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Internal categories as models of homotopy types

by

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Introduction

A *homotopy n-type* is a topological space which has trivial homotopy groups above degree n. Every space can be constructed from a sequence of such *homotopy types*, in a sense made precise by the theory of Postnikov towers, yielding improving 'approximations' to the space by encoding information about the first n homotopy groups for increasing n. Thus the study of homotopy types, and the search for models of such spaces that can be fruitfully investigated, has been a central problem in homotopy theory.

Of course, a homotopy 0-type is, up to weak homotopy equivalence (isomorphism of homotopy groups), a discrete set. It is well-known that a connected 1-type can be represented, again up to weak homotopy equivalence, as the *classifying space* of its fundamental group: this is the geometric realization of the simplicial set that is the nerve of the group regarded as a category with one object. Another way to phrase this is that the *homotopy category* of 1-types obtained by localizing at maps which are weak homotopy equivalences — formally adding inverses for these — is equivalent to the skeleton of the category of groups.

In [**Mac Lane and Whitehead**] it was proved that connected homotopy 2-types can be modeled, in the sense described above, by *crossed modules* of groups. A crossed module is equivalently what in [**Loday**] is called a 1-*cat-group*, but now often referred to as a $cat^1 - group$, which is a group G with two homomorphisms $s, t : G \to G$ such that $st = t$, $ts = s$ and $[\ker\, s, \ker\, t] = 1$. Most importantly for the project pursued here, the category $\mathbf{Cat^1 - Group}$, with morphisms those homomorphisms that are equivariant with respect to the structure maps s and t, is equivalent to internal categories in groups [**Loday**]. From the latter point of view, the homotopy 2-type associated to a $cat^1 - group$ G is the geometric realization $|\overline{W}NG|$, the simplicial group NG being the (internal) nerve of G.

Loday introduced the generalized notion of a $cat^n - group$, a group with homomorphisms $s_i, t_i : G \to G$, $1 \leq i \leq n$, such that, as well as the foregoing conditions on each pair s_k and t_k, two homomorphisms with distinct indices commute. In [**Loday**] certain diagrams of topological spaces — called *n-cubes of fibrations* — are constructed from $cat^n - groups$; indeed, it is shown that the two notions coincide. There is a ready relation of weak homotopy equivalence for these diagrams of spaces, defined pointwise, and it is established that the resulting homotopy category is equivalent to that of connected $(n+1)$-types, the demonstration working directly in terms of 'formal' localization as described above.

It is known that $cat^n - groups$ as defined by Loday are, extending the case $n = 1$, equivalent to n-fold categories internal to groups [**Paoli**], where a 2-fold category is a category internal to categories internal to groups, etc. The $(n+1)$-type associated to a $cat^n - group$ is, from this point of view, the geometric realization of the diagonal of $|\overline{W}X|$, with X the diagonal of the n-simplicial group obtained by iterating the internal nerve construction [**Bullejos, Cegarra and Duskin**].

In modern homotopy theory the abstract theory and terminology of Quillen model categories is pervasive, providing regimented and systematic techniques for addressing questions about localization and relations between *homotopical categories*. The aim of this work is to present another approach to the above-mentioned result, within the conceptual framework of abstract homotopy theory.

After reviewing the required topics from the homotopy theory of simplicial sets and model categories, we will show that internal categories in groups can be given the structure of a model category, using an adjunction between these and the category of simplicial groups to transfer the *cofibrantly generated* model structure on the latter. We then proceed to use a classical adjunction between simplicial groups and reduced simplicial sets — seen as a model for connected spaces — to relate internal categories to a Bousfield localization of reduced simplicial sets; this localization is one which has $(n+1)$-types as the skeleton of the homotopy category. Our main theorem proves that the adjunction between n-fold internal categories and connected $(n+1)$-types induces an equivalence of homotopy categories, expressed in terms of Quillen functors. In the appendix we will briefly indicate a generalization of these results to all homotopy types, not necessarily connected.

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Notation guide

Δ	The simplicial category of finite ordinals $[\mathbf{n}]$
$\mathbf{s}\mathcal{C}$	The category of simplicial objects in a category $\mathcal{C}$
$\mathbf{s}^n\mathcal{C}$	The category of n-fold simplicial objects in a category $\mathcal{C}$
$\mathbf{s}\mathcal{C}_{\leq n}$	The category of n-truncated simplicial objects in a category $\mathcal{C}$
tr_n	The n-truncation functor $\mathbf{s}\mathcal{C} \to \mathbf{s}\mathcal{C}_{\leq n}$
$sk_n/cosk_n$	The n-skeleton/n-coskeleton functor $\mathbf{s}\mathcal{C}_{\leq n} \to \mathbf{s}\mathcal{C}$
$Sk_n/Cosk_n$	The composite functor $sk_n tr_n/cosk_n tr_n$
$\mathbf{sSet}$	The category of simplicial sets
$\mathbf{sSet_0}$	The category of reduced simplicial sets
$\mathbf{sSet}_*$	The category of pointed simplicial sets
$\mathbf{hom}_{\mathcal{C}}(a,b)$, $\mathcal{C}(a,b)$	Set of morphisms between objects a and b of a category $\mathcal{C}$
$\mathbf{Hom}_{\mathcal{C}}(a,b)$	Morphism object between a and b in an enriched category $\mathcal{C}$
$X \otimes A$	Tensor of $X \in M$ and $A \in \mathcal{C}$ in a tensored $\mathcal{C}$-enriched category M
Δ^n	The representable simplicial set $\Delta(\quad, [\mathbf{n}])$
$\lvert X \rvert$	Geometric realization of a simplicial set X
N	The nerve functor $\mathbf{Cat} \to \mathbf{sSet}$;
	the multi-nerve functor $\mathbf{Cat}^n(\mathbf{Set}) \to \mathbf{s}^n\mathbf{Set}$
τ_1	The fundamental category functor $\mathbf{sSet} \to \mathbf{Cat}$
$\mathbf{Ho}(\mathcal{C})$	Homotopy category of a category $\mathcal{C}$ with weak equivalences
$\mathrm{L}_f\mathcal{M}$	Left (Bousfield) localization of a model category $\mathcal{M}$ with
	respect to a map f
$sr(X)$	Simplicial resolution of an object X of a model category
$Map^r(X,Y)$	Right homotopy function complex from X to Y in a model category
$\mathbf{Grp}(\mathcal{C})$	The category of internal group objects in a category $\mathcal{C}$
$\mathbf{Cat}(\mathcal{C})$	The category of internal category objects in a category $\mathcal{C}$
$\mathbf{Cat}^n(\mathcal{C})$	The category of n-fold categories internal to a category $\mathcal{C}$
σ	The ordinal sum functor $\Delta^{\times n} \to \Delta$, $([k_1], \ldots, [k_m]) \mapsto [k_1] \boxplus \ldots \boxplus [k_m]$
σ^*	The functor $\mathbf{sGrp} \to \mathbf{s}^n\mathbf{Grp}$ induced by σ
σ_*	The right adjoint of σ^*

$\mathbf{A}$	An object of $\mathbf{s}^n\mathbf{Set}$ or $\mathbf{s}^n\mathbf{Grp}$
$\mathbb{B}$	An object of $\mathbf{Cat}^n\mathbf{Set}$ or $\mathbf{Cat}^n\mathbf{Grp}$
τ	The fundamental multi-category functor $\mathbf{s}^n\mathbf{Set} \to \mathbf{Cat}^n(\mathbf{Set})$
G	The Kan loop group functor $\mathbf{sSet_0} \to \mathbf{sGrp}$
$\overline{W}$	The classifying complex functor $\mathbf{sGrp} \to \mathbf{sSet_0}$
$\mathcal{C} - \mathbf{Grpd}$	The category of groupoids enriched over a category $\mathcal{C}$

Contents

Chapter 1

Simplicial sets

1. Simplicial and cosimplicial objects

The *simplicial category* Δ has objects finite ordinals $[\mathbf{n}] = \{0 \to 1 \to \ldots \to n\}$ and morphisms weakly monotone functions between these. We shall sometimes consider the totally ordered set $[\mathbf{n}]$ as a category itself; morphisms in Δ are then simply functors. Two types of morphism in Δ are particularly significant: *coface* maps

$$d^i : [\mathbf{n-1}] \to [\mathbf{n}] \qquad 0 \leq i \leq n$$
$$d^i(k) = k \quad \text{for} \quad k \leq i-1 \quad \text{and} \quad d^i(k) = k+1 \quad \text{for} \quad k \geq i$$

and *codegeneracy* maps

$$s^j : [\mathbf{n+1}] \to [\mathbf{n}] \qquad 0 \leq i \leq n$$
$$s^j(k) = k \quad \text{for} \quad k \leq j \quad \text{and} \quad s^j(k) = k-1 \quad \text{for} \quad k \geq j+1$$

which satisfy the following relations, known as the *cosimplicial identities*.

$$\begin{cases} d^j d^i = d^i d^{j-1} & \text{if } i < j \\ s^j d^i = d^i s^{j-1} & \text{if } i < j \\ s^j d^j = 1 = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & \text{if } i > j+1 \\ s^j s^i = s^i s^{j+1} & \text{if } i \leq j \end{cases}$$

Every morphism ϕ in Δ is the composite of a surjection σ followed by an injection η, in one way only. The surjection σ can be further presented as a (possibly empty) sequence $s^{i_1} \ldots s^{i_k}$ of codegeneracy maps; this presentation is unique if we require $i_1 < \ldots < i_k$. Similarly, η is the composite $d^{j_1} \ldots d^{j_l}$ of coface maps, where $j_1 > \ldots > j_l$.

Let $\mathcal{C}$ be a category. A functor $Y : \Delta \to \mathcal{C}$ is called a *cosimplicial object* in $\mathcal{C}$. A contravariant functor $X : \Delta^{op} \to \mathcal{C}$ is called a *simplicial object* in $\mathcal{C}$. It follows from the above that Y is specified by a sequence of objects $\{Y^n \in \mathcal{C}\}_{n\geq 0}$ and morphisms $d^i : Y^n \to Y^{n+1}$, $0 \leq i \leq n+1$, and $s^j : Y^n \to Y^{n-1}$, $0 \leq j \leq n-1$ —also called cofaces and codegeneracies — subject to the cosimplicial identities. Dually X is specified by a sequence of objects $\{X_n \in \mathcal{C}\}_{n\geq 0}$ and morphisms $d_i : X_{n+1} \to X_n$, $0 \leq i \leq n+1$, and $s_j : X_{n-1} \to X_n$, $0 \leq j \leq n-1$; these morphisms are called *face* and *degeneracy* maps respectively and must satisfy the *simplicial identities* below.

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j \\ d_i s_j = s_{j-1} d_i & \text{if } i < j \\ d_j s_j = 1 = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \\ s_i s_j = s_{j+1} s_i & \text{if } i \leq j \end{cases}$$

(Note the notational convention of using subscripts for simplicial objects and superscripts for cosimplicial objects)

Henceforth we shall phrase most definitions and results in terms of simplicial objects, with the dual statements about cosimplicial objects left implicit.

The functor category $[\Delta^{op}, \mathcal{C}]$ of simplicial objects in $\mathcal{C}$ and natural transformations between them is denoted by $\mathbf{s}\mathcal{C}$. Limits and colimits of diagrams of simplicial objects, if they exist, are computed pointwise as in any functor category. Therefore $\mathbf{s}\mathcal{C}$ is complete or cocomplete if $\mathcal{C}$ has the corresponding property.

Simplicial objects in $\mathbf{s}\mathcal{C}$, equivalently functors $\Delta^{op} \times \Delta^{op} \to \mathcal{C}$, are called *bisimplicial* objects. More generally we may consider *multisimplicial* objects $X \in [(\Delta^{op})^{\times n}, \mathcal{C}]$ for any $n \geq 2$ and write $\mathbf{s}^n\mathcal{C}$ for this functor category; thus $\mathbf{s}^n\mathcal{C} = \mathbf{s}(\mathbf{s}^{n-1}\mathcal{C})$. Such a multisimplicial object is specified by objects $X_{i_1,\ldots,i_n} \in \mathcal{C}$ indexed by tuples of natural numbers standing for the arguments of the functor, and face and degeneracy maps in each coordinate. Of course any two such structure maps obey the simplicial identities if they are in the same coordinate, and commute otherwise. In the case of a bisimplicial object it is customary to use the adjectives horizontal and vertical for the two coordinates, with the picture of the lattice of integers in the plane in mind.

The diagonal functor $\delta : \Delta^{op} \to (\Delta^{op})^{\times n}$, $[n] \mapsto ([n], \ldots, [n])$, yields by pre-composition a functor $d = \delta^* : [(\Delta^{op})^{\times n}, \mathcal{C}] \to [\Delta^{op}, \mathcal{C}]$ from multisimplicial to simplicial objects, also referred to as the diagonal; explicitly $(dX)_n = X_{n,\ldots,n}$. The theory of Kan extensions

[**Mac Lane**, §X.3] implies that if $\mathcal{C}$ is cocomplete then d has a left adjoint, and if $\mathcal{C}$ is complete then it has a right adjoint. (These are respectively the left and right Kan extensions along δ)

Simplicial sets

The category **sSet** of simplicial sets, i.e. simplicial objects in **Set**, is of central importance in our study. Here we define the basic concepts and notation that will be used frequently.

2. Representables

The representable presheaf $\mathbf{hom}_\Delta(\quad, [n])$ is denoted by Δ^n. By the Yoneda lemma the set X_n of *n-simplices* of a simplicial set X is naturally isomorphic to maps from Δ^n to X: $\mathbf{sSet}(\Delta^n, X) \cong X_n$: a $\phi \in \mathbf{sSet}(\Delta^n, X)$ is uniquely specified by the image $\phi(\iota_n) \in X_n$, where ι_n represents the identity morphism $1_n : [\mathbf{n}] \to [\mathbf{n}]$.

Since there cannot be an injective map $[m] \to [n]$ when $m > n$ every *m-simplex* $x \in (\Delta^n)_m$ of dimension greater than n is in the image of a (not necessarily unique — but see the next paragraph) degeneracy map; we say that x is *degenerate.*

The Eilenberg-Zilber lemma [**Joyal and Tierney**, Proposition 1.2.2] states that any simplex of a simplicial set is either non-degenerate, i.e. not equal to $s_i(y)$ for any i or y, or else can be written as $\theta^*(z)$ where θ is a surjection in Δ and z is non-degenerate; moreover θ and z are unique with this property. It follows that there are three classes of simplices of Δ^n: non-degenerate simplices, necessarily of dimension less than or equal to n; degenerate simplices in the degeneracy image of the unique non-degenerate simplex $\iota_n \in (\Delta^n)_n$; and degenerate simplices in the degeneracy image of simplices of dimensions below n.

3. Boundaries and horns

When $n > 0$ every non-degenerate simplex of dimension less than n in Δ^n is either equal to one of the faces $d_i(\iota_n)$ or results from these by further action of face maps. It follows that the sets of all simplices of dimension less than n and their degeneracy images in higher dimensions form the smallest simplicial subset or sub-complex — that is, sub-object in the category **sSet** — of Δ^n that includes the $d_i(\iota_n)$, and is generated by them. This sub-complex is denoted by $\partial\Delta^n$ and called the *boundary* of Δ^n. For the terminal simplicial set Δ^0 we put $\partial\Delta^0 = \emptyset$, the empty simplicial set which is the initial object of **sSet**; this intuitive stipulation provides for generality in statement of most results involving boundary sub-complexes.

Any simplicial set, being a presheaf, is the colimit of the diagram of representables determined by its simplices [**Mac Lane**, §V.7] — generally called the category of elements of the presheaf. Working through the definition in the case of $\partial\Delta^n$ one finds that it can be presented as a coequalizer

$$\coprod_{0\le i<j\le n} \Delta^{n-2} \rightrightarrows \coprod_{0\le k\le n} \Delta^{n-1} \longrightarrow \partial\Delta^n$$

where the top and bottom arrows on the left represent respectively the maps that on the summand in the domain corresponding to the pair (i,j) are determined by the simplices $d_i(\iota_n)$ of the j-th summand and $d_{j-1}(\iota_n)$ of the i-th summand in the codomain. Intuitively, $\partial\Delta^n$ is obtained from the codimension-1 non-degenerate simplices of Δ^n, incident along their faces — codimension-2 non-degenerate simplices of Δ^n — according to the identification relations in this coequalizer. Hence a map $\partial\Delta^n \to X$ is specified by an $(n+1)$-tuple of simplices $x_0,\dots,x_n \in X_{n-1}$ such that $d_i(x_j) = d_{j-1}(x_i)$ for $i<j$.

Other important sub-complexes of Δ^n are the *horns* Λ^n_k, $0\le k\le n$; the k-th horn Λ^n_k is the smallest sub-complex containing the face simplices $d_i(\iota_n)$ for $i\ne k$. Each horn is also a sub-object of the boundary $\partial\Delta^n$, and it is easy to see that when $n\ge 2$ we have $(\Lambda^n_k)_m = (\Delta^n)_m$ for $m\le n-2$. An argument parallel to the foregoing shows that a map $\Lambda^n_k \to X$ is specified by an n-tuple of simplices $x_0,\dots,x_{k-1},x_{k+1},\dots,x_n \in X_{n-1}$ such that $d_i(x_j) = d_{j-1}(x_i)$ for $i<j$ both different from k.

4. The skeleton and coskeleton of a simplicial set

We have seen that $\partial\Delta^n$ can be described as the simplicial subset of Δ^n that contains all m-simplices for $m\le n-1$ and the degenerate simplices in higher dimensions obtained from these. This way of obtaining sub-complexes is applicable to any simplicial set X and produces what is called the *n-skeleton* of X, denoted by Sk_nX. Thus Sk_nX is the degeneracy closure of all simplices of X of dimension less than n. The n-skeleton construction can also be characterized by a universal property. To see this we shall reformulate it in a more abstract setting and generalize it.

Let $\mathcal{C}$ be a (small-) cocomplete category and X a simplicial object in $\mathcal{C}$. Writing $\Delta_{\le n}$ for the full subcategory of Δ consisting of ordinals less than or equal to $[\mathbf{n}]$, we obtain an object tr_nX in the category $[\Delta^{op}_{\le n}, \mathcal{C}] = \mathbf{s}\mathcal{C}_{\le n}$ of *n-truncated simplicial objects* by composing

$X : \Delta^{op} \to \mathcal{C}$ with the inclusion $\Delta^{op}_{\leq n} \hookrightarrow \Delta^{op}$, which we call the *n-truncation* of X. In other words the operation $tr_n : \mathbf{s}\mathcal{C} \to \mathbf{s}\mathcal{C}_{\leq n}$ restricts $X \in \mathbf{s}\mathcal{C}$ to the objects X_i of degree $i < n+1$ and the face and degeneracy morphisms between them.

Now, since $\mathcal{C}$ has all colimits, tr_n has a left adjoint $sk_n : \mathbf{s}\mathcal{C}_{\leq n} \to \mathbf{s}\mathcal{C}$, given as a left Kan extension along $\Delta^{op}_{\leq n} \hookrightarrow \Delta^{op}$. We write Sk_n (with an uppercase 'S') for the composite $sk_n tr_n$; it is customary to refer to both sk_n and Sk_n as the n-skeleton functor, relying on the context to disambiguate. It is not difficult to verify, by working through the formula for computing Kan extensions, that for a simplicial set this functor Sk_n is isomorphic to the n-skeleton described above, justifying our use of the same notation. In this case the universal property implied by the adjunction $sk_n \dashv tr_n$ can be simply expressed as the fact that specifying the image of a simplex under a natural transformation uniquely determines the image of its degeneracy span. The property that $(Sk_n X)_m = X_m$ for $m \leq n$, which can thus be seen to hold for simplicial sets, is valid for simplicial objects generally as a consequence of the underlying theory [**Mac Lane**, Corollary X.3.3].

If $\mathcal{C}$ is also complete then tr_n has a right adjoint $cosk_n$ called the *n-coskeleton* functor, as is the composite $Cosk_n = cosk_n tr_n$. By this adjunction,

$$\mathbf{s}\mathcal{C}(X, Cosk_n Y) \cong \mathbf{s}\mathcal{C}_{\leq n}(tr_n X, tr_n Y) \cong \mathbf{s}\mathcal{C}(Sk_n X, Y);$$

therefore $Sk_n \dashv Cosk_n$. Moreover, using the Yoneda lemma, for a simplicial set X

$$(Cosk_n X)_m \cong \mathbf{sSet}(\Delta^m, Cosk_n X) \cong \mathbf{sSet}_{\leq n}(tr_n \Delta^m, tr_n X) \cong \mathbf{sSet}(Sk_n \Delta^m, X).$$

Therefore, $(Cosk_n X)_m = X_m$ when $m \leq n$ (as for Sk_n this can be derived abstractly for simplicial objects in general) and $(Cosk_n X)_{n+1}$ can be identified with maps $\partial\Delta^{n+1} \to X$.

5. Geometric realization

The *topological n-simplex* $|\Delta^n|$ is the convex hull of the points $e_i = (0, \ldots, 1, \ldots, 0)$ corresponding to the standard basis vectors in $\mathbb{R}^{n+1}$; that is, it is the set $\{(t_0, \ldots, t_n) \in [0,1]^{n+1} | \Sigma t_i = 1\}$. To any function $\theta : [\mathbf{n}] \to [\mathbf{m}]$ we can associate a continuous map $\theta_* : |\Delta^n| \to |\Delta^m|$ by sending e_i to $e_{\theta(i)}$ and extending linearly. This correspondence defines a functor $\Delta \to \mathbf{Top}$ to the category of topological spaces. By a general result about presheaf categories [**Mac Lane and Moerdijk**, Corollary I.5.4] this functor extends uniquely to a functor $| \,.\, | : \mathbf{sSet} \to \mathbf{Top}$, called the *geometric realization*; it is an extension in the sense that its restriction along the Yoneda embedding $\mathbf{y} : \Delta \to \mathbf{sSet}$

takes $[\mathbf{n}]$ to $|\Delta^n|$. Moreover, $|\ .\ |$ is left adjoint to the functor $S : \mathbf{Top} \to \mathbf{sSet}$ given by $S(A)_n = \mathbf{Top}(|\Delta^n|, A)$; that is, $S(A)$ is the familiar *singular complex* of A used to calculate singular homology.

For $X \in \mathbf{sSet}$, $|X|$ can be presented as a colimit

$$|X| = \varinjlim_{\Delta \downarrow X} |\Delta^n|$$

of a diagram defined by the category of simplices of X; it is the diagram that results from projecting the comma category $\Delta \downarrow X$ of the Yoneda embedding over X onto Δ followed by the functor $[\mathbf{n}] \mapsto |\Delta^n|$. An object of the comma category $\Delta \downarrow X$ is a map $\Delta^n \to X$, and a morphism from $u : \Delta^n \to X$ to $v : \Delta^m \to X$ corresponds to a $\alpha : [\mathbf{m}] \to [\mathbf{m}]$ such that

$$\begin{tikzcd} \Delta^n \arrow[rr, "\alpha_*"] \arrow[dr, "u"'] & & \Delta^m \arrow[dl, "v"] \\ & X & \end{tikzcd}$$

commutes. Thus the colimit diagram in $\mathbf{Top}$ defining $|X|$ has a copy of $|\Delta^n|$ for each n-simplex in X, with morphisms representing its simplicial structure. Perspicuously, $|X|$ is obtained as the quotient

$$\coprod_{m \geq 0} \coprod_{x \in X_m} |\Delta^m| / \sim$$

of the space consisting of cells corresponding to the simplices of X by identifying the summand of $x \in X_m$ with that of $\alpha^*(x) \in X_n$ along $\alpha_* : |\Delta^n| \to |\Delta^m|$ for every $\alpha \in \Delta([\mathbf{n}], [\mathbf{m}])$. Each such α can be decomposed as a surjection followed by an injection, and the identification of cells related to a surjection implies that removing its domain makes no difference to the quotient space; thus it is not difficult to convince oneself of the following even more succinct description of $|X|$: it is the cell-complex built from the non-degenerate simplices of X with attaching maps given by face morphisms.

The geometric realization functor exemplifies the concept of a *Kan extension* — in this case, a **left** Kan extension — which will be encountered in later arguments too [**Mac Lane**, §X.3]. In general, given a functor $S : \mathcal{C} \to \mathcal{D}$ and a cocomplete category $\mathcal{E}$, the pre-composition $S^* : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$, $K \mapsto KS$, has a left adjoint L_S given by a construction involving colimits of diagrams of comma categories, similar to that above. (The existence of these colimits is sufficient for an adjunction, so cocompleteness of $\mathcal{E}$ is in fact a stronger condition than needed.) For $F : \mathcal{C} \to \mathcal{E}$, $L_S F$ is called the *left Kan extension of F along S*. Thus $|\ .\ |$ is the left Kan extension of $[\mathbf{n}] \mapsto |\Delta^n|$ along the Yoneda embedding $\mathbf{y}$. Dually, if $\mathcal{E}$ is complete then S^* has a right adjoint R_S — computed using limit diagrams — that takes $G : \mathcal{G} \to \mathcal{E}$ to its right Kan extension along S, $R_S G : \mathcal{D} \to \mathcal{E}$. The adjunction $L_S \dashv S^*$ is means that there is a natural isomorphism

$$\mathcal{E}^{\mathcal{C}}(F, KS) \cong \mathcal{E}^{\mathcal{D}}(L_S F, K);$$

hence in particular the statement that the geometric realization functor is a left Kan extension means that for an arbitrary functor $T : \mathbf{sSet} \to \mathbf{Top}$, a natural transformation $\mu : |\ .\ | \to T$ is uniquely determined by its components $\mu_{\Delta^n} : |\Delta^n| \to T(\Delta^n)$. This is implied by the description of $|X|$ as a cell-complex: every point $x \in |X|$ is in the image of a (not necessarily unique) map $|z| : |\Delta^n| \to |X|$ corresponding to an n-simplex $z : \Delta^n \to X$, and so by naturality $\mu_X(x) = T(z)\mu_{\Delta^n}(x)$.

The functor $|\ .\ |$, being a left adjoint, preserves colimits. If we construe $|\ .\ |$ as taking image in the category **CGTop** of *compactly generated* spaces [**May99**, Chapter 5], where the topology of a product space is a refinement of the product topology in **Top**, then it has the further important property of being strong monoidal; that is, $|X \times Y| \cong |X| \times |Y|$ when the product on the right is in **CGTop**. (One way to phrase this is that the topology of a product of CW-complexes as compactly generated spaces is the same as the weak topology on the product complex.)

Throughout this work we use the product-preserving property of the geometric realization functor in many arguments, with the assumption of a suitable background category of spaces (compact generation is a sufficient condition) left implicit.

6. Exponentials

Being a functor category, **sSet** is complete and cocomplete, with limits and colimits of diagrams calculated point-wise — that is, there is a natural isomorphism $(\varprojlim X^\alpha)_n \cong \varprojlim (X^\alpha)_n$ and similarly for colimits; thus, in particular, products exist and are given by $(X \times Y)_n = X_n \times Y_n$.

Moreover, as **sSet** is a presheaf category, it is cartesian closed: the product functor $(-) \times X$ has a right adjoint $(-)^X$, called the exponential. The general formula for exponentials of presheaves [**Mac Lane and Moerdijk**] in this case yields

$$(Y^X)_n = \mathbf{sSet}(X \times \Delta^n, Y)$$

with $\theta^* : (Y^X)_n \to (Y^X)_m$ for $\theta : [\mathbf{m}] \to [\mathbf{n}]$ obtained by composition with $1 \times \theta_* : X \times \Delta^m \to X \times \Delta^n$.

Using the naturality of the isomorphism $\mathbf{sSet}(Z \times X, Y) \cong \mathbf{sSet}(Z, Y^X)$ in Z and Y, there is a unique way to define a functor $Y^{(-)}$ for fixed Y such that the isomorphism is natural in all three variables. It follows that the exponential can be regarded as a bifunctor $\mathbf{sSet}^{op} \times \mathbf{sSet} \to \mathbf{sSet}$; it is also called the *internal hom* and written $\mathbf{Hom}_{\mathbf{sSet}}(-,-)$ or $\underline{\mathbf{sSet}}(-,-)$. With $\mathbf{Hom}_{\mathbf{sSet}}(X, Y)$ as the mapping object between X and Y, the closed symmetric monoidal category **sSet** is enriched over itself, hence the adjective 'internal'; see [**Kelly**, 1.6]).

Definition 6.1. A category that is enriched over **sSet** is said to be *simplicially enriched* or an **sSet**-*category*. A simplicially enriched category $\mathcal{C}$ is *tensored* if, denoting the simplicial set of morphisms between objects of $\mathcal{C}$ by $\mathbf{Hom}_{\mathcal{C}}(-,-)$, there exists a *tensor product* bifunctor $- \otimes - : \mathbf{sSet} \times \mathcal{C} \to \mathcal{C}$ with isomorphisms

$$\mathbf{Hom}_{\mathcal{C}}(K \otimes A, B) \cong \mathbf{Hom}_{\mathbf{sSet}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B))$$

natural in all three variables; $\mathcal{C}$ is *cotensored* if there exists a *cotensor product* bifunctor $(-)^{(-)} : \mathbf{sSet}^{op} \times \mathcal{C} \to \mathcal{C}$ with isomorphisms

$$\mathbf{Hom}_{\mathcal{C}}(A, B^K) \cong \mathbf{Hom}_{\mathbf{sSet}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B))$$

natural in all three variables.

sSet is tensored and cotensored, as is the case with any cartesian closed symmetric monoidal category enriched over itself by the internal hom (with the above definition modified in the obvious way) [**Riehl**, §I.3.7]: the tensor is the monoidal product and the cotensor is the exponential; thus the cotensor is the same as the mapping object. Note that this is a stronger statement, one at the level of enriched mapping objects, than that **sSet** is cartesian closed; the latter is a consequence derived by restricting an isomorphism of simplicial sets to the 0-simplices.

7. The nerve and fundamental category functors

Let us denote by **Cat** the category of small categories. Regarding each $[\mathbf{n}] \in \Delta$ as a category with a unique morphism $i \to j$ if and only if $i \leq j$, Δ is a full subcategory of **Cat**. For any small category $\mathcal{C}$, restricting the contravariant hom-functor $\mathbf{Cat}(\quad, \mathcal{C})$ to the subcategory Δ yields a simplicial set which is denoted by $N(\mathcal{C})$; thus we obtain a functor $N : \mathbf{Cat} \to \mathbf{sSet}$ called the *nerve*. The definition immediately implies that N is monoidal, i.e. $N(\mathcal{C} \times \mathcal{D}) \cong N\mathcal{C} \times N\mathcal{D}$: products in **sSet** are calculated level-wise, and

$$\begin{aligned} ((N\mathcal{C}) \times (N\mathcal{D}))_n &= (N\mathcal{C})_n \times (N\mathcal{D})_n \\ &\cong \mathbf{Cat}([\mathbf{n}], \mathcal{C}) \times \mathbf{Cat}([\mathbf{n}], \mathcal{D}) \\ &\cong \mathbf{Cat}([\mathbf{n}], \mathcal{C} \times \mathcal{D}) \\ &= (N(\mathcal{C} \times \mathcal{D}))_n. \end{aligned}$$

An element of $(N\mathcal{C})_m$ is a functor $[\mathbf{m}] \to \mathcal{C}$ and can be identified by its image which is a sequence

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} a_m$$

of m composable morphisms of $\mathcal{C}$. How are the face and degeneracy maps then represented? Considering, for brevity, a 2-simplex $x = a_0 \xrightarrow{f} a_1 \xrightarrow{g} a_2$ will make the general case clear: we have

$$\begin{aligned} d_0(x) &= a_1 \xrightarrow{g} a_2 \\ d_1(x) &= a_0 \xrightarrow{g \circ f} a_2 \\ d_2(x) &= a_0 \xrightarrow{f} a_1 \\ s_0(x) &= a_0 \xrightarrow{1_{a_0}} a_0 \xrightarrow{f} a_1 \xrightarrow{g} a_2 \\ s_1(x) &= a_0 \xrightarrow{f} a_1 \xrightarrow{1_{a_1}} a_1 \xrightarrow{g} a_2 \\ s_2(x) &= a_0 \xrightarrow{f} a_1 \xrightarrow{g} a_2 \xrightarrow{1_{a_2}} a_2 \, . \end{aligned}$$

As a simple example we have $N([\mathbf{m}]) = \Delta^m$.

Proposition 7.1. *The nerve functor is full and faithful.*

Proof. It is clear that N is faithful. That it is full follows from another useful property of $N\mathcal{C}$, for any category $\mathcal{C}$: it is 2-coskeletal, that is the unit map of the adjunction $tr_2 \dashv cosk_2$ is an isomorphism $N\mathcal{C} \cong Cosk_2 N\mathcal{C}$. This holds if and only if

$$\mathbf{sSet}(X, N\mathcal{C}) \cong \mathbf{sSet}(X, Cosk_2 N\mathcal{C}) \cong \mathbf{sSet}_{\leq 2}(tr_2 X, tr_2 N\mathcal{C});$$

in other words a simplicial map $X \to N\mathcal{C}$ is uniquely determined by its 2-truncation — its restriction to 0-, 1- and 2-simplices. But this is a consequence of the fact that any simplex $\Delta^m \to N\mathcal{C}$ is a sequence of morphisms, thus specified by its vertices and edges $tr_1\Delta^m \to tr_1N\mathcal{C}$, and on the other hand a set of objects and morphisms of $\mathcal{C}$ corresponding to $tr_1\Delta^m$ gives an m-simplex of $N\mathcal{C}$ if the image of a composite edge $i \to i+2$ is the composite of the images of $i \to i+1$ and $i+1 \to i+2$, which can be expressed as requiring that at the level of 2-simplices this assignment commutes with face maps. The claim that N is full is now easy to verify, since a map $tr_2N\mathcal{C} \to tr_2N\mathcal{D}$ being simplicial is simply a concise way of stating the conditions for having a functor. □

The nerve functor has a right adjoint $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ called the *fundamental category*, as noting the similarity between N and the singular functor from topological spaces would lead one to expect. Indeed, this left adjoint is given by the same general construction as used there: it is the left Kan extension of $\Delta \hookrightarrow \mathbf{Cat}$ along the Yoneda embedding $[\mathbf{n}] \mapsto \Delta^n$. We can describe $\tau_1 X$ directly in terms of the simplicial set X as follows. (In fact it will be seen that the fundamental category depends only on the 2-truncation of the simplicial set, which is compatible with the remarks in the preceding paragraph.)

Consider the directed graph with X_0 as its set of vertices and X_1 as edges, each $x \in X_1$ beginning at $d_1(x)$ and ending at $d_0(x)$, and let FX be the free category on this graph (see [**Mac Lane**, II.7]). Now let $\sim$ be the congruence on the morphisms of FX generated by $d_1(u) \sim d_0(u) \circ d_2(u)$ for every $u \in X_2$, and define $\tau_1 X = FX/\sim$. Thus a morphism $f \in \tau_1 X(a,b)$ can be represented by a path $x_1 \ldots x_n$ from a to b in the directed graph, and two such representations are equivalent if one can be obtained from the other by one or more applications of the congruence relation $\sim$. Note that the identity morphism of a can be represented by $s_0(a)$.

It is straightforward to verify that this construction yields an isomorphism $\mathbf{Cat}(\tau_1 X, \mathcal{C}) \cong \mathbf{sSet}(X, N\mathcal{C})$; we have already remarked that a map on the right is determined by its 2-truncation, which can be used to define the adjunct map on the left. Furthermore, the counit of the adjunction $\tau_1 \dashv N$ is an isomorphism, that is $\tau_1 N\mathcal{C} \cong \mathcal{C}$. (This can also be deduced from, in fact is equivalent to, the fact that N is full and faithful.)

The following lemma proves several important facts about the functors τ_1 and N. Note, in the statement of part (c), that we use the predicate 'directed' for the concept 'filtered' as defined in [**Mac Lane**, IX.2].

Lemma 7.2.

(a) τ_1 *preserves products:* $\tau_1(X \times Y) \cong \tau_1 X \times \tau_1 Y$.

(b) N preserves exponentials: $N(Z^Y) \cong (NZ)^{(NY)}$.

(c) N preserves directed colimits: if $\mathcal{D}$ is directed then $N \varinjlim_{\mathcal{D}} F \cong \varinjlim_{\mathcal{D}} NF$ for any $F : \mathcal{D} \to \mathbf{Cat}$.

Proof.

(a) Writing X and Y as colimits of representables, $X = \varinjlim_{\mathcal{I}} \Delta^m$ and $Y = \varinjlim_{\mathcal{J}} \Delta^n$, and using the fact that τ_1 and the product functors in **sSet** and **Cat**, being left adjoints, preserve colimits, we have

$$\begin{aligned}
\tau_1(X \times Y) &= \tau_1(\varinjlim_{\mathcal{I}} \Delta^m \times \varinjlim_{\mathcal{J}} \Delta^n) \cong \tau_1 \varinjlim_{\mathcal{I}}(\Delta^m \times \varinjlim_{\mathcal{J}} \Delta^n) \\
&\cong \tau_1 \varinjlim_{\mathcal{I}} \varinjlim_{\mathcal{J}}(\Delta^m \times \Delta^n) \cong \tau_1 \varinjlim_{\mathcal{I}\times\mathcal{J}} (\Delta^m \times \Delta^n) \\
&= \tau_1 \varinjlim_{\mathcal{I}\times\mathcal{J}} (N[\mathbf{m}] \times N[\mathbf{n}]) = \tau_1 \varinjlim_{\mathcal{I}\times\mathcal{J}} N([\mathbf{m}] \times [\mathbf{n}]) \\
&\cong \varinjlim_{\mathcal{I}\times\mathcal{J}} \tau_1 N([\mathbf{m}] \times [\mathbf{n}]) \cong \varinjlim_{\mathcal{I}\times\mathcal{J}} ([\mathbf{m}] \times [\mathbf{n}]) \\
&\cong \varinjlim_{\mathcal{I}} [\mathbf{m}] \times \varinjlim_{\mathcal{J}} [\mathbf{n}] = \varinjlim_{\mathcal{I}} \tau_1 \Delta^m \times \varinjlim_{\mathcal{J}} \tau_1 \Delta^n \\
&\cong \tau_1(\varinjlim_{\mathcal{I}} \Delta^m) \times \tau_1(\varinjlim_{\mathcal{J}} \Delta^n) \cong \tau_1 X \times \tau_1 Y.
\end{aligned}$$

(b) There are isomorphisms

$$\begin{aligned}
\mathbf{sSet}(X, N(Z^Y)) &\cong \mathbf{Cat}(\tau_1 X, Z^Y) \cong \mathbf{Cat}(\tau_1 X \times Y, Z) \\
&\cong \mathbf{Cat}(\tau_1 X \times \tau_1 NY, Z) \cong \mathbf{Cat}(\tau_1(X \times NY), Z) \\
&\cong \mathbf{sSet}(X \times NY, NZ) \cong \mathbf{Cat}(X, (NZ)^{NY})
\end{aligned}$$

natural in $X \in \mathbf{sSet}$, hence the claim follows from the Yoneda lemma.

(c) It is sufficient to establish isomorphisms $(N \varinjlim_{\mathcal{D}} F)_i \cong (\varinjlim_{\mathcal{D}} NF)_i$ for $0 \leq i \leq 2$. The 0-simplices of the simplicial set on the left are the objects of $\varinjlim_{\mathcal{D}} F$, which form the colimit of the object sets and functions in diagram F; this is the same as the set of 0-simplices on the right, since colimits in **sSet** are determined point-wise. An analogous reasoning applies to 1-simplices. As for 2-simplices, after pondering the colimit on the left one sees that what is needed is to show that if two morphisms $f \in F_\alpha$ and $g \in F_\beta$ represent the same morphism in $\varinjlim_{\mathcal{D}} F$ then $\bullet \xrightarrow{f} \bullet \xrightarrow{h} \bullet$ and

$\bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ represent the same element in $(\varinjlim_{\mathcal{D}} NF)_2$ whenever the composites are defined, and similarly for $\bullet \xrightarrow{k} \bullet \xrightarrow{f} \bullet$ and $\bullet \xrightarrow{k} \bullet \xrightarrow{g} \bullet$. Note that for any $\mathcal{C} \in \mathbf{Cat}$, $(N\mathcal{C})_2$ can be written as a pullback

$$\begin{tikzcd} (N\mathcal{C})_2 \arrow[r] \arrow[d] & (N\mathcal{C})_1 \arrow[d, "d_1"] \\ (N\mathcal{C})_1 \arrow[r, "d_1"'] & (N\mathcal{C})_0 \end{tikzcd} .$$

Therefore, since in **Set** finite limits and directed colimits commute [**Mac Lane**, Theorem IX.2.1],

$$\begin{aligned} (\varinjlim_{\mathcal{D}} NF)_2 &\cong \varinjlim_{\mathcal{D}} \big((NF_\alpha)_1 \times_{(NF_\alpha)_0} (NF_\alpha)_1\big) \\ &\cong \varinjlim_{\mathcal{D}} (NF_\alpha)_1 \times_{\varinjlim_{\mathcal{D}} (NF_\alpha)_0} \varinjlim_{\mathcal{D}} (NF_\alpha)_1 \\ &\cong N(\varinjlim_{\mathcal{D}} F)_1 \times_{N(\varinjlim_{\mathcal{D}} F)_0} N(\varinjlim_{\mathcal{D}} F)_1 \end{aligned}$$

from which the desired result follows.

$\square$

8. Fibrations and weak equivalences

(What follows is a brief overview of the homotopy theory of simplicial sets; for an in-depth study and proofs of most claims made here, see [**Goerss and Jardine**, Chapter 1] or [**Joyal and Tierney**, Chapter 3])

A map $f : X \to Y$ between simplicial sets is called a *Kan fibration*, or a fibration for short when the context of **sSet** is clear, if it has the right-lifting property against all horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$, $1 \leq n$, $0 \leq k \leq n$. This means that in every square commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

formed by the solid lines, a map, not necessarily unique, shown by the dotted arrow can be found rendering the entire diagram commutative. A simplicial set X is said to be *fibrant* or a *Kan complex* if the unique map $X \to *$ to the terminal object $* = \Delta^0$ of **sSet** is a fibration; otherwise stated, every map $\Lambda^n_k \to X$ can be extended to Δ^n.

Proposition 8.1. *$\mathcal{G}$ is a groupoid if and only if $N\mathcal{G}$ is fibrant.*

Proof. See [**Goerss and Jardine**, Lemma I.3.5] □

It is a simple exercise to show, directly from the definition, that the pullback (base change) of a fibration is again a fibration. Also, if $L \dashv R$ is an adjunction, then $R(g)$ has the right-lifting property against f if and only if g has this property against $L(f)$. One consequence of the latter fact is that if p is a Serre fibration in **Top** then $S(p)$ is a Kan fibration, and in particular the singular complex $S(Y)$ of a topological space Y is fibrant [**Goerss and Jardine**, Lemma I.3.3].

We define a reflexive relation on the set of 0-simplices of any $X \in \mathbf{sSet}$ by

$$y \sim z \qquad \text{if and only if} \qquad (\exists u \in X_1 \quad \text{with} \quad d_0 u = y \quad \text{and} \quad d_1 u = z)$$

and say y and z are *homotopic*. If X is a Kan complex this relation is symmetric and transitive [**Goerss and Jardine**, Lemma I.6.1]. In general, however, this is not the case and we consider the equivalence relation generated by the above definition. Then $y \sim z$ can be expressed also by saying that the points $y, z \in |X|$ are path-connected. The set of equivalence classes of 0-simplices under the homotopy relation is denoted by $\pi_0(X)$; these correspond to the path components of $|X|$. Another way to describe $\pi_0(X)$ is that it is the coequalizer of the two face maps $d_0, d_1 : X_1 \to X_0$, or equivalently the colimit of the diagram X in **Set**. It is not difficult to show that π_0 is right adjoint to the functor $\mathbf{Set} \to \mathbf{sSet}$ that sends a set S to the discrete simplicial set — that is, with all simplicial morphisms the identity — constant on S. We say X is *connected* if $\pi_0(X)$ is a singleton. The smallest sub-complex of X containing the simplices in an equivalence class is called a *component* of X. Each component is obviously connected, and X is the disjoint union of its *components.*

A very important result in the theory of simplicial sets is the following:

Theorem 8.2. *If $p : X \to Y$ is a fibration and $i : B \to A$ a monomorphism, then the dotted arrow in the diagram*

$$\begin{tikzcd}
X^A \arrow[dr, dashed] \arrow[drr, bend left, "p_*"] \arrow[ddr, bend right, "i^*"'] & & \\
& Y^A \times_{Y^B} X^B \arrow[r] \arrow[d] & Y^A \arrow[d, "i^*"] \\
& X^B \arrow[r, "p_*"'] & Y^B
\end{tikzcd}$$

given by the universal property of the pullback is also a fibration; it is also a weak equivalence — a notion introduced below — if p or i is.

Proof. [**Goerss and Jardine**, Proposition I.5.2] □

Letting Y be the terminal object $*$ and B the initial object $\emptyset$ of **sSet**, it follows that if X is a Kan complex then so is X^A for any simplicial set A. Thus the homotopy relation on $(X^A)_0 = \mathbf{sSet}(A, X)$ is an equivalence relation when X is fibrant. Explicitly, two simplicial maps $f, g : A \to X$ are *homotopic* — in the same component of X^A — if there is a map $h : A \times \Delta^1 \to X$ such that the diagram

$$\begin{tikzcd} A \times \Delta^0 \cong A \arrow[d, "1\times(d^1)_*"'] \arrow[dr, "f"] & \\ A \times \Delta^1 \arrow[r, "h"] & X \\ A \times \Delta^0 \cong A \arrow[u, "1\times(d^0)_*"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. We say that h is a *simplicial homotopy* from f to g and write $h : f \sim g$. If f has a *homotopy inverse*, that is there exists $j : X \to A$ such that $jf \sim 1_A$ and $fj \sim 1_X$, it is called a *simplicial homotopy equivalence*. For instance, if $\varphi \in \mathbf{Cat}(\mathcal{C}, \mathcal{D})$ is an equivalence then $N\varphi$ is a homotopy equivalence by the following lemma.

Lemma 8.3. *If there exists a natural transformation $\phi \to \psi$ between functors $\phi, \psi \in$ $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ then $N\phi, N\psi \in \mathbf{sSet}(N\mathcal{C}, N\mathcal{D})$ are homotopic.*

Proof. A natural transformation $\phi \to \psi$ is a morphism $[\mathbf{1}] \to \mathcal{D}^{\mathcal{C}}$ in the functor category, so it can also be written as a functor $\mathcal{C} \times [\mathbf{1}] \to \mathcal{D}$ that is ϕ when restricted to $\mathcal{C} \times \{0\}$ and ψ when restricted to $\mathcal{C} \times \{1\}$. Applying N and using the fact that it preserves products gives a homotopy $N\mathcal{C} \times \Delta^1 \to N\mathcal{D}$ from $N\phi$ to $N\psi$. □

Furthermore, denoting by $u \in (X^B)_0$ the constant map to the sub-complex (isomorphic to Δ^0) generated by $u \in X_0$, the right vertical map and hence also the left one in the pullback diagram

$$\begin{tikzcd} \mathbf{Hom}(A, X)/(B, u) \arrow[r] \arrow[d] & X^A \arrow[d, "i^*"] \\ * \arrow[r, "u"'] & X^B \end{tikzcd}$$

are fibrations. A 1-simplex $k : A \times \Delta^1 \to X$ of the Kan complex $\mathbf{Hom}(A, X)/(B, u)$ yields a *homotopy relative* B, one for which the restriction $k|_B : B \times \Delta^1 \hookrightarrow A \times \Delta^1 \to X$ is the constant map at u. Again, this is an equivalence relation.

We denote by $\pi_n(X, u)$, for $u \in X_0$ and $n \geq 1$, the set of homotopy classes of maps $\Delta^n \to X$ relative $u : \partial\Delta^n \to X$, i.e. $\pi_n(X, u) = \pi_0\big(\mathbf{Hom}(\Delta^n, X)/(\partial\Delta^n, u)\big)$. It is proved in [**Goerss and Jardine**, I.7] that $\pi_n(X, u)$ can be given a group structure, abelian if $n \geq 2$; so we call $\pi_n(X, u)$ the *nth simplicial homotopy group of* X *at vertex* u. Moreover, a simplicial map between Kan complexes $f : X \to Y$ gives group homomorphisms $f_* : \pi_n(X, u) \to \pi_n(Y, f(u))$.

A map $f \in \mathbf{sSet}(X, Y)$ is called a *weak equivalence* if $S|f| : S|X| \to S|Y|$ induces isomorphisms of simplicial homotopy groups for any vertex of X. Weak equivalences satisfy the 2 *out of* 3 property: given a composable pair of maps f and g, if any two members of $\{f, g, gf\}$ are weak equivalences then so is the third [**Goerss and Jardine**]. The remark above about lifting properties under adjunctions, together with the fact that the inclusion $|\partial\Delta^n| \to |\Delta^n|$ is homeomorphic to $S^{n-1} \hookrightarrow D^n$ in **Top**, imply that the simplicial homotopy groups of $S|X|$ can be naturally identified with the homotopy groups of $|X|$. Indeed, if X is fibrant the unit $\eta_X : X \to S|X|$ of the adjunction $| \,.\, | \dashv S$ is a weak equivalence; thus for maps between Kan complexes the preceding definition of a weak equivalence is equivalent to requiring the condition on $S|f|$ for f itself. Furthermore, it follows that (any) f is a weak equivalence if and only if $|f|$ is a weak homotopy equivalence, meaning that there are isomorphisms $\pi_*(f) : \pi_*(|X|, x) \to \pi_*(|Y|, f(x))$ for any base-point $x \in |X|$. (By Whitehead's theorem $|f|$ is then a homotopy equivalence, since $|X|$ and $|Y|$ are CW-complexes.)

The definition of weak equivalence given above is internal to **sSet**. In fact, it is possible to give a characterization of weak equivalences not involving the geometric realization functor at all: $f : X \to Y$ is a weak equivalence if and only if $f^* : K^Y \to K^X$ is a weak equivalence for every Kan complex K [**Joyal and Tierney**, Proposition 4.6.3]. (Since K^X and K^Y are fibrant it is sufficient to consider the action of f^* on simplicial homotopy groups.) Nevertheless, the connection between simplicial sets and topological spaces has been an important source of ideas in the study of the former, and there is a deep relationship between homotopy theories of the two categories: the adjunction $| \,.\, | \dashv S$ induces an equivalence of *homotopy categories*. Let us elaborate this statement a little further.

At a fundamental level, the homotopy theory of a (locally-small) category $\mathcal{C}$ with a notion of weak equivalence associated to some morphisms (including the isomorphisms) is concerned with the study of functors which are 'invariant' with respect to weak equivalences W, that is take these morphisms to isomorphisms. Thus a natural object to consider is the homotopy category $\mathrm{Ho}(\mathcal{C})$, the localization of $\mathcal{C}$ at W, also denoted by $L_W\mathcal{C}$ or $\mathcal{C}[W^{-1}]$. As the last notation suggests, $\mathrm{Ho}(\mathcal{C})$ is obtained by formally inverting the morphisms in W. This description of the homotopy category raises set-theoretic questions pertaining to whether, in general, the construction is possible within the category of *locally-small* categories; we will see, in discussing *model categories* in the next section, that in many cases of interest no such obstacles arise.

The definition of $\mathrm{Ho}(\mathcal{C})$ implies that a functor that is invariant with respect to weak equivalences has a unique factorization through the homotopy category (see **Proposition II.2.1**). We have seen that geometric realization preserves weak equivalences; therefore the composite functor

$$\mathbf{sSet} \xrightarrow{|\ .\ |} \mathbf{Top} \longrightarrow \mathrm{Ho}(\mathbf{Top})$$

induces a functor $\mathrm{Ho}(|\ .\ |) : \mathrm{Ho}(\mathbf{sSet}) \to \mathrm{Ho}(\mathbf{Top})$. It is not difficult to show that the singular functor S also has this property, hence yielding a functor $\mathrm{Ho}(S) : \mathrm{Ho}(\mathbf{Top}) \to \mathrm{Ho}(\mathbf{sSet})$.

The key fact is that the functors $\mathrm{Ho}(|\ .\ |)$ and $\mathrm{Ho}(S)$ are also adjoints. (**Proposition II.2.6** subsumes this under the stronger statement that $|\ .\ | \dashv S$ is a *Quillen adjunction*, conveying more detailed homotopy-theoretic information.) Once this is known, the observation that the unit $\eta_X : X \to S|X|$ and counit $\epsilon_Y : |SY| \to Y$ maps of the adjunction $|\ .\ | \dashv S$ are weak equivalences [**Hovey**, Theorem 3.6.7], and consequently project to natural isomorphisms in $\mathrm{Ho}(\mathbf{sSet})$ and $\mathrm{Ho}(\mathbf{Top})$ respectively, the equivalence of homotopy categories mentioned above can be concluded.

9. Pointed and reduced simplicial sets

A *pointed simplicial set* is an object of the comma category $\Delta^0 \downarrow \mathbf{sSet}$ of simplicial sets under Δ^0, denoted by $\mathbf{sSet}_*$; it is just a pair (X, x) with $x \in X_0$, a simplicial set with a *basepoint*. A morphism $X \to Y$ in $\mathbf{sSet}_*$ is a simplicial map that takes the basepoint of X to that of Y. The forgetful functor $\mathbf{sSet}_* \to \mathbf{sSet}$, that is the projection $(\Delta^0 \to X) \mapsto X$, has a left adjoint which 'freely' adds a basepoint to a simplicial set, $S \mapsto S_+ = S \coprod \Delta^0$.

Recall from §**I.6** that **sSet** is a tensored and cotensored simplicially enriched category. The above adjunction can be used to define for $\mathbf{sSet}_*$ the same structure [**Riehl**, Theorem I.3.7.11 & Example I.3.7.13]. For pointed simplicial sets $\Delta^0 \xrightarrow{x} X$ and $\Delta^0 \xrightarrow{y} Y$ the mapping simplicial set has n-simplices of

$$\mathbf{Hom}_{\mathbf{sSet}_*}(X,Y)_n = \mathbf{sSet}(X \times \Delta^n / x \times \Delta^n, Y)$$

and basepoint the constant map $X \xrightarrow{y} Y$. The tensor $X \otimes K$ for $K \in \mathbf{sSet}$ is $X \times K / x \times K$ — the sub-complex $x \times K$ identified to the basepoint — and the cotensor Y^K is given by $\mathbf{Hom}_{\mathbf{sSet}_*}(K_+, Y)$ with the constant map at y as basepoint. Note that the cotensor as a simplicial set — forgetting its basepoint — is isomorphic to that in **sSet**.

A simplicial set X is said to be *reduced* if it has a single 0-simplex. The subcategory of reduced simplicial sets is denoted by $\mathbf{sSet_0}$. The following result implies that the inclusion $\mathbf{sSet_0} \hookrightarrow \mathbf{sSet}$ induces an equivalence of homotopy categories, in the sense discussed above in relation to simplicial sets and topological spaces.

Lemma 9.1. *If $X \in \mathbf{sSet}$ is connected then there exists a reduced simplicial set X_r and a weak equivalence $X \to X_r$.*

Proof. There is a *fibrant approximation* to X, that is a weak equivalence $X \to K$ to a Kan complex K, which is therefore connected too; for example we can take $\eta_X : X \to S|X|$, the unit of the adjunction $|\,.\,| \dashv S$. Furthermore, K is homotopy equivalent to a Kan sub-complex $K' \subseteq K$ that is *minimal* [**Goerss and Jardine**, Proposition I.10.3], which means that two **distinct** n-simplices are not homotopic relative $\partial\Delta^n$. For 0-simplices this condition says that two different $x, y \in K'_0$ are not homotopic. But then, as K' is also connected, it must have a single 0-simplex. □

There is also an obvious embedding $\mathbf{sSet_0} \to \mathbf{sSet}_*$, identifying a reduced simplicial set with the pointed simplicial set with basepoint its unique 0-simplex. This embedding has a right adjoint E_1 called the *first Eilenberg sub-complex* [**May92**, Definition 8.3] — that is, $\mathbf{sSet_0}$ is a coreflective subcategory of $\mathbf{sSet}_*$: for $X \in \mathbf{sSet}_*$ with basepoint x, E_1X is defined by the pullback

$$\begin{tikzcd} E_1X \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^0 \arrow[r, "x"'] & Cosk_0X \end{tikzcd}$$

(the bottom arrow maps to the vertex x of $(Cosk_0 X)_0 = X_0$, and the right arrow is the counit of the adjunction $tr_n \dashv cosk_n$) and can be seen to be the sub-complex of X consisting of those simplices with 0-skeleton the basepoint x.

There is a functor $G : \mathbf{sSet_0} \to \mathbf{sGrp}$, called the *Kan loop group*, from reduced simplicial sets to the category of simplicial objects in groups. It has a right adjoint $\overline{W} : \mathbf{sGrp} \to \mathbf{sSet_0}$, related to the bar construction. Let us say a map of simplicial groups is a weak equivalence if it is so as a morphism in **sSet**. An important fact then is that $G : \mathbf{sSet_0} \rightleftarrows \mathbf{sGrp} : \overline{W}$ induces an equivalence of homotopy categories. As with the adjunction $| \, . \, | : \mathbf{sSet} \to \mathbf{Top}$, this equivalence can be expressed in a more precise and informative statement in terms of *Quillen model structures* (**Proposition II.2.6**).

We do not include details of the definitions of the functors $\overline{W}$ and G here, but will mention and use their significant homotopy-theoretic properties in several core results. For a thorough discussion, the reader can refer to [**Goerss and Jardine**, §V]; [**Stevenson**] also provides an illuminating point of view.

Chapter 2

Model categories and localization

This chapter provides a concise review of important concepts and results from the theory of model categories that will be used in subsequent parts. Our aim being to record the facts for later reference, most are stated without proof and at a level of generality adequate for our purposes. More comprehensive treatments of these topics and in-depth arguments can be found in the standard textbooks on the subject, [**Hirschhorn**] and [**Hovey**].

1. Model categories

Definition 1.1. A *Quillen model structure* on a category $\mathcal{M}$ that is small complete and cocomplete consists of three classes of morphisms of $\mathcal{M}$, called *cofibrations*, *fibrations* and *weak equivalences*, such that

1. Weak equivalences satisfy the 2 *out of* 3 property: given a composable pair of maps f and g, if any two members of $\{f, g, gf\}$ are weak equivalences then so is the third.

2. Each of the three classes of morphisms is closed under retracts in the arrow category $\mathcal{M}^{\cdot \to \cdot}$; this means that if in the commutative diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & c \arrow[r] \arrow[d, "g"'] & a \arrow[d, "f"] \\ b \arrow[r] & d \arrow[r] & b \end{tikzcd}$$

the composites of the horizontal maps are identities, then if g is a cofibration, fibration or weak equivalence f is too.

3. Every morphism in $\mathcal{M}$ can be factorized as a cofibration followed by a fibration that is also a weak equivalence; and also as a cofibration that is also a weak equivalence

followed by a fibration.

4. In every square commutative diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d, "j"'] & c \arrow[d, "p"] \\ b \arrow[r] \arrow[ur, dashed] & d \end{tikzcd}$$

formed by the solid lines, with j a cofibration and p a fibration, if j or p is also a weak equivalence then a map represented by the dotted arrow can be found rendering the entire diagram commutative.

A morphism that is a cofibration, or a fibration, and a weak equivalence is called, respectively, a *trivial cofibration* or a *trivial fibration.* An object $A \in \mathcal{M}$ is called *cofibrant* if the unique map $\emptyset \to A$ from the initial object is a cofibration; it is called *fibrant* if the map $A \to *$ to the terminal object is a fibration.

A small complete and cocomplete category with a Quillen model structure is called a *model category.*

Axiom 4 can be phrased concisely by saying cofibrations have the left lifting property against trivial fibrations, as do trivial cofibrations against fibrations.

The following consequences of these axioms are easy to verify [**Hirschhorn**, §7.2].

- Each class of morphisms in a model structure includes all isomorphisms.
- The class of cofibrations is closed under pushouts and coproducts.
- The class of fibrations is closed under pullbacks and products.
- A map is a cofibration or a trivial cofibration if and only if it has the left lifting property against, respectively, all trivial fibrations or all fibrations.
- A map is a fibration or a trivial fibration if and only if it has the right lifting property against, respectively, all trivial cofibrations or all cofibrations.

The last two statements imply, in particular, that if a model structure is known to exist it can be presented by identifying only the cofibrations and weak equivalences, fibrations then being determined by their right lifting property against trivial cofibrations; and similarly for fibrations and weak equivalences. Dually, it is also possible to specify only the cofibrations and the fibrations, or the cofibrations and trivial cofibrations, or the fibrations and trivial fibrations.

Example 1.1

(a) The category **sSet** of simplicial sets with weak equivalences and (Kan) fibrations as defined in §**I.8** is a model category. Cofibrations are those maps that have the left lifting property against all trivial fibrations. Note that since Kan fibrations are characterized by their lifting property against horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$, $1 \leq n$, $0 \leq k \leq n$, the latter are trivial cofibrations.

It can be shown that a morphism in **sSet** is a trivial fibration if and only if it has the right lifting property against all boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, and hence cofibrations are the same as monomorphisms [**Goerss and Jardine**, Theorem I.11.2],[**Joyal and Tierney**, Example 2.4.3]. Therefore in this model structure every simplicial set is cofibrant.

(b) There is a model structure on the category **Top** of topological spaces in which the weak equivalences are weak homotopy equivalences and the fibrations are Serre fibrations [**Dwyer and Spalinski**, §8]. In this model structure every space is fibrant. Cofibrations are those maps that have the left lifting property against all trivial fibrations. Since Serre fibrations are characterized by having the right lifting property against all inclusions $|\Lambda^n_k| \hookrightarrow |\Delta^n|$, $1 \leq n$, $0 \leq k \leq n$, the latter are trivial cofibrations.

By the adjunction $|\ .\ | \dashv S$ and the example (a) a map of topological spaces is a trivial fibration if and only if it has the right lifting property against $|\partial\Delta^n| \hookrightarrow |\Delta^n|$, $n \geq 0$ [**Hirschhorn**, Proposition 7.2.17].

(c) [**Rezk**] There is a model structure on **Cat**, the category of (small) categories, in which the weak equivalences are categorical equivalences and cofibrations are functors that are injective on objects. Fibrations are *isofibrations*, functors which have the right lifting property against $0 \hookrightarrow \mathcal{J}$, where $\mathcal{J} = (0 \rightleftarrows 1)$ is the groupoid with a unique non-identity isomorphism. Trivial fibrations are determined by having the right lifting property against $\emptyset \to 0$, $\{0,1\} \hookrightarrow \mathcal{J}$ and $(0 \rightrightarrows 1) \to \mathcal{J}$.

(d) The category $\mathbf{sSet_0}$ of reduced simplicial sets with weak equivalences and cofibrations as in **sSet** is a model category [**Quillen69**, Theorem 2.2][**Goerss and Jardine**, §V.6]. Fibrations are those maps that have the right lifting property against all trivial cofibrations.

A model category $\mathcal{M}$ is said to be **cofibrantly generated** if, as in the first three examples above, there exist sets $\mathcal{I}$ of *generating cofibrations* and $\mathcal{J}$ of *generating trivial cofibrations* which determine, respectively, trivial fibrations and fibrations by their right lifting property. Cofibrantly generated model categories have many important properties, for instance in relation to the homotopy theory of diagrams [**Hirschhorn**, Theorem 11.6.1 & §15.6]. Their significance was highlighted by Daniel Quillen in his original presentation and application of the concept of model structures [**Quillen67**][**Quillen69**]. In view of the central importance of the categories **sSet** and **Top** the following result is particularly useful.

Theorem 1.2. *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations $\mathcal{I}$ and generating trivial cofibrations $\mathcal{J}$. Let $\mathcal{N}$ be a small complete and cocomplete category, and $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ a pair of adjoint functors such that the right adjoint G preserves directed colimits and takes transfinite composites of pushouts of $F\mathcal{J} = \{Fu : u \in \mathcal{J}\}$ to weak equivalences. Then there is a model structure on $\mathcal{N}$ with weak equivalences and fibrations those maps that are taken by G to weak equivalences and fibrations in $\mathcal{M}$ respectively. Moreover with this model structure $\mathcal{N}$ is cofibrantly generated with generating cofibrations $F\mathcal{I}$ and generating trivial cofibrations $F\mathcal{J}$.*

Proof. This is a consequence of [**Hirschhorn**, Theorem 11.3.2]. See also [**Goerss and Jardine**, Theorem II.4.1]. □

Let us record, for later reference, two cases of application of this theorem.

Corollary 1.3.

*(i) [**Quillen67**, Theorem II.3.2] The category* **sGrp** *of simplicial groups is a cofibrantly generated model category in which a morphism is a weak equivalence or a fibration if it is so as a map of simplicial sets. There ar e generating cofibrations $F\partial\Delta^n \to F\Delta^n$, $0 \leq n$, and generating trivial cofibrations $F\Lambda^n_k \to F\Delta^n$, $1 \leq n$, $0 \leq k \leq n$, where $F :$* **sSet** $\to$ **sGrp** *is the free group functor (acting level-wise). Moreover, in this model structure every simplicial group is fibrant.*

(ii) There is a model structure on the category $\mathbf{sSet}_*$ *of pointed simplicial sets in which a morphism is a weak equivalence or a fibration if it is so as a map of simplicial sets.*

Proof.

(i) The (level-wise) forgetful functor $U : \mathbf{sGrp} \to \mathbf{sSet}$ is right adjoint to F and satisfies the conditions of **Theorem 1.2**. See [**May92**, Theorem 17.1] for a proof of the fact that every simplicial group, as a simplicial set, is a Kan complex.

(ii) Refer to §**I.9**.

□

2. The homotopy category, Quillen adjunctions and Quillen equivalences

In §**I.8** the notion of the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of a category $\mathcal{C}$ with a class $\mathcal{W}$ of morphisms called weak equivalences, satisfying the 2-out-of-3 property and containing all isomorphisms, was introduced; it is the localization $\mathcal{C}[\mathcal{W}^{-1}]$ of $\mathcal{C}$ at $\mathcal{W}$, obtained by formally adding inverses for morphisms in $\mathcal{W}$. Thus the functor $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ that takes each morphism to its representative in the homotopy category, called the *localization functor*, takes $\mathcal{W}$ to isomorphisms. This description makes it evident that the homotopy category can also be characterized by the following universal property (stated in terms of model categories).

Proposition 2.1. *Let $\mathcal{M}$ be a model category and denote by $\mathcal{W}$ its class of weak equivalences. The homotopy category $\mathrm{Ho}(\mathcal{M}) = \mathcal{M}[\mathcal{W}^{-1}]$, the localization of $\mathcal{M}$ at weak equivalences, together with the localization functor $\gamma_{\mathcal{M}} : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$, is initial in the full subcategory of $\mathcal{M} \downarrow \mathbf{Cat}$ consisting of functors $F : \mathcal{M} \to \mathcal{D}$ that take $\mathcal{W}$ to isomorphisms; that is, given any such F there exists a unique $\widetilde{F} : \mathrm{Ho}(\mathcal{M}) \to \mathcal{D}$ such that $\widetilde{F}\gamma_{\mathcal{M}} = F$.*

In §**I.8** it was mentioned that describing the homotopy category as the formal localization at the class of weak equivalences raises the question whether it is a locally-small category. But it is implicit in the statement of the above proposition, specifically in viewing $\mathrm{Ho}(\mathcal{M})$ as an object in $\mathcal{M} \downarrow \mathbf{Cat}$, that no set-theoretic problems arise in constructing the homotopy category of a model category. This is indeed the case. In fact, the conceptual framework of a Quillen model structure provides a more concrete perspective on the homotopy category that enables a methodical approach to questions such as the image of maps under localization, or when functors between model categories allow their homotopy categories to be compared. Significantly, these techniques often involve the notion of homotopy, which can be formulated in any model category generalizing the ideas familiar from examples such as **sSet** and **Top**:

Definition 2.2. Let $\mathcal{M}$ be a model category and $X, Y \in \mathcal{M}$.

1. An object $\widetilde{X} \in \mathcal{M}$ is called a *cylinder object for* X if there exists a factorization

$$X \amalg X \xrightarrow{i} \widetilde{X} \xrightarrow{r} X$$

of the codiagonal map $\nabla : X \amalg X \to X$, with i a cofibration and r a weak equivalence.

Two maps $f, g \in \mathcal{M}(X, Y)$ are said to be *left homotopic* if $f \amalg g : X \amalg X \to Y$ can be factorized through a cylinder object:

$$X \amalg X \xrightarrow{i} \widetilde{X} \xrightarrow{h} Y \ ;$$

h is then called a *left homotopy* from f to g.

2. An object $\widehat{Y} \in \mathcal{M}$ is called a *path object for* Y if there exists a factorization

$$Y \xrightarrow{s} \widehat{Y} \xrightarrow{p} Y \times Y$$

of the diagonal map $\Delta : Y \to Y \times Y$, with p a fibration and s a weak equivalence.

Two maps $f, g \in \mathcal{M}(X, Y)$ are said to be *right homotopic* if $f \times g : Y \to Y \times Y$ can be factorized through a path object:

$$Y \xrightarrow{k} \widehat{Y} \xrightarrow{p} Y \times Y \ ;$$

k is then called a *right homotopy* from f to g.

Note that by the factorization axiom for a model structure (**Definition 1.1**) every object has both a cylinder object and a path object.

The term 'cylinder object' is motivated, of course, by the archetypical construction of the cylinder $S \times |\Delta^1|$ over a topological space S. (Similarly, the space $S^{|\Delta^1|}$ of paths in S is the model for the definition of path objects.) For another instance, for any $X \in$ **sSet** the codiagonal map $\nabla : X \amalg X \to X$ can be factorized as

$$X \amalg X \cong X \times \partial\Delta^1 \longrightarrow X \times \Delta^1 \longrightarrow X$$

where the first map is induced by the inclusion $\iota : \partial\Delta^1 \to \Delta^1$, and the second is the projection of the product and a weak equivalence since $|X|$ is a deformation retract of $|X| \times |\Delta^1|$. Therefore $X \times \Delta^1$ is a cylinder object for X. It is in terms of this cylinder object that homotopy of simplicial maps is usually defined, as was done in §**I.8**.

If $X \in \mathbf{sSet}$ is a Kan complex then the diagonal map $\Delta : X \to X \times X$ can be factorized as

$$X \cong X^{\Delta^0} \xrightarrow{(s^0)^*} X^{\Delta^1} \xrightarrow{\iota^*} X^{\partial\Delta^1} \cong X \times X$$

with the second map a fibration by **Theorem I.8.2**, and the first map a weak equivalence since it is right inverse to $(d^0)^*$ which is a trivial fibration by the same theorem. Thus X^{Δ^1} is a path object X.

Proposition 2.3. *Let $\mathcal{M}$ be a model category and $X, Y \in \mathcal{M}$.*

1. *Left homotopy is an equivalence relation on $\mathcal{M}(X, Y)$ if X is cofibrant.*
2. *Right homotopy is an equivalence relation on $\mathcal{M}(X, Y)$ if Y is fibrant.*
3. *If both X is cofibrant and Y is fibrant then $f, g \in \mathcal{M}(X, Y)$ are left homotopic if and only if they are right homotopic; thus we may unambiguously speak of the homotopy relation on $\mathcal{M}(X, Y)$. Moreover, two maps in $\mathcal{M}(X, Y)$ that are homotopic are left homotopic for any choice of cylinder object for X and right homotopic for any choice of path object for Y.*

Proof. [**Hirschhorn**, Lemma 7.4.2, Proposition 7.4.5 & Proposition 7.4.7] □

The fact that homotopy of simplicial maps into a Kan complex is an equivalence relation can now be seen as a special case of part (3) of this proposition. (Note that although every simplicial set is cofibrant this result cannot be deduced from part (1) — the fibrancy condition cannot be removed — since that statement does not guarantee that transitivity of the left homotopy relation holds for the same cylinder object; that is, it is not necessarily the case that if $h_1 : X \times \Delta^1 \to Y$ is a homotopy from f to g and $h_2 : X \times \Delta^1 \to Y$ is a homotopy from g to h, then homotopy f is homotopic to h *also by a map from* $X \times \Delta^1$).

Given an object X in a model category an object QX is called a *cofibrant approximation* to X if QX is cofibrant and there is a weak equivalence $QX \to X$. Such a QX always exists, for instance by the cofibration-trivial fibration factorization of the unique map $\emptyset \to X$ from the initial object. Dually, an object RY is called a *fibrant approximation* to Y if RY is fibrant and there is a weak equivalence $RY \to Y$.

Theorem 2.4. *Let $\mathcal{M}$ be a model category and $\gamma_{\mathcal{M}} : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ its localization functor. The homotopy category* $\mathrm{Ho}(\mathcal{M})$ *of $\mathcal{M}$ is locally-small and the* set *of maps*

$\mathrm{Ho}(\mathcal{M})(\gamma_{\mathcal{M}}X, \gamma_{\mathcal{M}}Y)$ can be naturally identified with the set $\mathcal{M}(QX, RY)/\sim$ of homotopy equivalence classes of maps between a cofibrant approximation to X and a fibrant approximation to Y.

Proof. [**Hovey**, Theorem 1.2.10] □

Definition 2.5. An adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ — F left adjoint to G — between model categories is called a *Quillen adjunction*, F a *left Quillen* and G a *right Quillen* functor, if F preserves cofibrations and G preserves fibrations. Equivalently, F preserves cofibrations and trivial cofibrations; or, G preserves fibrations and trivial fibrations [**Hovey**, Lemma 1.3.4].

Proposition 2.6.

1. *Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be an adjunction between model categories such that the right adjoint G satisfies the conditions stated in **Theorem 1.2**, and N has the model structure described there. Then $F \dashv G$ is a Quillen adjunction.*

2. *$|\,.\,| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : S$ is a Quillen adjunction.*

3. *$G : \mathbf{sSet_0} \rightleftarrows \mathbf{sGrp} : \overline{W}$ is a Quillen adjunction.*

Proof.

1. G preserves fibrations and trivial fibrations by the definition of the model structure on $\mathcal{N}$.

2. [**Hovey**, Theorem 3.6.7]

3. [**Goerss and Jardine**, Proposition V.6.3]

□

If $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ is a Quillen adjunction and every object of $\mathcal{M}$ is cofibrant then it can be shown that F preserves weak equivalences [**Hirschhorn**, Corollary 7.7.2]. Hence, denoting by $\delta_{\mathcal{N}}$ the localization functor of $\mathcal{N}$, $\delta_{\mathcal{N}}F$ takes weak equivalences in $\mathcal{M}$ to isomorphisms. By the universal property of $\mathrm{Ho}(\mathcal{M})$ (**Proposition 2.1**) $\delta_{\mathcal{N}}F$ induces a functor $\widetilde{F} : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$. Dually, if every object of $\mathcal{N}$ is fibrant there is an induced functor $\widetilde{G} : \mathrm{Ho}(\mathcal{N}) \to \mathrm{Ho}(\mathcal{M})$. The adjunction $F \dashv G$ is called a *Quillen equivalence* if $\widetilde{F}$ and $\widetilde{G}$ yield an adjoint equivalence.

Theorem 2.7. *Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen adjunction, with every object of $\mathcal{M}$ cofibrant and every object of $\mathcal{N}$ fibrant, and denote by $\gamma_{\mathcal{M}}$ and $\delta_{\mathcal{N}}$ the localization functors of $\mathcal{M}$ and $\mathcal{N}$ respectively. Then $\widetilde{F} : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N}) : \widetilde{G}$ is an adjunction. Thus there is a natural isomorphism*

$$\mathrm{Ho}(\mathcal{N})(\delta_{\mathcal{N}}FX, \delta_{\mathcal{N}}Y) \cong \mathrm{Ho}(\mathcal{M})(\gamma_{\mathcal{M}}X, \gamma_{\mathcal{M}}GY).$$

If, further, for every $X \in \mathcal{M}$ and $Y \in \mathcal{N}$ a map $FX \to Y$ is a weak equivalence in $\mathcal{M}$ if and only if its transpose $X \to GY$ is a weak equivalence in $\mathcal{N}$ — or, equivalently, both the unit map $X \to GFX$ and counit map $FGY \to Y$ are weak equivalences — then the adjunction $\widetilde{F} \dashv \widetilde{G}$ is an equivalence of categories, that is $F \dashv G$ is a Quillen equivalence.

Proof. [**Hovey**, Proposition 1.3.13] □

Corollary 2.8.

1. $|\ .\ | : \mathbf{sSet} \rightleftarrows \mathbf{Top} : S$ *is a Quillen equivalence.*

2. $G : \mathbf{sSet_0} \rightleftarrows \mathbf{sGrp} : \overline{W}$ *is a Quillen equivalence.*

Proof.

1. [**Hovey**, Theorem 3.6.7]

2. [**Goerss and Jardine**, Proposition V.6.3]

□

3. Left Bousfield localization

In studying model categories and functors between them it is often useful to localize at a class of maps in addition to weak equivalences; that is, one is interested in understanding functors from a model category that, as well as taking weak equivalences to isomorphisms, are 'invariant' with respect to other maps. The particular case in which there is a single map will be a key part of the argument in the next section. We now proceed to discuss how, in many cases, it is possible to define a new model structure such that the associated homotopy category has the universal property of localization at a map f.

Definition 3.1. Let $\mathcal{M}$ be a model category in which every object is cofibrant and f a map in $\mathcal{M}$. A *left localization of $\mathcal{M}$ with respect to f* is a model category $\mathrm{L}_f\mathcal{M}$ together with a left Quillen functor (**Definition 2.5**) $j : \mathcal{M} \to \mathrm{L}_f\mathcal{M}$ such that the induced functor $\widetilde{j} : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathrm{L}_f\mathcal{M})$ takes the image in $\mathrm{Ho}(\mathcal{M})$ of f to an isomorphism in $\mathrm{Ho}(\mathrm{L}_f\mathcal{M})$, and j is initial among left Quillen functors with this property. That is, if $\mathcal{N}$ is a model category and $F : \mathcal{M} \to \mathcal{N}$ a left Quillen functor such its induced functor takes the image in $\mathrm{Ho}(\mathcal{M})$ of f to an isomorphism in $\mathrm{Ho}(\mathcal{N})$ then there exists a unique left Quillen functor $\varphi : \mathrm{L}_f\mathcal{M} \to \mathcal{N}$ such that $\varphi j = F$.

To define $L_f\mathcal{M}$ we need to introduce simplicial resolutions and right homotopy function complexes.

Definition 3.2. Let $\mathcal{M}$ be a model category. A *simplicial resolution* of an object X is a simplicial object $sr(X) \in \mathcal{M}^{\Delta^{op}}$ such that

1. $sr(X)_0$ is a fibrant approximation to X;
2. All the structure maps of $sr(X)$ are weak equivalences; and
3. Denoting by $M_n sr(X)$ the limit of the diagram indexed by the category of elements of $\partial\Delta^n$ — *as a covariant functor from* Δ^{op} — the canonical map $sr(X)_n \to M_n sr(X)$ is a fibration for every $n \geq 0$. $M_n sr(X)$ is called the *matching object* and the map $sr(X)_n \to M_n sr(X)$ the *matching map* of $sr(X)$.

Let us elaborate the third condition in this definition in the case $\mathcal{M}$ is **sSet** — the explanation also holds also for $\mathbf{sSet}_*$ or $\mathbf{sSet_0}$, since limits in these categories are the same as in **sSet**. A simplicial object Z' in **sSet** can be viewed as a bisimplicial set $Z_{\bullet,\bullet}$ that is a 'vertical' object in simplicial sets, i.e. with $Z_{m,n} = (Z'_n)_m$. Then $M_n Z'$ is the subcomplex of Z'_{n-1} consisting in each degree m of $(n+1)$-tuples of simplices $z_0, \ldots, z_n \in Z_{n,m}$ which *match* under the vertical face maps, $d^v_i(z_j) = d^v_{j-1}(z_i)$ for $i < j$. The matching map $Z_{\bullet,n} \to Z_{\bullet,n-1}$ sends a simplex in $Z_{m,n}$ to the $(n+1)$-tuple of its vertical faces in $Z_{m,n-1}$. Another way to describe the matching map is that in degree m it is isomorphic to the map

$$\mathbf{sSet}(\Delta^n, Z_{m,\bullet}) \to \mathbf{sSet}(\partial\Delta^n, Z_{m,\bullet})$$

induced by the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$.

One situation in which there is a natural candidate for a simplicial resolution of each object is when the model category $\mathcal{M}$ is simplicially enriched, tensored, cotensored (§**I.6**), and satisfies the following condition familiar from **sSet**:

Definition 3.3. A model category $\mathcal{M}$ that is simplicially enriched, tensored and cotensored is called a *simplicial model category* if, for $p : X \to Y$ a fibration in $\mathcal{M}$ and $i : B \to A$ a monomorphism of simplicial sets, the dotted arrow in the diagram

$$\begin{tikzcd}
X^A \arrow[dr, dashed] \arrow[drr, bend left, "p_*"] \arrow[ddr, bend right, "i^*"'] & & \\
& Y^A \times_{Y^B} X^B \arrow[r] \arrow[d] & Y^A \arrow[d, "i^*"] \\
& X^B \arrow[r, "p_*"'] & Y^B
\end{tikzcd}$$

given by the universal property of the pullback is a fibration that is also a weak equivalence if p or i is.

Example 3.1 By **Theorem I.8.2 sSet** is a simplicial model category. Recall that the category $\mathbf{sSet}_*$ is also simplicially enriched, tensored and cotensored, and the cotensor, as a simplicial *set*, is the same as in **sSet**. Since a map of pointed simplicial sets is a fibration or a weak equivalence if it is so as a morphism in **sSet**, it follows that $\mathbf{sSet}_*$ too is a simplicial model category.

If $\mathcal{M}$ is a simplicial model category and $Y \in \mathcal{M}$, consider the simplicial object $(\widehat{Y}^{\Delta})_n = \widehat{Y}^{\Delta^n}$ where $\widehat{Y}$ is a fibrant approximation to Y. The first condition in **Definition 3.2** is then satisfied since $\widehat{Y}^{\Delta^0} \cong \widehat{Y}$. As for the second condition we note that each face inclusion $d^i : \Delta^{n-1} \to \Delta^n$ is a trivial cofibration of simplicial sets and so $(d^i)^* : \widehat{Y}^{\Delta^n} \to \widehat{Y}^{\Delta^{n-1}}$ is a trivial fibration (let $Y = *$ in **Definition 3.3**). The degeneracy map $(s^i)^* : \widehat{Y}^{\Delta^n} \to \widehat{Y}^{\Delta^{n+1}}$ is a weak equivalence since it is right inverse to $(d^i)^*$. It is not difficult to verify, using the fact that $\widehat{Y}^{(-)}$ is a right adjoint and hence preserves limits, that $M_n\widehat{Y}^{\Delta} \cong \widehat{Y}^{\partial\Delta^n}$ and the matching map is isomorphic to $\widehat{Y}^{\Delta^n} \to \widehat{Y}^{\partial\Delta^n}$ obtained from the cofibration $\partial\Delta^n \hookrightarrow \Delta^n$, therefore is a fibration. We conclude that $\widehat{Y}^{\Delta}$ is a simplicial resolution of Y.

Definition 3.4. Let $\mathcal{M}$ be a model category. A *functorial right homotopy function complex* on $\mathcal{M}$ is a functor $\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}^{op} \times \mathcal{M}^{\Delta^{op}}$, $(X, Y) \mapsto (\widetilde{X}, sr(Y))$, such that $\widetilde{X}$ is a cofibrant approximation to X and $sr(Y)$ is a simplicial resolution of Y for every $X, Y \in \mathcal{M}$. The simplicial set $\mathcal{M}(\widehat{X}, sr(Y))$, with n-simplices $\mathcal{M}(\widehat{X}, sr(Y)_n)$, is denoted by $Map^r(X, Y)$ and called a *right homotopy function complex from X to Y*. A map $(f, g) : (X, Y) \to (X', Y')$ in $\mathcal{M}^{op} \times \mathcal{M}$ defines a simplicial map $Map^r(X, Y) \to Map^r(X', Y')$ of right homotopy function complexes, taking $h \in \mathcal{M}(\widehat{X}, sr(Y)_n)$ to $sr(g)_n \circ h \circ f$ where $sr(g) : sr(Y) \to sr(Y')$ is the induced map of simplicial resolutions.

Definition 3.5. Let $\mathcal{M}$ be a model category with a functorial right homotopy function complex, and $f : A \to B$ a map in $\mathcal{M}$.

- An object $W \in \mathcal{M}$ is f*-local* if W is fibrant and the map of right homotopy function complexes $Map^r(B, W) \to Map^r(A, W)$ induced by f is a weak equivalence.
- A map $g : X \to Y$ is an f*-local equivalence* if for every f-local object W the map of right homotopy function complexes $Map^r(Y, W) \to Map^r(X, W)$ induced by g is a weak equivalence.

Theorem 3.6. *Let $\mathcal{M}$ be one of the model categories* **sSet**, $\mathbf{sSet}_*$ *or* $\mathbf{sSet_0}$, *with a functorial right homotopy function complex, and f a map in $\mathcal{M}$. There is a model category structure $\mathrm{L}_f\mathcal{M}$ on the underlying category of $\mathcal{M}$, called the* left Bousfield localization of $\mathcal{M}$ with respect to f, *in which a morphism is a*

1. weak equivalence *if it is an f-local equivalence;*
2. cofibration *if it is a cofibration in $\mathcal{M}$;*
3. fibration *it it has the right lifting property against those morphisms in $\mathcal{M}$ that are both cofibrations and f-local equivalences.*

*Moreover, $\mathrm{L}_f\mathcal{M}$ together with the identity functor $1_{\mathcal{M}} : \mathcal{M} \to \mathrm{L}_f\mathcal{M}$ is a left localization of $\mathcal{M}$ with respect to f (**Definition 3.1**).*

Proof. See [**Hirschhorn**, Theorem 3.3.19 & Theorem 4.1.1] for a more general result; as the technical notions needed have not been discussed in this work the statement of the theorem here is narrower, but adequate for our purposes. □

Theorem 3.7. *Let $\mathcal{M}$ be one of the model categories* **sSet**, $\mathbf{sSet}_*$ *or* $\mathbf{sSet_0}$, *with a functorial right homotopy function complex, f a map in $\mathcal{M}$, and $\mathrm{L}_f\mathcal{M}$ the left Bousfield localization of $\mathcal{M}$ with respect to f. If $F : \mathcal{M} \rightleftarrows \mathcal{N}$ is a Quillen adjunction such that G takes every fibrant object of $\mathcal{N}$ to an f-local object of $\mathcal{M}$, then F is a left Quillen functor when considered as a functor $\mathrm{L}_f\mathcal{M} \to \mathcal{N}$.*

Proof. [**Hirschhorn**, Propositions 3.1.6 & 3.3.18] □

Chapter 3

Internal categories and homotopy types

1. Model structure on categorical groups

An **internal group** or **group object** in a category $\mathcal{C}$ with finite products — including the empty product, a terminal object — is an object $G \in \mathcal{C}$ with a *multiplication* map $\mu : G \times G \to G$ and a *unit* map from the terminal object $\iota : * \to G$ which satisfy the usual properties of groups (group objects in **Set**) familiar from algebra; for instance the diagram

$$\begin{tikzcd} G \arrow[r, "\cong"] \arrow[drr, "1_G"'] & * \times G \arrow[r, "\iota \times 1_G"] & G \times G \arrow[d, "\mu"] \\ & & G \end{tikzcd}$$

must commute — in an ordinary group stating that the unit object acts on the left as the identity function. Internal groups in $\mathcal{C}$ are the objects of a category denoted by $\mathbf{Cat}(\mathcal{C})$, with $\mathbf{Cat}(\mathcal{C})(G, H)$ consisting of those morphisms in $\mathcal{C}(G, H)$ which commute with the structural maps of G and H, again as with groups in **Set**.

Recall (§**I.7**) that the fundamental category and nerve functors $\tau_1 : \mathbf{sSet} \rightleftarrows \mathbf{Cat} : N$ are adjoints, preserve products, and N is full and faithful. The adjunction $\tau_1 \dashv N$ restricts to an adjunction between the subcategories of simplicial groups and categorical groups — group objects in **Cat** — denoted by $\mathbf{Cat}(\mathbf{Grp})$. with N still full and faithful:

Lemma 1.1. *If $L \dashv R : \mathcal{C} \rightleftarrows \mathcal{D}$ are product-preserving functors between categories with finite products then there is an induced adjunction $L \dashv R : \mathbf{Grp}(\mathcal{C}) \rightleftarrows \mathbf{Grp}(\mathcal{D})$ between* *the corresponding categories of internal groups. Moreover if $R : \mathcal{D} \to \mathcal{C}$ is full and faithful so is its induced functor $R : \mathbf{Grp}(\mathcal{D}) \to \mathbf{Grp}(\mathcal{C})$.*

Proof. Observe first that since internal groups are defined by diagrams involving only finite products the restrictions of the functors L and R as indicated are well-defined.

Given $G \in \mathbf{Grp}(\mathcal{C})$ with multiplication μ_G and unit ι_G, and $H \in \mathbf{Grp}(\mathcal{D})$ with multiplication μ_H and unit ι_H, by definition $\varphi \in \mathcal{D}(LG, H)$ is in $\mathbf{Grp}(\mathcal{D})(LG, H)$ if the diagram below commutes.

$$\begin{tikzcd} LG \times LG \arrow[r, "L\mu_G"] \arrow[d, "\varphi\times\varphi"'] & LG \arrow[d, "\varphi"] \\ H \times H \arrow[r, "\mu_H"'] & H \end{tikzcd}$$

In other words, $\mathbf{Grp}(\mathcal{D})(LG, H)$ is the equalizer

$$\mathbf{Grp}(\mathcal{D})(LG, H) \longrightarrow \mathcal{D}(LG, H) \underset{\mu_H\langle -,- \rangle}{\overset{\langle -,- \rangle L\mu_G}{\rightrightarrows}} \mathcal{D}(LG \times LG, H)$$

where one of the parallel arrows on the right is induced by $LG \times LG \to LG \to H$ and the other by $LG \times LG \to H \times H \to H$. Thus there are natural isomorphisms

$$\begin{aligned} \mathbf{Grp}(\mathcal{D})(LG, H) &\cong \operatorname{eq}\left(\mathcal{D}(LG, H) \underset{\mu_H\langle -,- \rangle}{\overset{\langle -,- \rangle L\mu_G}{\rightrightarrows}} \mathcal{D}(LG \times LG, H)\right) \\ &\cong \operatorname{eq}\left(\mathcal{D}(LG, H) \rightrightarrows \mathcal{D}(L(G \times G), H)\right) \\ &\cong \operatorname{eq}\left(\mathcal{C}(G, RH) \underset{\mu_H\langle -,- \rangle}{\overset{\langle -,- \rangle L\mu_G}{\rightrightarrows}} \mathcal{C}(G \times G, RH)\right) \\ &\cong \mathbf{Grp}(\mathcal{C})(G, RH) \end{aligned}$$

the parallel arrows in the third line obtained like before from the morphism $G \to RH$ adjunct to $LG \to H$ in the previous line. (There are coherence conditions with respect to the group operations that the adjunction $L \dashv R$ has to satisfy, as can be seen if one tries to write the above isomorphisms explicitly; this is the case in most examples, including the one we are considering — we have omitted the details.)

The second claim in the statement of the lemma is a consequence of the preceding description of morphisms of internal groups and the fact that R, being a right adjoint, preserves equalizers. □

Lemma 1.2. *For $G \in \mathbf{Cat}(\mathbf{Grp})$ and A any small category, the functor category G^A is also in $\mathbf{Cat}(\mathbf{Grp})$.*

Proof. As **Cat** is cartesian closed, the exponential $(-)^A$ preserves finite limits, and therefore group objects since these are defined by such limits. Explicitly, if μ_G and ι_G are the multiplication and unit maps of G respectively, then G^A has multiplication and unit maps

$$G^A \times G^A \cong (G \times G)^A \xrightarrow{\mu_G^A} G^A$$

$$* \cong *^A \xrightarrow{\iota_G^A} G^A$$

for some fixed choice of isomorphisms $G^A \times G^A \cong (G \times G)^A$ and $*^A \cong *$. $\square$

Theorem 1.3. *There exists a model category structure on* **Cat**(**Grp**) *in which a morphism* $f : G \to H$ *is*

- *a* weak equivalence *if* Nf *is a weak equivalence of simplicial groups*
- *a* fibration *if* Nf *is a fibration of simplicial groups (equivalently a fibration of the underlying simplicial sets)*
- *a* cofibration *if it has the left lifting property with respect to all trivial fibrations.*

Proof. Let us verify the conditions stated in **Theorem II.1.2** to conclude this result as a special case.

Cat(**Grp**) is complete and cocomplete [**nLab2**]. Furthermore the nerve functor commutes with directed colimits (**Lemma I.7.2**). By the definition of a fibration and the adjunction $\tau_1 \dashv N$ a pushout of the image $\tau_1 F\partial\Delta^n \to \tau_1 F\Delta^n$ of a generating cofibration in simplicial groups (**Corollary 1.3(i)**) is a cofibration with the left lifting property against all fibrations; the following variation on Quillen's argument [**Goerss and Jardine**, Lemma II.6.1] establishes that such a cofibration is a weak equivalence.

Suppose $f : G \to H$ is such a cofibration. Observe that, since a simplicial group is a Kan complex [**May92**, Theorem 17.1], by the definition of a fibration every $G \in$ **Cat**(**Grp**) is fibrant; thus the dotted arrow exists in the diagram below

$$\begin{tikzcd} G \arrow[r, "1"] \arrow[d, "f"'] & G \\ H \arrow[ur, dashed, "s"'] & \end{tikzcd}$$

Moreover, by **Lemma 1.2** any functor category G^A is also a categorical group and, denoting by $[\mathbf{1}]$ the poset $\{0 \to 1\}$, in the commutative diagram

$$\begin{tikzcd} G \arrow[r, "r"] \arrow[d, "f"'] & H^{[\mathbf{1}]} \arrow[d, "q"] \\ H \arrow[r, "p"'] & H \times H \end{tikzcd}$$

where

- r is the composite $G \to H = H^{[\mathbf{0}]} \to H^{[\mathbf{1}]}$ - the second map induced by the unique functor $[\mathbf{1}] \to [\mathbf{0}]$,
- q is induced by the inclusion of the discrete subcategory of objects of $[\mathbf{1}]$, and
- $p = (1, fs)$,

the right-hand map is a fibration since Nq is isomorphic to the map $(NH)^{\Delta^1} \to (NH)^{\partial\Delta^1}$ and NH is fibrant. By assumption there exists a lift $H \to H^{[\mathbf{1}]}$, i.e. a natural transformation $1 \to fs$. Applying the nerve functor, we see that Nf is a simplicial homotopy equivalence and therefore a weak equivalence of simplicial sets. □

2. Model structure on n-fold categorical groups

As with the notion of a group, the definition of a category can be expressed 'internally' inside any category $\mathcal{D}$ in which finite limits exist: such an **internal category C** consists of objects $C_0, C_1 \in \mathcal{D}$ together with *domain* and *codomain* morphisms $d_1, d_0 : C_1 \to C_0$, an *identity* morphism $\iota : C_0 \to C_1$ and a *composition* morphism $C_1 \times_{C_0} C_1 \to C_1$ from the pullback

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r] \arrow[d] & C_1 \arrow[d, "d_1"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd}$$

satisfying the usual properties, given by commutative diagrams. Thus, if $\mathcal{D}$ is **Set**, C_1 would be the set of morphisms and C_0 the set of objects of a small category.

An *internal functor* $F : \mathbf{B} \to \mathbf{C}$ between internal categories $\mathbf{B} = (B_0, B_1)$ and $\mathbf{C} = (C_0, C_1)$ in $\mathcal{D}$ — the structural morphisms not included in the notation — is a pair of morphisms $F_0 \in \mathcal{D}(B_0, C_0)$ and $F_1 \in \mathcal{D}(B_1, C_1)$ that commute with the domain, codomain, identity and composition morphisms of $\mathbf{B}$ and $\mathbf{C}$. (Compare the usual definition of a functor when $\mathcal{D}$ is **Set**.) The category with objects internal categories in $\mathcal{D}$ and internal functors as morphisms is denoted by $\mathbf{Cat}(\mathcal{D})$.

$\mathbf{Cat}(\mathcal{D})$ is small complete [**nLab2**]. So we may consider internal categories in $\mathbf{Cat}(\mathcal{D})$; we thus obtain *double categories* in $\mathcal{D}$, $\mathbf{Cat}(\mathbf{Cat}(\mathcal{D}))$, denoted by $\mathbf{Cat}^2(\mathcal{D})$. Generalizing, for any $n \geq 2$, $\mathbf{Cat}^n(\mathcal{D})$ is the category of *n-fold categories* in $\mathcal{D}$, defined recursively as internal categories in $\mathbf{Cat}^{n-1}(\mathcal{D})$ and internal functors between these. An n-fold category $\mathbf{C} \in \mathbf{Cat}^n(\mathcal{D})$ can be presented combinatorially in terms of objects and morphisms in $\mathcal{D}$. For instance, if $n = 2$ (see [**Fiore and Paoli**, Definition 2.2] for the general case), $\mathbf{C}$ is specified by a commutative diagram in $\mathcal{D}$ of shape

$$\begin{tikzcd} C_{0,1} \arrow[r] \arrow[r, leftarrow, shift left=2] \arrow[r, leftarrow, shift right=2] \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] & C_{1,1} \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] \\ C_{0,0} \arrow[r] \arrow[r, leftarrow, shift left=2] \arrow[r, leftarrow, shift right=2] & C_{1,0} \end{tikzcd}$$

together with morphisms $C_{i,1} \times_{C_{i,0}} C_{i,1} \to C_{i,1}$, for $i \in \{0,1\}$, such that both $\mathbf{C}_0 = (C_{0,0}, C_{0,1})$ and $\mathbf{C}_1 = (C_{1,0}, C_{1,1})$ are in $\mathbf{Cat}(\mathcal{D})$, with vertical arrows as structural morphisms. Moreover, the corresponding horizontal arrows at top and bottom define domain, codomain and composition internal functors between $\mathbf{C}_0$ and $\mathbf{C}_1$ so that $\mathbf{C} = (\mathbf{C}_0, \mathbf{C}_1)$ is an internal category in $\mathbf{Cat}(\mathcal{D})$; therefore there are also morphisms $C_{1,i} \times_{C_{0,i}} C_{1,i} \to C_{1,i}$, for $i \in \{0,1\}$, commuting with the structure maps of $\mathbf{C}_0$ and $\mathbf{C}_1$) and making $(\mathbf{C}_{0,1}, \mathbf{C}_{1,1})$ and $(\mathbf{C}_{0,0}, \mathbf{C}_{1,0})$ objects of $\mathbf{Cat}(\mathcal{D})$ too.

We denote by $\mathbf{Cat}^n(\mathbf{Grp})$ the category of n-fold categories internal to the category of groups — equivalently group objects in the category $\mathbf{Cat}^n(\mathbf{Set})$ of n-fold categories — and by $\mathbf{s}^n\mathbf{Grp}$ the category of n-simplicial groups, that is $[(\Delta^{\times n})^{\mathrm{op}}, \mathbf{Grp}]$. There are adjunctions, for $n > 1$,

$$\mathbf{sGrp} \underset{\sigma_*}{\overset{\sigma^*}{\rightleftarrows}} \mathbf{s}^n\mathbf{Grp} \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{Cat}^n(\mathbf{Grp})$$

Here $\sigma^* : \mathbf{sGrp} \to \mathbf{s}^n\mathbf{Grp}$ is induced by the ordinal sum functor $\Delta^{\times n} \to \Delta$, $([k_1], \ldots, [k_m]) \mapsto [k_1] \boxplus \ldots \boxplus [k_m]$ (the posets are concatenated, [**Mac Lane**, §VII.5]), so

$$(\sigma^* G)_{k_1, \ldots, k_m} = G_{k_1 + \ldots + k_m + m - 1};$$

in the case $n = 2$ this is the standard *total décalage* construction due to Illusie [**Illusie**]. The theory of Kan extensions (§**I.5**) states that σ^* has a right adjoint σ_*; it generalizes the *total simplicial set*, or *Artin-Mazur codiagonal*, of bisimplicial sets ([**Artin and Mazur**], [**Stevenson**, §2.1, §3]), which for $X \in \mathbf{s}^2\mathbf{Set}$ gives as n-simplices of $\sigma_* X$ those $(n+1)$-tuples

$$(x_0, \ldots, x_n) \in \prod_{i=0}^{i=n} X_{i,n-i}$$

satisfying $d_0^v x_i = d_{i+1}^h x_{i+1}$, $0 \le i < n$. ($d_\bullet^v$ and $d_\bullet^h$ are, respectively, the vertical and horizontal face maps.) This can be derived from the formula a right Kan extension in terms of limits. Similar calculations yield the simplicial structure of X is in $\mathbf{s}^n\mathbf{Set}$, $n > 2$; however, the explicit description is somewhat complicated combinatorially and will not be needed in what follows. (For the face and degeneracy operators of $\sigma_* X$ when X is a bisimplicial set, refer to [**Cegarra and Remedios**, §2.1].)

The functors τ and N are, respectively, the *fundamental n-fold category* and *multi-nerve* [**Fiore and Paoli**, §2], which for $n = 1$ specialize to the familiar fundamental category τ_1 and nerve functors (§**I.7**). The multi-nerve functor $\mathbf{Cat}^n(\mathbf{Grp}) \to \mathbf{s}^n\mathbf{Grp}$ can be defined, as in §**I.7**, in terms of hom-functors $\mathbf{Cat}^n(\mathbf{Set})(\quad, G)$ [**Fiore and Paoli**, Definition 2.14] (clearly an n-fold category internal to $\mathbf{Grp}$ is also an object of $\mathbf{Cat}^n(\mathbf{Set})$); it can be deduced then that the multi-nerve functor has a right adjoint given as a right Kan extension, as for $n = 1$. More generally, however, the multi-nerve functor must be regarded as the iteration of the nerve functor, defined by pullbacks (see the proof of **Lemma I.7.2(c)**) acting level-wise on simplicial objects in internal categories [**nLab3**]:

$$\mathbf{Cat}^n(\mathbf{Grp}) \to \mathbf{sCat}^{n-1}(\mathbf{Grp}) \to \mathbf{s^2Cat}^{n-2}(\mathbf{Grp}) \to \ldots \to \mathbf{s}^n\mathbf{Grp}.$$

Referring to the above discussion of a double category $\mathbf{C} \in \mathbf{Cat}^2(\mathcal{D})$, for example, applying the nerve functor once yields a simplicial object in $\mathbf{Cat}(\mathcal{D})$, which in degree 2 is $\mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1$. The multi-nerve is obtained as the level-wise nerve of each category constituting this simplicial object; the bi-degree $(2, 1)$ of the resulting bisimplicial object in $\mathcal{D}$, for instance, is $C_{1,1} \times_{C_{0,1}} C_{1,1}$. (The multi-simplicial object given by the multi-nerve can be indexed variously — in this case it can be construed either as a horizontal or a vertical simplicial object in $s\mathcal{D}$, and our description is based on the second choice; it is important in arguments that a convention is fixed and followed consistently, in particular that the right adjoint of each level-wise nerve functor is calculated 'in the same direction'.)

All the functors in the above adjunctions are monoidal (indeed this is why they restrict to group objects as indicated). This is clear for σ^*, σ_* and N. The proof for τ proceeds similarly to that for $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ (**Lemma I.7.2(a)**); it uses the facts that $N : \mathbf{Cat}^n(\mathbf{Grp}) \to \mathbf{s}^n\mathbf{Grp}$ is fully faithful (it is so on $\mathbf{Cat}^n(\mathbf{Set})$ [**Fiore and Paoli**, Proposition 2.17], and being product-preserving it retains this property when restricted to $\mathbf{Cat}^n(\mathbf{Grp})$ - see **Lemma 1.1**) and hence the counit of $\tau \dashv N$ is an isomorphism, and that $\mathbf{Cat}^n(\mathbf{Set})$ is cartesian closed [**nLab1**].

Definition 2.1. Define a morphism $f \in \mathbf{Cat}^n(\mathbf{Grp})$ to be

- a *weak equivalence* if $\sigma_* N f$ is a weak equivalence of simplicial groups (that is, a weak equivalence of the underlying simplicial sets)
- a *fibration* if $\sigma_* N f$ is a fibration of simplicial groups (equivalently, a fibration of the underlying simplicial sets)
- a *cofibration* if it has the left lifting property with respect to all trivial fibrations.

Our aim is to prove that these classes of maps define a model structure on $\mathbf{Cat}^n(\mathbf{Grp})$.

Now, $\mathbf{Cat}^n(\mathbf{Grp})$ is complete and cocomplete [**nLab2**]. Furthermore, σ_* preserves directed colimits, since it is constructed by finite limits. The nerve functor does too: if we view an n-fold categorical group, combinatorially, as a diagram of a certain shape in the category **Set** with extra data given in terms of finite limits [**Fiore and Paoli**, Definition 2.2, Proposition 2.5], then a 'pointwise' directed colimit of such diagrams (with maps between them respecting the extra structure) also satisfies the conditions for an n-fold categorical group, hence is a colimit in $\mathbf{Cat}^n(\mathbf{Grp})$; the nerve is also constructed by finite limit operations on the sets in the diagram, hence commutes with this colimit. Thus in order to verify that the above definition yields a model structure it is sufficient to know that pushouts of images $\tau\sigma^* F\Lambda^n_k \to \tau\sigma^* F\Delta^n$ of the generating trivial cofibrations $F\Lambda^n_k \to F\Delta^n$ of **sGrp** ($F : \mathbf{sSet} \to \mathbf{sGrp}$ is induced by the free group functor) are weak equivalences (**Theorem II.1.2**).

The category **sGrp** is enriched, tensored and cotensored over simplicial sets (**Definition I.6.1**) [**Goerss and Jardine**, §II.2]. The tensor is given, for $G \in \mathbf{sGrp}$ and $K \in \mathbf{sSet}$, by

$$(G \otimes K)_n = \coprod_{K_n} G_n$$

(the coproduct is in the category of groups) and, therefore, the mapping object has n-simplices

$$\mathbf{Hom}_{\mathbf{sGrp}}(G, H)_n \cong \mathbf{sSet}(\Delta^n, \mathbf{Hom}_{\mathbf{sGrp}}(G, H)) \cong \mathbf{sGrp}(G \otimes \Delta^n, H).$$

(The mapping object must be so defined for the tensor to satisfy the required adjoint property.)

Lemma 2.2. *The cotensor in* **sGrp***, as a simplicial set, is the same as in* **sSet**.

Proof. Observe first that, for $H \in \mathbf{sGrp}$ and $K \in \mathbf{sSet}$, H^K as defined for simplicial sets is also a simplicial group (**Lemma 1.1**). For H^K to be a cotensor, there should be an isomorphism

$$\mathbf{Hom_{sGrp}}(G \otimes K, H) \cong \mathbf{Hom_{sGrp}}(G, H^K);$$

written in terms of n-simplices, this means an isomorphism

$$\mathbf{sGrp}((G \otimes K) \otimes \Delta^n, H) \cong \mathbf{sGrp}(G \otimes \Delta^n, H^K)$$

natural in all G, H, K and n. It is easy to see from the definition of the tensor that $(G \otimes K) \otimes \Delta^n \cong G \otimes (K \times \Delta^n) \cong (G \otimes \Delta^n) \otimes K$, natural in n. Hence it is sufficient to prove that

$$\mathbf{sGrp}(G \otimes K, H) \cong \mathbf{sGrp}(G, H^K).$$

So let $\varphi \in \mathbf{sGrp}(G \otimes K, H)$. The components of φ are group homomorphisms $\varphi_n : \coprod_{K_n} G_n \to H_n$. Such a group homomorphism is uniquely determined by a function $G_n \times K_n \to H_n$ between *sets* that is a group homomorphism $G_n \to H_n$ for each $k \in K_n$ — this is the universal property of the coproduct. Now, considering φ as a map $G \times K \to H$ of simplicial sets the natural isomorphism

$$\mathbf{sSet}(G \times K, H) \cong \mathbf{sSet}(G, H^K)$$

gives a simplicial map $\theta : G \to H^K$ and, working through this correspondence explicitly [**Goerss and Jardine**, Proposition I.5.1], it is straightforward to verify that the components of θ are group homomorphisms if φ satisfies the above-mentioned homomorphic property. Similarly, the inverse isomorphism yields such a φ if θ is a map of simplicial groups. Thus the lemma is established. □

Corollary 2.3. *The simplicial enrichment defined above makes* **sGrp** *a simplicial model category (**Definition II.3.3**).*

Proof. Since a morphism in **sGrp** is a fibration or weak equivalence if it is so as a map of simplicial sets, and **sSet** is a simplicial model category (**Theorem I.8.2**), the claim follows from the lemma. □

Let us show that $F\Lambda^n_k$ is a 'deformation retract' of $F\Delta^n$, with a simplicial deformation homotopy.

If $i : G \to H$ is a trivial cofibration of simplicial groups then, since G is fibrant, the dotted arrow in the diagram below exists

$$\begin{tikzcd} G \arrow[r, "1"] \arrow[d, "i"'] & G \\ H \arrow[ur, dashed, "r"'] & \end{tikzcd}$$

so G is a retract of H. Consider the commutative solid-arrow diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "i"'] & H^{\Delta^1} \arrow[d] \\ H \arrow[r, "(1,ir)"'] \arrow[ur, dashed] & H^{\partial\Delta^1} \end{tikzcd}$$

in which the top morphism is adjoint to the constant homotopy $G \otimes \Delta^1 \to G \otimes \Delta^0 \cong G \to H$ and the isomorphism $H^{\partial\Delta^1} \cong H \times H$ is implicit in the bottom arrow. Since **sGrp** is a simplicial model category the right-hand morphism is a fibration and hence the diagonal dotted arrow can be filled, yielding a simplicial right homotopy $1 \simeq ir$. For the purpose of what follows we can and will rewrite this as a simplicial left homotopy in terms of $G \otimes \Delta^1$ instead of the path object H^{Δ^1}.

Our objective now is to define an enrichment of $\mathbf{Cat}^n(\mathbf{Grp})$ over $\mathbf{s}^n\mathbf{Grp}$ and a tensor operation such that there are 'deformation retraction' maps analogous to the above for a pushout of $\tau\sigma^* F\Lambda^n_k \to \tau\sigma^* F\Delta^n$, in the sense that they are taken by $\sigma_* N$ to diagrams in **sGrp** that can be used to construct simplicial homotopies.

The discussion in [**Goerss and Jardine**, §II.2] can be generalized to show that $\mathbf{s}^n\mathbf{Grp}$ is enriched, tensored and cotensored over $\mathbf{s}^n\mathbf{Set}$, with

$$(\mathbf{G} \otimes \mathbf{K})_{m_1,\dots,m_n} = \coprod_{\mathbf{K}_{m_1,\dots,m_n}} \mathbf{G}_{m_1,\dots,m_n}$$

for $\mathbf{G} \in \mathbf{s}^n\mathbf{Grp}$ and $\mathbf{K} \in \mathbf{s}^n\mathbf{Set}$ and the exponential in $\mathbf{s}^n\mathbf{Set}$ (as a presheaf category) being the underlying multisimplicial set of the cotensor. It is then easy to see that σ^* preserves tensors; for instance, to keep the notation concise, in the case $n = 2$ we have isomorphisms, natural in p and q,

$$\sigma^*(\mathbf{G} \otimes \mathbf{K})_{p,q} = (\mathbf{G} \otimes \mathbf{K})_{p+q+1} = \coprod_{\mathbf{K}_{p+q+1}} \mathbf{G}_{p+q+1} = \coprod_{(\sigma^*\mathbf{K})_{p,q}} (\sigma^*\mathbf{G})_{p,q} = (\sigma^*\mathbf{G} \otimes \sigma^*\mathbf{K})_{p,q}$$

Lemma 2.4. *For $\mathbb{G}, \mathbb{H} \in \mathbf{Cat}^n(\mathbf{Grp})$ there is a natural isomorphism*

$$N(\mathbb{H}^{\mathbb{G}}) \cong N\mathbb{H}^{N\mathbb{G}}$$

where $N\mathbb{G}^{N\mathbb{H}}$ denotes the cotensor of $N\mathbb{G} \in \mathbf{s}^n\mathbf{Grp}$ with $N\mathbb{H}$ as an object of $\mathbf{s}^n\mathbf{Set}$.

Proof. This can be see by considering multisimplicial level groups; for example, when $n = 2$,

$$\begin{aligned}(N\mathbb{H}^{N\mathbb{G}})_{p,q} &= \mathbf{s}^2\mathbf{Set}(N\mathbb{G} \times \Delta^{p,q}, N\mathbb{H}) \\ &\cong \mathbf{s}^2\mathbf{Set}(N(\mathbb{G} \times \tau\Delta^{p,q}), N\mathbb{H}) \\ &\cong \mathbf{Cat}^2\mathbf{Set}(\mathbb{G} \times \tau\Delta^{p,q}, \mathbb{H}) \\ &\cong \mathbf{Cat}^2\mathbf{Set}(\tau\Delta^{p,q}, \mathbb{H}^{\mathbb{G}}) \\ &\cong \mathbf{s}^2\mathbf{Set}(\Delta^{p,q}, N(\mathbb{H}^{\mathbb{G}})) \\ &\cong N(\mathbb{H}^{\mathbb{G}})_{p,q}\end{aligned}$$

(see also [**Fiore and Paoli**, Example 2.16, Proposition 2.17]) where we have used the fact that $\mathbf{Cat}^n(\mathbf{Grp})$ is cartesian closed ([**nLab1**]&**Lemma 1.1**) and, in going from the second to the third line, that N is fully faithful. It is not difficult to check that the natural isomorphisms above are isomorphisms of *groups*, the group structure on each morphism set being induced by that on the codomain. □

Define, for $\mathbb{C} \in \mathbf{Cat}^n(\mathbf{Set})$, a functor $(-) \otimes \mathbb{C} : \mathbf{Cat}^n(\mathbf{Grp}) \to \mathbf{Cat}^n(\mathbf{Grp})$ by

$$\mathbb{G} \otimes \mathbb{C} := \tau(N\mathbb{G} \otimes N\mathbb{C}).$$

Lemma 2.5. *There is an adjunction* $(-) \otimes \mathbb{C} \dashv (-)^{\mathbb{C}}$.

Proof.

$$\begin{aligned}\mathbf{Cat}^n(\mathbf{Grp})(\mathbb{G} \otimes \mathbb{C}, \mathbb{H}) &= \mathbf{Cat}^n(\mathbf{Grp})(\tau(N\mathbb{G} \otimes N\mathbb{C}), \mathbb{H}) \\ &\cong \mathbf{s}^n\mathbf{Grp}(N\mathbb{G} \otimes N\mathbb{C}, N\mathbb{H}) \\ &\cong \mathbf{s}^n\mathbf{Grp}(N\mathbb{G}, N\mathbb{H}^{N\mathbb{C}}) \\ &\cong \mathbf{s}^n\mathbf{Grp}(N\mathbb{G}, N(\mathbb{H}^{\mathbb{C}})) \\ &\cong \mathbf{Cat}^n(\mathbf{Grp})(\mathbb{G}, \mathbb{H}^{\mathbb{C}})\end{aligned}$$

□

Proposition 2.6. *For* $\mathbf{G} \in \mathbf{s}^n\mathbf{Grp}$ *and* $\mathbf{K} \in \mathbf{s}^n\mathbf{Set}$ *there is an isomorphism*

$$\tau(\mathbf{G} \otimes \mathbf{K}) \cong \tau\mathbf{G} \otimes \tau\mathbf{K}$$

natural in $\mathbf{G}$. *(To construe all occurrences of* τ *in this isomorphism uniformly, we regard* $\mathbf{G}$ *and* $\mathbf{G} \otimes \mathbf{K}$ *as objects of* $\mathbf{s}^n\mathbf{Set}$*; however the isomorphism holds in* $\mathbf{Cat}^n(\mathbf{Grp})$ *and not merely in* $\mathbf{Cat}^n\mathbf{Set}$*.)*

Proof.

$$\begin{aligned}\mathbf{Cat}^n(\mathbf{Grp})(\tau(\mathbf{G}\otimes\mathbf{K}),\mathbb{H}) &\cong \mathbf{s}^n\mathbf{Grp}(\mathbf{G}\otimes\mathbf{K},N\mathbb{H})\\ &\cong \mathbf{s}^n\mathbf{Grp}(\mathbf{G},N\mathbb{H}^{\mathbf{K}})\end{aligned}$$

Also, using the above lemma,

$$\begin{aligned}\mathbf{Cat}^n(\mathbf{Grp})(\tau\mathbf{G}\otimes\tau\mathbf{K},\mathbb{H}) &= \mathbf{Cat}^n(\mathbf{Grp})(\tau(N\tau\mathbf{G}\otimes N\tau\mathbf{K}),\mathbb{H})\\ &\cong \mathbf{s}^n\mathbf{Grp}(N\tau\mathbf{G}\otimes N\tau\mathbf{K},N\mathbb{H})\\ &\cong \mathbf{s}^n\mathbf{Grp}(N\tau\mathbf{G},N\mathbb{H}^{N\tau\mathbf{K}})\\ &\cong \mathbf{s}^n\mathbf{Grp}(N\tau\mathbf{G},N(\mathbb{H}^{\tau\mathbf{K}}))\\ &\cong \mathbf{Cat}^n(\mathbf{Grp})(\tau\mathbf{G},\mathbb{H}^{\tau\mathbf{K}})\\ &\cong \mathbf{s}^n\mathbf{Grp}(\mathbf{G},N\mathbb{H}^{N\tau\mathbf{K}})\end{aligned}$$

Thus it suffices to show that $N\mathbb{H}^{\mathbf{K}} \cong (N\mathbb{H})^{N\tau\mathbf{K}}$, naturally in $\mathbb{H}$, for then the left adjoints to these functors must be isomorphic too. Denoting by η the unit of the adjunction $\tau \dashv N$, there are isomorphisms (letting $n = 2$ for notational simplicity)

$$\begin{aligned}(N\mathbb{H}^{N\tau\mathbf{K}})_{p,q} &\cong \mathbf{s}^2\mathbf{Set}(N\tau\mathbf{K}\times\Delta^{p,q},N\mathbb{H})\\ &\cong \mathbf{s}^2\mathbf{Set}(N\tau\mathbf{K}\times N\tau\Delta^{p,q},N\mathbb{H})\\ &\cong \mathbf{s}^2\mathbf{Set}(N\tau(\mathbf{K}\times\Delta^{p,q}),N\mathbb{H}) \xrightarrow[\eta^*]{\cong} \mathbf{s}^2\mathbf{Set}(\mathbf{K}\times\Delta^{p,q},N\mathbb{H})\\ &\cong (N\mathbb{H}^{\mathbf{K}})_{p,q}\end{aligned}$$

where η^* being an isomorphism follows from the fact that η is a universal arrow to N. $\square$

As mentioned before our aim is to show that pushouts of images $\tau\sigma^*F\Lambda^n_k \to \tau\sigma^*F\Delta^n$ of the generating trivial cofibrations $F\Lambda^n_k \to F\Delta^n$ of $\mathbf{sGrp}$ are weak equivalences in $\mathbf{Cat}^n(\mathbf{Grp})$, i.e. are taken to weak equivalences of simplicial groups by σ_*N. Our argument proceeds in several steps:

(1) Apply $\tau\sigma^*$ to the simplicial homotopy $1 \simeq ir$ that gives a deformation retraction $F\Delta^n\otimes\Delta^1 \to F\Delta^n$ onto $F\Lambda^n_k$ (refer to the argument following **Corollary 2.3**).

Using **Proposition 2.6** we obtain a diagram

$$\begin{tikzcd}
\tau\sigma^* F\Delta^n \otimes \tau\sigma^*\Delta^0 \cong \tau\sigma^* F\Delta^n \arrow[d] \arrow[dr, "1"] & \\
\tau\sigma^* F\Delta^n \otimes \tau\sigma^*\Delta^1 \arrow[r] & \tau\sigma^* F\Delta^n \\
\tau\sigma^* F\Delta^n \otimes \tau\sigma^*\Delta^0 \cong \tau\sigma^* F\Delta^n \arrow[u] \arrow[ur, "(\tau\sigma^* i)(\tau\sigma^* r)"'] &
\end{tikzcd}$$

Furthermore, $\tau\sigma^* F\Lambda^n_k$ is a retract of $\tau\sigma^* F\Delta^n$ and the horizontal morphism above 'restricts', by composing with the map induced by $\tau\sigma^* i$, to

$$\tau\sigma^* F\Lambda^n_k \otimes \tau\sigma^*\Delta^1 \to \tau\sigma^* F\Lambda^n_k \otimes \tau\sigma^*\Delta^0 \cong \tau\sigma^* F\Lambda^n_k \xrightarrow{\tau\sigma^* i} \tau\sigma^* F\Delta^n.$$

(2) We would like to establish properties analogous to those expressed by the diagrams in **(1)** for any morphism that is a pushout of $\tau\sigma^* F\Lambda^n_k \to \tau\sigma^* F\Delta^n$. We will adopt a more general notation: suppose $u : \mathbb{A} \to \mathbb{B}$ and $r : \mathbb{B} \to \mathbb{A}$ in $\mathbf{Cat}^n(\mathbf{Grp})$ satisfy $ru = 1$, and there exists $\mathbb{I} \in \mathbf{Cat}^n(\mathbf{Grp})$ with morphisms $d_0, d_1 : * \to \mathbb{I}$ and $H : \mathbb{B} \otimes \mathbb{I} \to \mathbb{B}$ such that $H(1 \otimes d_0) = 1$, $H(1 \otimes d_1) = ur$ (the isomorphisms $\mathbb{A} \otimes * \cong \mathbb{A}$, etc, being implicit hereafter) and $H(u \otimes 1)$ is 'constant' on $\mathbb{A}$ as above.

If the solid-arrow diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "f"] \arrow[d, "u", shift left] & \mathbb{C} \arrow[d, "v", shift left] \\
\mathbb{B} \arrow[r, "g"'] \arrow[u, dashed, "r", shift left] & \mathbb{D} \arrow[u, dashed, "s", shift left]
\end{tikzcd}$$

is a pushout in $\mathbf{Cat}^n(\mathbf{Grp})$ then it follows easily from its universal property that there exists $s : \mathbb{D} \to \mathbb{C}$ such that $sv = 1$ and the entire diagram commutes. Now consider the commutative diagram of solid arrows

$$\begin{tikzcd}
\mathbb{A} \otimes \mathbb{I} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{C} \otimes \mathbb{I} \arrow[dr] \arrow[dd] & \\
& \mathbb{A} \arrow[rr] \arrow[dd] & & \mathbb{C} \arrow[dd] \\
\mathbb{B} \otimes \mathbb{I} \arrow[rr] \arrow[dr, "H"'] & & \mathbb{D} \otimes \mathbb{I} \arrow[dr, dashed, "K"] & \\
& \mathbb{B} \arrow[rr] & & \mathbb{D}
\end{tikzcd}$$

in which the front face is the above pushout square, the back face is obtained by applying $(-) \otimes \mathbb{I}$ to the front face, and the unlabelled arrows from the back to the front face are induced by $\mathbb{I} \to *$. Since $(-) \otimes \mathbb{I}$ is a left adjoint (**Lemma 2.5**), the back face is also a pushout square and hence the dotted arrow exists, yielding a commutative cube.

We can add to this diagram another copy of the front face, together with maps from it to the rear face induced by d_0 or $d_1 : * \to \mathbb{I}$. In the latter case we obtain the following composite diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "f"] \arrow[d, "u"'] & \mathbb{C} \arrow[d, "v"] \arrow[drr, "v"] & & \\
\mathbb{B} \arrow[r, "g"'] \arrow[rrr, bend right, "gur"'] & \mathbb{D} \arrow[r, "1\otimes d_1"'] & \mathbb{D}\otimes\mathbb{I} \arrow[r, "K"'] & \mathbb{D}
\end{tikzcd}$$

The universal property of the pushout square then implies that $K(1 \otimes d_1) = vs$. Similarly, one deduces that $K(1 \otimes d_0) = 1$.

(3) Letting $\mathbb{I} = \tau\sigma^*\Delta^1$ and u be the map $\tau\sigma^* F\Lambda^n_k \to \tau\sigma^* F\Delta^n$ in **(2)**, we have thus shown that if $v : \mathbb{C} \to \mathbb{D}$ is a pushout of the latter then there exists a retraction $s : \mathbb{D} \to \mathbb{C}$ and commutative diagrams (writing the functor $\otimes$ as defined)

$$\begin{tikzcd}
\mathbb{D}\otimes * \cong \mathbb{D} \arrow[d] \arrow[dr, "1"] & \\
\mathbb{D}\otimes\tau\sigma^*\Delta^1 \arrow[r, "K"] & \mathbb{D} \\
\mathbb{D}\otimes * \cong \mathbb{D} \arrow[u] \arrow[ur, "vs"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C}\otimes\tau\sigma^*\Delta^1 \arrow[r, "v\otimes 1"] \arrow[d] & \mathbb{D}\otimes\tau\sigma^*\Delta^1 \arrow[d, "K"] \\
\mathbb{C}\otimes\tau\sigma^*\Delta^0 \cong \mathbb{C} \arrow[r, "v"'] & \mathbb{D}
\end{tikzcd}$$

Rewriting $\otimes$ as defined, applying N and composing with the components of the unit η of $\tau \dashv N$ further gives

$$\begin{tikzcd}
& N\mathbb{D} \arrow[r, "\eta_{N\mathbb{D}}"] \arrow[dl] \arrow[d] & N\tau(N\mathbb{D}) \arrow[d] \arrow[dr, "1"] & \\
N\mathbb{D}\otimes\sigma^*\Delta^1 \arrow[r, "1\otimes\eta_{\sigma^*\Delta^1}"] & N\mathbb{D}\otimes N\tau\sigma^*\Delta^1 \arrow[r, "\eta_{N\mathbb{D}\otimes N\tau\sigma^*\Delta^1}"] & N\tau(N\mathbb{D}\otimes N\tau\sigma^*\Delta^1) \arrow[r, "NK"] & N\mathbb{D} \\
& N\mathbb{D} \arrow[r, "\eta_{N\mathbb{D}}"'] \arrow[ul] \arrow[u] & N\tau(N\mathbb{D}) \arrow[u] \arrow[ur, "(Nv)(Ns)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
N\mathbb{C} \otimes \sigma^*\Delta^1 \arrow[rr] \arrow[d, "1\otimes\eta_{\sigma^*\Delta^1}"'] & & N\mathbb{D} \otimes \sigma^*\Delta^1 \arrow[d, "1\otimes\eta_{\sigma^*\Delta^1}"] \\
N\mathbb{C} \otimes N\tau\sigma^*\Delta^1 \arrow[rr] \arrow[dr, "\eta_{N\mathbb{C}\otimes N\tau\sigma^*\Delta^1}"] \arrow[dd] & & N\mathbb{D} \otimes N\tau\sigma^*\Delta^1 \arrow[d, "\eta_{N\mathbb{D}\otimes N\tau\sigma^*\Delta^1}"] \\
 & N\tau(N\mathbb{C} \otimes N\tau\sigma^*\Delta^1) \arrow[r] \arrow[d] & N\tau(N\mathbb{D} \otimes N\tau\sigma^*\Delta^1) \arrow[d, "NK"] \\
N\mathbb{C} \arrow[r, "\eta_{N\mathbb{C}}"'] & N\tau(N\mathbb{C}) \arrow[r, "Nv"'] & N\mathbb{D}
\end{tikzcd}$$

(4) We will prove below that σ_* is naturally weakly equivalent to the diagonal functor $d : \mathbf{s}^n\mathbf{Grp} \to \mathbf{sGrp}$. Let us first use this fact to derive the intended result. Observing that

$$d(G \otimes K)_p = (G \otimes K)_{p,\dots,p} = \coprod_{K_{p,\dots,p}} G_{p,\dots,p} = \coprod_{(dK)_p} (dG)_p = (dG \otimes dK)_p$$

we obtain from the preceding diagrams

$$\begin{tikzcd}
dN\mathbb{D} \arrow[d] \arrow[dr, "1"] & \\
dN\mathbb{D} \otimes d\sigma^*\Delta^1 \arrow[r] & dN\mathbb{D} \\
dN\mathbb{D} \arrow[u] \arrow[ur, "(dNv)(dNs)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
dN\mathbb{C} \otimes d\sigma^*\Delta^1 \arrow[r] \arrow[d] & dN\mathbb{D} \otimes d\sigma^*\Delta^1 \arrow[d] \\
dN\mathbb{C} \arrow[r, "dNv"'] & dN\mathbb{D}
\end{tikzcd}$$

and therefore, using the tensor-cotensor adjunction,

$$\begin{tikzcd}
 & (dN\mathbb{D})^{d\sigma^*\Delta^1} \arrow[d] \\
dN\mathbb{D} \arrow[ur] \arrow[r, "((dNv)(dNs){,}1)"'] & dN\mathbb{D} \times dN\mathbb{D}
\end{tikzcd}$$

$$\begin{tikzcd}
dN\mathbb{D} \arrow[r] & (dN\mathbb{D})^{d\sigma^*\Delta^1} \arrow[d] \\
dN\mathbb{C} \arrow[u] \arrow[r, "dNv"'] & dN\mathbb{D}
\end{tikzcd}$$

Since every simplicial group is fibrant and $d\sigma^*\partial\Delta^1 \to d\sigma^*\Delta^1$ is a monomorphism of simplicial sets (d and σ^*, being right adjoints, preserve monomorphisms) the vertical arrow in the first diagram is a fibration. Moreover, $d\sigma^*\Delta^1$ is a (weakly) contractible simplicial set: there is a contracting simplicial homotopy $\Delta^1 \times \Delta^1 \to \Delta^1$ and therefore also

$$\sigma_*\sigma^*\Delta^1 \times \Delta^1 \to \sigma_*\sigma^*\Delta^1 \times \sigma_*\sigma^*\Delta^1 \to \sigma_*\sigma^*\Delta^1$$

by applying the monoidal functors σ^* and σ_* and composing with the unit of $\sigma^* \dashv \sigma_*$ in the second factor; we then again use the fact that d is weakly equivalent to σ_*.

It follows that we have a right homotopy $(dNv)(dNs) \simeq 1$, indeed a deformation retraction, in simplicial groups and hence in simplicial sets (recall that fibrations and weak equivalences of simplicial groups are those of the underlying simplicial sets, and the cotensor is the exponential). As $dN\mathbb{D}$ is fibrant this implies that $(dNv)(dNs)$, as a map of simplicial *sets*, is also left homotopic to the identity for any choice of cylinder object, in particular $dN\mathbb{D} \times \Delta^1$. Thus dNv is a simplicial homotopy equivalence and so a weak equivalence. Subject to proving the lemma below, which we have already used, we may conclude that σ_*Nv is a weak equivalence too.

Lemma 2.7. *For $n \geq 2$ the functors $d, \sigma_* : \mathbf{s}^n\mathbf{Set} \to \mathbf{sSet}$ are naturally weakly equivalent.*

Proof. As our argument is inductive, let us distinguish σ_* or σ^* and d for different values of n by using a subscript; so $\sigma^*_m : \mathbf{sSet} \to \sigma_* : \mathbf{s}^m\mathbf{Set}$ is induced by the ordinal sum functor $\sigma_m : \Delta^{\times m} \to \Delta$, $\sigma_m([k_1], \ldots, [k_m]) = [k_1] \boxplus \ldots \boxplus [k_m]$, and has right adjoint $(\sigma_m)_*$.

When $n = 2$ the generalized Cartier-Dold-Puppe theorem states that there is a natural weak equivalence $d_2X \to (\sigma_2)_*X$ for any bisimplicial set (see [**Stevenson**, §4] for a proof). Given that the claim is true for σ^*_m, consider $\sigma^*_{m+1}X$ for X a simplicial set.

We can think of $\sigma^*_{m+1}X$ as a simplicial object $Y_\bullet$ in $\mathbf{s}^m\mathbf{Set}$ specified by fixing the first index, that is $Y_k = (\sigma^*_{m+1}X)_{k,-,\ldots,-}$. Each Y_k is then the result of applying σ^*_m to the simplicial set $X_{[k]\boxplus -}$. Therefore $\sigma^*_{m+1} = (s\sigma^*_m)(\sigma^*_2)$, where $s\sigma^*_m$ denotes the operation of applying σ^*_m level-wise to a bisimplicial set (as a horizontal simplicial object in $\mathbf{sSet}$). Consequently the right adjoint $(\sigma_{m+1})_*$ of σ^*_{m+1} is (naturally isomorphic to) the composite $(\sigma_2)_*s(\sigma_m)_*$ (again, sF means apply F level-wise).

There is a natural weak equivalences $d_2s(\sigma_m)_* \to (\sigma_2)_*s(\sigma_m)_*$ and a level-wise natural weak equivalence $sd_m \to s(\sigma_m)_*$. The functor d_2 preserves level-wise weak equivalences, so by applying it to the latter map and composing it with the former we obtain a natural weak equivalence

$$d_{m+1} = d_2(sd_m) \to d_2s(\sigma_m)_* \to (\sigma_2)_*s(\sigma_m)_*$$

which proves the claim for $n = m + 1$. □

3. Homotopy types in sSet

A Kan complex is called a *homotopy n-type* if its (simplicial) homotopy groups, at any base-point, of degree greater than n are trivial. Here we establish that homotopy n-types in **sSet** are just the ∂^{n+2}-local spaces (**Definition II.3.5**), where ∂^{n+2} is the boundary inclusion $\partial\Delta^{n+2} \to \Delta^{n+2}$. This fact motivates the definition, in a later section, of a Bousfield localization of *reduced* simplicial sets such that homotopy types in $\mathbf{sSet_0}$ are the fibrant objects in the localization model structure (**Theorem II.3.6**).

We denote by $\overline{\mathcal{S}}$ the *saturated class* generated by the set of maps

$$\mathcal{S} = \{\Lambda^n_k \to \Delta^n : 0 \leq k \leq n, 1 \leq n\} \cup \{\partial\Delta^m \to \Delta^m : n + 2 \leq m\};$$

this is the smallest class of maps containing $\mathcal{S}$ and closed under retracts, coproducts, pushouts (along an arbitrary map) and (transfinite) composites. (Otherwise put, the saturated class is the collection of those morphisms that can be recursively obtained using these operations from the generating set.) Let $\mathcal{F}$ be the class of those simplicial sets that have the right lifting property against all maps in $\mathcal{S}$ and therefore $\overline{\mathcal{S}}$ [**Hirschhorn**, §7.2]. Since $\mathcal{S}$ contains all horn inclusions any $X \in \mathcal{F}$ is, in particular, a Kan complex.

Lemma 3.1. *A monomorphism $i : A \hookrightarrow B$ that is an isomorphism on $Sk_{n+1}A$ — that is, $i_k : A_k \hookrightarrow B_k$ is an isomorphism for $k \leq n + 1$ — is in $\overline{\mathcal{S}}$.*

Proof. The assumption on i implies that it is (isomorphic to) the composite

$$A \to A \amalg_{Sk_{n+1}A} Sk_{n+1}B \to A \amalg_{Sk_{n+2}A} Sk_{n+2}B \to \dots$$

where each inclusion $A \amalg_{Sk_mA} Sk_mB \to A \amalg_{Sk_{m+1}A} Sk_{m+1}B$ is a pushout

$$\begin{tikzcd}
\amalg_{N_{m+1}}\partial\Delta^{m+1} \arrow[r] \arrow[d] & \amalg_{N_{m+1}}\Delta^{m+1} \arrow[d] \\
A \amalg_{Sk_mA} Sk_mB \arrow[r] & A \amalg_{Sk_{m+1}A} Sk_{m+1}B
\end{tikzcd}$$

with N_{m+1} being the set of non-degenerate $(m + 1)$-simplices of B not in the image of i. (Compare [**Goerss and Jardine**, Proposition I.2.3].) □

Given $f : A \to B$ and $g : C \to D$ let us write $f \widehat{\times} g$ for the map indicated by the dotted arrow in the diagram

$$\begin{tikzcd}
A \times C \arrow[r, "f\times 1"] \arrow[d, "1\times g"'] & B \times C \arrow[d] \arrow[ddr, bend left, "1\times g"] & \\
A \times D \arrow[r] \arrow[drr, bend right, "1\times g"'] & (A \times D) \amalg_{(A\times C)} (B \times C) \arrow[dr, dashed] & \\
& & B \times D
\end{tikzcd}$$

induced by the universal property of the pushout. Note that, by the cartesian closure of **sSet**, finding a lift — the diagonal arrow — that makes

$$\begin{tikzcd}
(A \times D) \amalg_{(A\times C)} (B \times C) \arrow[r] \arrow[d, "f\widehat{\times} g"'] & X \arrow[d, "h"] \\
B \times D \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

is equivalent to the lifting problem

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "g"'] & X^A \arrow[d, "<f{,}h>"] \\
D \arrow[r] \arrow[ur, dashed] & Y^A \times_{Y^B} X^B
\end{tikzcd}$$

where $< f, h >$ is the dotted arrow in

$$\begin{tikzcd}
X^A \arrow[dr, dashed] \arrow[drr, bend left, "h_*"] \arrow[ddr, bend right, "f^*"'] & & \\
& Y^A \times_{Y^B} X^B \arrow[r] \arrow[d] & Y^A \arrow[d, "f^*"] \\
& X^B \arrow[r, "h_*"'] & Y^B
\end{tikzcd}$$

induced by the pullback square; symmetrically, f has the left lifting property against $< g, h >$. (See [**Hirschhorn**, Lemma 9.3.6])

Corollary 3.2. *If $i \in \overline{\mathcal{S}}$ and j is a monomorphism then $i \widehat{\times} j \in \overline{\mathcal{S}}$.*

Proof. The class of monomorphisms in **sSet** is the saturated class generated by the set $\{\partial\Delta^n \to \Delta^n\}$ of boundary inclusions (**Example II.1.1(a)**,[**Hirschhorn**, Proposition 11.2.1]). Moreover, a map is in $\overline{\mathcal{S}}$ if and only if it can be lifted on the *left* against all maps that have the *right* lifting property against the generators of $\mathcal{S}$ [**Hirschhorn**, Corollary 10.5.23]. Thus the observation preceding the statement of the corollary, and the fact that having the left lifting property against a given map is preserved under the operations involved in generating saturated classes, imply that it is sufficient to verify the claim for $i \in S$ and j a boundary inclusion. Corresponding to the two subsets involved in the definition of $\mathcal{S}$ there are two cases:

- $(\Lambda^n_k \to \Delta^n)\widehat{\times}(\partial\Delta^m \to \Delta^m)$, for any $n \geq 1$ and $k \leq n$ and m, is a cofibration and a weak equivalence, hence in the saturated class generated by horn inclusions [**Hirschhorn**, Proposition 11.2.1] and so in $\overline{\mathcal{S}}$;
- $(\partial\Delta^p \to \Delta^p)\widehat{\times}(\partial\Delta^m \to \Delta^m)$, for $p \geq n+2$ and any m, is in $\overline{\mathcal{S}}$ by the previous lemma.

□

Recall that $\mathcal{F}$ is the class of Kan complexes that have the right lifting property against $\{\partial\Delta^m \to \Delta^m : n+2 \leq m\}$.

Lemma 3.3. *If a monomorphism $i : A \hookrightarrow B$ is in $\overline{\mathcal{S}}$ then $i^* : X^B \to X^A$ is a weak equivalence, hence a trivial fibration (**Theorem I.8.2**), for all $X \in \mathcal{F}$.*

Proof. If $i \in \overline{\mathcal{S}}$ and $X \in \mathcal{F}$ then, since a morphism in **sSet** is a trivial fibration if and only if it has the right lifting property against all monomorphisms (**Example II.1.1(a)**), i^* is a trivial fibration by **Corollary 3.2** and the remarks before it. □

Proposition 3.4. *A Kan complex is in $\mathcal{F}$ if and only if $i^* : X^B \to X^A$ is a trivial fibration for all $i : A \to B$ in $\mathcal{S}$.*

Proof. If $X \in \mathcal{F}$ then i^* is a trivial fibration by **Lemma 3.3**. On the other hand, if i^* is a trivial fibration then it is surjective on 0-simplices (since a trivial fibration has the right lifting property against $\emptyset \to \Delta^0$), i.e. $\mathbf{sSet}(B, X) \to \mathbf{sSet}(A, X)$ is onto, hence every map $B \to X$ can be extended to A and so X has the right lifting property against i. □

Theorem 3.5. *A Kan complex X is an n-type if and only if the map $X^{\Delta^{n+2}} \to X^{\partial\Delta^{n+2}}$ induced by the inclusion $\partial\Delta^{n+2} \hookrightarrow \Delta^{n+2}$ is a trivial fibration.*

Proof. Note first that X is an n-type if and only if it is in $\mathcal{F}$, i.e. it has the right lifting property against $\{\partial\Delta^m \to \Delta^m : n+2 \leq m\}$. This can be seen by considering the geometric realization of X (but it is not difficult to give a direct combinatorial argument): X is simplicially homotopy equivalent to $S|X|$ (§**I.8**, [**Hirschhorn**, Theorem 7.5.10]), $|X|$

is a topological n-type and extending a map $\partial\Delta^m \to S|X|$ to Δ^m is equivalent to extending the adjunct map $|\partial\Delta^m| \to |X|$ to $|\Delta^m|$.

Thus it is sufficient to prove that $X \in \mathcal{F}$ if and only if $X^{\Delta^{n+2}} \to X^{\partial\Delta^{n+2}}$ is a trivial fibration. The 'only if' statement is an immediate consequence of **Proposition 3.4**. As for the other implication, observe that if $X^{\Delta^{n+2}} \to X^{\partial\Delta^{n+2}}$ is a trivial fibration then it has the right lifting property against $\{\partial\Delta^m \hookrightarrow \Delta^m : 0 \leq m\}$, hence every map

$$(\partial\Delta^m \times \Delta^{n+2}) \cup (\Delta^m \times \partial\Delta^{n+2}) \to X$$

can be extended to $\Delta^m \times \Delta^{n+2}$. But there is a simplicially homotopy equivalence of *pairs* between $(\Delta^m \times \Delta^{n+2}, (\partial\Delta^m \times \Delta^{n+2}) \cup (\Delta^m \times \partial\Delta^{n+2}))$ and $(\Delta^{m+n+2}, \partial\Delta^{m+n+2})$ (see the proof of **Proposition 5.1** below; alternatively one may consider geometric realizations again), therefore X has the right lifting property against $\{\partial\Delta^{m+n+2} \hookrightarrow \Delta^{m+n+2} : 0 \leq m\}$ and so is in $\mathcal{F}$. □

4. A functorial right homotopy function complex on $\mathbf{sSet}_0$

Recall from the discussion following **Example II.3.1** that in a simplicial model category $\mathcal{C}$ the mapping simplicial set $\mathbf{Hom}_{\mathcal{C}}(X, Y)$ given by the simplicial enrichment is a right homotopy function complex from X to Y (**Definition II.3.4**), when Y is fibrant. Thus **Theorem 3.5** above can be phrased as characterizing homotopy n-types in **sSet** as $(\partial\Delta^{n+2} \to \Delta^{n+2})$-local spaces. Our aim, realized in the next section, is to prove an analogous result for reduced simplicial sets. We cannot, however, exploit an existing structure of a simplicial model category in this case (we have not been able to find one; though nor have we established that none exists); therefore first we must construct a functorial right homotopy function complex on $\mathbf{sSet_0}$.

Let X and Y be reduced simplicial sets. Since every object of $\mathbf{sSet_0}$ is cofibrant, to obtain a right homotopy function complex from X to Y (**Definition II.3.4**) it is sufficient to find a simplicial resolution of Y.

Let $\widehat{Y}$ be a fibrant approximation to Y in $\mathbf{sSet}_*$, the category of pointed simplicial sets. Then $\widehat{Y}$ is also fibrant in **sSet** (**Corollary II.1.3**) and therefore the simplicial object $\widehat{Y}^{\Delta}$ in $\mathbf{sSet}_*^{\Delta^{\mathrm{op}}}$ given in degree n by $\widehat{Y}^{\Delta^n}$ is a simplicial resolution of $\widehat{Y}$ (refer to the argument following **Definition II.3.3**); however it is not a simplicial resolution in *reduced* simplicial sets. But we can construct a simplicial object $E_1\widehat{Y}^{\Delta}$ in $\mathbf{sSet_0}$ by applying the first Eilenberg subcomplex functor E_1 (§**I.9**) to each $\widehat{Y}^{\Delta^n}$ at the canonical basepoint determined by the unique 0-simplex $*$ of $\widehat{Y}$, that is the constant map $* : \Delta^n \to *$.

Lemma 4.1. *E_1 is a right Quillen functor: it preserves limits and maps fibrations and trivial fibrations in $\mathbf{sSet}_*$ to, respectively, fibrations and trivial fibrations in $\mathbf{sSet_0}$.*

Proof. Preservation of limits follows from the fact that E_1 is a right adjoint to the embedding $\mathbf{sSet_0} \to \mathbf{sSet}_*$ (§**I.9**).

To prove that E_1 preserves fibrations and trivial fibrations it is sufficient to show that the left adjoint preserves cofibrations and trivial cofibrations [**Hirschhorn**, Proposition 7.2.17]. To see this we recall, first, that in both $\mathbf{sSet_0}$ and $\mathbf{sSet}_*$ a morphism is a weak equivalence if it so as a map in **sSet**, hence a weak equivalence of reduced simplicial sets is also a weak equivalence of pointed simplicial sets. Furthermore, cofibrations in $\mathbf{sSet_0}$ are defined to be those of **sSet** — that is, they are monomorphisms — as are trivial fibrations in $\mathbf{sSet}_*$ ([**Goerss and Jardine**, §V.6], §**Corollary II.1.3**); since a map is a cofibration if and only if it has the left lifting property against all trivial fibrations a monomorphism of pointed simplicial sets is a cofibration in $\mathbf{sSet}_*$ too. □

Recall that a simplicial object is a resolution if all the structural morphisms are weak equivalences and each matching map, which is defined in terms of a finite limit, is a fibration (**Definition II.3.2**). Each $\widehat{Y}^{\Delta^n}$ is fibrant, because $\widehat{Y}$ is and $\mathbf{sSet}_*$ is a simplicial model category, and as E_1 preserves trivial fibrations it also preserves weak equivalences between fibrant objects [**Hirschhorn**, Corollary 7.7.2]. It follows that the simplicial object $E_1\widehat{Y}^{\Delta}$ is a simplicial resolution of Y in $\mathbf{sSet_0}$, and therefore $\mathbf{sSet_0}(X, E_1\widehat{Y}^{\Delta})$ is a right homotopy function complex from X to Y. Observe finally that since there is a natural isomorphism, for $K \in \mathbf{sSet_0}$ and $L \in \mathbf{sSet}_*$,

$$\mathbf{sSet_0}(K, E_1 L) \cong \mathbf{sSet}_*(K, L)$$

the simplicial set $\mathbf{sSet_0}(X, E_1\widehat{Y}^{\Delta})$ can be written as

$$\mathbf{sSet}_*(X, \widehat{Y}^{\Delta}) \cong \mathbf{Hom}_{\mathbf{sSet}_*}(X, \widehat{Y}).$$

5. $(\partial^{\mathbf{m}}_{\mathbf{r}} : \partial\boldsymbol{\Delta}^{\mathbf{m}}/\mathbf{Sk_0}\partial\boldsymbol{\Delta}^{\mathbf{m}} \to \boldsymbol{\Delta}^{\mathbf{m}}/\mathbf{Sk_0}\boldsymbol{\Delta}^{\mathbf{m}})$-local spaces and local equivalences

We determine local spaces and local equivalences in $\mathbf{sSet}_0$ for the map ∂^m_r using the homotopy function complex constructed above.

Proposition 5.1. *A fibrant reduced simplicial set W is ∂_r^m-local if and only if $\pi_i W \cong 0$ for $i > m-2$.*

Proof. By [**Hirschhorn**, Proposition 1.8.6] instead of the function complex $\mathbf{Hom}_{\mathbf{sSet}_*}(X,Y)$ we can work with the usual unpointed function complex on simplicial sets; thus W, fibrant in $\mathbf{sSet}_0$ and hence in $\mathbf{sSet}$ [**Goerss and Jardine**, Corollary V.6.8], is ∂_r^m-local if

$$(\partial_r^m)^* : \mathbf{Hom}_{\mathbf{sSet}}(\Delta^m/Sk_0\Delta^m, W) \to \mathbf{Hom}_{\mathbf{sSet}}(\partial\Delta^m/Sk_0\partial\Delta^m, W)$$

is a trivial fibration, i.e. W has the right lifting property against

$$(\Delta^m/Sk_0\Delta^m \ \times \partial\Delta^n) \cup (\partial\Delta^m/Sk_0\partial\Delta^m \ \times \Delta^n) \to (\Delta^m/Sk_0\Delta^m \ \times \Delta^n)$$

Suppose first that the homotopy groups of W are trivial above degree $m-2$. Then by **Theorem 3.5** W is $(\partial^m : \partial\Delta^m \to \Delta^m)$-local. Therefore the composite of a given

$$f : (\Delta^m/Sk_0\Delta^m \ \times \partial\Delta^n) \cup (\partial\Delta^m/Sk_0\partial\Delta^m \ \times \Delta^n) \to W$$

with the quotient map

$$(\Delta^m \times \partial\Delta^n) \cup (\partial\Delta^m \times \Delta^n) \to (\Delta^m/Sk_0\Delta^m \ \times \partial\Delta^n) \cup (\partial\Delta^m/Sk_0\partial\Delta^m \ \times \Delta^n)$$

can be extended to $(\Delta^m \times \Delta^n)$. But then, since $Sk_0\Delta^m = Sk_0\partial\Delta^m$ and $(Sk_0\Delta^m \ \times \partial\Delta^n) \cup (Sk_0\partial\Delta^m \ \times \Delta^n)$ is taken to a point, this extension must factor through $(\Delta^m/Sk_0\Delta^m \ \times \Delta^n)$ by a map which extends f.

Now assume that W is ∂_r^m-local and let $\alpha : \Delta^{n+m-1} \to W$ (rel $\partial\Delta^{n+m-1}$) be a representative of a simplicial homotopy class in $\pi_{n+m-1}W$, to begin with for $n > 0$. Construct a map

$$g : (\Delta^m/Sk_0\Delta^m \ \times \partial\Delta^n) \cup (\partial\Delta^m/Sk_0\partial\Delta^m \ \times \Delta^n) \to W$$

as follows:

- g is the constant map at the basepoint of W on the subcomplex $\partial\Delta^m/Sk_0\partial\Delta^m \ \times \Delta^n$;
- to define g on $(\Delta^m/Sk_0\Delta^m \ \times \partial\Delta^n)$ we shall present a map $\tilde{g} : (\Delta^m \times \partial\Delta^n) \to W$; it will be clear that it factors through the former. To do this observe first that by the standard presentation of $\partial\Delta^n$ as a coequalizer (§**I.3**), and the fact that the functor $\Delta^m \times (-)$ is a left adjoint, we can write $\Delta^m \times \partial\Delta^n$ as

$$\Delta^m \times \partial\Delta^n = \coprod(\Delta^m \times \Delta^{n-1})/\sim$$

where the coproduct ranges over the faces of Δ^n and the identifications result from the simplicial relations between these faces. Let $\tilde{g}$ be the constant map at the basepoint $* \in W$ on the first $n-1$ summands of the coproduct, corresponding to the first $n-1$ faces of Δ^n.

The poset map $[\mathbf{p}] \times [\mathbf{q}] \to [\mathbf{p}+\mathbf{q}]$ given by

$$\begin{aligned} (i,j) &\mapsto i && \text{if } i < p \\ (i,j) &\mapsto i+j && \text{if } i = p \end{aligned}$$

induces a simplicial homotopy equivalence $r_{p,q} : \Delta^p \times \Delta^q \to \Delta^{p+q}$ (see **Lemma I.8.3**) with the property that

$$r_{p,q}(\partial\Delta^p \times \Delta^q \cup \Delta^p \times \partial\Delta^q) \subseteq \partial\Delta^{p+q}.$$

The homotopy inverse $s_{p,q}$ of $r_{p,q}$ is induced by the map of posets $[\mathbf{p}+\mathbf{q}] \to [\mathbf{p}] \times [\mathbf{q}]$ which, geometrically described, takes the domain to the maximal chain along the left and top edges of the codomain lattice; it satisfies $rs = id$.

Define $\tilde{g}$ to be the composite $\alpha r_{m,n-1}$ on the last summand of the coproduct above. It is not difficult to verify that we obtain a well-defined map $\tilde{g} : (\Delta^m \times \partial\Delta^n) \to W$ which factors through $(\Delta^m/Sk_0\Delta^m \ \times \partial\Delta^n)$, and that this is compatible with the map on $(\partial\Delta^m/Sk_0\partial\Delta^m \ \times \Delta^n)$ and completes the definition of g.

By assumption g can be extended to $\hat{g} : (\Delta^m/Sk_0\Delta^m \ \times \Delta^n) \to W$. The composite

$$\Delta^{n+m} \xrightarrow{s_{m,n}} \Delta^m \times \Delta^n \xrightarrow{\hat{g}} W$$

then restricts to α on the last face and the constant map at $*$ on every other face. But this means that $[\alpha] \in \pi_{n+m-1}W$ is trivial, as was to be shown.

In the case $n = 0$ a simplified version of the foregoing argument establishes the same result. $\square$

Proposition 5.2. *A map $f : X \to Y$ in $\mathbf{sSet}_0$ is a ∂^m_r-local equivalence if and only if it induces isomorphisms $|f|_* : \pi_i|X| \cong \pi_i|Y|$ for $i \leq m-2$ and any basepoint in $|X|$. (Since X and hence $|X|$ is connected it is sufficient for this isomorphism to hold at the basepoint corresponding to the unique vertex of X.)*

Proof. As in the proof of **Proposition 5.1** above we can use the ordinary function complex on simplicial sets. If f induces isomorphisms of homotopy groups up to degree $m-2$ then by [**Hirschhorn**, Propositions 1.2.35 & 1.5.2] it is a $(\partial^m : \partial\Delta^m \to \Delta^m)$-local equivalence; this means that

$$f^* : \mathbf{Hom}_{\mathbf{sSet}}(Y, W) \to \mathbf{Hom}_{\mathbf{sSet}}(X, W)$$

is a weak equivalence for every ∂^m-local space W. The previous proposition and **Theorem 3.5** imply that a ∂^m_r-local space (in $\mathbf{sSet}_0$) is also ∂^m-local (in $\mathbf{sSet}$). It follows that f is also a ∂^m_r-local equivalence.

Conversely, suppose f is a ∂^m_r-local equivalence. If we show that f is also a ∂^m-local equivalence then it will follow from [**Hirschhorn**, Propositions 1.5.4] that it has the stated property. Since weak equivalences are always local equivalences [**Hirschhorn**, Proposition 1.2.16] we may, by taking a cofibration-trivial fibration factorization if necessary, assume that f is cofibration. So f is a ∂^m-local equivalence if for every ∂^m-local space Z the induced map

$$f^* : \mathbf{Hom}_{\mathbf{sSet}}(Y, Z) \to \mathbf{Hom}_{\mathbf{sSet}}(X, Z)$$

is a trivial fibration (**Theorem I.8.2**); that is, f^* has the right lifting property against all boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ (**Example II.1.1(a)**). As **sSet** is cartesian closed, an equivalent condition is that an extension (dotted arrow) exists in every diagram of the form

$$\begin{tikzcd}
(X \times \Delta^n) \cup (Y \times \partial\Delta^n) \arrow[r] \arrow[d] & Z \\
Y \times \Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

Given such an extension problem, the domain of the vertical map is connected (as can be seen, for instance, by considering its geometric realization and noting that it is path-connected, keeping in mind our assumption that f is a monomorphism of reduced simplicial sets) and therefore the image of the horizontal map lies in a connected component of Z, which must also be ∂^m-local. Thus the problem is reduced to the case in which Z is connected. But every connected Kan complex is weakly equivalent to one, call it $\tilde{Z}$, which is reduced, for instance by taking a minimal subcomplex (**Lemma I.9.1**). Being weakly equivalent to a ∂^m-local space $\tilde{Z}$ has the same property and hence, by the previous proposition and **Theorem 3.5**, is ∂^m_r-local too. Finally, in the diagram

$$\begin{tikzcd}
\mathbf{Hom}_{\mathbf{sSet}}(Y, Z) \arrow[r, "f^*"] \arrow[d] & \mathbf{Hom}_{\mathbf{sSet}}(X, Z) \arrow[d] \\
\mathbf{Hom}_{\mathbf{sSet}}(Y, \tilde{Z}) \arrow[r, "f^*"'] & \mathbf{Hom}_{\mathbf{sSet}}(Y, \tilde{Z})
\end{tikzcd}$$

the vertical maps are induced by the weak equivalence $Z \to \tilde{Z}$ and consequently are weak equivalences, and the bottom map is one since f is a ∂^m_r-local equivalence. It follows that the top map is also a weak equivalence, so the required extension exists. $\square$

6. A Quillen equivalence between $\mathbf{Cat^n(Grp)}$ and $\mathbf{L}_{\partial_r^{n+3}}(\mathbf{sSet_0})$

The left Bousfield localisation $\mathrm{L}_{\partial_r^m}(\mathbf{sSet}_0)$ of $\mathbf{sSet}_0$ at ∂_r^m is a model structure on the same category in which the ∂_r^m-local equivalences are the weak equivalences and cofibrations are, as in the model category $\mathbf{sSet_0}$, monomorphisms (**Theorem II.3.6**).

Recall that there is a Quillen equivalence $G : \mathbf{sSet_0} \rightleftarrows \mathbf{sGrp} : \overline{W}$, where G is the *Kan loop group functor* and $\overline{W}$ the *classifying complex* functor; $\overline{W}$ has the property that $\pi_{m+1}\overline{W}G \cong \pi_m G$ — an isomorphism of simplicial homotopy groups of Kan complexes (§**I.8**) — or equivalently $\pi_{n+1}|\overline{W}G| \cong \pi_n|G|$ for any basepoints.

With the model structures defined on $\mathbf{Cat(Grp)}$ in §**III.1** and $\mathbf{Cat^n(Grp)}$ for $n \geq 2$ in §**III.2**, there are also Quillen adjunctions $\mathbf{sGrp} \rightleftarrows \mathbf{Cat^n(Grp)}$ given by $\tau_1 : \mathbf{sGrp} \rightleftarrows \mathbf{Cat(Grp)} : N$ when $n = 1$ and the composite adjunction

$$\begin{tikzcd} \mathbf{sGrp} \arrow[r, shift left, "\sigma^*"] & \mathbf{s}^n\mathbf{Grp} \arrow[l, shift left, "\sigma_*"] \arrow[r, shift left, "\tau"] & \mathbf{Cat}^n(\mathbf{Grp}) \arrow[l, shift left, "N"] \end{tikzcd}$$

when $n \geq 2$ (**Theorem II.2.6**).

Theorem 6.1. *The composite adjunctions*

$$\begin{tikzcd} \mathbf{sSet_0} \arrow[r, shift left, "G"] & \mathbf{sGrp} \arrow[l, shift left, "\overline{W}"] \arrow[r, shift left, "\tau_1"] & \mathbf{Cat}(\mathbf{Grp}) \arrow[l, shift left, "N"] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbf{sSet_0} \arrow[r, shift left, "G"] & \mathbf{sGrp} \arrow[l, shift left, "\overline{W}"] \arrow[r, shift left, "\sigma^*"] & \mathbf{s}^n\mathbf{Grp} \arrow[l, shift left, "\sigma_*"] \arrow[r, shift left, "\tau"] & \mathbf{Cat}^n(\mathbf{Grp}) \arrow[l, shift left, "N"] \end{tikzcd}$$

for $n \geq 2$ induce Quillen equivalences

$$L_{\partial_r^{n+3}}(\mathbf{sSet_0}) \rightleftarrows \mathbf{Cat}^n(\mathbf{Grp})$$

for all $n \geq 1$.

Proof. Note first that the adjunction $\mathbf{sSet_0} \rightleftarrows \mathbf{Cat}^n(\mathbf{Grp})$ is a Quillen adjunction, since both $\mathbf{sGrp} \rightleftarrows \mathbf{Cat}^n(\mathbf{Grp})$ and $\mathbf{sSet_0} \rightleftarrows \mathbf{sGrp}$ are (**Proposition II.2.6**) and it is clear from **Definition II.2.5** that the composite of two left or right Quillen functors is another such functor.

By **Theorem II.3.7** there is an induced adjunction $\mathrm{L}_{\partial_r^{n+3}}(\mathbf{sSet_0}) \rightleftarrows \mathbf{Cat}^n(\mathbf{Grp})$ if the right adjoint — $\overline{W}N$ when $n = 1$ and $\overline{W}\sigma_*N$ when $n > 1$ — takes every $\mathbb{G} \in \mathbf{Cat}^n(\mathbf{Grp})$ to a ∂_r^{n+3}-local object in $\mathrm{L}_{\partial_r^{n+3}}(\mathbf{sSet_0})$; that is, if $\pi_k(\overline{W}N\mathbb{G}) \cong 0$ for $k > 2$ if $n = 1$, and $\pi_k(\overline{W}\sigma_*N\mathbb{G}) \cong 0$ for $k > n+1$ if $n > 1$ (**Proposition III.5.1**). The functor $\overline{W}$ preserves weak equivalences [**Goerss and Jardine**, Proposition V.6.3] and it was shown in **Lemma III.2.7** that σ_* is naturally weakly equivalent to the diagonal $d : \mathbf{s}^n\mathbf{Grp} \to \mathbf{sGrp}$, hence the preceding isomorphisms can be equivalently rewritten with σ_* replaced by d. Expressed that way, this result was first established in [**Loday**] (see also [**Paoli**, §2 & §5]); an alternative proof is given in [**Bullejos, Cegarra and Duskin**, §1].

It remains to show that the Quillen adjunction $\mathrm{L}_{\partial_r^{n+3}}(\mathbf{sSet_0}) \rightleftarrows \mathbf{Cat}^n(\mathbf{Grp})$ induces an adjoint equivalence of homotopy categories. A necessary and sufficient condition [**Hovey**, Corollary 1.3.16] for this is that the unit map of the adjunction is a weak equivalence in $\mathrm{L}_{\partial_r^{n+3}}(\mathbf{sSet_0})$, i.e. a ∂_r^{n+3}-local equivalence, for every reduced simplicial set. Denoting by η, δ and λ the unit natural transformations of $G \dashv \overline{W}$, $\sigma_* \dashv \sigma^*$ and $\tau \dashv N$ respectively, the unit of the composite adjunction has component at $X \in \mathbf{sSet_0}$ [**Mac Lane**, Theorem IV.8.1]

$$X \xrightarrow[\eta_X]{} \overline{W}GX \xrightarrow[\overline{W}\lambda_{GX}]{} \overline{W}N\tau GX$$

when $n = 1$ and

$$X \xrightarrow[\eta_X]{} \overline{W}GX \xrightarrow[\overline{W}\delta_{GX}]{} \overline{W}\sigma_*\sigma^*GX \xrightarrow[\overline{W}\sigma_*\lambda_{\sigma^*GX}]{} \overline{W}\sigma_*N\tau\sigma^*GX$$

for $n \geq 2$.

As $G \dashv \overline{W}$ is a Quillen equivalence, η_X is a weak equivalence of reduced simplicial sets and therefore a ∂_r^{n+3}-local equivalence [**Hirschhorn**, Proposition 1.2.16]. δ_Y is also a weak equivalence of simplicial sets for every $Y \in \mathbf{sSet}$ [**Cegarra and Remedios**, Proposition 7.1], hence, since $\overline{W}$ preserves weak equivalences, $\overline{W}\delta_{GX}$ is a ∂_r^{n+3}-local equivalence. Now, it is a standard result that λ_Y is a weak equivalence for $n = 1$, so $\overline{W}\lambda_{GX}$ is too. In the case $n \geq 2$, $\overline{W}\sigma_*\lambda_{\sigma^*GX}$ is naturally weakly equivalent to $\overline{W}d\lambda_{\sigma^*GX}$ (**Lemma III.2.7**) and the latter map is a ∂_r^{n+3}-local equivalence by [**Bullejos, Cegarra and Duskin**, Theorem 2.3]. Thus the unit of $\mathrm{L}_{\partial_r^{n+3}}(\mathbf{sSet_0}) \rightleftarrows \mathbf{Cat}^n(\mathbf{Grp})$ is the composite of ∂_r^{n+3}-local equivalences and so a weak equivalence in $\mathrm{L}_{\partial_r^{n+3}}(\mathbf{sSet_0})$ (**Proposition III.5.2**). $\square$

This theorem implies that up to isomorphism in $\mathrm{Ho}\big(\mathrm{L}_{\partial_r^{n+3}}(\mathbf{sSet_0})\big)$, in other words ∂_r^{n+3}-local equivalence, every reduced simplicial set is in the image of the composite right adjoint. In particular, every $(n+1)$-type can be represented by an n-fold categorical group, up to weak homotopy equivalence.

Chapter 4

Outlook: Enriched groupoids and homotopy types

*In **Theorem 6.1** it was established that the model category of n-fold categorical groups is Quillen equivalent to a Bousfield Loclization of the model category of reduced simplicial sets, a result which can be interpreted as showing that the former provides a model for homotopy types in connected spaces.*

To obtain categorical models for homotopy types that may be non-connected, i.e. simplicial sets that are not reduced, one possible guiding idea is to find objects to which the arguments in the previous chapter apply 'locally': since every simplicial set is a disjoint union of connected components, thus weakly equivalent to a coproduct of reduced simplicial sets, the objects which model these must be identifiable, in a suitable sense, with 'locally' determined n-fold categorical groups.

*Motivated by this thought, we consider groupoids enriched over n-fold categories. We will exploit a generalization, due to Dwyer and Kan [**Dwyer and Kan**], of the adjoint functors* $\overline{W} : \mathbf{sGrp} \to \mathbf{sSet_0} : G$ *to functors between* simplicially enriched groupoids *and* **sSet** *and their definition of a model structure on the former. Also denoted by* $\overline{W}$*, the functor* $\mathbf{sSet} - \mathbf{Grpd} \to \mathbf{sSet}$ *introduced by Dwyer and Kan is a right Quillen functor and specializes to the familiar* $\overline{W}$ *from simplicial groups when these are regarded as simplicially enriched groupoids with one object.*

We will propose a model structure on $\mathbf{Cat}^n(\mathbf{Set})$-$\mathbf{Grpd}$ *using the model structure on*

sSet – **Grpd** *and* ***Theorem II.1.2****, our argument elaborating on the sense in which a groupoid enriched over n-fold categories is 'skeletally' a coproduct of n-fold categorical groups, thus enabling application of the results in the previous sections. Although not detailed, we believe the following discussion will make it plausible that the statements and proofs of those results can be modified to establish a Quillen equivalence between the category of groupoids enriched over n-fold categories and a Bousfield localization of* **sSet***. Indeed, the reasoning involved will be somewhat simpler, since there is a ready functorial right homotopy function complex on* **sSet** *(§****II.3****) for Bousfield localization, and the local spaces and local equivalences are known directly (§****III.4****,[****Hirschhorn****, §1.5]).*

Given a monoidal category $\mathcal{U}$, we denote by $\mathcal{U}$-**Grpd** the category of groupoids *enriched* over $\mathcal{U}$ and $\mathcal{U}$-functors between them [**Kelly**, §1.2]. The cases of interest to us here are when $\mathcal{U}$ is one of **sSet**, $\mathbf{s}^n\mathbf{Set}$ or $\mathbf{Cat}^n(\mathbf{Set})$. The adjoint pairs of monoidal functors $\tau \dashv N$ and $\sigma^* \dashv \sigma_*$ (§**III.2**) induce functors and adjunctions between the corresponding enriched groupoids, for which we use the same notation:

$$\mathbf{sSet}\text{-}\mathbf{Grpd} \underset{\sigma_*}{\overset{\sigma^*}{\rightleftarrows}} \mathbf{s}^n\mathbf{Set}\text{-}\mathbf{Grpd} \underset{N}{\overset{\tau}{\rightleftarrows}} \mathbf{Cat}^n(\mathbf{Set})\text{-}\mathbf{Grpd}$$

Each of these functors takes a groupoid enriched over its domain to one with the same objects enriched over its codomain, by operating on the morphism objects. (Note that the two uses of the word *object* in the preceding sentence have different senses.) The transpose of an enriched functor under each adjunction is then the same function on objects and its action on morphisms is obtained by taking adjuncts. It is not difficult to verify that these constructions have the required properties. (To be precise, some coherence conditions on the adjunctions and monoidal structures are required, which are satisfied here; we omit the details.)

An object in one of the above enriched categories can be viewed equivalently as an object internal to **Grpd**, with some constraints; for instance a $\mathbf{Cat}^n(\mathbf{Set})$-enriched groupoid is the same as an n-fold category in groupoids such that all the internal categorical structural functors are the identity on objects. Then the functors under consideration can be written, in analogy to the unenriched case, in terms of finite limits (N and σ_*) or colimits (τ, a left Kan extension). This formulation makes it clear that their properties noted in the previous context are still true, e.g. $\tau N \cong 1$.

Theorem. *The category* **sSet-Grpd** *has a cofibrantly generated model structure, with generating trivial cofibrations* $F'\Lambda^n_k \to F'\Delta^n$, $0 \leq k \leq n$, *and generating cofibrations* $F'\partial\Delta^n \to F'\Delta^n$, $n \geq 0$, *and* $\emptyset \to *$. *Here* $(F'X)_n$ *for a simplicial set* X *is the free groupoid on the directed graph with two objects* a *and* b *and arrows from* a *to* b *corresponding to the simplices in* X_n.

Proof. [**Dwyer and Kan**]. See also [**Goerss and Jardine**, §V.7] □

Our aim is to transfer the model structure on **sSet-Grpd** to $\mathbf{Cat}^n(\mathbf{Set})$-**Grpd** using the adjunction displayed in the above diagram (see **Theorem II.1.2**), by defining a morphism in $\mathbf{Cat}^n(\mathbf{Set})$-**Grpd** to be a weak equivalence or fibration if its image under $\sigma_* N$ is so; cofibrations are then defined as usual to be those morphisms having the left lifting property against all trivial fibrations.

Thinking of $\mathbf{Cat}^n(\mathbf{Set})$-**Grpd** as n-fold categories internal to groupoids with the additional condition stated above, we see that the same reasoning as for $\mathbf{Cat}^n(\mathbf{Set})$ [**nLab2**] shows that it is (small) complete and cocomplete. Similarly, σ_*, regarded as acting on n-simplicial objects in **Grpd**, preserves directed colimits. The argument for N is still valid too: we note that directed colimits of groupoids, as for limits, are obtained by calculating the directed colimits of objects and morphisms (see the paragraph following **Definition III.2.1**). Then, as in the argument for $\mathbf{Cat}^n(\mathbf{Grp})$ in §**III.2**, we must show that pushouts of $\tau\sigma^* F'\Lambda^n_k \to \tau\sigma^* F'\Delta^n$ are weak equivalences.

Consider a diagram

$$\begin{tikzcd} \tau\sigma_* A \arrow[r, "s"] \arrow[d, "\tau u"'] & \mathbb{G} \\ \tau\sigma_* B & \end{tikzcd}$$

in $\mathbf{Cat}^n(\mathbf{Set})$-**Grpd**, where $A, B \in$ **sSet-Grpd** are connected, meaning that $A(a, a') \neq \emptyset$ for any $a, a' \in A$, and similarly for B, hence so are $\tau\sigma_* A$ and $\tau\sigma_* B$. (Equivalently, their underlying categories are connected.) Without loss of generality we may assume that $\mathbb{G}$ is connected too: pushouts commute with coproducts, and weak equivalences are stable under coproducts. Since τ, being a left adjoint, is cocontinuous and $\mathbb{G} \cong \tau N\mathbb{G}$, we can obtain the colimit of the above diagram by first producing the pushout

$$\begin{tikzcd} \sigma_* A \arrow[r, "Ns"] \arrow[d, "\sigma_* u"'] & N\mathbb{G} \arrow[d, "v"] \\ \sigma_* B \arrow[r] & \mathbf{H} \end{tikzcd}$$

in $\mathbf{s}^n\mathbf{Set}$-$\mathbf{Grpd}$ and then applying τ.

Now, according to the alternative point of view discussed above, A, B and $N\mathbb{G}$ are n-simplicial objects of connected groupoids with a constant set of objects. A connected groupoid $\mathcal{G}$ is equivalent to its skeleton [**Mac Lane**, §IV.4] which is a groupoid with one object, which can be identified with its *group* of morphisms: denoting by $\mathcal{I}$ the groupoid $0 \rightleftarrows 1$, there exists a 'deformation' retraction $r : \mathcal{G} \times \mathcal{I} \to \mathcal{G}$, $r|_{\mathcal{G}\times\{0\}} = id_{\mathcal{G}}$ and $r(g, 1) = *$ for all $g \in \mathcal{G}$, for an arbitrary choice of object $* \in \mathcal{G}$ and uniquely determined by associating a morphism $\alpha_g \in \mathcal{G}(g, *)$ to every $g \in \mathcal{G}$. We can construct such a retraction for the (level) groupoids constituting the n-simplicial objects in the diagram in a way compatible with the simplicial morphisms and the maps $\sigma_* u$ and Ns, as follows.

Choose an object $a \in A$ and a morphism $\mu_{A_0,a'} \in A_0(a', a)$ for every $a' \in A$; these determine an equivalence $\phi_{A_0} : A_0 \to A_0(a, a)$, using identical notation for the groupoid with the single object a and its group of morphisms $A_0(a, a)$. For any index m let $\phi_{A_m} : A_m \to A_m(a, a)$ be the equivalence given by morphisms $\mu_{A_m,a'} \in A_m(a', a)$ that are obtained by transporting the $\mu_{A_0,a'}$ from A_0 to A_m by action of degeneracy operators — any sequence of degeneracy maps will have the same image by the simplicial identities. The morphisms $\mu_{A_{\cdot},\cdot}$ then correspond under the structural functors of A and together yield a retraction $A \times I \to A(a, a)$ which is a point-wise equivalence, where $I \in \mathbf{sGrpd}$ is discrete on $\mathcal{I}$, or seen inside $\mathbf{sSet}$-$\mathbf{Grpd}$ the groupoid with two objects and every simplicial set of morphisms the terminal one. (Note that $A(a, a)$ is a simplicial group) Consequently, there exists a retraction and point-wise equivalence $\sigma_* A \times \mathbf{I} \to \sigma_* A(a, a)$, $\mathbf{I}$ being the discrete n-simplicial groupoid constant on $\mathcal{I}$.

To obtain a similar retraction for $\sigma_* B$ that is also 'equivariant' with respect to u, use u to transport the morphisms $\mu_{A_{\cdot},\cdot}$ for objects in its image, and for other objects of B proceed as for A. Likewise, we can obtain morphisms in each level groupoid of $N\mathbb{G}$ by using Ns or, for objects not in its range, as degeneracy images of some chosen morphisms in $N\mathbb{G}_{0,\ldots,0}$.

By an argument parallel to that given in the previous section, remembering that **Grpd** is cartesian closed, this construction gives for each $(i_1, \ldots, i_n)$ a diagram (identifying the object a and its images)

$$\begin{tikzcd}
(\sigma_*A)_{i_1,\dots,i_n} \times \mathbf{I} \arrow[rr, "Ns_{i_1,\dots,i_n}\times 1"] \arrow[dr] \arrow[dd, "(\sigma_*u)_{i_1,\dots,i_n}\times 1"'] & & (N\mathbb{G})_{i_1,\dots,i_n} \times \mathbf{I} \arrow[dr] \arrow[dd] & \\
& (\sigma_*A(a,a))_{i_1,\dots,i_n} \arrow[rr] \arrow[dd] & & (N\mathbb{G}(a,a))_{i_1,\dots,i_n} \arrow[dd] \\
(\sigma_*B)_{i_1,\dots,i_n} \times \mathbf{I} \arrow[rr] \arrow[dr] & & \mathbf{H}_{i_1,\dots,i_n} \times \mathbf{I} \arrow[dr, dashed] & \\
& (\sigma_*B(a,a))_{i_1,\dots,i_n} \arrow[rr] & & (\mathbf{H}(a,a))_{i_1,\dots,i_n}
\end{tikzcd}$$

in which the front and back faces are pushout squares, the dotted arrow is induced by the universal property of the pushout, and all oblique arrows are equivalences.

The colimit of a diagram in $\mathbf{s}^n\mathbf{Grpd}$ consisting of n-simplicial groupoids with a constant set of objects also has the same property. Thus it is also the colimit in $\mathbf{s}^n\mathbf{Set\text{-}Grpd}$. Since colimits $\mathbf{s}^n\mathbf{Grpd}$ are calculated point-wise, the diagrams depicted above yield, for all indices together,

$$\begin{tikzcd}
\sigma_*A \times \mathbf{I} \arrow[rr, "Ns\times 1"] \arrow[dr] \arrow[dd, "\sigma_*u\times 1"'] & & N\mathbb{G} \times \mathbf{I} \arrow[dr] \arrow[dd] & \\
& \sigma^*A(a,a) \arrow[rr] \arrow[dd] & & N\mathbb{G}(a,a) \arrow[dd] \\
\sigma_*B \times \mathbf{I} \arrow[rr] \arrow[dr] & & \mathbf{H} \times \mathbf{I} \arrow[dr, dashed] & \\
& \sigma^*A(a,a) \arrow[rr] & & \mathbf{H}(a,a)
\end{tikzcd}$$

in which the front and back faces are again pushout squares and $\mathbf{I}$ is as above. Furthermore, the front face is a pushout in $\mathbf{Cat}^n(\mathbf{Grp})$. Consider the right-hand face of this cube and apply τ and then σ_*N to obtain

$$\begin{tikzcd}
\sigma_*N\mathbb{G} \times I \arrow[r] \arrow[d] & \sigma_*N\mathbb{G}(a,a) \arrow[d] \\
\sigma_*N\tau\mathbf{H} \times I \arrow[r] & \sigma_*N\tau\mathbf{H}(a,a)
\end{tikzcd}$$

in $\mathbf{sSet\text{-}Grpd}$, using the fact that τ and σ_*N preserve products (§**III.2**). The horizontal arrows are point-wise equivalences, deformation retracting each level groupoid onto its skeleton group.

Now, recall the problem in the background of our discussion: showing that pushouts of $\tau\sigma^*F'\Lambda^n_k \to \tau\sigma^*F'\Delta^n$ are weak equivalences. So let $A = F'\Lambda^n_k$ and $A = F'\Delta^n$ in the preceding argument. It can be shown [**Goerss and Jardine**, Lemma V.7.1, Corollary V.7.3] that $F'\Lambda^n_k(a,a) \to F'\Delta^n(a,a)$ is a trivial cofibration of simplicial groups. Therefore $\mathbb{G}(a,a) \to \tau\mathbf{H}(a,a)$ is the pushout, in $\mathbf{Cat}^n(\mathbf{Grp})$, of $\tau\sigma^*F'\Lambda^n_k(a,a) \to \tau\sigma^*F'\Delta^n(a,a)$.

We proved in §**III.2** that this implies $\sigma_* N\mathbb{G}(a,a) \to \sigma_* N\tau\mathbf{H}(a,a)$ is a weak equivalence in **sGrp**.

The right Quillen functor $\overline{W} : \mathbf{sSet}\text{-}\mathbf{Grpd} \to \mathbf{sSet}$ preserves products, and $\overline{W}(I) = \mathrm{nerve}(I)$ (see [**Goerss and Jardine**, §V.7]); 'nerve' is of course the familiar functor $\mathbf{Cat} \to \mathbf{sSet}$, notated differently from usual to avoid confusion with N as used here. Applying $\overline{W}$ to the above diagram and composing with the map induced by the inclusion $[\mathbf{1}] \hookrightarrow I$ gives

$$\begin{tikzcd}
\overline{W}\sigma_* N\mathbb{G} \times \Delta^1 \arrow[r] \arrow[d] & \overline{W}\sigma_* N\mathbb{G} \times \mathrm{nerve}(I) \arrow[r] \arrow[d] & \overline{W}\sigma_* N\mathbb{G}(a,a) \arrow[d] \\
\overline{W}\sigma_* N\mathbf{H} \times \Delta^1 \arrow[r] & \sigma_* N\tau\mathbf{H} \times \mathrm{nerve}(I) \arrow[r] & \overline{W}\sigma_* N\tau\mathbf{H}(a,a)
\end{tikzcd}$$

where the right vertical map is a weak equivalence. Our argument has shown that the composites of the horizontal maps are homotopy equivalences. It follows that $\overline{W}\sigma_* NG \to \overline{W}\sigma_* N\tau\mathbf{H}(a,a)$ is a weak equivalence, and so, since $G \dashv \overline{W}$ is a Quillen equivalence, $\sigma_* NG \to \sigma_* N\tau\mathbf{H}(a,a)$ is too.

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[**nLab2**] nLab; sketch — Remark 3. http://nlab.mathforge.org/nlab/show/sketch [**nLab3**] nLab; Internal category — Internal nerve. http://nlab.mathforge.org/nlab/show/internal+category

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