\left[L(\Delta_t, \tau_{pt+1})\right] - F_t\right]\\ &+ \lambda_{3t}\left[\frac{N_t^{\varphi+1}}{(1-\epsilon^{-1})\Delta_t} + \theta\beta E_t\left[M(\Delta_t, \tau_{pt+1})\right] - S_t\right]\\ &+ \lambda_{4t}\left[(1-\theta)\left(\frac{1-\theta\Pi_t{}^{\epsilon-1}}{1-\theta}\right)^{\frac{\epsilon}{\epsilon-1}} + \theta\Pi_t^{\epsilon}\Delta_{t-1} - \Delta_t\right]\\ &+ \lambda_{5t}\left[F_t\left(\frac{1-\theta\Pi_t{}^{\epsilon-1}}{1-\theta}\right)^{\frac{1}{1-\epsilon}} - S_t\right]\end{aligned}$$

where λ_{jt} $(j = 1, 2, 3, 4, 5)$ are the Lagrange multipliers, and

$$L(\Delta_t, \tau_{pt+1}) \equiv \Pi_{t+1}^{\epsilon-1}F_{t+1}$$

$$M(\Delta_t, \tau_{pt+1}) \equiv \Pi_{t+1}^{\epsilon}S_{t+1}$$

The equilibrium conditions for time-consistent policy are,

$$C_t = N_t/\Delta_t$$

$$F_t = (1-\tau_{pt})C_t^{1-\sigma} + \theta\beta E_t\left[\Pi_{t+1}^{\epsilon-1}F_{t+1}\right]$$

$$S_t = \frac{N_t^{\varphi+1}}{(1-\epsilon^{-1})\Delta_t} + \theta\beta E_t\left[\Pi_{t+1}^{\epsilon} S_{t+1}\right]$$

$$\Delta_t = (1-\theta)\left(\frac{1-\theta{\Pi_t}^{\epsilon-1}}{1-\theta}\right)^{\frac{\epsilon}{\epsilon-1}} + \theta\Pi_t^{\epsilon}\Delta_{t-1}$$

$$S_t = F_t\left(\frac{1-\theta{\Pi_t}^{\epsilon-1}}{1-\theta}\right)^{\frac{1}{1-\epsilon}}$$

$$0 = 1 - \lambda_{1t}C_t^{\sigma} - \sigma(1-\tau_{pt})\lambda_{2t}$$

$$0 = \Delta_t C_t^{\sigma}{N_t}^{\varphi} - C_t^{\sigma}\lambda_{1t} - (1-\tau_{pt})\lambda_{2t} - \frac{(\varphi+1)\,C_t^{\sigma}N_t^{\varphi}\lambda_{3t}}{(1-\epsilon^{-1})}$$

$$0 = \lambda_{2t} - \lambda_{5t}\left(\frac{1-\theta{\Pi_t}^{\epsilon-1}}{1-\theta}\right)^{\frac{1}{1-\epsilon}}$$

$$0 = \lambda_{3t} + \lambda_{5t}$$

$$\begin{aligned} 0 = \epsilon &\left(\left(\frac{1-\theta{\Pi_t}^{\epsilon-1}}{1-\theta}\right)^{\frac{1}{\epsilon-1}} - \Delta_{t-1}\Pi_t\right)\lambda_{4t} \\ &- \frac{1}{1-\theta}\left(\frac{1-\theta{\Pi_t}^{\epsilon-1}}{1-\theta}\right)^{\frac{\epsilon}{1-\epsilon}}\lambda_{5t}F_t \end{aligned}$$

$$\begin{aligned} 0 = &\frac{C_t}{\Delta_t}\lambda_{1t} + (1-\tau_{pt})\frac{C_t^{1-\sigma}}{\Delta_t}\lambda_{2t} + \frac{N_t^{\varphi}C_t}{(1-\epsilon^{-1})\Delta_t}\lambda_{3t} \\ &+ \lambda_{4t} - \theta\beta\lambda_{2t}E_t\left[L_1(\Delta_t,\tau_{pt+1})\right] - \theta\beta\lambda_{3t}E_t\left[M_1(\Delta_t,\tau_{pt+1})\right] - \theta\beta E_t\left[\Pi_{t+1}^{\epsilon}\lambda_{4t+1}\right] \end{aligned}$$

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4 Optimal Time-Consistent Monetary, Fiscal and Debt Maturity Policy*

We develop a New Keynesian model with bonds of mixed maturity and solve for optimal time-consistent policy using global solution techniques. This reveals several ignored non-linearities in LQ models with one-period debt. Firstly, the steady-state balances an inflation and debt stabilization bias to generate a small negative debt value with a slight undershooting of the inflation target. This falls far short of the first-best asset levels. Secondly, starting from debt levels consistent with currently observed debt-GDP ratios, the optimal policy will gradually reduce that debt, but the policy mix changes radically along the transition path. At high debt levels, there is a reliance on a relaxation of monetary policy to reduce debt through an expanded tax base and reduced debt service costs, while tax rates are used to moderate the increases in inflation. However, as debt levels fall, the use of monetary policy in this way diminishes and the authority turns to fiscal policy to continue debt reduction. This endogenous switch from passive to active monetary policy, possibly accompanied by an active to passive fiscal policy, occurs at higher debt levels the longer the average debt maturity. This is largely because policy makers are tempted to use surprise inflation to erode the real value of debt. This temptation is severe when debt is of shorter maturity, since the inflationary effects of raising taxes to reduce debt become increasingly costly as debt levels rise. Finally, we consider how to optimally vary debt maturity in response to shocks and across varying levels of debt and show that variations in maturity are largely used to support variations in the underlying time-consistent policy mix.

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4.1 Introduction

The recent global financial crisis has led to unprecedented buildups of government debt especially in advanced economies. Figure 4.1 shows that the debt-GDP ratios in advanced economies steadily increased from 73% in 2007 to 105.4% in 2015. This development has prompted an interest from both policymakers and researchers in rethinking the appropriate relationship among monetary policy, fiscal policy and debt management policy. The conventional policy assignment calls upon monetary authorities to determine the level of short-term interest rates in order to control demand and inflation, while the fiscal authorities choose the level of the budget deficit to ensure fiscal sustainability and a debt management office undertakes the technical issue of choosing the maturity and form in which federal debt is issued. With the onset of the 2007/2008 financial crisis and the subsequent easing of monetary policy, the clean lines between these domains have blurred. With short-term interest rates at the zero lower bound, conventional monetary policies have lost their impact. As a result, central banks have resorted to quantitative easing (QE) policies to support aggregate demand. Because QE shortens the maturity structure of debt instruments that private investors have to hold, central banks have effectively entered the domain of debt management policy. For example, the series of open market operations by the Federal Reserve between 2008 and 2014 and the expansion in excess reserves reduced the average duration of U.S. government liabilities by over 20%, from 4.6 years to 3.6 years (Corhay et al., 2014).[27] At the same time, fiscal authorities' debt management offices have been extending the average maturity of the debt to mitigate fiscal risks associated with the government's growing debt burden. These fiscal actions have operated as a kind of reverse quantitative easing, replacing money-like short-term debt with longer-term debt. For instance, the stock of US government debt with a maturity over 5 years that is held by the public (excluding the holdings by the Federal Reserve) has risen from 8 percent of GDP at the end of 2007 to 15 percent at the middle of 2014 (Greenwood et al., 2014) . The observation that monetary and fiscal policies with regard to government debt have been pushing in opposite directions suggests the need to reconsider the principles underlying government debt management policy.

[27] The Fed has been paying interest on reserves since 2008, so that reserves are effectively the same as short maturity treasuries (e.g., Cochrane, 2014).

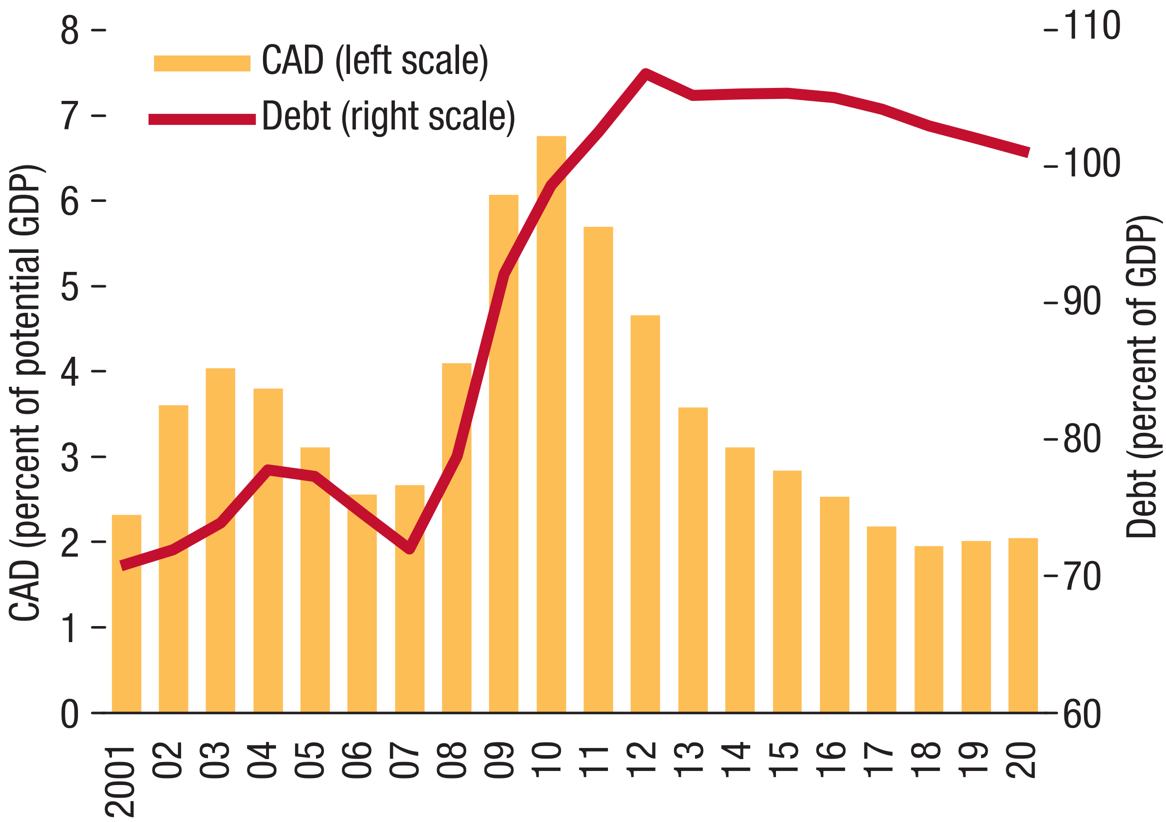

Figure 4.1: The average debt-GDP ratio and the cyclically adjusted deficit as percent of potential GDP in advanced economies. CAD = cyclically adjusted deficit. Source: IMF Fiscal Monitor, 2015.

Against this background, this chapter studies the jointly optimal monetary and fiscal policy when the policymakers can issue a portfolio of bonds of multiple maturities, but cannot commit. A major focus of the chapter is on how debt maturity affects the optimal time-consistent policy mix and equilibrium outcomes in the presence of distortionary taxes and sticky prices. From this normative analysis, we can draw some implications on questions like whether surprise inflation and interest rates other than taxes and government spending should be used to reduce and stabilize debt. Given the magnitude of the required fiscal consolidation in so many advanced economies, the issue as to the optimal policy mix in stabilizing the debt is highly relevant.

In sticky price New Keynesian models with one-period government debt, Schmitt-Grohe and Uribe (2004a) show that even a mild degree of price stickiness implies nearly constant inflation and near random walk behaviour in government debt and tax rates when policymakers can commit to time-inconsistent monetary and fiscal policies, in response to government spending disturbances. In other words, monetary policy should not be used to stabilize debt. However, Sims (2013) questions the robustness of this result when government can issue long-term nominal bonds. With only short term government debt, unexpected current inflation or deflation is the only way to change its market value in cushioning fiscal shocks. In contrast, if debt is long term, large changes in the value of debt can be produced by sustained changes in the nominal interest rate (or asset prices), with much smaller changes in current inflation. Based on these considerations, Sims sketches out a theoretical argument for using nominal debt - of which the real value can be altered with surprise changes in inflation and interest rates - as a cushion against fiscal disturbances to substitute for large movements in distorting taxes. This mechanism is explored further in Leeper and Leith (2017).

Our chapter contributes to the literature along at least three dimensions. Firstly, we take both non-state-contingent short-term and long-term nominal bonds into account. The consideration of long-term debt and the maturity structure is motivated by Sims' theoretical insights as well as the empirical facts. Figure 4.2 (right panel) shows the average debt maturity in a selection of advanced countries is between 2 and 14 years. Secondly, we focus on the time-consistent policy problem which is less studied in the literature. In a linear-quadratic approximation to the policy problem, Leith and Wren-Lewis (2013) show that the time inconsistency inherent in commitment policy means that the optimal time-consistent discretionary policy for debt is quite different. The random walk result, typically, no longer holds, and instead debt returns to its steady-state level following shocks. In addition, time-consistent policy regime is arguably the more appropriate description of policymaking around the world. While the Ramsey policy implies it is optimal to induce a random walk in steady state debt as a result of standard tax smoothing argument, ex ante fiscal authorities typically want to adopt fiscal rules which are actually quite aggressive in stabilising debt. They then typically abandon these rules in the face of adverse shocks. There is therefore a clear failure to adopt fiscal rules which mimic commitment policy. Understanding how optimal time-consistent discretionary policy differs from its time-inconsistent counterpart, in particular the implications for debt, has particular empirical relevance today as governments assess the extent to which they need to reverse the large increases in debt caused by the severe recession, in a context where fiscal policy commitments are often far from credible. Finally, we solve the model non-linearly using the global solution methods which enable us to analyze episodes with sharp increases in debt-GDP ratios as observed in many countries during the global recession. Leith and Wren-Lewis (2013) adopt traditional linear-quadratic (LQ) methods, that is, using some artificial devices to ensure the steady-state being efficient and then linearizing the model around this steady state, while Schmitt-Grohe and Uribe (2004a) adopt log-linear approximation and second-order approximation to the first-order conditions of the Ramsey problem.[28]

[28]In LQ models with long-term debt, Leeper and Zhou (2013) ask some similar questions and they solve the optimal monetary and fiscal policies from the timeless perspective, while Bhattarai et al. (2014) study optimal time consistent monetary and fiscal policies, taking the zero lower bound

In contrast, we are not imposing any kind of approximation around a steady-state so that we can fully explore the effects of non-linearities inherent in the New Keynesian model.[29] In fact, the results under discretion in Leith and Wren-Lewis (2013) suggest that there are massive nonlinearities - for example, there is an overshooting in the debt correction in a single period at higher steady state debt levels, but a more gradual debt reduction following shocks at lower ones. This implies that the optimal speed of debt stabilization is likely to be highly state dependent.

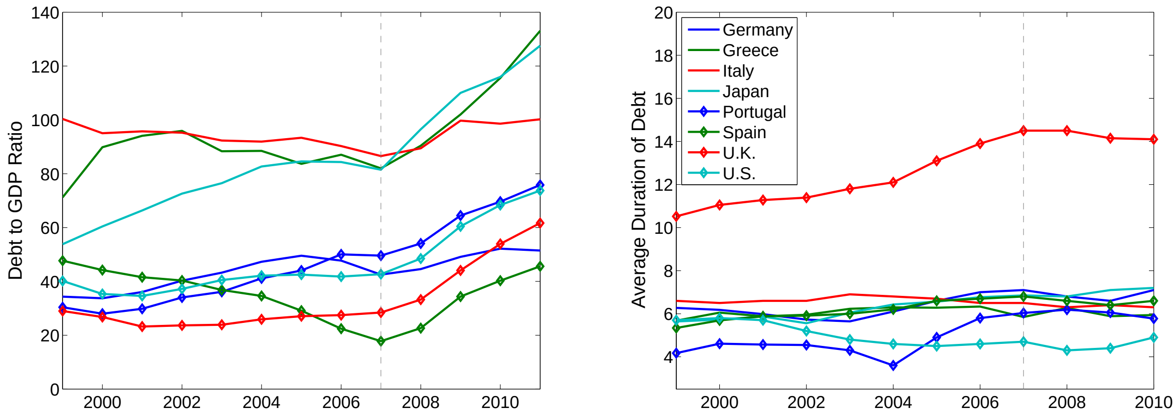

Figure 4.2: This figure shows the evolution of debt-to-GDP ratios and average maturity of debt for a selected group of countries. The debt-to-GDP time series is measured as net financial liabilities as a percentage of nominal GDP; the average maturity of debt is measured as the average term to maturity of total outstanding government debt. Taken from Eusepi and Preston (2013).

To address these non-linearities and the time inconsistency problem which depends on the incentives to induce inflation surprises to deflate debt, we develop a New Keynesian DSGE model augmented with fiscal policy and a portfolio of mixed maturity bonds and solve the optimal time-consistent policy problem using global non-linear solution techniques. In particular, we study how the change in nominal government debt maturity affects optimal monetary and fiscal policy decisions and equilibrium outcomes in the presence of distortionary taxes and sticky prices. In the model, the government cannot commit, and would like to use unexpected inflation to erode the real value of nominal debt. In this way, the government can minimize the need to vary distortionary income taxes over the business cycle. Anticipating this, economic agents raise their inflationary expectations such that high debt levels deliver a state-dependent inflationary bias problem which is particularly costly, due to nominal rigidities. As debt levels fall, the efficacy of surprise inflation as a means of reducing the debt burden also falls,

constraint on nominal interest rate into consideration.

[29] There are some recent papers using global solution techniques which consider optimal discretionary monetary policy with trivial fiscal policy in the New Keynesian models, see Anderson et al. (2010), Van Zandweghe and Wolman (2011), Nakata (2013), Leith and Liu (2014), Ngo (2014) and among others.

reducing the induced inflationary bias problem and influencing the optimal policy mix.

We find the following key results. Firstly, the steady-state balances the inflation and debt stabilization bias to generate a small negative long-run optimal value for debt, which implies a slight undershooting of the inflation target in steady state. This falls far short of the accumulated level of assets that would be needed to finance government consumption and eliminate tax and other distortions. Secondly, starting from levels of debt consistent with currently observed debt-GDP ratios, the optimal policy will gradually reduce that debt, but with radical changes in the policy mix along the transition path. At high debt levels, there is a reliance on a relaxation of monetary policy to reduce debt through an expansion in the tax base and reduced debt service costs, while tax rates are used to moderate the increases in inflation. However, as debt levels fall, the use of monetary policy in this way is diminished and the policy maker turns to fiscal policy to continue the reduction in debt. This is akin to a switch from an active to passive fiscal policy in rule based descriptions of policy, which occurs endogenously under the optimal policy as debt levels fall. It can also be accompanied by a switch from passive to active monetary policy. This switch in the policy mix occurs at higher debt levels the longer the average maturity of government debt. This is largely because high debt levels induce an inflationary bias problem as policy makers face the temptation to use surprise inflation to erode the real value of that debt. This temptation is then more acute when debt is of shorter maturity, since the inflationary effects of raising taxes to reduce debt become increasingly costly as debt levels rise. Finally, we consider how to optimally vary debt maturity in response to shocks and across varying levels of debt. We show that variations in the maturity structure are optimally used to support alterations in the time-consistent policy mix rather than support significantly different speeds of fiscal correction.

Related Literature. This chapter is related to several strands of the optimal fiscal policy literature. We will discuss those that are most closely related in terms of topics and numerical methods.

Our contribution is most closely related to the literature that studies optimal fiscal and monetary policy in sticky price New Keynesian models using non-linear solution techniques. Following the work of Schmitt-Grohe and Uribe (2004a) and Siu (2004), Faraglia et al. (2013) solve a Ramsey problem using global solution methods to examine the implications for optimal inflation of changes in the level and maturity of government debt. We study the discretionary equivalent of this policy, which is radically different. Niemann and Pichler (2011) globally solve for optimal fiscal and monetary policies under both commitment and discretion in an economy exposed to large adverse shocks. Using the same method, Niemann et al. (2013) study time consistent policy in the model of Schmitt-Grohe and Uribe (2004a) and identify a simple mechanism that generates inflation persistence. Government spending is exogenous in the latter two papers which also do not consider long-term debt. Similarly, abstracting from long-term debt, Matveev (2014) compares the efficacy of discretionary government spending and labor income taxes for the purpose of fiscal stimulus at the liquidity trap. The value function iteration method is adopted to deal with the zero lower bound problem. In contrast to these papers, debt of different maturities, time-consistent optimal policy making and endogenous government expenditure are all essential elements in our model.

Aside from the relatively small literature exploring optimal monetary and fiscal policy in non-linear New Keynesian models, there is a vast literature on Ramsey fiscal and monetary policy in the tradition of Lucas and Stokey (1983) which tends to focus on real or flexible price economies. In flexible-price environments, the government's problem consists in financing an exogenous stream of public spending by choosing the least disruptive combination of inflation and distortionary income taxes. In an incomplete-markets version of Lucas and Stokey (1983), Aiyagari et al. (2002) simulate the model globally and show that the level of welfare in Ramsey economies with and without real state-contingent debt is virtually the same. In addition, they reaffirm the random-walk results of debt and taxes from Barro (1979). Angeletos et al. (2013) introduce collateral constraints and a liquidity role for government bonds into Aiyagari et al. (2002). They use the value function iteration method to globally solve the modified model and find that the steady-state level of debt is no longer indeterminate, when government bonds can serve as collateral. Cao (2014) extends Angeletos et al. (2013) with long-term debt and studies how the cost of inflation to commercial banks affects the design of fiscal and monetary policy. Likewise, Faraglia et al. (2014) use global solution methods solve a Ramsey problem with incomplete markets and long-term bonds. They show that many features of optimal policy are sensitive to the introduction of long bonds, in particular tax variability and long run behaviour of debt. Our findings convey the same message that maturity lengths like those observed in actual economies can substantially alter the nature of optimal policies, but the policy problem in our sticky price economy where the policy maker is unable to commit is fundamentally different.

There is also a literature on optimal time consistent fiscal and monetary policy in real models. This literature typically focuses on Markov-perfect policy, where households' and the government's policy rules are functions of payoff-relevant variables only. Local approximations around a non-stochastic steady state are typically infeasible for these models, since optimal behaviour is characterized by generalized Euler equations that involve the derivatives of some equilibrium decision rules, and thus it is impossible to compute the steady state independent of these rules. Hence, as in our contribution, global solution methods are required. Klein and Rios-Rull (2003) compare the stochastic properties of optimal fiscal policy without commitment with those properties under a full-commitment policy in a neoclassical growth model with a balanced government budget, see also Krusell et al. (2006) and Klein et al. (2008). Ortigueira (2006) studies Markov-perfect optimal taxation under a balanced-budget rule, while Ortigueira et al. (2012) deal with the case of unbalanced budgets. In a version of Lucas and Stokey (1983) model with endogenous government expenditure, Debortoli and Nunes (2013) find that when governments cannot commit, debt is no longer indeterminate and often converges to a steady-state with no debt accumulation at all. This is a quite striking difference in the behaviour of debt between the full commitment and the no-commitment cases. Similarly, Grechyna (2013) also considers endogenous government spending in the environment of Lucas and Stokey (1983) with only one-period debt and shows that around the steady state, the properties of the fiscal variables are very similar, regardless of commitment assumptions. More recently, Debortoli et al. (2015) consider a Lucas and Stokey (1983) economy without state-contingent bonds and commitment, and show that the government actively manages its debt positions and can approximate optimal policy by confining its debt instruments to consols. Our chapter shares the same technical problem due to the presence of generalized Euler equations, but nominal rigidities make our model setup quite different from these papers.

Finally, there is also a literature on optimal fiscal and monetary policy in monetary models, which do not contain nominal interia, but which may contain a cost to inflation. Schmitt-Grohe and Uribe (2004b) study Ramsey policy in a flexible-price model with cash-in-advance constraint, which essentially extends the model of Lucas and Stokey (1983) to an imperfectly competitive environment. A global numerical method is used to characterize the dynamic properties of the Ramsey allocation. In a cash-in advance model, Martin (2009) studies the time consistency problems that arise from the interaction between debt and monetary policy, since inflation reduces the real value of nominal liabilities. He uses global solution methods to deal with the generalized Euler equations, see also Martin (2011), Martin (2013) and Martin (2014) where time consistent policies are studied in variants of the monetary search model of Lagos and Wright (2005). In contrast, we abstract from monetary frictions and emphasize nominal price stickiness which is the conventional approach to generating real effects from monetary policy.

Roadmap. The chapter proceeds as follows. We describe the benchmark model in section 4.2. The first best allocations are characterized in 4.3. We study the optimal time consistent policy problem in section 4.4. In section 4.5, we describe the solution method and present the numerical results. In section 4.6 we conclude.

4.2 The Model

Our model is a standard New Keynesian model, but augmented to include the government's budget constraint where government spending is financed by distortionary taxation and/or borrowing. This basic set-up is similar to that in Benigno and Woodford (2003) and Schmitt-Grohe and Uribe (2004a) but with some differences. Firstly, we allow the government to optimally vary government spending in the face of shocks, rather than simply treating government spending as an exogenous flow which must be financed. This is a necessary modification to answer questions like the relative effectiveness of government spending cuts and tax increases in debt stabilization.[30] Secondly, our nominal debt is not of single-period maturity, but consists of a portfolio of bonds of mixed maturities. In reality, most countries issue long-term nominal debt in overwhelming proportions of total debt. This is an important consideration in highly indebted economies, since even modest surprise changes in inflation and interest rates can have substantial effects on the market value of debt, and hence become a sizeable source of fiscal revenue.[31] This fact suggests that the maturity structure of debt is an essential element in characterizing the joint optimal monetary and fiscal policy. Thirdly, we not only take the average debt maturity as exogenously given, but also allow it to optimally vary over the business cycle, see section 4.A.4 in the appendix.

4.2.1 Households

There are a continuum of households of size one. Households appreciate private consumption as well as the provision of public goods and dislike labor. We shall assume complete asset markets, such that, through risk sharing, they will face the same budget constraint. As a result the typical household will seek to maximise the following objective function

[30] International Monetary Fund (2012) reports that current fiscal consolidation efforts rely heavily on government spending cuts. In addition, Bi et al. (2013) introduce ex ante uncertainty over the composition of the fiscal consolidation, either tax based or spending based, and show that the macroeconomic consequences of spending cuts can be quite different from tax increases, even if the direct fiscal consequences are similar.

[31] See Hall and Sargent (2011) and Sims (2013) for the empirical findings on the contribution of this kind of fiscal financing to the decline in the U.S. debt-GDP ratio from 1945 to 1974.

$$E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t, G_t\right) \tag{4.1}$$

where C, G and N are a consumption aggregate, a public goods aggregate, and labour supply respectively.

The consumption aggregate is defined as

$$C_t = \left(\int_0^1 C_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}} \tag{4.2}$$

where j denotes the good's type or variety and $\epsilon > 1$ is the elasticity of substitution between varieties. The public goods aggregate takes the same form

$$G_t = \left(\int_0^1 G_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}} \tag{4.3}$$

The budget constraint at time t is given by

$$\int_0^1 P_t(j)C_t(j)dj + P_t^S B_t^S + P_t^M B_t^M \leq \Xi_t + (1+\rho P_t^M)B_{t-1}^M + B_{t-1}^S + W_t N_t(1-\tau_t)$$

where $P_t(j)$ is the price of variety j , Ξ is the representative household's share of profits in the imperfectly competitive firms, W are wages, and τ is an wage income tax rate[32]. Households hold two basic forms of government bond. The first is the familiar one period debt, B_t^S which has the price equal to the inverse of the gross nominal interest rate, $P_t^S = R_t^{-1}$. The second type of bond is actually a protfolio of many bonds which, following Woodford (2001) pay a declining coupon of ρ^j dollars $j+1$ periods after they were issued where $0 < \rho \leq \beta^{-1}$. The duration of the bond $(1-\beta\rho)^{-1}$, which allows us to vary ρ as a means of changing the implicit maturity structure of government debt. By using such a simple structure we need only price a single bond, since any existing bond issued j periods ago is worth ρ^j new bonds. In the special case where $\rho = 1$ these bonds become infinitely lived consols. When introducing steady state gross inflation of Π^* we assume that the nominal payments on the bonds decline at rate $\widetilde{\rho}\Pi^*$, such that the real rate of coupon decline is $\widetilde{\rho}$, and the duration of the bond without price suprises is, $(1-\beta\widetilde{\rho})^{-1}$. This enables us to vary the steady state rate of inflation

[32]Since fiscal policy is one important element of this chapter, we do not assume any kind of lump-sum-tax-financed subsidy to offset the distortion arising from monopolistic competition, which is a typical assumption in the optimal fiscal and monetary policy literature using New Keynesian models. Thus, the steady-state of the model economy is not efficient. In addition, in the presence of the zero lower bound constraint, policy functions have kinks, therefore an accurate evaluation of optimal policy and welfare requires a global solution method.

without altering the implict maturity structure of government debt.[33]

Households must first decide how to allocate a given level of expenditure across the various goods that are available. They do so by adjusting the share of a particular good in their consumption bundle to exploit any relative price differences - this minimises the costs of consumption. Optimisation of expenditure for any individual good implies the demand function given below,

$$C_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} C_t$$

where we have price indices given by

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}$$

The budget constraint can therefore be rewritten as

$$P_t^S B_t^S + P_t^M B_t^M \leq \Xi_t + (1 + \rho P_t^M) B_{t-1}^M + B_{t-1}^S + W_t N_t (1 - \tau_t) - P_t C_t \tag{4.4}$$

where $\int_0^1 P_t(j) C_t(j) dj = P_t C_t$. P_t is the current price level. The constraint says that total financial wealth in period t can be worth no more than the value of financial wealth brought into the period plus nonfinancial income during the period net of taxes and the value of consumption spending.

For much of the analysis the one period government bond B_t^S is in zero net supply with beginning-of-period price P_t^S, while the general portfolio of government bond B_t^M is in non-zero net supply with beginning-of-period price P_t^M. Higher ρ raises the maturity of the bond portfolio. In extreme case $\rho = 0$, the debt portfolio collapses to one-period debt and if $\rho = 1$, it becomes a consol. We cannot allow the rate of decay on bonds to become time varying without either implictly allowing the government to renege on existing bond contracts or tracking the distribution of bond of different maturities that have been issued in the past. Therefore in order to allow the policy maker to tractably vary the maturity structure we shall consider the case where both B_t^S and B_t^M are potentially in non-zero net supply so that the policy maker can vary the overall maturity of the outstanding debt stock by varying the relative proportion of short and longer-term bonds in that portfolio.

Similarly, the allocation of government spending across goods is determined by minimising total costs, $\int_0^1 P_t(j) G_t(j) dj$. Given the form of the basket of public goods

[33] This way of modeling long-term debt is quite elegant, since it allows us to study long-duration bonds without increasing the dimensionality of the state space, and it is commonly adopted in the literature (e.g., Chen et al., 2012; Eusepi and Preston, 2012).

this implies,

$$G_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} G_t$$

Households' Intertemporal Consumption Problem

The first of the households intertemporal problems involves allocating consumption expenditure across time. For tractability assume that (4.1) takes the specific form

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} + \chi \frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) \tag{4.5}$$

Note that the assumption of a separable utility function is standard in models with political disagreement, such that the provisions of public good does not affect private decisions.

We can then maximise utility subject to the budget constraint (4.4) to obtain the optimal allocation of consumption across time, based on the pricing of one period bonds,

$$\beta R_t E_t \left\{ \left(\frac{C_t}{C_{t+1}}\right)^{\sigma} \left(\frac{P_t}{P_{t+1}}\right) \right\} = 1 \tag{4.6}$$

and the declining payoff consols,

$$\beta E_t \left\{ \left(\frac{C_t}{C_{t+1}}\right)^{\sigma} \left(\frac{P_t}{P_{t+1}}\right) \left(1 + \rho P_{t+1}^M\right) \right\} = P_t^M \tag{4.7}$$

Combining (4.6) and (4.7) yields the no-arbitrage condition between one-period and long-term bonds,

$$P_t^M = E_t \left[P_t^S \left(1 + \rho P_{t+1}^M\right) \right] \tag{4.8}$$

where $P_t^S = R_t^{-1}$ is used. Notice that when these reduce to single period bonds, $\rho = 0$, the price of these bonds is also given by $P_t^M = R_t^{-1}$. However, outside of this special case the longer term bonds introduce the term structure of interest rates to the model. It is convenient to define the stochastic discount factor (for nominal payoffs) for later use,

$$\beta \left(\frac{C_t}{C_{t+1}}\right)^{\sigma} \left(\frac{P_t}{P_{t+1}}\right) = Q_{t,t+1}$$

The second FOC relates to their labour supply decision and is given by,

$$(1 - \tau_t) \left(\frac{W_t}{P_t}\right) = N_t^{\varphi} C_t^{\sigma}$$

That is, the marginal rate of substitution between consumption and leisure equals the after-tax wage rate. Besides these FOCs, necessary and sufficient conditions for household optimization also require the household's budget constraints to bind with equality. In addition, there is an associated no-Ponzi-game condition derived as follows. Define household wealth brought into period t as,

$$D_t = (1 + \rho P_t^M) B_{t-1}^M + B_{t-1}^S$$

the no-Ponzi-game condition can be written as,

$$\lim_{T \to \infty} E_t \left[\frac{1}{R_{t,T}} \frac{D_T}{P_T} \right] \geq 0 \tag{4.9}$$

where

$$R_{t,T} = \prod_{s=t}^{T-1} \left(\frac{1 + \rho P_{s+1}^M}{P_s^M} \frac{P_s}{P_{s+1}} \right)$$

for $T \geq 1$ and $R_{t,t} = 1$, also see Eusepi and Preston (2011). The no-Ponzi-game says that the present discounted value of household's real wealth at infinity is non-negative, that is, there is no overaccumulation of debt. In equilibrium, the equality holds.

4.2.2 Firms

The production function is linear, so for firm j

$$Y_t(j) = A_t N_t(j) \tag{4.10}$$

where $a_t = \ln(A_t)$ is AR(1) such that $a_t = \rho_a a_{t-1} + e_{at}$, with $0 \leq \rho_a < 1$ and $e_{at} \overset{i.i.d}{\sim} N(0, \sigma_a^2)$. The real marginal costs of production is defined as $mc_t = W_t / (P_t A_t)$. The demand curve they face is given by,

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t$$

where $Y_t = \left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right]^{\frac{\epsilon}{\epsilon-1}}$. Firms are also subject to quadratic adjustment costs in changing prices, as in Rotemberg (1982).

We define the Rotemberg price adjustment costs for a monopolistic firm j as,

$$\upsilon_t(j) = \frac{\phi}{2} \left(\frac{P_t(j)}{\Pi^* P_{t-1}(j)} - 1 \right)^2 Y_t \tag{4.11}$$

where $\phi \geq 0$ measures the degree of nominal price rigidity. The adjustment cost, which accounts for the negative effects of price changes on the customer–firm relationship, increases in magnitude with the size of the price change and with the overall scale of economic activity Y_t. Π^* is the targeted rate of inflation.

The problem facing firm j is to maximise the discounted value of profits,

$$\max_{P_t(j)} E_t \sum_{z=0}^{\infty} Q_{t,t+z} \Xi_{t+z}(j)$$

where profits are defined as,

$$\begin{aligned}
\Xi_t(j) &= P_t(j)Y_t(j) - mc_t Y_t(j) P_t - \frac{\phi}{2}\left(\frac{P_t(j)}{\Pi^* P_{t-1}(j)} - 1\right)^2 P_t Y_t \\
&= P_t(j)^{1-\epsilon} P_t^{\epsilon} Y_t - mc_t P_t(j)^{-\epsilon} P_t^{1+\epsilon} Y_t - \frac{\phi}{2}\left(\frac{P_t(j)}{\Pi^* P_{t-1}(j)} - 1\right)^2 P_t Y_t
\end{aligned}$$

So that, in a symmetric equilibrium where $P_t(j) = P_t$ the first order conditions are given by,

$$\begin{aligned}
0 &= (1-\epsilon) + \epsilon mc_t - \phi \frac{\Pi_t}{\Pi^*}\left(\frac{\Pi_t}{\Pi^*} - 1\right) \\
&+ \phi \beta E_t \left[\left(\frac{C_t}{C_{t+1}}\right)^{\sigma} \frac{Y_{t+1}}{Y_t} \frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*} - 1\right)\right]
\end{aligned} \tag{4.12}$$

which is the Rotemberg version of the Phillips curve relationship.

Goods market clearing requires, for each good j,

$$Y_t(j) = C_t(j) + G_t(j) + v_t(j)$$

which allows us to write,

$$Y_t = C_t + G_t + v_t$$

with $v_t = \int_0^1 v_t(j)\, dj$. In a symmetrical equilibrium,

$$Y_t \left[1 - \frac{\phi}{2}\left(\frac{\Pi_t}{\Pi^*} - 1\right)^2\right] = C_t + G_t$$

There is also market clearing in the bonds market where we assume the one period bonds are in zero net supply, $B_t^S = 0$ and the remaining longer term portfolio evolves according to the government's budget constraint which we will now describe.

4.2.3 Government Budget Constraint

The government consists of two authorities. First, there is a monetary authority which controls the nominal interest rates on short-term nominal bonds. Second, there is a fiscal authority deciding on the level of government expenditures, labor income taxes and on debt policy. Government expenditures consist of spending for the provision of public goods and for interest payments on outstanding debt. The level of public goods provision is a choice variable of the government. We assume that monetary and fiscal policy is coordinated by a benevolent policymaker who seeks to maximize household welfare, and the government can credibly commit to repay its debt.

The government expenditures G_t are financed by levying labor income taxes at the rate τ_t, and by issuing one-period, risk free (non-contingent), nominal obligations B_t^S, and long term bonds B_t^M. The government's sequential budget constraint is then given by

$$P_t^M B_t^M + P_t^S B_t^S + \tau_t W_t N_t = P_t G_t + B_{t-1}^S + (1 + \rho P_t^M) B_{t-1}^M$$

Note that the one-period bond is assumed in zero net supply, that is, $B_t^S = 0$, hence the flow budget constraint of the government is simplified into

$$P_t^M B_t^M = (1 + \rho P_t^M) B_{t-1}^M - W_t N_t \tau_t + P_t G_t \tag{4.13}$$

Note that $(1+\rho P_t^M) B_{t-1}^M$ is outstanding government liabilities in period t. Distortionary taxation and spending adjustments are required to service government debt as well as stabilize the economy. Rewriting in real terms

$$P_t^M b_t = (1 + \rho P_t^M) \frac{b_{t-1}}{\Pi_t} - \frac{W_t}{P_t} N_t \tau_t + G_t \tag{4.14}$$

where real debt is defined as, $b_t \equiv B_t^M / P_t$.

Given the nominal nature of debt, monetary policy decisions affect the government budget through two channels: first, the nominal interest rate policy of monetary authority influences directly the nominal return the government has to offer on its instruments; second, nominal interest rate decisions also affect the price level and thereby the real value of outstanding government debt.

In particular, the role of the maturity of government debt can be seen clearly from the government budget constraint. In (4.14), the amount of outstanding real government debt is $P_t^M b_t$, and the period real return on holding government debt is $(1 + \rho P_t^M) / \left(\Pi_t P_{t-1}^M\right)$. If $\rho = 0$, government debt b_t is reduced into one-period debt, and then the only way to adjust real return on bonds ex post is through inflation in the current period Π_t. Large fluctuations in prices can be very costly in the presence of nominal rigidities. However, if government debt has a longer maturity, $0 < \rho < 1$, adjustments in the ex post real return can be engineered via changes in the bond price P_t^M, which depends on inflation in future periods. This means that changes in the real debt return can be produced by a small, but sustained inflation, which is less costly than equivalent large fluctuations in inflation. As a result, long-term debt helps the policy maker achieve the desired adjustment in the ex post real return at a smaller cost.

That completes the description of our model which contains the usual resource constraint, consumption Euler equation and New Keynesian Phillips curve as well as the government's budget constraint and the bond pricing equation for longer-term bonds. These equations and the debt-dependent steady state are described in the Appendix 4.A.2.

4.3 First-Best Allocation

In some analyses of of optimal fiscal policy (e.g., Aiyagari et al., 2002), it is desirable for the policy maker to accumulate a 'war chest' which pays for government consumption and/or fiscal subsidies to correct for other market imperfections. In order to assess to what extent our optimal, but time-consistent policy attempts to do so it is helpful to define the level of government accumulated assets that would be necessary to mimic the social planner's allocation under the decentralised solution. The first step in doing so is defining the first-best allocation that would be implemented by the social planner. The social planner ignores the nominal inertia and all other inefficiencies and chooses real allocations that maximize the representative consumer's utility subject to the aggregate resource constraint and the aggregate production function. That is, the first-best allocation $\{C_t^*, N_t^*, G_t^*\}$ is the one that maximizes utility (4.38), subject to the technology constraint (4.37), and aggregate resource constraint $Y_t = C_t + G_t$.

The Lagrangian is

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left(\frac{(C_t^*)^{1-\sigma}}{1-\sigma} + \chi \frac{(G_t^*)^{1-\sigma_g}}{1-\sigma_g} - \frac{(N_t{}^*)^{1+\varphi}}{1+\varphi} \right) + \lambda_t \left[A_t N_t{}^* - C_t^* - G_t^* \right] \right\}$$

The first order conditions imply that

$$(C_t^*)^{-\sigma} = \chi \left(G_t^*\right)^{-\sigma_g} = \left(N_t{}^*\right)^{\varphi} / A_t = \left(Y_t{}^*\right)^{\varphi} A_t^{-(1+\varphi)}$$

That is, given the resource constraints, it is optimal to equate the marginal utility of private and public consumption to the marginal disutility of labor effort.

The optimal share of government consumption in output is

$$\frac{G_t^*}{Y_t^*} = \chi^{\frac{1}{\sigma_g}} \left(\frac{Y_t^*}{A_t}\right)^{-\frac{\varphi+\sigma_g}{\sigma_g}} A_t^{\frac{1-\sigma_g}{\sigma_g}}$$

In steady state (technology level A normalized to unity),

$$\frac{G^*}{Y^*} = \chi^{\frac{1}{\sigma_g}} \left(Y^*\right)^{-\frac{\varphi+\sigma_g}{\sigma_g}}$$

where Y^* can be solved from the aggregate resource constraint

$$\left(Y^*\right)^{-\frac{\varphi}{\sigma}} + \chi^{\frac{1}{\sigma_g}} \left(Y^*\right)^{-\frac{\varphi}{\sigma_g}} = Y^*.$$

In particular, when $\sigma = \sigma_g$,

$$Y^* = \left(1 + \chi^{\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma+\varphi}}$$

and hence,

$$\frac{G^*}{Y^*} = \left(1 + \chi^{-\frac{1}{\sigma}}\right)^{-1}$$

It is illuminating to contrast the allocation achieved in the steady state of the decentralized equilibrium with the first best allocation. We do this by finding policies and prices that make the first-best allocation and the decentralized equilibrium collide. Note that

$$\left(Y^*\right)^{\varphi+\sigma} \left(1 - \frac{G^*}{Y^*}\right)^{\sigma} = 1 \tag{4.15}$$

Comparing (4.39) and (4.15), and assuming the steady state share of government consumption is the same, then the two allocations will be identical when the labor income tax rate is set optimally to be,

$$\tau^* = 1 - \frac{\epsilon}{\epsilon - 1} = \frac{-1}{\epsilon - 1} \tag{4.16}$$

Notice that the optimal tax rate is negative, that is, it is effectively a subsidy which offsets the monopolistic competition distortion. This in turn, implies that the government accumulate a stock of assets which imply,

$$b^* = \frac{\Pi^* - \beta\rho}{1-\beta} \left[\frac{-1}{\epsilon} - \left(1 + \chi^{-\frac{1}{\sigma}}\right)^{-1}\right] \left(1 + \chi^{\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma+\varphi}}$$

and

$$\frac{P^{M*} b^*}{4Y^*} = \frac{\beta}{4\left(1-\beta\right)} \left[\frac{-1}{\epsilon} - \left(1 + \chi^{-\frac{1}{\sigma}}\right)^{-1}\right]$$

Using our benchmark calibration below this would imply $b^* = -2.47$. That is, a stock of assets of 843.75% of GDP would be required to generate sufficient income to pay for government consumption and a labor income subsidy which undoes the effects of the monopolistic competition distortion. We shall see that the steady-state level of debt in our optimal policy problem while negative, falls far short of this 'war chest' value.

4.4 Optimal Policy Under Discretion

We assume that the policymaker cannot credibly commit to particular future policy actions. Instead, the policymaker reassesses his policy response each period, that is, this policy is time-consistent. In our model, the presence of government debt makes the optimal time-consistent policy history dependent, that is, the future path of the policy instruments depends on today's level of government debt.

The policy under discretion can be described as a set of decision rules for

$$\{C_t, Y_t, \Pi_t, b_t, \tau_t, G_t\}$$

which maximise,

$$V(b_{t-1}, A_t) = \max \left\{ \frac{C_t^{1-\sigma}}{1-\sigma} + \chi \frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi} + \beta E_t \left[V(b_t, A_{t+1}) \right] \right\}$$

subject to (4.34), (4.35), and

$$\begin{aligned} &\beta E_t \left\{ \left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \left(\frac{P_t}{P_{t+1}} \right) \left(1 + \rho P^M_{t+1} \right) \right\} b_t \\ &= \left(1 + \rho \beta E_t \left\{ \left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \left(\frac{P_t}{P_{t+1}} \right) \left(1 + \rho P^M_{t+1} \right) \right\} \right) \frac{b_{t-1}}{\Pi_t} \\ &- \left(\frac{\tau_t}{1-\tau_t} \right) \left(\frac{Y_t}{A_t} \right)^{1+\varphi} C_t^{\sigma} + G_t \end{aligned} \tag{4.17}$$

where we have used the bond pricing equation (4.33) to eliminate the current value of the bond in (4.36).

Defining auxilliary functions,

$$M(b_t, A_{t+1}) = (C_{t+1})^{-\sigma} Y_{t+1} \frac{\Pi_{t+1}}{\Pi^*} \left(\frac{\Pi_{t+1}}{\Pi^*} - 1 \right)$$

$$L(b_t, A_{t+1}) = (C_{t+1})^{-\sigma} (\Pi_{t+1})^{-1} (1 + \rho P^M_{t+1})$$

we can write the constraints (4.35) and (4.17) facing the policy maker as,

$$(1-\epsilon)+\epsilon(1-\tau_t)^{-1}Y_t^{\varphi}C_t^{\sigma}A_t^{-1-\varphi}-\phi\frac{\Pi_t}{\Pi^*}\left(\frac{\Pi_t}{\Pi^*}-1\right)+\phi\beta C_t^{\sigma}Y_t^{-1}E_t\left[M(b_t,A_{t+1})\right]=0 \quad (4.18)$$

$$\begin{aligned}0 &= \beta b_t C_t^{\sigma}E_t\left[L(b_t,A_{t+1})\right]-\frac{b_{t-1}}{\Pi_t}\left(1+\rho\beta C_t^{\sigma}E_t\left[L(b_t,A_{t+1})\right]\right) \\ &+\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi}C_t^{\sigma}-G_t\end{aligned} \quad (4.19)$$

By using the auxiliary functions in this way, we take account of the fact that the policy maker recognises the impact their actions have on the endogenous state, but that they cannot commit to future policy actions beyond that i.e. we have a time-consistent policy. Therefore the Lagrangian for the policy problem can be written as,

$$\begin{aligned}\mathcal{L} &= \left\{\frac{C_t^{1-\sigma}}{1-\sigma}+\chi\frac{G_t^{1-\sigma_g}}{1-\sigma_g}-\frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi}+\beta E_t[V(b_t,A_{t+1})]\right\} \\ &+\lambda_{1t}\left[Y_t\left(1-\frac{\phi}{2}\left(\frac{\Pi_t}{\Pi^*}-1\right)^2\right)-C_t-G_t\right] \\ &+\lambda_{2t}\left[\begin{array}{c}(1-\epsilon)+\epsilon(1-\tau_t)^{-1}Y_t^{\varphi}C_t^{\sigma}A_t^{-1-\varphi}-\phi\frac{\Pi_t}{\Pi^*}\left(\frac{\Pi_t}{\Pi^*}-1\right) \\ +\phi\beta C_t^{\sigma}Y_t^{-1}E_t\left[M(b_t,A_{t+1})\right]\end{array}\right] \\ &+\lambda_{3t}\left[\begin{array}{c}\beta b_t C_t^{\sigma}E_t\left[L(b_t,A_{t+1})\right]-\frac{b_{t-1}}{\Pi_t}\left(1+\rho\beta C_t^{\sigma}E_t\left[L(b_t,A_{t+1})\right]\right) \\ +\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi}C_t^{\sigma}-G_t\end{array}\right]\end{aligned}$$

We can write the first order conditions for the policy problem as follows:

Consumption,

$$\begin{aligned}&C_t^{-\sigma}-\lambda_{1t}+\lambda_{2t}\left[\sigma\epsilon(1-\tau_t)^{-1}Y_t^{\varphi}C_t^{\sigma-1}A_t^{-1-\varphi}+\sigma\phi\beta C_t^{\sigma-1}Y_t^{-1}E_t\left[M(b_t,A_{t+1})\right]\right] \\ &+\lambda_{3t}\left[\begin{array}{c}\sigma\beta b_t C_t^{\sigma-1}E_t\left[L(b_t,A_{t+1})\right]-\rho\sigma\beta\frac{b_{t-1}}{\Pi_t}C_t^{\sigma-1}E_t\left[L(b_t,A_{t+1})\right] \\ +\sigma\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi}C_t^{\sigma-1}\end{array}\right]=0\end{aligned} \quad (4.20)$$

which says that higher consumption increases utility, tightens the resource constraint ($\lambda_{1t} \geq 0$), has adverse effects on the inflation-output tradeoffs at time t ($\lambda_{2t} \leq 0$), and relaxes the government budget constraint ($\lambda_{3t} \geq 0$);

Government spending,

$$\chi G_t^{-\sigma_g}-\lambda_{1t}-\lambda_{3t}=0 \quad (4.21)$$

which says that higher government spending increases utility, tightens the resource constraint ($\lambda_{1t} \geq 0$), and tightens the government budget constraint ($\lambda_{3t} \geq 0$);

Output,

$$-Y_t^{\varphi} A_t^{-1-\varphi} + \lambda_{1t} \left[1 - \frac{\phi}{2} \left(\frac{\Pi_t}{\Pi^*} - 1 \right)^2 \right]$$

$$+\lambda_{2t} \left[\epsilon\varphi(1-\tau_t)^{-1} Y_t^{\varphi-1} C_t^{\sigma} A_t^{-1-\varphi} - \phi\beta C_t^{\sigma} Y_t^{-2} E_t \left[M(b_t, A_{t+1}) \right] \right]$$

$$+ \lambda_{3t} \left[(1+\varphi) Y_t^{\varphi} C_t^{\sigma} \left(\frac{\tau_t}{1-\tau_t} \right) A_t^{-1-\varphi} \right] = 0 \tag{4.22}$$

which says that higher output (requiring higher labor) decreases utility, relaxes the resource constraint ($\lambda_{1t} \geq 0$), has adverse effects on the inflation-output tradeoffs at time t ($\lambda_{2t} \leq 0$), and relaxes the government budget constraint ($\lambda_{3t} \geq 0$);

Taxation,

$$\lambda_{2t} \left[\epsilon(1-\tau_t)^{-2} Y_t^{\varphi} C_t^{\sigma} A_t^{-1-\varphi} \right] + \lambda_{3t} \left[Y_t^{1+\varphi} C_t^{\sigma} (1-\tau_t)^{-2} A_t^{-1-\varphi} \right] = 0$$

simplifying,

$$\epsilon\lambda_{2t} + \lambda_{3t} Y_t = 0 \tag{4.23}$$

which says that higher tax rate has adverse effects on the inflation-output tradeoffs at time t ($\lambda_{2t} \leq 0$), and relaxes the government budget constraint ($\lambda_{3t} \geq 0$);

Inflation,

$$-\lambda_{1t} \left[Y_t \frac{\phi}{\Pi^*} \left(\frac{\Pi_t}{\Pi^*} - 1 \right) \right] - \lambda_{2t} \left[\frac{\phi}{\Pi^*} \left(\frac{2\Pi_t}{\Pi^*} - 1 \right) \right]$$

$$+ \lambda_{3t} \left[\frac{b_{t-1}}{\Pi_t^2} \left(1 + \rho\beta C_t^{\sigma} E_t \left[L(b_t, A_{t+1}) \right] \right) \right] = 0 \tag{4.24}$$

which says that higher inflation rate tightens the resource constraint ($\lambda_{1t} \geq 0$), has positive effects on the inflation-output tradeoffs at time t ($\lambda_{2t} \leq 0$), and relaxes the government budget constraint ($\lambda_{3t} \geq 0$);

Government debt,

$$\beta E_t[V_1(b_t, A_{t+1})] + \lambda_{2t} \left[\phi\beta C_t^{\sigma} Y_t^{-1} E_t \left[M_1(b_t, A_{t+1}) \right] \right]$$

$$+\beta\lambda_{3t} \left[C_t^{\sigma} E_t \left[L(b_t, A_{t+1}) \right] + b_t C_t^{\sigma} E_t \left[L_1(b_t, A_{t+1}) \right] - \rho \frac{b_{t-1}}{\Pi_t} C_t^{\sigma} E_t \left[L_1(b_t, A_{t+1}) \right] \right] = 0$$

where

$$V_1(b_t, A_{t+1}) \equiv \partial V(b_t, A_{t+1}) / \partial b_t$$

$$L_1(b_t, A_{t+1}) \equiv \partial L(b_t, A_{t+1})/\partial b_t$$

$$M_1(b_t, A_{t+1}) \equiv \partial M(b_t, A_{t+1})/\partial b_t$$

Note that by the envelope theorem,

$$V_1(b_{t-1}, A_t) = -\lambda_{3t}\frac{1}{\Pi_t}\left(1 + \rho\beta C_t^{\sigma} E_t\left[L(b_t, A_{t+1})\right]\right)$$

we can write the FOC for government debt as,

$$\lambda_{2t}\left[\phi\beta C_t^{\sigma} Y_t^{-1} E_t\left[M_1(b_t, A_{t+1})\right]\right] + \beta\lambda_{3t}\left[\begin{array}{c} C_t^{\sigma} E_t\left[L(b_t, A_{t+1})\right] + b_t C_t^{\sigma} E_t\left[L_1(b_t, A_{t+1})\right] \\ -\rho\frac{b_{t-1}}{\Pi_t} C_t^{\sigma} E_t\left[L_1(b_t, A_{t+1})\right] \end{array}\right]$$

$$-\beta E_t\left[\lambda_{3t+1}\frac{1}{\Pi_{t+1}}(1 + \rho\beta C_{t+1}^{\sigma} E_{t+1}\left[L(b_{t+1}, A_{t+2})\right])\right] = 0 \tag{4.25}$$

which says that higher stock of debt b_t has adverse effects on the inflation-output tradeoffs at time t ($\lambda_{2t} \leq 0$), tightens the time t government budget constraint ($\lambda_{3t} \geq 0$), while relaxes the government budget constraint at time $t+1$ ($\lambda_{3t+1} \geq 0$).

The discretionary equilibrium is determined by the system given by the FOCs, (4.20), (4.21), (4.22), (4.23), (4.24), (4.25), and the constraints, (4.34), (4.18) and (4.19), and finally the exogenous process for the technology shock,

$$a_t = \rho_a a_{t-1} + e_{at}$$

where $a_t = \ln A_t$, and $e_{at} \overset{i.i.d}{\sim} N(0, \sigma_a^2)$.

Note there is a two period ahead expectation implicit in (4.25), related to the forward pricing of future longer term bonds. This can be simplified as,

$$\underbrace{\beta\lambda_{3t} C_t^{\sigma} E_t\left[L(b_t, A_{t+1})\right] - \beta E_t\left[\frac{\lambda_{3t+1}}{\Pi_{t+1}}(1 + \rho P_{t+1}^M)\right]}_{\text{tradeoff between current and future distortions}}$$

$$\underbrace{+\lambda_{2t}\phi\beta C_t^{\sigma} Y_t^{-1} E_t\left[M_1(b_t, A_{t+1})\right] + \beta\lambda_{3t}\left[b_t C_t^{\sigma} E_t\left[L_1(b_t, A_{t+1})\right] - \rho\frac{b_{t-1}}{\Pi_t} C_t^{\sigma} E_t\left[L_1(b_t, A_{t+1})\right]\right]}_{\text{additional terms due to lack of commitment}} = 0 \tag{4.26}$$

since (4.7) implies that

$$P_t^M = \beta C_t^{\sigma} E_t\left[L(b_t, A_{t+1})\right] \tag{4.27}$$

Note that (4.26) is a generalized Euler equation, which involves the derivatives of the equilibrium policy rules with respect to the state variable, inherited debt stock. The standard tradeoff between current and future distortions, reflecting in the wedge between λ_{3t} and λ_{3t+1}, leads to the tax-smoothing argument as in Barro (1979). The presence of partial derivative of debt is due to a time-consistency problem which is caused by the fact that future government will not internalize how its policy affects current actions. This tradeoff between tax-smoothing and the time-consistency problem determines the equilibrium level of debt.

We can solve the nonlinear system consisting of these six first order conditions, the three constraints and (4.27) to yield the time-consistent optimal policy. Specifically, we need to find these ten time-invariant Markov-perfect equilibrium policy rules which are functions of the two state variables $\{b_{t-1},\ A_t\}$. That is, we need to find policy functions such as $b_t = b\left(b_{t-1}, A_t\right)$, $\tau_t = \tau\left(b_{t-1}, A_t\right)$, and $\Pi_t = \Pi\left(b_{t-1}, A_t\right)$.

4.5 Numerical Analysis

4.5.1 Solution Method and Calibration

For the model described in the previous section, the equilibrium policy functions cannot be computed in closed form. We thus resort to computational methods and derive numerical approximations to the policy rules. Local approximation methods are not applicable for this purpose because the model's steady state around which local dynamics should be approximated is endogenously determined as part of the model solution and thus a priori unknown. In light of this difficulty, we resort to a global solution method. Specifically, We use the Chebyshev collocation with time iteration method to solve the model.[34] The detailed algorithm is presented in section 4.A.3 in the appendix. In general, optimal discretionary policy problems can be characterized as a dynamic game between the private sector and successive governments. Multiplicity of equilibria is a common problem in dynamic games. One strategy has been to focus on equilibria with continuous strategies, see Judd (2004) for a discussion. Since we use polynomial approximations, we are searching only for continuous Markov-perfect equilibria where agents condition their strategies only on payoff-relevant state variables.

Before solving the model numerically, the benchmark values of structural parameters must be specified. The calibration of parameters is summarized in Table 4.1. We set $\beta = (1/1.02)^{1/4} = 0.995$, which is a standard value for models with quarterly data and implies 2% annual real interest rate. The intertemporal elasticity of substitution

[34]See Judd (1998) for a textbook treatment.

is set to one half ($\sigma = \sigma^g = 2$) which is in the middle of the parameter range typically considered in the literature. Labor supply elasticity is set to $\varphi^{-1} = 1/3$. The elasticity of substitution between intermediate goods is chosen as $\epsilon = 21$, which implies a monopolistic markup of approximately 5%, similar to Siu (2004). The decaying parameter of coupon rate $\rho = 0.95$, corresponds to $4 \sim 5$ years of debt maturity, consistent with US data. The scaling parameter $\chi = 0.055$ ensures that the share of government spending in output is about 19%. The technology parameters are set to $\rho_a = 0.95$ and $\sigma_a = 0.01$. The price adjustment cost parameter $\phi = 32.5$ - implying that on average firms re-optimize prices every four to six months - is well in line with empirical evidence. Finally, annual rate of inflation target is chosen to be 2%, which is current target adopted by most of the inflation targeting economies.

With this benchmark parameterization, we solve the fully nonlinear models via the Chebyshev collocation method. The maximum Euler equation error over the full range of the grid is of the order of 10^{-6}. As suggested by Judd (1998), this order of accuracy is reasonable.

Table 4.1: Parameterization

Parameter	Value	Definition
β	0.995	Quarterly discount factor
σ	2	Relative risk aversion coefficient
σ^g	2	Relative risk aversion coefficient for government spending
φ	3	Inverse Frish elasticity of labor supply
ϵ	21	Elasticity of substitution between varieties
ρ	0.95	Debt maturity structure
χ	0.055	Scaling parameter associated with government spending
ρ_a	0.95	AR-coefficient of technology shock
σ_a	0.01	Standard deviation of technology shock
ϕ	32.5	Rotemberg adjustment cost coefficient
Π^*	2%	Annual inflation rate target

4.5.2 Numerical Results

In this section we explore the properties of the equilibrium under the optimal time-consistent policy. We begin by considering the steady-state under our benchmark calibration, before turning to the transitional dynamics that drive the economy to that steady-state which highlights the state-dependent nature of the optimal policy mix. We then turn to consider the role of debt maturity in these results, focusing on the impact of debt maturity on the inflationary bias problem and the sensitivity of the policy to the level of government debt. We conclude by allowing the policy making to issue both short and longer-term bonds and show that this enables the policy maker to import some of the policy mix associated with short-term debt even though the bulk of its debt portfolio is longer-term debt.

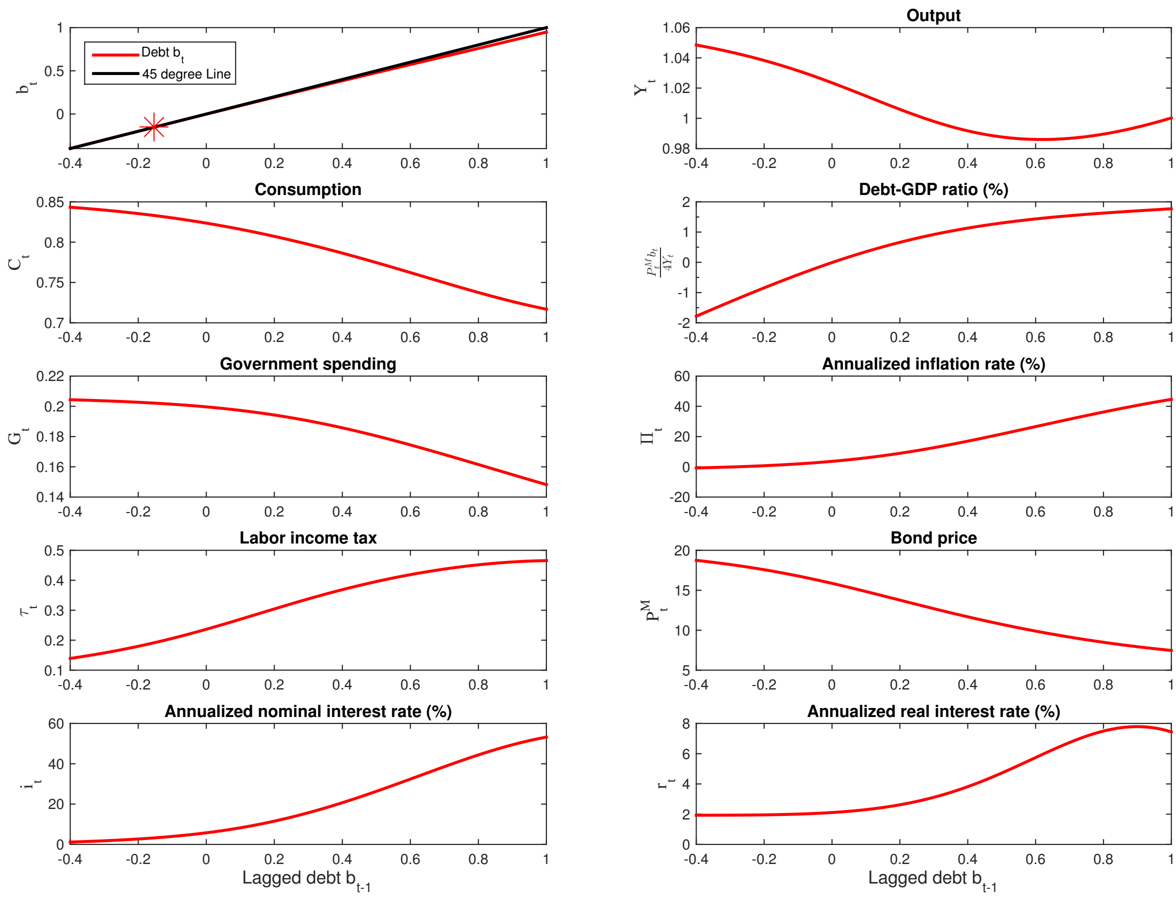

Figure 4.3: Under the benchmark parameters, the policy rules as functions of lagged debt, when the grid for technology is fixed at A=1.

Table 4.2: The steady state under the benchmark parameterization

Variable	Steady State Value	Definition
b	-0.1523	real long term debt
Y	1.0352	output
C	0.8330	consumption
$P^M b/(4Y) \times 100$	-63.34%	debt-GDP ratio in terms of annual output
G	21	government spending
$(\Pi^4 - 1) \times 100$	1.2455	annualized inflation rate
τ	0.1918	income tax rate
P^M	17.2146	long-term bond price
i	3.2441%	annualized nominal interest rate
r	1.9741%	annualized real interest rate

Steady State

We begin by plotting the policy functions for our benchmark calibration in order to assess the steady-state of our optimal policy problem. Figure 4.3 plots policy rules against lagged debt, with the grid for A_t fixed at 1. The first subplot illustrates how to find the steady state debt associated with the time consistent equilibrium. We should note that the long-run debt under the benchmark parameters is negative (-0.1523) which is bigger than the first best value (-2.47) implying a stock of assets of 63.34% of GDP rather than the 'war chest' value of 843.75% of GDP. The interaction of inflation and debt stabilization bias generate a small negative long-run optimal value for debt, which falls far short of the accumulated level of assets that would be needed to finance government consumption and eliminate tax and other distortions. In addition, the annualized inflation rate in the steady state is 1.25% less than the target of 2%. That is, there is a slight undershooting of the inflation target in steady state. Table 4.2 summarizes the steady state values.

In standard analysis of the inflationary bias problem the magnitude of the bias is determined by the exogenously given degree of monopolistic competition which implies that the equilibrium level of output is inefficiently low. In the presence of debt and distortionary taxation, at higher debt/tax levels the inefficiency is more pronounced and the desire to generate a surprise inflation is greater, ceteris paribus. At the same time, any surprise inflation reduces the real value of government debt and mitigates the costs of distortionary taxation and the associated inflationary bias in the future - we follow Leith and Wren-Lewis (2013) in labelling this the "debt stabilization bias". As a result, the policy maker will seek to reduce debt levels to mitigate the costs of distortionary taxation and the endogenous inflationary bias problem. However, once debt turns negative, the policy maker faces a trade-off. Any surprise inflation will boost output, moving it closer to the efficient level. However, any surprise inflation when the government is holding a net stock of nominal assets rather than liabilities will reduce the real value of those assets and thereby worsen the future inefficiencies in the economy. The steady-state then balances these opposing forces, such that there is a small stock of positive government assets and a mild deflationary bias beyond which the government is not tempted to induce further deflationary surprises, as this would worsen output levels in the short-run, even though they would lead to a greater stock of assets in the longer run.

In addition, Table 4.3 shows that debt maturity has a nonlinear effect on the steady state debt-GDP ratio and inflation rate. The time-consistent level of accumulated assets held by the government first decreases, and then increases, as the average maturity of debt lengthens. Correspondingly, the undershooting of inflation target becomes less severe initially, while deteriorates afterwards. The intuition can be understood as follows. As noted above, the policy maker essentially faces two biases - the conventional inflation bias where the policy maker wishes to induce a surprise inflation to boost activity in a sub-optimally small economy, and a debt stabilization bias where the policy maker wishes to use surprise inflation to reduce the value of debt or increase the real value of its nominal assets. At low maturity levels, in steady state the inflationary bias dominates such that inflation lies above its target value. As maturity levels rise slightly, the inflationary bias falls, the government accumulates a larger stock of assets which support lower tax rates, even though government consumption as a share of GDP rises slightly. As maturity rises more, the debt stabilization bias starts to outweigh the inflation bias and steady state inflation lies below target, due to the stock of nominal assets the government has accumulated. These nominal asset stocks, along with falls in government consumption relative to GDP help support the reduced tax rates. It should be noted that the movements in debt to GDP ratios, tax rates and the share of public consumption in output are not entirely monotonic as maturity levels change, reflecting the balancing of the two forms of bias and their associated impact on the policy mix differences emphasised elsewhere in the chapter. However, the overwhelming tendency is for the debt stabilization bias to prevent the policy maker accumulating a war chest of nominal assets sufficient to finance all government activities, and this is particularly the case when debt is of a shorter maturity.

Table 4.3: The steady states under alternative maturities

Parameter		Steady State Values							
maturity		real value of debt	debt-GDP ratio	government spending	consumption	annualized inflation rate	income tax rate	tax base	welfare
ρ	years	b	$\frac{100P^M b}{4Y}$	$\frac{G}{Y}$	$\frac{C}{Y}$	$100\,(\Pi^4 - 1)$	τ	wN	$\frac{V}{1-\beta}$
0	0.25	−0.74	−17.80%	19.94%	80.05%	2.53%	20.57%	0.7820	−352.1533
0.1	0.28	−0.79	−21.01%	19.96%	80.04%	2.44%	20.53%	0.7826	−352.1438
0.2	0.31	−0.85	−25.29%	19.99%	80.01%	2.32%	20.46%	0.7834	−352.1314
0.3	0.35	−0.92	−31.17%	20.01%	79.99%	2.17%	20.37%	0.7846	−352.1146
0.4	0.41	−1.00	−39.45%	20.04%	79.96%	1.98%	20.23%	0.7864	−352.0913
0.5	0.49	−1.08	−51.22%	20.06%	79.94%	1.72%	20.00%	0.7891	−352.0586
0.6	0.61	−1.13	−66.61%	20.06%	79.94%	1.40%	19.68%	0.7929	−352.0151
0.7	0.81	−1.01	−79.26%	19.99%	80.00%	1.11%	19.36%	0.7966	−351.9716
0.7588	1	−0.83	−80.87%	19.92%	80.07%	1.03%	19.24%	0.7977	−351.9551
0.8844	2	−0.36	−70.43%	19.67%	80.33%	1.12%	19.19%	0.7973	−351.9496
0.9263	3	−0.22	−65.10%	19.57%	80.42%	1.22%	19.20%	0.7968	−351.9550
0.9472	4	−0.16	−63.52%	19.53%	80.46%	1.24%	19.18%	0.7968	−351.9565
0.9598	5	−0.13	−63.59%	19.50%	80.49%	1.23%	19.15%	0.7971	−351.9555
0.9681	6	−0.11	−64.47%	19.49%	80.51%	1.20%	19.11%	0.7975	−351.9531
0.9741	7	−0.09	−66.24%	19.47%	80.52%	1.14%	19.05%	0.7981	−351.9493
0.9786	8	−0.08	−67.98%	19.46%	80.53%	1.08%	19.01%	0.7986	−351.9456
0.9821	9	−0.07	−70.27%	19.45%	80.54%	1.01%	18.95%	0.7992	−351.9412
0.9849	10	−0.07	−72.52%	19.45%	80.54%	0.94%	18.90%	0.7998	−351.9373

Transition Dynamics and the Policy Mix

Before plotting the transition dynamics, it is helpful to consider the non-linearities implied by the policy functions plotted in Figure 4.3. Here we can see that inflation is rising steeply with the level of debt, as the endogenous inflationary bias problem worsens with rising debt levels. We can also see how the policy response varies with debt levels - as debt levels rise we see reduced government consumption, higher tax rates and as higher debt levels raise inflation, a rise in real interest rates. However, once debt levels rise sufficiently we can see that the rise in labor income tax rates slows and real interest rates start to fall. This suggests that we may start to see a change in the policy mix as we transition from high levels of debt towards the steady state.

Figure 4.4 plots the transition dynamics starting from a high level of debt given the benchmark calibration. Here we can see the non-linearities implicit in the policy functions plotted in Figure 4.3. At vary high initial levels of debt, we have a massive inflationary bias problem (with annualized inflation in excess of 40%), and as a result, the policy maker is acting to reduce the level of debt fairly rapidly. To do so, they cut government spending and raise labor income tax rates. As as result of the high inflation, they also raise real interest rates. This is in line with the conventional monetary and fiscal policy assignment - fiscal policy is stabilising debt and monetary policy is raising real interest rates to reduce aggregate demand and, thereby, inflation. However, looking closely at the start of the transition when debt levels are particularly high, we see a different policy mix - real interest rates are rising in the first few periods as inflation and debt fall. Essentially in the first few periods debt levels are so high that monetary policy is slightly moderated to mitigate the effects of raised debt service costs. We shall now show that these changes in the policy mix are highly dependent on the maturity structure.

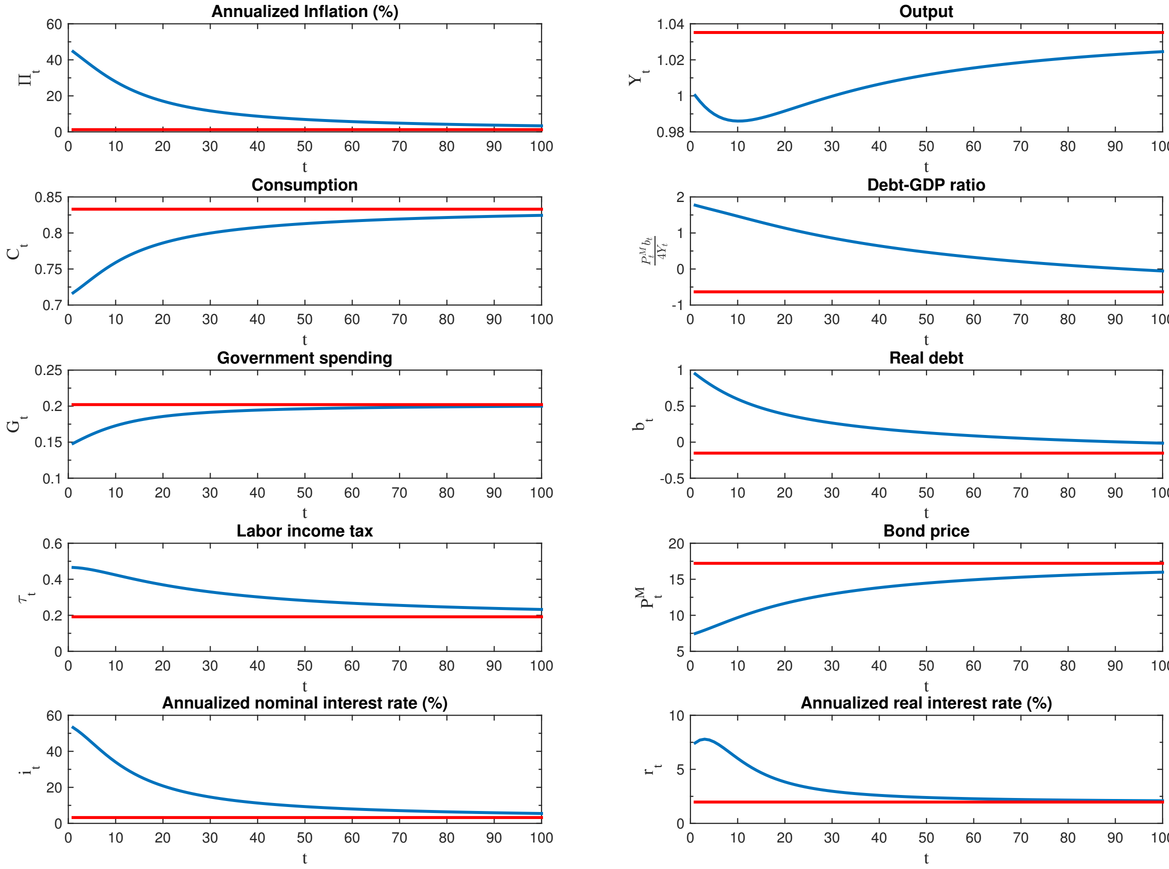

Figure 4.4: Under the benchmark parameters, this figure plots the transition paths of policy variables when debt starts from levels consistent with currently observed debt-GDP ratios, and technology is fixed at A=1. The red lines indicate steady states.

The Role of Debt Maturity

To illustrate the importance of the maturity structure on the optimal policy mix, we plot the policy functions for inflation, real interest rates, the labor income tax and government spending as a function of debt levels for the conventional single period debt ($\rho = 0$) and longer maturity debt ($\rho = 0.7588, 0.9598, 0.9786$), as shown in Figure 4.5. In the case of single period debt, we obtain a large endogenous inflationary bias problem, but find that even when inflation is high as a result, real interest rates fall at higher debt levels. Moreover, although tax rates initially rise with the level of debt, they start to fall once the debt to GDP ratio passes 30%. Therefore, we find that real interests are lower as debt levels rise, since monetary policy seeks to reduce debt service costs and expand the tax base. This would look like a passive monetary policy, if one was to estimate a standard policy rule. At the same time, tax policy looks conventional at low to moderate debt levels, but once debt levels rise above 30% of GDP higher debt is associated with lower tax rates - an apparently active fiscal policy.

When we turn to the longer maturity debt ($\rho = 0.9598$ as an example), we have

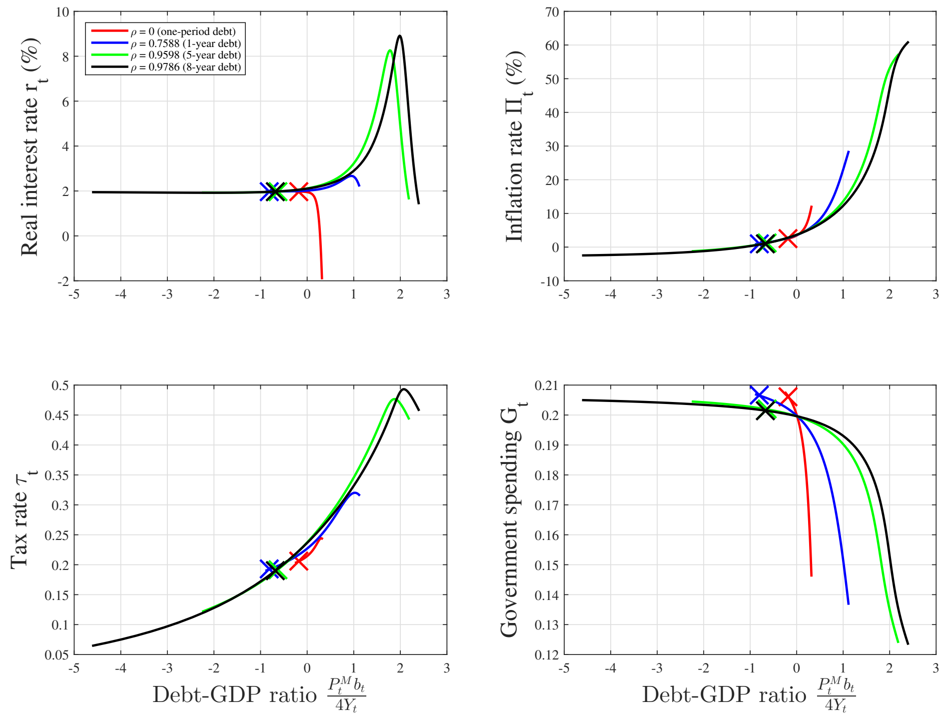

Figure 4.5: This figure illustrates the relationship between policy mix and the debt-GDP ratio under alternative maturities.

conventional policies in place for a wider range of debt to GDP ratios. As debt levels rise, we have a worsening of the inflationary bias problem, although not as pronounced as in the case of shorter maturity debt. However, unlike the case of single period debt, monetary policy raises real interest rates in response to this rise in inflation until debt to GDP ratios exceed 175% of GDP at which point they start falling sharply as debt levels rise further. At the same point, labor income tax rates start falling with rising debt levels too. Therefore, we have a policy mix which looks like the conventional policy assignment at lower debt levels - real interest rates rise to fight inflation and tax rates rise and government consumption fall to stabilize debt - but at higher debt levels we observe a reversal in the policy mix - monetary policy reduces real interest rates to stabilize debt and fiscal policy moderates the increases in tax rates to mitigate the rise in inflation.

We can then see the role of debt maturity on the transition dynamics by plotting the transition paths for four cases of ρ: $\rho = 0$ (single periood debt), $\rho = 0.7588$ (1 year debt maturity), $\rho = 0.9598$ (5 year debt maturity), and $\rho = 0.9786$ (8 year debt maturity) where we begin from the same debt to GDP ratio, as shown in Figure 4.6. Here we can see the radically higher inflationay bias problem when debt maturity is low, and the unconventional policy mix this engenders - real interest rates are cut to help reduce debt when debt is single period, tax increases and moderated and government consumption is markedly reduced. As debt maturity is increased, we both reduce the inflationary bias problem and the conventional policy mix is applied a lower debt to GDP ratios.

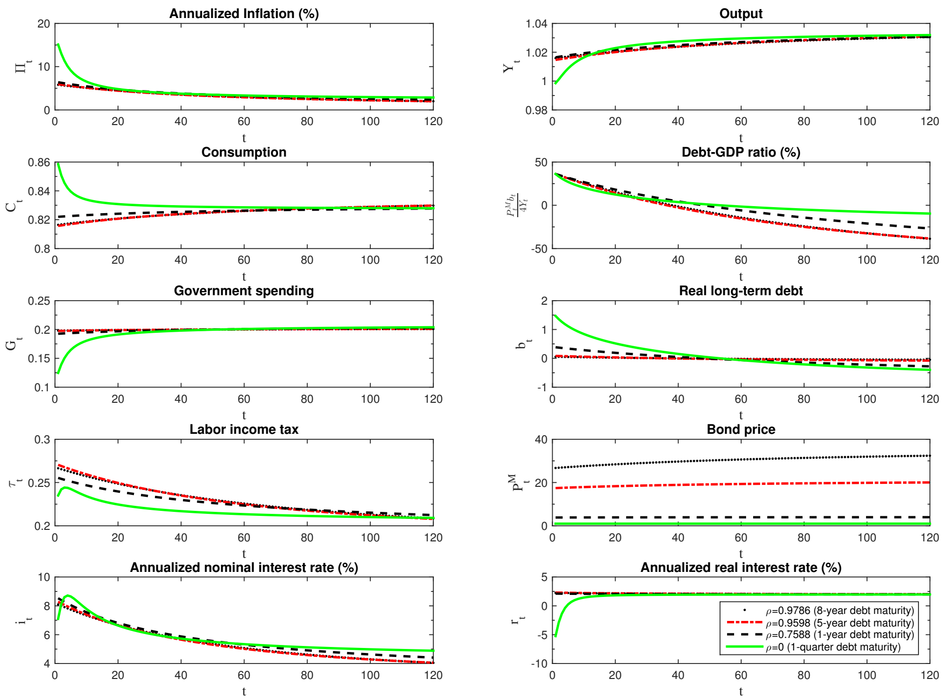

Figure 4.6: This figure plots the transition paths under different maturities, when the debt-GDP ratio starts from the same level.

Endogenizing Debt Maturity

Up until this point, we have held the level of debt maturity fixed by controlling ρ. We now allow the policy maker to have some control over the maturity structure by allowing them to issue a mixture of single period and longer-maturity debt of a given ρ. By varying the relative proportions of these two types of bonds, the policy maker can influence the average maturity of the outstanding stock of debt. We plot the transition dynamics for the benchmark calibration in Figure 4.7 where we start from the same initial overall debt to GDP ratio. Here it is important to stress that despite the high overall debt to GDP ratio, the quantity of short-term debt issued is very low. We do not observe the extreme portfolios made up of issuing long-term debt to purchase short-term assets which have been used as hedging devices when policy makers can commit (see Debortoli et al. (2015) and Leeper and Leith (2017)). Instead, there is an extremely modest issuance of short-term debt when overall debt levels are very high, which serves to support small changes in the time-consistent policy mix. Specifically, we do not observe significant changes in the paths for inflation or overall indebtedness,

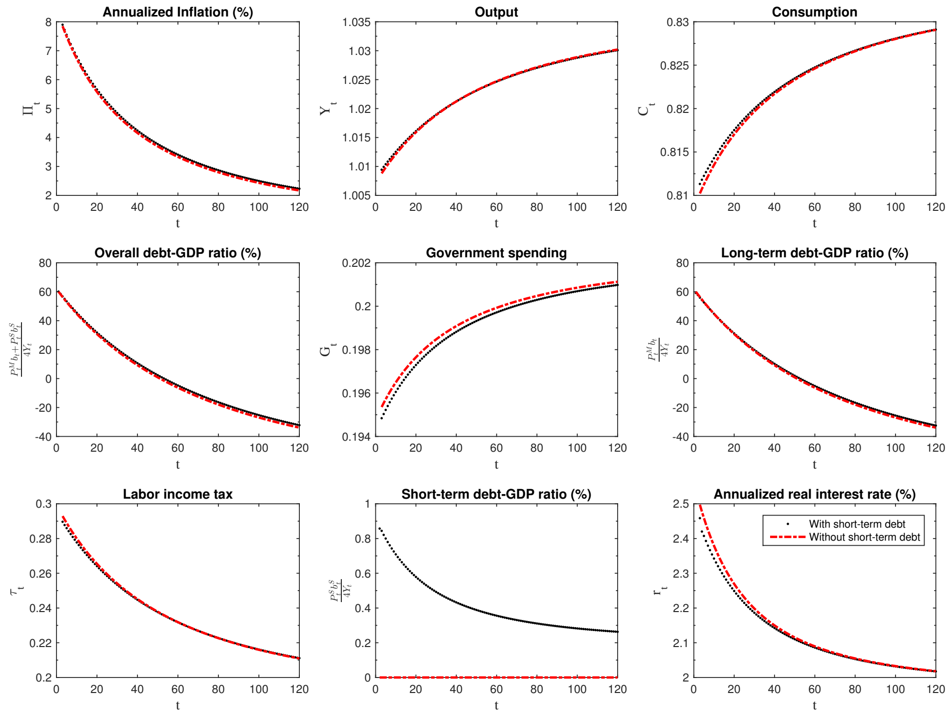

Figure 4.7: This figure compares the transition paths under the benchmark case with and without short-term debt, when the debt-GDP ratio starts from the same level.

suggesting that the availability of short-term debt is not used to radically alter the speed of fiscal correction. Instead, the policy mix underpinning those dynamics does change - real interest rates, government consumption and tax rates are lower when the policy maker can issue short-term debt and overall debt levels are high. In other words, the issuance of short-term debt tilts the policy mix towards the unconventional policies pursued at lower maturity levels with more adjustment being borne by monetary policy and cuts in government spending and less in tax increases. This tilting in the policy mix produces a very modest welfare gain (equivalent to 1.5% of one-period's steady state consumption). If we turn to a lower maturity structure (an average medium-term debt maturity of two years) then the effects are qualitatively similar, but quantitatively much smaller - see Figure 4.8.

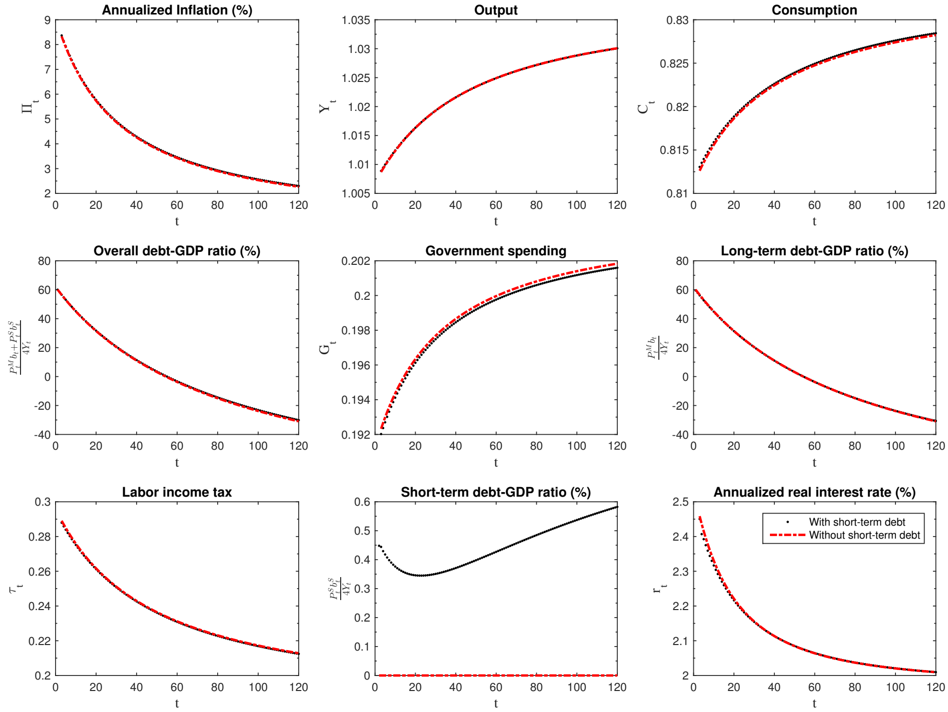

Figure 4.8: This figure compares the transition paths under the case with debt maturity of two years, with and without short-term debt, when the debt-GDP ratio starts from the same level.

4.6 Conclusions

In this chapter we have considered the optimal monetary and fiscal policy mix in a New Keynesian economy with long-term debt. The existence of nominal debt induces a substantial endogenous inflation bias problem as the policy maker faces the temptation to reduce the real value of debt through inflation surprises - a debt stabilization bias. In fact, this temptation results in a steady state where the government accumulates a small stock of assets (falling well short of the 'war chest' needed to finance all of the government's activities without recourse to taxation) and suffers a mild deflationary undershooting of the inflation target. Moreover, we find that the policy equilibrium is highly non-linear depending crucially on both the level of debt and the maturity structure of that debt. Adopting single period debt implies a policy mix which can look quite unconventional, especially as debt levels rise. Specifically, monetary policy will seek to stabilize debt through lower debt interest payments while tax policy attempts to stabilize inflation. With longer debt maturities, optimal policy looks more like the conventional policy assignment - monetary policy raises real interest rates to fight inflation, while taxes are raised to stabilize debt, unless debt level rise sufficiently high that we reverse the policy assignment as in the case of single period debt. This policy mix reversal occurs at far higher debt levels, as we move from single period debt to plausibly calibrated debt maturities.

Finally, we consider the role of endogenous maturity by allowing the policy maker to issue both single-period and medium maturity debt. We find that this does little to affect the underlying inflation bias problems and debt dynamics, but that a modest issuance of short-term debt allows the policy maker to shift the policy mix to be more like that of the single period debt case with lower real interest rates, government consumption and tax rates. This is mildly welfare improving. It is also interesting to note that the implicit government debt portfolio does not attempt to achieve any of the hedging effects associated with some optimal policy exercises when the policy maker can commit.

Appendix

4.A Technical Appendix

4.A.1 Derivation of Household's FOCs

First, let's clarify how to derive (4.4). one-period securities B_t^S purchased at time t pay a nominal return R_t at time $t+1$. Long-term bonds B_t^M cost P_t^M at time t and pay an exponentially decaying coupon ρ^j at time $t+j+1$. By definition, the yield to maturity is the discount rate that makes the current bond price equal to the present value of the payments (coupon payments and maturity value). Then,

$$\begin{aligned} P_t^M &= \frac{1}{R_t^m} + \frac{\rho}{(R_t^m)^2} + \frac{\rho^2}{(R_t^m)^3} + \cdots \\ &= \frac{1}{R_t^m - \rho} \end{aligned}$$

Equivalently, the gross yield to maturity at time t on the long-term bond is given by

$$R_t^m = \frac{1}{P_t^M} + \rho \tag{4.28}$$

see Woodford (2001). Hence, the duration of this bond is

$$\frac{R_t^m}{(R_t^m - \rho)^2 P_t^M} = \frac{R_t^m}{(R_t^m - \rho)}$$

see the proof below. Also note that the price of a bond issued j periods before is given by

$$P_t^M(j) = \rho^j P_t^M \tag{4.29}$$

which says that any existing bonds issued j periods ago is equivalent to ρ^j new bonds. This enables us to write the household's flow budget constraint as a function of the stock of total long-term debt, B_t^M, instead of the current period's purchases of long-term debt.

The flow budget constraint of households is

$$P_t C_t + P_t^S B_t^S + P_t^M B_t^M \leq \Xi_t + \sum_{j=1}^{\infty} \rho^{j-1} B_{t-j}^M + B_{t-1}^S + W_t N_t (1-\tau_t) \tag{4.30}$$

Using (4.29), we can rewrite (4.30) in a more convenient recursive formulation. Imposing the no-arbitrage condition at time $t-1$,

$$P_{t-1}^M B_{t-1}^M = \sum_{j=1}^{\infty} P_t^M(j) B_{t-j}^M = \sum_{j=1}^{\infty} \rho^{j-1} P_{t-1}^M B_{t-j}^M$$

that is,

$$B_{t-1}^M = \sum_{j=1}^{\infty} \rho^{j-1} B_{t-j}^M$$

Using (4.28), at time t, B_{t-1}^M is worth $B_{t-1}^M \left(1+\rho P_t^M\right) = P_t^M R_t^m B_{t-1}^M$. This gives us (4.4). See Chen et al. (2012) for a similar model setup.

We can derive optimality conditions for the agent by writing the Lagrangian expression,

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \left(\frac{C_t^{1-\sigma}}{1-\sigma} + \chi \frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{N_t{}^{1+\varphi}}{1+\varphi} \right) \\ +\lambda_t \left[\Xi_t + (1+\rho P_t^M) B_{t-1}^M + B_{t-1}^S + W_t N_t (1-\tau_t) - P_t C_t - P_t^S B_t^S - P_t^M B_t^M \right] \end{array} \right\}$$

where λ_t is the Lagrangian multiplier. The FOCs are given as follows:

$$\frac{\partial \mathcal{L}}{\partial C_t} = C_t^{-\sigma} - P_t \lambda_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = -N_t^{\varphi} + \lambda_t (1-\tau_t) W_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial B_t^S} = -\lambda_t P_t^S + \beta E_t \left(\lambda_{t+1}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial B_t^M} = -\lambda_t P_t^M + \beta E_t \left[\left(1+\rho P_{t+1}^M\right) \lambda_{t+1} \right] = 0$$

Note that, $P_t^S = R_t^{-1}$, then these conditions can be simplified further into

$$\beta R_t E_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \frac{P_t}{P_{t+1}} \right] = 1$$

$$(1-\tau_t) \frac{W_t}{P_t} = N_t^{\varphi} C_t^{\sigma}$$

$$\beta E_t \left[(1 + \rho P_{t+1}^M) \left(\frac{C_t}{C_{t+1}} \right)^\sigma \frac{P_t}{P_{t+1}} \right] = P_t^M$$

Using (4.28), the Euler equation for the long-term bond can also be written as,

$$\beta E_t \left[\left(\frac{R_{t+1}^m}{R_{t+1}^m - \rho} \right) \left(\frac{C_t}{C_{t+1}} \right)^\sigma \frac{P_t}{P_{t+1}} \right] = \frac{1}{R_t^m - \rho}$$

that is,

$$\beta \left(R_t^m - \rho \right) E_t \left[\left(\frac{R_{t+1}^m}{R_{t+1}^m - \rho} \right) \left(\frac{C_t}{C_{t+1}} \right)^\sigma \frac{P_t}{P_{t+1}} \right] = 1$$

Combining the Euler equations for short- and long-term bonds yields the no-arbitrage restriction between these two bonds

$$R_t = E_t \left(\frac{1 + \rho P_{t+1}^M}{P_t^M} \right) \equiv E_t \left(R_{t+1}^M \right) \tag{4.31}$$

which means that the expected one-period returns from date t to $t+1$ on the two bonds should be equal. Equivalently, the no-arbitrage condition between the two bonds can be written as

$$R_t = E_t \left[\frac{R_{t+1}^m \left(R_t^m - \rho \right)}{R_{t+1}^m - \rho} \right]$$

Derivation of No-Ponzi-Game Condition

The no-Ponzi game constraint in our model is the same as the one in Eusepi and Preston (2011). Define nominal household wealth brought into period t as,

$$D_t = (1 + \rho P_t^M) B_{t-1}^M + B_{t-1}^S$$

then the real beginning-of-period financial wealth in period t is

$$\frac{D_t}{P_t} = \frac{(1 + \rho P_t^M) B_{t-1}^M + B_{t-1}^S}{P_t}$$

and the no-Ponzi game condition is

$$\lim_{T \to \infty} E_t \left[q_{t,T} \frac{D_T}{P_T} \right] = 0$$

Note that $q_{t,T} = \prod_{s=t+1}^{T} q_{s-1,s} = \prod_{s=t+1}^{T} \left(\beta \left(\frac{C_{s-1}}{C_s}\right)^{\sigma}\right) = \beta^{T-t} \left(\frac{C_t}{C_T}\right)^{\sigma}$, hence the no-Ponzi game condition can also be written as

$$\lim_{T\to\infty} E_t \left[\beta^{T-t} \left(\frac{C_t}{C_T}\right)^{\sigma} \frac{D_T}{P_T}\right] = 0$$

Note that

$$\lambda_t = \frac{C_t^{-\sigma}}{P_t}$$

then, the no-Ponzi game condition can be rewritten as

$$\lim_{T\to\infty} E_0 \left[\beta^T \lambda_T D_T\right] = 0$$

which is the typical form used in the literature.

We can also get the same form of no-Ponzi game condition in Eusepi and Preston (2011) as follows. The no-arbitrage condition between one-period and long-term bonds

$$P_t^M = E_t \left[P_t^S \left(1 + \rho P_{t+1}^M\right)\right]$$

implies that

$$P_t^M = E_t \left[Q_{t,t+1} \left(1 + \rho P_{t+1}^M\right)\right]$$

that is,

$$E_t \left[Q_{t,t+1} \left(\frac{1 + \rho P_{t+1}^M}{P_t^M}\right)\right] = 1$$

As a result,

$$q_{t,T} = q_{t,t+1} q_{t+1,t+2} \cdots q_{T-1,T} = Q_{t,t+1} \left(\frac{P_{t+1}}{P_t}\right) Q_{t+1,t+2} \left(\frac{P_{t+2}}{P_{t+1}}\right) \cdots Q_{T-1,T} \left(\frac{P_T}{P_{T-1}}\right)$$

$$= \left(\frac{P_t^M}{1+\rho P_{t+1}^M}\right) \left(\frac{P_{t+1}}{P_t}\right) \left(\frac{P_{t+1}^M}{1+\rho P_{t+2}^M}\right) \left(\frac{P_{t+2}}{P_{t+1}}\right) \cdots \left(\frac{P_{T-1}^M}{1+\rho P_T^M}\right) \left(\frac{P_T}{P_{T-1}}\right)$$

$$= \prod_{s=t}^{T-1} \left(\frac{P_s^M}{1+\rho P_{s+1}^M}\right) \left(\frac{P_{s+1}}{P_s}\right)$$

$$= \prod_{s=t}^{T-1} \frac{1}{R_{s,s+1}} = \frac{1}{R_{t,T}}$$

where

$$R_{s,s+1} = \left(\frac{1+\rho P_{s+1}^M}{P_s^M}\right) \left(\frac{P_s}{P_{s+1}}\right)$$

is the realized one-period gross real return on long-term bonds. Hence, the no-Ponzi game condition can be rewritten as

$$\lim_{T\to\infty} E_t \left[\frac{1}{R_{t,T}} \frac{D_T}{P_T} \right] = 0$$

4.A.2 Summary of Model

The model can be summarized by the following equations:

Consumption Euler equation,

$$\beta R_t E_t \left\{ \left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \left(\frac{P_t}{P_{t+1}} \right) \right\} = 1 \tag{4.32}$$

Pricing of longer-term bonds,

$$\beta E_t \left\{ \left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \left(\frac{P_t}{P_{t+1}} \right) \left(1 + \rho P^M_{t+1} \right) \right\} = P^M_t \tag{4.33}$$

Labour supply,

$$N_t^{\varphi} C_t^{\sigma} = (1-\tau_t) \left(\frac{W_t}{P_t} \right) \equiv (1-\tau_t) w_t$$

Resource constraint,

$$Y_t \left[1 - \frac{\phi}{2} \left(\frac{\Pi_t}{\Pi^*} - 1 \right)^2 \right] = C_t + G_t \tag{4.34}$$

Phillips curve,

$$\begin{aligned} 0 = (1-\epsilon) + \epsilon mc_t - \phi \frac{\Pi_t}{\Pi^*} \left(\frac{\Pi_t}{\Pi^*} - 1 \right) \\ + \phi \beta E_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \frac{Y_{t+1}}{Y_t} \frac{\Pi_{t+1}}{\Pi^*} \left(\frac{\Pi_{t+1}}{\Pi^*} - 1 \right) \right] \end{aligned} \tag{4.35}$$

Government budget constraint,

$$P^M_t b_t = (1+\rho P^M_t) \frac{b_{t-1}}{\Pi_t} - \frac{W_t}{P_t} N_t \tau_t + G_t$$

$$= (1+\rho P^M_t) \frac{b_{t-1}}{\Pi_t} - \left(\frac{\tau_t}{1-\tau_t} \right) N_t^{1+\varphi} C_t^{\sigma} + G_t$$

$$= (1+\rho P^M_t) \frac{b_{t-1}}{\Pi_t} - \left(\frac{\tau_t}{1-\tau_t} \right) \left(\frac{Y_t}{A_t} \right)^{1+\varphi} C_t^{\sigma} + G_t \tag{4.36}$$

Technology,

$$Y_t = A_t N_t \tag{4.37}$$

Marginal costs,

$$mc_t = W_t / \left(P_t A_t\right) = (1-\tau_t)^{-1} Y_t^{\varphi} C_t^{\sigma} A_t^{-1-\varphi}$$

The objective function for social welfare is given by,

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} + \chi \frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi} \right) \tag{4.38}$$

There are two state variables, real debt b_t and productivity $a_t = \ln\left(A_t\right)$.

The Deterministic Steady State

Given the system of non-linear equations, the corresponding steady state system can be written as follows:

$$A = 1$$

$$\frac{\beta R}{\Pi} = 1$$

$$\frac{\beta}{\Pi}\left(1 + \rho P^M\right) = P^M$$

$$(1-\tau)w = N^{\varphi} C^{\sigma}$$

$$Y\left[1 - \frac{\phi}{2}\left(\frac{\Pi}{\Pi^*} - 1\right)^2\right] = C + G$$

$$(1-\epsilon) + \epsilon mc + \phi\left(\beta - 1\right)\left[\frac{\Pi}{\Pi^*}\left(\frac{\Pi}{\Pi^*} - 1\right)\right] = 0$$

$$P^M b = (1 + \rho P^M)\frac{b}{\Pi} - \left(\frac{\tau}{1-\tau}\right) Y^{1+\varphi} C^{\sigma} + G$$

$$Y = N$$

$$mc = w = (1-\tau)^{-1} Y^{\varphi} C^{\sigma}$$

Hence, when $\Pi = \Pi^*$,

$$P^M = \frac{\beta}{\Pi^* - \beta\rho}$$

$$mc = w = \frac{\epsilon - 1}{\epsilon}$$

$$\frac{C}{Y} = \left[(1-\tau)\left(\frac{\epsilon-1}{\epsilon}\right)\right]^{1/\sigma} Y^{-\frac{\varphi+\sigma}{\sigma}}$$

$$\frac{G}{Y} = 1 - \frac{C}{Y} = 1 - \left[(1-\tau)\left(\frac{\epsilon-1}{\epsilon}\right)\right]^{1/\sigma} Y^{-\frac{\varphi+\sigma}{\sigma}}$$

$$P^M b = \frac{\beta}{1-\beta}\left[\tau\left(\frac{\epsilon-1}{\epsilon}\right) - \frac{G}{Y}\right] Y$$

Note that,

$$Y^{\varphi+\sigma}\left(1-\frac{G}{Y}\right)^{\sigma} = (1-\tau)\left(\frac{\epsilon-1}{\epsilon}\right) \tag{4.39}$$

which will be used to contrast with the allocation that would be chosen by a social planner.

4.A.3 Numerical Algorithm

Let $s_t = (b_{t-1}, a_t)$ denote the state vector at time t, where real stock of debt b_{t-1} is endogenous and technology $A_t = \exp(a_t)$ is exogenous and respectively, with the following law of motion:

$$P_t^M b_t = (1+\rho P_t^M)\frac{b_{t-1}}{\Pi_t} - w_t N_t \tau_t + G_t$$

$$a_t = \rho_a a_{t-1} + e_{at}$$

where $0 \leq \rho_a < 1$ and technology innovation e_{at} is an *i.i.d.* normal random variable, which has a zero mean and a finite standard deviation σ_a.

There are 7 endogenous variables and 3 Lagrangian multipliers. Correspondingly, there are 10 functional equations associated with the 10 varaibles

$$\left\{C_t, Y_t, \Pi_t, b_t, \tau_t, P_t^M, G_t, \lambda_{1t}, \lambda_{2t}, \lambda_{3t}\right\}$$

. Let's define a new function $X : \mathbb{R}^2 \to \mathbb{R}^{10}$, in order to collect the policy functions of endogenous variables as follows:

$$X(s_t) = \left(C_t(s_t), Y_t(s_t), \Pi_t(s_t), b_t(s_t), \tau_t(s_t), P_t^M(s_t), G_t(s_t), \lambda_{1t}(s_t), \lambda_{2t}(s_t), \lambda_{3t}(s_t)\right)$$

Given the specification of the function X, the equilibrium conditions can be written more compactly as,

$$\Gamma\left(s_t, X(s_t), E_t\left[Z\left(X(s_{t+1})\right)\right], E_t\left[Z_b\left(X(s_{t+1})\right)\right]\right) = 0$$

where $\Gamma : \mathbb{R}^{2+10+3+3} \to \mathbb{R}^{10}$ summarizes the full set of dynamic equilibrium relationship, and

$$Z\left(X(s_{t+1})\right) = \begin{bmatrix} Z_1\left(X(s_{t+1})\right) \\ Z_2\left(X(s_{t+1})\right) \\ Z_3\left(X(s_{t+1})\right) \end{bmatrix} \equiv \begin{bmatrix} M(b_t, A_{t+1}) \\ L(b_t, A_{t+1}) \\ (\Pi_{t+1})^{-1}\left(1+\rho P^M_{t+1}\right)\lambda_{3t+1} \end{bmatrix}$$

with

$$M(b_t, A_{t+1}) = (C_{t+1})^{-\sigma} Y_{t+1} \frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*} - 1\right)$$

$$L(b_t, A_{t+1}) = (C_{t+1})^{-\sigma}(\Pi_{t+1})^{-1}(1+\rho P^M_{t+1})$$

and

$$Z_b\left(X(s_{t+1})\right) = \begin{bmatrix} \frac{\partial Z_1(X(s_{t+1}))}{\partial b_t} \\ \frac{\partial Z_2(X(s_{t+1}))}{\partial b_t} \\ \frac{\partial Z_3(X(s_{t+1}))}{\partial b_t} \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial M(b_t, A_{t+1})}{\partial b_t} \\ \frac{\partial L(b_t, A_{t+1})}{\partial b_t} \\ \frac{\partial\left[(\Pi_{t+1})^{-1}\left(1+\rho P^M_{t+1}\right)\lambda_{3t+1}\right]}{\partial b_t} \end{bmatrix}$$

More specifically,

$$L_1(b_t, A_{t+1}) = \frac{\partial\left[(C_{t+1})^{-\sigma}(\Pi_{t+1})^{-1}(1+\rho P^M_{t+1})\right]}{\partial b_t}$$

$$= -\sigma(C_{t+1})^{-\sigma-1}(\Pi_{t+1})^{-1}(1+\rho P^M_{t+1})\frac{\partial C_{t+1}}{\partial b_t}$$
$$-(C_{t+1})^{-\sigma}(\Pi_{t+1})^{-2}(1+\rho P^M_{t+1})\frac{\partial \Pi_{t+1}}{\partial b_t} + \rho(C_{t+1})^{-\sigma}(\Pi_{t+1})^{-1}\frac{\partial P^M_{t+1}}{\partial b_t}$$

and

$$M_1(b_t, A_{t+1}) = \frac{\partial\left[(C_{t+1})^{-\sigma} Y_{t+1}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)\right]}{\partial b_t}$$

$$= -\sigma(C_{t+1})^{-\sigma-1}Y_{t+1}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)\frac{\partial C_{t+1}}{\partial b_t} + (C_{t+1})^{-\sigma}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)\frac{\partial Y_{t+1}}{\partial b_t}$$
$$+(C_{t+1})^{-\sigma}\frac{Y_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)\frac{\partial \Pi_{t+1}}{\partial b_t} + (C_{t+1})^{-\sigma}\frac{Y_{t+1}}{\Pi^*}\frac{\Pi_{t+1}}{\Pi^*}\frac{\partial \Pi_{t+1}}{\partial b_t}$$

$$= -\sigma(C_{t+1})^{-\sigma-1}Y_{t+1}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)\frac{\partial C_{t+1}}{\partial b_t} + (C_{t+1})^{-\sigma}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)\frac{\partial Y_{t+1}}{\partial b_t}$$
$$+(C_{t+1})^{-\sigma}\frac{Y_{t+1}}{\Pi^*}\left(\frac{2\Pi_{t+1}}{\Pi^*}-1\right)\frac{\partial \Pi_{t+1}}{\partial b_t}$$

Note we are assuming $E_t\left[Z_b\left(X(s_{t+1})\right)\right] = \partial E_t\left[Z\left(X(s_{t+1})\right)\right]/b_t$, which is normally valid using the Interchange of Integration and Differentiation Theorem. Then the problem is to find a vector-valued function X that Γ maps to the zero function. Projection methods, hence, can be used.

Following the notation convention in the literature, we simply use $s = (b, a)$ to denote the current state of the economy $s_t = (b_{t-1}, a_t)$, and s' to represent next period state that evolves according to the law of motion specified above. The Chebyshev collocation method with time iteration which we use to solve this nonlinear system can be described as follows:

1. Define the collocation nodes and the space of the approximating functions:

   - Choose an order of approximation (i.e., the polynomial degrees) n_b and n_a for each dimension of the state space $s = (b, a)$, then there are $N_s = (n_b + 1) \times (n_a + 1)$ nodes in the state space. Let $S = (S_1, S_2, ..., S_{N_s})$ denote the set of collocation nodes.
   - Compute the $n_b + 1$ and $n_a + 1$ roots of the Chebychev polynomial of order $n_b + 1$ and $n_a + 1$ as

     $$z_b^i = \cos\left(\frac{(2i-1)\pi}{2(n_b+1)}\right), \text{ for } i = 1, 2, ..., n_b + 1.$$
     $$z_a^i = \cos\left(\frac{(2i-1)\pi}{2(n_a+1)}\right), \text{ for } i = 1, 2, ..., n_a + 1.$$

   - Compute collocation points a_i as

     $$a_i = \frac{\overline{a} + \underline{a}}{2} + \frac{\overline{a} - \underline{a}}{2} z_a^i = \frac{\overline{a} - \underline{a}}{2}\left(z_a^i + 1\right) + \underline{a}$$

     for $i = 1, 2, ..., n_a + 1$. Note that the number of collocation nodes is $n_a + 1$. Similarly, compute collocation points b_i as

     $$b_i = \frac{\overline{b} + \underline{b}}{2} + \frac{\overline{b} - \underline{b}}{2} z_b^i = \frac{\overline{b} - \underline{b}}{2}\left(z_b^i + 1\right) + \underline{b}$$

     for $i = 1, 2, ..., n_b + 1$, which map $[-1, 1]$ into $[\underline{b}, \overline{b}]$. Note that

     $$S = \{(b_i, a_j) \mid i = 1, 2, ..., n_b + 1,\ j = 1, 2, ..., n_a + 1\}$$

     that is, the tensor grids, with $S_1 = (b_1, a_1)$, $S_2 = (b_1, a_2)$, ..., $S_{N_s} = (b_{n_b+1}, a_{n_a+1})$.
   - The space of the approximating functions, denoted as Ω, is a matrix of

two-dimensional Chebyshev polynomials. More specifically,

$$\Omega(S) = \begin{bmatrix} \Omega(S_1) \\ \Omega(S_2) \\ \vdots \\ \Omega(S_{n_a+1}) \\ \vdots \\ \Omega(S_{N_s}) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & T_0(\xi(b_1)T_1(\xi(a_1)) & T_0(\xi(b_1)T_2(\xi(a_1)) & \cdots & T_{n_b}(\xi(b_1)T_{n_a}(\xi(a_1)) \\ 1 & T_0(\xi(b_1)T_1(\xi(a_2)) & T_0(\xi(b_1)T_2(\xi(a_2)) & \cdots & T_{n_b}(\xi(b_1)T_{n_a}(\xi(a_2)) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & T_0(\xi(b_1)T_1(\xi(a_{n_a+1})) & T_0(\xi(b_1)T_2(\xi(a_{n_a+1})) & \cdots & T_0(\xi(b_1)T_{n_a}(\xi(a_{n_a+1})) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & T_0(\xi(b_{n_b+1})T_1(\xi(a_{n_a+1})) & T_0(\xi(b_{n_b+1})T_2(\xi(a_{n_a+1})) & \cdots & T_0(\xi(b_{n_b+1})T_{n_a}(\xi(a_{n_a+1})) \end{bmatrix}_{N_s \times N_s}$$

where $\xi(x) = 2(x - \underline{x}) / (\overline{x} - \underline{x}) - 1$ maps the domain of $x \in [\underline{x}, \overline{x}]$ into $[-1, 1]$.

- Then, at each node $s \in S$, policy functions $X(s)$ are approximated by $X(s) = \Omega(s)\Theta_X$,

  where

  $$\Theta_X = \left[\theta^c, \theta^y, \theta^\pi, \theta^b, \theta^\tau, \theta^{\widetilde{p}}, \theta^g, \theta^{\lambda_1}, \theta^{\lambda_2}, \theta^{\lambda_3}\right]$$

  is a $N_s \times 10$ matrix of the approximating coefficients.

2. Formulate an initial guess for the approximating coefficients, Θ_X^0, and specify the stopping rule ϵ_{tol}, say, 10^{-6}.

3. At each iteration j, we can get an updated Θ_X^j by implement the following time iteration step:

- At each collocation node $s \in S$, compute the possible values of future policy functions $X(s')$ for $k = 1, ..., q$. That is,

  $$X(s') = \Omega(s')\Theta_X^{j-1}$$

  where q is the number of Gauss-Hermite quadrature nodes. Note that

  $$\Omega(s') = T_{j_b}(\xi(b'))T_{j_a}(\xi(a'))$$

  is a $q \times N_s$ matrix, with $b' = \widehat{b}(s; \theta^b)$, $a' = \rho_a a + z_k\sqrt{2\sigma_a^2}$, $j_b = 0, ..., n_b$, and $j_a = 0, ..., n_a$. The hat symbol indicates the corresponding approximate policy functions, so $\widehat{b}$ is the approximate policy for real debt, for example.

Similarly, the two auxilliary functions can be calculated as follows:

$$M(s') \approx \left(\widehat{C}(s';\theta^c)\right)^{-\sigma} \widehat{Y}(s';\theta^y)\frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*}\left(\frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*}-1\right)$$

and,

$$L(s') \approx \left(\widehat{C}(s';\theta^c)\right)^{-\sigma}\left(\widehat{\Pi}(s';\theta^\pi)\right)^{-1}\left(1+\frac{\rho\widehat{P^M}\left(s';\theta^{\widetilde{p}}\right)}{\Pi^*-\rho\beta}\right)$$

Note that we use $\widetilde{P}_t^M = (\Pi^* - \rho\beta)\, P_t^M$ rather than P_t^M in numerical analysis, since the former is far less sensitive to maturity structure variations.

- Now calculate the expectation terms $E\left[Z\left(X(s')\right)\right]$ at each node s. Let ω_k denote the weights for the quadrature, then

$$E\left[M(s')\right] \approx \frac{1}{\sqrt{\pi}}\sum_{k=1}^{q}\omega_k\left(\widehat{C}(s';\theta^c)\right)^{-\sigma}\widehat{Y}(s';\theta^y)\frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*}\left(\frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*}-1\right) \equiv \overline{M}\left(s',q\right)$$

$$E\left[L(s')\right] \approx \frac{1}{\sqrt{\pi}}\sum_{k=1}^{q}\omega_k\left(\widehat{C}(s';\theta^c)\right)^{-\sigma}\left(\widehat{\Pi}(s';\theta^\pi)\right)^{-1}\left(1+\frac{\rho\widehat{P^M}\left(s';\theta^{\widetilde{p}}\right)}{\Pi^*-\rho\beta}\right) \equiv \overline{L}\left(s',q\right)$$

and

$$E_t\left[\left(\frac{1+\rho P_{t+1}^M}{\Pi_{t+1}}\right)\lambda_{3t+1}\right] \approx \frac{1}{\sqrt{\pi}}\sum_{k=1}^{q}\omega_k\left(\frac{1+\frac{\rho\widehat{P^M}\left(s';\theta^{\widetilde{p}}\right)}{\Pi^*-\rho\beta}}{\widehat{\Pi}(s';\theta^\pi)}\right)\widehat{\lambda_3}(s';\theta^{\lambda_3}) \equiv \Lambda\left(s',q\right).$$

Hence,

$$E\left[Z\left(X(s')\right)\right] \approx E\left[\widehat{Z}\left(X(s')\right)\right] = \begin{bmatrix} \overline{M}\left(s',q\right) \\ \overline{L}\left(s',q\right) \\ \Lambda\left(s',q\right) \end{bmatrix}$$

- Next calculate the partial derivatives under expectation $E\left[Z_b\left(X(s')\right)\right]$.

- Note that we only need to compute $\partial C_{t+1}/\partial b_t$, $\partial Y_{t+1}/\partial b_t$, $\partial \Pi_{t+1}/\partial b_t$ and $\partial P_{t+1}^M/\partial b_t$, which are given as follows:

$$\frac{\partial C_{t+1}}{\partial b} \approx \sum_{j_b=0}^{n_b}\sum_{j_a=0}^{n_a}\frac{2\theta^c_{j_b j_a}}{\overline{b}-\underline{b}}T'_{j_b}(\xi(b'))T_{j_a}(\xi(a')) \equiv \widehat{C}_b\left(s'\right)$$

$$\frac{\partial Y_{t+1}}{\partial b_t} \approx \sum_{j_b=0}^{n_b}\sum_{j_a=0}^{n_a}\frac{2\theta^y_{j_b j_a}}{\overline{b}-\underline{b}}T'_{j_b}(\xi(b'))T_{j_a}(\xi(a')) \equiv \widehat{Y}_b\left(s'\right)$$

$$\frac{\partial \Pi_{t+1}}{\partial b_t} \approx \sum_{j_b=0}^{n_b} \sum_{j_a=0}^{n_a} \frac{2\theta^{\pi}_{j_b j_a}}{\overline{b} - \underline{b}} T'_{j_b}(\xi(b')) T_{j_a}(\xi(a')) \equiv \widehat{\Pi}_b\left(s'\right)$$

$$\frac{\partial P^M_{t+1}}{\partial b_t} \approx \sum_{j_b=0}^{n_b} \sum_{j_a=0}^{n_a} \frac{2\theta^{\widetilde{p}}_{j_b j_a}}{\left(\overline{b} - \underline{b}\right)\left(\Pi^* - \rho\beta\right)} T'_{j_b}(\xi(b_i)) T_{j_a}(\xi(a_j)) \equiv \widehat{P}^M_b\left(s'\right)$$

Hence, we can approximate the two partial derivatives under expectation

$$\frac{\partial E\left[M(s')\right]}{\partial b}$$

$$\approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{q} \omega_k \begin{bmatrix} -\sigma\left(\widehat{C}(s';\theta^c)\right)^{-\sigma-1} \widehat{Y}(s';\theta^y) \frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*}\left(\frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*} - 1\right) \widehat{C}_b\left(s'\right) \\ +\left(\widehat{C}(s';\theta^c)\right)^{-\sigma} \frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*}\left(\frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*} - 1\right) \widehat{Y}_b\left(s'\right) \\ +\left(\widehat{C}(s';\theta^c)\right)^{-\sigma} \frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*}\left(\frac{2\widehat{\Pi}(s';\theta^\pi)}{\Pi^*} - 1\right) \widehat{\Pi}_b\left(s'\right) \end{bmatrix}$$

$$\equiv \widehat{M_b}\left(s', q\right),$$

$$\frac{\partial E\left[L(s')\right]}{\partial b}$$

$$\approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{q} \omega_k \begin{bmatrix} -\sigma\left(\widehat{C}(s';\theta^c)\right)^{-\sigma-1}\left(\widehat{\Pi}(s';\theta^\pi)\right)^{-1}\left(1 + \frac{\rho\widehat{P^M}\left(s';\theta^{\widetilde{p}}\right)}{\Pi^* - \rho\beta}\right)\widehat{C}_b\left(s'\right) \\ -\left(\widehat{C}(s';\theta^c)\right)^{-\sigma}\left(\widehat{\Pi}(s';\theta^\pi)\right)^{-2}\left(1 + \frac{\rho\widehat{P^M}\left(s';\theta^{\widetilde{p}}\right)}{\Pi^* - \rho\beta}\right)\widehat{\Pi}_b\left(s'\right) \\ +\rho\left(\widehat{C}(s';\theta^c)\right)^{-\sigma}\left(\widehat{\Pi}(s';\theta^\pi)\right)^{-1} \widehat{P}^M_b\left(s'\right) \end{bmatrix}$$

$$\equiv \widehat{L}_b\left(s', q\right).$$

That is,

$$E\left[Z_b\left(X(s')\right)\right] \approx E\left[\widehat{Z}_b\left(X(s')\right)\right] = \begin{bmatrix} \widehat{M_b}\left(s', q\right) \\ \widehat{L}_b\left(s', q\right) \end{bmatrix}$$

4. At each collocation node s, solve for $X(s)$ such that

$$\Gamma\left(s, X(s), E\left[\widehat{Z}\left(X(s')\right)\right], E\left[\widehat{Z}_b\left(X(s')\right)\right]\right) = 0$$

The equation solver *csolve* written by Christopher A. Sims is employed to solve the resulted system of nonlinear equations. With $X(s)$ at hand, we can get the corresponding coeffcient

$$\widehat{\Theta}^j_X = \left(\Omega\left(S\right)^T \Omega\left(S\right)\right)^{-1} \Omega\left(S\right)^T X(s)$$

5. Update the approximating coefficients, $\Theta^j_X = \eta\widehat{\Theta}^j_X + (1-\eta)\,\Theta^{j-1}_X$, where $0 \leq \eta \leq 1$ is some dampening parameter used for improving convergence.

6. Check the stopping rules. If $\left\|\Theta_X^j - \Theta_X^{j-1}\right\| < \epsilon_{tol}$, then stop, else update the approximation coefficients and go back to step 3.

When implementing the above algorithm, we start from lower order Chebyshev polynomials and some reasonable initial guess. Then, we increase the order of approximation and take as starting value the solution from the previous lower order approximation. This informal homotopy continuation idea ensures us to find a solution.

Remark. Given the fact that the price P_t^M fluctuates significantly for larger ρ, in numerical analysis, we scale rule for P_t^M by $(\Pi^* - \rho\beta)$, that is, $\widetilde{P}_t^M = (\Pi^* - \rho\beta)P_t^M$. In this way, the steady state of $\widetilde{P}_t^M$ is very close to β, and $\widetilde{P}_t^M$ does not differ hugely as we change the maturity structure.

4.A.4 Optimal Policy Under Discretion With Endogenous Short-Term Debt

In this case, the government is allowed to issue new bonds of a different maturity and swap these for existing bonds, in a way which does not affect the wealth of the bond holders at the time of the swap, such that the exchange is voluntary.

The policy under discretion in this case can be described as a set of decision rules for $\{C_t, Y_t, \Pi_t, b_t, \tau_t, G_t, b_t^S\}$ which maximise,

$$V(b_{t-1}, A_t, b_{t-1}^S) = \max \left\{ \frac{C_t^{1-\sigma}}{1-\sigma} + \chi \frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi} + \beta E_t \left[V(b_t, A_{t+1}, b_t^S) \right] \right\}$$

subject to the following constraints:

$$Y_t \left[1 - \frac{\phi}{2} \left(\frac{\Pi_t}{\Pi^*} - 1 \right)^2 \right] = C_t + G_t$$

$$\begin{aligned} 0 = (1-\epsilon) + \epsilon mc_t - \phi \frac{\Pi_t}{\Pi^*} \left(\frac{\Pi_t}{\Pi^*} - 1 \right) \\ + \phi\beta E_t \left[\left(\frac{C_t}{C_{t+1}} \right)^\sigma \frac{Y_{t+1}}{Y_t} \frac{\Pi_{t+1}}{\Pi^*} \left(\frac{\Pi_{t+1}}{\Pi^*} - 1 \right) \right] \end{aligned}$$

$$\beta E_t\left\{\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{P_t}{P_{t+1}}\right)(1+\rho P^M_{t+1})\right\}b_t+\beta E_t\left\{\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{P_t}{P_{t+1}}\right)\right\}b^S_t$$
$$=\left(1+\rho\beta E_t\left\{\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{P_t}{P_{t+1}}\right)(1+\rho P^M_{t+1})\right\}\right)\frac{b_{t-1}}{\Pi_t}+\frac{b^S_{t-1}}{\Pi_t}$$
$$-\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi}C^{\sigma}_t+G_t$$

where b^S_t is the level of real short-term debt.

Defining auxilliary functions,

$$M(b_t,A_{t+1},b^S_t)=(C_{t+1})^{-\sigma}\,Y_{t+1}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)$$

$$L(b_t,A_{t+1},b^S_t)=(C_{t+1})^{-\sigma}(\Pi_{t+1})^{-1}(1+\rho P^M_{t+1})$$
$$K\left(b_t,A_{t+1},b^S_t\right)=C^{-\sigma}_{t+1}\Pi^{-1}_{t+1}$$

we can rewrite the NKPC and government budget constraints as, respectively,

$$(1-\epsilon)+\epsilon(1-\tau_t)^{-1}Y^{\varphi}_tC^{\sigma}_tA^{-1-\varphi}_t-\phi\frac{\Pi_t}{\Pi^*}\left(\frac{\Pi_t}{\Pi^*}-1\right)+\phi\beta C^{\sigma}_tY^{-1}_tE_t\left[M(b_t,A_{t+1},b^S_t)\right]=0$$

$$0=\beta b_tC^{\sigma}_tE_t\left[L(b_t,A_{t+1},b^S_t)\right]+\beta b^S_tC^{\sigma}_tE_t\left[K\left(b_t,A_{t+1},b^S_t\right)\right]$$
$$-\frac{b_{t-1}}{\Pi_t}\left(1+\rho\beta C^{\sigma}_tE_t\left[L(b_t,A_{t+1},b^S_t)\right]\right)-\frac{b^S_{t-1}}{\Pi_t}+\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi}C^{\sigma}_t-G_t$$

The Lagrangian for the policy problem can be written as,

$$\begin{aligned}\mathcal{L}=&\left\{\frac{C^{1-\sigma}_t}{1-\sigma}+\chi\frac{G^{1-\sigma_g}_t}{1-\sigma_g}-\frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi}+\beta E_t[V(b_t,A_{t+1},b^S_t)]\right\}\\
&+\lambda_{1t}\left[Y_t\left(1-\frac{\phi}{2}\left(\frac{\Pi_t}{\Pi^*}-1\right)^2\right)-C_t-G_t\right]\\
&+\lambda_{2t}\left[\begin{array}{c}(1-\epsilon)+\epsilon(1-\tau_t)^{-1}Y^{\varphi}_tC^{\sigma}_tA^{-1-\varphi}_t-\phi\frac{\Pi_t}{\Pi^*}\left(\frac{\Pi_t}{\Pi^*}-1\right)\\+\phi\beta C^{\sigma}_tY^{-1}_tE_t\left[M(b_t,A_{t+1},b^S_t)\right]\end{array}\right]\\
&+\lambda_{3t}\left[\begin{array}{c}\beta b_tC^{\sigma}_tE_t\left[L(b_t,A_{t+1},b^S_t)\right]+\beta b^S_tC^{\sigma}_tE_t\left[K\left(b_t,A_{t+1},b^S_t\right)\right]\\-\frac{b_{t-1}}{\Pi_t}\left(1+\rho\beta C^{\sigma}_tE_t\left[L(b_t,A_{t+1},b^S_t)\right]\right)\\-\frac{b^S_{t-1}}{\Pi_t}+\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi}C^{\sigma}_t-G_t\end{array}\right]\end{aligned}$$

We can write the first order conditions for the policy problem as follows:

Consumption,

$$C_t^{-\sigma} - \lambda_{1t} + \lambda_{2t}\left[\sigma\epsilon(1-\tau_t)^{-1}Y_t^{\varphi}C_t^{\sigma-1}A_t^{-1-\varphi} + \sigma\phi\beta C_t^{\sigma-1}Y_t^{-1}E_t\left[M(b_t, A_{t+1}, b_t^S)\right]\right]$$

$$+\lambda_{3t}\left[\begin{array}{c} \sigma\beta b_t C_t^{\sigma-1}E_t\left[L(b_t, A_{t+1}, b_t^S)\right] + \sigma\beta b_t^S C_t^{\sigma-1}E_t\left[K\left(b_t, A_{t+1}, b_t^S\right)\right] \\ -\rho\sigma\beta\frac{b_{t-1}}{\Pi_t}C_t^{\sigma-1}E_t\left[L(b_t, A_{t+1}, b_t^S)\right] + \sigma\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi}C_t^{\sigma-1} \end{array}\right] = 0$$

Government spending,

$$\chi G_t^{-\sigma_g} - \lambda_{1t} - \lambda_{3t} = 0$$

Output,

$$-Y_t^{\varphi}A_t^{-1-\varphi} + \lambda_{1t}\left[1 - \frac{\phi}{2}\left(\frac{\Pi_t}{\Pi^*} - 1\right)^2\right]$$

$$+\lambda_{2t}\left[\epsilon\varphi(1-\tau_t)^{-1}Y_t^{\varphi-1}C_t^{\sigma}A_t^{-1-\varphi} - \phi\beta C_t^{\sigma}Y_t^{-2}E_t\left[M(b_t, A_{t+1}, b_t^S)\right]\right]$$

$$+\lambda_{3t}\left[(1+\varphi)Y_t^{\varphi}C_t^{\sigma}\left(\frac{\tau_t}{1-\tau_t}\right)A_t^{-1-\varphi}\right] = 0$$

Taxation,

$$\epsilon\lambda_{2t} + \lambda_{3t}Y_t = 0$$

Inflation,

$$-\lambda_{1t}\left[Y_t\frac{\phi}{\Pi^*}\left(\frac{\Pi_t}{\Pi^*} - 1\right)\right] - \lambda_{2t}\left[\frac{\phi}{\Pi^*}\left(\frac{2\Pi_t}{\Pi^*} - 1\right)\right]$$

$$+\lambda_{3t}\left[\frac{b_{t-1}}{\Pi_t^2}\left(1 + \rho\beta C_t^{\sigma}E_t\left[L(b_t, A_{t+1}, b_t^S)\right]\right) + \frac{b_{t-1}^S}{\Pi_t^2}\right] = 0$$

Government debt, b_t,

$$\beta E_t[V_1(b_t, A_{t+1}, b_t^S)] + \lambda_{2t}\left[\phi\beta C_t^{\sigma}Y_t^{-1}E_t\left[M_1(b_t, A_{t+1}, b_t^S)\right]\right]$$

$$+\beta C_t^{\sigma}\lambda_{3t}\left[\begin{array}{c} E_t\left[L(b_t, A_{t+1}, b_t^S)\right] + b_tE_t\left[L_1(b_t, A_{t+1}, b_t^S)\right] + b_t^SE_t\left[K_1\left(b_t, A_{t+1}, b_t^S\right)\right] \\ -\rho\frac{b_{t-1}}{\Pi_t}E_t\left[L_1(b_t, A_{t+1}, b_t^S)\right] \end{array}\right] = 0$$

where

$$V_1(b_t, A_{t+1}, b_t^S) \equiv \partial V(b_t, A_{t+1}, b_t^S)/\partial b_t$$

$$L_1(b_t, A_{t+1}, b_t^S) \equiv \partial L(b_t, A_{t+1}, b_t^S)/\partial b_t$$

$$M_1(b_t, A_{t+1}, b_t^S) \equiv \partial M(b_t, A_{t+1}, b_t^S)/\partial b_t$$

$$K_1(b_t, A_{t+1}, b_t^S) \equiv \partial K(b_t, A_{t+1}, b_t^S)/\partial b_t$$

Short-term government debt, b_t^S,

$$\beta E_t[V_3(b_t, A_{t+1}, b_t^S)] + \lambda_{2t}\left[\phi\beta C_t^{\sigma} Y_t^{-1} E_t\left[M_3(b_t, A_{t+1}, b_t^S)\right]\right]$$

$$+\beta C_t^{\sigma}\lambda_{3t}\left[\begin{array}{c} b_t E_t\left[L_3(b_t, A_{t+1}, b_t^S)\right] + E_t\left[K\left(b_t, A_{t+1}, b_t^S\right)\right] + b_t^S E_t\left[K_3\left(b_t, A_{t+1}, b_t^S\right)\right] \\ -\rho\frac{b_{t-1}}{\Pi_t} E_t\left[L_3(b_t, A_{t+1}, b_t^S)\right] \end{array}\right] = 0$$

where

$$V_3(b_t, A_{t+1}, b_t^S) \equiv \partial V(b_t, A_{t+1}, b_t^S)/\partial b_t^S$$

$$L_3(b_t, A_{t+1}, b_t^S) \equiv \partial L(b_t, A_{t+1}, b_t^S)/\partial b_t^S$$

$$M_3(b_t, A_{t+1}, b_t^S) \equiv \partial M(b_t, A_{t+1}, b_t^S)/\partial b_t^S$$

$$K_3(b_t, A_{t+1}, b_t^S) \equiv \partial K(b_t, A_{t+1}, b_t^S)/\partial b_t^S$$

Note that by the envelope theorem,

$$V_1(b_{t-1}, A_t, b_{t-1}^S) = -\frac{\lambda_{3t}}{\Pi_t}\left(1 + \rho\beta C_t^{\sigma} E_t\left[L(b_t, A_{t+1}, b_t^S)\right]\right)$$

$$= -\frac{\lambda_{3t}}{\Pi_t}\left(1 + \rho P_t^M\right)$$

$$V_3(b_{t-1}, A_t, b_t^S) = -\frac{\lambda_{3t}}{\Pi_t}$$

hence,

$$V_1(b_t, A_{t+1}, b_t^S) = \frac{\lambda_{3t+1}}{\Pi_{t+1}}\left(1 + \rho P_{t+1}^M\right)$$

$$V_3(b_t, A_{t+1}, b_t^S) = -\frac{\lambda_{3t+1}}{\Pi_{t+1}}$$

and the FOCs for government debt b_t and b_t^S can be rewritten as, respectively,

$$-\beta E_t\left[\frac{\lambda_{3t+1}}{\Pi_{t+1}}(1 + \rho P_{t+1}^M)\right] + \lambda_{2t}\phi\beta C_t^{\sigma} Y_t^{-1} E_t\left[M_1(b_t, A_{t+1}, b_t^S)\right]$$

$$+\beta C_t^{\sigma}\lambda_{3t}\left[\begin{array}{c} E_t\left[L(b_t, A_{t+1}, b_t^S)\right] + b_t E_t\left[L_1(b_t, A_{t+1}, b_t^S)\right] + b_t^S E_t\left[K_1\left(b_t, A_{t+1}, b_t^S\right)\right] \\ -\rho\frac{b_{t-1}}{\Pi_t} E_t\left[L_1(b_t, A_{t+1}, b_t^S)\right] \end{array}\right] = 0$$

and

$$-\beta E_t[\frac{\lambda_{3t+1}}{\Pi_{t+1}}] + \lambda_{2t}\phi\beta C_t^{\sigma} Y_t^{-1} E_t\left[M_3(b_t, A_{t+1}, b_t^S)\right]$$

$$+\beta C_t^{\sigma}\lambda_{3t}\left[\begin{array}{c} b_t E_t\left[L_3(b_t, A_{t+1}, b_t^S)\right] + E_t\left[K\left(b_t, A_{t+1}, b_t^S\right)\right] + b_t^S E_t\left[K_3\left(b_t, A_{t+1}, b_t^S\right)\right] \\ -\rho\frac{b_{t-1}}{\Pi_t}E_t\left[L_3(b_t, A_{t+1}, b_t^S)\right] \end{array}\right] = 0$$

A Note on Numerical Analysis

Given the solution from the case with exogenously given short-term debt, we can use a penalty term to gradually get the solution for the case with endogenous short-term debt. Specifically, we modify the period utility function into the following form,

$$\frac{C_t^{1-\sigma}}{1-\sigma} + \chi\frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi} - \zeta\left(b_t^S - \overline{b}^S\right)^2$$

where ζ is a scale parameter of penalty. Note that the modified period utility function is reduced into the original form, when $\zeta = 0$. This fact motivates a homotopy continuation idea. Initially, we can set ζ large enough to make sure b_t^S close to $\overline{b}^S$. Then, we incrementally decrease ζ down to zero.

Now, the FOC for b_t^S becomes,

$$-2\zeta\left(b_t^S - \overline{b}^S\right) - \beta E_t[\frac{\lambda_{3t+1}}{\Pi_{t+1}}] + \lambda_{2t}\left[\phi\beta C_t^{\sigma}Y_t^{-1}E_t\left[M_3(b_t, A_{t+1}, b_t^S)\right]\right]$$

$$+\beta C_t^{\sigma}\lambda_{3t}\left[\begin{array}{c} b_t E_t\left[L_3(b_t, A_{t+1}, b_t^S)\right] + E_t\left[K\left(b_t, A_{t+1}, b_t^S\right)\right] + b_t^S E_t\left[K_3\left(b_t, A_{t+1}, b_t^S\right)\right] \\ -\rho\frac{b_{t-1}}{\Pi_t}E_t\left[L_3(b_t, A_{t+1}, b_t^S)\right] \end{array}\right] = 0$$

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