Dickson, Liam (2014) *Topics regarding close operator algebras.* PhD thesis.

Topics regarding close operator algebras

by

Liam Dickson

A thesis submitted to the
College of Science and Engineering
at the University of Glasgow
for the degree of
Doctor of Philosophy

October 2014

In this thesis we focus on two topics. For the first we introduce a row version of Kadison and Kastler's metric on the set of C*-subalgebras of $\mathbb{B}(\mathcal{H})$. By showing C*-algebras have row length (in the sense of Pisier) of at most two we show that the row metric is equivalent to the original Kadison-Kastler metric. We then use this result to obtain universal constants for a recent perturbation result of Ino and Watatani, which states that sufficiently close intermediate subalgebras must occur as small unitary perturbations, by removing the dependence on the structure of inclusion.

Roydor has recently proved that injective von Neumann algebras are Kadison-Kastler stable in a non-self adjoint sense, extending seminal results of Christensen. We prove a one-sided version, showing that an injective von Neumann algebra which is nearly contained in a weak*-closed non-self adjoint algebra can be embedded by a similarity close to the natural inclusion map. This theorem can then be used to extend results of Cameron et al. by demonstrating Kadison-Kastler stability of certain crossed products in the non self-adjoint setting. These crossed products can be chosen to be non-amenable.

I would like to thank:

The School of Mathematics and Statistics for their support throughout the PhD.

The analysis group, Roger Smith and Erik Christensen.

Jorge, Gabriele, Neil and Stuart for performing the unenviable task of proof reading sections of this document, I very much appreciate it.

Dave and Giorgia, Joan, Robbie and Nell, and the uncles for being extremely accommodating and hospitable.

My parents, the IC, Robbie and Nell for their unfaltering moral support during the previous four years.

All my office mates for their camaraderie.

All the people that I have played with for the Glasgow Uni squash team for their commitment and most importantly for the way they conduct themselves. In particular, Neil McClean, whose bravery is an absolute inspiration.

My friends, especially: Dave, Neil, Robbie, Pete, Joan and Halliday.

I'd also like to acknowledge the contribution of some institutions which have provided welcome relief from Mathematics; Stirling Albion Football Club (in particular their magnificent 2013-2014 League 2 Playoff Championship run), all the players and the bar staff at Western Squash Club, Withered Hand (in particular the lyrical mastery on *Love in the time of ecstasy*) and the esteemed Glasgow establishment: Tennent's Bar.

Finally, thanks to Stuart for his inestimable contribution to this PhD and his guidance, patience and friendship over the past four years- it's been a total privilege.

I declare, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Liam Dickson

Introduction

In [39] Kadison and Kastler initiated the study of the uniform perturbations of operator algebras. They introduced a metric on the set of linear subspaces of operators on a given Hilbert space, and conjectured that sufficiently close operator algebras are necessarily spatially isomorphic. This conjecture was the focus of much research in the 1970's with seminal results being obtained by Christensen [9], Johnson [33], Phillips and Raeburn [55], establishing a strong form of the conjecture in the case of injective von Neumann algebras: a von Neumann algebra N sufficiently close to an injective von Neumann algebra M arises as a small unitary perturbation of M, i.e. $N = uMu^*$ for a unitary with $u \approx 1$.

Analogous questions in the context of C^*-algebras have recently been the focus of considerable research activity, and remarkable progress has been made by Christensen et al. [13] using point-norm techniques to verify the conjecture in the separable nuclear case, vastly generalising the AF case established in [11] and the continuous trace case of [50]. Both separability and point norm techniques are necessary due to the counterexamples in the non-separable case of Choi and Christensen [7] and Johnson's construction of close representations of $C[0,1] \otimes \mathcal{K}$ which are not small unitary perturbations of each other [34]. An embedding theorem for near inclusions involving separable nuclear C^*-algebras has also been obtained by combining these ideas with a strengthening of the completely positive approximation property, [28].

Amenability in various guises plays a central role in many perturbation results. Whether averaging over the unitary group or an approximate diagonal, it provides a key tool for refining certain almost multiplicative maps into homomorphisms (for example completely positive maps on injective von Neumann algebras which are close to the identity may be perturbed into genuine *-homomorphisms). In Chapter 1 we recall Johnson's definition of amenability for Banach algebras and present a deep characterisation of amenability in the C*-context; a C*-algebra is amenable if and only if it is nuclear. The appropriate formulation of the concept for von Neumann algebras is a weaker notion: Connes amenability. This fits into a hugely important equivalence of von Neumann properties including injectivity, hyperfiniteness, property P and semi-discreteness. The aforementioned results make these classes of operator algebras particularly susceptible to various averaging techniques.

Given a finite factor there is a particular natural representation, standard form, in which it is often very convenient to work (for example the von Neumann algebra and its commutant are canonically anti-isomorphic here). We discuss this representation in Chapter 2 and present some facts that will be utilised in Chapter 7. We briefly define the index of a subfactor and the basic construction introduced by Jones in [37] to study subfactors of II_1 factors. This definition motivates Watatani's definition of the index and basic construction in the C*-context which are central in Chapter 5.

In Chapter 3 we define the Kadison-Kastler metric and the notion of near containment. This allows us to make a precise statement of the conjecture due to the same authors [39]. We present the main positive results and counterexamples from the literature, mentioned above, and remark on the current state of the conjecture. We also set out notation and collect some technical results that will be required in the sequel. Finally in this section we give an outline of Christensen's proof of the Kadison-Kastler stability of injective von Neumann algebras [9] as it serves as a prototype for a number of arguments in this thesis.

The remainder of the thesis focuses on two topics. The first is to study close subalgebras of a fixed C^*-algebra. In [10] Christensen shows that sufficiently close von Neumann subalgebras of a finite von Neumann algebra arise from small unitary perturbations, and his work gives uniform estimates valid for all finite von Neumann algebras. In their recent paper Ino and Watatani [30] (see also [29]) prove an analogous theorem in the context of intermediate C^*-subalgebras [30, Proposition 3.6]: given an inclusion of C*-algebras $C \subseteq D$ with a finite index conditional expectation, and intermediate subalgebras A, B with $C \subseteq A, B \subseteq D$, if A and B are sufficiently close then they are necessarily small unitary perturbations of each other. In contrast to Christensen's work, the estimates obtained by Ino and Watatani are given in terms of the basis for $C \subseteq D$ and so depend on the inclusion $C \subseteq D$. Our first main theorem in this thesis (Theorem 5.3.2) is to obtain uniform estimates valid for all finite index inclusions.

The main ingredient in the proof of Theorem 5.3.2 is to work with a "row" version of the Kadison-Kastler metric. Natural variants of the metric have been considered previously. In particular there is a completely bounded version of the metric, where the distance between A and B is obtained as the supremum of the distances between the matrix amplifications $M_n(A)$ and $M_n(B)$. There is a deep connection between the Kadison-Kastler perturbation conjecture and Kadison's similarity problem (which asks whether every bounded unital homomorphism $A \to \mathbb{B}(H)$ from a C^*-algebra is similar to a *-homomorphism) dating back to [11] which is exemplified by the characterisation that the similarity problem is true for all C^*-algebras if and only if the completely bounded Kadison-Kastler metric is equivalent to the Kadison-Kastler metric, [6,13]. The row metric naturally fits between the usual metric and the completely bounded metric, and our first main technical observation, which may be of interest in its own right, is that the row metric is equivalent to the Kadison-Kastler metric. With this it is shown that Ino and Watatani's techniques can be refined using the row metric, displacing the need for the constant to depend on the inclusion $C \subseteq D$.

In Chapter 4 we present a survey of Pisier's intrinsic characterisation of the similarity property in terms of matrix factorisation length [51]. We then demonstrate the equivalence of the row metric and the Kadison-Kastler metric by combining the ideas in Pisier's work with Haagerup's Little Groethendieck Inequality (see [24, Lemma 3.2]) which enables us to demonstrate that rows over C^*-algebras can always be factorised in a uniform fashion, with factorisation length at most 2. The equivalence of the two metrics then follows using an argument from [53].

In the final two chapters we aim to extend some existing perturbation results to the non self-adjoint setting. We call a closed algebra of operators on a Hilbert space a non-self adjoint operator algebra. Explicitly, given an operator in such an algebra we do not assume that its adjoint is also contained in the algebra (as we would for a C* or von Neumann algebra). These algebras have been studied in detail (for example [19]), nest and limit algebras provide well known examples. In [56] Roydor shows that if N is in a certain class of non self-adjoint operator algebras and is sufficiently close to an injective von Neumann algebra M, then the algebras are similar; there exists a invertible operator (of course one cannot expect to find a unitary) $S \approx 1$ such that $S^{-1}MS = N$. This verifies a non self-adjoint version the Kadison-Kastler conjecture in this case and can be seen as the non self-adjoint analogue of Christensen's result for injective von Neumann algebras described above. In Chapter 6 we provide a one-sided version of Roydor's work (in the spirit of [28]). We show that if an injective von Neumann algebra M is approximately contained in a non self-adjoint algebra N, then there exists an invertible operator $S \approx 1$ implementing a genuine containment $S^{-1}MS \subseteq N$.

In [5] Cameron et al. construct certain von Neumann crossed products with the property that any sufficiently close von Neumann algebra must arise as a small unitary perturbation. Furthermore these crossed products can be chosen to be non-amenable thus providing the first verification of the Kadison Kastler conjecture outside the class of amenable operator algebras. We aim to extend this result to the non self-adjoint setting. Given an action $\alpha : \Gamma \curvearrowright P$ of a discrete group Γ on an amenable von Neumann algebra P we consider the crossed product $M = P \rtimes_\alpha \Gamma$ represented in standard form. We aim to show that any non self-adjoint operator algebra N satisfying $N \approx M$ and $N' \approx M'$ must be similar to M. The embedding theorem from Chapter 6 allows us to reduce to the case where both M and N contain a copy of P. We then modify the techniques developed in [5] to transfer the crossed product structure to N under a vanishing cohomology assumption on the action. This then allows us to write down an invertible operator $S \approx 1_{L^2(M)}$ such that $SMS^{-1} = N$.

Contents

Chapter 1

Averaging and amenability

1.1 Amenable Banach algebras

The notion of amenability for a Banach algebra was defined by Johnson in [32] in terms of the vanishing of a certain cohomology group. This class of amenable algebras are closely related to amenable groups and we will use the latter to provide us with examples in this chapter. We will present characterisations of amenability in the C* and von Neumann setting, these characterisations will play a fundamental role in the approximation arguments used in the remainder of this thesis.

We start by presenting Johnson's original definition of amenability. Let A be a Banach algebra. A Banach space $\mathcal{E}$ with a left and right A action is called a Banach A-bimodule if there exists a $\kappa > 0$ such that

$$\|a \cdot x\|_{\mathcal{E}} \leq \kappa \|a\|_A \|x\|_{\mathcal{E}} \qquad \text{and} \qquad \|x \cdot a\|_{\mathcal{E}} \leq \kappa \|x\|_{\mathcal{E}} \|a\|_A \qquad (a \in A, x \in \mathcal{E}). \tag{1.1}$$

A *derivation* is a linear map D from a Banach algebra A to a Banach A-bimodule that satisfies the Leibniz rule

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A). \tag{1.2}$$

We say $D : A \to \mathcal{E}$ is *inner* if there exists an element $x \in \mathcal{E}$ such that $D(a) = a \cdot x - x \cdot a$. Given a Banach algebra A and a Banach A-bimodule $\mathcal{E}$ the dual Banach space $\mathcal{E}^*$ also has a Banach A-bimodule structure. For $f \in \mathcal{E}^*$ and $a \in A$ then $a \cdot f$ is given by

$$\langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle \qquad (x \in \mathcal{E}). \tag{1.3}$$

The right action is defined similarly:

$$\langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle \qquad (x \in \mathcal{E}). \tag{1.4}$$

Definition 1.1.1. A Banach algebra A is amenable if for any Banach A-bimodule $\mathcal{E}$ all derivations from A to $\mathcal{E}^*$ are inner.

In [31] Johnson gives an intrinsic characterisation of amenable Banach algebras in terms of the existence of approximate and virtual diagonals. These constructions will be used in the sequel so we include the definitions.

Let A be a Banach algebra. The *projective tensor* product denoted $A\widehat{\otimes}A$ is formed by completing the algebraic tensor product $A \odot A$ with respect to the following norm

$$\|x\| = \inf\{\sum_j \|a_j\|\|b_j\| \quad : \quad x = \sum a_j \otimes b_j\} \qquad (x \in A \odot A) \tag{1.5}$$

The maps defined on elementary tensor as follows:

$$a \cdot (b \otimes c) = (ab \otimes c) \qquad \text{and} \qquad (b \otimes c) \cdot a = b \otimes ca \qquad (a, b, c \in A), \tag{1.6}$$

may be extended to maps from $A\widehat{\otimes}A$ to $A\widehat{\otimes}A$ by continuity (see [58, Proposition 2.3]). This gives $A\widehat{\otimes}A$ a left and right A action and it becomes a Banach A-bimodule. This action extends to its bidual $(A\widehat{\otimes}A)^{**}$.

Define $\Delta : A\widehat{\otimes}A \to A$ to be the extension of the multiplication $a \otimes b \to ab$.

Definition 1.1.2. A bounded net $(m_\alpha)_\alpha \in A\hat{\otimes}A$ is an *approximate diagonal* if

- $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ in norm for all $a \in A$, and
- $(\Delta m_\alpha)a \to a$ in norm for all $a \in A$.

An element $M \in (A\hat{\otimes}A)^{**}$ is a *virtual diagonal* if

- $a \cdot M = M \cdot a$ for all $a \in A$, and
- $a\Delta^{**}M = a$ for all $a \in A$.

Remark. If A has a virtual diagonal, then one may find an approximate diagonal $(m_\alpha)_\alpha \in A\widehat{\otimes}A$ for A that converges to M in the weak* topology [31, Proof of Lemma 1.2].

Theorem 1.1.3 (Johnson, Theorem 1.3 of [31])**.** *Let A be a Banach algebra. Then the following are equivalent:*

- *A is amenable,*
- *A has a virtual diagonal,*
- *A has an approximate diagonal.*

For a positive real number L we say A has *amenability constant* strictly less than L if there exists an approximate diagonal $(m_\alpha)_\alpha$ such that $\|m_\alpha\| < L$ for all α. If this is the case we can (and do) write $m_\alpha = \sum_j a_{\alpha,j} \otimes b_{\alpha,j}$ where $\sum_j \|a_{\alpha,j}\|\|b_{\alpha,j}\| < L$ for all α. The *amenability constant* of A is defined to be the infimum of all such L.

One source of amenable Banach algebras are those constructed from amenable groups. A discrete group G is amenable if there exists a left-invariant mean on $\ell^\infty(G)$. That is an element $m \in \ell^\infty(G)^*$ such that

- $\langle 1, m\rangle = \|m\| = 1$ and
- $\langle g \cdot \phi, m\rangle = \langle \phi, m\rangle$ for all $\phi \in \ell^\infty(G)$,

where $g \cdot \phi(t) = \phi(g^{-1}t)$.

One may construct a virtual diagonal for $\ell^1(G)$ as follows. We identify $(\ell^1(G)\widehat{\otimes}\ell^1(G))^{**}$ with $(\ell^\infty(G \times G))^*$. Then define $M \in (\ell^\infty(G \times G))^*$ as

$$\langle M, \phi\rangle = \langle m, \tilde{\phi}\rangle \qquad (\phi \in \ell^\infty(G \times G)) \tag{1.7}$$

where $\tilde{\phi} \in \ell^\infty(G)$ is defined a follows

$$\tilde{\phi}(g) = \phi(g, g^{-1}) \qquad (g \in G). \tag{1.8}$$

It can then be shown that M satisfies the properties of a virtual diagonal witnessing the amenability of $\ell^1(G)$ as a Banach algebra.

1.2 (Connes) Amenable von Neumann algebras

The above definition of amenability is too strong to yield many interesting examples in the case of von Neumann algebras. However, there is a weaker property which fits more naturally in the W^*-context. Given a von Neumann algebra M, a normal dual Banach M-module $\mathcal{E}^*$ is a dual M-module with the property that the maps $a \mapsto a \cdot \xi$ and $a \mapsto \xi \cdot a$ are weak*-weak* continuous from M to $\mathcal{E}^*$ for all $\xi \in \mathcal{E}^*$. We say a von Neumann algebra M is *Connes amenable* if every derivation from M to a normal dual M-module is inner. When there is no ambiguity we will refer to a Connes amenable von Neumann as amenable. Connes amenable von Neumann algebras arise as the bidual of amenable C*-algebras [1, Prop IV.3.3.12].

Proposition 1.2.1. *Let A be an amenable C*-algebra. Then A^{**} is Connes amenable.*

We will use the following theorem due to Johnson, Kadison and Ringrose (see [1, Theorem IV.2.5.2]) as a tool for constructing Connes amenable von Neumann algebras.

Theorem 1.2.2. *Let M be a von Neumann algebra and suppose that there exists a discrete amenable subgroup G of $\mathcal{U}(M)$ such that $G'' = M$. Then M is Connes amenable.*

It follows immediately that the group von Neumann algebra generated by an amenable group is Connes amenable. Since the finite group of direct sums of permutation matrices and diagonal matrices with entries in $\{1, -1\}$ generate finite dimensional von Neumann algebras, Theorem 1.2.2 implies that all finite dimensional von Neumann algebras are Connes amenable. We can now build up more interesting amenable von Neumann algebras from finite dimensional pieces.

Definition 1.2.3. A von Neumann algebra M is *hyperfinite* if there exists a directed collection $(M_\lambda)_\lambda$ of finite dimensional *-subalgebras such that $\cup_\lambda M_\lambda$ is weak*-dense in M. If M has separable predual then $(M_\lambda)_\lambda$ can be chosen to be an increasing sequence.

Given an inclusion of a finite dimensional C*-algebra A in a larger finite dimensional C*-algebra B we may construct the matrix units of B in such a way so that the matrix units of A are sums of the matrix units in B.

Let F be a finite dimensional algebra, the perturbation matrices and matrices diagonal matrices with entries is $\{-1, 1\}$ form a finite subgroup of the unitary group of a F which generates F as a von Neumann algebra. The previous sentences imply that one may construct finite generating sets for each finite dimensional subalgebra of a hyperfinite von Neumann algebra with separable predual in this way. This yields a locally finite (and hence amenable) generating subgroup of unitaries and hence a separable hyperfinite von Neumann algebra is Connes amenable by Theorem 1.2.2. As well as Connes amenability the hyperfinite von Neumann algebras enjoy a number of other properties. We explore some of these in the remainder of this section and see how they relate to one another.

Given an inclusion of C*-algebras $A \subseteq B$ and a completely positive map $\phi : A \to \mathbb{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, there exists, by Arveson's extension theorem, a completely positive map $\psi : B \to \mathbb{B}(\mathcal{H})$ extending ϕ. We will be interested in other target algebras that enjoy this extension property. We say a C*-algebra C is *injective* if the following condition holds: if $A \subseteq B$ is an inclusion of C*-algebras and $\phi : A \to C$ is a completely positive map, then there exists a completely positive map $\psi : B \to C$ extending ϕ. If C is represented on $\mathcal{H}$ this property is equivalent to the existence of a conditional expectation of $\mathbb{B}(\mathcal{H})$ onto C (because of Arveson's extension theorem). The latter characterisation will be more useful to us and so we use it to define injectivity for von Neumann algebras.

Definition 1.2.4. A von Neumann algebra $M \subseteq \mathbb{B}(\mathcal{H})$ is *injective* if there exists a norm one projection from $\mathbb{B}(\mathcal{H})$ to M. Such a projection is automatically a completely positive contraction (in fact it is a conditional expectation [1, Theorem II.6.10.2]).

One important result is that a von Neumann algebra is injective if and only if its commutant is [1, Theorem IV.2.2.7].

Theorem 1.2.5. *Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Then M is injective if and only if M' is injective.*

We use the above theorem to show that hyperfinite von Neumann algebras are injective. Firstly suppose $M = \cap M_i$ is the intersection of a net of injective von Neumnan algebras $(M_i)_i$ on a Hilbert space $\mathcal{H}$ with norm one projections $\theta_i : \mathbb{B}(\mathcal{H}) \to M_i$. Then taking a point weak*-limit of the θ_i's gives a conditional expectation from $\mathbb{B}(\mathcal{H})$ to M and so M is injective. Now suppose M is hyperfinite. We write $M = (\cup_i M_i)''$ for a net of finite dimensional algebras M_i. Since each finite dimensional algebra is the direct sum of matrix algebra and since direct sums of injective algebras are injective [1, IV.2.1.2] we have that the M_i are injective for all i. It follows M_i' are injective by Theorem 1.2.5 for all i and so $M' = \cap M_i'$ is injective by the first part of the paragraph. Applying Theorem 1.2.5 again shows that M is also injective.

This result may be strengthened by showing that given a Connes amenable algebra M on a Hilbert space $\mathcal{H}$ one may construct a conditional expectation from $\mathbb{B}(\mathcal{H})$ to M' (see [1, IV.2.5.4]) witnessing the injectivity of M' and hence M (again by Theorem 1.2.5).

A C*-algebra A is said to have the *Dixmier approximation property* if for every element $T \in A$, the norm closure of the convex hull of elements of the form uTu^*, for $u \in \mathcal{U}(\mathcal{H})$, has non-empty intersection with the centre $\mathcal{Z}(A)$. Every von Neumann algebra has the Dixmier approximation property (see for example [60, Theorem 2.2.2]). *Schwartz's property P* builds on these ideas.

Definition 1.2.6. A von Neumann algebra $M \subseteq \mathbb{B}(\mathcal{H})$ has *property P* if for every element $T \in \mathbb{B}(\mathcal{H})$ the weak*-closure of the convex hull of the operators of the form uTu^* with u a unitary operator in M has non-empty intersection with the commutant M';

$$\overline{\mathrm{conv}_{u\in\mathcal{U}(M)} uTu^*}^{\mathrm{w}^*} \cap M' \neq \emptyset. \tag{1.9}$$

It may be shown that this property does not depend on the representation of M (see the paragraph following [1, IV.2.2.20]) Property P may be thought of as an averaging procedure over the unitary group of M into the commutant M'.

We will sketch an argument which shows that a hyperfinite von Neumann M algebra has property P. Write $M = (\cup_i M_i)'' \subseteq \mathbb{B}(\mathcal{H})$ with each $M_i \subseteq M_j$ for $i \leq j$. Let F_i be a finite subgroup of $\mathcal{U}(M_i)$ that spans M_i. (for example pick permutation matrices and diagonal matrices with entries in $\{1, -1\}$ as before). Given an operator $T \in \mathbb{B}(\mathcal{H})$ the net $(S_i = \frac{1}{|F_i|}\sum_{u\in F_i} uTu^*)_i$ is bounded and so has a weak*-accumulation point, S say. Now for $x \in M_i$ we have $xS_j = S_jx$ for $j \geq i$ and hence $S \in M_i'$ for every i and therefore, by the density of $\cup_i M_i$, belongs to M' as required.

Along with hyperfinite and Connes amenable algebras, von Neumann algebras satisfying Property P are also injective [59].

Theorem 1.2.7. *Let M be a von Neumann algebra satisfying Property P on a Hilbert space $\mathcal{H}$, then M' is injective.*

The following remarkable result is of inestimable importance in the field of von Neumann algebras and C*-algebras. It underpins many results in this thesis.

Theorem 1.2.8. *Let M be an injective von Neumann algebra. Then M is hyperfinite.*

Connes proved Theorem 1.2.8 with the assumption that M has separable predual [15]. Subsequently, Elliott was able to remove this assumption [20] and hence obtain the theorem as stated. We collect Theorem 1.2.8 together with the results discussed in this chapter for reference in the remainder of the thesis.

Theorem 1.2.9. *Let M be a von Neumann algebra, the following are equivalent:*

- *M is Connes amenable,*
- *M is injective,*
- *M is hyperfinite,*
- *M has Property P.*

1.3 Amenable C*-algebras

Firstly we will provide an example of an amenable C*-algebra. As above we will use an amenable group to construct these algebras. This time, however, we will use an alternative characterisation of group amenability (see [22]).

Proposition 1.3.1. *A countable discrete group G is amenable if and only if there exists a sequence of finite sets $(F_n)_{n=1}^{\infty}$ of G such that for an element $g \in G$ we have*

$$\lim_{n\to\infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0 \tag{1.10}$$

where Δ is the symmetric difference of sets. We call the net $(F_n)_{n=1}^{\infty}$ a Følner sequence.

A Følner sequence can be used to construct an approximate diagonal, witnessing the amenability of $C^*(G) = C^*_r(G)$. Indeed consider the element $m_n = \frac{1}{|F_n|}\sum_{g\in F_n} g \otimes g^{-1} \in C^*(G) \odot C^*(G)$. For any $h \in G$ we have

$$\|h \cdot m_n - m_n \cdot h\|_{C^*(G)\widehat{\otimes} C^*(G)} \leq \frac{|hF_n \Delta F_n|}{|F_n|} \tag{1.11}$$

since if we have $g \in F_n$ and $hg = g' \in F_n$, then $hg \otimes g^{-1} - g' \otimes g'^{-1}h = 0$. The second condition in Definition 1.1.2 is immediate.

A C*-algebra A is *nuclear* if it satisfies the *completely positive approximation property*: for every finite set $\mathcal{F} \subseteq A$ and positive constant $\epsilon > 0$ there exists a matrix algebra $M_n(\mathbb{C})$ and completely positive contractive maps $\phi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ such that

$$\|\psi(\phi(x)) - x\| < \epsilon \qquad (x \in \mathcal{F}). \tag{1.12}$$

The protypical examples of nuclear C*-algebras are *AF-algebras* which in the separable case is the inductive limit of finite dimensional C*-algebras. The nuclearity of a group C*-algebra is characterised by the amenability of the group; $C^*(G)$ is nuclear if and only if G is amenable and in this case $C^*(G) = C^*_r(G)$.

The above example is certainly nuclear and is, in fact, a special case of the fact that nuclearity characterises amenability in the C*-setting.

Theorem 1.3.2. *Let A be a C*-algebra. Then A is amenable if and only if it is nuclear.*

The 'only if' direction is due to Connes [16]. Given an amenable C*-algebra A then Theorem 1.2.1 implies A^{**} is Connes amenable, so it follows by Theorem 1.2.9 that A^{**} is injective and hence A is nuclear. The 'if' direction is a deep theorem due to Haagerup [23].

Remark. In fact, Haagerup proves that all nuclear C*-algebras are amenable with amenability constant 1.

We briefly show how a Følner sequence may be used to witness nuclearity for C*-algebras associated with a discrete amenable group G (see [2, Theorem 2.6.8]). Let $\lambda : \mathbb{C}G \to \mathbb{B}(\ell^2(G))$ denote the left regular representation;

$$(\lambda(t)f)(s) = f(t^{-1}s) \qquad (s, t \in G \ \ f \in \ell^2(G)). \tag{1.13}$$

For each F_n in a Følner sequence let P_n be the projection in $\mathbb{B}(l^2(G))$ onto the closed linear span of $(\delta_g)_{g \in F_n}$. Let ϕ_n be the compression to the finite dimensional subalgebra $P_n\mathbb{B}(l^2(G))P_n$ and let $(e_{p,q})_{p,q \in F_n}$ be the set of matrix units. Observe that $e_{p,p}\lambda(s)e_{q,q} = e_{p,q}$ if $sq = p$ and 0 otherwise. It follows that

$$P_n\lambda(s)P_n = \sum_{p,q \in F_n} e_{p,p}\lambda(s)e_{q,q} = \sum_{p \in F_n \cap sF_n} e_{p,s^{-1}p}. \tag{1.14}$$

Next define $\psi_n : P_n\mathbb{B}(l^2(G))P_n \to C^*_\lambda(G)$ which sends $e_{p,q}$ to $\frac{1}{|F_n|}\lambda(pq^{-1})$. This map is unital completely positive (since $(\psi(e_{pq}))_{p,q\in F_n} = (\lambda(p)\lambda(q)^*)_{p,q\in F_n}$ is positive). For $s \in G$ we compute

$$\psi_n(\phi_n(\lambda(s))) = \psi_n\Big(\sum_{p\in F_n\cap sF_n} e_{p,s^{-1}p}\Big) = \frac{1}{|F_n|}\sum_{p\in F_n\cap sF_n} \lambda(s). \tag{1.15}$$

The Følner condition implies that for any $s \in G$ then we have $\|\lambda(s) - \psi_n(\phi_n(\lambda(s)))\| \to 0$ and hence that $C^*_\lambda(G)$ has the completely positive approximation property.

Chapter 2

Standard form, index and the basic construction

2.1 Standard form for finite von Neumann algebras

For a finite von Neumann algebra M with a faithful normal tracial state τ the representation given by the GNS construction with respect to τ enjoys some important properties. We sketch some of the details in this section. Define a sesquilinear form on M by

$$\langle x, y \rangle = \tau(xy^*) \qquad (x, y \in M) \tag{2.1}$$

and write $L^2(M)$ for the completion of M with respect to the norm $\|\cdot\|_2$ where $\|x\|_2^2 = \langle x, x \rangle = \tau(xx^*)$. Write ξ for the vector $1_M \in L^2(M)$. Left multiplication gives an action of M on the dense subspace $M\xi$;

$$x(y\xi) = xy\xi \qquad (x, y \in M). \tag{2.2}$$

This may be extended by continuity to give a representation of M is represented on the Hilbert space $L^2(M)$. When M is represented in this way we say it is in *standard form*. The vector ξ is cyclic and separating for M: by construction $M\xi$ is dense in $L^2(M)$ and $x\xi = 0$ implies $\tau(xx^*) = 0$ which in turn implies $x = 0$ as τ was faithful. It follows from [36, Lemma 1.2.3] that ξ is cyclic and separating for M'.

The *modular conjugation* operator J_M is defined by extending the bounded operator

$$J_M(x\xi) = x^*\xi \qquad (x \in M) \tag{2.3}$$

to $L^2(M)$ by continuity. The operator J_M is a conjugate linear isometry. We summarise some of its properties below (see [60, Section 3.6]).

Lemma 2.1.1. *Let M be a finite von Neumann algebra represented on $L^2(M)$. Then*

- $J_M^2 = 1_{L^2(M)}$
- $\langle J_M\eta, \eta'\rangle = \langle J_M\eta', \eta\rangle$ *and* $\langle J_M\eta, J_M\eta'\rangle = \langle \eta, \eta'\rangle \qquad (\eta, \eta' \in L^2(M))$
- *The map defined by $T \mapsto J_M T^* J_M$ for $T \in \mathbb{B}(\mathcal{H})$ is an anti-isomorphism.*
- $J_M M J_M = M'$

2.2 Index and the basic construction (von Neumann algebras)

The basic construction and index were introduced by Jones in [37] to study of subfactors of II_1 factors. We do not attempt a systematic exposition of the theory, instead, we restrict ourselves the definitions and some facts that will be useful in Chapter 7. Although we will not use the index in the von Neumann context we hope that the definition will help motivate the C*-counterpart introduced by Watatani [63]. This will play a central role in Chapter 5. The reader is referred to [36] for a much more complete account of this theory.

Fix a von Neumann algebra M. A *left M-module* is a Hilbert space $\mathcal{H}$ together with a unital normal *-homomorphism $\pi : M \to \mathbb{B}(\mathcal{H})$. A *right M-module* is also a Hilbert space but with a unital normal *-preserving linear map $\rho : M \to \mathbb{B}(\mathcal{H})$ that reverses the order of multiplication;

$$\rho(xy) = \rho(y)\rho(x) \qquad (x, y \in M). \tag{2.4}$$

Let M be a finite factor. Then $L^2(M)$ is a left and right M module. The left action is given by the representation of M in standard form. The map $\pi_r : M \to \mathbb{B}(L^2(M))$ defined as

$$\pi_r(x) = J_M x^* J_M \tag{2.5}$$

is a normal unital *-preserving map that reverses the order of multiplication so $L^2(M)$ is a right M-module. We will write ξx for $\pi_r(x)\xi$. It follows from Lemma 2.1.1 that $\pi_l(M)' = \pi_r(M)$.

The Hilbert space $\mathcal{H}_\infty = L^2(M) \otimes \ell^2$ is a left M-module with a unital normal *-homomorphism given by

$$\pi_l^{(\infty)} : x \mapsto \pi_l(x) \otimes \mathrm{id}_{\ell^2} \in \mathbb{B}(\mathcal{H}_\infty) \qquad (x \in M). \tag{2.6}$$

Let $M_\infty(M) = M \overline{\otimes} \mathbb{B}(\ell^2)$. Then $\mathcal{H}_\infty$ is a right $M_\infty(M)$-module under the map given by right matrix multiplication

$$\pi_r^{(\infty)}(x)\xi = \xi x = (\sum_{k=1}^{\infty} \xi_k x_{k1}, \sum_{k=1}^{\infty} \xi_k x_{k2}, \dots) \tag{2.7}$$

for $x = (x_{ij})_{ij} \in M_\infty(M)$ and $\xi = (\xi_1, \xi_2, \dots) \in \mathcal{H}_\infty$. By Tomita's commutation theorem (see [36, Lemma 2.1.5]) we have $\pi_l^{(\infty)}(M)' = \pi_r^{(\infty)}(M_\infty(M))$. The following proposition is the key tool in classifying left modules of a II_1 factor (see [36, Theorem 2.1.6]).

Proposition 2.2.1. *Let $\mathcal{K}$ be a separable left M-module. Then there exists a projection $p \in \pi_l^{(\infty)}(M)'$ such that $\mathcal{K}$ is isomorphic to $p(\mathcal{H}_\infty)$ as left M-modules.*

We can therefore write a separable left M-module $\mathcal{K}$ as $\mathcal{K} = p\mathcal{H}_\infty = \mathcal{H}_\infty q$ for some $q \in M_\infty(M)$. There is a natural trace Tr on $M_\infty(M)$ defined by setting $\mathrm{Tr}((x_{ij})_{ij}) = \sum_j \tau(x_{jj})$. The projection q above is uniquely determined up to the trace Tr and so we may define the *dimension* of $\mathcal{K}$ over M as

$$\dim_M(\mathcal{K}) = \mathrm{Tr}(q). \tag{2.8}$$

One of Jones' main goals was to study the inclusion of II_1 factors $N \subseteq M$. In such a situation there is always a unique trace preserving conditional expectation from M to N denoted $\mathbb{E}_P^M$. Jones defined the *index* of an inclusion $N \subseteq M$ of II_1 factors to be

$$[M : N] = \dim_N(L^2(M)). \tag{2.9}$$

We will not pursue the subject of index of subfactors any further. We only note the following proposition due to Jones (see [36, Proposition 4.3.3]) which will be of particular relevance in the analogue C*-theory.

Proposition 2.2.2. *Let $N \subseteq M$ be an inclusion of II_1 factors with $[M : N] < \infty$ then there exists* *a finite set $(v_1, \dots v_l)$ of M such that*

$$x = \sum_{i=1}^{l} v_i \mathbb{E}_N^M(v_i^* x) \qquad (x \in D). \tag{2.10}$$

We now turn to the basic construction. Let $P \subseteq M$ be an inclusion of an arbitrary von Neumann algebra in a II_1 factor. Write e_P for the orthogonal projection of $L^2(M)$ onto the closed subspace $\overline{P\xi}$. The basic construction is the von Neumann algebra generated by M and e_P in $\mathbb{B}(L^2(M))$ and is written $\langle M, e_P \rangle = (M \cup P)''$. The range of e_P when restricted to $M\xi \subseteq L^2(M)$ is $P\xi$ and the resulting map is the unique trace preserving conditional expectation from M to P (see [60, Lemma 3.6.2]). We collect some basic facts that will be relevant in the sequel (see for example Chapter 3.1 of [36]).

Lemma 2.2.3. *With $P \subseteq M$ an inclusion of an arbitrary von Neumann algebra in a II_1 factor. We have the following:*

- $P = M \cap \{e_p\}'$;
- $\langle M, e_P \rangle' = J_M P J_M$;
- $p \mapsto pe_P$ *is an algebraic homomorphism from P to Pe_P and hence is isometric;*
- *for $x \in M$ we have $e_P x e_P = \mathbb{E}_P^M(x) e_P$ and $e_P \langle M, e_P \rangle e_P = Pe_P$.*

If $[M : P] < \infty$ then $\langle M, e_P \rangle$ is also a II_1 factor. This process can be iterated to construct *towers* which were studied extensively by Jones (see Section 3.3 in [36] for more details).

2.3 Finite index and the basic construction (C*-algebras)

In this section we develop an analogous theory of index and the basic construction in the context of C*-algebras. These concepts were introduced by Watatani in [63].

As Jones studied the inclusion $M \subseteq N$ of II_1 factors we focus on an inclusion of $B \subseteq D$ of C*-algebras. Recall that in the II_1 factor case there exists a normal faithful tracial state on N and a unique trace preserving conditional expectation $\mathbb{E}_M^N$.

In general, given an arbitrary inclusion of C*-algebras, a conditional expectation need not even exist. As $C_r^*(\mathbb{F}_2)$ is exact it may be embedded by Kirchberg's embedding theorem [43] in $\mathcal{O}_2$. There cannot exist a conditional expectation from $\mathcal{O}_2$ to the embedded copy of $C_r^*(\mathbb{F}_2)$ as this would imply that $C_r^*(\mathbb{F}_2)$ is nuclear which is false as $\mathbb{F}_2$ is not amenable.

To define the basic construction for C*-algebras we assume the existence of a faithful conditional expectation $E_B : D \to B$ and use this to construct a Hilbert module relative to the conditional expectation. We can then represent D as adjointable operators on the Hilbert module and define an analogy of the Jones projection.

Suppose $B \subseteq D$ is an inclusion of C*-algebras with a faithful conditional expectation $E_B : D \to B$. We assign a B-valued sesquilinear form to D as follows

$$\langle x, y\rangle_B = E_B(x^*y) \qquad (x, y \in D). \tag{2.11}$$

The completion $\mathcal{E}$ of D with respect to the norm $\|\cdot\|_B = \|\langle\cdot,\cdot\rangle_B\|^{1/2}$ is a Hilbert B-module when equipped with the inner product above. Let $\eta : D \to \mathcal{E}$ be the natural inclusion map which is injective as E_B was assumed to be faithful.

A linear map $T : \mathcal{E} \to \mathcal{E}$ is adjointable if there exists another linear map $T^* : \mathcal{E} \to \mathcal{E}$ which satisfies

$$\langle Tx, y\rangle_B = \langle x, T^*y\rangle_B \qquad (x, y \in \mathcal{E}). \tag{2.12}$$

Such maps are automatically linear, bounded and B-modular (see the paragraph following Definition 3.2.8 in [46]). We write $\mathbb{B}(\mathcal{E})$ to denote the C*-algebra of adjointable maps on $\mathcal{E}$.

The Jones projection $e_B \in \mathbb{B}(\mathcal{E})$ is defined by extending

$$e_B(\eta(x)) = \eta(E_B(x)) \qquad (x \in D) \tag{2.13}$$

to $\mathcal{E}$ by continuity. For $x, y \in D$ we have

$$\begin{aligned}\langle e_B(\eta(x)), \eta(y)\rangle_B &= E_B((E_B(x)^*y) \\ &= E_B(x^*)E_B(y) \\ &= E_B(x^*E_B(y)) \\ &= \langle \eta(x), e_B(\eta(y))\rangle_B \end{aligned} \tag{2.14}$$

and so e_B is self adjoint by density. The left regular representation is given by the *-homomorphism $\lambda : D \to \mathbb{B}(\mathcal{E})$ where, for $x \in D$, $\lambda(x)$ is defined by extending

$$\lambda(x)(\eta(y)) = \eta(xy) \qquad (y \in D) \tag{2.15}$$

to $\mathcal{E}$ by continuity. A direct calculation shows that the adjoint of $\lambda(x)$ is given by $\lambda(x^*)$. We recall the following facts relating the left regular representation and the Jones projection [63, Lemma 2.1.1.] (compare with Lemma 2.2.3).

Lemma 2.3.1. *With E_B, e_B and λ as above we have;*

1. *for all $x \in D$ we have $\lambda(x)e_B = e_B\lambda(x)$ if and only if $x \in B$,*
2. *$e_B\lambda(x)e_B = \lambda(E_B(x))e_B$ for all $x \in D$,*
3. *$x \mapsto \lambda(x)e_B$ is an injective *-homomorphism of B into $\mathbb{B}(\mathcal{E})$.*

We now turn to defining finite index inclusions of C*-algebras. Again we lack the structure to proceed as in the von Neumann case. Instead we say an inclusion $C \subseteq D$ is *finite index* if there exists a finite index conditional expectation E_C^D.

Definition 2.3.2. Let $C \subseteq D$ be an inclusion of C*-algebras. A conditional expectation $E : D \to C$ is of *finite index* if there exists a finite set $v_1, \ldots, v_n \in D$ such that

$$x = \sum_{i=1}^{n} v_i E(v_i^* x) = \sum_{i=1}^{n} E(xv_i)v_i^* \qquad (x \in D). \tag{2.16}$$

The set $\{v_1, \ldots, v_n\}$ is called a quasi-basis for E. We define the *index* of E as $\sum_{i=1}^{n} v_i v_i^*$.

This definition mirrors the identity (2.10) in the II_1 case. It follows immediately from Lemma 2.2.2 that an inclusion $N \subseteq M$ of II_1 factors with $[M : N] < \infty$ is also finite index inclusion of C*-algebras with the conditional expectation provided by $\mathbb{E}_N^M$ (it can be shown [63, Proposition 2.5.1] that the C* index of $\mathbb{E}_N^M$ is equal to $[M : N]1_M$).

Finite index conditional expectations will play a central role in Chapter 5 and we record some properties which will be required in the sequel. The following observations are standard and can be found in [63] although we sketch some details.

A finite index conditional expectation is necessarily faithful [63].

The value of the index is independent of the choice of quasi-bases. Indeed, let $\{v_1, \dots, v_n\}$ and $\{u_1, \dots, u_m\}$ be two quasi-bases for E then

$$\sum_{i=1}^{n} v_i v_i^* = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} u_j E(u_j^* v_i) \right) v_i^* = \sum_{j=1}^{m} u_j \left(\sum_{i=1}^{n} E(u_j^* v_i) v_i^* \right) = \sum_{j=1}^{m} u_j u_j^*. \tag{2.17}$$

The index is central in D. If $x \in D$ we have

$$\sum_{i=1}^{n} x v_i v_i^* = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} v_j E(v_j^* x v_i) \right) v_i^* = \sum_{j=1}^{n} v_j \left(\sum_{i=1}^{n} E(v_j^* x v_i) v_i^* \right) = \sum_{j=1}^{n} v_j v_j^* x \tag{2.18}$$

Finally, the index is positive and invertible. Let $v = (v_1, \dots v_n) \in M_{1,n}(D)$ so that vv^* is the index and $1_D = vE(v^*)$. Using Kadison's inequality we have

$$1_D = vE^{(n,1)}(v^*)E^{(1,n)}(v)v^* \leq vE^{(n)}(v^*v)v^* \leq \|v\|^2 (vv^*). \tag{2.19}$$

Example 2.3.3. One example of a finite index conditional expectation is

$$\tau \otimes \mathrm{id} : M_n(\mathbb{C}) \otimes A \to \mathbb{C}1 \otimes A \tag{2.20}$$

where A is a unital C*-algebra and τ is the normalised trace. Let $\{e_{ij}\}_{i,j=1}^n$ be a set of matrix units for $M_n(\mathbb{C})$, the elements $\{\sqrt{n} e_{ij} \otimes 1_A\}_{i,j=1}^n$ form a quasi-basis for $\tau \otimes \mathrm{id}$. Indeed, for an elementary tensor $e_{kl} \otimes x \in M_n(\mathbb{C}) \otimes A$ we have

$$n \sum_{i,j=1}^{n} (e_{ij} \otimes 1_A)(\tau \otimes \mathrm{id}) \left((e_{ji} \otimes 1_A)(e_{kl} \otimes x) \right) \tag{2.21}$$

$$= n \sum_{j=1}^{n} (e_{kj} \otimes 1_A)(\tau \otimes \mathrm{id})((e_{jl} \otimes x)) \tag{2.22}$$

$$= n(e_{kl} \otimes 1_A)(\frac{1}{n} 1_{M_n(\mathbb{C})} \otimes x) \tag{2.23}$$

$$= e_{kl} \otimes x. \tag{2.24}$$

The index of $\tau \otimes \mathrm{id}$ is $n^2(1_{M_n(\mathbb{C})} \otimes 1_A)$.

Example 2.3.4. Let G and H be discrete groups with $[G : H] < \infty$ then $C_r^*(H) \subseteq C_r^*(G)$ is finite index with index $[G : H] 1_{\ell^2(G)}$.

If $C \subseteq A \subseteq D$ is an inclusion of C*-algebras so that there exists a conditional expectation E_C^D of finite index and a conditional expectation E_A^D then the restriction $E_C^A := E_C^D|_A : A \to C$ is a conditional expectation of finite index. In fact if $(v_i)_{i=1}^n$ is a quasi-basis for E_C^D then $(E_A^D(v_i))_{i=1}^n$ is a quasi-basis for E_C^A.

Chapter 3

Perturbation theory

3.1 The Kadison-Kastler metric

Before proceeding we introduce some notation. Let A be an operator algebra. We write A_1 to denote the unit ball of A;

$$A_1 = \{x \in A : \ \ \|x\| \leq 1\}. \tag{3.1}$$

The set of all projections in A and unitaries in A will be denoted by A_p and A_u respectively.

There is a natural metric on the collection of operators on a Hilbert space induced by the norm. We will show how this metric is used to induce a metric on the class of operator algebras on a fixed Hilbert space. We study the structural implications of operator algebras that are close in this sense and to what extent such C*-algebras are *-isomorphic.

We start by recalling the definition of the *Hausdorff distance* d_H between two closed subspaces U and V of a metric space X with metric d_X,

$$d_H(U, V) = \max\{\sup_{y \in V} \inf_{x \in U} d_X(x, y), \sup_{x \in U} \inf_{y \in V} d_X(y, x)\}. \tag{3.2}$$

Intuitively this means that if $d_H(U, V)$ is small then every point in the space U (respectively V) may be 'approximated' by a point in V (respectively U) that is close with respect to the metric d_X. This Hausdorff distance may be thought of as the minimum tolerance with which one can approximate (using the metric) every point of one subspace with a point in the other (and vice versa).

In [39] Kadison and Kastler introduced a metric on the class of all operator algebras on a fixed Hilbert space by measuring the Hausdorff distance between their unit balls, we will refer to this as the *Kadison-Kastler metric.*

Definition 3.1.1. Let A and B be closed subspaces of bounded operators on a fixed Hilbert space, then

$$d(A, B) = \max\{\sup_{y\in B_1}\inf_{x\in A_1}\|x-y\|, \sup_{x\in A_1}\inf_{y\in B_1}\|y-x\|\} \tag{3.3}$$

Remark. It is straightforward to check that d is in fact a metric and $d(A, B) \le 1$ since any element in the unit ball of any operator algebra may always be approximated by 0.

Subsequently, in [11], Christensen introduced a one-side version of the Kadison-Kastler metric.

Definition 3.1.2. Let A and B be operator algebras on a fixed Hilbert space and let $\gamma > 0$ be a constant. We write $A \subseteq_\gamma B$ if for every element $x \in A_1$ there exists an element $y \in B$ such that $\|x - y\| < \gamma$.

If the above situation does occur we refer to it as a *near inclusion* or we say that A is *nearly contained* in B. Notice we do not require the approximating element in B to belong to the unit ball. However, it is true that if $A \subseteq_\gamma B$ and $B \subseteq_\gamma A$ then $d(A, B) \le 2\gamma$, for if $x \in A_1$ then we may find an element $y \in B$ such that $\|x - y\| \le \gamma$ and we may assume $1 < \|y\| \le 1 + \gamma$. Then we have

$$\|x - \frac{y}{\|y\|}\| \le \|x - y\| + (1 - \frac{1}{\|y\|})\|y\| \le \gamma + (1 - \frac{1}{1+\gamma})(1 + \gamma) = 2\gamma. \tag{3.4}$$

The next lemma deals with the situation where we have a near containment on one side and a genuine containment on the other. It will be used frequently in the sequel to demonstrate surjectivity. It is folklore but we provide a proof for completeness.

Lemma 3.1.3. *Let A and B be closed subspaces of a Banach algebra and $0 < \gamma < 1$ a constant. If $A \subseteq_\gamma B$ and $B \subseteq A$ then $A = B$.*

Proof. Fix $x \in A_1$, we will show that x belongs to B. To this end, we will inductively construct a sequence of elements $(y_n)_n$ in B such that $\|x - y_n\| \le \gamma^n$ and the result will follow since B is closed. By hypothesis we may find an element $y_1 \in B$ such that $\|x - y_1\| \le \gamma$ which verifies the inductive hypothesis at the first level. Assume the hypothesis is true at the n^{th} level so $\|x - y_n\| \le \gamma^n$ for some $y_n \in B$. Then $x' = \frac{(x-y_n)}{\|x-y_n\|}$ is an element in the unit ball of A so we may find an element y'

such that $\|x' - y'\| \leq \gamma$. Then

$$\|x - y_n - \|x - y_n\|y'\| = \|x - y_n\|\|x' - y'\| \leq \|x - y_n\|\gamma \leq \gamma^{n+1} \tag{3.5}$$

which verifies the hypothesis as the $(n+1)^{th}$ level with $y_{n+1} = y_n + \|x - y_n\|y' \in B$. □

It is often useful to work with the Hausdorff distance between two algebras after tensoring with the compact operators. This leads to the *completely bounded metric*.

Definition 3.1.4. Let A and B be operator algebras on a Hilbert space $\mathcal{H}$ then the *completely bounded metric* is given by

$$d_{\mathrm{cb}}(A, B) = \sup_n d(M_n(A), M_n(B)). \tag{3.6}$$

Remark. Equivalently one may define $d_{\mathrm{cb}}(A, B) = d(A \otimes \mathbb{K}, B \otimes \mathbb{K})$. To see that the right-hand side is at most the left-hand side set $\epsilon > 0$, $d_{\mathrm{cb}}(A, B) = \gamma$ and let x be in the unit ball of $A \otimes \mathbb{K}$. Using a density argument we may find an element y in the unit ball of $A \otimes M_n \cong M_n(A)$ for some $n \in \mathbb{N}$ such that $\|x - y\|_{A \otimes \mathbb{K}} < \epsilon$. Then find an element z in the unit ball of $M_n(B)$ such that $\|y - z\| \leq \gamma$. Applying the triangle inequality and a symmetric argument gives $d(A \otimes \mathbb{K}, B \otimes \mathbb{K}) \leq d_{\mathrm{cb}}(A, B) + \epsilon$ and the claim follows since ϵ was arbitrary.

Clearly $d(A, B) \leq d_{\mathrm{cb}}(A, B)$, we investigate when these metrics are equivalent in Chapter 4 (see also [13, 14] for properties of d_{cb}). We also introduce an intermediate row metric d_{row} which sits naturally between d and d_{cb}. It seems that this metric provides the most appropriate framework in some situations; it will be used extensively in Chapter 5. We will see (Theorem 4.5.2) that d and d_{row} are equivalent metrics on the class of C*-algebras.

Definition 3.1.5. Let A and B be operator algebras on a Hilbert space $\mathcal{H}$ then the *row metric* is given by

$$d_{\mathrm{row}}(A, B) = \sup_n d(M_{1,n}(A), M_{1,n}(B)). \tag{3.7}$$

Where we consider the natural inclusion of $M_{1,n}(A)$ and $M_{1,n}(B)$ in $\mathbb{B}(\mathcal{H}^n)$ in order to compute $d(M_{1,n}(A), M_{1,n}(B))$.

With these metrics in hand we develop some tools to deal with close operators and outline some facts regarding close operator algebras. The following lemma is a standard calculation.

Lemma 3.1.6. *Let $\mathcal{H}$ be a Hilbert space and $p \in \mathbb{B}(\mathcal{H})$ be a projection. If $r \in \mathbb{B}(\mathcal{H})$ is a self adjoint element satisfying $\|p - r\| \leq \gamma$, there exists a projection $q \in C^*(r)$ such that $\|p - q\| \leq 2\gamma$.*

Proof. Assume $\gamma < 1/2$. For any $\lambda \in \mathbb{C}$ such that $d(\lambda, \{0,1\}) > \gamma$ we have $\|(p - \lambda 1_{\mathcal{H}})^{-1}\| \leq \frac{1}{d(\lambda,\{0,1\})}$ and so

$$\|(p - \lambda 1_{\mathcal{H}})^{-1}(r - \lambda 1_{\mathcal{H}}) - 1_{\mathcal{H}}\| = \|(p - \lambda 1_{\mathcal{H}})^{-1}(r - p)\| < 1. \tag{3.8}$$

Therefore $r - \lambda 1_{\mathcal{H}}$ is left invertible and a similar argument shows it is right invertible so that $\lambda \notin \mathrm{sp}(r)$. The characteristic function $\chi_{[1-\gamma,1+\gamma]}$ is continuous on the spectrum of r. By continuous functional calculus $q = \chi_{[1-\gamma,1+\gamma]}(r) \in C^*(r)$ is a projection and $\|r - q\| \leq \gamma$. The stated bound follows from the triangle inequality. □

Remark. It is immediate from this that close C*-algebras have close lattices of projections (see [8, Lemma 2.1]). If $d(A, B) \leq \gamma$ and $p \in A_p$ we may find an element s in B such that $\|p - s\| \leq \gamma$. The self-adjoint element $r = \frac{s+s^*}{2}$ satisfies the hypothesis of Lemma 3.1.7 and so there exists a projection $q \in C^*(1_{\mathcal{H}}, r) \subseteq B$ such that $\|p - q\| \leq 2\gamma$. Therefore

$$d(A_p, B_p) \leq 2d(A, B). \tag{3.9}$$

An important tool, often used in conjunction with the previous lemma, is the unitary equivalence of close projections (for example see [47, Lemma 6.2.1]).

Lemma 3.1.7. *Let p and q be two projections on a Hilbert space $\mathcal{H}$ such that $\|p - q\| < 1$ there exists a unitary $u \in C^*(p, q, 1_{\mathcal{H}})$ such that $\|1_{\mathcal{H}} - u\| \leq \sqrt{2}\|p - q\|$ and $upu^* = q$.*

The following lemma is due to Christensen [8, Lemma 2.7] and is the key tool used by Khoshkam [40, Lemma 1.10] who established that close algebras have close unitary groups;

$$d(A_u, B_u) \leq \sqrt{2}d(A, B). \tag{3.10}$$

Lemma 3.1.8. *Let x be a operator on a Hilbert space $\mathcal{H}$ and let $x = u|x|$ be its polar decomposition. If $\|1_{\mathcal{H}} - x\| \leq \gamma < 1$ then $\|1_{\mathcal{H}} - u\| \leq \sqrt{2}\gamma$.*

Proof. By hypothesis x is invertible and u is a unitary so

$$\||x| - u\| = \||x| - u^*\| = \|x - 1_{\mathcal{H}}\| \leq \gamma. \tag{3.11}$$

Let

$$V := \{\lambda \in \mathbb{C} : -\gamma \leq \mathrm{Re}(\lambda) \leq \|x\| + \gamma \text{ and } -\gamma \leq \mathrm{Im}(\lambda) \leq \gamma\}. \tag{3.12}$$

Suppose $\lambda \notin V$ and find $\alpha > 0$ such that $d(\lambda, V) > \alpha$. Fix a unit vector $\xi \in \mathcal{H}$ and let p be the projection onto $\mathbb{C}\xi$. We have $p|x|p\xi = \beta\xi$ for $0 \leq \beta \leq \|x\|$. Therefore

$$\|(\lambda 1_{\mathcal{H}} - p|x|p)\xi\| = \|p(\lambda 1_{\mathcal{H}} - |x|)\xi\| \leq \|(\lambda 1_{\mathcal{H}} - |x|)\xi\| \tag{3.13}$$

so

$$\begin{aligned} \|(\lambda 1_{\mathcal{H}} - u)\xi\| &\geq \|(\lambda 1_{\mathcal{H}} - |x|)\xi\| - \|(|x| - u)\xi\| \\ &\geq \|(\lambda 1_{\mathcal{H}} - p|x|p)\xi\| - \||x| - u\| \\ &\geq |\lambda - \beta|\|\xi\| - \gamma > \alpha \end{aligned} \tag{3.14}$$

by the choice of γ . Therefore $\lambda 1_{\mathcal{H}} - u$ is bounded below and as it also has dense range it is invertible. We have established $\mathrm{sp}(u) \subseteq V$ and the bound follows by a spectral radius calculation using the normality of u. □

3.2 The Kadison-Kastler conjecture

This study of close operator algebras was initiated by Kadison and Kastler in [39]. They investigated the type decomposition of neighbouring von Neumann algebras and were able to show that two sufficiently close factors which contain the identity operator have the same type. A number of other properties are inherited by close operator algebras; we collect some of these below. The first and third statements are due to Christensen [8, Lemma 2.3] and [11, Theorem 6.5] respectively. The second is folklore and can be found as Proposition 2.9 [14]. The last claims follows by the Kaplansky density theorem and Lemma 3.1.3.

Proposition 3.2.1. *Suppose A and B are C*-algebras on a fixed Hilbert space. Then*

- *if $d(A, B) \leq 1/4$ and A is abelian then B is abelian,*
- *if $d(A, B) \leq 1/2$ and A is separable then B is separable,*
- *if $d(A, B) \leq 1/101$ and A is nuclear then B is nuclear.*
- *if $d(A, B) \leq \gamma < 1/2$ and A is a von Neumann algebra then B is a von Neumann algebra.*

Under suitable hypothesis Khoskham proved in [40] and [41] that completely close C*-algebras have isomorphic K-theories and are KK-equivalent. Perera et al. proved in [49] that completely close C*-algebras have isomorphic Cuntz semigroups.

Theorem 3.2.2. *Let A and B be C*-algebras on a fixed Hilbert space such that $d_{cb}(A, B) \leq 1/42$ then $Cu(A) \cong Cu(B)$.*

We now turn to the question of when close operator algebras occur. One way of constructing a close operator algebra is by a *small unitary perturbation*. Namely, if A is a C*-algebra on a Hilbert space $\mathcal{H}$ and $u \in \mathbb{B}(\mathcal{H})$ is a unitary such that $\|u - 1_{\mathcal{H}}\| \leq \gamma$ then $d(uAu^*, A) \leq 2\gamma$. To see this let $x \in A_1$, then uxu^* is in the unit ball of uAu^* and

$$\|x - uxu^*\| = \|(1_{\mathcal{H}} - u)x\| + \|ux(1_{\mathcal{H}} - u^*)\| \leq 2\gamma. \tag{3.15}$$

In 1972 Kadison and Kastler [39] conjectured that all sufficiently close C*-algebras on the same Hilbert space are isomorphic. One may strengthen this conjecture by asking for the isomorphism to be implemented by a unitary and an even stronger form would ask for the unitary to be close to the identity operator; do all close C*-algebras occur via small unitary perturbations? Cameron et al. [5, Definition 2.1.2] make this precise by introducing terminology to describe these three conditions in a space independent setting. The following is based on this definition but we consider C*-algebras on a fixed Hilbert space and attempt to classify neighbouring C*-algebras which are also represented on this Hilbert space.

Definition 3.2.3. Let A be a C*-algebra on a Hilbert space $\mathcal{H}$.

- We say that A is *strongly Kadison-Kastler stable* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that given any C*-algebra $B \subseteq \mathbb{B}(\mathcal{H})$ with $d(A, B) < \delta$ there exists a unitary operator u on $\mathcal{H}$ such that $uAu^* = B$ and $\|1_{\mathcal{H}} - u\| \leq \epsilon$.

- We say that A is *Kadison-Kastler stable* if there exists a $\delta > 0$ such that for any C*-algebra B on $\mathcal{H}$ with $d(A, B) < \delta$ there exists a unitary operator u on $\mathcal{H}$ such that $uAu^* = B$.

- We say that A is *weakly Kadison-Kastler stable* if there exists $\delta > 0$ such that any C*-algebra B on $\mathcal{H}$ satisfying $d(A, B) < \delta$ satisfies $A \cong B$.

Remark. In Chapter 6 and 7 we consider an extension of this conjecture to the non self-adjoint setting: for a fixed algebras A on $\mathcal{H}$ we ask whether a sufficiently close closed (in an appropriate sense) subalgebra of $\mathbb{B}(\mathcal{H})$ is isomorphic (or indeed similar) to A.

Proposition 3.2.1 and Theorem 3.2.2 certainly give some weight to the conjecture and, in fact, it has been established for a number of classes of operator algebras. However, we first present the two main counterexamples which demonstrate its limitations.

Theorem 3.2.4. *For any $\epsilon > 0$ there exist Hilbert spaces $\mathcal{H}, \mathcal{K}$ and C*-algebras A and B on $\mathcal{H}$ that are isomorphic to $C[0,1] \otimes \mathbb{K}(\mathcal{K})$ such that $d(A, B) < \epsilon$ but there does not exist a *-homomorphism $\phi : A \to B$ that satisfies*

$$\|\phi(x) - x\| \leq \frac{1}{70}\|x\| \qquad (x \in A). \tag{3.16}$$

The above theorem due to Johnson [34] demonstrates the existence of close non strongly Kadison-Kastler stable C*-algebras (isomorphic to $C([0,1], \mathbb{K})$) and so the conjecture cannot be true in its strongest form. Subsequently in [7] Choi and Christensen construct arbitrarily close non-isomorphic C*-algebras disproving the weakest form of the conjecture. However, the algebras in this construction are non-separable and so the conjecture that all separable C*-algebras are Kadison-Kastler stable remains open.

Theorem 3.2.5. *For any $\epsilon > 0$ there exist C*-algebras A and B on a Hilbert space $\mathcal{H}$ such that $d(A, B) < \epsilon$ but A and B are not *-isomorphic.*

The first main positive result was to establish the strongest form of the conjecture for injective von Neumann algebras. The result stated below is as presented by Christensen in [11, Corollary 4.2 (c)] however Phillips and Raeburn obtain a similar result in [50]. Christensen was also able to remove the assumption that N is injective although with worse estimates (Theorem 4.3 and Corollary 4.4 of [11]). This demonstrates the strong Kadison-Kastler stability of injective von Neumann algebras.

Theorem 3.2.6. *Let M and N be injective von Neumann algebras on a Hilbert space $\mathcal{H}$ such that $d(M, N) < \frac{1}{8}$. Then there exists a unitary $u \in (M \cup N)''$ such that $uMu^* = N$ and $\|1_{\mathcal{H}} - u\| \leq 12d(M, N)$.*

Christensen also verified the conjecture for abelian and for separable AF C*-algebras in [11]. These results were generalised to all separable nuclear C*-algebras by Christensen et al. in their seminal paper [14].

Theorem 3.2.7 (Theorem 5.4 of [14]). *Let A and B be C*-algebras acting on a separable Hilbert space $\mathcal{H}$. Suppose that A is separable and nuclear, and that $d(A, B) < 10^{-11}$. Then there exists a unitary $u \in (A \cup B)''$ such that $uAu^* = B$*

The algebras in Theorem 3.2.4 are nuclear so one cannot, in general, choose the unitary in Theorem 3.2.7 to be close to the identity.

The first positive result outside the class of amenable operator algebras was given by Cameron et al. in [5], we record it below.

Theorem 3.2.8. *Let $n \geq 3$ and let $\alpha : SL_n(\mathbb{Z}) \curvearrowright (X, \mu)$ be a free ergodic and measure preserving action of $SL_n(\mathbb{Z})$ on a standard nonatomic probability space (X, μ). Write $M = (L^\infty(X, \mu) \rtimes_\alpha SL_n(\mathbb{Z}))\overline{\otimes}R$, where R is the hyperfinite II_1 factor. For $\epsilon > 0$, there exists $\delta > 0$ with the following property: given a normal unital representation $M \subseteq \mathbb{B}(\mathcal{H})$ and another von Neumann algebra N on $\mathcal{H}$ with $d(M, N) < \delta$, there exists a unitary $u \in \mathbb{B}(\mathcal{H})$ with $\|u - 1_{\mathcal{H}}\| < \epsilon$ and $uMu^* = N$.*

We will extend this result (Theorem 7.5.2) to non-self adjoint algebras in Chapter 7.

3.3 Some remarks on Christensen's proof of the Kadison-Kastler conjecture in the injective case

For the remainder of this chapter we will sketch some of the ideas behind the proof of Theorem 3.2.6 as they will play a crucial role in the sequel. The proof of the theorem (and most perturbation arguments) may be roughly divided into three main steps

1. Construct a completely positive map from M to N that is uniformly close to the inclusion of M into $\mathbb{B}(\mathcal{H})$.
2. Perturb this map into a *-homomorphism from M to N while still retaining a uniform bound relative to the inclusion map $\iota_M : M \to \mathbb{B}(\mathcal{H})$.
3. Show that this *-homomorphism is surjective and spatially implemented.

We focus on the first two steps here. Since N is injective there exists a norm one projection Φ (which is automatically completely positive) from $\mathbb{B}(\mathcal{H})$ to N. To complete the first step we need only restrict Φ to M for if $x \in M_1$ there exists an element $y \in N_1$ such that $\|x - y\| \leq \gamma$ so that

$$\|\Phi_M(x) - x\| \leq \|\Phi(x - y)\| + \|y - x\| \leq 2\gamma. \tag{3.17}$$

This witnesses $\|\Phi_M - \iota_M\| \leq 2\gamma$ where $\iota_M : M \to \mathbb{B}(\mathcal{H})$ is the canonical inclusion map.

Next we show how Christensen perturbs a unital normal completely positive map ϕ that is close to the natural inclusion into a genuine *-homomorphism. Since ϕ is normal, by Stinespring's Theorem for normal completely positive maps there exists a Hilbert space $\mathcal{K}$, a projection $p \in \mathbb{B}(\mathcal{K})$ and a normal *-homomorphism $\pi : M \to \mathbb{B}(\mathcal{K})$ such that $\phi(x) = p\pi(x)p$ for all $x \in M$. As a unital completely positive map is a *-homomorphism precisely when the Stinespring projection lies in commutant of the image of the representation, our strategy will be to perturb the projection p into $\pi(M)'$ while ensuring the range of the resulting map still lies in N. Since π is normal it follows that $\pi(M)$ is also an injective von Neumann algebra and hence has property P (see [27]) and so we may find an element

$$r \in \overline{\mathrm{conv}_{u \in \mathcal{U}(M)} \pi(u)p\pi(u^*)}^{\mathrm{w}^*} \cap \pi(M)' \subseteq (p \cup \pi(M))'' \cap \pi(M)'. \tag{3.18}$$

Since ϕ is close to being a *-homomorphism (it is close to the natural inclusion) a calculation (see the first displayed equation block of Lemma 3.3 in [9]) shows that p almost commutes with the range of π and so r is close to p. We will see versions of this calculation in Section 5.2 and Section 6.5.

We then apply Lemma 3.1.6 to find a projection $q \in (p \cup \pi(M))'' \cap \pi(M')$ that is close to p and Lemma 3.1.7 to find a unitary u close to the identity, lying in $(p \cup \pi(M))''$ and satisfying $upu^* = q$. We then define a map ψ as follows

$$\psi : x \mapsto pu^*\pi(x)up. \tag{3.19}$$

It is clear that ψ is *-preserving and for x and y in M we have

$$\begin{aligned}
\psi(x)\psi(y) &= pu^*\pi(x)upu^*\pi(y)up \\
&= pu^*\pi(x)q\pi(y)up \\
&= pu^*q\pi(x)\pi(y)up \qquad (\text{as } q \in \pi(M)') \\
&= pu^*\pi(xy)up = \psi(xy) \qquad (\text{as } pu^*q = pu^*)
\end{aligned} \tag{3.20}$$

which verifies that ψ is a *-homomorphism. Finally, $p\big(\mathrm{Alg}(p \cup \pi(M))\big)p$ lies in N since it just contains algebraic combinations of elements in the range of ϕ. The map $T \mapsto pTp$ from $\mathbb{B}(\mathcal{K})$ to $\mathbb{B}(\mathcal{K})$ is normal so $p(p \cup \pi(M))''p \subseteq N$ since N is weak*-closed and so the range of ψ lies in N. Furthermore, since u is close to the identity the map ψ is close to ϕ and hence the natural inclusion.

Of course we cannot immediately apply this argument to the map $\Phi|_M$ as we have assumed that ϕ is normal. Christensen resolves this by taking the normal part of $\Phi|_M$ and showing that this too is close to the natural inclusion map.

Chapter 4

The similarity problem

4.1 The similarity problem

In [38] Kadison raised the similarity problem, asking if every bounded homomorphism from a C*-algebra into the bounded operators on a Hilbert space is necessarily similar to a *-representation. Explicitly, given a C*-algebra A, a Hilbert space $\mathcal{H}$ and bounded homomorphism $\phi : A \to \mathbb{B}(\mathcal{H})$, does there exist an invertible operator $S \in \mathbb{B}(\mathcal{H})$ such that $\tilde{\phi} : A \to \mathbb{B}(\mathcal{H})$, defined by

$$\tilde{\phi}(x) = S\phi(x)S^{-1} \qquad x \in A, \tag{4.1}$$

is a *-homomorphism? If this condition holds for all Hilbert spaces and bounded homomorphisms $\phi : A \to \mathbb{B}(\mathcal{H})$ then A is said to satisfy the similarity property. It is still unknown whether all C*-algebras have this property however there are a number of known characterisations. We state a selection of these here.

Theorem 4.1.1. *Let A be a C*-algebra. The following statements are equivalent.*

1. *A has the similarity property.*

2. *Every bounded homomorphism $\phi : A \to \mathbb{B}(\mathcal{H})$, from A to some Hilbert space $\mathcal{H}$, is completely bounded.*

3. *Let $\pi : A \to \mathbb{B}(\mathcal{H})$ be a faithful non-degenerate *-homomorphism. Every bounded derivation $\delta : \pi(A) \to \mathbb{B}(\mathcal{H})$ (a linear map satisfying $\delta(ab) = a\delta(b) + b\delta(a)$ for $a, b \in \pi(A)$) is inner (that is there is an operator $T \in \mathbb{B}(\mathcal{H})$ such that $\delta(a) = \mathrm{ad}(T)(a) := Ta - aT$ for all $a \in \pi(A)$).*

4. *A has finite length (in the sense we will define below).*

5. *Taking commutants is a continuous operation at A with respect to the Kadison-Kastler metric d: for every $\epsilon > 0$ there exists a $\delta > 0$ such that given any faithful non-degenerate *-homomorphism $\pi : A \to \mathbb{B}(\mathcal{H})$, a C*-algebra $B \subseteq \mathbb{B}(\mathcal{H})$ satisfying $d(B, \pi(A)) < \delta$ must also satisfy $d(B', \pi(A)') < \epsilon$.*

Remark. For a fixed positive real number k we say A has property *Property D_k* (this was introduced by Christensen in [11]) if for every representation $\pi : A \to \mathbb{B}(\mathcal{H})$ the following inequality holds

$$d(T, \pi(A)') \leq k\|\mathrm{ad}(T)|_{\pi(A)}\|. \tag{4.2}$$

Therefore property D_k provides a quantitative version of condition (5). We say A has property D_k^* if every normal representation π of A satisfies (4.2). Christensen used property P to show [11, Theorem 2.4] that an injective von Neumann algebra has D_1^*. Although it is unknown whether all C*-algebras have property D_k for some k, we do have *Arveson's distance formula* (for example [10, Proposition 2.1]). For a C*-algebra A on a Hilbert space $\mathcal{H}$ the following holds:

$$d(x, A') = \frac{1}{2}\|\mathrm{ad}(x)|_A\|_{\mathrm{cb}} \qquad (x \in \mathbb{B}(\mathcal{H})). \tag{4.3}$$

The equivalence of (1) and (2) is obtained by combining results of Hadwin and Wittstock [26,64], an alternative proof was independently obtained by Haagerup [24, Theorem 1.10] and also by Paulsen. In fact, these authors give a 'quantitative' version of the equivalence: for any unital bounded homomorphism $\phi : A \to \mathbb{B}(\mathcal{H})$ we have

$$\|\phi\|_{\mathrm{cb}} = \inf\{\|S\|\|S^{-1}\|\} \tag{4.4}$$

where the supremum runs over all $S \in \mathbb{B}(\mathcal{H})$ such that $a \mapsto S^{-1}\phi(a)S$ is a *-homomorphism. The $\leq$ inequality in (4.4) can be seen as follows , given a bounded homomorphism ϕ, we may find a *-homomorphism $\tilde{\phi}$ and an operator S as in (4.1). Therefore

$$\phi^{(n)}(x) = \mathrm{diag}(S^{-1}, \ldots, S^{-1})\tilde{\phi}^{(n)}(x)\mathrm{diag}(S, \ldots, S) \qquad (x \in M_n(A)). \tag{4.5}$$

The norm of the right hand side is at most $\|S\|\|S^{-1}\|\|x\|$.

The implication (1) $\implies$ (3) follows from the observation that, using the notation in (3), the bounded homomorphism

$$\phi(a) = \begin{pmatrix} \pi(a) & \delta(\pi(a)) \\ 0 & \pi(a) \end{pmatrix} \tag{4.6}$$

is similar to a *-homomorphism if and only if δ is inner. The reverse implication is a deep theorem due to Kirchberg [42]. Christensen et al. [6] then prove that (3) is equivalent to (5). Another characterisation of the similarity property closely related to (5) will be discussed in Section 4.5.

We will provide the definition of finite length in this section and prove part of Pisier's equivalence of (2) and (4) (see [51, Theorem 4.1]) in Section 4.3.

A positive answer to the similarity problem has been obtained in a number of settings.

Theorem 4.1.2. *The following classes of C*-algebras have the similarity property:*

1. *Nuclear C*-algebras;*
2. *C*-algebras without tracial states;*
3. *Type II_1 factors with property Gamma.*

The nuclear case is part of a remarkable characterisation of nuclearity which we present below (Theorem 4.1.6), the third case was settled by Christensen in [12]. The second is due to Haagerup [24, Theorem 1.1], the key ingredient is the automatic complete boundedness of bounded homomorphisms in the presence of a cyclic vector. Given a C*-algebra A and a bounded homomorphism $\phi : A \to \mathbb{B}(\mathcal{H})$, a vector ξ is cyclic if $\overline{\phi(A)\xi} = \mathcal{H}$ (see [24, Theorem 1.1]).

Theorem 4.1.3. *Let A be a C*-algebra and $\phi : A \to \mathbb{B}(\mathcal{H})$ be a bounded homomorphism with a cyclic vector. Then*

$$\|\phi\|_{cb} \leq \|\phi\|^4. \tag{4.7}$$

The length of an operator algebra was defined by Pisier in [51]. Finite length (property (4) of Theorem 4.1.1) reflects the existence of factorisations into diagonal matrices and scalar matrices, with length and norm controlled simultaneously in a uniform manner, across all matrix amplifications of an algebra.

Definition 4.1.4. An operator algebra A has *length at most d* if there exists a positive constant K such that for every $n, m \in \mathbb{N}$ and $x \in M_{m,n}(A)$ there exists an integer $N \in \mathbb{N}$ and matrices

$C_1, \ldots, C_{d+1}, D_1, \ldots, D_d$ where

- $C_1 \in M_{m,N}(\mathbb{C})$;
- $C_k \in M_N(\mathbb{C})$ for $2 \le k \le d$;
- $C_{d+1} \in M_{N,n}(\mathbb{C})$;
- $D_l \in M_N(A)$ are diagonal matrices with entries in the unit ball of A for $1 \le l \le d$

which satisfy

$$x = C_1 D_1 C_2 D_2 \cdots C_d D_d C_{d+1} \tag{4.8}$$

and

$$\prod_{i=1}^{d+1} \|C_i\| \le K\|x\|. \tag{4.9}$$

To encapsulate these factorisations we define a norm $\|\cdot\|_{(d)}$ for each positive integer d on matrix amplifications of A. For $x \in M_{m,n}(A)$ set

$$\|x\|_{(d)} = \inf\{\prod_{i=1}^{d+1} \|C_i\| : x = C_1 D_1 C_2 D_2 \ldots C_d D_d C_{d+1}\} \tag{4.10}$$

where the infimum is taken over factorisations into scalar and diagonal matrices as described above.

Remark. Note that there is no bound on the size of N, in particular we may choose rectangular initial and final matrices to allow intermediate matrices of arbitrary size.

In the next section we will present another characterisation of $\|\cdot\|_{(d)}$ which demonstrates that it is a norm but we observe here that for an arbitrary operator algebra A, an integer $n \in \mathbb{N}$ and an element $x \in M_n(A)$, then $\|x\|_{(d)}$ is finite. Indeed, fix a bijection $\sigma : \{1, \ldots, n\}^2 \to \{1, \ldots, n^2\}$. Set

- $C_1 = (\|x\|^{1/2} \sum_{k=1}^n \delta_{\sigma(i,k),j})_{ij} \in M_{n,n^2}(\mathcal{C})$,
- D_1 the diagonal matrix with k^{th} diagonal entry $\|x\|^{-1} x_{\sigma^{-1}(k)}$ and
- $C_2 = (\|x\|^{1/2} \sum_{k=1}^n \delta_{\sigma(k,j),i})_{ij} \in M_{n^2,n}(\mathbb{C})$.

We have $C_1 C_1^* = C_2^* C_2 = n 1_{M_n(A)}$. Therefore $x = C_1 D_1 C_2$ with $\|C_1\|\|C_2\|\|x\| = n\|x\|$ so $\|x\|_{(1)} \le n\|x\|$. Clearly this bound depends upon the size of matrix amplification to which x belongs so is not universal. On the other hand for every $\epsilon > 0$ there exists a d such that $\|x\|_{(d)} \le \|x\| + \epsilon$ (see [48]). The former factorisation may be thought of as an efficient algebraic factorisation as the number of factors may be controlled across matrix amplifications (but not the norm) while the latter is analytically efficient in the sense that the norm may be controlled but the length d cannot be chosen independently of the matrix amplification. Finite length therefore reflects the tension between these two extremes and insists that there exists factorisations which are uniformly algebraically and analytically efficient across all matrix amplifications of the algebra in question.

For some algebras it is possible to construct explicit factorisations which witness finite length.

Example 4.1.5 (Pisier, *cf.* Lemma 5 of [52])**.** Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then for $n \in \mathbb{N}$ and $x = (x_{ij})_{ij} \in M_n(\mathbb{B}(\mathcal{H}))$ we have $\|x\|_{(3)} \leq \|x\|$ and hence $\mathbb{B}(\mathcal{H})$ has length at most 3. Indeed, find isometries $S_1, \ldots, S_n \in \mathbb{B}(\mathcal{H})$ such that $S_s^* S_t = \delta_{s,t} 1_{\mathcal{H}}$. Let $w = e^{2\pi i/n}$ and define a unitary matrix $W = (W_{st})_{st} = (\frac{1}{\sqrt{n}} w^{st})_{st}$. Set $D_1 = \mathrm{diag}(S_1^*, \ldots, S_n^*), D_3 = \mathrm{diag}(S_1, \ldots, S_n) \in M_n(\mathbb{B}(\mathcal{H}))$ and $D_2 \in M_n(\mathbb{B}(\mathcal{H}))$ the diagonal matrix with k^{th} entry given by

$$\begin{aligned} D_2(k) &= n \sum_{i,j=1}^{n} S_i \overline{W_{ik}} x_{ij} \overline{W_{kj}} S_j^* \\ &= n \begin{pmatrix} S_1 \overline{W_{1k}} & \ldots & S_n \overline{W_{nk}} \end{pmatrix} x \begin{pmatrix} \overline{W_{1k}} S_1^* \\ \vdots \\ \overline{W_{nk}} S_n^* \end{pmatrix} \end{aligned} \tag{4.11}$$

for $1 \leq k \leq n$. So, since

$$\| \begin{pmatrix} S_1 \overline{W_{1k}} & \ldots & S_n \overline{W_{kn}} \end{pmatrix}^* \begin{pmatrix} S_1 \overline{W_{1k}} & \ldots & S_n \overline{W_{kn}} \end{pmatrix} \| = \| \frac{1}{n} 1_{\mathcal{H}^n} \| = \frac{1}{n} \tag{4.12}$$

and, by a similar calculation, the column matrix has the same norm, we obtain $\|D_2\| \leq \|x\|$. Furthermore we have $x = D_1 W D_2 W D_3$. Indeed, for $1 \leq s, t \leq n$, we have

$$\begin{aligned} (D_1 W D_2 W D_3)_{st} &= \sum_{k=1}^{n} D_1(s) W_{sk} D_2(k) W_{kt} D_3(t) \\ &= n \sum_{k=1}^{n} S_s^* W_{sk} (\sum_{i,j=1}^{n} S_i \overline{W_{ik}} x_{ij} \overline{W_{kj}} S_j^*) W_{kt} S_t \\ &= n x_{st} \sum_{k=1}^{n} |W_{sk}|^2 |W_{kt}|^2 = x_{st}. \end{aligned} \tag{4.13}$$

In [54] Pisier was able to give a complete characterisation of nuclear C*-algebras in terms of length.

Theorem 4.1.6 (Pisier)**.** *Let A be a C*-algebra. Then A is nuclear if and only if A has length at most 2. Moreover if A is nuclear then for $n \in \mathbb{N}$ and $x \in M_n(A)$ then $\|x\|_{(2)} \leq \|x\|$.*

The 'only if' statement is easier and was well known before Pisier's result (for example [3, Theorem 3.5]). One way to see this is as follows. Since A^{**} is injective it has property D_1^* (see the remark following Theorem 4.1.1). This implies A has property D_1 by the Kaplansky density theorem and the fact that any representation of A may be extended to a normal representation of A^{**}. It then follows from [13, Proposition 2.8] that A has length 2.

Theorem 4.1.1 tells us finite length is equivalent to the similarity property. In particular, finite length provides an intrinsic characterisation of the similarity property; it depends only on the internal structure of the algebra and, in contrast with the other characterisations, we are not required to understand the structure of a class of maps (derivations or bounded homomorphisms) from the algebra into $\mathbb{B}(\mathcal{H})$. This equivalence was established by Pisier [51] by showing that an operator algebra satisfies condition (2) of 4.1.1 if and only if it has finite length. The 'only if' statement may be seen directly. Suppose that A has length at most d so that there exists a $K > 0$ such that $\|x\|_{(d)} \leq K\|x\|$ for $n \in \mathbb{N}$ and $x \in M_n(A)$. Let $\phi : A \to \mathbb{B}(\mathcal{H})$ be a bounded unital homomorphism. Fix $n \in \mathbb{N}$ and $x \in M_n(A)$. Write $x = C_1D_1 \dots D_dC_{d+1}$; a factorisation into scalar and diagonal matrices with

$$\prod_{i=1}^{d+1} \|C_i\| \leq K\|x\|. \tag{4.14}$$

Then,

$$\begin{aligned} \|\phi^{(n)}(x)\| &= \|\phi^{(n,N)}(C_1)\phi^{(N)}(D_1)\dots\phi^{(N)}(D_d)\phi^{(N,n)}(C_{d+1})\| && (4.15)\\ &\leq \|\phi\|^d \prod_{i=1}^{d+1} \|C_i\| \leq K\|\phi\|^d\|x\| && (4.16)\end{aligned}$$

and therefore $\|\phi\|_{\text{cb}} \leq K\|\phi\|^d$. This calculation witnesses diagonal matrices being 'good' in the sense that the norm bounds don't increase with the size of the amplification.

We will present Pisier's proof of the 'if' direction in Section 4.3. In Section 4.4 we will see how these ideas combined with Haagerup's version of the Groethendieck's little inequality can be used to show that any row of an arbitrary C*-algebra may be factorised with length 2.

4.2 Some technical operator space results

We outline the constructions used in Section 4.3 which we extract from Chapters 14, 17 and 19 of [48].

If S is a closed linear subspace of a C*-algebra A then it inherits a norm structure on its matrix amplifications from A. We call such an S a *concrete operator space* and attempt to characterise these objects abstractly. The precise conditions for an arbitrary vector space with norms on its matrix amplifications to be completely isometrically represented as a concrete operator space are determined by Ruan's Theorem. We refer to such vector spaces as *abstract operator spaces* or sometimes just *operator spaces*.

Definition 4.2.1. An abstract operator space is a vector space V with norms on its matrix amplifcations, $\|\cdot\|_{m,n}$ on $M_{m,n}(V)$ for all m, n, which satisfy two conditions:

1. $\|A \cdot X \cdot B\|_{p,q} \leq \|A\|\|X\|_{m,n}\|B\|$ for all $m, n \in \mathbb{N}$ and $X \in M_{m,n}(V)$; and all $p, q \in \mathbb{N}$ and $A \in M_{p,m}(\mathbb{C})$ and $B \in M_{n,q}(\mathbb{C})$ and
2. $\|X \oplus Y\|_{m+p,n+q} = \max\{\|X\|_{m,n}, \|Y\|_{p,q}\}$ for all $m, n, p, q \in \mathbb{N}$ and $X \in M_{m,n}(V)$ and $Y \in M_{p,q}(V)$

where we use the notation

$$X \oplus Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in M_{m+p,n+q}(V). \tag{4.17}$$

We refer to norms satisfying the second condition as L^∞-norms and those satisfying the first as matrix-norms. A family of norms satisfying both conditions are said to be L^∞*-matrix norms*.

Ruan's Theorem [57] says that any L^∞-matrix normed space is completely isometric to a concrete operator space. Given an arbritrary normed vector space we may assign a number of norm structures on its matrix amplifications to give it an operator space structure. Of particular importance will be the maximal operator space norm.

Definition 4.2.2. Let V be a normed vector space, the maximal norm $\|\cdot\|_{\mathrm{MAX(V)}}$ is defined for $m, n \in \mathbb{N}$ and $(x_{ij})_{ij} \in M_{m,n}(V)$ by

$$\|(x_{ij})_{ij}\|_{\mathrm{MAX}(V)} = \sup\{\|(\phi(x_{ij}))_{ij}\|_{M_{m,n}(\mathbb{B}(\mathcal{H}))}\}, \tag{4.18}$$

where the supremum is taken over all Hilbert spaces $\mathcal{H}$ and linear isometries $\phi : V \to \mathbb{B}(\mathcal{H})$.

These norms are a family of L^∞-matrix norms and hence make V into an operator space. By Ruan's Theorem it follows that $\|\cdot\|_0 \leq \|\cdot\|_{\mathrm{MAX}(V)}$ for any other family of L^∞-matrix norms $\|\cdot\|_0$ extending the norm on V to its amplifications.

For use in the sequel we define a scaling of the maximal operator space. For $c > 0$ define the operator space $\mathrm{MAX}(A)_c$ to be $\mathrm{MAX}(A)$ but with a scaled norm $\|\cdot\|_{\mathrm{MAX}(A)_c}$ on its matrix amplification defined as follows; for $m, n \in \mathbb{N}$ and $x \in M_{m,n}(A)$ then

$$\|x\|_{\mathrm{MAX}(A)_c} = c\|x\|_{\mathrm{MAX}(A)}. \tag{4.19}$$

The maximal operator space stucture has an important characterisation in the context of factorisations.

Proposition 4.2.3. *Let V be a vector space and let $(x_{ij})_{ij} \in M_{m,n}(V)$. Then*

$$\begin{aligned}\|(x_{ij})_{ij}\|_{MAX(V)} = \inf\{\|A\|\|B\| : \quad & (x_{ij})_{ij} = A \cdot diag(y_1, \ldots, y_k) \cdot B \\ & k \in \mathbb{N}, \quad A \in M_{m,k}(\mathbb{C}), \quad B \in M_{k,n}(\mathbb{C}), \quad y_i \in V, \quad \|y_i\| \leq 1\}\end{aligned} \tag{4.20}$$

Proof. Let $\|\cdot\|_{m,n}$ be the norm on $M_{m,n}(V)$ obtained by computing the right hand side of equation (4.20). It may be shown [48, proof of Theorem 14.2] that $\|\cdot\|_{m,n}$ is an L^∞-matrix norm. Furthermore, if $x = (x_{ij})_{ij} \in M_{m,n}(V)$ and $\phi : V \to \mathbb{B}(\mathcal{H})$ is a linear isometry, then for any factorisation $x = A \cdot \mathrm{diag}(y_1, \ldots, y_k) \cdot B$ we have

$$\|(\phi(x_{ij}))_{ij}\| = \|A \cdot \mathrm{diag}(\phi(y_1), \ldots, (y_k)) \cdot B\| \leq \|A\|\|B\|, \tag{4.21}$$

hence $\|x\|_{\mathrm{MAX}(V)} \leq \|x\|_{m,n}$ and the claim follows by the maximality of $\|\cdot\|_{\mathrm{MAX}(V)}$. □

The Haagerup tensor norm allows us to assign a norm to the algebraic tensor product of operator spaces. It will play a key role in proof of Pisier's theorem so we include the details.

Definition 4.2.4 (Haagerup tensor product)**.** Let E and F be operator spaces with algebraic tensor product $E \odot F$. For $m, n \in \mathbb{N}$ and $(x_{ij})_{ij} \in M_{m,n}(E \odot F)$ the *Haagerup tensor norm* is defined by

$$\|(x_{ij})_{ij}\|_h = \inf\{\|(e_{ij})_{ij}\|\|(f_{ij})_{ij}\|\} \tag{4.22}$$

where the infimum is over all $k \in \mathbb{N}$ and matrices $(e_{ij})_{ij} \in M_{m,k}(E)$ and $(f_{ij})_{ij} \in M_{k,n}(F)$ such that

$$(x_{ij})_{ij} = (e_{ij})_{ij} \odot (f_{ij})_{ij} := (\sum_{l=1}^{k} e_{il} \otimes f_{lj})_{ij}. \tag{4.23}$$

The completion of $E \odot F$ with respect to the Haagerup tensor norm is called the *Haagerup tensor product* and is denoted $E \otimes_h F$.

It is shown in [48, Proposition 17.2] that the Haagerup tensor norm is an L^∞-matrix norm. We will show that it is finite by constructing a factorisation as above. Let $Z = (\sum_{k=1}^{l_{ij}} x_{ij}^{(k)} \otimes y_{ij}^{(k)})_{ij} \in M_{m,n}(E \odot F)$. Set

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x_{21} & \cdots & x_{2n} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & x_{m1} & \cdots & x_{mn} \end{pmatrix} \tag{4.24}$$

and

$$Y = \begin{pmatrix} y_{11} & 0 & \cdots & 0 & 0 \\ 0 & y_{12} & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & y_{1n} \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ y_{m1} & 0 & \cdots & 0 & 0 \\ 0 & y_{m2} & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & y_{mn} \end{pmatrix}, \tag{4.25}$$

where x_{ij} represents the row matrix $(x_{ij}^{(1)}, \cdots, x_{ij}^{(l_{ij})})$ and y_{ij} represents the row column matrix $(y_{ij}^{(1)}, \cdots y_{ij}^{(l_{ij})})^T$. Then $Z = X \odot Y$.

Analogously to abstract operator spaces, we aim to characterise closed subalgebras of $\mathbb{B}(\mathcal{H})$ (concrete operator algebras) abstractly.

Definition 4.2.5. Let A be an algebra which also has an operator space structure. If the multiplication is completely contractive, that is if for all $k, m, n \in \mathbb{N}$ and all $(a_{ij})_{ij} \in M_{n,k}(A)$ and

$(b_{ij})_{ij} \in M_{k,m}(A)$ the map

$$((a_{ij})_{ij}, (b_{ij})_{ij}) \mapsto (\sum_{l=1}^{k} a_{il} b_{lj})_{ij} \tag{4.26}$$

is contractive, then A is an *abstract operator algebra.*

The GNS construction allows us to concretely represent C*-algebras on a Hilbert space by *-homomorphisms. The following theorem due to Bletcher, Ruan and Sinclair (see [48, Corollary 16.7]) allows us to concretely represent abstract operator algebras on a Hilbert space by complete isometries.

Theorem 4.2.6 (The BRS Theorem)**.** *Let A be an abstract unital operator algebra. Then there exists a Hilbert space $\mathcal{H}$ and a completely isometric homomorphism $\pi : A \to \mathbb{B}(\mathcal{H})$.*

We may always view an abstract operator space as a closed subspace of its universal operator algebra which is defined as follows (see Chapter 19 of [48] for more details).

Definition 4.2.7. Let V be a normed vector space and let $\mathcal{F}(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ be the vector space of finite direct sums of elements from the algebraic tensor powers of V. Define a multiplication $\odot$ on $\mathcal{F}(V)$ as follows; if $u = u_1 \otimes \cdots \otimes u_k$ and $v = v_1 \otimes \cdots \otimes v_l$ are elementary tensors in $\mathcal{F}(V)$ then

$$u \odot v = (u_1 \otimes \cdots \otimes u_k) \odot (v_1 \otimes \cdots \otimes v_l) = u_1 \otimes \cdots \otimes u_l \otimes v_1 \otimes \cdots \otimes v_l. \tag{4.27}$$

We view V as a subspace of $\mathcal{F}(V)$. The above multiplication gives $\mathcal{F}(V)$ an algebra structure and it satisfies the following universal property: if $\mathcal{A}$ is an algebra and $\phi : V \to \mathcal{A}$ a linear map then there exists a unique algebra homomorphism $\pi_\phi : \mathcal{F}(V) \to \mathcal{A}$ extending ϕ. It is defined on elementary tensors $u_1 \otimes \cdots \otimes u_k$ as

$$\pi_\phi(u_1 \otimes \cdots \otimes u_k) = \phi(u_1) \ldots \phi(u_k). \tag{4.28}$$

Now if V is also an operator space we may define a norm $\|\cdot\|_1$ on all matrix amplifications of $\mathcal{F}(V)$ as follows, if $m, n \in \mathbb{N}$ and $x = (x_{ij})_{ij} \in M_{m,n}(\mathcal{F}(V))$ then

$$\|x\|_1 = \sup\{\|(\pi_\phi(x_{ij}))_{ij}\| : \|\phi\|_{\mathrm{cb}} \leq 1\}, \tag{4.29}$$

where the supremum is taken over all Hilbert spaces $\mathcal{H}$ and all linear maps $\phi : V \to \mathbb{B}(\mathcal{H})$ with $\|\phi\|_{cb} \leq 1$. Let $\mathrm{OA}_1(V)$ be the operator algebra obtained by completing $\mathcal{F}(V)$ with respect to the norm $\|\cdot\|_1$.

The universal operator algebra $\mathrm{OA}_1(V)$ has the property that if A is an operator algebra and $\phi : V \to A$ is completely contractive then π_ϕ extends to a completely contractive map $\pi_\phi : \mathrm{OA}_1(V) \to A$. We note two important properties of the universal operator algebra (the proof of the second proposition may be found in [48, Proposition 19.3]).

Proposition 4.2.8. *Let V be an operator space. For each $n \geq 1$ the map $\Gamma_n : \mathcal{F}(V) \to \mathcal{F}(V)$, defined on an element $x = \oplus_j x_j \in \mathcal{F}(V)$ with $x_j \in V^{\otimes j}$ as*

$$\Gamma_n(x) = x_n, \tag{4.30}$$

is completely contractive with respect to the norm $\|\cdot\|_1$ and so extends to a complete contraction $\Gamma_n : OA_1(V) \to OA_1(V)$.

Proof. Define a map Γ_n on $\mathcal{F}(V)$ by the formula given in (4.30). Let $x = (x_{st})_{st} \in M_{m,n}(\mathcal{F}(V))$ where $x_{st} = \oplus_k x_{st}^{(k)}$ is a finite direct sum with $x_{st}^{(k)} \in V^{\otimes k} \subseteq \mathcal{F}(V)$. Fix $\phi : V \to \mathbb{B}(\mathcal{H})$ with $\|\phi\|_{\mathrm{cb}} \leq 1$. For $0 \leq \theta \leq 2\pi$ set $\phi_\theta(v) = e^{i\theta}\phi(v)$. We have

$$\frac{1}{2\pi i}\int_0^{2\pi} (e^{-in\theta}\pi_{\phi_\theta}(x_{st}))_{st} d\theta = (\pi_\phi(x_{st}^{(n)}))_{st}. \tag{4.31}$$

It follows that

$$\|(\pi_\phi(\Gamma_n(x_{st})))_{st}\| \leq \|(\pi_\phi(x_{st}^{(n)}))_{st}\| \leq \frac{1}{2\pi}\int_0^{2\pi} \|(\pi_{\phi_\theta}(x_{st}))_{st}\| d\theta \leq \|x\|_1 \tag{4.32}$$

so that Γ_n is completely contractive on $\mathcal{F}(V)$. We may now extend this map to a completely contractive map on $\mathrm{OA}_1(V)$ by continuity (see [48, Proposition 19.3]). ☐

Proposition 4.2.9. *Let V be a normed vector space and let $k \in \mathbb{N}$. The inclusion of the k-fold Haagerup tensor product into the universal operator algebra $\iota : V^{\otimes_h k} \hookrightarrow OA_1(V)$ is a complete isometry: for all $m, n \in \mathbb{N}$ and $x \in M_{m,n}(V^{\otimes_h k})$ the following inequality holds*

$$\|x\|_{OA_1(V)} = \|x\|_h. \tag{4.33}$$

We assign one further family of norms $|||\cdot|||_{(d)}$ to an abstract operator algebra. We will show this family of norms is equal to $\|\cdot\|_{(d)}$ (see Definition 4.1.4) and use the properties of $|||\cdot|||_{(d)}$ to deduce some crucial properties for $\|\cdot\|_{(d)}$.

Definition 4.2.10. Let A be an abstract operator algebra and let $k \in \mathbb{N}$. Let

$$V_k = \mathrm{MAX}(A) \otimes_h \cdots \otimes_h \mathrm{MAX}(A) \tag{4.34}$$

be the k-fold Haagerup tensor product of the operator space $\mathrm{MAX}(A)$. The product map $\gamma_k : V_k \to A$ defined on an elementary tensor $a_1 \otimes \cdots \otimes a_k \in V_k$ as

$$\gamma_k(a_1 \otimes \cdots \otimes a_k) = a_1 \ldots a_k \tag{4.35}$$

is completely contractive. Let $K_k \subseteq V_k$ be the kernel of this map so the induced map $\tilde{\gamma_k} : V_k/K_k \to A$ is a linear isomorphism. Define a norm $|||\cdot|||_{(k)}$ on matrix amplifications of A as follows. Let $m, n \in \mathbb{N}$ and $x = (x_{ij})_{ij} \in M_{m,n}(A)$ then

$$|||x|||_{(k)} = \|(\tilde{\gamma_k}^{-1}(x_{ij}))_{ij}\|_{V_k/K_k}. \tag{4.36}$$

It is immediate that if $|||(x_{ij})|||_{(k)} < 1$ then there exists an element $(y_{ij})_{ij}$ in the open unit ball of $M_{m,n}(V_k)$ with $(x_{ij})_{ij} = (\gamma_k(y_{ij}))_{ij}$.

Proposition 4.2.11. *Let A be a unital operator algebra, let $m, n \in \mathbb{N}$, $x = (x_{ij}) \in M_{m,n}(A)$ and let $k \in \mathbb{N}$ we have*

1. $\|x\|_{(k)} = |||x|||_{(k)}$ *and*
2. $\|x\| \leq \|x\|_{(k+1)} \leq \|x\|_{(k)}$.

Proof. To prove the first assertion suppose that $x = (x_{ij}) \in M_{m,n}(A)$ satisfies $|||x|||_{(k)} < 1$, then we may find elements $(y_{ij}) \in M_{m,n}(V_k)$ such that $(x_{ij}) = (\gamma_n(y_{ij}))$ and $\|(y_{ij})\|_{M_{m,n}(V_k)} < 1$. Using the definition of the Haagerup tensor product we may find matrices $(y_{ij}^{(1)}), \ldots, (y_{ij}^{(k)})$ such that

$$(y_{ij}) = (y_{ij}^{(1)}) \odot \cdots \odot (y_{ij}^{(k)}). \tag{4.37}$$

and $\|(y_{ij}^{(l)})\|_{\mathrm{MAX}(A)} < 1$ for $1 \leq l \leq k$. Applying Proposition 4.2.3, we obtain scalar matrices R_l, S_l with $\|R_l\| < 1, \|S_l\| < 1$ and diagonal matrix D_l with entries in the unit ball of A such that $(y_{ij}^{(l)}) = R_l D_l S_l$. Therefore

$$\begin{aligned} x =& (\gamma_k(y_{ij}))_{ij} = (\gamma_k((R_1 D_1 S_1 \odot R_2 D_2 S_2 \odot \cdots \odot R_k D_k S_k)_{ij}))_{ij} \\ =& R_1 D_1 S_1 R_2 D_2 S_2 \ldots R_k D_k S_k \end{aligned} \tag{4.38}$$

which is a factorisation witnessing $\|x\|_{(k)} < 1$.

On the other hand suppose that $x = (x_{ij}) \in M_{m,n}(A)$ satisfies $\|x\|_{(k)} < 1$ and let

$$x = C_1 D_1 C_2 \dots C_k D_k C_{k+1} \tag{4.39}$$

be a factorisation into scalar and diagonal matrices with

$$\prod_{l=1}^{k+1} \|C_l\| < 1. \tag{4.40}$$

Let $y = C_1 D_1 \odot C_2 D_2 \odot \cdots \odot C_k D_k C_{k+1}$. Then

$$\|y\|_{V_k} \leq \|C_1 D_1\| \|C_2 D_2\| \cdots \|C_l D_l C_{k+1}\| \leq \prod_{l=1}^{k+1} \|C_l\| < 1 \tag{4.41}$$

and so $x = \gamma^{(m,n)}(y)$ witnesses $|||x|||_{(k)} < 1$. The second inequality in the second assertion is trivial. This concludes the proof of the first assertion.

To see the first inequality in the second assesrtion, one may characterise the *maximal operator algebra norm* of A (see the paragraph following Theorem 18.8 in [48]) as

$$\|x\|_{\mathrm{MAXA}(A)} = \inf\{\prod_{i=1}^{k} \|C_i\| \quad : \quad x = C_1 D_1 C_2 \cdots C_k D_k C_{k+1}\} \tag{4.42}$$

where the infimum is taken over all factorisations into scalar and diagonals of arbitrary length. Using the first assertion we have

$$\|x\| \leq \|x\|_{\mathrm{MAXA}(A)} = \inf_k \|x\|_{(k)}. \tag{4.43}$$

$\square$

Finally, we require a technical lemma, this is a weakening of Lemma 19.10 of [48] (the proof of which appears as Exercise 19.5 in [48] which the author was unable to solve).

Lemma 4.2.12. *Let V be a vector space equipped with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Suppose that there exists constants $\lambda, \mu > 0$ satisfying $0 < \lambda < 1$ such that all vectors $v \in V$ decompose as $v = v' + v''$ with*

- $\|v'\|_2 \leq \mu \|v\|_1$ *and*
- $\|v''\|_1 \leq \lambda \|v\|_1$.

Furthermore, suppose that the identity map from $(V, \|\cdot\|_1)$ to $(V, \|\cdot\|_2)$ is continuous. Then for every $v \in V$ we have

$$\|v\|_2 \leq \frac{\mu}{1-\lambda}\|v\|_1. \tag{4.44}$$

Proof. Fix $v \in V$. We claim that for every $n \in \mathbb{N}$ there exists $v_n \in V$ which satisfies

- $\|v\|_2 \leq \mu\|v\|_1(1 + \lambda + \cdots + \lambda^{n-1}) + \|v_n\|_2$ and
- $\|v_n\|_1 \leq \lambda^n\|v\|_1$.

For the first step write $v = v' + v''$ as in the hypothesis and set $v_1 := v''$.

Now we assume that the claim has been verified at the n^{th} level. Find $v', v'' \in V$ such that $v_n = v' + v''$ with

$$\|v'\|_2 \leq \ \mu\|v_n\|_1 \leq \mu\lambda^n\|v\|_1 \tag{4.45}$$

and

$$\|v''\|_1 \leq \lambda\|v_n\|_1 \leq \lambda^{n+1}\|v\|_1. \tag{4.46}$$

It follows that

$$\begin{aligned}\|v\|_2 &\leq \mu\|v\|_1(1 + \lambda + \cdots + \lambda^{n-1}) + \|v_n\|_2 \\ &\leq \mu\|v\|_1(1 + \lambda + \cdots + \lambda^{n-1}) + \|v'\|_2 + \|v''\|_2 \\ &\leq \mu\|v\|_1(1 + \lambda + \cdots + \lambda^{n}) + \|v''\|_2,\end{aligned} \tag{4.47}$$

which verifies the inductive hypothesis with $v_{n+1} := v''$.

Finally, fix $\epsilon > 0$, find a $\delta > 0$ so $\|x\|_1 < \delta \implies \|x\|_2 < \epsilon$ and find an integer $n \in \mathbb{N}$ such that $\|v_n\|_1 < \delta$. Then

$$\|v\|_2 \leq \frac{\mu}{1-\lambda}\|v\|_1 + \varepsilon. \tag{4.48}$$

Since ε was arbitrary, this completes the proof. $\square$

4.3 Pisier's theorem

We will now present Pisier's proof that condition (2) of Theorem 4.1.1 implies finite length. We follow Paulsen's exposition which we divide into two parts. The first, Proposition 4.3.1, shows that factorisations exist given the lifting property described in the following paragraph. The second part, Proposition 4.3.2, shows that if condition (2) of Theorem 4.1.1 is satisfied then such liftings exist. We have adopted this structure as it allows us to apply Proposition 4.3.1 directly in the following section where we restrict to the factorisation of rows of C*-algebras.

Let A be a unital operator algebra and fix $c > 1$. Let $\iota : \mathrm{MAX}(A)_c \to A$ be the identity map. For $m, n \in \mathbb{N}$ and $x \in M_{m,n}(A)$ we have

$$\|x\|_{\mathrm{MAX}(A)_c} = c\|x\|_{\mathrm{MAX}(A)} \geq \|x\|_{\mathrm{MAX}(A)} \geq \|x\| \tag{4.49}$$

and hence ι is completely contractive. Therefore ι extends to a homomorphism

$$\pi_{\iota,c} : \mathrm{OA}_1(\mathrm{MAX}(A)_c) \to A \tag{4.50}$$

which satisifes $\pi_{\iota,c}(a) = a$ for $a \in A \subseteq \mathrm{OA}_1(\mathrm{MAX}(A)_c)$ and is a complete contraction by the universal property described in the paragraph preceding Definition 4.2.7.

Proposition 4.3.1. *Let $M > 0$, $c > 1$ and A be a unital operator algebra. For $m, n \in \mathbb{N}$ suppose that for each element a of $M_{m,n}(A)$ with $\|a\| \leq 1/M$, there exists an element $y \in M_{m,n}(\mathcal{F}(MAX(A)_c))$ in the open unit ball of $OA_1(MAX(A)_c)$ satisfying $\pi_{\iota,c}^{(m,n)}(y) = a$. If d is an integer satisfying $c^d(c-1) > M$ then each element in $x \in M_{m,n}(A)$ satisfies*

$$\|x\|_{(d)} \leq \frac{M(c^{d+1}-1)}{c^{d+1}-c^d-M}\|x\|. \tag{4.51}$$

Proof. Let a be an element of $M_{m,n}(A)$ such that $\|a\| < 1/M$. We use the hypothesis to find an element y in the open unit ball of $\mathrm{OA}_1(\mathrm{MAX}(A)_c)$ satisfying $\pi_{\iota,c}^{(m,n)}(y) = a$. Let $y = y_1 + \cdots + y_l$ be the decomposition of y into homogeneous parts with $y_k \in M_{m,n}((\mathrm{MAX}(A)_c)^{\otimes k})$. Then by Proposition 4.2.8 each y_k is in the open unit ball of $\mathrm{OA}_1(\mathrm{MAX}(A)_c)$ and, applying Proposition 4.2.9, we have

$$\begin{aligned} 1 > \|y_k\|_{\mathrm{OA}_1(\mathrm{MAX}(A)_c)} &= \inf\{\|C_1\|_{\mathrm{MAX}(A)_c} \cdots \|C_k\|_{\mathrm{MAX}(A)_c}\} \\ &= c^k \inf\{\|C_1\|_{\mathrm{MAX}(A)} \cdots \|C_k\|_{\mathrm{MAX}(A)}\} \\ &= c^k \|y_k\|_{V_k} \end{aligned} \tag{4.52}$$

where the infimum is over matrices $y_k = C_1 \odot \cdots \odot C_k$ as in the definition of the Haagerup tensor norm.

Set $a_k = \pi_{\iota,c}^{(m,n)}(y_k) = \gamma_k^{(m,n)}(y_k)$ so that $a = a_0 + \cdots + a_l$ where a_0 is a scalar matrix multiplied by 1_A with norm at most 1 and $\|a_k\|_{(k)} \leq c^{-k}$ for $1 \leq k \leq l$. Then, using the second property in Proposition 4.2.11, it follows that

$$\begin{aligned} \|a_0 + \cdots + a_d\|_{(d)} &\leq \|a_0\|_{(d)} + \|a_1\|_{(d)} + \cdots + \|a_d\|_{(d)} \\ &\leq \|a_0\| + \|a_1\|_{(1)} + \cdots + \|a_d\|_{(d)} \\ &\leq 1 + c^{-1} + \cdots + c^{-d} \end{aligned} \tag{4.53}$$

and

$$\|a_{d+1} + \cdots + a_l\| \leq c^{-(d+1)} + \cdots + c^{-l} \leq \frac{c^{-(d+1)}}{1 - c^{-1}} = \frac{c^{-d}}{c-1}. \tag{4.54}$$

We will show that the norm $\|\cdot\|$ and $\|\cdot\|_{(d)}$ satisfy the hypothesis of Lemma 4.2.12. Firstly, to show continuity, let x be an arbitrary element of $M_{m,n}(A)$. Using the second assertion in Proposition 4.2.11 and the bound obtained in the remark following Definition 4.1.4 we have $\|x\|_{(d)} \leq \|x\|_1 \leq N\|x\|$ where $N = \max\{m, n\}$.

To show the existence of a decomposition as in the statement, set $a = \frac{x}{\|x\|M}$ then $\|a\| < 1/M$. By the above calculation we may decompose $a = a' + a''$ where

$$\|a'\|_{(d)} \leq 1 + c^{-1} + \cdots + c^{-d} \text{ and } \|a''\| \leq \frac{c^{-d}}{c-1}. \tag{4.55}$$

Set

$$x_1 = \|x\|Ma' \qquad \text{and} \qquad x_2 = \|x\|Ma''. \tag{4.56}$$

Then $x = x_1 + x_2$ satisifes the hypothesis of Lemma 4.2.12 with

- $\|\cdot\|_1 = \|\cdot\|$,
- $\|\cdot\|_2 = \|\cdot\|_{(d)}$,
- $\mu = M(1 + c^{-1} + \cdots + c^{-d})$ and
- $\lambda = M\frac{c^{-d}}{c-1} < 1$.

The result follows. □

The following proposition is the main step in Pisier's proof, it shows that if every bounded homomorphism is completely bounded then a lifting as in the hypothesis of Proposition 4.3.1 exists.

Proposition 4.3.2. *Let $c > 1$ and A be a unital operator algebra with the property that there exists a constant $M > 0$ such that for every unital homomorphism $\rho : A \to \mathbb{B}(\mathcal{H})$ satisfying $\|\rho\| \leq c$ implies $\|\rho\|_{cb} \leq M$. For each $m, n \in \mathbb{N}$ and every element $a \in M_{m,n}(A)$ satisfying $\|a\| < \frac{1}{M}$ there exists an element $y \in M_{m,n}(\mathcal{F}(MAX(A)_c))$ in the open unit ball of $OA_1(MAX(A)_c)$ satisfying $\pi_{\iota,c}^{(m,n)}(y) = a$.*

Proof. Using the notation of the paragraph preceding Proposition 4.3.1, the restriction $\pi_{\iota,c}|_{\mathcal{F}(\mathrm{MAX}(A)_c)} : \mathcal{F}(\mathrm{MAX}(A)_c) \to A$ is completely contractive.

Let $I = \ker(\pi_{\iota,c}) \cap \mathcal{F}(\mathrm{MAX}(A)_c)$ which is a closed ideal in $\mathcal{F}(\mathrm{MAX}(A)_c)$. The algebra $\mathcal{F}(\mathrm{MAX}(A)_c)$ inherits the operator space structure from $\mathrm{OA}_1(\mathrm{MAX}(A)_c)$ and hence is a non-complete operator algebra and so is the quotient $Z := \mathcal{F}(\mathrm{MAX}(A)_c)/I$. We have restricted before taking the quotient as we will require the property $I \subseteq \mathcal{F}(\mathrm{MAX}(A)_c)$ later in the proof.

In [51, Proof of Theorem 1.7] Pisier observes the completion $\tilde{Z}$ is an abstract operator algebra, we include some details for completeness. It is certainly a unital Banach algebra and has an operator space structure induced by $\mathrm{OA}_1(\mathrm{MAX}(A)_c)$. We will show that multiplication is completely contractive. Fix $x = (x_{ij} + I)_{ij}, y = (y_{ij} + I)_{ij} \in M_n(Z)$ and $\epsilon > 0$. Set

$$\eta = \min\{\epsilon\|x\|/3, \epsilon\|y\|/3, \sqrt{\epsilon/3}\}. \tag{4.57}$$

We may find $(x'_{ij})_{ij}, (y'_{ij})_{ij} \in M_n(I)$ such that

$$\|(x_{ij} + x'_{ij})_{ij}\| \leq \|(x_{ij} + I)_{ij}\| + \eta \qquad \text{and} \qquad \|(y_{ij} + y'_{ij})_{ij}\| \leq \|(y_{ij} + I)_{ij}\| + \eta. \tag{4.58}$$

Since $\mathrm{OA}_1(\mathrm{MAX}(A)_c)$ is an abstract operator algebra, it has completely contractive multiplication. It follows that

$$\begin{aligned}\|xy\| &= \|(\sum_{k=1}^{n} x_{ik}y_{kj} + I)_{ij}\| \\ &\leq \|(\sum_{k=1}^{n} (x_{ik} + x'_{ik})(y_{kj} + y'_{kj}))_{ij}\| \\ &\leq \|(x_{ij} + x'_{ij})_{ij}\|\|(y_{ij} + y'_{ij})_{ij}\| \\ &\leq (\|x\| + \eta)(\|y\| + \eta) \leq \|x\|\|y\| + \epsilon. \end{aligned} \tag{4.59}$$

Finally let $a = (a_{ij})_{ij}, b = (b_{ij})_{ij} \in M_n(\tilde{Z})$ and $\epsilon > 0$. Set $K_1 = \max\{\|a\|, \|b\|\}$ and $K_2 = \max_{i,j}\{\|b_{ij}\|\}$. Find $a'_{ij} \in Z$ such that

$$\|a_{ij} - a'_{ij}\| \leq \min\{\frac{\epsilon^{1/2}}{2n}, \frac{\epsilon}{8nK_1}, \frac{\epsilon}{4n^2K_2}\} \tag{4.60}$$

Set $K_3 = \max_{ij}\{\|a'_{ij}\|\}$. Find $b'_{ij} \in Z$ such that

$$\|b_{ij} - b'_{ij}\| \leq \min\{\frac{\epsilon^{1/2}}{2n}, \frac{\epsilon}{8nK_1}, \frac{\epsilon}{4n^2K_3}\} \tag{4.61}$$

Set $a' = (a'_{ij})$ and $b' = (b'_{ij})$. Let C_1 and C_2 be matrices as in the remark following Definition 4.1.4. Then

$$\begin{aligned}
\|ab\| &\leq \|(a-a')b\| + \|a'(b-b')\| + \|a'b'\| \\
&\leq \|(a-a')b\| + \|a'(b-b')\| + \|a'\|\|b'\| \\
&\leq \|(a-a')b\| + \|a'(b-b')\| + (\|a'-a\| + \|a\|)(\|b'-b\| + \|b\|) \\
&\leq \|(a-a')b\| + \|a'(b-b')\| + \|a'-a\|\|b'-b\| + \|b'-b\|\|a\| + \|a'-a\|\|b\| + \|a\|\|b\| \\
&= \|C_1D_1C_2\| + \|C_1D_2C_2\| + \|C_1D_3C_2\|\|C_1D_4C_2\| + \|C_1D_4C_2\|\|a\| + \|C_1D_3C_2\|\|a\| + \|a\|\|b\| \\
&\leq \|a\|\|b\| + \epsilon
\end{aligned} \tag{4.62}$$

where:

- $D_1 \in M_{n^2}(\tilde{Z})$ is the matrix with diagonal entries: $\sum_{k=1}^n (a_{ik} - a'_{ik})b_{kj}$;
- $D_2 \in M_{n^2}(\tilde{Z})$ is the matrix with diagonal entries: $\sum_{k=1}^n a'_{ik}(b_{kj} - b'_{kj})$;
- $D_3 \in M_{n^2}(\tilde{Z})$ is the matrix with diagonal entries: $a_{ij} - a'_{ij}$;
- $D_4 \in M_{n^2}(\tilde{Z})$ is the matrix with diagonal entries: $b_{ij} - b'_{ij}$;

for $1 \leq i, j \leq n$. The stated bound follow from the choice of a'_{ij} and b'_{ij} and because $\tilde{Z}$ has an L^∞-matrix norm. Since ϵ was arbitrary, this establishes that multiplication is completely contractive in $\tilde{Z}$.

The map induced by $\pi_{\iota,c}$,

$$\tilde{\pi} : \tilde{Z} \to A, \tag{4.63}$$

is completely contractive, onto and injective on Z. Let $\rho : A \to \tilde{Z}$ be its left inverse. For $a \in A$ we have

$$\rho(a) = \rho(\pi_{\iota,c}(a)) = \rho(\tilde{\pi}(a + I)) = a + I, \tag{4.64}$$

which demonstrates that ρ is a genuine inverse. Applying (4.64) gives

$$\|\rho(a)\| = \|a + I\|_{\tilde{Z}} \leq \|a\|_c = c\|a\|. \tag{4.65}$$

Therefore the unital homorphism ρ satisfies $\|\rho\| \leq c$. Now let $m, n \in \mathbb{N}$ be fixed and let $a = (a_{ij})_{ij} \in M_{m,n}(A)$ such that $\|a\| < \frac{1}{M}$. By Theorem 4.2.6 we may assume that $\tilde{Z}$ is isometrically represented on $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and so by hypothesis

$$\|\rho^{(m,n)}(a)\| \leq M\|a\| < 1. \tag{4.66}$$

On the other hand

$$\rho^{(m,n)}(a) = (\rho(a_{ij}))_{ij} = (a_{ij} + I)_{ij}. \tag{4.67}$$

For $1 \leq i \leq m$ and $1 \leq j \leq n$ we may find elements $z_{ij} \in I$ such that

$$\|(a_{ij} + z_{ij})_{ij}\|_{\mathrm{OA}_1(\mathrm{MAX}(A)_c)} < 1. \tag{4.68}$$

Let $y = (a_{ij} + z_{ij})_{ij} \in M_{m,n}(\mathcal{F}(\mathrm{MAX}(A)_c)$ then $\pi_{\iota,c}^{(m,n)}(y) = a$ as required. $\square$

The main theorem now follows by applying Proposition 4.3.2 followed by Proposition 4.3.1.

Theorem 4.3.3. *Let $c > 1$ and A be a unital operator algebra with the property that there exists a constant $M > 0$ such that for every unital homomorphism $\rho : A \to \mathbb{B}(\mathcal{H})$ with $\|\rho\| \leq c$ we have $\|\rho\|_{cb} \leq M$. If d is an integer d satisfying $M < c^d(c-1)$ then A has length at most d. In fact, for any $m, n \in \mathbb{N}$ and any $x \in M_{m,n}(A)$,*

$$\|x\|_{(d)} \leq \frac{M(c^{d+1} - 1)}{c^{d+1} - c^d - M}\|x\|. \tag{4.69}$$

By [51, Theorem 4.1] the hypothesis of the above lemma is equivalent to Condition (2) of Theorem 4.1.1 and hence this shows Condition (2) implies Condition (4).

4.4 Row factorisations

We now restrict our attention to factorising rows of C*-algebras.

Definition 4.4.1. Let A be a C*-algebra. We say that A has *row length at most k* if there exists $K > 0$ such that for all $n \in \mathbb{N}$ and $x \in M_{1,n}(A)$ we have

$$\|x\|_{(k)} \leq K\|x\|. \tag{4.70}$$

Finite row length only requires factorisations to exist for row amplifications of the algebra so is clearly a weaker condition than finite length. The aim of this section is to establish the following theorem [17].

Theorem 4.4.2. *Let A be a unital C*-algebra. Then A has a row length at most 2. In fact, for $x \in M_{1,n}(A)$ we have*

$$\|x\|_{(2)} \leq \inf_{c>\sqrt{2}+1}\left(\frac{\sqrt{2}(c^3-1)}{c-\sqrt{2}-1}\right)\|x\| < 55\|x\| \tag{4.71}$$

where the final inequality follows by taking $c = 3.5$.

As we may phrase the finite length in terms of completely bounded homomorphisms, we expect row bounded homomorphisms to play a similar role when considering row factorisations.

Definition 4.4.3. Let A and B be C*-algebras and let $\phi : A \to B$ be a linear map. Then we define

$$\|\phi\|_{\mathrm{row}} = \sup_{n\in\mathbb{N}} \|\phi^{(1,n)}\|. \tag{4.72}$$

The first step in establishing Theorem 4.4.2 will be to show that any unital bounded homomorphism from any unital C*-algebra is bounded in the row norm. Once we have achieved this we will modify Proposition 4.3.2 to work with unital C*-algebras where the automatic row boundedness allows to remove the 'bounded implies row bounded' hypothesis. Finally, we apply Proposition 4.3.1 to establish the theorem.

One way of establishing automatic row boundedness is to use the following lemma of Haagerup [24, Lemma 1.5].

Lemma 4.4.4. *Let π be a bounded, non-degenerate representation of a unital C*-algebra A on a Hilbert space $\mathcal{H}$. Then for integers $n, m \in \mathbb{N}$ and elements $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ in A, the following implication holds*

$$\sum_{i=1}^{m} x_i^* x_i \leq \sum_{j=1}^{n} y_j^* y_j \implies \sum_{i=1}^{m} \pi(x_i)^*\pi(x_i) \leq \|\pi\|^6 \sum_{j=1}^{n} \pi(y_j)^*\pi(y_j) \tag{4.73}$$

The following lemma is a direct consequence of the previous lemma. The proof is sketched in the first paragraph of [12, Proposition 2.1], we will include the details for completeness.

Lemma 4.4.5. *Let A be a unital C*-algebra. Then every unital bounded homomorphism $\phi : A \to \mathbb{B}(\mathcal{H})$ is 'row bounded' with*

$$\|\phi\|_{row} \leq \|\phi\|^3. \tag{4.74}$$

Proof. Let $\pi : A \to \mathbb{B}(\mathcal{H})$ be defined for $x \in A$ as

$$\pi(x) = \phi(x^*)^*. \tag{4.75}$$

Since ϕ is a homomorphism so is π and furthermore $\|\pi\| = \|\phi\|$. Now fix an $m \in \mathbb{N}$ and $x = (x_1, \ldots, x_m)$ in the unit ball of $M_{1,m}(A)$. Applying Lemma 4.4.4 to the bounded homomorphism π with $y = 1_A$ and $n = 1$ we have

$$\begin{aligned}
\sum_{i=1}^{m} x_i x_i^* \leq 1 \implies & \sum_{i=1}^{m} \pi(x_i^*)^* \pi(x_i^*) \leq \|\phi\|^6 \\
\implies & \sum_{i=1}^{m} \phi(x_i)\phi(x_i)^* \leq \|\phi\|^6 \\
\implies & (\phi(x_1), \ldots, \phi(x_m))(\phi(x_1), \ldots, \phi(x_m))^* \leq \|\phi\|^6.
\end{aligned} \tag{4.76}$$

Therefore

$$\|\phi^{(1,m)}(x)\| = \|(\phi(x_1), \ldots, \phi(x_m))\| \leq \|\phi\|^3. \tag{4.77}$$

Since m and x were arbitrary the proof is complete. $\square$

However Erik Christensen suggested using Haagerup's Little Groethendieck inequality [24, Lemma 3.2] to obtain the following quadratic bound (although with a slightly worse constant than in Lemma 4.4.7).

Lemma 4.4.6 (Haagerup's Little Groethendieck inequality)**.** *Let A be a C*-algebra, let $\mathcal{H}$ be a Hilbert space, and let $T : A \to \mathcal{H}$ be a bounded linear map. There exist two states f and g on A, such that*

$$\|T(x)\|^2 \leq \|T\|^2 (f(x^*x) + g(xx^*)) \qquad (x \in A). \tag{4.78}$$

Lemma 4.4.7. *Let A be a unital C*-algebra then every unital bounded homorphism $\phi : A \to \mathbb{B}(\mathcal{H})$ satisfies*

$$\|\phi\|_{row} \leq \sqrt{2}\|\phi\|^2. \tag{4.79}$$

Proof. Let $x \in A$ have polar decomposition $x = hv$ in A^{**} (with $h = (xx^*)^{1/2}$). Let $\bar{\phi} : A^{**} \to \mathbb{B}(\mathcal{H})^{**}$ denote the extension of ϕ to the bidual of A. We will check that $\phi(x) = \phi(h)\bar{\phi}(v)$. Use the Kaplansky density theorem to pick a bounded net $(v_\lambda)_\lambda$ in A such that $v_\lambda \to v$. For a Banach space X and element $x \in X$ let $L_x : X^* \to X^*$ be defined by

$$\langle y, L_x(f)\rangle = \langle xy, f\rangle \qquad (f \in X^*, y \in X). \tag{4.80}$$

Fix $f \in \mathbb{B}(\mathcal{H})^*$. Then

$$\begin{aligned}
\langle \phi(hv), f\rangle =& \langle hv, \phi^*(f)\rangle \\
=& \langle v, L_h\phi^*(f)\rangle \\
=& \lim_\lambda \langle v_\lambda, L_h\phi^*(f)\rangle \\
=& \lim_\lambda \langle hv_\lambda, \phi^*(f)\rangle \\
=& \lim_\lambda \langle \phi(h)\phi(v_\lambda), f\rangle \\
=& \lim_\lambda \langle v_\lambda, \phi^*(L_{\phi(h)}f)\rangle \\
=& \langle v, \phi^*(L_{\phi(h)}f)\rangle \\
=& \langle \phi(h)\bar{\phi}(v), f\rangle
\end{aligned} \tag{4.81}$$

and hence $\phi(x) = \phi(h)\bar{\phi}(v)$. So we have

$$\phi(x)\phi(x)^* = \phi(h)\bar{\phi}(v)\bar{\phi}(v)^*\phi(h)^* \leq \|\phi\|^2\phi(h)\phi(h)^*. \tag{4.82}$$

Let $\xi \in \mathcal{H}$ be a unit vector. Applying Lemma 4.4.6 to the bounded linear map $x \mapsto \phi(x^*)^*\xi$ gives states f and g on A, such that

$$\|\phi(x)^*\xi\|^2 \leq \|\phi\|^2\|\phi(h)^*\xi\|^2 \leq \|\phi\|^4(f(h^2) + g(h^2)) = \|\phi\|^4(f(xx^*) + g(xx^*)). \tag{4.83}$$

Fix $n \in \mathbb{N}$, let $x = (x_1, \ldots, x_n) \in M_{1,n}(A)$ then,

$$\|\phi^{(1,n)}(x)^*\xi\|^2 = \sum_{j=1}^n \|\phi(x_j)^*\xi\|^2 \leq \|\phi\|^4\sum_{j=1}^n (f(x_jx_j^*) + g(x_jx_j^*)) \leq 2\|\phi\|^4\|xx^*\| = 2\|\phi\|^4\|x\|^2. \tag{4.84}$$

It follows that $\|\phi^{(1,n)}(x)\| = \|\phi^{(1,n)}(x)^*\| \leq \sqrt{2}\|\phi\|^2\|x\|$. $\square$

We are now in a position to modify Proposition 4.3.2 to deal with rows of unital C*-algebras.

Lemma 4.4.8. *Let $c > 1$ and A be a unital C*-algebra.Then for each $n \in \mathbb{N}$ and element $a \in M_{1,n}(A)$ satisfying $\|a\| < \frac{1}{\sqrt{2}c^2}$ there exists an $y \in M_{1,n}(\mathcal{F}(MAX(A)_c))$ in the open unit ball of $OA_1(MAX(A)_c)$ satisfying $\pi_{\iota,c}^{(1,n)}(y) = a$.*

Proof. We construct the abstract operator algebra $\tilde{Z}$ and a map $\rho : A \to \tilde{Z}$ exactly as in the proof of Proposition 4.3.2. Now let $n \in \mathbb{N}$ be fixed and let $a = (a_1, \ldots a_n) \in M_{1,n}(A)$ such that $\|a\| < \frac{\sqrt{2}}{2c^2}$. As before we may assume, by the Blecher-Ruan-Sinclair theorem (Theorem 4.2.6), that $\tilde{Z}$ is isometrically represented on $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. As A is a unital C*-algebra we may apply Lemma 4.4.7 to yield

$$\|\rho^{(1,n)}(a)\| \leq \sqrt{2}c^2\|a\| < 1. \tag{4.85}$$

We are now in a position to follow the proof of Proposition 4.3.2 from (4.67). We have

$$\rho^{(1,n)}(a) = (\rho(a_1), \ldots, \rho(a_n)) = (a_1 + I, \ldots, a_n + I) \tag{4.86}$$

and so we may find elements $z_1, \ldots, z_n \in I$ such that

$$\|(a_1 + z_1, \ldots, a_n + z_n)\|_{\mathrm{OA}_1(\mathrm{MAX}(A)_c)} < 1. \tag{4.87}$$

Let $y = (a_1 + z_1, \ldots, a_n + z_n) \in M_{1,n}(\mathcal{F}(\mathrm{MAX}(A)_c))$ then $\pi_{\iota,c}^{(1,n)}(y) = a$ as required. □

To complete the proof of Theorem 4.4.2 we firstly restrict ourselves to row elements of small norm. We use Lemma 4.4.8 to find a lift into the universal operator algebra and then directly apply Lemma 4.3.1 to yield row factorisations.

Proof of Theorem 4.4.2. First fix an integer $n \in \mathbb{N}$; we will show that we can factorise any element in $M_{1,n}(A)$. Now fix $c > \sqrt{2} + 1$ and set $M = \sqrt{2}c^2$. Let $a \in M_{1,n}(A)$ such that $\|a\| < \frac{1}{M}$. By applying Lemma 4.4.2 we can find an element $y \in M_{1,n}(\mathcal{F}(\mathrm{MAX}(A)_c))$ in the unit ball of $\mathrm{OA}_1(\mathrm{MAX}(A)_c)$ satisfying $\pi_{\iota,c}^{(1,n)}(y) = a$. The inequality $c > \sqrt{2} + 1$ implies that

$$c^2(c - 1) > \sqrt{2}c^2 = M. \tag{4.88}$$

Therefore the hypothesis of Lemma 4.3.1 are satisfied with $d = 2$. It follows that any element of $x \in M_{1,n}(A)$ may be factorised with length 2, in particular

$$\|x\|_{(2)} \leq \frac{M(c^3 - 1)}{c^3 - c^2 - M}\|x\| = \frac{\sqrt{2}(c^3 - 1)}{c - \sqrt{2} - 1}\|x\|. \tag{4.89}$$

Since n was arbitrary the proof is complete. □

4.5 The similarity problem and the Kadison-Kastler metric

Property 5 of Theorem 4.1.1 witnesses a deep connection between the similarity problem and the perturbation theory of C*-algebras. It is often useful to assume that close C*-algebras also have close commutants. For example in [5] Cameron et al. are able to use this assumption to prove the following ingredient in their proof of Theorem 3.2.8. Suppose two close von Neumann algebras M and N have (completely) close commutants. Then if π is a normal *-representation of M on a Hilbert space $\mathcal{K}$ one may find a normal *-representation ρ of N on $\mathcal{K}$ such that $\pi \approx \rho$. We have seen that Theorem 3.2.2 allows us to transfer K-theoretic data between completely close C*-algebras. The following proposition (see [6]) can be used to show (by finding a universal length and length constant and applying an argument similar to the proof of Theorem 4.5.2) that close algebras satisfying the similarity property are automatically completely close.

Proposition 4.5.1. *Every C*-subalgebra of $\mathbb{B}(\mathcal{H})$ has the similarity property if and only if d and d_{cb} are equivalent metrics on C*-subalgebras of $\mathbb{B}(\mathcal{H})$.*

The same argument can also be used to show that if A and B have finite length then d_{cb} is bounded by a function of d and the maximum length and length constant of A and B.

We sketch a proof of the 'if' direction of Proposition 4.5.1. Fix a von Neumann algebra A on a Hilbert space $\mathcal{H}$. Firstly, we claim that for any von Neumann algebra $B \subseteq \mathbb{B}(\mathcal{H})$ we have

$$d_{\mathrm{cb}}(A', B') \leq 2d_{\mathrm{cb}}(A, B). \tag{4.90}$$

Suppose that $d_{\mathrm{cb}}(A, B) \leq \gamma$. Let $T \in A' \otimes \mathbb{M}_n$ for some $n \in \mathbb{N}$ it follows from Arveson's distance formula (see the remark following Theorem 4.1.1) that

$$d(T, B' \otimes \mathbb{M}_n) \leq \frac{1}{2}\|\mathrm{ad}(T)|_{B\otimes\mathbb{C}I_n}\|_{\mathrm{cb}}. \tag{4.91}$$

Then given $s \in \mathbb{N}$ and $x \in (B \otimes \mathbb{M}_n) \otimes \mathbb{M}_s$ we may find an $y \in (A \otimes \mathbb{M}_n) \otimes \mathbb{M}_s$ such that $\|x - y\| \leq \gamma\|x\|$. Therefore,

$$\|(\mathrm{ad}(T) \otimes \mathrm{id}_{M_s})(x)\| = \|\mathrm{ad}(T \otimes 1_s)(x)\| \leq 2\|T \otimes 1_s\|\|x - y\| \leq 2\|T\|\gamma. \tag{4.92}$$

It then follows that $d(T, B' \otimes \mathbb{M}_n) \leq \|T\|\gamma$. By applying a symmetric argument and normalising to obtain elements in the unit ball the bound in (4.90) is established. The general case of Proposition

4.5.1 (when A and B are only assumed to be C*-algebras) follows by the Kaplanksy density theorem. Assuming the equivalence of d and d_{cb} we may find a $K > 0$ such that

$$d(A', B') \leq d_{\mathrm{cb}}(A', B') \leq 2d_{\mathrm{cb}}(A, B) \leq Kd(A, B) \tag{4.93}$$

demonstrating the continuity of commutants at A when represented on the Hilbert space $\mathcal{H}$. From the equivalence of conditions (1) and (5) of Theorem 4.1.1 it follows that A has the similarity property.

The 'only if' direction (see [13, Proposition 2.10]) uses the finite length characterisation of the similarity property to establish the equivalence of the metrics. Since all unital C*-algebras have finite row length, the same argument may be used to prove that d and d_{row} are equivalent metrics on unital C*-subalgebras of $\mathbb{B}(\mathcal{H})$.

Theorem 4.5.2. *The metrics d and d_{row} are equivalent on unital C*-algebras. In particular if $A, B \subseteq \mathbb{B}(\mathcal{H})$ are unital C*-algebras, the following inequality holds*

$$d(A, B) \leq d_{row}(A, B) \leq 220d(A, B). \tag{4.94}$$

Proof. Fix $m \in \mathbb{N}$ and let x be in the unit ball of $M_{1,m}(A)$. We may apply Theorem 4.4.2 to find a factorisation of x:

$$x = C_1 D_1 C_2 D_2 C_3 \tag{4.95}$$

with $N \in \mathbb{N}$ and $C_1 \in M_{n,N}(\mathbb{C})$, $C_2 \in M_N(\mathbb{C})$ and $C_3 \in M_{N,n}(\mathbb{C})$ scalar matrices satisfying

$$\prod_{i=1}^{3} \|C_i\| \leq 55 \tag{4.96}$$

and D_1, D_2 are diagonal matrices with entries $D_i^{(j)}$ in the unit ball of A for $1 \leq j \leq N$. For each $i = 1, 2$ and $1 \leq j \leq N$ let $E_i^{(j)}$ be an element of the unit ball of B such that $\|D_i^{(j)} - E_i^{(j)}\| \leq \gamma$ using the hypothesis $d(A, B) \leq \gamma$. Let E_i be the diagonal matrix in $M_N(B)$ with $E_i^{(j)}$ in the (j, j) entry. Then for $i = 1, 2$ we have

$$\|D_i - E_i\| \leq \gamma. \tag{4.97}$$

By construction the element

$$y' = C_1 E_1 C_2 E_2 C_3. \tag{4.98}$$

is in $M_{1,m}(B)$. Furthermore

$$\|x - y'\| \leq \|C_1(D_1 - E_1)C_2D_2C_3\| + \|C_1E_1C_2(D_2 - E_2)C_3\| \leq 110\gamma \tag{4.99}$$

Finally the element $y = y'/\|y'\|$ is in the unit ball of $M_{1,m}(B)$ and satisfies $\|x - y\| \leq 220\gamma$. The same argument may be repeated to approximate elements in the unit ball of $M_{1,m}(B)$ with those in $M_{1,m}(A)$ and, since m was arbitrary, the bound is as claimed. □

Chapter 5

Ino and Watatani's theorem

5.1 Intermediate subalgebras

In this section we investigate the Kadison-Kastler stability of certain subalgebras of a fixed C*-algebra. Christensen obtained a positive result in the von Neumann context, showing that close von Neumann subalgebras of a finite von Neumann algebra arise as small unitary perturbations [10].

In their recent paper Ino and Watatani consider an analogous situation in the C*-context: the Kadison-Kastler stability of intermediate subalgebras between a finite index inclusion (see Section 2.3 for the definition and properties of finite index inclusions).

Definition 5.1.1. Let $C \subseteq D$ be a unital inclusion of C*-algebras with a finite index conditional expectation $E_C^D : D \to C$. A C*-algebra A is said to be an *intermediate subalgebra* if there is a unital inclusion $C \subseteq A \subseteq D$ and a conditional expectation $E_A^D : D \to A$ so that $E_C^D = E_C^A \circ E_A^D$ where $E_C^A = E_C^D|_A$. We write $\mathrm{IMS}(C, D, E_C^D)$ to denote the set of all such intermediate subalgebras.

In [30] Ino and Watatani prove the following perturbation result for intermediate subalgebras.

Proposition 5.1.2. *Let D be a unital C*-algebra and let C be a C*-subalgebra of D with a common unit with a conditional expectation $E_C^D : D \to C$ of finite index. Then there exists a positive contstant $\gamma > 0$ such that if $A, B \in IMS(C, D, E_C^D)$ satisfy $d(A, B) < \gamma$ there exists a unitary $u \in C^*(A, B)$ such that $uAu^* = B$. Furthermore, we can choose the unitary in the relative commutant $C' \cap D$.*

Explicitly, close intermediate subalgebras as above are spatially isomorphic. The condition $u \in C' \cap D$ guarantees that this isomorphism acts as the identity on C.

Given a finite index inclusion $C \subseteq D$ as in the statement above, the constant γ obtained by Ino and Watatani depends upon the properties of the conditional expectation E_C^D; more precisely, on the number of elements in the quasi-basis. The aim of this chapter is to remove the dependence on the structure of E_C^D from Theorem 5.1.2. All results in this chapter are part of [17].

Our strategy is to work with an a priori stronger hypothesis: we assume that A and B are close in the row metric. We are then able to modify Ino and Watatani's techniques to obtain constants independent of the quasi-basis. Finally, using the equivalence of the row and Kadison-Kastler metrics (Theorem 4.5.2), we can recast the hypothesis in terms of the Kadison-Kastler metric.

Our method closely follows that of [30] with slight alterations to deal with estimates involving row amplifications. As with Ino and Watatani's work our proof is divided into two steps. Firstly, we prove the existence of a *-homomorphism from A to B which preserves the subalgebra C and is close to the natural inclusion $\iota : A \to D$. This step is based on Christensen's theorem for injective von Neumann algebra (see Steps 1 and 2 of Section 3.3). As with Christensen's argument the conditional expectation E_B^D can be restricted to A to obtain a completely positive map from A to B close to the identity. By working in the C*-basic construction (see Section 2.3) and using the finite index property to 'average' the projection e_B into $\lambda(A)'$ we obtain a *-homomorphism from A to B close to the natural inclusion.

The next step is to show that close *-homomorphisms from A to D, which preserve C, are unitarily conjugate by a unitary commuting with C. This result can be used to show the *-homomorphism from the previous step is implemented by a unitary close to the identity. We are then in a position to apply Lemma 3.1.3 to demonstrate surjectivity.

5.2 Existence

We start by providing row versions of the estimates from [30] starting with a 'row version' of [30, Lemma 3.2].

Lemma 5.2.1. *Suppose that A and B are C*-subalgebras of a C*-algebra D and suppose that*

$E_B : D \to B$ is a conditional expectation. Let $\iota_A : A \to D$ be the inclusion map, then

$$\|E_B|_A - \iota_A\|_{\text{row}} \leq 2d_{\text{row}}(A, B). \tag{5.1}$$

Further, for $m \in \mathbb{N}$ and an element x in the unit ball of $M_{1,m}(A)$ we have

$$\|E_B^{(1,m)}(x)E_B^{(m,1)}(x^*) - E_B(xx^*)\| \leq 4d_{\text{row}}(A, B) \quad \textit{and} \quad \|E_B^{(m,1)}(x^*)E_B^{(1,m)}(x) - E_B^{(m)}(x^*x)\| \leq 4d_{\text{row}}(A, B). \tag{5.2}$$

Proof. Set $\gamma = d_{\text{row}}(A, B)$. Fix $m \in \mathbb{N}$ and x in the unit ball of $M_{1,m}(A)$. By the hypothesis $d_{\text{row}}(A, B) = \gamma$ there exists x' in the unit ball of $M_{1,m}(B)$ such that $\|x - x'\| \leq \gamma$. We have

$$\|E_B^{(1,m)}(x) - x\| \leq \|E_B^{(1,m)}(x - x')\| + \|x' - x\| \leq 2\gamma \tag{5.3}$$

which proves (5.1) and

$$\|xx^* - x'x'^*\| \leq \|(x - x')x^*\| + \|x'(x^* - x'^*)\| \leq 2\gamma. \tag{5.4}$$

Since x', x'^* and $x'x'^*$ are in $M_{1,m}(B)$, $M_{m,1}(B)$ and B respectively, we have $E_B(x'x'^*) = x'x'^* = E_B^{(1,m)}(x')E_B^{(m,1)}(x'^*)$, combining this with (5.4) yields

$$\begin{aligned}\|E_B^{(1,m)}(x)E_B^{(m,1)}(x^*) - E_B(xx^*)\| \leq{}& \|E_B^{(1,m)}(x)(E_B^{(m,1)}(x^*) - E_B^{(m,1)}(x'^*))\| \\ &+ \|((E_B^{(1,m)}(x) - E_B^{(1,m)}(x'))E_B^{(m,1)}(x'^*)\| \\ &+ \|E_B(x'x'^*) - E_B(xx^*)\| \leq 4\gamma. \end{aligned}\tag{5.5}$$

The final estimate follows in a similar fashion. □

Next we show how statements about the approximate multiplicativity of the conditional expectation can be translated to statements about the norm of operators in $\mathbb{B}(\mathcal{E})$.

Lemma 5.2.2. *Suppose that D is a C^*-algebra and B is a C^*-subalgebra with faithful conditional expectation $E_B : D \to B$ with $e_B, \lambda \in \mathbb{B}(\mathcal{E})$ as defined in Section 2.3. Let $m \in \mathbb{N}$ and $x = (x_1, \ldots, x_m) \in M_{1,m}(D)$. Then the following identities hold*

$$\|E_B^{(1,m)}(x)E_B^{(m,1)}(x^*) - E_B(xx^*)\| = \|e_B\lambda^{(1,m)}(x)(\text{diag}^{(m)}(1 - e_B))\lambda^{(m,1)}(x^*)e_B\|_{\mathbb{B}(\mathcal{E})}$$

and

$$\|E_B^{(m,1)}(x^*)E_B^{(m,1)}(x) - E_B^{(m)}(x^*x)\|_{M_m(D)} = \|\text{diag}^{(m)}(e_B)\lambda^{(m,1)}(x^*)(1 - e_B)\lambda^{(1,m)}(x)\text{diag}^{(m)}(e_B)\|_{M_m(\mathbb{B}(\mathcal{E}))}$$

Proof. Since the map $b \mapsto \lambda(b)e_B$ is a *-isomorphism (see Lemma 2.3.1) so is its amplification

$$(x_{ij})_{ij} \mapsto (\lambda(x_{ij})e_B)_{ij} \tag{5.6}$$

which takes $M_m(B)$ into $M_m(\mathbb{B}(\mathcal{E}))$. Writing $x = (x_1, \ldots, x_m)$ we use the observation in the previous sentence and condition 2 of Lemma 2.3.1 to compute

$$\begin{aligned}
\|E_B^{(1,m)}(x)E_B^{(m,1)}(x^*) - E_B(xx^*)\| &= \|\lambda(E_B^{(1,m)}(x)E_B^{(m,1)}(x^*) - E_B(xx^*))e_B\|_{\mathbb{B}(\mathcal{E})} \\
&= \|\sum_{j=1}^{m} \lambda(E_B(x_j)E_B(x_j^*) - E_B(x_jx_j^*))e_B\|_{\mathbb{B}(\mathcal{E})} \\
&= \|\sum_{j=1}^{m} e_B\lambda(x_j)e_B\lambda(x_j^*)e_B - e_B\lambda(x_jx_j^*)e_B\|_{\mathbb{B}(\mathcal{E})} \\
&= \|\sum_{j=1}^{m} e_B\lambda(x_j)(1-e_B)\lambda(x_j^*)e_B\|_{\mathbb{B}(\mathcal{E})} \\
&= \|e_B\lambda^{(1,m)}(x)(\mathrm{diag}^{(m)}(1-e_B))\lambda^{(m,1)}(x^*)e_B\|_{\mathbb{B}(\mathcal{E})}
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
&\|E_B^{(m,1)}(x^*)E_B^{(1,m)}(x) - E_B^{(m)}(x^*x)\|_{M_m(D)} \\
=&\|(E_B(x_i^*)E_B(x_j) - E_B(x_i^*x_j))_{ij}\|_{M_m(\mathbb{B}(\mathcal{E}))} \\
=&\|(\lambda(E_B(x_i^*)E_B(x_j) - E_B(x_i^*x_j))e_B)_{ij}\|_{M_m(\mathbb{B}(\mathcal{E}))} \\
=&\|(e_B\lambda(x_i^*)(1-e_B)\lambda(x_j)e_B)_{ij}\|_{M_m(\mathbb{B}(\mathcal{E}))} \\
=&\|\mathrm{diag}^{(m)}(e_B)\lambda^{(m,1)}(x^*)(1-e_B)\lambda^{(1,m)}(x)\mathrm{diag}^{(m)}(e_B)\|_{M_m(\mathbb{B}(\mathcal{E}))}.
\end{aligned} \tag{5.8}$$

□

We now modify [30, Lemma 3.3] to work with the row metric obtaining universal constants independent of the inclusion $C \subseteq D$. The argument is based on techniques developed by Christensen in [9] and [10] which we have sketched in Section 3.3.

Lemma 5.2.3. *Let $C \subseteq D$ be a unital inclusion of C^*-algebras. Suppose that $B \subseteq D$ is a C^*-algebra containing C such that there exists a faithful conditional expectation $E_B^D : D \to B$. Suppose that $A \subseteq D$ is another C^*-algebra containing C with a finite index conditional expectation $E_C^A : A \to C$ such that $d_{\mathrm{row}}(A, B) \leq \gamma < 1/16$. Let $\iota_A : A \to D$ denote the inclusion map. Then there exists a *-homomorphism $\phi : A \to B$ such that $\|\phi - \iota_A\|_{\mathrm{row}} \leq 8\sqrt{2}\gamma^{\frac{1}{2}} + 2\gamma$ and $\phi|_C = id_c$.*

Proof. Let $\mathcal{E}$ be the completion of D with the norm derived from E_B^D as described in the paragraph preceding Lemma 2.3.1 and Jones projection $e_B \in \mathbb{B}(\mathcal{E})$. Let $(v_i)_{i=1}^n$ be a quasi-basis for E_C^A in A with $T = \sum_{i=1}^n v_i v_i^*$ the index of E_A, which we recall is central in A and invertible. It follows that $T^{-1/2}$ also belongs to the centre of A. We set

$$t = \sum_{i=1}^{n} \lambda(T^{-1/2}v_i)e_B\lambda(T^{-1/2}v_i^*), \tag{5.9}$$

a symmetrised version of the element defined in [30, Lemma 3.3]. For $x \in A$, using (2.16) and condition 1 from Lemma 2.3.1, we have

$$\begin{aligned}
\lambda(x)t &= \sum_{i=1}^{n} \lambda(T^{-1/2})\lambda(xv_i)e_B\lambda(v_i^*)\lambda(T^{-1/2}) \qquad (\text{as } T^{-1/2} \in \mathcal{Z}(A))\\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} \lambda(T^{-1/2})\lambda(v_j E_C^A(v_j^* x v_i))e_B\lambda(v_i^*)\lambda(T^{-1/2})\\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} \lambda(T^{-1/2})\lambda(v_j)e_B\lambda(E_C^A(v_j^* x v_i)v_i^*)\lambda(T^{-1/2}) \qquad (\text{as } e_B \in \lambda(C)')\\
&= \sum_{j=1}^{n} \lambda(T^{-1/2})\lambda(v_j)e_B\lambda(v_j^* x)\lambda(T^{-1/2})\\
&= \sum_{j=1}^{n} \lambda(T^{-1/2})\lambda(v_j)e_B\lambda(v_j^*)\lambda(T^{-1/2})\lambda(x) = t\lambda(x) \qquad (\text{as } T^{-1/2} \in \mathcal{Z}(A)).
\end{aligned} \tag{5.10}$$

This establishes $t \in \lambda(A)'$. The row

$$M := (T^{-1/2}v_1, \ldots, T^{-1/2}v_n) \tag{5.11}$$

is in the unit ball of $M_{1,n}(A)$ since $MM^* = 1_A$. By modifying the estimates in the first displayed equation on page 6 of [30] to work with rows we have

$$\begin{aligned}
\|t - e_B\| &= \|\sum_{i=1}^{n} \lambda(T^{-1/2}v_i)(e_B\lambda(T^{-1/2}v_i^*) - \lambda(T^{-1/2}v_i^*)e_B)\|\\
&= \|\lambda^{(1,n)}(M)(\mathrm{diag}^{(n)}(e_B)\lambda^{(n,1)}(M^*) - \lambda^{(n,1)}(M^*)e_B)\|\\
&\leq \|\mathrm{diag}^{(n)}(e_B)\lambda^{(n,1)}(M^*) - \lambda^{(1,n)}(M^*)e_B\|\\
&= \|(\mathrm{diag}^{(n)}(e_B))\lambda^{(n,1)}(M^*)(1_{\mathcal{E}} - e_B) - (\mathrm{diag}^{(n)}(1_{\mathcal{E}} - e_B))\lambda^{(n,1)}(M^*)e_B\|\\
&= \max\{\|(\mathrm{diag}^{(n)}(e_B))\lambda^{(n,1)}(M^*)(1_{\mathcal{E}} - e_B)\lambda^{(1,n)}(M)(\mathrm{diag}^{(n)}(e_B))\|^{\frac{1}{2}},\\
&\qquad \|e_B\lambda^{(1,n)}(M)(\mathrm{diag}^{(n)}(1_{\mathcal{E}} - e_B))\lambda^{(n,1)}(M^*)e_B\|^{\frac{1}{2}}\} \leq 2\gamma^{\frac{1}{2}},
\end{aligned} \tag{5.12}$$

The last equality is obtained by applying the C*-identity to the expression on the 4th line and then using the L^∞ property of the norm. The final bound is obtained by applying Lemma 5.2.2 and Lemma 5.2.1 to each expression. We are now in a position to closely follow [30, Lemma 3.3] for the rest of the proof. Set $\delta = 2\gamma^{\frac{1}{2}}$. By applying Lemma 3.1.6 with the bound (5.12) we may find a projection

$$q \in C^*(t, 1_{\mathcal{E}}) \subseteq \lambda(A)' \cap C^*(\lambda(A), e_B) \tag{5.13}$$

with

$$\|q - e_B\| \leq 2\delta < 1. \tag{5.14}$$

Now by applying Lemma 3.1.7 to q and e_B we may find a unitary

$$w \in C^*(q, e_B) \subseteq C^*(\lambda(A), e_B) \tag{5.15}$$

such that $wqw^* = e_B$ and

$$\|w - 1_{\mathcal{E}}\| \leq 2\sqrt{2}\delta. \tag{5.16}$$

We note that since q and e_B commutes with $\lambda(C)$ we also have $w \in C^*(q, e_B) \subseteq \lambda(C)'$.

By the choice of q and w the map $\tilde{\phi} : A \to \lambda(B)e_B$ defined for $x \in A$ by

$$\tilde{\phi}(x) = wq\lambda(x)qw^* = e_B w\lambda(x)w^* e_B \tag{5.17}$$

is a *-homomorphism (this is exactly the same calculation as described in Section 3.3). It follows from condition 2 of Lemma 2.3.1 that $e_B C^*(\lambda(A), e_B)e_B \subseteq \lambda(B)e_B$ and so the range is as stated.

The map $\theta : B \to \lambda(B)e_B$ defined by $b \mapsto \lambda(b)e_B = e_B\lambda(b)e_B$ is a *-isomorphism (by condition 3 of Lemma 2.3.1) so

$$\phi := \theta^{-1} \circ \tilde{\phi} : A \to B \tag{5.18}$$

is a *-homomorphism. For $c \in C$ we have

$$\begin{aligned}\tilde{\phi}(c) &= e_B w\lambda(c)w^* e_B \\ &= e_B\lambda(c)e_B \qquad (\text{as } w \in \lambda(C)') \\ &= \lambda(c)e_B \qquad (\text{as } e_B \in \lambda(C)')\end{aligned} \tag{5.19}$$

and it follows that $\phi(c) = c$.

For $m,n \in \mathbb{N}$ and x in the unit ball of $M_{m,n}(A)$

$$\begin{aligned}\|\phi^{(m,n)}(x) - E_B^{(m,n)}(x)\| &= \|\mathrm{diag}^{(m)}(e_B w)\lambda^{(m,n)}(x)\mathrm{diag}^{(n)}(w^* e_B) - \mathrm{diag}^{(m)}(e_B)\lambda^{(m,n)}(x)\mathrm{diag}^{(n)}(e_B)\| \\ &\leq 2\|1_{\mathcal{E}} - w\| \leq 4\sqrt{2}\delta. \end{aligned} \tag{5.20}$$

Thus $\|\phi - E_B\|_{cb} \leq 4\sqrt{2}\delta$ hence, by Lemma 5.2.1 we have the following estimate

$$\|\phi - \iota_A\|_{\mathrm{row}} \leq 4\sqrt{2}\delta + 2\gamma = 8\sqrt{2}\gamma^{\frac{1}{2}} + 2\gamma. \tag{5.21}$$

□

Remark. If we had assumed that $d_{\mathrm{cb}}(A, B)$ was small then ι_A would be completely close to E_B and hence we would obtain a bound on $\|\phi - \iota_A\|_{\mathrm{cb}}$.

5.3 Uniqueness and the main theorem

Finally we modify [30, Lemma 3.4] again working with the row norm to obtain universal constants. Note that we are automatically improving a row bound to a complete bound.

Lemma 5.3.1. *Let $C \subseteq D$ be a unital inclusion of C*-algebras and suppose $A \subseteq D$ is a C*-subalgebra containing C with a finite index conditional expectation $E_C^A : A \to C$. Let $\phi_1, \phi_2 : A \to D$ be unital *-homomorphisms such that $\phi_1|_C = id_C = \phi_2|_C$ and there exists a constant $0 \leq \gamma < 1$ such that $\|\phi_1 - \phi_2\|_{row} \leq \gamma$. Then there exists a unitary $u \in C' \cap D$ such that $Ad(u) \circ \phi_1 = \phi_2$ and $\|1 - u\| \leq 2\gamma$, in particular, $\|\phi_1 - \phi_2\|_{cb} \leq 4\gamma$.*

Proof. Let $(v_i)_{i=1}^n$ be a quasi-basis for E_C^A and $T = \sum_{i=1}^n v_i v_i^*$ be the index. Again, symmetrising the element defined in [30, Lemma 3.4], set

$$s = \sum_{i=1}^n \phi_1(T^{-1/2}v_i)\phi_2(T^{-1/2}v_i^*). \tag{5.22}$$

For $a \in A$, using the finite index property (2.16) and the hypothesis that ϕ_1 and ϕ_2 are *-homomorphisms that act as the identity on C, we have

$$
\begin{aligned}
\phi_1(a)s &= \sum_{i=1}^{n} \phi_1(T^{-1/2}av_i)\phi_2(T^{-1/2}v_i^*) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} \phi_1(T^{-1/2})\phi_1(v_jE_C^A(v_j^*av_i))\phi_2(T^{-1/2}v_i^*) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} \phi_1(T^{-1/2})\phi_1(v_j)E_C^A(v_j^*av_i)\phi_2(T^{-1/2}v_i^*) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} \phi_1(T^{-1/2})\phi_1(v_j)\phi_2(E_C^A(v_j^*av_i)T^{-1/2}v_i^*) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} \phi_1(T^{-1/2})\phi_1(v_j)\phi_2(T^{-1/2})\phi_2(E_C^A(v_j^*av_i)v_i^*) \\
&= \sum_{j=1}^{n} \phi_1(T^{-1/2}v_j)\phi_2(T^{-1/2}v_j^*a) \\
&= s\phi_2(a). \qquad (5.23)
\end{aligned}
$$

As in the previous lemma, the row

$$M := (T^{-1/2}v_1, \ldots, T^{-1/2}v_n) \qquad (5.24)$$

is in the unit ball of $M_{1,n}(A)$. Since ϕ_1 is unital, using the row norm estimate in the hypothesis we have

$$
\begin{aligned}
\|1 - s\| &= \| \sum_{i=1}^{n} \phi_1(T^{-1/2}v_i)(\phi_1(T^{-1/2}v_i^*) - \phi_2(T^{-1/2}v_i^*))\| \\
&= \|\phi_1^{(1,n)}(M)(\phi_1^{(n,1)}(M^*) - \phi_2^{(n,1)}(M^*))\| \leq \gamma < 1 \qquad (5.25)
\end{aligned}
$$

and hence s is invertible in D. The polar decomposition $s = u|s|$ has unitary $u \in D$ such that $\|1 - u\| \leq \sqrt{2}\gamma$ (see Lemma 3.1.8). We have

$$\phi_1(a^*)s = s\phi_2(a^*) \qquad (a \in A) \qquad (5.26)$$

from (5.23). By taking adjoints we have

$$s^*\phi_1(a) = \phi_2(a)s^* \qquad (a \in A). \qquad (5.27)$$

Combining this with (5.23) we obtain

$$\phi_2(a)s^*s = s^*\phi_1(a)s = s^*s\phi_2(a) \qquad (a \in A). \qquad (5.28)$$

Therefore s^*s and consequently $|s|$ lie in $\phi_2(A)'$. So

$$\phi_1(a) = s\phi_2(a)s^{-1} = u|s|\phi_2(a)|s|^{-1}u^* = u\phi_2(a)u^* \qquad (a \in A). \tag{5.29}$$

It remains to show that u lies in C'. We have $u\phi_1(c) = \phi_2(c)u$ for all $c \in C$ but since $\phi_1(c) = \phi_2(c) = c$ the claim follows. $\square$

Finally, we turn to the proof of our version of [30, Proposition 3.5].

Theorem 5.3.2. *Let $C \subseteq D$ be an inclusion of C*-algebras. Suppose that $B \subseteq D$ is a C*-algebra containing C such that there exists a faithful conditional expectation $E_B^D : D \to B$. Suppose that $A \subseteq D$ is another C*-algebra containing C with a finite index conditional expectation $E_C^A : A \to C$ such that $d(A, B) \leq \gamma < 10^{-6}$. Then there exists a unitary $u \in C' \cap D$ such that $uAu^* = B$ with bound $\|u - 1_D\| \leq 16\sqrt{110}\gamma^{\frac{1}{2}} + 880\gamma$.*

The hypothesis of Theorem 5.1.2 implis that there exists a conditional expectation of finite index from A to C (see the paragraph following Example 2.3.4) and that E_B^D is a faithful (this follows because E_C^D is a finite index conditional expectation and hence faithful) conditional expectation from D to B. Therefore Theorem 5.3.2 are implied by the set up in Thoerem 5.1.2 and hence provides a generalisation of by removing the dependency on the number of elements in the quasi-basis.

In the diagram below we have assumed the existence of E_B^D and E_C^A in the hypothesis of our theorem but by the result of the theorem we have the existence of conditional expectations f and g as shown. To see this let $f : x \mapsto u^*E_B^D(uxu^*)u$, with u as in the result of the theorem, this provides a conditional expectation from D to A and the existence of g is similar.

$$\begin{tikzcd} D \arrow[r, "E_B^D"'] \arrow[d, "f"] & B \arrow[d, "g"] \\ A \arrow[r, "E_C^A"'] & C \end{tikzcd}$$

Proof. Set $\gamma' = 220\gamma$ and apply Theorem 4.5.2 to the hypothesis $d(A, B) \leq \gamma < 10^{-6}$ which establishes $d_{\text{row}}(A, B) \leq \gamma' < 1/2066$. The hypothesis of Lemma 5.2.3 are satisfied, hence there exists a *-homomorphism $\phi : A \to B$ with $\phi|_C = \mathrm{id}_C$ such that

$$\|\phi - \iota_A\|_{\mathrm{row}} \leq 8\sqrt{2}\gamma'^{\frac{1}{2}} + 2\gamma' < 1 \tag{5.30}$$

by the choice of γ'. We apply Lemma 5.3.1 to the *-homomorphisms ϕ and ι_A to yield a unitary $u \in C^*(A, B) \cap C'$ such that $\phi = \mathrm{Ad}(u)$, in particular, $uAu^* \subseteq B$ and

$$\|1_D - u\| \leq 16\sqrt{2}\gamma'^{\frac{1}{2}} + 4\gamma' = 16\sqrt{110}\gamma^{\frac{1}{2}} + 880\gamma. \tag{5.31}$$

Let $b \in B_1$. Using the hypothesis $d(A, B) \leq \gamma$ we may find an element $a \in A_1$ such that $\|a - b\| \leq \gamma \leq \gamma'$. Applying the triangle inequality and using the bound (5.31) we obtain

$$\begin{aligned} \|uau^* - b\| &\leq \|(u - 1_D)au^*\| + \|a(u^* - 1_D)\| + \|a - b\| \\ &\leq 32\sqrt{2}\gamma'^{\frac{1}{2}} + 9\gamma' < 1 \end{aligned} \tag{5.32}$$

by the choice of γ. Since $b \in B_1$ was arbitrary and $\|uau^*\| = \|a\| \leq 1$ we have $d(uAu^*, B) < 1$ and so it follows from Lemma 3.1.3 that $uAu^* = B$. $\square$

Notice that Theorem 3.2.6 implies that close injective von Neumann algebras are automatically completely close (because $d_{\mathrm{cb}}(uAu^*, A) \leq 2\|1 - u\|$). By the same argument the above theorem demonstrates that close intermediate subalgebras are automatically completely close.

Chapter 6

A one-sided version of Roydor's theorem

6.1 Introduction

In [56, Theorem 3.12] Roydor showed that if an injective von Neumann algebra on a Hilbert space $\mathcal{H}$ is sufficiently close to a weak*-closed subalgebra of $\mathbb{B}(\mathcal{H})$ with a normal virtual diagonal, then the algebras are necessarily similar and, furthermore, the implementing invertible element may be chosen to be close to the identity. This establishes a non self-adjoint version of strong Kadison-Kastler stability for injective von Neumann algebras.

The first step in Roydor's argument is similar to the one described at the start of Section 3.3; he restricts the norm one projection (from $\mathbb{B}(\mathcal{H})$ onto the injective algebra) to the non self-adjoint operator algebra and obtains a completely bounded map close to the natural inclusion. Roydor is then able to modify techniques developed by Johnson in [35] in order to use the normal virtual diagonal to refine the completely bounded map into a bounded homomorphism (the analogy of step 2 in Christensen's original argument).

A one-sided version of the Kadison-Kastler conjecture asks whether, given a sufficiently close near containment of a C*-algebra A in a C*-algebra B (see Definition 3.1.2), one may find a unitary $u(\approx 1)$ implementing a genuine embedding $uAu^* \subseteq B$. Christensen showed that this is the case when A and B are von Neumann algebras on the same Hilbert space with A injective [11] and in [28]

Hirshberg, Kirchberg and White obtain a positive solution if A and B are separable C*-algebras with A nuclear, providing one-sided versions of Theorem 3.2.6 and Theorem 3.2.7 respectively (although, like the two-sided version, $u \not\approx 1$ in general in the nuclear case). It is important to assume amenability on A as the known techniques involve averaging maps with A as a domain to obtain an embedding from A to B.

In this chapter we investigate a one-sided version of Roydor's work and consider a near inclusion of an injective von Neumann algebra A on $\mathcal{H}$ in a non-self adjoint weak*-closed subalgebra $N \subseteq \mathbb{B}(\mathcal{H})$. We cannot expect to find an implementing unitary, instead we aim to find an invertible operator $S \approx 1_{\mathcal{H}}$ implementing a genuine containment $SAS^{-1} \subseteq N$.

We will not be able to proceed using Roydor's method, sketched above, because in our situation the first step must entail finding a map from A to N onto which there is not, in general, a norm one projection. Instead, in Section 6.3, we exploit the hyperfiniteness of A (see Definition 1.2.8 and Theorem 1.2.9) which allows us to consider A as the weak*-closure of a net of finite dimensional algebras. On each finite dimensional piece we are able to construct a completely bounded map into N which is close to the natural inclusion. We then find a weak*-accumulation point of such maps. The resulting map is a completely bounded map from A to N however, in contrast with Roydor and Christensen's work, this map is only close to the identity on a weak*-dense AF-subalgebra of A.

Next we turn to a version of Step 2 of Christensen's argument in which we refine the completely bounded map described in the paragraph above into a bounded homomorphism. One possible method (Section 6.5) is to modify techniques developed by Christensen (presented at the end of Section 3.3) to work in the completely bounded setting. We do obtain an analogous result (Lemma 6.5.2) under the hypothesis that the completely bounded map is normal. However, we were unable to reduce to this case by taking normal parts as Christensen does in the paragraph following Lemma 3.5 in [9]. One can take the normal part of a completely bounded map but in our case, as we only know that the map is close to the inclusion on a dense subalgebra, the resulting normal map may not be close to the inclusion map anywhere.

Instead, in Section 6.4 we consider the restriction of the map constructed in Section 6.3 to the dense AF-algebra on which it is bounded close to the inclusion. Since the map is close to the inclusion, it is almost multiplicative and so we may use Johnson's result [35] to refine it into a bounded homomorphism. We are then able to prove that this bounded homomorphism is spatially implemented and hence may be extended to A, this establishes the result Theorem 6.1.1.

Finally in Section 6.6 we adapt Johnson's theorem to work with completely bounded norms to obtain Lemma 6.6.3 which improves the estimates which are obtained in Theorem 6.1.1 by applying Johnson's theorem directly. The following is joint work with Stuart White [18].

Theorem 6.1.1. *Let $0 < \gamma < 0.0017$. Suppose that A is a countably generated injective von Neumann algebra on $\mathcal{H}$ with $1_A = 1_{\mathcal{H}}$ and suppose that N is a weak*-closed subalgebra of $\mathbb{B}(\mathcal{H})$ which satisfies $A \subseteq_\gamma N$. Then there exists a bounded homomorphism $\rho : A \to N$ such that*

$$\|\rho - \iota_A\| \leq 4\gamma^6 + 24\gamma^5 + 64\gamma^4 + 96\gamma^3 + 77\gamma^2 + 26\gamma. \tag{6.1}$$

Furthermore, we may find a invertible operator S with

$$\|1_{\mathcal{H}} - S\| \leq 4\gamma^6 + 24\gamma^5 + 64\gamma^4 + 96\gamma^3 + 77\gamma^2 + 26\gamma \tag{6.2}$$

and with $S, S^{-1} \in \overline{Alg(A, N, 1_{\mathcal{H}})}^{w^*}$ such that

$$\rho(x) = SxS^{-1} \qquad (x \in A). \tag{6.3}$$

The proof of Theorem 6.1.1 as stated will be given in Section 6.4. This theorem strengthens Roydor's results by removing the assumption that there exists a normal virtual diagonal. Indeed, if A is an injective von Neumann algebra and N is a neighbouring non self-adjoint weak*-closed algebra then by the above theorem one may find an operator S, close to the identity, such that $SAS^{-1} \subseteq N$. It follows that N is nearly contained in SAS^{-1} and we have $SAS^{-1} = N$ by applying Lemma 3.1.3.

Remark. The case where $1_A \neq 1_{\mathcal{H}}$ follows from Theorem 6.1.1 with worse constants. By the functional calculus argument of the proof of Lemma 6.5.2 we may find an idempotent $q \in N$ that is close to 1_A. Providing A and N were sufficiently close we may find an invertible element $z \in \mathbb{B}(\mathcal{H})$ with $z^{-1}1_A z = q$. Then A is nearly contained in $1_A zNz^{-1}1_A$. By Theorem 6.1.1 there exists an invertible element S that is close to the identity and satisfies $SAS^{-1} \subseteq 1_A zNz^{-1}1_A$. Therefore

$$z^{-1}SAS^{-1}z \subseteq z^{-1}1_A zNz^{-1}1_A z = qNq \subseteq N. \tag{6.4}$$

6.2 Notation and technical preliminaries

We record some technical lemmas that will be used throughout this chapter. The following proposition follows by noting that the proof of Proposition 2.10 and Corollary 2.12 of [13] do not require B to be a C*-algebra, only to be closed under multiplication.

Proposition 6.2.1. *Let $A \subseteq \mathbb{B}(\mathcal{H})$ be a nuclear C*-algebra and suppose N is a subalgebra of $\mathbb{B}(\mathcal{H})$ such that $A \subseteq_\gamma N$ for some $\gamma > 0$. Given any nuclear C*-algebra D, we have $A \otimes D \subseteq_{2\gamma+\gamma^2} N \otimes D$ inside $\mathbb{B}(\mathcal{H}) \otimes D$.*

We will require the following calculation a number of times so we will record it here.

Lemma 6.2.2. *Let A and B be C^*-algebras acting on a Hilbert space $\mathcal{H}$. Suppose ϕ and ψ are elementary maps from A to B which are implemented by elements of $\mathbb{B}(\mathcal{H})$ as follows; there exists $S, S', T, T' \in \mathbb{B}(\mathcal{H})$ such that*

$$\phi(x) = SxS' \qquad \text{and} \qquad \psi(x) = TxT' \tag{6.5}$$

for all $x \in A$. Then

$$\|\phi - \psi\|_{cb} \leq \min\{\|S - T\|\|S'\| + \|T\|\|S' - T'\|, \|S\|\|S' - T'\| + \|S - T\|\|T'\|\}. \tag{6.6}$$

Proof. Let $n \in \mathbb{N}$ and $x = (x_{ij})_{ij}$ be an element of the unit ball of $M_n(A)$. Then

$$\begin{aligned}
\|\phi^{(n)}(x) - \psi^{(n)}(x)\| =& \|(\phi(x_{ij}))_{ij} - (\psi(x_{ij}))_{ij}\| \\
=& \|(Sx_{ij}S')_{ij} - (Tx_{ij}T')_{ij}\| \\
=& \|\mathrm{diag}(S, \dots, S)x\mathrm{diag}(S', \dots, S') - \mathrm{diag}(T, \dots, T)x\mathrm{diag}(T', \dots, T')\| \\
\leq& \|\mathrm{diag}(S - T, \dots, S - T)x\mathrm{diag}(S', \dots, S')\| \\
&+ \|\mathrm{diag}(T, \dots, T)x\mathrm{diag}(S - T, \dots, S - T)\| \\
\leq& (\|S - T\|\|S'\| + \|T\|\|S' - T'\|)\|x\|.
\end{aligned} \tag{6.7}$$

The other bound is obtained in a similar fashion. $\square$

It follows from [55, Theorem 2] that a bounded homomorphism from a finite dimensional C*-algebra into $\mathbb{B}(\mathcal{H})$ which is sufficiently close to the natural inclusion is spatially implemented. We provide a direct proof for completeness.

Lemma 6.2.3. *Let $0 < \gamma < 1$ and let A be a finite dimensional C*-algebra on $\mathcal{H}$ such that $1_A = 1_{\mathcal{H}}$ with natural inclusion map $\iota_A : A \to \mathbb{B}(\mathcal{H})$. Suppose $\phi : A \to \mathbb{B}(\mathcal{H})$ is a bounded homomorphism such that*

$$\|\phi - \iota_A\| \leq \gamma. \tag{6.8}$$

Then there exists an invertible element $S \in Alg(A, \phi(A))$ such that for all $x \in A$

$$\phi(x) = S^{-1}xS \tag{6.9}$$

and $\|1_{\mathcal{H}} - S\| \leq \gamma$.

Proof. Write $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$. For each $1 \leq k \leq r$ let G_k be the finite subgroup of the unitary group of $M_{n_k}(\mathbb{C})$ generated by the union of the set of permutation matrices and the set of all diagonal matrices with entries in $\{-1, 1\}$. The linear span of G_k is $M_{n_k}(\mathbb{C})$. Let $G = G_1 \oplus \cdots \oplus G_r$ and set

$$S := \frac{1}{|G|} \sum_{g \in G} g\phi(g^{-1}) \in \mathrm{Alg}(A, \phi(A)). \tag{6.10}$$

Then for any $h \in G$ we have

$$hS = \frac{1}{|G|} \sum_{g \in G} hg\phi(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} g\phi(g^{-1}h) = S\phi(h) \tag{6.11}$$

Since elements of G span A this implies that $xS = S\phi(x)$ for all $x \in A$. By hypothesis we have $\|1_{\mathcal{H}} - S\| \leq \gamma < 1$ and so S is invertible. □

6.3 Constructing completely bounded maps

The first step in our proof of Theorem 6.1.1 is to construct a map from a finite dimensional algebra F on $\mathcal{H}$ to a nearly containing subalgebra $N \subseteq \mathbb{B}(\mathcal{H})$ that is completely close to the natural inclusion map. The proof is based on that of Theorem 6.4 of [14].

Lemma 6.3.1. *Let $F \cong M_{n_1} \oplus \cdots \oplus M_{n_r}$ be a finite dimensional C*-algebra on a Hilbert space $\mathcal{H}$. Suppose that N is a subalgebra of $\mathbb{B}(\mathcal{H})$ such that $F \subseteq_\gamma N$. Then there exists a completely bounded linear map $\phi : F \to N$ satisfying $\|\phi - \iota_F\|_{cb} \leq (2\gamma + \gamma^2)(2 + 2\gamma + \gamma^2)$.*

Proof. For $1 \le k \le r$ let $(e_{ij}^{(k)})_{i,j=1}^{n_k}$ be matrix units for M_{n_k}. Set $m := \max\{n_1, \dots, n_r\}$ and let $(f_{ij})_{i,j=1}^{m}$ be matrix units for M_m. For $1 \le k \le r$ define maps $\theta_k : M_{n_k} \to M_m$

$$\theta_k(e_{ij}^{(k)}) = f_{ij} \tag{6.12}$$

for $1 \le i, j \le n_k$.

Define a *-homomorphism $\theta : F \to M_r(\mathbb{B}(\mathcal{H}) \otimes M_m)$ as follows

$$\theta(x_1, \dots, x_r) = \begin{pmatrix} 1_F \otimes \theta_1(x_1) & 0 & \dots & 0 \\ 0 & 1_F \otimes \theta_2(x_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1_F \otimes \theta_r(x_r) \end{pmatrix}. \tag{6.13}$$

For $1 \le k \le r$ define elements

$$u_k := \sum_{i,j=1}^{n_k} e_{ij}^{(k)} \otimes f_{ji} \in F \otimes M_m(\mathbb{C}). \tag{6.14}$$

For $1 \le k \le r$ and $1 \le i, j \le n_k$ we have

$$u_k\big(1_F \otimes \theta_k(e_{ij}^{(k)})\big)u_k^* = e_{ij}^{(k)} \otimes \sum_{j=1}^{n_k} f_{jj}. \tag{6.15}$$

Set $p := 1_{\mathcal{H}} \otimes f_{11} \in \mathbb{B}(\mathcal{H}) \otimes M_m$ and define the completely contractive linear map $w_{p,p}$ on $\mathbb{B}(\mathcal{H}) \otimes M_m$ by

$$w_{p,p}(x) = pxp \qquad (x \in \mathbb{B}(\mathcal{H}) \otimes M_m). \tag{6.16}$$

Let u be the row vector $(u_1, \dots, u_r) \in M_{1,r}(F \otimes M_m)$. For $(x_1, \dots, x_r) \in F$ we use (6.15) to obtain

$$\begin{aligned} w_{p,p} \circ Ad(u) \circ \theta(x_1, \dots, x_r) =& p\left(u_1\big(1_F \otimes \theta_1(x_1)\big)u_1^* + \dots + u_r\big(1_F \otimes \theta_1(x_1)\big)u_r^*\right)p \\ =& p(x_1 \otimes \sum_{j=1}^{n_1} f_{jj} + \dots + x_r \otimes \sum_{j=1}^{n_r} f_{jj})p \\ =& (x_1 \oplus \dots \oplus x_r) \otimes f_{11}. \end{aligned} \tag{6.17}$$

Using the identification $\mathbb{B}(\mathcal{H}) \cong \mathbb{B}(\mathcal{H}) \otimes f_{11}M_m f_{11}$ we have

$$w_{p,p} \circ Ad(u) \circ \theta(x) = x \qquad (x \in F). \tag{6.18}$$

With $u_1, \ldots, u_r$ as above, define the element $\tilde{u} \in M_r(F \otimes M_m)$ as follows

$$\tilde{u} := \begin{pmatrix} u_1 & u_2 & \ldots & u_r \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{6.19}$$

We identify $M_r(\mathbb{B}(\mathcal{H}) \otimes M_m)$ with $\mathbb{B}(\mathcal{H}) \otimes (M_m \otimes M_r)$. Since F and $M_m \otimes M_r$ are nuclear we may apply Proposition 6.2.1 to yield the following near containment in $\mathbb{B}(\mathcal{H}) \otimes (M_m \otimes M_r)$

$$F \otimes (M_m \otimes M_r) \subseteq_{2\gamma+\gamma^2} N \otimes (M_m \otimes M_r). \tag{6.20}$$

Since $\tilde{u}\tilde{u}^* \leq 1$ we have $\|\tilde{u}\| \leq 1$ and so we may use (6.20) to find an element $\tilde{S} \in N \otimes (M_m \otimes M_r)$ such that

$$\|\tilde{u} - \tilde{S}\| \leq 2\gamma + \gamma^2. \tag{6.21}$$

Using the same identification, we view $\tilde{S}$ as an element of $M_r(\mathbb{B}(\mathcal{H}) \otimes M_m)$ and let

$$e := (1_{\mathcal{H}} \otimes 1_{M_m}, 0, \ldots, 0) \in M_{1,r}(\mathbb{B}(\mathcal{H}) \otimes M_m), \tag{6.22}$$

we have $e\tilde{u} = u$ and so, setting $S := e\tilde{S} \in M_{1,r}(N \otimes M_m)$, it follows from (6.21) that

$$\|u - S\| = \|e(\tilde{u} - \tilde{S})\| \leq 2\gamma + \gamma^2. \tag{6.23}$$

Since u is in the unit ball of $\mathbb{B}(\mathcal{H})$ it follows from (6.23) that

$$\|S\| \leq 1 + 2\gamma + \gamma^2. \tag{6.24}$$

In a similar fashion we can find an element $S' \in M_{r,1}(N \otimes M_m)$ such that

$$\|u^* - S'\| \leq 2\gamma + \gamma^2. \tag{6.25}$$

In accordance with our convention, define the map $w_{S,S'}(x) : \mathbb{B}(\mathcal{H}) \otimes M_m \to \mathbb{B}(\mathcal{H}) \otimes M_m$ by

$$w_{S,S'}(x) := SxS' \tag{6.26}$$

for $x \in M_r(\mathbb{B}(\mathcal{H}) \otimes M_m)$. With p and the map $w_{p,p}$ as above, set $\phi(x) := w_{p,p} \circ w_{S,S'} \circ \theta$. For $x \in F$ we have $\theta(x) \in M_r(1_{\mathcal{H}} \otimes M_m)$ and $S \in M_{1,r}(N \otimes M_m)$, $S' \in M_{r,1}(N \otimes M_m)$ it follows that

$w_{S,S'} \circ \theta(x)$ lies in $N \otimes M_m$. Under the identification $p(\mathbb{B}(\mathcal{H}) \otimes M_m)p \cong \mathbb{B}(\mathcal{H})$ it follows that ϕ takes F to N.

Using (6.23),(6.24), (6.25) and Lemma 6.2.2 we obtain

$$\|Ad(u) - w_{S,S'}\|_{\mathrm{cb}} \leq (2\gamma + \gamma^2)(2 + 2\gamma + \gamma^2). \tag{6.27}$$

Using (6.27) and the fact that $w_{p,p}$ and θ are completely contractive we have the following

$$\begin{aligned} \|\phi - \iota_F\|_{\mathrm{cb}} &= \|w_{p,p} \circ (w_{S,S'} - Ad(u)) \circ \theta\|_{\mathrm{cb}} \\ &\leq \|Ad(u) - w_{S,S'}\|_{\mathrm{cb}} \\ &\leq (2\gamma + \gamma^2)(2 + 2\gamma + \gamma^2). \end{aligned}$$

□

We now turn to the situation of an injective von Neumann algebra A on $\mathcal{H}$ that is nearly contained in a weak*-closed subalgebra $N \subseteq \mathbb{B}(\mathcal{H})$. Using the hyperfiniteness of A and Theorem 6.3.1 we are able to construct a net of maps from A to N that are completely close to the identity on the collection of finite dimensional subalgebras of A. We then take a point-weak* accumulation point of this net and show that it is completely close to the identity on a weak*-dense AF-subalgebra of A.

Lemma 6.3.2. *Let A be a countably generated injective von Neumann algebra on $\mathcal{H}$ with inclusion $\iota_A : A \to \mathbb{B}(\mathcal{H})$. If N is a weak*-closed subalgebra of $\mathbb{B}(\mathcal{H})$ such that $A \subseteq_\gamma N$ then we may find a completely bounded map $\psi : A \to N$ and a weak*-dense approximately finite-dimensional C*-subalgebra $A_0 \subseteq A$ such that*

$$\|\psi|_{A_0} - \iota_{A_0}\|_{cb} \leq (2\gamma + \gamma^2)(2 + 2\gamma + \gamma^2) = \gamma'. \tag{6.28}$$

Proof. By [20, Corollary 5] we may find a directed set $(F_\lambda)_\lambda$ of finite dimensional C*-subalgebras of A such that $\bigcup_{\lambda \in \Lambda} F_\lambda$ is weak*-dense in A.

Fix $\lambda \in \Lambda$. Since F_λ is finite dimensional it is an injective von Neumann algebra and so we may extend the identity map on F_λ to a completely positive map $E_\lambda : A \to F_\lambda$.

Since $A \subseteq_\gamma N$ it follows that $F_\lambda \subseteq_\gamma N$ so we may use Lemma 6.3.1 to find a completely bounded map $\phi_\lambda : F_\lambda \to N$ such that $\|\phi_\lambda - \iota_{F_\lambda}\|_{\mathrm{cb}} \leq (2\gamma + \gamma^2)(2 + \gamma + \gamma^2) = \gamma'$. Define $\psi_\lambda := \phi_\lambda \circ E_\lambda : A \to N$.

For $n \in \mathbb{N}$ and x in the unit ball of $M_n(F_\lambda)$ we have $E_\lambda^{(n)}(x) = x$, therefore we have

$$\|\psi_\lambda^{(n)}(x) - x\| = \|\phi_\lambda^{(n)}(x) - x\| \leq (2\gamma + \gamma^2)(2 + 2\gamma + \gamma^2) = \gamma'. \tag{6.29}$$

In this manner we can construct a net $(\psi_\lambda)_{\lambda\in\Lambda}$ of maps from A to N with

$$\|\psi_\lambda|_{F_\lambda} - \iota_{F_\lambda}\|_{\mathrm{cb}} \leq \gamma'. \tag{6.30}$$

We now construct a point-weak* limit of the these maps, this is a standard technique (see [2, Theorem 1.3.7]), however we include the details for completeness. Let $\mathbb{B}(A, \mathbb{B}(\mathcal{H}))$ denote the bounded linear maps from A to $\mathbb{B}(\mathcal{H})$. Set

$$X = \overline{\mathrm{span}\{x \otimes \xi : \ \ x \in A, \xi \in \mathbb{B}(\mathcal{H})_*\}} \subseteq \mathbb{B}(A, \mathbb{B}(\mathcal{H}))^* \tag{6.31}$$

with

$$(x \otimes \xi)(T) = \xi(T(x)) \qquad (T \in \mathbb{B}(A, \mathbb{B}(\mathcal{H}))). \tag{6.32}$$

For $T \in \mathbb{B}(A, \mathbb{B}(\mathcal{H}))$ define $\tilde{T} \in X^*$ as follows

$$\tilde{T}(x \otimes \xi) = \xi(T(x)). \tag{6.33}$$

The map $T \mapsto \tilde{T}$ is an isometric isomorphism, we will only show that this map is a surjection. For $f \in X^*$ define $T_f \in \mathbb{B}(A, \mathbb{B}(\mathcal{H}))$ by

$$\xi(T_f(x)) = f(x \otimes \xi) \qquad (x \in A, \xi \in \mathbb{B}(\mathcal{H})_*). \tag{6.34}$$

Then

$$\tilde{T_f}(x \otimes \xi) = \xi(T_f(x)) = f(x \otimes \xi) \qquad (x \in A, \xi \in \mathbb{B}(\mathcal{H})_*) \tag{6.35}$$

which verifies surjectivity. It follows that $\mathbb{B}(A, \mathbb{B}(\mathcal{H}))$ has a predual from which it receives the weak*-topology.

The net $(\psi_\lambda)_{\lambda\in\Lambda}$ is norm bounded in $\mathbb{B}(A, \mathbb{B}(\mathcal{H}))$ and so has a weak*-convergent subnet by the Banach-Alaoglu theorem. Therefore there exists a directed set $M = (\mu)_\mu$ and a monotonic function $h : M \to \Lambda$ such that $(\psi_{h(\mu)})_\mu$ is point weak*-convergent. Let ψ denote the point-weak* limit so that

$$\lim_\mu \xi(\psi_{h(\mu)}(x)) = \xi(\psi(x)) \qquad (x \in A, \xi \in \mathbb{B}(\mathcal{H})_*). \tag{6.36}$$

For all $x \in A$ and $\mu \in M$ we have $\psi_{h(\mu)}(x) \in N$ by the construction of the $\psi'_\lambda s$. Hence $\psi(x) \in N$ since N is weak*-closed.

We claim that for each $\lambda \in \Lambda$ we have $\|\psi_{F_\lambda} - \iota_{F_\lambda}\|_{cb} \leq \gamma'$. Indeed, fix $\lambda \in \Lambda$ and an element $x = (x_{ij})_{ij}$ in the unit ball of $M_n(F_\lambda)$. Find μ' such that $h(\mu') \geq \lambda$ then (6.30) implies that for all $\mu \geq \mu'$ we have $\|\psi^{(n)}_{h(\mu)}(x) - x\| \leq \gamma'$. As $\psi(x_{ij}) = \text{w*-}\lim_\mu \psi_{h(\mu)}(x_{ij})$ it follows that $\psi^{(n)}(x) = \text{w*-}\lim_\mu \psi^{(n)}_{h(\mu)}(x)$. Set

$$B(x, \gamma') := \{T \in M_n(\mathbb{B}(\mathcal{H})) \ : \ \|x - T\| \leq \gamma'\}. \tag{6.37}$$

By the Hahn-Banach theorem $B(x, \gamma')$ is weak*-closed. Since the net $(\psi^{(n)}_{h(\mu)}(x))_{\mu \geq \mu'}$ lies in in $B(x, \gamma')$ its weak*-limit $\psi^{(n)}(x)$ must also lie in $B(x, \gamma')$, so we have

$$\|\psi^{(n)}(x) - x\| \leq \gamma' \tag{6.38}$$

which verifies the claim.

We show that the above bound holds for elements in the weak*-dense AF-subalgebra

$$A_0 = \overline{\cup_{\lambda \in \Lambda} F_\lambda}^{\|\cdot\|} \subseteq A. \tag{6.39}$$

Fix $n \in \mathbb{N}$ and $\epsilon > 0$. Let $x = (x_{ij})_{ij}$ be an element of the unit ball of $M_n(A_0)$ we may find $\lambda \in \Lambda$ and $x' = (x'_{ij})_{ij} \in M_n(F_\lambda)$ such that $\|x_{ij} - x'_{ij}\| \leq \frac{\epsilon}{n^2\|\psi\|+1}$ for $1 \leq i, j \leq n$. Using (6.38) we obtain the following estimate

$$\begin{aligned} \|\psi^{(n)}(x) - x\| &\leq \|\psi^{(n)}(x) - \psi^{(n)}(x')\| + \|\psi^{(n)}(x') - x'\| + \|x' - x\| \\ &\leq \|(\psi(x_{ij} - x'_{ij}))_{ij}\| + \gamma' + \frac{\epsilon}{n^2\|\psi\| + 1} \\ &\leq n^2 \max_{i,j} \|\psi(x_{ij} - x'_{ij})\| + \gamma' + \frac{\epsilon}{n^2\|\psi\| + 1} \\ &\leq n^2\|\psi\| \frac{\epsilon}{n^2\|\psi\| + 1} + \gamma' + \frac{\epsilon}{n^2\|\psi\| + 1} = \gamma' + \epsilon. \end{aligned} \tag{6.40}$$

Since ϵ was arbitrary this calculation verifies (6.28) as claimed. □

6.4 Constructing a completely bounded homomorphism

We have constructed a map from A to N that is close to the identity in norm on a weak*-dense subalgebra A_0. If this map was normal we would obtain the same bound on all of A, however, this does not seem to follow. To get round this problem we restrict to A_0 and refine this map into a homomorphism close to the identity from A_0 to N by a result of Johnson. We then show that the resulting homomorphism is spatially implemented and hence normal, and so extends to a bounded homomorphism from A to N with the same bound relative to the inclusion map. We start by introducing some notation due to Johnson [35].

Definition 6.4.1. Let A and B be Banach algebras. For $T \in \mathbb{B}(A, B)$ define the map $T^\vee : A \times A \to B$ as follows

$$T^\vee(a, b) := T(ab) - T(a)T(b) \qquad (a \in A, b \in B). \tag{6.41}$$

The norm of the bilinear map $T^\vee$ measures the defect in multiplicity of T.

We now state Johnson's theorem [35, Theorem 3.1]

Theorem 6.4.2 (Johnson)**.** *Let A be an amenable Banach algebra and suppose that B is a Banach algebra such that there is a Banach B-bimodule B_* so that B is isomorphic as a B=bimodule with $(B_*)^*$. Then (A, B) is AMNM.*

By Theorem 1.2.9 all nuclear (and hence AF) algebras are amenable and have amenability constant 1 (L in the statement of [35, Theorem 3.1]). A weak*-closed subalgebra of $\mathbb{B}(\mathcal{H})$ has a predual with a natural module action as described in the statement of [35, Theorem 3.1]. We combine these facts and extracting the constants in the proof to phrase the theorem for AF algebras.

Theorem 6.4.3 (Johnson)**.** *Let K, γ be positive constants satisfying $\delta < \min\{(4+8K)^{-1}, (4\max\{K, 1\})^{-1}, 1\}$. Let A be an AF-algebra and let B be a weak*-closed sublagebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Suppose $\psi : A \to B$ is a linear map satisfying $\|\psi\| \leq K$ and $\|\psi^\vee\| \leq \delta$. Then there exists a bounded homomorphism $\rho : A \to B$ with $\|\rho - \psi\| \leq 4K\delta$.*

We now show that a bounded homomorphism defined on an AF-algebra which is sufficiently close to the identity map is spatially implemented and so extends to its weak*-closure.

Lemma 6.4.4. *Set $0 < \gamma < 1$ and let $A = (\cup_{\lambda\in\Lambda} F_\lambda)''$ be a hyperfinite von Neumann algebra on $\mathcal{H}$ with $(F_\lambda)_\lambda$ an increasing net of finite dimensional algebras. Let $A_0 = \overline{\cup_{\lambda\in\Lambda} F_\lambda}^{\|\cdot\|}$ and suppose that $1_A = 1_{\mathcal{H}} \in A_0$. If N is a weak*-closed subalgebra of $\mathbb{B}(\mathcal{H})$ and $\rho : A_0 \to N$ is a bounded* *homomorphism with*

$$\|\rho - \iota_{A_0}\| \leq \gamma. \tag{6.42}$$

Then there is an element $S \in \overline{Alg(\rho(A_0), A_0)}^{w^*}$ *such that*

$$\rho(x) = SxS^{-1} \qquad (x \in A_0) \tag{6.43}$$

with $\|1 - S\| \leq \gamma$*. Therefore* ρ *extends to a bounded homomorphism* $\tilde{\rho}$ *from* A *to* N *and this map satisfies*

$$\|\tilde{\rho} - \iota_A\| \leq \gamma. \tag{6.44}$$

Proof. For each $\lambda \in \Lambda$ we have $\|\rho|_{F_\lambda} - \iota_{F_\lambda}\| \leq \gamma < 1$. We may apply Lemma 6.2.3 to find an element $S_k \in \mathbb{B}(\mathcal{H})$ such that

$$xS_\lambda = S_\lambda \rho(x) \qquad (x \in F_\lambda) \tag{6.45}$$

and $\|1_{\mathcal{H}} - S_\lambda\| \leq \gamma$. The net $(S_\lambda)_{\lambda \in \Lambda}$ is bounded so by the Banach-Alaoglu theorem we may find a weak*-convergent subnet $(S_{h(\mu)})_{\mu \in M} \subseteq \mathbb{B}(\mathcal{H})$ where M is a directed set and $h : M \to \Lambda$ is a monotone function. Let $S \in \mathbb{B}(\mathcal{H})$ be the limit point of this subnet in the weak*-topology. For a fixed $\lambda \in \Lambda$ and $x \in F_\lambda$ there exists a μ' such that $h(\mu') \geq \lambda$ so for $\mu \geq \mu'$ we have $xS_{h(\mu)} = S_{h(\mu)}\rho(x)$ and therefore

$$xS = S\rho(x). \tag{6.46}$$

Since weak*-limits preserve norm bounds (Hahn-Banach) we have $\|1_{\mathcal{H}} - S\| \leq \gamma < 1$ and so S is invertible. Since λ was arbitrary, we may rearrange (6.46) to get

$$\rho(x) = S^{-1}xS \qquad (x \in A_0). \tag{6.47}$$

Therefore ρ is normal so it extends to A. Set

$$\tilde{\rho}(x) = S^{-1}xS \qquad (x \in A). \tag{6.48}$$

It follows from the weak*-density of A_0 in A that the range of $\tilde{\rho}$ is contained in N. Since $\tilde{\rho}$ is spatially implemented it is normal and since it extends ρ it must satisfy

$$\|\tilde{\rho} - \iota_A\| \leq \gamma. \tag{6.49}$$

□

We are now in a position to assemble the steps described above.

Proof of Theorem 6.1.1. Set $\gamma' = (2\gamma + \gamma^2)(2 + 2\gamma + \gamma^2)$. Use Lemma 6.3.2 to find a completely bounded map $\psi : A \to N$ and a weak*-dense AF-algebra $A_0 \subseteq A$ with

$$\|\psi|_{A_0} - \iota_{A_0}\|_{\mathrm{cb}} \leq \gamma'. \tag{6.50}$$

We have $\|\psi|_{A_0}\| \leq 1 + \gamma'$. For $x, y \in A_0$ we have

$$\begin{aligned}\|\psi(xy) - \psi(x)\psi(y)\| &\leq \|\psi(xy) - xy\| + \|(x - \psi(x))y\| + \|\psi(x)(y - \psi(y)\| &(6.51)\\ &\leq (3\gamma' + \gamma'^2)\|x\|\|y\| &(6.52)\end{aligned}$$

and therefore $\|(\psi|_{A_0})^\vee\| \leq (3\gamma' + \gamma'^2)$. The bound on γ in the hypothesis of Theorem 6.1.1 guarantees that Theorem 6.4.3 applies to ψ so there exists a bounded homomorphism $\rho : A_0 \to N$ such that

$$\|\rho - \psi|_{A_0}\| \leq 4(1 + \gamma')(3\gamma' + \gamma'^2). \tag{6.53}$$

Combining (6.50) and (6.53), and by the choice of γ in Theorem 6.1.1, we have

$$\|\rho - \iota_{A_0}\| \leq 4(1 + \gamma')(3\gamma' + \gamma'^2) + \gamma' < 1. \tag{6.54}$$

We use Lemma 6.4.4 to extend this map to a bounded homomorphism $\phi : A \to N$ with

$$\|\phi - \iota_A\| \leq 4(1 + \gamma')(3\gamma' + \gamma'^2) + \gamma' = 4\gamma^6 + 24\gamma^5 + 64\gamma^4 + 96\gamma^3 + 77\gamma^2 + 26\gamma. \tag{6.55}$$

It particular,

$$\phi(x) = S^{-1}xS \qquad (x \in A) \tag{6.56}$$

with $S, S^{-1} \in \overline{\mathrm{Alg}(A, N, 1_{\mathcal{H}})}^{\mathrm{w}^*}$ as in the proof of Lemma 6.4.4. □

Remark. As the map ϕ is spatially implemented by an element close to the identity operator we obtain a complete bound relative to the inclusion map. Set $\gamma'' = 4\gamma^6 + 24\gamma^5 + 64\gamma^4 + 96\gamma^3 + 77\gamma^2 + 26\gamma$. By Lemma 6.4.4 we have $\|1_{\mathcal{H}} - S\| \leq \gamma''$, it follows that $\|S^{-1}\| \leq \frac{1}{1-\gamma''}$ and $\|1_{\mathcal{H}} - S^{-1}\| \leq \frac{\gamma''}{1-\gamma''}$. Finally, applying Lemma 6.2.2 gives

$$\|\phi - \iota\|_{cb} \leq \frac{2\gamma''}{1 - \gamma''}. \tag{6.57}$$

6.5 A Christensen type theorem

In the discussion at the end of Section 3.3 we outlined some ideas in Christensen's proof of the perturbation theorem for injective von Neumann algebras. In his argument the first step was to construct a completely positive map uniformly close to the identity. Comparing this with our counterpart, Theorem 6.3.2, it seems natural that completely bounded maps play the role of completely positive maps in the non self-adjoint setting. However, where Christensen was able to achieve a uniform bound on all of the domain, we were only able to obtain such a bound on a weak*-dense subalgebra.

The next step we discussed was Christensen's method for perturbing a normal unital completely positive map into a *-homomorphism. We show how to prove a version of this theorem for completely bounded maps in the remainder of this section. However, as we remarked in the closing paragraph of Section 3.3, Christensen was able to take normal parts of his original completely positive map, maintaining the norm bound relative to the inclusion, in order to apply this perturbation method. In our case we are not able to apply a similar argument because we only know our original map is close to the inclusion map on a weak*-dense subalgebra of the domain. This gap notwithstanding we include our version of Christensen's perturbation theorem.

Lemma 6.5.1. *Suppose that q is an idempotent operator on $\mathcal{H}$ such that $\|q\| \leq 1+\delta$. Let p be the range projection of q, then $\|q-p\| \leq \epsilon$ where $\epsilon = (\delta^2+2\delta)^{1/2}$.*

Proof. Suppose that $\|q-p\| > \epsilon$. Since $p\mathcal{H}$ is in the kernel of $q-p$ there exists a unit vector $\eta \in p\mathcal{H}^\perp$ such that $\|q\eta\| = c > \epsilon$. Let $\xi = c\eta + \frac{1}{c}q\eta$ then

$$\begin{aligned}\|\xi\|^2 &= \langle c\eta + \frac{1}{c}q\eta, c\eta + \frac{1}{c}q\eta\rangle \\ &= c^2\langle \eta,\eta\rangle + \langle pq\eta,\eta\rangle + \langle \eta, pq\eta\rangle + \frac{1}{c^2}\langle q\eta, q\eta\rangle \\ &= c^2+1. \end{aligned} \tag{6.58}$$

Now

$$\|q\xi\|^2 = \|cq\eta + \frac{1}{c}q\eta\|^2 = ((c+\frac{1}{c})c)^2 = (c^2+1)^2. \tag{6.59}$$

So

$$\frac{\|q\xi\|^2}{\|\xi\|^2} = c^2+1 > \epsilon^2+1 = (1+\delta)^2 \tag{6.60}$$

which contradicts the assumption that $\|q\| \leq 1 + \delta$. $\square$

The following is a completely bounded version of [9, Lemma 3.3].

Lemma 6.5.2. *Let γ be a constant satisfying $0 < \gamma < 0.00016$. Then there exists a function $f : [0, \infty) \to [0, \infty)$ which is continuous at 0 and $f(0) = 0$ and satisfies the following property. If A is an injective von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, N is a weak*-closed subalgebra of $\mathbb{B}(\mathcal{H})$ and $\phi : A \to N$ is a normal unital completely bounded map with $\|\phi - \iota_A\|_{cb} \leq \gamma$ then there exists a completely bounded homomorphism $\psi : A \to N$ with*

$$\|\phi - \psi\|_{cb} \leq f(\gamma). \tag{6.61}$$

Proof. The Stinespring dilation theorem can be generalised to completely bounded maps (see [48, Theorem 8.4]). In [25, Theorem 2.4] Haagerup and Musat show that if a completely bounded map is normal then the representation in the Stinespring dilation may also be chosen to be normal. Therefore, there exists a Hilbert space $\mathcal{K}$, a normal unital *-homomorphism $\pi : A \to \mathbb{B}(\mathcal{K})$ and operators $V, W : \mathcal{H} \to \mathcal{K}$ such that $\phi(x) = V^*\pi(x)W$, for all $x \in A$, and $\|\phi\|_{cb} = \|V\|\|W\|$.

Since N is an algebra and ϕ takes A to N it follows that

$$V^*\mathrm{Alg}(\pi(A) \cup WV^*)W \subseteq N \tag{6.62}$$

by the choice of V, W and π. Set

$$M := \overline{\mathrm{Alg}(\pi(A) \cup WV^*)}^{w^*}. \tag{6.63}$$

The inclusion $V^*MW \subseteq N$ follows from taking weak*-closures in (6.62) since the map

$$x \mapsto V^*xW \qquad (x \in A) \tag{6.64}$$

is normal.

Since ϕ and π are unital $\phi(1_{\mathcal{H}}) = V^*W = 1_{\mathcal{H}}$ and so $WV^*WV^* = WV^*$. We have

$$\|WV^*\| \leq \|W\|\|V\| = \|\phi\|_{cb} \leq 1 + \gamma. \tag{6.65}$$

Set $\alpha = (\gamma^2 + 2\gamma)^{1/2}$. It follows from Lemma 6.5.1 that there exists a projection $p \in \mathbb{B}(\mathcal{K})$ with

$$\|p - WV^*\| \leq \alpha. \tag{6.66}$$

Since A is an injective von Neumann algebra it has property P (see Definition 1.2.6 and Theorem 1.2.9). As π is a normal non-degenerate representation $\pi(A)$ is also a von Neumann algebra with property P (see [27]). We may therefore find an element $r \in \mathbb{B}(\mathcal{K})$ such that

$$r \in \pi(A)' \cap \overline{\mathrm{conv}_{u\in\mathcal{U}(A)}\{\pi(u)WV^*\pi(u^*)\}}^{\mathrm{w}^*} \subseteq \pi(A)' \cap M. \tag{6.67}$$

We use (6.66) to establish the following bound

$$\begin{aligned}
\|r - WV^*\| &\leq \sup_{u\in\mathcal{U}(A)} \{\|\pi(u)WV^* - WV^*\pi(u)\|\} \\
&\leq \sup_{u\in\mathcal{U}(A)} \{\|\pi(u)WV^* - \pi(u)p\| + \|\pi(u)p - p\pi(u)\| + \|p\pi(u) - WV^*\pi(u)\|\} \\
&\leq \sup_{u\in\mathcal{U}(A)} \{\|\pi(u)p - p\pi(u)\|\} + 2\alpha
\end{aligned} \tag{6.68}$$

Applying the calculation of the first displayed equation of Lemma 3.3 in [9] we have

$$\|r - WV^*\| \leq \sup_{u\in\mathcal{U}(A)} \{\max(\|p\pi(u)(1_{\mathcal{K}} - p)\pi(u^*)p\|^{1/2}, \|p\pi(u^*)(1_{\mathcal{K}} - p)\pi(u)p\|^{1/2})\} + 2\alpha. \tag{6.69}$$

Using (6.66) again we have

$$\begin{aligned}
\|p\pi(u)(1_{\mathcal{K}} - p)\pi(u^*)p\| =& \|(p - WV^*)\pi(u)(1_{\mathcal{K}} - p)\pi(u^*)p\| \\
&+ \|WV^*\pi(u)(1_{\mathcal{K}} - (p - WV^*))\pi(u^*)p\| \\
&+ \|WV^*\pi(u)(1_{\mathcal{K}} - WV^*)\pi(u^*)(p - WV^*)\| \\
&+ \|WV\pi(u^*)(1_{\mathcal{K}} - WV^*)\pi(u)WV^*\| \\
\leq& \|WV\pi(u^*)(1_{\mathcal{K}} - WV^*)\pi(u)WV^*\| + \alpha + \alpha(1+\alpha) + \alpha(1+\alpha)^2.
\end{aligned} \tag{6.70}$$

We use the estimate (6.65) and the hypothesis $\|\phi - \iota\| \leq \gamma$ to obtain the following bound

$$\begin{aligned}
\|WV\pi(u^*)(1 - WV^*)\pi(u)WV^*\| &= \|W\big(1_{\mathcal{H}} - V^*\pi(u)WV^*\pi(u^*)\big)V^*\| \\
&\leq \|W\|\|V\|\|1_{\mathcal{H}} - \phi(u)\phi(u^*)\| \\
&\leq \|W\|\|V\|\Big(\|(u - \phi(u))u^*\| + \|\phi(u)(u^* - \phi(u^*))\|\Big) \\
&\leq (1+\gamma)\big(\gamma + \gamma(1+\gamma)\big).
\end{aligned} \tag{6.71}$$

Combining this with (6.66) establishes

$$\|p\pi(u)(1_{\mathcal{K}} - p)\pi(u^*)p\| \leq (1+\gamma)\big(\gamma + \gamma(1+\gamma)\big) + \alpha + \alpha(1+\alpha) + \alpha(1+\alpha)^2. \tag{6.72}$$

A similar calculation gives the same bound

$$\|p\pi(u^*)(1_{\mathcal{K}} - p)\pi(u)p\| \leq (1+\gamma)\big(\gamma + \gamma(1+\gamma)\big) + \alpha + \alpha(1+\alpha) + \alpha(1+\alpha)^2. \tag{6.73}$$

Set

$$\beta' = \big((1+\gamma)\big(\gamma + \gamma(1+\gamma)\big) + \alpha + \alpha(1+\alpha) + \alpha(1+\alpha)^2\big)^{1/2} + 2\alpha. \tag{6.74}$$

Combining (6.69) with (6.72) we have

$$\|r - WV^*\| \leq \beta'. \tag{6.75}$$

Then, with $\beta = \beta' + \alpha$, another application of (6.66) gives

$$\|r - p\| \leq \beta < 1/2. \tag{6.76}$$

We now show that the spectrum of r is contained in the union of balls of radius β centred at 0 and 1. Indeed, let $\lambda \in \mathbb{C}$ such that $d(\lambda, \{0,1\}) > \beta$. Since p is normal we may use functional calculus to establish the following bound

$$\|(p - \lambda 1_{\mathcal{K}})^{-1}\| = \frac{1}{d(\lambda, \{0,1\})}.$$

It follows that

$$\|1_{\mathcal{K}} - (p - \lambda 1_{\mathcal{K}})^{-1}(r - \lambda 1_{\mathcal{K}})\| = \|(p - \lambda 1_{\mathcal{K}})^{-1}\big((p - \lambda 1_{\mathcal{K}}) - (r - \lambda 1_{\mathcal{K}})\big)\| \leq \frac{\beta}{d(\lambda, \{0,1\})} < 1$$

and so $(r - \lambda 1_{\mathcal{K}})$ is invertible, verifying $\lambda \notin \mathrm{Sp}(r)$. Let χ be the characteristic function of the ball of radius β centered at 0 and 1. The function χ is holomorphic on the spectrum of r. By holomorphic functional calculus $q = \chi(\mathrm{r}) \in M \cap \pi(A)'$ is an idempotent operator which satisfies

$$\|q - WV^*\| \leq 2\beta + \alpha. \tag{6.77}$$

Set $z = WV^*q + (1_{\mathcal{K}} - WV^*)(1_{\mathcal{K}} - q) \in M$. As WV^* and q are idempotent we have

$$WV^*z = WV^*q = zq. \tag{6.78}$$

Combining (6.77) and (6.65) we have

$$\begin{aligned}
\|1_{\mathcal{K}} - z\| \leq& \|WV^* - WV^*q\| + \|(1_{\mathcal{K}} - WV^*) - (1_{\mathcal{K}} - WV^*)(1_{\mathcal{K}} - q)\| \\
\leq& \|WV^*\big(WV^* - q\big)\| + \|(1_{\mathcal{K}} - WV^*)\big((1_{\mathcal{K}} - WV^*) - (1_{\mathcal{K}} - q)\big)\| \\
\leq& 2(2\beta + \alpha)(1+\gamma) < 1
\end{aligned} \tag{6.79}$$

so z is invertible with

$$\|z^{-1}\| \leq \frac{1}{1-2(2\beta+\alpha)(1+\gamma)}. \tag{6.80}$$

Therefore

$$\|1_{\mathcal{K}} - z^{-1}\| \leq \|z^{-1}\|\|(z-1_{\mathcal{K}})\| \leq \frac{2(2\beta+\alpha)(1+\gamma)}{1-2(2\beta+\alpha)(1+\gamma)}. \tag{6.81}$$

Rearranging (6.78) gives

$$z^{-1}WV^*z = q. \tag{6.82}$$

Define a map ψ as follows

$$\psi(x) = V^*z\pi(x)z^{-1}W \qquad (x \in A). \tag{6.83}$$

Since $z, \pi(A)$ and z^{-1} belong to M so does $z\pi(A)z^{-1}$. It follows that $\psi(x) \in V^*MW \subseteq N$ by the argument of the second paragraph of the proof. Since $V^*W = 1_{\mathcal{H}}$ we have

$$V^*zq = V^*WV^*z = V^*z. \tag{6.84}$$

It follows that, for $x, y \in A$, the following identity holds

$$\begin{aligned}
\psi(x)\psi(y) &= V^*z\pi(x)z^{-1}WV^*z\pi(y)z^{-1}W \\
&= V^*z\pi(x)q\pi(y)z^{-1}W \qquad \text{(by (6.82))} \\
&= V^*zq\pi(xy)z^{-1}W \qquad \text{(since } q \in \pi(A)'\text{)} \\
&= V^*z\pi(xy)z^{-1}W = \psi(xy) \qquad \text{(by (6.84))}
\end{aligned} \tag{6.85}$$

and so ψ is a homomorphism. Finally, the bound stated in the lemma follows from

$$\|\psi - \phi\|_{cb} \leq \|V\|\|W\|(\|z-1_{\mathcal{K}}\|\|z^{-1}\| + \|1_{\mathcal{K}} - z^{-1}\|\|z\|). \tag{6.86}$$

$\square$

6.6 Completely bounded version of Johnson's theorem

We now prove a 'completely bounded' version of Johnson's theorem [35, Theorem 3.1]: a completely bounded maps from amenable (see Section 1.3) algebras to weak*-closed subalgebra of $\mathbb{B}(\mathcal{H})$ that are completely close to the natural inclusion are completely close to a homomorphism with the same domain and target algebra. We exploit the existence of an approximate diagonal for amenable Banach algebras, the reader is referred to Chapter 1.1 for the definition of this concept.

The following is a direct calculation in [35, Theorem 3.1] which will be needed and so we compute explicitly.

Lemma 6.6.1. *Let A and B be Banach algebras and $T : A \to B$ be a bounded linear map. For a finite linear sum of elementary tensors $m = \sum_j a_j \otimes b_j \in A \widehat{\otimes} A$ define a map $S_m : A \to B$ as follows*

$$S_m(x) = \sum_{j=1}^{n} T(a_j) T^{\vee}(b_j, x). \tag{6.87}$$

Then for $x, y \in A$ we have

$$\begin{aligned} S_m(xy) - T(x)S_m(y) - S_m(x)T(y) = &\sum_{j=1}^{n} \Big((T^{\vee}(a_j, b_j) - T(a_j b_j)) T^{\vee}(x, y) \Big) \\ &+ \sum_{j=1}^{n} \Big(T^{\vee}(x, a_j) T^{\vee}(b_j, y) \Big) \\ &+ \sum_{j=1}^{n} \Big((T(x a_j) T(b_j) T(y) - T(a_j) T(b_j x)) T(y) \Big) \\ &+ \sum_{j=1}^{n} \Big(T(a_j) T(b_j x y) - T(x a_j) T(b_j y) \Big). \end{aligned} \tag{6.88}$$

Note that this map does not depend on how m is represented as a sum of elementary tensors.

Proof. Firstly, using the definition of S_m we expand

$$\begin{aligned} S_m(xy) - T(x)S_m(y) - S_m(x)T(y) = &\sum_{j=1}^{n} \Big(T(a_j) T^{\vee}(b_j, xy) - T(x) T(a_j) T^{\vee}(b_j, y) - T(a_j) T^{\vee}(b_j, x) T(y) \Big) \\ = &\sum_{j=1}^{n} \Big(T(a_j) T(b_j x y) - T(a_j) T(b_j) T(xy) \Big) \\ &- \sum_{j=1}^{n} \Big(T(x) T(a_j) T(b_j y) - T(x) T(a_j) T(b_j) T(y) \Big) \\ &- \sum_{j=1}^{n} \Big(T(a_j) T(b_j x) T(y) - T(a_j) T(b_j) T(x) T(y) \Big), \end{aligned} \tag{6.89}$$

adding and subtracting $\sum_{j=1}^{n} T(x a_j) T(b_j y) - T(x a_j) T(b_j) T(y)$ to the previous equation block and rearranging gives

$$
\begin{aligned}
S_m(xy) - T(x)S_m(y) - S_m(x)T(y) = &\sum_{j=1}^{n} \Big(T(a_j)T(b_j)(T(x)T(y) - T(xy)) \Big) \\
&+ \sum_{j=1}^{n} \Big(T(xa_j)T(b_jy) - T(xa_j)T(b_j)T(y) \Big) \\
&+ \sum_{j=1}^{n} \Big(- T(x)T(a_j)T(b_jy) + T(x)T(a_j)T(b_j)T(y) \Big) \\
&+ \sum_{j=1}^{n} \Big(T(xa_j)T(b_j)T(y) - T(a_j)T(b_jx)T(y) \Big) \\
&+ \sum_{j=1}^{n} \Big(T(a_j)T(b_jxy) - T(xa_j)T(b_jy) \Big). \qquad (6.90)
\end{aligned}
$$

The 1st line of the previous equation block may be rewritten in terms of $T^\vee$ while the four terms on the 2nd and 3rd line of the previous equation block may be factorised in a quadratic fashion with each terms in the factorisation of the form $T^\vee$ as follows

$$
\begin{aligned}
S_m(xy) - T(x)S_m(y) - S_m(x)T(y) = &\sum_{j=1}^{n} \Big((T^\vee(a_j, b_j) - T(a_jb_j))T^\vee(x, y) \Big) \\
&+ \sum_{j=1}^{n} \Big(T^\vee(x, a_j)T^\vee(b_j, y) \Big) \\
&+ \sum_{j=1}^{n} \Big((T(xa_j)T(b_j)T(y) - T(a_j)T(b_jx))T(y) \Big) \\
&+ \sum_{j=1}^{n} \Big(T(a_j)T(b_jxy) - T(xa_j)T(b_jy) \Big). \qquad \square
\end{aligned}
$$

We now modify [35, Theorem 3.1] to deal with completely bounded norms. The following lemma shows how to reduce the defect in multiplicity of a completely bounded map.

Lemma 6.6.2. *Let A be a unital amenable operator algebra with amenability constant L, let B be a unital weak*-closed subalgebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and let $T : A \to B$ be a unital map satisfying $\|T\|_{cb} \le K$. Define a map $T_a^\vee : A \to B$ as follows:*

$$T_a^\vee(b) := T^\vee(a, b) \qquad (b \in A). \tag{6.91}$$

Suppose that $\delta > 0$ is chosen so that the following inequality holds

$$\|T_a^\vee\|_{cb} \le \delta\|a\|. \tag{6.92}$$

Then there exists a map $S : A \to B$ such that

$$\|S\|_{cb} \leq KL\delta \tag{6.93}$$

and

$$\|(T+S)_a^\vee\|_{cb} \leq (K^2L^2\delta^2 + 2L\delta^2)\|a\|. \tag{6.94}$$

Proof. A bounded bilinear map $R : A \times A \to B$ induces a linear map $R : A \odot A \to B$ which extends to the projective tensor product $A\widehat{\otimes}A$ by continuity. Let $M \in (A\widehat{\otimes}A)^{**}$ be a virtual diagonal for A and $(m_\alpha)_\alpha$ be an approximate diagonal which converges to M in the weak* topology (see the remark following Definition 1.1.2) with $m_\alpha = \sum_j a_{\alpha,j} \otimes b_{\alpha,j} \in A \odot A$ a finite sum of elementary tensors such that $\sum_j \|a_{\alpha,j}\|\|b_{\alpha,j}\| \leq L$.

Since B is a weak*-closed operator algebra it has a predual which we denote B_*. Let $\iota : B_* \to B^*$ be the natural inclusion of B_* into its bidual. We will show that $(R(m_\alpha))_\alpha$ is weak*-convergent in B and the limit is equal to $\iota^* R^{**}(M) \in B$ (the element $R^{**}(M)$ is in B^{**} and is then mapped into B by ι^*). Indeed for $\eta \in B_*$ we have

$$\begin{aligned}\langle \iota^* R^{**}(M), \eta\rangle &= \langle M, R^*\iota(\eta)\rangle \\ &= \lim_\alpha \langle m_\alpha, R^*\iota(\eta)\rangle \\ &= \lim_\alpha \langle R(m_\alpha), \eta\rangle \end{aligned} \tag{6.95}$$

since $R^*(\iota(\eta)) \in (A\widehat{\otimes}A)^*$.

In particular, for a fixed $x \in A$ we may define a bilinear map $R_x : A \times A \to B$ as follows

$$R_x(a,b) = T(a)T^\vee(b,x) \qquad (a,b \in A). \tag{6.96}$$

From the preceding paragraph R_x induces a map on $A\widehat{\otimes}A$ and the weak* limit of $R_x(m_\alpha)_\alpha$ exists. Define a map $S : A \to B$ by evaluating the weak*-limit of R_x at each point

$$S(x) = \lim_\alpha R_x(m_\alpha) = \lim_\alpha \sum_j T(a_{\alpha,j})T^\vee(b_{\alpha,j}, x) \qquad (x \in A). \tag{6.97}$$

Note that in the notation of Lemma 6.6.1 that $S(x)$ is the weak*-limit of $S_{m_\alpha}(x)$. We will show that S is competely bounded. Firstly, we show that for a fixed $a \in A$ the map $T_a^\vee : A \to B$ is completely bounded.

For $n \in \mathbb{N}$ and $x \in M_n(A)$ then, for a fixed α, we have

$$\|S_{m_\alpha}^{(n)}(x)\| = \|(\sum_j T(a_{\alpha,j})T^\vee(b_{\alpha,j}, x_{kl}))_{kl}\| \leq K\delta\|x\| \sum_j \|a_{\alpha,j}\|\|b_{\alpha,j}\| \leq KL\delta\|x\|. \tag{6.98}$$

Since $S^{(n)}(x)$ is the weak*-limit of $S_{m_\alpha}^{(n)}(x)$ and weak*-limits preserve norm bounds, (6.98) establishes

$$\|S\|_{cb} \leq KL\delta. \tag{6.99}$$

The next step is to compute the complete bound of the map $(T+S)_a^\vee : A \to B$ as (defined as in (6.91)). We have

$$\begin{aligned}
((T+S)_a^\vee)^{(n)}(x) =& \big((T+S)_a^\vee(x_{kl})\big)_{kl} \\
=& \big((T+S)^\vee(a, x_{kl})\big)_{kl} \\
=& \big((T+S)(ax_{kl}) - (T+S)(a)(T+S)(x_{kl})\big)_{kl} \\
=& \big(T(ax_{kl}) - T(a)T(x_{kl}) - S(a)S(x_{kl})\big)_{kl} \\
& + \big(S(ax_{kl}) - T(a)S(x_{kl}) - S(a)T(x_{kl})\big)_{kl} \\
=& \big(T^\vee(a, x_{kl}) - S(a)S(x_{kl})\big)_{kl} \\
& + \lim_\alpha \big(S_{m_\alpha}(ax_{kl}) - T(a)S_{m_\alpha}(x_{kl}) - S_{m_\alpha}(a)T(x_{kl})\big)_{kl}.
\end{aligned} \tag{6.100}$$

Expanding the final term in the previous equation block using Lemma 6.6.1 we have

$$\begin{aligned}
((T+S)_a^\vee)^{(n)}(x) =& \big(T^\vee(a, x_{kl}) - S(a)S(x_{kl})\big)_{kl} \\
& + \lim_\alpha \big(\sum_j T^\vee(a_{\alpha,j}, b_{\alpha,j})T^\vee(a, x_{kl}) - T(a_{\alpha,j}b_{\alpha,j})T^\vee(a, x_{kl})\big)_{kl} \\
& + \lim_\alpha \big(\sum_j T^\vee(aa_{\alpha,j})T^\vee(b_{\alpha,j}, x_{kl})\big)_{kl} \\
& \lim_\alpha \Big(\sum_j \big(T(aa_{\alpha,j})T(b_{\alpha,j})T(x_{kl}) - T(a_{\alpha,j})T(b_{\alpha,j}a)\big)T(x_{kl}) \\
& + \lim_\alpha \big(\sum_j T(a_{\alpha,j})T(b_{\alpha,j}ax_{kl}) - T(aa_{\alpha,j})T(b_{\alpha,j}x_{kl})\big)\Big)_{kl}.
\end{aligned} \tag{6.101}$$

As $(\sum_j a_{\alpha_j} \otimes b_{\alpha_j})_\alpha$ is an approximate diagonal we have $\sum_j a_{\alpha,j}b_{\alpha,j} \to 1$ and since T is unital it follows that

$$(1 - \sum_j T(a_{\alpha,j}b_{\alpha,j}))T^\vee(a, x_{kl}) \to 0 \qquad (x_{kl} \in A) \tag{6.102}$$

in norm along α and therefore also in weak*-topology.

Again, using the approximate diagonal property we have $a(\sum_j a_{\alpha,j} \otimes b_{\alpha,j}) - (\sum_j a_{\alpha,j} \otimes b_{\alpha,j})a \to 0$ in norm. Therefore

$$\sum_j aa_{\alpha,j} \otimes b_{\alpha,j}x_{kl} - \sum_j a_{\alpha,j} \otimes b_{\alpha,j}ax_{kl} \to 0 \tag{6.103}$$

and so

$$\lim_\alpha \sum_j \Big(T(aa_{\alpha,j})T(b_{\alpha,j}x_{kl}) - T(a_{\alpha,j})T(b_{\alpha,j}ax_{kl})\Big) = 0 \tag{6.104}$$

in norm and, therefore, in the weak*-topology. A similar calculation yields

$$\lim_\alpha \sum_j \Big(\big(T(aa_{\alpha,j})T(b_{\alpha,j})T(x_{kl}) - T(a_{\alpha,j})T(b_{\alpha,j}a)\big)T(x_{kl})\Big) = 0. \tag{6.105}$$

We now simplify (6.101). Firstly (6.102) implies that the first term in the first line and the second term in the second line cancel in the limit, while the fourth and fifth line vanish by (6.104) and (6.105) respectively. Therefore

$$((T+S)_a^\vee)^n(x) = \Big(-S(a)S(x_{kl}) + \lim_\alpha \sum_j \big(T^\vee(a_{\alpha,j}, b_{\alpha,j})T^\vee(a, x_{kl}) + T^\vee(aa_{\alpha,j})T^\vee(b_{\alpha,j}, x_{kl})\big)\Big)_{kl}. \tag{6.106}$$

We will compute the norm of each of the summands in (6.106). Firstly, using (6.99), we have

$$\|(S(a)S(x_{kl}))_{kl}\| = \|\mathrm{diag}(S(a), \ldots, S(a))S^{(n)}(x)\| \le \|S\|_{cb}^2 \|a\|\|x\| \le K^2L^2\delta^2\|a\|\|x\|. \tag{6.107}$$

For a fixed α, we use (6.114) and the bound $\sum_j \|a_{\alpha,j}\|\|b_{\alpha,j}\| \le L$, to obtain

$$\begin{aligned}
&\|\big(\sum_j T^\vee(a_{\alpha,j}, b_{\alpha,j})T^\vee(a, x_{kl})\big)_{kl}\| \\
=&\|\mathrm{diag}\big(\sum_j T^\vee(a_{\alpha,j}, b_{\alpha,j}), \ldots, \sum_j T^\vee(a_{\alpha,j}, b_{\alpha,j})\big)(T_a^\vee)^{(n)}(x)\| \\
\le&\|T_a^\vee\|_{cb}\|x\|\|T^\vee\| \sum_j \|a_{\alpha,j}\|\|b_{\alpha,j}\| \\
\le&\delta^2 L\|a\|\|x\|
\end{aligned} \tag{6.108}$$

and

$$
\begin{aligned}
&\|(\sum_j T^\vee(a, a_{\alpha,j})T^\vee(b_{\alpha,j}, x_{kl}))_{kl}\| \\
=&\|\sum_j \operatorname{diag}(T^\vee(a, a_{\alpha,j}), \ldots, T^\vee(a, a_{\alpha,j}))(T^\vee_{b_{\alpha,j}})^{(n)}(x)\| \\
\leq&\sum_j \|T^\vee\|\|a\|\|a_{\alpha,j}\|\|T^\vee_{b_{\alpha,j}}\|_{cb}\|x\| \\
\leq&\delta^2 L\|a\|\|x\|. \qquad (6.109)
\end{aligned}
$$

Combining the estimate (6.107); the estimates (6.108), (6.109), the fact that weak*-limits respect norm bounds and, as n and x were arbitrary, we have

$$\|(T+S)^\vee_a\|_{cb} \leq (K^2L^2\delta^2 + 2L\delta^2)\|a\|. \qquad (6.110)$$

□

Lemma 6.6.3. *Let A be a unital amenable operator algebra with amenability constant L and suppose that B is a unital weak*-closed operator algebra. Let γ be a positive real numbers satisfying*

$$0 < \gamma < \min\left(\frac{1}{32(1+\gamma)^2L^2 + 16L}, \frac{1}{16L(1+\gamma)}, 1\right). \qquad (6.111)$$

Suppose that T is a unital completely bounded linear map from A to B with $\|T - \iota_A\|_{cb} \leq \gamma$. Then there exists a completely bounded homomorphism $T' : A \to B$ such that

$$\|T - T'\|_{cb} \leq 4\gamma(1+\gamma)(3+\gamma)L. \qquad (6.112)$$

Proof. For $n \in \mathbb{N}$ and $x = (x_{kl})_{kl} \in M_n(A)$ we have

$$
\begin{aligned}
\|(T^\vee_a)^{(n)}(x)\| =&\|(T^\vee_a(x_{kl}))_{kl}\| \\
=&\|(T(ax_{kl}) - T(a)T(x_{kl}))_{kl}\| \\
\leq&\|(T(ax_{kl}) - ax_{kl})_{kl}\| + \|(ax_{kl} - aT(x_{kl}))_{kl}\| \\
&+ \|(aT(x_{kl}) - T(a)T(x_{kl}))_{kl}\| \\
=&\|(T-\iota_A)^{(n)}(\operatorname{diag}(a, \ldots, a)x)\| + \|(\operatorname{diag}(a, \ldots, a))(\iota_A - T)^{(n)}(x)\| \\
&+ \|(\iota_A - T)^{(n)}(\operatorname{diag}(a, \ldots, a))T^{(n)}(x)\| \\
\leq&2\gamma\|a\|\|x\| + (1+\gamma)\gamma\|a\|\|x\| = \gamma(3+\gamma)\|a\|\|x\|. \qquad (6.113)
\end{aligned}
$$

Set $\delta := 3\gamma + \gamma^2$. The above calculation establishes

$$\|T_a^{\vee}\|_{cb} \leq \delta\|a\|. \tag{6.114}$$

It follows from (6.114) that $\|T^{\vee}\| \leq \delta$.

Set $K = 1 + \gamma$.

For $n \in \mathbb{N}_0$, set $K_n = (2 - 2^{-n})K$ and $\delta_n = 2^{-n}\delta$. We will inductively construct a sequence of maps $(T_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{B}(A, B)$ satisfying the following properties:

1. $\|T_n\|_{cb} \leq K_n$,

2. $\|(T_n)_a^{\vee}\|_{cb} \leq \delta_n\|a\|$ for $a \in A$ (in particular $\|(T_n)^{\vee}\| \leq \delta_n$) and

3. $\|T_{n+1} - T_n\|_{cb} \leq 4KL\delta_n$.

To start the induction set $T_0 = T$. The hypothesis of this lemma verifies condition 1 and (6.114) verifies condition 2. We may then apply Lemma 6.6.2 to find a operator S so so that $T_1 = T_0 + S$ satisfies condition 3.

Suppose T_n satisfies the inductive hypothesis at the n^{th} stage. We may apply Lemma 6.6.2 to find a map S_n with

$$\|S_n\|_{cb} \leq K_nL\delta_n \leq 4KL\delta_n. \tag{6.115}$$

(using $K_n \leq 2K$ and $\delta_n = \frac{\delta_{n+1}}{2}$) and

$$\|(T_n + S_n)_a^{\vee}\|_{cb} \leq (K_n^2L^2\delta_n^2 + 2L\delta_n^2)\|a\|. \tag{6.116}$$

By the hypothesis $\gamma \leq \frac{1}{32K^2L^2+16L}$ in the statement of the lemma we have

$$\delta = 3\gamma + \gamma^2 \leq 4\gamma \leq (8K^2L^2 + 4L)^{-1}. \tag{6.117}$$

Combining this estimate with the right hand side of (6.116) and $K_n \leq 2K$ establishes

$$\begin{aligned}\|(T_n + S_n)_a^{\vee}\|_{cb} &\leq \frac{1}{2}(8K^2L^2 + 4L)\frac{\delta}{2^n}\delta_n\|a\| \\ &\leq \frac{1}{2}\delta_n\|a\| = \delta_{n+1}\|a\|\end{aligned} \tag{6.118}$$

which verifies condition 2 at the $n+1$ stage. Finally, using (6.115)

$$\begin{aligned}\|T_{n+1}\|_{cb} &= \|T_n + S_n\|_{cb} \\ &\leq K_n + K_n L\delta_n. \end{aligned} \tag{6.119}$$

Using the definition of K_n, δ_n and the hypothesis $\gamma \leq \frac{1}{16L}$, we have

$$\delta = 3\gamma + \gamma^2 \leq 4\gamma \leq \frac{1}{4L} \tag{6.120}$$

and therefore

$$\begin{aligned}\|T_{n+1}\|_{cb} &\leq (2-2^{-n})K + (2-2^{-n})KL\delta 2^{-n} \\ &\leq (2-2^{-n})K + 2KL\delta 2^{-n} \\ &\leq (2-2^{-n})K + 2^{-n-1}K = (2-2^{-n-1})K. \end{aligned} \tag{6.121}$$

This establishes condition 1 at the $n+1$ stage and completes the induction.

Condition 3 shows that the sequence $(T_n)_n$ converges in $\mathbb{B}(A,B)$, let T' be the limit. Condition 2 implies that $T'^{\vee} = 0$ and so T' is a bounded homomorphism with

$$\|T - T'\|_{cb} \leq \sum_{i=1}^{\infty} \|S_n\| \leq 4KL\delta = 4L\gamma(1+\gamma)(3+\gamma) \tag{6.122}$$

establishing the lemma.

□

Remark. The preceding lemma allows us to improve the complete bound given in the remark following the proof of Theorem 6.1.1. In fact we obtain

$$\|\rho - \iota_A\|_{\mathrm{cb}} \leq 4\gamma^6 + 24\gamma^5 + 64\gamma^4 + 96\gamma^3 + 77\gamma^2 + 26\gamma. \tag{6.123}$$

We proceed in the same manner as in the proof of Theorem 6.1.1 but use the full strength of Lemma 6.3.2 to obtain

$$\|\psi|_{A_0} - \iota_{A_0}\|_{\mathrm{cb}} \leq (2\gamma+\gamma^2)(2+2\gamma+\gamma^2) = \gamma' \tag{6.124}$$

and hence $\|\psi\|_{\mathrm{cb}} \leq 1+\gamma'$. We may then apply Lemma 6.6.3 to this map with $L=1$. Continuing in the same fashion as in the proof Theorem 6.1.1 we show that the resulting homomorphism is spatially implemented and hence extends to all of A with the same complete bound as (6.124).

Chapter 7

Non self-adjoint algebras close to certain crossed products

7.1 Crossed product construction

In Section 6 of [5] Cameron et al. prove that certain crossed product von Neumann algebras are strongly Kadison-Kastler stable (see Definition 3.2.3 and Theorem 3.2.8). These crossed products can be chosen to be non-amenable and thus provide the first positive verification of the Kadison Kastler conjecture outside the class of amenable operator algebras. In this section we aim to extend these results to the non self-adjoint setting (where one of the algebras is not assumed to be closed under taking adjoints). All results in this chapter are joint work with Stuart White.

Crossed product von Neumann algebras were introduced by von Neumann in [62] and have been studied extensively (see, for example [61, Chapter X]). We recall some of the details of their construction for the reader's convenience.

Let P be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and Γ be a discrete group with an action on P given by a homomorphism $\alpha : \Gamma \to \mathrm{Aut}(P)$ (we will write α_g for $\alpha(g)$).

We represent P on the Hilbert space $\mathcal{H} \otimes \ell^2(\Gamma) \cong \ell^2(\Gamma, \mathcal{H})$ via the *-homomophism π defined as follows

$$[\pi(x)(f)](s) = \alpha_{s^{-1}}(x)(f(s)) \qquad (x \in P, \quad f \in \ell^2(\Gamma, \mathcal{H}), \quad s \in \Gamma). \tag{7.1}$$

Let λ be the unitary representation of Γ on $\ell^2(\Gamma, \mathcal{H})$ defined by

$$[\lambda(t)(f)](s) = f(t^{-1}s) \qquad (f \in \ell^2(\Gamma, \mathcal{H}); s, t \in \Gamma). \tag{7.2}$$

This representation of Γ implements the action α on P in the following sense

$$\lambda(t)\pi(x)\lambda(t^{-1}) = \pi(\alpha_t(x)) \qquad (x \in P, \ \ t \in \Gamma). \tag{7.3}$$

To see this fix $f \in \ell^2(\Gamma, \mathcal{H})$ and write $h_{x,t} = \pi(x)\lambda(t^{-1})f$ then

$$h_{x,t}(s) = \alpha_{s^{-1}}(x)f(ts) \qquad (s \in \Gamma) \tag{7.4}$$

and

$$[\lambda(t)\pi(x)\lambda(t^{-1})f](s) = [\lambda(t)h_{x,t}](s) = h_{x,t}(t^{-1}s) = \alpha_{s^{-1}t}(x)f(s) = [\pi(\alpha_t(x))f](s). \tag{7.5}$$

The crossed product $M = P \rtimes_\alpha \Gamma$ is the von Neumann algebra generated by $\lambda(G)$ and $\pi(P)$ on $l^2(\Gamma) \otimes \mathcal{H}$. We will write u_g for $\lambda(g)$ and identify P with its range under π so that (7.3) becomes $u_g x u_g^* = \alpha_g(x)$. The space of finite sums $\sum_{g \in \Gamma} u_g x_g$ with $x_g \in P$ is weak operator topology dense in M. For the remainder of this chapter P will be a finite von Neumann algebra equipped with tracial state τ_P and represented in standard form $L^2(P)$. There is a conditional expectation E from M to P with $E(u_g) = 0$ for $g \neq e$ so $\tau_M = \tau_P \circ E$ defines a trace on M and we have $M = P \rtimes_\alpha \Gamma$ is represented on $L^2(P) \otimes \ell^2(\Gamma) \cong L^2(M)$. The subspaces $(\overline{u_g P \xi})_{g \in \Gamma}$ are orthogonal and we may write $L^2(M) = \oplus_{g \in \Gamma} \overline{u_g P \xi}$. We have seen that the following hold for the choice of conditional expectation

- $u_g x u_g^* = \alpha_g(x) \qquad (g \in \Gamma, x \in P)$;
- $u_g u_h = u_{gh} \qquad (g \in \Gamma, h \in \Gamma)$;
- $E(u_g) = 0$ for $g \neq e$.

Our goal is to show that under certain conditions on P, Γ and if a certain vanishing bounded cohomology group related to α vanishes, the crossed product $M = P \rtimes_\alpha \Gamma$, represented in standard form on $L^2(M)$, has the following property: if N is a non self-adjoint weak*-closed subalgebra of $\mathbb{B}(L^2(M))$ satisfying $M \approx N$ and $M' \approx N'$, then there exists an invertible operator $S \approx 1_{L^2(M)}$ such that $SMS^{-1} = N$. In [5] Cameron et al. do not need to assume either that M is represented in standard form or that the commutants are close. However, due to non self-adjoint technicalities we were unable to remove these hypotheses. Indeed, if N is a von Neumann algebra on $L^2(M)$

with $M \approx N$, then it is true that $M' \approx N'$ (see [5, Lemma 4.1]). If N is an arbitrary weak*-closed subalgebra of $\mathbb{B}(L^2(M))$ one can use the same argument to show that N' is nearly contained in M'. However, we do not know if M' is automatically nearly contained in N'. In fact, we do not even know if N' must be non-trivial (the upper triangular matrices proved an example of a weak*-closed operator algebra which has trivial commutant).

There are a number of steps in the proof which we perform separately. In Section 7.2 we reduce to the case where P is contained in N as well as M (alongside another technical condition). After imposing certain conditions on the crossed product we are able to construct a sequence of normalisers for P in N. In Section 7.3 we construct a sequence of normalisers $(v_g)_{g\in\Gamma}$ for P in N so that $v_g \approx u_g$ and u_g and v_g implement the automorphism of P. These normalisers automatically satisfy $v_g v_h \approx v_{gh}$. In Section 7.4 we use the vanishing boundary cohomology condition to construct normalisers as above with $v_g v_h = v_{gh}$. Once this has been achieved we can define an invertible operator, close to the identity on $L^2(M)$, which implements the inclusion $SMS^{-1} \subseteq N$ and then use a containment-near containment argument to demonstrate equality.

In this chapter we have decided not to include the value of constants. Although it is possible to compute constants explicitly, we do not feel they are particularly enlightening (or nice). Instead we use an (ϵ, δ) approach throughout. Where we obtain such qualitative bounds in intermediary steps we introduce functions $\delta_1, \delta_2, \ldots$ with each δ_i continuous and having $\delta_i(0) = 0$.

7.2 Reductions

The first lemma uses Theorem 6.1.1 to reduce to the situation where both M and N contain a copy of P and that $J_M P J_M$ is contained in N' as well as in M'. It is crucial that we are able to appeal to the embedding result of Chapter 6 at this point as we only have a one-sided near-inclusion of P in N. We assume that M is a finite von Neumann algebra represented in standard form defined in Chapter 2.

Lemma 7.2.1. *Let M be a finite von Neumann algebra represented on $L^2(M)$ containing an amenable subalgebra $P \subseteq M$. Let $\epsilon > 0$ be given. There exists a constant $\gamma > 0$ such that if N is* *a weak*-closed subalgebra of* $\mathbb{B}(L^2(M))$ *with*

$$d(M,N) \leq \gamma \qquad \text{and} \qquad d(M',N')) \leq \gamma, \tag{7.6}$$

then there exists an invertible element $S \in \mathbb{B}(L^2(M))$ *satisfying*

- $\|1_{L^2(M)} - S\| \leq \epsilon$,
- $P \subseteq S^{-1}NS$ *and*
- $J_M P J_M \subseteq (S^{-1}NS)'$.

As described above, we start with a fixed constant γ (assumed to be taken sufficiently small) and show an operator S, as above, may be found with distance from the identity given by a fixed function of γ. Given an arbitrary ϵ one may therefore find a suitable γ

Proof. Temporarily fix $\gamma > 0$. The hypothesis $d(M,N) \leq \gamma$ implies that $P \subseteq_\gamma N$ and so we may apply Theorem 6.1.1 (provided we have chosen γ to be small enough) to find an invertible element $t_1 \in \overline{\mathrm{Alg}(N \cup P)}^{\mathrm{w}^*}$ that implements the containment $t_1^{-1}Pt_1 \subseteq N$ and satisfies the bound

$$\|1_{L^2(M)} - t_1\| \leq \delta_1 = \delta_1(\gamma) < 1. \tag{7.7}$$

Consequently $\|1_{L^2(M)} - t_1^{-1}\| \leq \frac{\delta_1}{1-\delta_1}$. For T in the unit ball of $\mathbb{B}(L^2(M))$ we have

$$\begin{aligned} \|T - t_1^{-1}Tt_1\| &\leq \|T - t_1^{-1}T\| + \|t_1^{-1}T - t_1^{-1}Tt_1\| \\ &= \|(1_{L^2(M)} - t_1^{-1})T\| + \|t^{-1}T(1_{L^2(M)} - t_1)\| \\ &\leq \frac{2\delta_1}{1-\delta_1}. \end{aligned} \tag{7.8}$$

Applying a similar argument to get the same bound for $\|T - t_1Tt_1^{-1}\|$ and normalising the approximating elements if necessary (see the argument following Definition 3.1.2) gives $d(N', t_1N't_1^{-1}) \leq \frac{4\delta_1}{1-\delta_1}$. Combining this with the hypothesis $d(M',N') \leq \gamma$ gives

$$d(M', t_1N't_1^{-1}) \leq \gamma + \frac{4\delta_1}{1-\delta_1} = \delta_2(\gamma). \tag{7.9}$$

If x lies in N' then

$$t_1xt_1^{-1}t_1yt_1^{-1} = t_1yt_1^{-1}t_1xt_1^{-1} \qquad (y \in N), \tag{7.10}$$

so $t_1N't_1^{-1} \subseteq (t_1Nt_1^{-1})'$. On the other hand, for $x \in (t_1Nt_1^{-1})'$ we have

$$t_1^{-1}xt_1y = t_1^{-1}xt_1yt_1^{-1}t_1 = t_1^{-1}t_1yt_1^{-1}xt_1 = yt_1^{-1}xt_1 \qquad (y \in N), \tag{7.11}$$

so $t_1^{-1}(t_1Nt_1^{-1})'t_1 \subseteq N'$ and consequently $(t_1Nt_1^{-1})' = t_1N't_1^{-1}$. Since $J_MPJ_M \subseteq J_MMJ_M = M'$ (see Lemma 2.1.1) (7.9) implies that $J_MPJ_M \subseteq_{\delta_2} t_1N't_1^{-1} = (t_1Nt_1^{-1})'$. As J_MPJ_M is also an injective von Neumann algebra we may apply Theorem 6.1.1 again (providing γ is taken to be small enough) to find a second invertible element $t_2 \in \overline{\mathrm{Alg}(J_MPJ_M, (t_1Nt_1^{-1})')}^{\mathrm{w}^*} \subseteq P'$ such that $t_2J_MPJ_Mt_2^{-1} \subseteq (t_1Nt_1^{-1})'$ and satisfying

$$\|1_{L^2(M)} - t_2\| \leq \delta_3 = \delta_3(\gamma) < 1. \tag{7.12}$$

By the same argument that shows $t_1^{-1}(t_1Nt_1^{-1})'t_1 \subseteq N'$, we have $J_MPJ_M \subseteq t_2^{-1}(t_1Nt_1^{-1})'t_2 = (t_2^{-1}t_1Nt_1^{-1}t_2)'$. Furthermore, $P \subseteq t_2^{-1}t_1Nt_1^{-1}t_2$ since $P \subseteq t_1Nt_1^{-1}$ and t_2 lies in P'. The lemma follows by setting $S = t_1^{-1}t_2$ and adjusting the initial choice of γ to ensure that $\|1_{L^2(M)} - S\| \leq \epsilon$. $\square$

7.3 Transferring normalisers

Given an inclusion of von Neumann algebras $P \subseteq M$ we say a unitary $u \in M$ is a *unitary normaliser* if $uPu^* = P$, we write $\mathcal{UN}(P \subseteq M)$ to denote the collection of all such unitaries. Similarly, given an arbitrary operator algebra N containing P we call an invertible element $S \in N$ a *similarity normaliser* if $S^{-1} \in N$ and $SPS^{-1} = P$, we write $\mathcal{SN}(P \subseteq N)$ to denote the collection of these elements. The next lemma shows that if M, N and P are as above with M close to N then given a unitary normaliser of P in M we can find a nearby similarity normaliser of P in N.

Lemma 7.3.1. *Let $\epsilon > 0$ be given, there exists a constant $\gamma > 0$ with the following property: Suppose $P \subseteq M$ is an inclusion of an injective von Neumann algebra inside an arbitrary von Neumann algebra on a Hilbert space $\mathcal{H}$, if N is a weak*-closed subalgebra of $\mathbb{B}(\mathcal{H})$ containing $1_{\mathcal{H}}$ and P with $d(M,N) \leq \gamma$, then for every $u \in \mathcal{UN}(P \subseteq M)$ there exists $S \in \mathcal{SN}(P \subseteq N)$ such that $\|u - S\| \leq \epsilon$.*

Proof. Again, we temporarily fix $\gamma > 0$. For $u \in \mathcal{UN}(P \subseteq M)$ there exists an element $v \in N$ such that $\|u - v\| \leq \gamma$ by the hypothesis $d(M,N) \leq \gamma$. If $\gamma < 1$ then we have $\|1_{\mathcal{H}} - vu^*\| \leq \gamma < 1$ and so v is invertible with $\|v^{-1}\| \leq \frac{1}{1-\gamma}$. We will show v^{-1} lies in N. Indeed, find an element $w \in N$ such that $\|u^* - w\| \leq \gamma$. Then for a suitable choice of γ we have

$$\|wv - 1_{\mathcal{H}}\| \leq \|(w - u^*)v\| + \|u^*(v - u)\| < 1. \tag{7.13}$$

So wv is invertible with inverse in $\sum_n (1_{\mathcal{H}} - wv)^n \in N$ and therefore $v^{-1} = (wv)^{-1}w$ also lies in N.

Now

$$\|u^* - v^{-1}\| = \|1_{\mathcal{H}} - uv^{-1}\| = \|uv^{-1}(vu^* - 1_{\mathcal{H}})\| \leq \frac{\gamma}{1-\gamma}. \tag{7.14}$$

For $x \in P_1$ we have

$$\begin{aligned}\|x - v^{-1}uxu^*v\| &\leq \|(1_{\mathcal{H}} - v^{-1}u)x\| + \|v^{-1}ux(1_{\mathcal{H}} - u^*v)\| \\ &\leq \|x\|\|1_{\mathcal{H}} - v^{-1}u\| + \|v^{-1}u\|\|x\|\|1_{\mathcal{H}} - u^*v\| \\ &\leq \|x\|\|u^* - v^{-1}\| + \|v^{-1}\|\|x\|\|u - v\| \leq \frac{2\gamma}{1-\gamma}. \end{aligned} \tag{7.15}$$

Normalising the approximating element if necessary and applying a symmetric argument it follows that

$$d(v^{-1}Pv, P) \leq \frac{4\gamma}{1-\gamma}. \tag{7.16}$$

Provided γ was chosen such that the right hand side of (7.16) is as small as required by the hypothesis of [56, Theorem 3.12] (cf. Theorem 6.1.1) we may find an invertible element $t \in \overline{\mathrm{Alg}(v^{-1}Pv, P)}^{\mathrm{w}^*} \subseteq N$ with $t^{-1}v^{-1}Pvt = P$ and with $\|1_{\mathcal{H}} - t\| \leq \delta_1 = \delta_1(\gamma) < 1$. It follows that $t^{-1} \in \overline{\mathrm{Alg}(v^{-1}Pv, P, 1_{\mathcal{H}})}^{\mathrm{w}^*} \subseteq N$ and therefore $S = vt$ lies in $\mathcal{SN}(P \subseteq N)$. We return to choose γ so that

$$\|u - S\| \leq \|u - v\| + \|v(1_{\mathcal{H}} - t)\| \leq \gamma + (1+\gamma)\delta_1 \leq \epsilon \tag{7.17}$$

which completes the proof. □

If M and N are as above and P is amenable, then we may perturb a similarity normaliser close to a unitary normaliser so that it implements the same automorphism.

Lemma 7.3.2. *Let $\epsilon > 0$ be given. Suppose $P \subseteq M$ is an inclusion of von Neumann algebras represented on $\mathcal{H}$ with P amenable and $1_P = 1_{\mathcal{H}}$. There exists a constant $\gamma > 0$ with the following property: if N is a weak*-closed subalgebra of $\mathbb{B}(\mathcal{H})$ containing P, and if $u \in \mathcal{UN}(P \subseteq M)$ and $v \in \mathcal{SN}(P \subseteq N)$ satisfy $\|u - v\| \leq \gamma$; then there exists $w \in \mathcal{SN}(P \subseteq N)$ with $\|w - u\| \leq \epsilon$ and*

$$wxw^{-1} = uxu^* \qquad (x \in P). \tag{7.18}$$

Proof. By the same calculation as was carried out in (7.15) we have

$$\|x - v^{-1}uxu^*v\| \leq \frac{2\gamma}{1-\gamma}\|x\| \qquad (x \in P). \tag{7.19}$$

Therefore the map $\phi : x \mapsto v^{-1}uxu^*v$ is a bounded homomorphism mapping P to P with

$$\|\mathrm{id}_P - \phi\| \leq \frac{2\gamma}{1-\gamma} = \delta < 1 \tag{7.20}$$

(for a suitable γ).

Since P is amenable it is hyperfinite (see Definition 1.2.3 and Theorem 1.2.9). Write $P = (\cup_{\lambda\in\Lambda}F_\lambda)''$ with F_λ finite dimensional.

Use Lemma 6.2.3 to find elements $S_\lambda \in P$ such that $\|1_{\mathcal{H}} - S_\lambda\| \leq \delta$ and

$$S_\lambda\phi(x) = xS_\lambda \qquad (x \in F_\lambda) \tag{7.21}$$

for each $\lambda \in \Lambda$. The net $(S_\lambda)_\lambda$ is bounded and so we may find a weak*-accumulation point S which also satisfies $\|1_{\mathcal{H}} - S\| \leq \delta < 1$. It follows that S is invertible and that $S\phi(x) = xS$ (see the argument in the proof of Lemma 6.4.4) for all $x \in P$. Therefore

$$v^{-1}uxu^*v = \phi(x) = S^{-1}xS \qquad (x \in P). \tag{7.22}$$

Set $w = vS^{-1}$. Then w and w^{-1} lie in N and satisfy

$$wxw^{-1} = vS^{-1}xSv^{-1} = uxu^* \qquad (x \in P) \tag{7.23}$$

and (by choosing γ appropriately)

$$\|u - w\| \leq \|u - v\| + \|v(1_{\mathcal{H}} - S^{-1})\| \leq \epsilon. \tag{7.24}$$

□

Remark. If $P \cong L^\infty(X)$ for some measure space X then the map ϕ defined in the proof above is equal to the identity and v automatically implements the same action as u.

We now prove two technical lemmas, the first is essentially Lemma 2.10 of [5]. The second appeared in an earlier version of the same work. We include the proofs for completeness.

Lemma 7.3.3. *Let $P \subseteq M$ be an inclusion of von Neumann algebras on $L^2(M)$ with $\xi \in L^2(M)$ the cyclic and separating vector coming from $1_M \in M$. Suppose that $P' \cap M \subseteq P$. Given an element $v \in P' \cap \langle M, e_P\rangle$ there exists an element $s \in \mathcal{Z}(P)$ such that $v\xi = s\xi$*

Proof. The condition $P' \cap M \subseteq P$ means that [21, Lemma 3.2] applies and so we have $e_P \in \mathcal{Z}(P' \cap \langle M, e_p \rangle)$. By Lemma 2.2.3 we have $Pe_P = e_P \langle M, e_P \rangle e_P$ so we may find an element $s \in P$ such that $se_P = e_P v e_P = v e_P$. Therefore

$$s\xi = se_P\xi = e_P v e_P \xi = v e_P \xi = v\xi. \tag{7.25}$$

For $p \in P$ we have

$$(sp - ps)e_P = spe_P - pe_P v e_P = spe_P - e_P v e_P p e_P = 0. \tag{7.26}$$

Since the map $p \mapsto pe_P$ is injective (Lemma 2.2.3), it follows that s is in $\mathcal{Z}(P)$. $\square$

Lemma 7.3.4. *Suppose that $P \subseteq M$ is an inclusion of von Neumann algebras on $\mathcal{H}$ with P amenable and satisfying $M \cap P' \subseteq P$. If N is a weak*-closed subalgebra of $\mathbb{B}(\mathcal{H})$ containing P such that $d(M, N) \leq 1/4$ then $N \cap P' \subseteq P$ or equivalently $N \cap P' = \mathcal{Z}(P)$.*

Proof. Suppose that $P' \cap N$ strictly contains $\mathcal{Z}(P)$, then we may find an element $x \in N_1 \cap P'$ satisfying

$$\|x - z\| > 3/4 \qquad (z \in \mathcal{Z}(P)). \tag{7.27}$$

By the hypothesis $d(M, N) \leq 1/4$ there exists an element $y \in M_1$ such that $\|x - y\| \leq 1/4$. Since $x \in P'$, for a unitary $u \in P$, we have the following bound

$$\|uyu^* - y\| \leq \|u(y - x)u^*\| + \|x - y\| \leq 2/4. \tag{7.28}$$

Since P is amenable it has property P (see Definition 1.2.6 and Theorem 1.2.9) so there exists an element $z \in \overline{\mathrm{conv}_{u \in \mathcal{U}(P)}(uyu^*)}^{\mathrm{w}^*} \cap P'$ which lies in $M \cap P' \subseteq \mathcal{Z}(P)$. Since z is the weak*-limit of convex sums of the form uyu^*, (7.28) implies that $\|z - y\| \leq 1/2$ and consequently

$$\|z - x\| \leq \|z - y\| + \|y - x\| \leq 3/4 \tag{7.29}$$

which contradicts (7.27). $\square$

7.4 Vanishing cohomology

We now turn to transferring normalisers from crossed product von Neumann algebras to nearby non self-adjoint algebras. If α is an action of a discrete group on a finite von Neumann algebra

P we show that, under certain conditions on the action, once we have constructed a sequence of normalisers $(v_g)_{g\in\Gamma}$ in N which are close to the canonical normalisers in $M = P \rtimes_\alpha \Gamma$ and implement the same action, then the normalisers may be adjusted so that they also satisfy $v_g v_h = v_{gh}$ for all $g, h \in \Gamma$

The first condition we require is that the relative commutant of P in M is contained in P;

$$P' \cap M \subseteq P. \tag{7.30}$$

We say α is *properly outer* if for any element $g \in \Gamma$, not equal to the identity, any central projection $z \in \mathcal{Z}(P)$ satisfying $\alpha_g(z) = z$ has the property that the action α restricted to Pz is not inner (not implemented by a unitary in Pz). We say that α is *centrally ergodic* if the fixed point algebra of the action restricted to $\mathcal{Z}(P)$ ($z \in \mathcal{Z}(P)$ such that $\alpha_g(z) = z$ for all $g \in \Gamma$) is trivial. It is folklore (see [5, Proposition 2.19]) that under these hypotheses (7.30) holds.

Secondly we require the vanishing of the bounded cohomology group $H_b^2(\Gamma, \mathcal{Z}(P))$. We discuss which groups and actions satisfy this condition in Appendix A.

Lemma 7.4.1. *Let P be a von Neumann algebra represented on $\mathcal{H}$ and let $\alpha : \Gamma \to Aut(P)$ be an action of a discrete group Γ on P. Let $M = P \rtimes_\alpha \Gamma$ with a sequence of unitary normalisers $(u_g)_{g\in\Gamma}$ generating M and satisfying*

- $u_g x u_g^* = \alpha_g(x) \qquad (g \in \Gamma, x \in P)$ *and*
- $u_g u_h = u_{gh} \qquad (g, h \in \Gamma)$.

Suppose $P' \cap M \subseteq P$ and $H_b^2(\Gamma, \mathcal{Z}(P)) = 0$. For any $\epsilon > 0$ there exists a $\gamma > 0$ with the following property: suppose N is a weak-closed subalgebra of $\mathbb{B}(\mathcal{H})$ satisfying $d(M, N) \leq \gamma$ and containing both P and a collection $(v_g)_{g\in\Gamma} \subseteq \mathcal{SN}(P \subseteq N)$ with*

$$v_g x v_g^{-1} = \alpha_g(x) \qquad (g \in \Gamma, x \in P) \tag{7.31}$$

and $\sup_{g\in\Gamma} \|v_g - u_g\| \leq \gamma$. Then there exists $(v'_g)_{g\in\Gamma} \subseteq \mathcal{SN}(P \subseteq N)$ such that

- $v'_g x v'^{-1}_g = \alpha_g(x) \qquad (x \in P, g \in \Gamma)$,
- $v'_g v'_h = v'_{gh} \qquad (g, h \in \Gamma)$ *and*
- $\sup_{g\in\Gamma} \|v'_g - u_g\| \leq \epsilon$.

Proof. The action of Γ restricts to an action on the invertible elements in the center of P (which we write as $\mathrm{Inv}(\mathcal{Z}(P))$). A map $\omega : \Gamma \times \Gamma \to \mathrm{Inv}(\mathcal{Z}(P))$ is a 2-cocycle (see Appendix A) if the following identity holds

$$\partial\omega(g,h,k) = \alpha_g(\omega(h,k))\omega(gh,k)^{-1}\omega(g,hk)\omega(g,h)^{-1} = 1_P \qquad (g,h,k \in \Gamma). \tag{7.32}$$

We will show that $\omega(g,h) = v_g v_h v_{gh}^{-1}$ is a cocycle. For $g,h \in \Gamma$ we have

$$v_g v_h x v_h^{-1} v_g^{-1} = \alpha_g(\alpha_h(x)) = \alpha_{gh}(x) = v_{gh} x v_{gh}^{-1} \qquad (x \in P) \tag{7.33}$$

so $v_{gh}^{-1} v_g v_h \in N \cap P'$. Applying Lemma 7.3.4 to the hypothesis $M \cap P' \subseteq \mathcal{Z}(P)$ we have $N \cap P' \subseteq \mathcal{Z}(P)$ (providing we have chosen γ small enough). Therefore $v_{gh}^{-1} v_g v_h \in \mathcal{Z}(P)$ and consequently $v_{gh}(v_{gh}^{-1} v_g v_h) v_{gh}^{-1} = v_g v_h v_{gh}^{-1} \in \mathcal{Z}(P)$. For $g,h,k \in \Gamma$ we have

$$\begin{aligned}
\partial\omega(g,h,k) &= \alpha_g(\omega(h,k))\omega(gh,k)^{-1}\omega(g,hk)\omega(g,h)^{-1} \\
&= \alpha_g(v_h v_k v_{hk}^{-1})(v_{gh} v_k v_{ghk}^{-1})^{-1}(v_g v_{hk} v_{ghk}^{-1})(v_g v_h v_{gh}^{-1})^{-1} \\
&= (v_g v_h v_k v_{hk}^{-1} v_g^{-1})(v_{ghk} v_k^{-1} v_{gh}^{-1})(v_g v_{hk} v_{ghk}^{-1})(v_{gh} v_h^{-1} v_g^{-1}) \\
&= (v_g v_h v_k v_{hk}^{-1} v_g^{-1})(v_g v_{hk} v_{ghk}^{-1})(v_{ghk} v_k^{-1} v_{gh}^{-1})(v_{gh} v_h^{-1} v_g^{-1}) = 1_P
\end{aligned} \tag{7.34}$$

as the second and third bracketed terms in the second line both lie in $\mathcal{Z}(P)$.

For $g,h \in \Gamma$ we have

$$\begin{aligned}
\|w(g,h) - 1_{\mathcal{H}}\| \leq \quad & \|(v_g - u_g)v_h v_{gh}^{-1}\| + \|u_g(v_h - u_h)v_{gh}^{-1}\| \\
& + \|u_{gh}(v_{gh}^{-1} - u_{gh}^{-1})\| = \delta_1 = \delta_1(\gamma) < 1
\end{aligned} \tag{7.35}$$

for a suitable choice of γ. It follows that the spectrum of $w(g,h)$ lies in the ball of radius 1 centered at 1 where the continuous logarithm (denoted log) with $\log(1) = 0$ is holomorphic. We use holomorphic functional calculus to define $\psi : \Gamma \times \Gamma \to \mathcal{Z}(P)$, setting

$$\psi(g,h) = \log(\omega(g,h)) \qquad (g,h \in \Gamma). \tag{7.36}$$

For any polynomial p we have $\alpha_g\big(p(\omega(h,k))\big) = p\big(\alpha_g(\omega(h,k))\big)$ since α_g is an automorphism of P. It follows by a Taylor series approximation that $\log \alpha_g(\omega(h,k))) = \alpha_g(\log(\omega(h,k)))$. Since $\mathcal{Z}(P)$ is abelian, using a Taylor series approximation again to simplify the logarithm on the 2nd line, we have

$$
\begin{aligned}
&\alpha_g(\psi(h,k)) - \psi(gh,k) + \psi(g,hk) - \psi(g,h) \\
=& \log(\alpha_g(\omega(h,k)) - \log\omega(gh,k) + \log\omega(g,hk) - \log\omega(w(g,h)) \\
=& \log(\alpha_g(\omega(h,k))\omega(gh,k)^{-1}\omega(g,hk)\omega(g,h)^{-1}) = \log(1_P) = 0 \qquad (7.37)
\end{aligned}
$$

where the last two identities are valid as the spectrum of all of the arguments of the logarithms are contained in the unit ball around 1_P where log is holomorphic. This demonstrates that ψ is an additive cocycle. Furthermore, since $\sup_{g,h\in\Gamma}\|\omega(g,h)-1_P\| = \delta_1$ there exists a constant $\delta_2 = \delta_2(\gamma)$ such that $\sup_{g,h\in\Gamma}\|\psi(g,h)\| \leq \delta_2$ by the continuity of log at 1 providing γ is taken to be small enough.

The hypothesis that $H_b^2(\Gamma,\mathcal{Z}(P)) = 0$ means the map

$$
\partial : C_b^1(\Gamma,\mathcal{Z}(P)) \to Z_b^2(\Gamma,\mathcal{Z}(P)) \qquad (7.38)
$$

is surjective and so by the open mapping theorem (using the bound on ψ) we may find a cochain $\phi : \Gamma \to \mathcal{Z}(P)$ and a constant $\delta_3 = \delta_3(\gamma, K)$, where K is fixed the constant in the open mapping theorem so that δ_3 goes to 0 as γ does, such that $\partial\phi = \psi$, that is

$$
\psi(g,h) = \alpha_g(\phi(h)) - \phi(gh) + \phi(g), \qquad (7.39)
$$

and $\|\phi\| = \sup_{g\in\Gamma}\|\phi(g)\| \leq \delta_3$ (see Appendix A).

Since exp is holomorphic everywhere we may define $\eta(g) = \exp(\phi(g))$ for $g \in \Gamma$ by holomorphic functional calculus to obtain the map $\eta = \exp(\phi) : \Gamma \to \mathrm{Inv}(\mathcal{Z}(P))$. Approximating with polynomials as before, it follows that exp commutes with α_g. Since $\mathcal{Z}(P)$ is commutative we may approximate with polynomials again to simplify the exp term on the second line, giving

$$
\begin{aligned}
\partial\eta(g,h) &= \alpha_g(\eta(h))\eta(gh)^{-1}\eta(g) \\
&= \alpha_g(\exp(\phi(h))(\exp(\phi(gh)))^{-1}\exp(\phi(g)) \\
&= \exp(\alpha_g(\phi(h)))\exp(-\phi(gh))\exp(\phi(g)) \\
&= \exp(\alpha_g(\phi(h)) - \phi(gh) + \phi(g)) \\
&= \exp((\partial\phi)(g,h)) = \exp(\psi(g,h)) = \omega(g,h). \qquad (7.40)
\end{aligned}
$$

Since $\sup_{g\in\Gamma}\|\phi(g)\| \leq \delta_3$ it follows from the continuity of the exponential function at zero that there exists some δ_4 (again depending continuously on γ and K) such that $\sup_{g\in\Gamma}\|\eta(g)-1_P\| \leq \delta_4$.

Returning to the definition of ω we have

$$\begin{aligned} v_g v_h v_{gh}^{-1} &= \omega(g,h) = (\partial\eta)(g,h) = \alpha_g(\eta(h))\eta(gh)^{-1}\eta(g) \\ &= v_g\eta(h)v_g^{-1}\eta(g)\eta(gh)^{-1} \end{aligned} \tag{7.41}$$

so

$$\eta(g)^{-1}v_g\eta(h)^{-1}v_h = \eta(gh)^{-1}v_{gh}. \tag{7.42}$$

Since $\sup_{g\in\Gamma}\|\eta(g)-1_P\| \leq \delta_4$, there exists a constant δ_5 such that $\sup_{g\in\Gamma}\|\eta(g)^{-1}-1_P\| \leq \delta_5$. Set $v'_g = \eta(g)^{-1}v_g$ for $g \in \Gamma$. Then the following hold (provided we have chosen γ appropriately):

- $v'_g v'_h = v'_{gh} \qquad (g,h \in \Gamma)$
- $\sup_{g\in\Gamma}\|v_g - v'_g\| \leq \epsilon$
- $v'_g x v'^{-1}_g = \eta(g)^{-1}\alpha_g(x)\eta(g) = \alpha_g(x) \qquad (g \in \Gamma, x \in P)$.

The second statement is obtained by making a suitable choice of γ (relative to ϵ and the constant in the open mapping theorem) and the third follows because $\eta(g) \in \mathcal{Z}(P)$. □

7.5 Twisting and the main theorem

The final step is to show that once we have constructed similarity normalisers $(v_g)_{g\in\Gamma}$ for P in N satisfying the conditions described in the conclusion of Lemma 7.4.1 we can define an invertible operator $S \approx 1_{L^2(M)}$ which implements the inclusion $SMS^{-1} \subseteq N$.

Lemma 7.5.1. *Let $M = P \rtimes_\alpha \Gamma \subseteq \mathbb{B}(L^2(M))$ be represented on $L^2(M)$ with $P \cap M' \subseteq P$ with a canonical sequence of unitaries $(u_g)_{g\in\Gamma}$ implementing the action α_g. Suppose $0 < \gamma < 1$ is a positive constant and N is a weak*-closed subalgebra of $\mathbb{B}(L^2(M))$ satisfying*

- $P \subseteq N$,
- $J_M P J_M \subseteq N'$

and containing a sequence $(v_g)_{g\in\Gamma}$ in $\mathcal{SN}(P \subseteq N)$ such that for all $x \in P$ and $g, h \in \Gamma$ the following hold:

- $v_g x v_g^{-1} = u_g x u_g^* = \alpha_g(x)$,
- $\|u_g - v_g\| \leq \gamma$,
- $v_g v_h = v_{gh}$.

Then there exists an invertible element $S \in \mathbb{B}(L^2(M))$ such that $\|1_{L^2(M)} - S\| \leq \gamma$ and $SMS^{-1} \subseteq N$.

Proof. Let ξ denote the cyclic vector for M when represented in standard form (see Chapter 2). We will show that $\overline{v_g P\xi} = \overline{u_g P\xi}$ for all $g \in \Gamma$. Fix $g \in \Gamma$ and $x \in P$.

By hypothesis we have $v_g^{-1} u_g x u_g^* v_g = x$ so that $v_g^{-1} u_g$ lies in P'. Write $u_g = v_g w_g$ with $w_g \in P'$. Taking commutants of the inclusion $J_M P J_M \subseteq N'$ and applying Lemma 2.2.3 gives

$$N \subseteq N'' \subseteq (J_M P J_M)' = \langle M, e_P\rangle. \tag{7.43}$$

It follows that $w_g = v_g^{-1} u_g$ is actually in $P' \cap \langle M, e_P\rangle$ so we may apply Lemma 7.3.3 to find an element $z_g \in \mathcal{Z}(P)$ such that $w_g\xi = z_g\xi$. This gives

$$\begin{aligned} u_g x\xi &= u_g x u_g^* u_g \xi \\ &= \alpha_g(x) v_g w_g \xi \\ &= \alpha_g(x) v_g z_g \xi \\ &= v_g x v_g^{-1} v_g z_g \xi \\ &= v_g x z_g \xi \in \overline{v_g P\xi}. \end{aligned} \tag{7.44}$$

The reverse inclusion follows from a similar argument after writing $v_g = u_g w_g'$ with $w_g' \in P'$.

Since $L^2(M) = \oplus_{g\in\Gamma} \overline{u_g P\xi}$ we also have that $L^2(M) = \oplus_{g\in\Gamma} \overline{v_g P\xi}$. Define an operator S on finite sums as follows

$$S \sum u_g x_g \xi = \sum v_g x_g \xi \qquad (g \in \Gamma, x_g \in P). \tag{7.45}$$

Then, for a fixed $g \in \Gamma$ and $x_g \in P$, we have

$$\begin{aligned}
\|(S - 1_{L^2(M)})u_g x_g \xi\|_2^2 &\leq \|(v_g - u_g)x_g\xi\|_2^2 \\
&= \tau(x_g^*(v_g - u_g)^*(v_g - u_g)x_g) \\
&\leq \ \tau(x_g^* x_g)\|v_g - u_g\|^2 \\
&= \gamma^2 \|u_g x_g\|_2^2. \qquad (7.46)
\end{aligned}$$

Since S and $1_{L^2(M)}$ are diagonal operators with respect to direct sum decomposition $L^2(M) = \oplus_{g\in\Gamma}\overline{v_g P\xi} = \oplus_{g\in\Gamma}\overline{u_g P\xi}$, then (7.46) implies that $S - 1_{L^2(M)}$ extends to $L^2(M)$ with uniform bound $\|S - 1_{L^2(M)}\| \leq \gamma$ which is invertible by the choice of γ. Now fix $g \in \Gamma$ and $x_g \in P$, for any $h \in \Gamma$ and $x_h \in P$ we have

$$\begin{aligned}
S u_g x_g S^{-1} v_h x_h \xi &= S u_g x_g u_h x_h \xi \\
&= S u_g u_h u_h^* x_g u_h x_h \xi \\
&= S u_{gh} \alpha_{h^{-1}}(x_g) x_h \xi \\
&= v_{gh} \alpha_{h^{-1}}(x_g) x_h \xi \\
&= v_g v_h v_h^{-1} x_g v_h x_h \xi \\
&= v_g x_g v_h x_h \xi. \qquad (7.47)
\end{aligned}$$

Therefore $S u_g x_g S^{-1} = v_g x_g \in N$ and since M is generated by such elements we have that $SMS^{-1} \subseteq N$. □

We are now in a position to assemble the steps described above. Once we have found a similarity implementing a genuine containment as in Lemma 7.5.1, we will use Lemma 3.1.3 to demonstrate surjectivty, that is $SMS^{-1} = N$ (see [18]).

Theorem 7.5.2. *Let $\alpha : \Gamma \to Aut(P)$ be a trace preserving, centrally ergodic and properly outer action of a countable discrete group Γ on a finite amenable von Neumann P with separable predual. Suppose that $H_b^2(\Gamma, \mathcal{Z}(P)) = 0$ and $M = P \rtimes_\alpha \Gamma$ is represented on $L^2(M)$. Let $\epsilon > 0$ be fixed. Then there exists a constant $\gamma > 0$ with the following property: if N is a weak*-closed subalgebra of $\mathbb{B}(L^2(M))$ with*

$$d(M, N) \leq \gamma \qquad \text{and} \qquad d(M', N') \leq \gamma, \qquad (7.48)$$

then there exists an invertible operator S on $L^2(M)$ with $\|1_{L^2(M)} - S\| \leq \epsilon$ and $SMS^{-1} = N$.

Remark. By the results of Appendix A we may choose Γ to be $SL_n(\mathbb{Z})$ for $n \geq 3$. This ensures that M is a non-amenable von Neumann algebra.

Proof. By [5, Proposition 2.19] it follows that M is a II_1 factor satisfying $M \cap P' \subseteq P$. Fix $\epsilon > 0$. Use Lemma 7.5.1 to pick $\gamma_1 > 0$ so that the following property holds: if $N_1 \subseteq \mathbb{B}(L^2(M))$ is weak*-closed operator algebra with

- $P \subseteq N_1$ and
- $J_M P J_M \subseteq N_1'$

and if there exists $(v_g)_{g \in \Gamma} \subseteq N_1$ satisfying

- $v_g x v_g^{-1} = \alpha_g(x) \qquad (x \in P, g \in \Gamma)$,
- $\|v_g - u_g\| \leq \gamma_1 \qquad (g \in \Gamma)$,
- $v_g v_h = v_{gh} \qquad (g, h \in \Gamma)$;

then there exists an invertible operator s on $L^2(M)$ that implements the inclusion $sMs^{-1} \subseteq N$ and such that $\|1 - s\| \leq \epsilon/3$ and is also small enough to ensure $d(sMs^{-1}, M) < 1/3$.

By using Lemma 7.4.1, Lemma 7.3.2 and Lemma 7.3.1 we may find a $\gamma_2 > 0$ such that if $d(M, N_1) \leq \gamma_2$ then there exists $(v_g)_{g \in \Gamma} \subseteq N_1$ satisfying the conditions described in the previous paragraph.

The theorem is now proved by applying Lemma 7.2.1 to find a $0 < \gamma < \min\{\gamma_2/2, 1/3\}$ such that for an arbitrary weak*-closed subalgebra N on $L^2(M)$ satisfying

$$d(M, N) \leq \gamma \qquad \text{and} \qquad d(M', N') \leq \gamma \tag{7.49}$$

there exists an invertible operator t on $L^2(M)$ which satisfies

- $P \subseteq tNt^{-1}$ and
- $J_M P J_M \subseteq (tNt^{-1})'$

and such that $\|1_{L^2(M)} - t\|$ is small enough to ensure that

- $\|1_{L^2(M)} - t^{-1}\| \leq \epsilon/3$,
- $\|t^{-1}\| \leq 2$ and

- $d(N, tNt^{-1}) < \min\{\gamma_2/2, 1/3\}$.

Now suppose N satisfies (7.49), find t as above and set $N^{(1)} = tNt^{-1}$. We have

$$d(M, N^{(1)}) \leq d(M, N) + d(N, N') \leq \gamma_2/2 + \gamma_2/2 = \gamma_2. \tag{7.50}$$

Therefore we may find s, as above, so that $sMs^{-1} \subseteq N_1$. However, we also have

$$d(sMs^{-1}, N^{(1)}) \leq d(sMs^{-1}, M) + d(M, N) + d(N, tNt^{-1}) < 1/3 + 1/3 + 1/3 = 1 \tag{7.51}$$

so by Lemma 3.1.3 it follows that $sMs^{-1} = N^{(1)}$. Set $S = t^{-1}s$ so rearranging gives $SMS^{-1} = t^{-1}sMs^{-1}t = N$ and

$$\|1_{L^2(M)} - t^{-1}s\| \leq \|1_{L^2(M)} - t^{-1}\| + \|t^{-1}(1_{L^2(M)} - s)\| \leq \epsilon. \tag{7.52}$$

$\square$

Appendix A

Group cohomology

We give a very brief outline of bounded group cohomology. Our exposition follows that given by Cameron et al. in [5].

Let Γ be a countable discrete group and let X be an abelian group with an action $\alpha : \Gamma \curvearrowright X$. We define the n^{th} *cochain complex* to be the collection of maps

$$C^n(\Gamma, X) = \{f : \Gamma^n \to X\}. \tag{A.1}$$

The n^{th} coboundary map $\partial^n : C^n(\Gamma, X) \to C^{n+1}(\Gamma, X)$ is given by

$$\begin{aligned} \partial^n(f)(g_0, \dots, g_n) =& \alpha_{g_0}(f(g_1, \dots, g_n)) + \sum_{i=0}^{n-1}(-1)^{i+1} f(g_0, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots g_n) \\ &+ (-1)^{n+1} f(g_0, \dots, g_{n-1}). \end{aligned} \tag{A.2}$$

We supress the n and write ∂ when it is obvious which coboundary map is being used. It follows from a calculation that $\partial^n \circ \partial^{n-1} = 0$ for $n \geq 1$.

We say $\psi \in C^n(\Gamma, X)$ is an *n-cocycle* if $\psi \in \ker \partial^n$, that is $\partial^n \psi = 0$, and we denote the collection of cocycles as $Z^n(\Gamma, X)$. We say $\psi \in C^n(\Gamma, X)$ is an *n-coboundary* if $\psi \in \text{im } \partial^{n-1}$, that is there exists a cochain $\phi \in C^{n-1}(\Gamma, X)$ with $\partial^{n-1}\phi = \psi$, and write $B^n(\Gamma, X)$ for the collection of such maps. Since $\partial^n \circ \partial^{n-1} = 0$ it follows that $B^n(\Gamma, X)$ is a subgroup of $Z^n(\Gamma, X)$ and we define the n^{th} cohomology group $H^n(\Gamma, X)$ to be quotient $Z^n(\Gamma, X)/B^n(\Gamma, X)$.

If X is an abelian Banach algebra, a *bounded coboundary* is a coboundary $\psi \in C^n(\Gamma, X)$ such that

$$\|\psi\| = \sup_{g_0,\ldots,g_{n-1}\in\Gamma} \|\psi(g_0, \ldots, g_{n-1})\| < \infty. \tag{A.3}$$

We denote these maps by $C_b^n(\Gamma, X)$. This naturally induces the n^{th} coboundary group

$$H_b^n(\Gamma, X) = Z_b^n(\Gamma, X)/B_b^n(\Gamma, X) \tag{A.4}$$

where

$$Z_b^n(\Gamma, X) = \{\psi \in C_b^n(\Gamma, X) : \quad \partial\psi = 0\} \tag{A.5}$$

and

$$B_b^n(\Gamma, X) = \{\psi \in C_b^n(\Gamma, X) : \exists\phi \in C_b^{n-1}(\Gamma, X) \text{ so that } \partial\phi = \psi\}. \tag{A.6}$$

When this vanishes ∂^{n-1} maps $C_b^{n-1}(\Gamma, X)$ surjectively onto $Z_b^n(\Gamma, X)$ so by the open mapping theorem their is a positive constant K such that for all $\psi \in Z_b^n(\Gamma, X)$ there exists a $\phi \in C^{n-1}(\Gamma, X)$ with $\partial^{n-1}\phi = \psi$ and such that $\|\phi\| \leq K\|\psi\|$.

In our situation the abelian Banach algebra will be the centre of a von Neumann algebra P written $\mathcal{Z}(P)$. An action $\alpha : \Gamma \curvearrowright P$ restricts to an action on $\mathcal{Z}(P)$. Fix $g \in \Gamma$ and $x \in \mathcal{Z}(P)$ then

$$\alpha_g(x)y = \alpha_g(x\alpha_{g^{-1}}(y)) = \alpha_g(\alpha_{g^{-1}}(y)x) = y\alpha_g(x) \qquad (y \in P) \tag{A.7}$$

and so $\alpha_g(x) \in \mathcal{Z}(P)$.

Of particular importance will be when the group $H_b^2(\Gamma, \mathcal{Z}(\mathcal{P}))$ vanishes. In [5, Theorem 2.7.1] Cameron et al. describe how results of Burger, Monod and Shalom [4, 44, 45] can be combined to show the following.

Theorem A.0.1. *Let $\Gamma = SL_n(\mathbb{Z})$ for $n \geq 3$. Then, for any properly outer, centrally ergodic, trace preserving action of Γ on a finite von Neumann algebra P with separable predual, the cohomology group $H_b^2(\Gamma, \mathcal{Z}(P)_{\mathrm{sa}})$ vanishes.*

We will show that under the same conditions we also have $H_b^2(\Gamma, \mathcal{Z}(P)) = 0$.

We write $\mathcal{Z}(P) = \mathcal{Z}(P)_{\mathrm{sa}} \oplus i\mathcal{Z}(P)_{\mathrm{sa}}$. Suppose $\psi \in Z_b^2(\Gamma, \mathcal{Z}(P))$ and write $\psi_{\mathrm{Re}}(g, h) = \mathrm{Re}(\psi(g, h))$ and $\psi_{\mathrm{Im}}(g, h) = \mathrm{Im}(\psi(g, h))$. So

$$\partial(\psi_{\mathrm{Re}}) + i\partial(\psi_{\mathrm{Im}}) = \partial(\psi_{\mathrm{Re}} + i\psi_{\mathrm{Im}}) = \partial(\psi) = 0. \tag{A.8}$$

However, $\partial(\psi_{\mathrm{Re}})(g,h,k) \in \mathcal{Z}(P)_{\mathrm{sa}}$ for $\partial\psi = 0$ and so $\partial(\psi_{\mathrm{Re}}) = 0$ and $\partial(\psi_{\mathrm{Im}}) = 0$. By hypothesis there exists ϕ_{Re} and ϕ_{Im} in $C_b^1(\Gamma, \mathcal{Z}(P)_{\mathrm{sa}})$ such that $\partial(\phi_{\mathrm{Re}}) = \psi_{\mathrm{Re}}$ and $\partial(\phi_{\mathrm{Im}}) = \psi_{\mathrm{Im}}$. It follows that $\partial(\phi_{\mathrm{Re}} + i\phi_{\mathrm{Im}}) = \psi$, demonstrating $H_b^2(\Gamma, \mathcal{Z}(\mathcal{P})) = 0$.

References

[1] Bruce Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[2] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[3] John W. Bunce. The similarity problem for representations of C^*-algebras. *Proc. Amer. Math. Soc.*, 81(3):409–414, 1981.

[4] Marc Burger and Nicolas Monod. Continuous bounded cohomology and applications to rigidity theory. *Geom. Funct. Anal.*, 12(2):219–280, 2002.

[5] Jan Cameron, Erik Christensen, Allan Sinclair, Roger R. Smith, Stuart White, and Alan Wiggins. Kadison- Kastler stable factors. Duke. Math. J., to appear. arXiv:1209.4116, 2012.

[6] Jan Cameron, Erik Christensen, Allan M. Sinclair, Roger R. Smith, Stuart White, and Alan D. Wiggins. A remark on the similarity and perturbation problems. *C. R. Math. Acad. Sci. Soc. R. Can.*, 35(2):70–76, 2013.

[7] Man Duen Choi and Erik Christensen. Completely order isomorphic and close C^*-algebras need not be *-isomorphic. *Bull. London Math. Soc.*, 15(6):604–610, 1983.

[8] Erik Christensen. Perturbations of type I von Neumann algebras. *J. London Math. Soc. (2)*, 9:395–405, 1974/75.

[9] Erik Christensen. Perturbations of operator algebras. *Invent. Math.*, 43(1):1–13, 1977.

[10] Erik Christensen. Perturbations of operator algebras. II. *Indiana Univ. Math. J.*, 26(5):891–904, 1977.

[11] Erik Christensen. Near inclusions of C^*-algebras. *Acta Math.*, 144(3-4):249–265, 1980.

[12] Erik Christensen. Similarities of II_1 factors with property Γ. *J. Operator Theory*, 15(2):281–288, 1986.

[13] Erik Christensen, Allan Sinclair, Roger R. Smith, and Stuart White. Perturbations of C^*-algebraic invariants. *Geom. Funct. Anal.*, 20(2):368–397, 2010.

[14] Erik Christensen, Allan M. Sinclair, Roger R. Smith, Stuart A. White, and Wilhelm Winter. Perturbations of nuclear C^*-algebras. *Acta Math.*, 208(1):93–150, 2012.

[15] Alain Connes. Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$. *Ann. of Math. (2)*, 104(1):73–115, 1976.

[16] Alain Connes. On the cohomology of operator algebras. *J. Functional Anal.*, 28(2):248–253, 1978.

[17] Liam Dickson. A Kadison-Kastler row metric and intermediate subalgebras. *Internat. J. Math.*, 25(8):1450082 (16 pages), 2014.

[18] Liam Dickson and Stuart White. An embedding theorem for injective von Neumann algebras.

[19] Edward G. Effros and Zhong-Jin Ruan. On nonselfadjoint operator algebras. *Proc. Amer. Math. Soc.*, 110(4):915–922, 1990.

[20] George A. Elliott. On approximately finite-dimensional von Neumann algebras. II. *Canad. Math. Bull.*, 21(4):415–418, 1978.

[21] Junsheng Fang, Roger R. Smith, Stuart A. White, and Alan D. Wiggins. Groupoid normalizers of tensor products. *J. Funct. Anal.*, 258(1):20–49, 2010.

[22] Erling Følner. On groups with full Banach mean value. *Math. Scand.*, 3:243–254, 1955.

[23] Uffe Haagerup. All nuclear C^*-algebras are amenable. *Invent. Math.*, 74(2):305–319, 1983.

[24] Uffe Haagerup. Solution of the similarity problem for cyclic representations of C^*-algebras. *Ann. of Math. (2)*, 118(2):215–240, 1983.

[25] Uffe Haagerup and Magdalena Musat. Classification of hyperfinite factors up to completely bounded isomorphism of their preduals. *J. Reine Angew. Math.*, 630:141–176, 2009.

[26] Donald W. Hadwin. Dilations and Hahn decompositions for linear maps. *Canad. J. Math.*, 33(4):826–839, 1981.

[27] Jôsuke Hakeda. On property P of von Neumann algebras. *Tôhoku Math. J. (2)*, 19:238–242, 1967.

[28] Ilan Hirshberg, Eberhard Kirchberg, and Stuart White. Decomposable approximations of nuclear C^*-algebras. *Adv. Math.*, 230(3):1029–1039, 2012.

[29] Shoji Ino. Perturbations of von Neumann subalgebras with finite index. arXiv:1407.6097., 2014.

[30] Shoji Ino and Yasuo Watatani. Perturbations of intermediate C*-subalgebras for simple C*-algebras. *Bull. Lond. Math. Soc.*, 46(3):469–480, 2014.

[31] Barry E. Johnson. Approximate diagonals and cohomology of certain annihilator Banach algebras. *Amer. J. Math.*, 94:685–698, 1972.

[32] Barry E. Johnson. *Cohomology in Banach algebras.* American Mathematical Society, Providence, R.I., 1972. Memoirs of the American Mathematical Society, No. 127.

[33] Barry E. Johnson. Perturbations of Banach algebras. *Proc. London Math. Soc. (3)*, 34(3):439–458, 1977.

[34] Barry E. Johnson. A counterexample in the perturbation theory of C^*-algebras. *Canad. Math. Bull.*, 25(3):311–316, 1982.

[35] Barry E. Johnson. Approximately multiplicative maps between Banach algebras. *J. London Math. Soc. (2)*, 37(2):294–316, 1988.

[36] Vaughan Jones and Vaikalathur S. Sunder. *Introduction to subfactors*, volume 234 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1997.

[37] Vaughan F. R. Jones. Index for subfactors. *Invent. Math.*, 72(1):1–25, 1983.

[38] Richard V. Kadison. On the orthogonalization of operator representations. *Amer. J. Math.*, 77:600–620, 1955.

[39] Richard V. Kadison and Daniel Kastler. Perturbations of von Neumann algebras. I. Stability of type. *Amer. J. Math.*, 94:38–54, 1972.

[40] Mahmood Khoshkam. Perturbations of C^*-algebras and K-theory. *J. Operator Theory*, 12(1):89–99, 1984.

[41] Mahmood Khoshkam. Close C^*-algebras are KK-equivalent. In *Operator theory: operator algebras and applications, Part 2 (Durham, NH, 1988)*, volume 51 of *Proc. Sympos. Pure Math.*, pages 159–162. Amer. Math. Soc., Providence, RI, 1990.

[42] Eberhard Kirchberg. The derivation problem and the similarity problem are equivalent. *J. Operator Theory*, 36(1):59–62, 1996.

[43] Eberhard Kirchberg and N. Christopher Phillips. Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$. *J. Reine Angew. Math.*, 525:17–53, 2000.

[44] Nicolas Monod. On the bounded cohomology of semi-simple groups, S-arithmetic groups and products. *J. Reine Angew. Math.*, 640:167–202, 2010.

[45] Nicolas Monod and Yehuda Shalom. Cocycle superrigidity and bounded cohomology for negatively curved spaces. *J. Differential Geom.*, 67(3):395–455, 2004.

[46] Hitoshi Moriyoshi and Toshikazu Natsume. *Operator algebras and geometry*, volume 237 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2008. Translated from the 2001 Japanese original by the authors.

[47] Gerard J. Murphy. *C^*-algebras and operator theory*. Academic Press Inc., Boston, MA, 1990.

[48] Vern Paulsen. *Completely bounded maps and operator algebras*, volume 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.

[49] Francesc Perera, Andrew Toms, Stuart White, and Wilhelm Winter. The Cuntz semigroup and stability of close C^*-algebras. *Anal. PDE*, 7(4):929–952, 2014.

[50] John Phillips and Iain Raeburn. Perturbations of C^*-algebras. II. *Proc. London Math. Soc. (3)*, 43(1):46–72, 1981.

[51] Gilles Pisier. The similarity degree of an operator algebra. *St. Petersburg Math. J.*, 10(1):103–146, 1999.

[52] Gilles Pisier. Remarks on the similarity degree of an operator algebra. *Internat. J. Math.*, 12(4):403–414, 2001.

[53] Gilles Pisier. *Similarity problems and completely bounded maps*, volume 1618 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 2001. Includes the solution to "The Halmos problem".

[54] Gilles Pisier. A similarity degree characterization of nuclear C^*-algebras. *Publ. Res. Inst. Math. Sci.*, 42(3):691–704, 2006.

[55] Iain Raeburn and Joseph L. Taylor. Hochschild cohomology and perturbations of Banach algebras. *J. Funct. Anal.*, 25(3):258–266, 1977.

[56] Jean Roydor. Dual operator algebras with normal virtual h-diagonal. *Integral Equations Operator Theory*, 73(3):365–382, 2012.

[57] Zhong-Jin Ruan. Subspaces of C^*-algebras. *J. Funct. Anal.*, 76(1):217–230, 1988.

[58] Raymond A. Ryan. *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2002.

[59] Jacob Schwartz. Two finite, non-hyperfinite, non-isomorphic factors. *Comm. Pure Appl. Math.*, 16:19–26, 1963.

[60] Allan M. Sinclair and Roger R. Smith. *Finite von Neumann algebras and masas*, volume 351 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2008.

[61] Masamichi Takesaki. *Theory of operator algebras. II*, volume 125 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 6.

[62] John von Neumann. *Collected works. Vol. III: Rings of operators*. General editor: A. H. Taub. Pergamon Press, New York-Oxford-London-Paris, 1961.

[63] Yasuo Watatani. Index for C^*-subalgebras. *Mem. Amer. Math. Soc.*, 83(424):vi+117, 1990.

[64] Gerd Wittstock. Ein operatorwertiger Hahn-Banach Satz. *J. Funct. Anal.*, 40(2):127–150, 1981.