String Rewriting Systems and Associated Finiteness Conditions

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Abstract

We begin with an introduction which describes the thesis in detail, and then a preliminary chapter in which we discuss rewriting systems, associated complexes and finiteness conditions. In particular, we recall the graph of derivations Γ and the 2-complex $\mathcal{D}$ associated to any rewriting system, and the related geometric finiteness conditions FDT and FHT. In §1.4 we give basic definitions and results about finite *complete* rewriting systems, that is, rewriting systems which rewrite any word in a finite number of steps to its normal form, the unique irreducible word in its congruence class.

The main work of the thesis begins in Chapter 2 with some discussion of rewriting systems for groups which are confluent on the congruence class containing the empty word. In §2.1 we characterize groups admitting finite λ-complete rewriting systems as those with a λ-Dehn presentation, and in §2.2 we give some examples of finite rewriting systems for groups which are λ-complete but not complete.

For the remainder of the thesis, we study how the properties of finite complete rewriting systems which are discussed in the first chapter are mirrored in higher dimensions. In Chapter 3 we extend the 2-complex $\mathcal{D}$ to form a new 3-complex $\overline{\mathcal{D}^{\mathrm{P}}}$, and in Chapter 4 we define new finiteness conditions FDT_2 and FHT_2 based on the homotopy and homology of this complex. In §4.4 we show that if a monoid admits a finite complete rewriting system, then it is of type FDT_2.

The final chapter contains a discussion of alternative ways to define such higher dimensional finiteness conditions. This leads to the introduction, in §5.2, of a variant of the Guba-Sapir homotopy reduction system which can be associated to any complete rewriting system. This is a rewriting system operating on paths in Γ, and is complete in the sense that it rewrites paths in a finite number of steps to a unique "normal form".

Statement

Chapter 1 covers some basic material including group presentations and rewriting systems, related complexes and finiteness conditions, and also a discussion of complete rewriting systems. Apart from a new proof of Theorem 1.3.4, the material in this chapter can be found elsewhere - references are given in the text.

Chapters 2, 3, 4 and 5 are the original work of the author, with the exception of instances mentioned in the text. The main results of Chapters 3, 4 and 5 are contained in a paper which has been submitted for publication.

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Notation

		page
$\mathbb{Z}$	the set of integers	
$\mathbb{N}$	the set of natural numbers	
$\mathbb{N}_0$	the set of non-negative integers	
$F = F(\mathbf{x})$	the free monoid on the set $\mathbf{x}$	7
$\mathcal{G} = \langle \mathbf{x}; \mathbf{r} \rangle$	group presentation	8
$\mathcal{P} = [\mathbf{x}; \mathbf{r}]$	rewriting system	9
$\rightarrow_{\mathbf{r}}$	the single-step relation induced by $\mathbf{r}$	9
$\rightarrow_{\mathbf{r}}^{*}$	the reflexive, transitive closure of $\rightarrow_{\mathbf{r}}$	9
$\longleftrightarrow_{\mathbf{r}}^{*}$	the Thue congruence generated by $\mathbf{r}$	9
$\overline{w}$	the congruence class of a word w	9
$S = S(\mathcal{P})$	the monoid defined by $\mathcal{P}$	9
$>_{llex}$	the length-lexicographic reduction ordering	10
$\mathcal{F} = \mathcal{F}(\mathbf{x})$	the free group with basis $\mathbf{x}$	11
$\succ_{\mathbf{r}}$	the reduction ordering induced by $\mathbf{r}$	11
$\mathbb{Z}T$	the integral monoid ring of a monoid T	12
J	the kernel of the ring map $\mathbb{Z}F \rightarrow \mathbb{Z}S$ induced by $\mathcal{P}$	12
$\Gamma = \Gamma(\mathcal{P})$	the graph of derivations	14
$\mathbf{e}^{+}$	the set of all positive edges in Γ	14
$star^{+}(w)$	the set of positive edges at a vertex w	15
$<_{w}$	the order of edges in $star^{+}(w)$	15
$\vartheta = \vartheta_{\mathcal{P}}$	the height function associated to $\mathcal{P}$	16
$\mathcal{D} = \mathcal{D}(\mathcal{P})$	the 2-complex of monoid pictures	16
$[e, f]$	2-cell in $\mathcal{D}$ $(e, f \in \mathbf{e}^{+})$	16
$\Pi = \Pi(\mathcal{P})$	the homology bimodule of $\mathcal{P}$	3, 18
$\Pi^{l}(\mathcal{P})$	the left homology bimodule of $\mathcal{P}$	4
$\mathcal{M} = \mathcal{M}(\mathcal{P})$	the relation module of $\mathcal{P}$	4, 18
$\mathcal{M}^{l}$	the left relation module of $\mathcal{P}$	4

		page
$\mathbb{Z}T.\mathbf{X}.\mathbb{Z}T$	the $(\mathbb{Z}T, \mathbb{Z}T)$-bimodule with basis $\mathbf{X}$	18
$\mathcal{D}^{\mathbf{p}}$	the 2-complex based on the pair $(\mathcal{D}, \mathbf{p})$, where $\mathbf{p}$ is a set of closed paths	20
$Irr(w)$	the normal form of w	25
$[w; (e, f)]$	2-cell in $\mathcal{D}^{\mathbf{p}}$, where $\mathbf{p}$ is a set of closed paths composed of resolutions of critical pairs	35
$G*_{A\cong B}$	an HNN group with base G and associated subgroups A and B	49
$G_{(p,q)}$	the (p, q)-Baumslag-Solitar group	53
$[w, p, w']$	2-cell in $\mathcal{D}^{\mathbf{p}}$ with boundary $w.p.w'$, where $p\in\mathbf{p}$ and $w, w'\in F$	56
$\overline{\mathcal{D}^{\mathbf{p}}}$	the 3-complex based on the pair $(\mathcal{D}, \mathbf{p})$	56
$[\sigma, e]$, $[e, \sigma]$	3-cells in $\overline{\mathcal{D}^{\mathbf{p}}}$ (σ a 2-cell and $e\in\mathbf{e}^+$)	56
$\hat{\mathbf{p}}$	the set of 2-cells $\{\hat{p} = [1, p, 1] : p\in\mathbf{p}\}$ in $\mathcal{D}^{\mathbf{p}}$	59
$\Pi_2 = \Pi_2(\mathcal{P}, \mathbf{p})$	the second homology bimodule of the pair $(\mathcal{P}, \mathbf{p})$	61
$K^{\mathbf{p}}$	the kernel of the bimodule homomorphism (3.6)	62
Π_2^l	the left $\mathbb{Z}S$-module $\Pi_2\otimes_{\mathbb{Z}S}\mathbb{Z}$	74
$p_l(w)$	the leftmost path from a vertex w	74
$\mathbb{P}$	a picture over $\overline{\mathcal{D}^{\mathbf{p}}}$	90
$\rightsquigarrow$	single-step homotopy reduction	107
$\rightsquigarrow^*$	the reflexive, transitive closure of the relation $\rightsquigarrow$	107

Introduction

Every group is the homomorphic image of a free group, and as such can be presented as a quotient of a free group by some normal subgroup. Specifying defining relations as elements of the free group which generate this kernel will then give a concise description of the group in terms of *generators and relations.* Combinatorial group theory is concerned with the study of such presentations which may in fact provide very little group theoretic information. The *word problem* asks whether we are even able to determine whether or not two different words in the generators of the group represent the same group element.

It was the problem of deciding whether or not two knots are the same that led Dehn to the first statement of the word problem [19], and he was able to give a positive solution for the fundamental groups of closed orientable surfaces by using what is now known as the *Dehn algorithm.* This result has two well-known generalizations. Firstly, Magnus [46] showed that the word problem was solvable for all one-relator groups, and later Greendlinger [25] began the study of more general groups allowing a Dehn algorithm solution of their word problem - these are the *word-hyperbolic* groups of Gromov [26].

Solutions to the word problem of other important classes of groups have also been given, for example for abelian groups and for braid groups [62], but the most famous result is that of Novikov [50] and, independently, Boone [9] that there are groups with unsolvable word problem. Moreover, it is possible to write down presentations of such groups - a presentation involving 10 generators and 29 relations with unsolvable word problem is given in [15, §7.2].

In his original paper, Dehn had shown that a knot was trivial if and only if the corresponding knot group is abelian[1], a fact which can be easily ascertained if a solution to the word problem is given. Much later Waldhausen [64] showed that knot groups have solvable word problem, and so it is indeed possible to decide whether a given knot is trivial.

String rewriting systems have been much studied in theoretical computer science - in semigroup and group theory they can be used to give presentations of monoids. Of particular interest are finite *complete* rewriting systems, that is to say finite rewriting systems that are both terminating and confluent, and so have normal form algorithms which solve their word problem. For many years it was an open question as to whether, conversely, any finitely presented monoid with solvable word problem had a finite complete rewriting system, but Squier [59] proved this to be false. He showed that monoids with a finite complete rewriting system were of type left and right FP_3, and then gave an example of a finitely presented monoid with solvable word problem which was not of type left FP_3. In fact, it was implicit in an earlier paper of Anick [3] that monoids admitting finite complete rewriting systems satisfied the stronger finiteness conditions left and right FP_∞. This result has also been proved by Groves [27], Kobayashi [36], and Brown [12].

This thesis begins with a preliminary chapter, Chapter 1, which reviews these concepts in detail, together with other finiteness conditions for rewriting systems which are discussed below. Proofs of results in this chapter are given either where similar ideas are used later in the thesis, or where references are not easily available. Chapter 2 is a short chapter which contains some new results on *λ-complete rewriting systems*, which contain a solution to their word problem when they define groups. In particular, Proposition 2.1.14 characterizes groups with finite λ-complete rewriting systems as those with *λ-Dehn presentations*, a generalization of the standard notion of a Dehn presentation. We also use Britton's Lemma to construct some interesting

[1]Dehn's result relied on an incorrect proof of what is now called Dehn's Lemma, and which was finally proved much later by Papakyriakopoulos [53].

examples of such rewriting systems.

However, the main aim of this thesis is to develop the following ideas one dimension higher. Squier [60] introduced the class of monoids of *finite derivation type (FDT)* which contains each monoid with a finite complete rewriting system. It was then shown by various authors [17, 43, 54] that FDT monoids were necessarily of type left and right FP_3, and also [18, 55] that for finitely presented *groups* these conditions were equivalent. A related property *finite homological type* (FHT) was introduced by Wang and Pride [66].

Roughly speaking, the properties FDT and FHT are defined as follows. If $\mathcal{P}$ is a finite rewriting system for a monoid S consisting of a set of rules $\mathbf{r}$ on an alphabet $\mathbf{x}$, then there is a certain 2-complex $\mathcal{D} = \mathcal{D}(\mathcal{P})$ (the *Squier complex* or *2-complex of monoid pictures*) derived from $\mathcal{P}$ on which the free monoid F on the set $\mathbf{x}$ acts on both the left and the right. We require that the first homotopy or first homology of $\mathcal{D}$ is finitely based, that is to say, there is a finite set $\mathbf{p}$ of closed paths in $\mathcal{D}$ such that attaching 2-cells to the paths $F.\mathbf{p}.F$ gives a 2-complex $\mathcal{D}^{\mathbf{p}}$ which is simply-connected (FDT) or has trivial first homology (FHT).

Both these properties are monoid invariants, that is, they are independent of the choice of finite rewriting system for S [60, 66]. Also, a retract of an FDT or FHT monoid will have the same property [66]. (Recall that a *retract* R of a monoid S is (up to isomorphism) a submonoid of S such that there is a homomorphism (a *retraction*) of S onto R which fixes R elementwise.) From the definitions, it is immediate that an FDT monoid is FHT. An FHT monoid is also of type left and right FP_3 [66], but, on the other hand, Kobayashi and Otto [42] have given an example of a finitely presented monoid of type left and right FP_3 which is not FHT. It remains a major open question as to whether FHT is equivalent to FDT[2] [41, 66] - for *groups* the two are equivalent.

The first homology group of $\mathcal{D}$ is in fact a natural $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule (the *homology bimodule* of $\mathcal{P}$) which we denote by $\Pi = \Pi(\mathcal{P})$, and the FHT property is just the

[2] Pride and Otto have recently found an example of a monoid which is FHT but not FDT.

assertion that this bimodule is finitely generated. There is an important short exact sequence

$$0 \longrightarrow \Pi \xrightarrow{\eta} \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S \longrightarrow \mathcal{M}(\mathcal{P}) \longrightarrow 0 \tag{1}$$

of $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules involving Π. Here $\mathbb{Z}S.\mathbf{r}.\mathbb{Z}S$ denotes the free $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule with basis $\mathbf{r}$, and $\mathcal{M}(\mathcal{P})$ is the *relation bimodule* of $\mathcal{P}$ introduced by Ivanov [32]. The sequence (1) was introduced by Pride [54], apart from the injectivity of η which was proved by Guba and Sapir [29]. An alternative proof has been given by Otto and Kobayashi [40], where it is also shown that the sequence remains exact upon killing either the right or left S-action. To be more specific, applying the tensor $-\otimes_{\mathbb{Z}S}\mathbb{Z}$ to (1) will preserve the injectivity of η, that is, the left $\mathbb{Z}S$-module homomorphism

$$(\eta\otimes 1) : \Pi^l \longrightarrow \mathbb{Z}S.\mathbf{r} \tag{2}$$

is injective, where $\Pi^l = \Pi\otimes_{\mathbb{Z}S}\mathbb{Z}$ is the *left homology module* of $\mathcal{P}$ and $\mathbb{Z}S.\mathbf{r}$ denotes the free left $\mathbb{Z}S$-module with basis $\mathbf{r}$ which is naturally isomorphic to $\mathbb{Z}S.\mathbf{r}.\mathbb{Z}S\otimes_{\mathbb{Z}S}\mathbb{Z}$ [40]. We therefore have a short exact sequence

$$0 \longrightarrow \Pi^l \xrightarrow{\eta\otimes 1} \mathbb{Z}S.\mathbf{r} \longrightarrow \mathcal{M}^l \longrightarrow 0 \tag{3}$$

of left $\mathbb{Z}S$-modules, where $\mathcal{M}^l(\mathcal{P}) = \mathcal{M}^l = \mathcal{M}\otimes_{\mathbb{Z}S}\mathbb{Z}$ is the *left relation module* [32]. An analogous exact sequence of right $\mathbb{Z}S$-modules is obtained by applying $\mathbb{Z}\otimes_{\mathbb{Z}S}-$.

For finitely presented *groups* the properties FDT, FHT and left-FP_3 are equivalent, and in §1.3.4 we give a new proof of the equivalence of FHT and left-FP_3.

Main results

We begin Chapter 3 by constructing a new 3-complex $\overline{\mathcal{D}^{\mathbf{P}}}$ by attaching 3-cells to certain obvious spherical subcomplexes in $\mathcal{D}^{\mathbf{P}}$. It turns out that the second homology group of this 3-complex is also a $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule, which we denote by $\Pi_2 = \Pi_2(\mathcal{P}, \mathbf{p})$, and this module will play the role of Π one dimension higher.

We obtain a short exact sequence analogous to (1):

Theorem 3.3.1 *If the homology classes of the paths in* $\mathbf{p}$ *give rise to a set of bimodule generators of* Π *then there is a short exact sequence*

$$0 \longrightarrow \Pi_2 \xrightarrow{\Phi} \mathbb{Z}S.\mathbf{p}.\mathbb{Z}S \xrightarrow{\nu} \Pi \longrightarrow 0 \tag{4}$$

of $(\mathbb{Z}S, \mathbb{Z}S)$*-bimodules.*

Furthermore, we shall show in §3.4 that the analogue of (2) is injective, giving the following short exact sequence analogous to (1)

Theorem 3.4.1 *If the homology classes of the paths in* $\mathbf{p}$ *give rise to a set of bimodule generators of* Π *then there is a short exact sequence*

$$0 \longrightarrow \Pi_2^l \longrightarrow \mathbb{Z}S.\mathbf{p} \longrightarrow \Pi^l \longrightarrow 0 \tag{5}$$

of left $\mathbb{Z}S$*-modules, where* $\Pi_2^l = \Pi_2 \otimes_{\mathbb{Z}S} \mathbb{Z}$*.*

In Chapter 4 we introduce new finiteness conditions analogous to FDT and FHT. Roughly speaking, S is FDT_2 (respectively FHT_2) if it is FDT (respectively, FHT) and if for some finite rewriting system $\mathcal{P}$ and finite homotopy (respectively, homology) trivializer $\mathbf{p}$ the second homotopy (respectively, homology) is finitely based. There are certain subtleties involved in these definitions which we shall discuss in §5.1, where we also describe alternatives.

In §4.2.1 and §4.2.2 respectively, we prove some invariance properties:

Theorem 4.2.1 *The properties* FDT_2 *and* FHT_2 *are monoid invariants (that is, they are independent of the choice of finite rewriting system and finite trivializer).*

Theorem 4.2.2 *Any retraction of an* FDT_2 *or* FHT_2 *monoid has the same property.*

It is clear that an FDT_2 monoid is also of type FHT_2, and it turns out that for FDT monoids the two properties FDT_2 and FHT_2 are in fact equivalent (see Remark 4.1.3). Consequently, if the properties FHT and FDT turn out to be equivalent, then the properties FDT_2 and FHT_2 will be also.

Note that for finitely presented *groups* the properties FDT_2 and FHT_2 *are* equivalent. In fact (see §4.3),

Theorem 4.3.1 *For finitely presented* groups *the properties FDT_2, FHT_2 and FP_4 are all equivalent.*

This result is obtained by repeating the proof of the analogous result of the equivalence of the properties FHT and left-FP_3 given in §1.3.4: here we use the fact that for an FDT group G, Π_2 is finitely generated as a $(\mathbb{Z}G, \mathbb{Z}G)$-bimodule if and only if Π_2^l is finitely generated as a left $\mathbb{Z}G$-module.

In §4.4 we give proofs of the following result.

Theorem 4.4.1 *A monoid S which admits a finite complete rewriting system is of type FDT_2 (and of type FHT_2).*

Thus our 3-complex $\overline{\mathcal{D}^{\mathcal{P}}}$ and the correspondingly defined finiteness conditions would seem to be correct, in that the results mirror exactly those one dimension lower, which concern the properties FDT and FHT. However, there are certain subtleties involved in these definitions of FDT_2 and FHT_2 which we discuss in §5.1, where we also describe alternatives. We also describe how the properties FDT_2 and FHT_2 relate to finiteness conditions introduced elsewhere, namely *n-dimensional homological finite derivation type* [2] and the property bi-FP_n [41]. In particular, Kobayashi and Otto have characterized FHT monoids as finitely presented monoids of type bi-FP_3 [41], and there is an analogous characterization of FHT_2 monoids as finitely presented monoids of type bi-FP_4.

Finally, the possibility of defining FDT_2 by studying the graph of derivations of Γ leads us to a discussion in §5.2 of a *complete homotopy reduction system* which we associate to complete rewriting systems. The critical pairs of this higher dimensional rewriting system (a variant of the Guba-Sapir reduction system [29, 39]) are shown to correspond to the critical triples of the original rewriting system.

Chapter 1

Preliminaries

1.1 Presentations and the word problem

A *word* w on an alphabet $\mathbf{x}$ is a finite sequence of elements of $\mathbf{x}$. The length $|w|$ of w is just the length of the sequence, and we shall use 1 to denote the word of length zero (the *empty word*). The concatenation of two words w_1 and w_2 is written w_1w_2, and the *free monoid* $F = F(\mathbf{x})$ on $\mathbf{x}$ consists of all words on $\mathbf{x}$ together with this multiplication.

1.1.1 Group presentations

In combinatorial group theory, to an alphabet $\mathbf{x}$ of *generators* we associate a set

$$\mathbf{x}^{-1} = \{x^{-1} : x \in \mathbf{x}\}$$

of formal inverses, and consider words on the alphabet $\mathbf{x} \cup \mathbf{x}^{-1}$. For any such word

$$w = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n},$$

where $x_i \in \mathbf{x}$ and $\varepsilon_i = \pm 1$ for each $j = 1, 2, \ldots, n$, we use w^{-1} to denote the word

$$x_n^{-\varepsilon_n} x_{n-1}^{-\varepsilon_{n-1}} \ldots x_1^{-\varepsilon_1}.$$

Also, we shall say that w is *freely reduced* if it contains no subword of the form $x^{\varepsilon} x^{-\varepsilon}$ ($x \in \mathbf{x}$, $\varepsilon = \pm 1$), and *cyclically reduced* if moreover $x_1^{\varepsilon_1}$ and $x_n^{-\varepsilon_n}$ are distinct.

A *group presentation* is a pair $\mathcal{G} = \langle \mathbf{x}; \mathbf{r} \rangle$ consisting of a set of generators $\mathbf{x}$ and a set $\mathbf{r}$ of words in $\mathbf{x}$, the *relators.* Two words w_1 and w_2 are said to be *equivalent* if one can be converted to the other by a finite sequence of operations of the following type, together with their inverses:

1. replace a word uv with $ux^{\varepsilon}x^{-\varepsilon}v$ where $x \in \mathbf{x}$ and $\varepsilon = \pm 1$;

2. replace a word uv with urv where $r \in \mathbf{r}$.

The equivalence classes form a group (see [47, Theorem 1.1]), the *group defined by* $\mathcal{G}$, where the multiplication is that induced by concatenation of words. We say that $\mathcal{G}$ is a presentation for any group which is in the same isomorphism class as the group it defines.

A group presentation is called finite if both $\mathbf{x}$ and $\mathbf{r}$ are finite sets, and its *word problem* is described as follows:

> *for any pair of words w_1 and w_2 in the generators $\mathbf{x}$, decide whether they are in the same equivalence class.*

A *Tietze transformation* of $\mathcal{G}$ takes one of the following forms:

type I: Replace $\mathcal{G}$ by $\langle \mathbf{x} \, ; \, \mathbf{r} \cup \{w\} \rangle$, where w is a word on $\mathbf{x} \cup \mathbf{x}^{-1}$ equivalent to 1.

type II: Replace $\mathcal{G}$ by $\langle \mathbf{x} \cup \{z\} \, ; \, \mathbf{r} \cup \{z^{-1}w\} \rangle$, where z is a new letter not in $\mathbf{x}$, and w is some word on $\mathbf{x} \cup \mathbf{x}^{-1}$.

The Tietze Theorem [33, p.49] states that two finite group presentations define isomorphic groups if and only if they are *Tietze equivalent*, that is one can be obtained from the other by a finite number of Tietze transformations and their inverses. If the word problem is solvable for some finite presentation of a group G, then it is also solvable for any other finite presentation of G [15, Corollary 1.1.10], and so we can speak of (finitely presented) *groups* with solvable word problem.

In this thesis, however, we shall mostly use *rewriting systems* to present groups and, more generally, monoids.

1.1.2 Rewriting systems

A *rewriting system* is a pair $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ where $\mathbf{x}$ is a set of letters and $\mathbf{r}$ is a set of *rules* of the form $r = (r_{+1}, r_{-1}) \in F \times F$, where $F = F(\mathbf{x})$ denotes the free monoid on $\mathbf{x}$. When displaying a rewriting system we shall generally write r in the form $r_{+1} = r_{-1}$. We say that $\mathcal{P}$ is *finite* if both $\mathbf{x}$ and $\mathbf{r}$ are finite. The corresponding single-step reduction relation

$$\rightarrow_{\mathbf{r}} = \{(wr_{+1}w', wr_{-1}w') : r \in \mathbf{r} \text{ and } w, w' \in F\}$$

on elements of F is given by rewriting words according to the set of rules. The reflexive and transitive closure of this relation is denoted by $\rightarrow^*_{\mathbf{r}}$, and the reflexive, transitive and symmetric closure $\leftrightarrow^*_{\mathbf{r}}$ is the *Thue congruence* generated by $\mathbf{r}$. Two rewriting systems on the same alphabet are said to be *equivalent* if the two sets of rules generate the same Thue congruence.

We denote the congruence class of $w \in F$ by $\overline{w}$, so that the monoid $S = S(\mathcal{P})$ defined by $\mathcal{P}$ consists of these congruence classes with the multiplication

$$\overline{u}.\overline{v} = \overline{uv} \qquad (u, v \in F).$$

We say that a monoid T is *presented* by $\mathcal{P}$ if T is isomorphic to S, and is *finitely presented* if it can be presented by some finite rewriting system.

The Tietze Theorem is easily extended to rewriting systems. A *Tietze transformation* of $\mathcal{P}$ takes one of the following forms:

type I: Replace $\mathcal{P}$ by $[\mathbf{x} \,;\, \mathbf{r} \cup \{u = v\}]$, where $u, v \in F$ satisfy $u \leftrightarrow^*_{\mathbf{r}} v$.

type II: Replace $\mathcal{P}$ by $[\mathbf{x} \cup \{z\} \,;\, \mathbf{r} \cup \{z = w\}]$ where z is a new symbol not in $\mathbf{x}$, and $w \in F(\mathbf{x})$.

Again, two finite rewriting systems present the same monoid if and only if they are *Tietze equivalent*, that is one can be obtained from the other by a finite number of Tietze transformations and their inverses (see, for example, [60, Proposition 4.2]).

Reduction orderings

We call an irreflexive and transitive relation $\succ$ on a set A a *partial order.* A *total order* of A is a partial order such that for distinct elements $a_1, a_2 \in A$ one of $a_1 \succ a_2$ or $a_2 \succ a_1$ holds. We say that a partial order is *noetherian* if there is no infinite sequence

$$a_1, a_2, a_3, \ldots$$

of elements of A satisfying $a_i \succ a_{i+1}$ for each $i = 1, 2, 3, \ldots$.

Let $\succ$ be some partial ordering of the free monoid F which respects the multiplication of F, so that for all $w, u, v, z \in F$ with $u \succ v$ we have $wuz \succ wvz$. If $\succ$ is noetherian then we say that it is a *reduction* order on F. We shall say that a set of rules $\mathbf{r}$ is *compatible* with a reduction ordering $\succ$ if for each $r \in \mathbf{r}$ the relation $r_{+1} \succ r_{-1}$ holds.

We shall need the following reduction orders on free monoids:

Definition 1.1.1 The *length-reducing order* is the reduction order on F given by writing $u \succ v$ if and only if $|u| > |v|$ for any $u, v \in F$.

Definition 1.1.2 Let $\rhd$ be a total order on the alphabet $\mathbf{x}$. The corresponding *length-lexicographical order* is the reduction ordering of F defined as follows. For $u, v \in F$ write $u >_{llex} v$ if and only if $|u| > |v|$, or $|u| = |v|$, $u = au'$ and $v = bv'$, where $a, b \in \mathbf{x}$ and $u', v' \in F$, and one of the following holds:

1. $a \rhd b$
2. $a = b$ and $u' >_{llex} v'$

Note that $>_{llex}$ is a total order.

Definition 1.1.3 (Dershowitz [21]) Let $\rhd$ be a noetherian order on a set $\mathbf{x}$. The corresponding *recursive path ordering from the left* is the reduction order on F defined as follows. For $u, v \in F$ write $u \succ v$ if and only if $u \neq 1$ and $v = 1$, or $u = au'$ and $v = bv'$, where $a, b \in \mathbf{x}$ and $u', v' \in F$, and one of the following holds:

1. $a \rhd b$ and $au' \succ v'$

2. $a = b$ and $u' \succ v'$

3. $u' \succeq bv'$

Noetherian and complete rewriting systems

A rewriting system $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ is said to be *noetherian* if there are no infinite chains $w \rightarrow_{\mathbf{r}} w' \rightarrow_{\mathbf{r}} \cdots$, and such a rewriting system is said to be *complete* if each congruence class contains a unique irreducible word, sometimes called a *normal form.* We can then view the monoid defined by the rewriting system as the set of normal forms, where the multiplication is given by concatenation followed by a reduction to the irreducible.

Example 1.1.4 (Newman [49]) The rewriting system

$$[x,\ x^{-1}\ (x \in \mathbf{x})\ ;\ xx^{-1} = 1,\ x^{-1}x = 1 \quad (x \in \mathbf{x})] \tag{1.1}$$

is a complete rewriting system presenting the free group $\mathcal{F}(\mathbf{x})$ with basis $\mathbf{x}$. The normal forms are just the set of words which are freely reduced. ⌟

It is clear that any monoid with a finite complete rewriting system has solvable word problem, as any two words are in the same congruence class if and only if they have the same normal form[1].

Definition 1.1.5 If $\mathcal{P}$ is noetherian, then the partial order $\succ_{\mathbf{r}}$ on F given by writing

$$wr_{+1}z \succ_{\mathbf{r}} wr_{-1}z$$

for each $r \in \mathbf{r}$ and $w, z \in F$ is a reduction order, *the reduction ordering induced by* $\mathbf{r}$.

[1]It was remarked in the Introduction that Knot Theory provided important motivation for the study of the word problem. It is interesting to note, therefore, that the Freyd-Yetter proof of the existence of the knot invariant known as the HOMFLY polynomial [23] uses a complete set of rewrite rules to obtain the polynomial "normal form".

Integral monoid rings, and the ideal J

For any monoid T, the *integral monoid ring* $\mathbb{Z}T$ consists of all formal sums

$$\sum_{t\in T}\alpha_t t \qquad (\alpha_t\in\mathbb{Z} \text{ non-zero for only finitely many } t\in T),$$

with the natural multiplication

$$(\sum_{t\in T}\alpha_t t)(\sum_{t\in T}\beta_t t) = \sum_{t,t'\in T}\alpha_t\beta_{t'}tt'$$

induced by that of T.

Let F be the free monoid on the alphabet of some rewriting system defining the monoid S. The natural epimorphism

$$F\longrightarrow S \qquad w\mapsto\bar{w}$$

extends linearly to a ring epimorphism

$$\mathbb{Z}F\longrightarrow\mathbb{Z}S$$

of the associated integral monoid rings. We will use J to denote the kernel of this epimorphism. The following lemma is well-known.

Lemma 1.1.6

1. *As an abelian group, J is generated by all elements of the form*

$$u(r_{+1}-r_{-1})v \qquad (u,v\in F, r\in\mathbf{r}). \tag{1.2}$$

2. *As a $(\mathbb{Z}F,\mathbb{Z}F)$-bimodule J is generated by the elements $(r_{+1}-r_{-1})$, where r is in the set of rules $\mathbf{r}$.*

Proof ([40]): Part *2* follows immediately from part *1*.

It is clear that each element of the form (1.2) is in the ideal J. Suppose on the other hand that ξ is an element of J. We can write ξ uniquely as

$$\xi = \sum_{i=1}^{n} \alpha_i w_i \ ,$$

where $w_i \in F$ and α_i is a non-zero integer for each $i = 1, 2, \ldots n$, and if $i \neq j$ then $w_i \neq w_j$. We shall proceed by induction on the value

$$T(\xi) = \sum_{i=1}^{n} |\alpha_i| \ .$$

Firstly, $T(\xi) = 0$ if and only if $\xi = 0$. Suppose that $T(\xi) > 0$. Since

$$\sum_{i=1}^{n} \alpha_i \overline{w}_i = 0$$

we can choose a pair of integers $0 < j, k \leq n$ such that $\alpha_j > 0$, $\alpha_k < 0$ and $\overline{w}_j = \overline{w}_k$, and so there is a sequence of the form

$$w_j = u_1(r_1)_{\varepsilon_1} v_1, \ u_1(r_1)_{-\varepsilon_1} v_1 = u_2(r_2)_{\varepsilon_2} v_2, \ \ldots, \ u_m(r_m)_{\varepsilon_m} v_m = w_k$$

describing a derivation from w_j to w_k, where $u_i, v_i \in F$, $r_i \in \mathbf{r}$ and $\varepsilon_i = \pm 1$ for $i = 1, 2, \ldots, m$. Then

$$w_j - w_k = \sum_{i=1}^{m} \varepsilon_i u_i ((r_i)_{+1} - (r_i)_{-1}) v_i$$

is in the subgroup generated by elements of the form (1.2). So is $\xi - (w_j - w_k)$ by induction, since $T(\xi - (w_j - w_k)) = T(\xi) - 2$, and therefore ξ is also. This completes the proof. ┘

In the special case where S is the trivial monoid we denote by I the kernel of the *augmentation mapping*

$$\mathbb{Z}F \longrightarrow \mathbb{Z} \qquad w \mapsto 1 \quad (w \in F).$$

1.2 Related complexes

1.2.1 The graph of derivations

The *graph of derivations* $\Gamma = \Gamma(\mathcal{P})$ associated to $\mathcal{P}$ is a geometric interpretation of the rewriting system, and is constructed as follows. The vertex set is the free monoid F, and the edge set consists of all quadruples of the form

$$e = (w, r, \varepsilon, w') \qquad w, w' \in F,\ r \in \mathbf{r} \text{ and } \varepsilon = \pm 1$$

with initial, terminal and inverse functions

$$\iota e = w r_\varepsilon w', \quad \tau e = w r_{-\varepsilon} w' \text{ and } e^{-1} = (w, r, -\varepsilon, w').$$

The edge e is called *positive* if $\varepsilon = 1$, and we shall denote by $\mathbf{e}^+$ the set of all positive edges. A *path* p of length n in Γ is a sequence $p = e_1 e_2 \ldots e_n$ of edges with $\iota e_{i+1} = \tau e_i \quad (i = 1, \ldots, n-1)$, and it is *positive* if it is composed of positive edges. We say that p is *closed* if $\iota e_1 = \tau e_n$.

Sometimes it will be convenient to depict the positive edge e as

$$w \underline{r_{+1}} w' \to w r_{-1} w' \ ,$$

where the subword being rewritten is underlined. As an illustration, in the graph Γ of the rewriting system (1.1) in Example 1.1.4 the edge $(x, (xx^{-1} = 1), +1, x)$ is more clearly described by writing

$$x\underline{xx^{-1}}x \to x^2 \, .$$

There is a natural two-sided action of F on Γ. The action on the vertices is given by left and right multiplication in F, and for an edge $e = (w, r, \varepsilon, w')$ and $z, z' \in F$ we define

$$z.e.z' = (zw, r, \varepsilon, w'z'),$$

and this action extends to paths.

Note that there is a positive edge from u to v $(u, v \in F)$ if and only if $u \rightarrow_{\mathbf{r}} v$, and a path in Γ from u to v describes a *derivation* in F rewriting u to v using the rules in $\mathbf{r}$ and their inverses. Thus u and v lie in the same connected component of Γ if and only if $u \leftrightarrow_{\mathbf{r}}^{*} v$, and so there is a one-to-one correspondence between the connected components of Γ and the elements of S. Note also that the abelian group generators (1.2) of J are precisely the elements $\iota e - \tau e$ $(e \in \mathbf{e}^{+})$.

We will denote the set of positive edges with initial vertex $w \in F$ by $star^{+}(w)$, and we order these edges as follows: If $e = (u, r, +1, v)$ and $f = (u', r', +1, v')$ where $u, u', v, v' \in F$ and $r, r' \in \mathbf{r}$ with $\iota e = \iota f = w$, then write $e <_{w} f$ if:

1. u is a proper prefix of u'; or

2. $u = u'$ and r_{+1} is a proper prefix of r'_{+1}; or

3. $u = u'$, $r_{+1} = r'_{+1}$ and $r_{-1} <_{llex} r'_{-1}$ in some chosen length-lexicographical order $<_{llex}$ on F.

In particular, each non-irreducible vertex has a least outgoing edge which, following Guba and Sapir [29, Definition 9.1], we call the *left principal* edge. We say that two positive edges e_1, e_2 with the same initial vertex are *disjoint* if they can be written in the form

$$e_1 = f_1 . \iota f_2, \qquad e_2 = \iota f_1 . f_2$$

for some positive edges f_1, f_2. Also, we say that an edge e is *left-reduced* (respectively, *right-reduced*) if it cannot be written in the form $u.f$ (respectively, $f.u$) for some non-empty word $u \in F$ and edge f.

Example 1.2.1 Consider the vertex $w = x^2 x^{-1} x^3 x^{-1}$ in the graph Γ of the rewriting system (1.1) in Example 1.1.4. The set $star^{+}(w)$ consists of the edges

$$x\underline{xx^{-1}}x^3x^{-1} \rightarrow x^4x^{-1}, \ x^2\underline{x^{-1}x}x^2x^{-1} \rightarrow x^4x^{-1}, \text{ and } x^2x^{-1}x^2\underline{xx^{-1}} \rightarrow x^2x^{-1}x^2$$

which for convenience we label e, f and g respectively. These edges are ordered as $e <_w f <_w g$, so that e is the left principal edge. The edge g is disjoint from both e and f, and is, moreover, right-reduced. ⌟

The following well-known result is related to König's Lemma:

Lemma 1.2.2 *If $\mathcal{P}$ is noetherian, then for any vertex w in Γ there is a bound on the length of any positive path originating at w.*

Proof: Otherwise, we may choose $w_1 \in F$ with $w \to_{\mathbf{r}} w_1$ and with no bound on the lengths of paths originating at w_1. In the same way we may choose some $w_2 \in F$ with $w_1 \to_{\mathbf{r}} w_2$ and no bound on the lengths of paths originating at w_2, and so on, producing an infinite reduction sequence $w \to_{\mathbf{r}} w_1 \to_{\mathbf{r}} w_2 \to_{\mathbf{r}} \ldots$, a contradiction. ⌟

Definition 1.2.3 By Lemma 1.2.2, we can define the following height function $\vartheta = \vartheta_{\mathcal{P}}$ (called the *disorder function* in [28]) , which we use in §4.4 and §5.2.

$$\begin{aligned} \vartheta \ : \ & F \longrightarrow \mathbb{N} \\ & w \longmapsto \text{max. length of all paths originating at } w. \end{aligned}$$

1.2.2 The 2-complex $\mathcal{D}$ and its homology

The 2-complex $\mathcal{D} = \mathcal{D}(\mathcal{P})$ associated to $\mathcal{P}$ is the combinatorial 2-complex with 1-skeleton Γ, to which, for each pair of positive edges e and f, a 2-cell $[e, f]$ is attached along the closed path

$$\partial[e,f] = (e.\iota f)(\tau e.f)(e^{-1}.\tau f)(\iota e.f^{-1}).$$

Remark: In the usual definition of the Squier complex, for each pair of positive edges e and f we also attach 2-cells $[e, f^{-1}]$, $[e^{-1}, f]$ and $[e^{-1}, f^{-1}]$ along closed paths

$$\begin{aligned} \partial[e,f^{-1}] &= (e.\tau f)(\tau e.f^{-1})(e^{-1}.\iota f)(\iota e.f) \\ \partial[e^{-1},f] &= (e^{-1}.\iota f)(\iota e.f)(e.\tau f)(\tau e.f^{-1}) \\ \partial[e^{-1},f^{-1}] &= (e^{-1}.\tau f)(\iota e.f^{-1})(e.\iota f)(\tau e.f). \end{aligned}$$

But the additional cells here are attached along paths which are cyclic permutations of $(\partial[e,f])^{\pm 1}$, and since we are interested in the homology and homotopy of this complex it is unnecessary to include them: In the presence of $[e,f]$ only, the attaching paths of the others are already null-homotopic. $\lrcorner$

Squier [60] in fact introduced a homotopy relation on Γ which coincides with homotopy of this 2-complex: $\mathcal{D}$ is properly "the 2-complex of monoid pictures" introduced independently by both Pride [54] and Kilibarda [34]. The term "Squier complex" was introduced by Guba and Sapir [29, Introduction].

The two sided action of F on Γ extends naturally to the 2-cells by

$$w.[e,f].w' = [w.e, f.w'] \quad (w, w' \in F,\ e, f \text{ positive edges}).$$

Homology

We have the chain complex

$$\mathbf{C}(\mathcal{D}): \qquad 0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0 \tag{1.3}$$

where the chain groups C_0, C_1 and C_2 are, respectively, the free abelian groups with bases F, the set of all positive edges, and the set of all 2-cells, and the boundary maps are

$$\partial_1 e = \iota e - \tau e \qquad (e \text{ a positive edge})$$

and

$$\partial_2[e,f] = e.(\iota f - \tau f) - (\iota e - \tau e)f \qquad (e, f \text{ positive edges}).$$

The chain groups are $(\mathbb{Z}F, \mathbb{Z}F)$-bimodules where the F-action is inherited from the two-sided action of F on $\mathcal{D}$ and, moreover, the boundary maps respect this action. It follows that the first homology group $H_1(\mathcal{D})$ is also a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule. It can be shown [54, Lemma 4.1] that

$$J.H_1(\mathcal{D}) = H_1(\mathcal{D}).J = 0 \tag{1.4}$$

and so $H_1(\mathcal{D})$ has an induced $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule structure. This is the *homology bimodule* of $\mathcal{P}$, denoted by $\Pi = \Pi(\mathcal{P})$. In the short exact sequence (1) of $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules Π is exhibited as the kernel of a presentation of the *relation bimodule* $\mathcal{M}(\mathcal{P}) = J/J^2$ of $\mathcal{P}$.

If we use $(\mathcal{D}, s)$ to denote the connected component of $\mathcal{D}$ corresponding to the element s of S, and write $\Pi_s = H_1(\mathcal{D}, s)$, then we have the (abelian group) decomposition

$$\Pi = \bigoplus_{s \in S} \Pi_s. \tag{1.5}$$

For $a, b \in S$ we then have $a.\Pi_s.b \subseteq \Pi_{asb}$, with equality if a and b are units.

In particular *if S is a group* then we have

$$a.\Pi_s.b = \Pi_{asb} \text{ for all } a, s, b \in S. \tag{1.6}$$

We shall require this in our proof of Theorem 1.3.4.

Bimodules

For a set $\mathbf{X}$ and monoid T, we shall use $\mathbb{Z}T.\mathbf{X}.\mathbb{Z}T$ to denote the free $(\mathbb{Z}T, \mathbb{Z}T)$-bimodule with basis $\mathbf{X}$. The underlying abelian group is free with basis

$$\{t.x.t' : t, t' \in T \text{ and } x \in \mathbf{X}\},$$

on which the two-sided T-action is given by writing

$$s.(t.x.t').s' = st.x.t's'$$

for any $s, s' \in T$.

The following lemma (and proof) is from a paper of Kobayashi and Otto [40, Lemma 6.1].

Lemma 1.2.4 *Let S be the monoid defined by some rewriting system, let F be the free monoid on its alphabet, and let $\mathbb{Z}F.\mathbf{X}.\mathbb{Z}F$ and $\mathbb{Z}S.\mathbf{X}.\mathbb{Z}S$ be the free $(\mathbb{Z}F, \mathbb{Z}F)$ and $(\mathbb{Z}S, \mathbb{Z}S)$ bimodules on some set $\mathbf{X}$. If we consider $\mathbb{Z}S.\mathbf{X}.\mathbb{Z}S$ as a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule, with F acting via the epimorphism $F \longrightarrow S$, then we have a natural epimorphism*

$$\nu : \mathbb{Z}F.\mathbf{X}.\mathbb{Z}F \longrightarrow \mathbb{Z}S.\mathbf{X}.\mathbb{Z}S \qquad w.x.z \longmapsto \overline{w}.x.\overline{z} \qquad (x \in \mathbf{X}, w, z \in F) \tag{1.7}$$

of $(\mathbb{Z}F, \mathbb{Z}F)$-bimodules. The kernel $Ker(\nu)$ of (1.7) can be written as

$$J.\mathbf{X}.\mathbb{Z}F + \mathbb{Z}F.\mathbf{X}.J\,.$$

Proof: It is clear that the inclusion $J.\mathbf{X}.\mathbb{Z}F + \mathbb{Z}F.\mathbf{X}.J \subseteq Ker(\nu)$ holds, so we must show that the converse inclusion is also true.

If we choose for each congruence class $\overline{w}$ a representative element $\hat{w}$, then we can define homomorphisms

$$\theta : \mathbb{Z}S.\mathbf{X}.\mathbb{Z}S \longrightarrow \mathbb{Z}F.\mathbf{X}.\mathbb{Z}F \qquad \overline{w}.x.\overline{z} \longmapsto \hat{w}.x.\hat{z} \qquad (x \in \mathbf{X}, w, z \in F)$$

and

$$\phi : \mathbb{Z}F.\mathbf{X}.\mathbb{Z}F \longrightarrow \mathbb{Z}F.\mathbf{X}.\mathbb{Z}F \qquad w.x.z \longmapsto w.x.z - \hat{w}.x.\hat{z} \qquad (x \in \mathbf{X}, w, z \in F).$$

of abelian groups. Note that $(\theta\nu + \phi)$ is the identity on $\mathbb{Z}F.\mathbf{X}.\mathbb{Z}F$, mapping a basis element $w.x.z$ $(w, z \in F, x \in \mathbf{X})$ to

$$\theta(\overline{w}.x.\overline{z}) + w.x.z - \hat{w}.x.\hat{z} = \hat{w}.x.\hat{z} + w.x.z - \hat{w}.x.\hat{z} = w.x.z.$$

Now because, for any $x \in \mathbf{X}$ and $w, z \in F$,

$$\begin{aligned} \phi(w.x.z) &= w.x.z - \hat{w}.x.\hat{z} \\ &= (w - \hat{w}).x.z + \hat{w}.x.(z - \hat{z}), \end{aligned}$$

we see that $Im(\phi) \subseteq J.\mathbf{X}.\mathbb{Z}F + \mathbb{Z}F.\mathbf{X}.J$. But for any $\xi \in Ker(\nu)$,

$$\xi = (\theta\nu + \phi)\xi = \phi\xi\,,$$

and therefore $\xi \in J.\mathbf{X}.\mathbb{Z}F + \mathbb{Z}F.\mathbf{X}.J$. ⌟

1.3 Finiteness conditions

1.3.1 Finite derivation type (FDT) and finite homological type (FHT)

Let $\mathbf{p}$ be a set of closed paths in $\mathcal{D}$. We shall denote by $\mathcal{D}^{\mathbf{p}}$ the 2-complex obtained by attaching 2-cells $[w, p, w']$ along all closed paths of the form

$$\partial[w, p, w'] = w.p.w' \qquad (p \in \mathbf{p},\ w, w' \in F).$$

A rewriting system $\mathcal{P}$ is said to be of *finite derivation type* (FDT) (respectively, *finite homological type* (FHT)) if it is finite, and if there is some *finite* set $\mathbf{p}$ (a *homotopy* (respectively, *homology*) *trivializer*) of closed paths in $\mathcal{D}$ such that the 2-complex $\mathcal{D}^{\mathbf{p}}$ has trivial fundamental groups (respectively, has trivial first homology). An equivalent formulation of the FHT property is that there is a finite set of closed paths in $\mathcal{D}$ whose homology classes generate Π as a $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule. Note that a homotopy trivializer is also a homology trivializer, and so if $\mathcal{P}$ is FDT then it is also FHT; as remarked in the introduction, whether the converse holds is an open question[2] [42, 66].

The properties FDT and FHT are in fact monoid invariants in the sense that if $\mathcal{P}$ and $\mathcal{P}'$ are two finite rewriting systems presenting isomorphic monoids, if one of them has the property FDT (respectively, FHT) then so does the other [60, 66]; thus we can talk about FDT and FHT *monoids*. These properties are also closed under retraction [66].

Squier [60] showed that monoids with finite complete rewriting systems are of type FDT (see §1.4.2), and hence also FHT [54, 60]. More recently, Kobayashi [38] has shown that monoids with a finite presentation containing one relation only are of type FDT; it is an open question as to whether such monoids have finite complete rewriting systems.

[2]Pride and Otto have recently found an example of a monoid which is FHT but not FDT.

1.3.2 Homological finiteness conditions

A monoid S is left-FP_n (respectively, right-FP_n) if $\mathbb{Z}$, regarded as a left (respectively, right) $\mathbb{Z}S$-module with trivial S-action

$$s.n = n \quad (\text{resp. } n.s = n) \qquad (s\in S, n\in\mathbb{Z}),$$

has a partial resolution

$$A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} A_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

by free left (respectively, right) $\mathbb{Z}S$-modules of finite rank.

The following is a standard result of homological algebra.

Lemma 1.3.1 (generalized Schanuel's Lemma [11, p.193]) *Let*

$$0 \longrightarrow K \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots P_0 \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow K' \longrightarrow P'_n \longrightarrow P'_{n-1} \longrightarrow \cdots P'_0 \longrightarrow M \longrightarrow 0$$

be exact sequences of modules (or bimodules) where P_i and P'_i are projective for each $i = 0, 1, \ldots, n$. If n is even then there is an isomorphism

$$P_0 \oplus P'_1 \oplus P_2 \oplus P'_3 \oplus \cdots \oplus P_n \oplus K' \quad \cong \quad P'_0 \oplus P_1 \oplus P'_2 \oplus P_3 \oplus \cdots \oplus P'_n \oplus K\ ,$$

otherwise there is an isomorphism

$$P_0 \oplus P'_1 \oplus P_2 \oplus P'_3 \oplus \cdots \oplus P'_n \oplus K \quad \cong \quad P'_0 \oplus P_1 \oplus P'_2 \oplus P_3 \oplus \cdots \oplus P_n \oplus K'\ .$$

Consequently, if P_i and P'_i are finitely generated for $i \leq n$, then K is finitely generated if and only if K' is finitely generated. ⌟

If we splice together the sequence (3) with sequences found in [54] we obtain the following complex

$$0 \longrightarrow \Pi^l \longrightarrow \mathbb{Z}S.\mathbf{r} \longrightarrow \mathbb{Z}S.\mathbf{x} \longrightarrow \mathbb{Z}S \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{1.8}$$

giving a partial resolution of $\mathbb{Z}$ up to dimension 2, and so if S is finitely presented, then by the generalized Schanuel Lemma (Lemma 1.3.1 above) S is of type left-FP_3 if and only if Π^l is finitely generated as a left $\mathbb{Z}S$-module. There is also the corresponding sequence of right $\mathbb{Z}S$-modules.

For groups the properties left- and right-FP_n are equivalent for all n. However, this is not the case for monoids in general. Cohen has given an example of a monoid which is right-FP_n for all $n\geq 0$ but is not left-FP_1 [13]; also a monoid of type left-FP_n for all $n\geq 0$ introduced by Squier [59] was later shown not to be right-FP_3 by Pride and Wang [56].

Another standard result of homological algebra is the Snake Lemma, which we shall use in Chapter 3.

Lemma 1.3.2 (Snake Lemma) *Suppose we have the commutative diagram*

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r, "\rho"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\iota"] & B' \arrow[r] & C' & \end{tikzcd}$$

where both rows are exact.

Then the map $D : ker\gamma \longrightarrow coker\alpha$ defined by $c \longmapsto \iota^{-1}\beta\rho^{-1}c + im\alpha$ is a homomorphism. Moreover, there is an exact sequence

$$ker\alpha \longrightarrow ker\beta \longrightarrow ker\gamma \xrightarrow{D} coker\alpha \longrightarrow coker\beta \longrightarrow coker\gamma.$$

1.3.3 FDT and FHT monoids are left and right FP_3

Let $\mathbf{p}$ be a set of closed paths in the 2-complex $\mathcal{D}$, and consider the sequence

$$\mathbb{Z}S.\mathbf{p} \xrightarrow{\partial_3} \mathbb{Z}S.\mathbf{r} \xrightarrow{\partial_2} \mathbb{Z}S.\mathbf{x} \xrightarrow{\partial_1} \mathbb{Z}S \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0 \tag{1.9}$$

of left $\mathbb{Z}S$-modules, where the homomorphisms are defined as follows. Firstly, ∂_0 is the *augmentation homomorphism*, mapping each $s \in S$ to 1, and ∂_1 is the homomorphism defined by the map

$$x \longmapsto (x-1) \qquad (x \in \mathbf{x}).$$

If for $w = x_1 x_2 \ldots x_n \in F$ we use $\langle w \rangle$ to denote the element

$$\overline{x_1 \ldots x_{n-1}}.x_n + \overline{x_1 \ldots x_{n-2}}.x_{n-1} + \ldots + x_1$$

of $\mathbb{Z}S.\mathbf{x}$, then ∂_2 is the homomorphism defined by the map

$$r \longmapsto \langle r_{-1} \rangle - \langle r_{+1} \rangle \qquad (r \in \mathbf{r}).$$

Lastly, ∂_3 is the homomorphism defined by the map

$$p \longmapsto \sum_{i=1}^{n} \varepsilon_i \overline{u_i}.r_i$$

for $p = (u_1, r_1, \varepsilon_1, v_1) \ldots (u_n, r_n, \varepsilon_n, v_n) \in \mathbf{p}$, where, for $i = 1, 2, \ldots, n$, $u_i, v_i \in F$, $\varepsilon_i = \pm 1$ and $r_i \in \mathbf{r}$.

It is straightforward to see that (1.9) is a complex, and in fact

Theorem 1.3.3 ([17, 54]) *If* $\mathbf{p}$ *is a homology trivializer then the complex* (1.9) *is exact.*

A partial resolution of right $\mathbb{Z}S$-modules can be constructed in the same way, and therefore FDT and FHT monoids are of type left and right FP_3.

1.3.4 The properties FDT and FHT for groups

It has been shown that *for finitely presented groups*, the properties FDT, FHT and FP_3 are equivalent. This follows from the equivalence of FDT and FP_3, shown by Cremanns and Otto [18] and Pride [55], together with the following result, for which we give a new proof:

Theorem 1.3.4 ([55]) *For finitely presented groups, the properties FHT and FP_3 are equivalent.*

Proof: We use the following general result. Let G be a group and let A be a $(\mathbb{Z}G, \mathbb{Z}G)$-bimodule with an abelian group decomposition

$$A = \bigoplus_{g \in G} A_g$$

such that $h_1.A_g.h_2 = A_{h_1 g h_2}$ for all $h_1, g, h_2 \in G$. Then A_1 has a left $\mathbb{Z}G$-module structure with G-action

$$g * a = g.a.g^{-1} \qquad (g \in G, a \in A_1).$$

Lemma 1.3.5 *1. $A \otimes_{\mathbb{Z}G} \mathbb{Z}$ and A_1 are isomorphic as left $\mathbb{Z}G$-modules.*

2. If A_1 is finitely generated as a left $\mathbb{Z}G$-module then A is finitely generated as a bimodule.

Proof:

1. Regarding $\mathbb{Z}$ as a left $\mathbb{Z}G$-module with trivial G-action, the map

$$\psi : A \times \mathbb{Z} \longrightarrow A_1 \qquad (\sum_{g \in G} a_g, n) \longmapsto n \sum_{g \in G} a_g.g^{-1} \qquad (a_g \in A_g, n \in \mathbb{Z})$$

is bilinear, since in particular for any $h \in G$,

$$\psi((\sum_{g \in G} a_g).h, n) = n \sum_{g \in G} a_g.h(gh)^{-1} = n \sum_{g \in G} a_g.g^{-1} = \psi(\sum_{g \in G} a_g, h.n).$$

Therefore ψ induces a homomorphism

$$\Psi : A \otimes_{\mathbb{Z}G} \mathbb{Z} \longrightarrow A_1 \qquad \sum_{g \in G} a_g \otimes 1 \longmapsto \sum_{g \in G} a_g.g^{-1}$$

which is the inverse of the homomorphism

$$A_1 \longrightarrow A \otimes_{\mathbb{Z}G} \mathbb{Z} \qquad a \longmapsto a \otimes 1$$

induced by the inclusion of A_1 into A. It is easily seen that these isomorphisms respect the left G-actions.

2. Suppose that the left module A_1 has finite generating set $B = \{b_1, \ldots, b_n\}$. Then for any $a \in A_g$ we can express $a.g^{-1} \in A_1$ as a sum

$$a.g^{-1} = \sum_{i=1}^{k} \varepsilon_i (h_i * b_{j_i}) = \sum_{i=1}^{k} \varepsilon_i h_i.b_{j_i}.h_i^{-1},$$

where, for $1 \leq i \leq k$, $\varepsilon_i = \pm 1, h_i \in G$ and $1 \leq j_i \leq n$, and so in the bimodule A we can write

$$a = \sum_{i=1}^{k} \varepsilon_i h_i b_{j_i} (h_i^{-1} g).$$

Therefore B generates A as a bimodule. ⌟

Theorem 1.3.4 now follows by applying the Lemma to the decomposition (1.5) of Π, taking account of (1.6): the bimodule Π is finitely generated if and only if Π^l is finitely generated as a left-module, and this is the case if and only if S is FP_3 (applying the generalized Schanuel's Lemma (Lemma 1.3.1) to the exact sequence (1.8)). ⌟

1.4 Complete rewriting systems

General references for complete rewriting systems are [8, 7, 14]. In §1.1.2, we gave the following definition: a rewriting system $\mathcal{P}$ is complete if it is noetherian and it has precisely one irreducible element in each congruence class. Therefore we can denote by $Irr(w)$ the irreducible word obtained by rewriting any word w, and if z is another word in the same congruence class, then $Irr(z) = Irr(w)$. We now give two equivalent definitions.

For any $w \in F$ we shall say that $\mathcal{P}$ is *confluent* (respectively, *locally confluent*) *at* w if whenever $u, v \in F$ satisfy $w \rightarrow_{\mathbf{r}}^{*} u$ and $w \rightarrow_{\mathbf{r}}^{*} v$ (respectively, $w \rightarrow_{\mathbf{r}} u$ and $w \rightarrow_{\mathbf{r}} v$) then there is some $z \in F$ such that $u \rightarrow_{\mathbf{r}}^{*} z$ and $v \rightarrow_{\mathbf{r}}^{*} z$. We shall say that $\mathcal{P}$ is *confluent* (respectively, *locally confluent*) if it is confluent at w (respectively locally confluent at w) for all $w \in F$.

In geometric terms, $\mathcal{P}$ is confluent (respectively, locally confluent) if whenever there is a pair of positive paths (respectively, positive edges) in Γ from $w \in F$ to vertices u and v, there are also positive paths in Γ from u and v to some common terminal vertex.

Lemma 1.4.1 (Newman [49]) *Any congruence class in a confluent rewriting system contains at most one irreducible word.*

Proof: For any pair of words u and u' lying in the same congruence class, there is a path in Γ from one to the other. We shall use an inductive argument on the path length to show that there is a pair of positive paths from u and u' to some common vertex.

If the path has length one, then the pair of positive paths consists of either the path itself or its inverse, together with the corresponding empty path at u or u'. If the path has length $n > 1$, it can be considered as the product of two non-empty paths of length $< n$: a path from u to some vertex v, say, together with a path from v to u'. By induction, there are positive paths p and q from u and v respectively to some common vertex w, say, and also positive paths q' and p' from v and u' to some common vertex w'. Because $\mathcal{P}$ is confluent, there are positive paths r and r' from w and w' to some common vertex z, say, and then pr and $p'r'$ are a pair of positive paths from u and u' to z.

The lemma is now straightforward to prove: if u and u' are irreducible words in the same congruence class, then there is a pair of positive paths from these words to some common vertex, and because these paths must be empty we see that u and u' are the common vertex, that is, $u = u'$. ⌟

We therefore have the following characterization, due to Newman [49]:

$$\textit{a rewriting system is complete if and only if it is noetherian and confluent.} \tag{1.10}$$

Lemma 1.4.2 (Principal of noetherian induction) *Let P be a predicate on a set A with a noetherian order $\succ$. Suppose that whenever $a \in A$ has the property that $P(a')$ holds for every $a' \in A$ with $a \succ a'$, then $P(a)$ holds. Then $P(a)$ holds for every $a \in A$.*

Proof. For suppose that $P(a)$ does not hold for some $a \in A$. Then by our supposition there will be some $a_1 \in A$ such that $a \succ a_1$ and $P(a_1)$ is false. Continuing this argument will give an infinite sequence

$$a \succ a_1 \succ a_2 \succ a_3 \succ \cdots \quad ,$$

a contradiction to our assumption that $\succ$ is a noetherian order. ⌟

This method of induction will be used in proving Proposition 3.3.4 and Theorem 4.4.1; we illustrate its use here by giving the well-known proof of the so-called Diamond Lemma:

Lemma 1.4.3 (Diamond Lemma [49]) *A noetherian and locally confluent rewriting system $\mathcal{P}$ is confluent.*

Proof: We use noetherian induction on F, where the order is the reduction order $\succ_{\mathbf{r}}$ induced by the rules.

Let w be an element of F, and inductively assume the rewriting system is confluent at any vertex $w' \in F$ with $w \succ_{\mathbf{r}} w'$. Suppose that $w \to^*_{\mathbf{r}} u$ and $w \to^*_{\mathbf{r}} v$ hold for some $u, v \in F$, so there are positive paths p and q in Γ from w to u and v respectively. We want to deduce that $\mathcal{P}$ is also confluent at w, that is, there is some $z \in F$ such that $u \to^*_{\mathbf{r}} z$ and $v \to^*_{\mathbf{r}} z$.

Firstly, if w is irreducible, then $u = v = w$ and we can just take z to be w.

Suppose on the other hand that w is not irreducible. If either of the paths is empty, say $w = u$, then we can again just take $z = v$, so we can assume that both paths are non-empty and can therefore be written as products $p = ep'$ and $q = fq'$, where e and f are positive edges and p', q' are positive paths. Since $\mathcal{P}$ is locally confluent, there is some $w' \in F$ such that $\tau e \to^*_{\mathbf{r}} w'$ and $\tau f \to^*_{\mathbf{r}} w'$.

$$\begin{tikzcd}
 & & w \arrow[dl] \arrow[dr] & & \\
 & \tau e \arrow[dl, "*"'] \arrow[dr, "*"] & & \tau f \arrow[dl, "*"'] \arrow[dr, "*"] & \\
u \arrow[dr, "*"] & & w' \arrow[dl, "*"'] \arrow[dr, "*"] & & v \arrow[dl, "*"'] \\
 & u' \arrow[dr, "*"] & & v' \arrow[dl, "*"'] & \\
 & & z & &
\end{tikzcd}$$

Figure 1.1: The Diamond Lemma

Because $\tau e \to^*_{\mathrm{r}} u$ and $\tau e \to^*_{\mathrm{r}} w'$ hold, by inductive assumption there is some $u' \in F$ such that $u \to^*_{\mathrm{r}} u'$ and $w' \to^*_{\mathrm{r}} u'$; similarly, there is some $v' \in F$ such that $v \to^*_{\mathrm{r}} v'$ and $w' \to^*_{\mathrm{r}} v'$. Also, because $w' \to^*_{\mathrm{r}} u'$ and $w' \to^*_{\mathrm{r}} v'$ hold, there is some $z \in F$ such that $u' \to^*_{\mathrm{r}} z$ and $v' \to^*_{\mathrm{r}} z$. We therefore have (see Figure 1.1)

$$u \to^*_{\mathrm{r}} u' \to^*_{\mathrm{r}} z \quad \text{and} \quad v \to^*_{\mathrm{r}} v' \to^*_{\mathrm{r}} z \; ;$$

thus $\mathcal{P}$ is confluent at w, and the lemma follows by noetherian induction (Lemma 1.4.2). ┘

Thus we have the alternative characterization:

a rewriting system is complete if and only if it is noetherian and locally confluent. (1.11)

The class of monoids which have finite complete rewriting systems has no known alternative characterization, although as we have seen such monoids must be finitely presented and have solvable word problem. We also have the following result:

Theorem 1.4.4 (Anick-Groves-Squier [3, 27, 59]) *A monoid with a finite complete rewriting system is of type left and right FP_∞.*

In fact Squier [59] showed that such monoids were of type left and right FP_3, and Groves [27] gave the more general result; later it was realized that this result was contained in work already published by Anick [3]. Other proofs have been given by Kobayashi [36], and Brown [12].

Corollary 1.4.5 (Squier [59]) *There are finitely presented monoids with solvable word problem which do not have finite complete rewriting systems.*

Squier proved this by appealing to known examples of finitely-presented groups with solvable word problems which are not of type FP_3, for example groups studied by Stallings [61] and Abels [1]. Abels' example is a group of matrices, which we shall consider in §A.1, and Stallings' group is shown to have solvable word problem in [4]. Squier also gave his own examples of finite rewriting systems with solvable word problem for monoids which are not left FP_3.

Critical pairs

The characterization (1.11) prompts the study of local confluence.

Definition 1.4.6 *A pair of positive edges with the same initial vertex form a* critical pair *if either:*

1. *One of the pair is both left- and right-reduced (a critical pair of* inclusion *type); or*

2. *One of the pair is left-reduced but not right-reduced, the other is right-reduced but not left-reduced, and they are not disjoint (a critical pair of* overlapping *type).*

Remark 1.4.7 We are trying here to emphasize the geometric interpretation of rewriting systems - the usual definition of a critical pair would only be the terminal vertices of one of our critical pairs (of edges), that is, the two possible results of rewriting the initial word.

Example 1.4.8 Consider the rewriting system (1.1)

$$[\mathbf{x} \cup \mathbf{x}^{-1}\ ;\ xx^{-1} = 1,\ x^{-1}x = 1 \quad (x \in \mathbf{x})]$$

of Example 1.1.4 defining the free group with basis $\mathbf{x}$. Each critical pair is of overlapping type and is in the star of a vertex $xx^{-1}x$ or $x^{-1}xx^{-1}$ $(x \in \mathbf{x})$:

$$\begin{array}{ccc} & & x \\ & \nearrow & \\ \underline{xx^{-1}}x & & \\ \| & & \\ x\underline{x^{-1}x} & & \\ & \searrow & \\ & & x \end{array} \qquad \text{or} \qquad \begin{array}{ccc} & & x^{-1} \\ & \nearrow & \\ \underline{x^{-1}x}\,x^{-1} & & \\ \| & & \\ x^{-1}\underline{xx^{-1}} & & \\ & \searrow & \\ & & x^{-1} \end{array}$$

$\lrcorner$

Note that any pair of edges in $star^{+}(w)$ $(w \in F)$ are either disjoint or are a translate of a critical pair by the two-sided action of F. A critical pair as defined above is a pair of edges corresponding to the two different ways of rewriting a word composed of the (non-disjoint) left-hand-sides of two rules.

Definition 1.4.9 *A* resolution *of a critical pair* (e, f) *is a pair of positive paths from* τe *and* τf *to some common vertex; we say that a critical pair is* resolvable *if it has a resolution.*

Example 1.4.8 is particularly simple as its critical pairs are resolved immediately (that is, with empty paths).

Example 1.4.10 The rewriting system given by the rules

$$ac = ca, \quad caaa = aa, \quad bacb = 1$$

on the alphabet a, b, c has one critical pair of inclusion type,

$$\begin{array}{ccc} & & 1 \\ & \nearrow & \\ \underline{bacb} & & \\ \| & & \\ b\underline{ac}\,b & & \\ & \searrow & \\ & & bcab \end{array} ,$$

and also three of overlapping type, for example

$$
\begin{tikzcd}[row sep=small]
 & aac \\
\underline{caaa}\,c \arrow[ur] \arrow[d, equal] & \\
caa\underline{ac} \arrow[dr] & \\
 & caaca
\end{tikzcd}
$$

(As mentioned previously (Remark 1.4.7), elsewhere these critical pairs would usually be written as the pairs $(1, bcab)$ and (a^2c, ca^2ca) respectively.)

The following two paths give a resolution of the critical pair of overlapping type, to the vertex ca^2:

$$
\begin{tikzcd}
a\underline{ac} \arrow[r] & \underline{ac}a \arrow[drr] & & \\
 & & & caa \\
ca\underline{ac}a \arrow[r] & c\underline{ac}aa \arrow[r] & c\underline{caaa} \arrow[ur] &
\end{tikzcd}
$$

The critical pair of inclusion type cannot be resolved, since the terminal vertices are distinct irreducibles. ⌟

Proposition 1.4.11 (Knuth and Bendix [35]) *A noetherian rewriting system is complete if and only if all the critical pairs are resolvable.*

Proof: A noetherian rewriting system is locally confluent if and only if each critical pair is resolvable. ⌟

Hence the result that the rewriting system (1.1) given in Example 1.1.4 is complete, since the critical pairs are resolved.

1.4.1 The Knuth-Bendix completion procedure

Let $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ be a finite rewriting system where $\mathbf{r}$ is compatible with some total reduction order $\succ$ on F - note that $\mathcal{P}$ is always equivalent to such a rewriting system, for example if we choose some length-lexicographical ordering of F, then we can replace each rule $r \in \mathbf{r}$ for which $r_{+1} \not\succ_{llex} r_{-1}$ with the rule (r_{-1}, r_{+1}). If $\mathcal{P}$ is not complete then applying the Knuth-Bendix procedure [35] will produce an equivalent complete rewriting system $\mathcal{P}_\infty$, but which in general will have an infinite set of rules.

The procedure is as follows. For each non-resolvable critical pair (e, f) we can choose a pair of positive paths p_e and p_f from τe and τf respectively to different irreducibles. Let $\mathbf{r}'$ denote the set of rules obtained from $\mathbf{r}$ by adding for each such critical pair (e, f) a rule $(\tau p_e, \tau p_f)$ if $\tau p_e \succ \tau p_f$, otherwise adding the rule $(\tau p_f, \tau p_e)$. Since $\mathcal{P}$ is not complete, $\mathbf{r}' \neq \mathbf{r}$, but the rewriting system $[\mathbf{x}; \mathbf{r}']$ is equivalent to $\mathcal{P}$. (More generally, we can add any finite set of rules

$$\{u_1 = v_1, u_2 = v_2, \ldots\}$$

which allows us to resolve the critical pairs, provided that $u_i \succ v_i$, and of course that $u_i \leftrightarrow^*_{\mathbf{r}} v_i$ for each $i = 1, 2, \ldots$.)

Put $\mathbf{r} = \mathbf{r}_0$, and then for all $n \geq 1$ we define $\mathbf{r}_n$ to be $\mathbf{r}'_{n-1}$, and

$$\mathbf{r}_\infty = \bigcup_{n \geq 0} \mathbf{r}_n.$$

The rewriting system $\mathcal{P}_\infty = [\mathbf{x}; \mathbf{r}_\infty]$ is equivalent to $\mathcal{P}$, and is compatible with the original noetherian order on F. But it is also complete, for any critical pair arises from a pair of rules found in $\mathbf{r}_n$ for some $n \geq 0$, and is therefore resolvable using a rule in $\mathbf{r}'_n = \mathbf{r}_{n+1} \subseteq \mathbf{r}_\infty$.

Example 1.4.12 Consider the rewriting system consisting of the set of rules

$$\mathbf{r} = \{\, xx^{-1} = 1, \quad x^{-1}x = 1, \quad xx = 1 \,\}$$

on the alphabet x, x^{-1}, which is compatible with the length-lexicographic ordering based on the ordering $x^{-1} \triangleright x$. There are four critical pairs, all of overlapping type: two arising from the rules $xx^{-1} = 1$ and $x^{-1}x = 1$ which are immediately resolved as in Example 1.4.8, and the critical pairs

$$\begin{array}{ccc} & & x \\ & \nearrow & \\ \underline{x^{-1}x}\,x & & \\ \| & & \\ x^{-1}\underline{xx} & & \\ & \searrow & \\ & & x^{-1} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & x^{-1} \\ & \nearrow & \\ \underline{xx}\,x^{-1} & & \\ \| & & \\ x\,\underline{xx^{-1}} & & \\ & \searrow & \\ & & x \end{array}$$

Applying the Knuth-Bendix procedure, we obtain the set $\mathbf{r}_1$ of rules by adding the rule $x^{-1} = x$ to $\mathbf{r}$. There are two new critical pairs (of inclusion type) which arise from the introduction of this rule, namely

$$\begin{array}{ccc} & & xx \\ & \nearrow & \\ x\,\underline{x}^{-1} & & \\ \| & & \\ \underline{xx^{-1}} & & \\ & \searrow & \\ & & 1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & xx \\ & \nearrow & \\ \underline{x}^{-1}x & & \\ \| & & \\ \underline{x^{-1}x} & & \\ & \searrow & \\ & & 1 \end{array} \quad ,$$

and they are both easily resolved by the path $\underline{xx} \to 1$.

Thus in this case the Knuth-Bendix procedure terminates with a complete rewriting system after adding a single rule. This rewriting system is a presentation of the group of order two, which in fact has the simpler complete rewriting system

$$[\,\vartheta\,;\ \vartheta\vartheta = 1\,]\,.$$

┘

Example 1.4.13 It has been shown that the rewriting system consisting of the set of rules

$$\mathbf{r} = \mathbf{r}_0 = \{\quad xa{=}atx,\quad xt{=}tx\,,\quad xy{=}1,\quad xb{=}bx,\quad ab{=}1\quad\}$$

on the alphabet a, b, t, x, y has no equivalent finite complete rewriting system [60, Corollary 6.8]. Therefore in this case the Knuth-Bendix procedure will not terminate in a finite number of steps. The rewriting system is compatible with the recursive path ordering $\succ$ from the left (Definition 1.1.3) induced by the ordering $x \rhd a \rhd b \rhd t \rhd y$:

1. $xa \succ atx$ holds because $x \rhd a$ and $xa \succ tx$ ($xa \succ tx$ holds because $x \rhd t$ and $xa \succ x$, and $xa \succ x$ because $a \succ 1$);

2. $xt \succ tx$ holds because $x \rhd t$ and $xt \succ x$ ($xt \succ x$ holds because $t \succ 1$);

3. $xb \succ bx$ holds because $x \rhd b$ and $xb \succ x$ ($xb \succ x$ holds because $b \succ 1$).

For $n = 1, 2, 3, \ldots$, let

$$\mathbf{r}_n = \mathbf{r}_{n-1} \cup \{at^n b = 1\}.$$

By [60, Lemma 6.2(b)], the relation $at^n b \leftrightarrow^*_{\mathbf{r}} 1$ holds for each n, and so the rules $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \ldots$ all generate the same Thue congruence. We use $\mathcal{P}_n$ to denote the rewriting system with the rules $\mathbf{r}_n$ on the alphabet a, b, t, x.

For $n = 0, 1, 2, 3, \ldots$, the rules $\mathbf{r}_n$ give rise to the single critical pair

$$\begin{tikzcd} & atxt^n b \\ \underline{xa}\, t^n b \arrow[ur] \arrow[d, equal] & \\ x\underline{at^n b} \arrow[dr] & \\ & x \end{tikzcd}$$

which (for $n \geq 1$) does not arise from the rules $\mathbf{r}_{n-1}$. In each case the critical pair can be resolved by the path

$$at\underline{xt}\, t^{n-1} b \to at^2 \underline{xt}\, t^{n-2} b \to \cdots \to at^{n+1} \underline{xb} \to \underline{at^{n+1} b}\, x \to x\,.$$

in $\Gamma(\mathcal{P}_{n+1})$.

Thus this sequence of rewriting systems can be produced by applying the Knuth-Bendix procedure to $\mathcal{P}_0$. It is only the union

$$\mathcal{P}_\infty = [a, b, t, x\ ;\ \bigcup_{n \geq 0} \mathbf{r}_n]$$

which is complete. ⌟

1.4.2 The property *FDT*

Throughout this section we shall assume that for a complete rewriting system $\mathcal{P}$, $\mathbf{p}$ is a set of closed paths in Γ obtained by choosing resolutions of each critical pair. The following observation is fundamental to our study of complete rewriting systems.

Remark 1.4.14 The boundary of each 2-cell σ in $\mathcal{D}^{\mathbf{p}}$ has a unique maximal vertex w_σ and minimal vertex z_σ with respect to the reduction ordering $\succ_{\mathbf{r}}$ on F, and the boundary of σ consists of two positive paths from w_σ to z_σ. These paths will come from resolutions of either critical or disjoint pairs, according as to whether σ arises from the trivializer $\mathbf{p}$ or not.

Moreover each 2-cell is uniquely determined by the two edges at its maximal vertex, and we can therefore denote each 2-cell σ by $[w_\sigma; (e, f)]$ where $e <_{w_\sigma} f$ are the corresponding edges in $star^+(w_\sigma)$. We shall use this notation when studying complete rewriting systems in §3.3, §4.4 and §5.2.

Example 1.4.15 Consider the vertex $w = xxx^{-1}x$ in the graph Γ of the complete rewriting system in Example 1.4.12 consisting of the rules $\mathbf{r}_1$ on the alphabet x, x^{-1}. The set $star^+(w)$ consists of the three edges

$$\underline{xx}x^{-1}x \to x^{-1}x, \qquad x\underline{xx^{-1}}x \to xx, \qquad xx\underline{x^{-1}x} \to xx$$

which for convenience we label e, f and g respectively. The 2-cell $[w; (e, g)]$ is in the 2-complex $\mathcal{D}$, and is attached along the closed path

$$\begin{tikzcd} & \underline{x^{-1}x} \arrow[dr] & \\ w \arrow[ur, "e"] \arrow[dr, "g"'] & & 1 \\ & \underline{xx} \arrow[ur] & \end{tikzcd}$$

If **p** is chosen to contain the closed paths

$$\begin{tikzcd}
 & \underline{x}^{-1} \arrow[dd] \\
\underline{xx}\,x^{-1} \arrow[ur] \arrow[d, equal] & \\
x\underline{xx^{-1}} \arrow[r] & x
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\underline{xx^{-1}}x \arrow[d, equal] \arrow[dr] & \\
x\underline{x^{-1}x} \arrow[r] & x
\end{tikzcd},$$

then the 2-cells $[w;(e,f)]$ and $[w;(f,g)]$ will be attached along the closed paths

$$\begin{tikzcd}
 & \underline{x}^{-1}x \arrow[d] \\
w \arrow[ur, "e"] \arrow[r, "f"'] & xx
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
w \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & xx
\end{tikzcd}$$

in $\mathcal{D}^{\mathbf{p}}$. ⌟

Theorem 1.4.16 (Squier [60]) *The set* **p** *is a homotopy trivializer, and therefore a* finite *complete rewriting system has the property FDT.*

We need the following two lemmas.

Lemma 1.4.17 *Any pair of positive paths p, q from a single vertex to its irreducible will give rise to a closed path pq^{-1} which is null-homotopic.*

Proof: This is proved by noetherian induction on F, where the order is the reduction order $\succ_{\mathbf{r}}$ induced by the rules.

Suppose that $p = e_1 \ldots e_m$ and $q = f_1 \ldots f_n$ are two positive paths from $w \in F$ to $Irr(w)$, and assume inductively that for any $w' \in F$ satisfying $w \succ_{\mathbf{r}} w'$ and positive paths r, s from w' to $Irr(w')$ $(= Irr(w))$ we have that rs^{-1} is null-homotopic. We want to deduce that the closed path pq^{-1} is also homotopically trivial.

If w is irreducible, then both paths are empty, and the result is obvious. Otherwise, *both* paths are non-empty. If $e_1 = f_1$ then by inductive assumption the closed path $e_2 \ldots e_m f_n^{-1} \ldots f_2^{-1}$ is null-homotopic and the result follows. If not, then there is a 2-cell

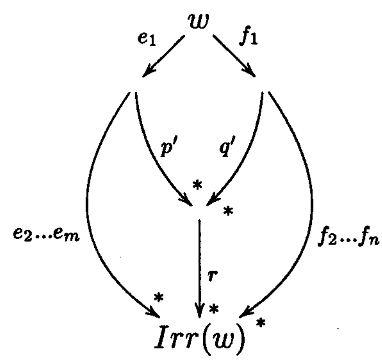

Figure 1.2: Construction used in Lemma 1.4.17

$[w; (e_1, f_1)]$ in $\mathcal{D}^{\mathrm{P}}$ with boundary $e_1 p' q'^{-1} f_1^{-1}$, say, where p' and q' are positive paths. Let us choose some positive path r from $\tau p' = \tau q'$ to $Irr(w)$. If we use $\simeq$ to denote the relation "homotopic with endpoints fixed" then we can write (see Figure 1.2)

$$\begin{aligned} p &\simeq f_1 q' p'^{-1} e_2 \ldots e_m \\ &\simeq f_1 q' p'^{-1} p' r && \text{(by inductive assumption)} \\ &\simeq f_1 q' r \\ &\simeq q && \text{(by inductive assumption).} \end{aligned}$$

Thus any pair of positive paths from w to $Irr(w)$ give rise to a null-homologous path, and the lemma now follows by noetherian induction (Lemma 1.4.2). ⌟

Lemma 1.4.18 *For any path p from u to w ($u, w \in F$),*

$$p \simeq q_u q_w^{-1}$$

holds for any pair q_u and q_w of positive paths from u and w respectively to $Irr(u) = Irr(w)$.

Proof: We use induction on the length n of p. If p is empty, then by Lemma 1.4.17 $p \simeq q_u q_w^{-1}$. If $n > 0$ then we can write $p = e^{\varepsilon} p'$, where $e \in \mathbf{e}^+$ and $\varepsilon = \pm 1$, and if we write $v = \tau e^{\varepsilon}$ and choose some positive path q_v from v to the irreducible (z, say), then we have

(a) $\quad \varepsilon = +1$

$$\begin{tikzcd}
u \arrow[r, "e"] \arrow[dr, bend right, "q_u"'] & v \arrow[r, "p'"] \arrow[d, "q_v"] & w \arrow[dl, bend left, "q_w"] \\
& z &
\end{tikzcd}$$

(b) $\quad \varepsilon = -1$

$$\begin{tikzcd}
u \arrow[dr, bend right, "q_u"'] & v \arrow[l, "e"'] \arrow[r, "p'"] \arrow[d, "q_v"] & w \arrow[dl, bend left, "q_w"] \\
& z &
\end{tikzcd}$$

Figure 1.3: Proof of Lemma 1.4.18

1. $e^{\varepsilon} \simeq q_u q_v^{-1}$, because

   (a) if $\varepsilon = +1$ then by Lemma 1.4.17 we have $q_u \simeq e q_v$;

   (b) if $\varepsilon = -1$ then again by Lemma 1.4.17 we have $e q_u \simeq q_v$; and

2. (by the inductive hypothesis) $p' \simeq q_v q_w^{-1}$.

Therefore (see Figure 1.3)

$$p = e^{\varepsilon} p' \simeq q_u q_v^{-1} q_v q_w^{-1} \simeq q_u q_w^{-1},$$

and the result follows by induction. ⌟

Proof of Theorem 1.4.16 If p_1 and p_2 are paths with the same initial and terminal vertices (u and v, say) then we can choose positive paths q_u and q_v to the irreducible. By Lemma 1.4.18 we have

$$p_1 \simeq q_u q_v^{-1} \simeq p_2.$$

It follows immediately that any closed path in $\mathcal{D}^{\mathbf{p}}$ is homotopically trivial, and so $\mathbf{p}$ is indeed a homotopy trivializer. ⌟

Example 1.4.19 Let $\mathcal{P}$ be the complete rewriting system in Example 1.4.13 with the infinite set of rules. Each critical pair is of the form

$$\begin{tikzcd}[column sep=small, row sep=small]
 & atxt^n b \\
\underline{xa}\, t^n b \arrow[ur] & \\
\Vert & \\
x\underline{at^n b} \arrow[dr] & \\
 & x
\end{tikzcd}$$

and can be resolved by the path

$$at\underline{xt}\, t^{n-1}b \to at^2\underline{xt}\, t^{n-2}b \to \cdots \to at^{n+1}\underline{xb} \to \underline{at^{n+1}b}\, x \to x\,.$$

By Theorem 1.4.16, the set of closed paths of the form

$$\begin{tikzcd}[column sep=small, row sep=small]
 & at\underline{xt}\, t^{n-1}b \arrow[r] & \cdots \arrow[r] & at^2\underline{xt}\, t^{n-2}b \arrow[dr] & \\
\underline{xa}\, t^n b \arrow[ur] & & & & at^{n+1}\underline{xb} \arrow[dl] \\
\Vert & & & \underline{at^{n+1}b}x \arrow[dll] & \\
x\underline{at^n b} \arrow[dr] & & & & \\
 & x & & &
\end{tikzcd}$$

for $n \geq 0$ is a homotopy trivializer of $\mathcal{D}$.

1.4.3 Groups with finite complete rewriting systems

There are many groups which are known to have finite complete rewriting systems, for example free groups of finite rank (Example 1.4.8), finitely generated abelian groups, the Heisenberg group [33], and a group studied by Greendlinger [51]. Dekov [20] has shown that torus-knot groups have such rewriting systems, as do one-relator groups with a finite presentation of the form

$$\langle\, \mathbf{x} \cup \{a, b\}\ ;\ a^{-1}b^{-1}abw \,\rangle \tag{1.12}$$

where w is a word on the generators $\mathbf{x} \cup \mathbf{x}^{-1}$.

Groves and Smith [28] have investigated how the property of being presented by a finite complete rewriting system behaves under various group-theoretic constructions (subgroups, quotient groups, group extensions and HNN-extensions) and used the results to prove:

Theorem 1.4.20 ([28]) *Constructible solvable groups have finite complete rewriting systems.*

The following questions have so far been unanswered:

Open Question 1.4.21 ([28]) *If a group G has a finite index subgroup with a finite complete rewriting system, then G itself has a finite complete rewriting system. Is the converse true?*

Open Question 1.4.22 ([65]) *Suppose that H_1 and H_2 are groups and that the free product $H_1 * H_2$ has a finite complete rewriting system. Is it always true that H_1 and H_2 themselves have finite complete rewriting systems?*

Because of the following result, a positive answer to the second question would also give a positive answer to the first:

Theorem 1.4.23 (Pride and Wang [56]) *Let H be a subgroup of finite index n in a group G. If G has a finite complete rewriting system, then so does the free product $H * \mathcal{F}_{n-1}$ of H with the free group of rank $n-1$.*

We also mention the following open questions, regarding two much-studied classes of groups which are of type FP_∞ and have solvable word problems:

Open Question 1.4.24 ([38]) *Do all finitely generated one-relator groups have a finite complete rewriting system?*

Open Question 1.4.25 ([52]) *Do all hyperbolic groups have finite complete rewriting systems? Hyperbolic groups are particular examples of automatic groups, which were introduced by Cannon and Thurston* [22]*. Do all automatic groups have a finite complete rewriting system?*

Chapter 2

λ-complete rewriting systems

Abstract

We introduce *λ-complete rewriting systems* which give solutions to *λ-word problems.* In §2.1.1 we characterize monoids which have a finite λ-complete rewriting system as those monoids with a *λ-Dehn presentation,* and in §2.2 we describe a method of obtaining λ-complete rewriting systems for certain groups which are HNN-extensions whose base groups have complete rewriting systems, and we use this to obtain some interesting examples.

2.1 λ-solvable word problems

Definition 2.1.1 The *λ-word problem* of a rewriting system will be to decide for any word w whether or not w is congruent to the empty word. We shall say that a rewriting system has a *λ-solvable word problem* if there is an algorithm which solves its λ-word problem.

The following proposition shows that this property is a monoid invariant, so we can extend the above definition to say that a finitely presented monoid has λ-solvable word problem if each of its finite rewriting systems does.

Proposition 2.1.2 *If $\mathcal{P}$ and $\mathcal{Q}$ are finite rewriting systems which define isomorphic monoids then $\mathcal{P}$ has λ-solvable word problem if and only if $\mathcal{Q}$ does.*

Proof: By the Tietze Theorem (Theorem 1.1.1), it is enough to show that the result is true whenever $\mathcal{Q}$ is obtained from $\mathcal{P}$ by a single Tietze transform.

1. Suppose that a type 1 Tietze transformation is applied to $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ to give the rewriting system $\mathcal{Q} = [\mathbf{x}; \mathbf{r}']$, where $\mathbf{r}'$ is obtained from $\mathbf{r}$ by adjoining a rule $u = v$ with $u \leftrightarrow^*_{\mathbf{r}} v$. Because the congruences $\leftrightarrow^*_{\mathbf{r}}$ and $\leftrightarrow^*_{\mathbf{r}'}$ are the same, any algorithm solving the λ-word problem for $\mathcal{P}$ also solves the λ-word problem for $\mathcal{Q}$, and vice versa.

2. Suppose that a type 2 Tietze transformation is applied to $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ to give the rewriting system $\mathcal{Q}$ by adding a new letter y and rule $y = z$ for some word z on the alphabet $\mathbf{x}$. Recall from §1.1.2 that the homomorphism of free monoids on the alphabets of the respective rewriting systems which identifies the letters in $\mathbf{x}$ and maps y to z induces an isomorphism

$$S(\mathcal{Q}) \longrightarrow S(\mathcal{P}).$$

A word w on the alphabet $\mathbf{x} \cup \{y\}$ of $\mathcal{Q}$ is therefore congruent to the empty word (with respect to the rules of $\mathcal{Q}$) if and only if $w' \leftrightarrow^*_{\mathbf{r}} 1$ holds, where w' denotes the word obtained from w by rewriting each instance of the letter y according to the rule $y = z$.

It follows easily that $\mathcal{Q}$ has λ-solvable word problem if and only if $\mathcal{P}$ does. ⌟

Definition 2.1.3 We shall say that a rewriting system is λ-*complete* if it is noetherian, and the empty word is the unique irreducible in its congruence class.

Proposition 2.1.4 *If a monoid has a finite λ-complete rewriting system then it has λ-solvable word problem.*

Proof: Suppose that $\mathcal{P}$ is a finite λ-complete rewriting system. Because $\mathcal{P}$ is noetherian, any word w can be rewritten in a finite number of steps to an irreducible word

w^*, say, and then since the empty word is the unique irreducible in its congruence class, $w \leftrightarrow_{\mathbf{r}}^* 1$ if and only if $w^* = 1$. ⌟

Obviously, if a rewriting system has solvable word problem then it has λ-solvable word problem. Example 2.1.5 shows that the converse does not hold in general; it is true, however, if we restrict our attention to the class of groups (Proposition 2.1.6 below).

Example 2.1.5 The following rewriting system was shown by Tsejtin [63] to have unsolvable word problem (the congruence class containing the word *aaa* is a non-recursive subset of the free monoid on the given alphabet).

alphabet: a, b, c, d, e

rules: $ac{=}ca,\quad ad{=}da\ ,\quad de{=}edb,\quad caaa{=}aa,\quad cdca{=}cdcae,$
$bc{=}cd,\quad dc{=}cd,\quad ce{=}eca,\quad daaa{=}aa.$

Because there are no rules whose left or right hand side is the empty word, the empty word is in the congruence class consisting only of itself; an algorithm to solve the λ-word problem is then just to check whether or not a word is the empty word.

Proposition 2.1.6 *A group G with λ-solvable word problem has solvable word problem.*

Proof: Suppose that G has a rewriting system $\mathcal{P}$ with some algorithm solving the λ-word problem. For each letter x in the alphabet $\mathbf{x}$ of $\mathcal{P}$ we can choose a word $w_x \in F$ such that $\overline{xw_x} = 1$: since G is a group we know that such words exist, and we can check using the algorithm whether or not any word $w \in F$ satisfies $\overline{xw} = 1$.

We wish to describe an algorithm which solves the word problem for G. We claim that any pair of words u and $v = x_1 x_2 \ldots x_n$ (where $x_i \in \mathbf{x}$ for $i = 1, 2, \ldots n$) are congruent with respect to the rules $\mathbf{r}$ if and only if $uw_{x_n} \ldots w_{x_1} \leftrightarrow_{\mathbf{r}}^* 1$. We can check this fact, since we have assumed an algorithm recognizing words that are congruent to the empty word.

For suppose that u and v are congruent. Then

$$\begin{aligned} uw_{x_n}\cdots w_{x_1} &\leftrightarrow_{\mathrm{r}}^{*} vw_{x_n}\cdots w_{x_1} \\ &= x_1\cdots x_n w_{x_n}\cdots w_{x_1} \\ &\leftrightarrow_{\mathrm{r}}^{*} x_1\cdots x_{n-1}w_{x_{n-1}}\cdots w_{x_1} \\ &\vdots \\ &\leftrightarrow_{\mathrm{r}}^{*} x_1 w_{x_1} \\ &\leftrightarrow_{\mathrm{r}}^{*} 1. \end{aligned}$$

On the other hand, suppose that $uw_{x_n}\cdots w_{x_1}$ is congruent to the empty word. Since G is a group, xw_x is congruent to the empty word if and only if $w_x x$ is also, and so

$$\begin{aligned} v &\leftrightarrow_{\mathrm{r}}^{*} uw_{x_n}\cdots w_{x_1}x_1\cdots x_n \\ &\leftrightarrow_{\mathrm{r}}^{*} uw_{x_n}\cdots w_{x_2}x_2\cdots x_n \\ &\vdots \\ &\leftrightarrow_{\mathrm{r}}^{*} uw_{x_n}x_n \\ &\leftrightarrow_{\mathrm{r}}^{*} u. \end{aligned}$$

This proves our claim. ⌟

Thus, as regards the word problem, finite complete rewriting systems for groups seem to give unnecessary information. Indeed, as mentioned in §1.4, there are known examples of finitely presented groups with solvable word problem but without any finite complete rewriting system. The following questions present themselves:

Open Question 2.1.7 ([8, 16]) *Does every finitely presented group with solvable word problem have a finite λ-complete rewriting system?*

Open Question 2.1.8 ([8, 16]) *Is there a group with a finite λ-complete rewriting system but with no finite complete rewriting system?*

We could also rephrase Open Questions 1.4.24 and 1.4.25 as follows.

Open Question 2.1.9 *Do all finitely generated one-relator groups have a finite λ-complete rewriting system?*

Open Question 2.1.10 *Do all hyperbolic groups have finite λ-complete rewriting systems? Do all automatic groups have finite λ-complete rewriting systems?*

2.1.1 λ-Dehn presentations

In the next section we discuss some examples of groups with finite λ-complete rewriting systems, but first we give an alternative description of such groups.

Definition 2.1.11 ([44]) *A* group *presentation* $\langle \mathbf{x} ; \mathbf{r} \rangle$ *is called a* Dehn presentation *if it is finite, and if the following holds:*

> *whenever a word w on the alphabet $\mathbf{x} \cup \mathbf{x}^{-1}$ is non-empty and satisfies $\overline{w} = 1$, then either w is not freely reduced, or w contains a subword u for which there is a word v of shorter length such that uv^{-1} is a cyclic conjugate of an element of $\mathbf{r} \cup \mathbf{r}^{-1}$.*

For any group presentation $\mathcal{G} = \langle \mathbf{x} ; \mathbf{r} \rangle$ we can construct a length-reducing (and therefore noetherian) rewriting system

$$\hat{\mathcal{G}}_{lr} = [\mathbf{x} \cup \mathbf{x}^{-1} \, ; \, \hat{\mathbf{r}}] \tag{2.1}$$

whose rules consist of all free reductions $xx^{-1} = 1, x^{-1}x = 1$ ($x \in \mathbf{x}$) together with all rules of the form $u = v$ where uv^{-1} is a cyclic permutation of some relator or its inverse, and v has strictly shorter length than u. If $\mathcal{G}$ is a Dehn presentation, the empty word is the only irreducible element in its congruence class, and therefore $\hat{\mathcal{G}}_{lr}$ is λ-complete. This rewriting system performs the *Dehn algorithm* [44] and, of course, any group with a Dehn presentation has a solvable word problem. The fact that groups with Dehn presentations have λ-complete rewriting systems has been noted before (see [8, 45] and the references therein). In [45] some results on the classes of groups which have particular types of length-reducing λ-complete rewriting systems are given.

Dehn [19] originally showed that the fundamental groups of closed orientable surfaces of genus $n{\geq}2$ have such presentations. More recently, Gromov [26] introduced the notion of *word hyperbolic* groups, and stated that this class of groups coincides with the class of groups which admit a Dehn presentation. Proofs of this result have been given elsewhere, for example in [58].

A characterization theorem for groups with a finite λ-complete rewriting system.

We shall generalize the idea of a Dehn presentation as follows.

Definition 2.1.12 *We shall call a* group *presentation*

$$\mathcal{G} = \langle \mathbf{x}\,;\, \mathbf{r} \rangle$$

a λ-Dehn presentation *if it is finite, and if for some reduction ordering $\succ$ on the free monoid $F(\mathbf{x}{\cup}\mathbf{x}^{-1})$ the following holds:*

1. $xx^{-1}, x^{-1}x{\succ}1$ *for all $x{\in}\mathbf{x}$; and*
2. *if $\overline{w} = 1$, where $w{\in}F(\mathbf{x}{\cup}\mathbf{x}^{-1})$ is non-empty, then either w is not freely reduced, or w contains a subword u for which there is a word v such that $u{\succ}v$ and uv^{-1} is a cyclic conjugate of an element of $\mathbf{r}{\cup}\mathbf{r}^{-1}$.*

Thus a Dehn presentation is a λ-Dehn presentation where we can choose the length reducing order (Definition 1.1.1). On the other hand the next example shows that there are groups which have a λ-Dehn presentation for some non-length reducing reduction order but which do not have any Dehn presentation.

Example 2.1.13 Consider the group with presentation

$$\mathcal{G} = \langle a, b, c\,;\, abc = cba \rangle.$$

Greendlinger [25] gave this as an example of a presentation that was not a Dehn presentation. In fact it is known that this group has no Dehn presentation; Gromov observed that any abelian subgroup of a word hyperbolic group containing an element of infinite order is finite-by-cyclic ([26] - for a proof of this see [58, Corollary 3.6]), whereas the subgroup generated by $\overline{ab}$ and $\overline{cb}$ is free abelian of rank two. However, it does have a finite complete rewriting system, discovered by Otto [51]:

$$\begin{array}{rllll} \textbf{alphabet:} & a, a^{-1}, b, b^{-1}, c, c^{-1} & & & \\ \textbf{rules:} & aa^{-1} = 1, & a^{-1}a = 1, & a^{-1}cb = bca^{-1}, & abc = cba, \\ & bb^{-1} = 1, & b^{-1}b = 1, & ac^{-1} = b^{-1}c^{-1}ab, & \\ & cc^{-1} = 1, & c^{-1}c = 1, & a^{-1}b^{-1} = c^{-1}b^{-1}a^{-1}c. & \end{array}$$

Dekov [20] gave another proof that this rewriting system is noetherian by showing that it is compatible with the recursive path ordering from the left $\succ$ induced by the partial ordering on the alphabet given by writing

$$a \rhd b^{-1} \rhd c, \qquad a \rhd c^{-1}, \qquad a^{-1} \rhd c^{-1} \rhd b, \qquad a^{-1} \rhd b^{-1}$$

of the alphabet. For example $ac^{-1} \succ b^{-1}c^{-1}ab$ holds because $a \rhd b^{-1}$ and $ac^{-1} \succ c^{-1}ab$; the latter holds because $a \rhd c^{-1}$ and $ac^{-1} \succ ab$; this time the latter holds because $c^{-1} \rhd b$.

It is easily checked that each rule is either a free reduction or can be obtained from a cyclic permutation of $abc = cba$, and it follows that $\mathcal{G}$ is a λ-Dehn presentation. ⌟

Proposition 2.1.14 *The class of groups with finite λ-complete rewriting systems coincides with the class of groups which have a λ-Dehn presentation.*

Proof: In the same way that a finite (length-reducing) λ-complete rewriting system (2.1) can be constructed from a Dehn presentation, we can construct a finite λ-complete rewriting system from any λ-Dehn presentation.

Conversely, suppose that G is a group with finite λ-complete rewriting system $\mathcal{P} = [\mathbf{x}\,;\,\mathbf{r}]$. Since G is a group, for each $x \in \mathbf{x}$ we can choose words w_x on $\mathbf{x}$ such that $\overline{xw_x} = \overline{w_x x} = 1$. Introducing new monoid generators $\mathbf{x}^{-1} = \{x^{-1} : x \in \mathbf{x}\}$ by type 2 Tietze transformations gives the equivalent finite rewriting system

$$\mathcal{P}' = [\mathbf{x} \cup \mathbf{x}^{-1}\,;\,\mathbf{r}'],$$

where $\mathbf{r}' = \mathbf{r} \cup \{x^{-1} = w_x : x \in \mathbf{x}\}$.

Then $\mathcal{P}'$ is also noetherian, since the new rules only operate on the new generators, replacing them with words on $\mathbf{x}$: any hypothetical infinite sequence

$$w_1 \rightarrow_{\mathbf{r}'} w_2 \rightarrow_{\mathbf{r}'} w_3 \rightarrow_{\mathbf{r}'} \cdots$$

involving words on $\mathbf{x} \cup \mathbf{x}^{-1}$ would involve only a finite number of applications of the new rules, and so we could easily derive a corresponding infinite reduction sequence for $\mathcal{P}$, a contradiction.

It is also λ-complete, because any word w on $\mathbf{x} \cup \mathbf{x}^{-1}$ can be rewritten in a finite number of steps to a word w' on $\mathbf{x}$, and then $w \leftrightarrow^*_{\mathbf{r}'} 1$ holds if and only if $w' \leftrightarrow^*_{\mathbf{r}} 1$. We can then rewrite w to the empty word as follows:

$$w \rightarrow^*_{\mathbf{r}'} w' \rightarrow^*_{\mathbf{r}} 1.$$

(We are again using the fact that the homomorphism of free monoids which identifies $\mathbf{x}$ and maps each $x^{-1} \in \mathbf{x}^{-1}$ to the word w_x on $\mathbf{x}$ induces an isomorphism

$$S(\mathcal{P}') \longrightarrow S(\mathcal{P})$$

-see §1.1.2.)

Now write

$$\mathcal{G} = \langle \mathbf{x} \, ; \, \hat{\mathbf{r}} \rangle,$$

where $\hat{\mathbf{r}} = \{r_{+1} r_{-1}^{-1} : r \in \mathbf{r}\} \cup \{w_x x : x \in \mathbf{x}\}$, and let $\succ_{\mathbf{r}'}$ be the reduction ordering on $F(\mathbf{x} \cup \mathbf{x}^{-1})$ induced by the set of rules $\mathbf{r}'$ of $\mathcal{P}'$ (Definition 1.1.5). If some non-empty word w on $\mathbf{x} \cup \mathbf{x}^{-1}$ is equivalent to the empty word, then, as $\mathcal{P}'$ is λ-complete, w can be rewritten to the empty word, and so:

1. $w \succ 1$: in particular, $x x^{-1} \succ_{\mathbf{r}'} 1$ and $x^{-1} x \succ_{\mathbf{r}'} 1$;

2. w must contain a subword r_{+1}, the left-hand side of some rule in $\mathbf{r}$, or a letter x^{-1} ($x \in \mathbf{x}$), where there is a relator $r_{+1} r_{-1}^{-1}$ or $x^{-1} w_x^{-1}$ in $\mathcal{G}$ with $r_{+1} \succ_{\mathbf{r}'} r_{-1}$ or $x^{-1} \succ_{\mathbf{r}'} w_x$ respectively.

Thus $\mathcal{G}$ is λ-Dehn, and we are done. ⌟

2.2 Producing λ-complete rewriting systems from complete rewriting systems

In this section we introduce a method of constructing λ-complete rewriting systems for HNN groups whose base groups have certain complete rewriting systems with nice properties. It relies on Britton's Lemma (Lemma 2.2.2), which describes the structure of words that are equivalent to 1 in HNN groups.

We use this method to construct some interesting examples, and remark on the possibility of using similar constructions to search for an answer to Open Question 2.1.8.

2.2.1 HNN-extensions

Let G be a group with group presentation $\mathcal{G} = \langle \mathbf{x}; \mathbf{r} \rangle$, and suppose that A and B are isomorphic subgroups. Let $a_1, a_2, \ldots$ be words on $\mathbf{x}$ representing a generating set of A, and let $b_1, b_2, \ldots$ be words representing the images of this generating set under some isomorphism $\varphi : A \rightarrow B$. If $t \notin \mathbf{x}$ is some new letter, then the group presented by

$$\langle \mathbf{x} \cup \{t\} \,;\, \mathbf{r} \cup \{t^{-1}a_1t = b_1,\, t^{-1}a_2t = b_2, \ldots\} \rangle \tag{2.2}$$

is an *HNN-extension of G with stable letter t, associating subgroups A and B (by the isomorphism φ).* We call G the base group, and we denote the HNN-extension by $G*_{\varphi:A\cong B}$, or just $G*_{A\cong B}$ if the isomorphism is understood. We shall generally abbreviate the presentation (2.2) to

$$\langle \mathcal{G} \mid t \,;\, t^{-1}a_1t = b_1,\, t^{-1}a_2t = b_2, \ldots \rangle.$$

Lemma 2.2.1 **([57, p.412])** *The group homomorphism induced by the inclusion of the presentation $\langle \mathbf{x}; \mathbf{r} \rangle$ of the base group into (2.2) is injective.* ⌟

This group construction has become very important since it was first introduced by Higman, Neumann and Neumann [30] (where the above Lemma was first proved, and then used to give some embedding theorems). In particular, the combinatorial arguments used by both Boone and Novikov to give groups with unsolvable word problems were later seen to be contained in the more general theory of HNN-extensions, and Britton [10] was able to use the following lemma to considerably simplify Boone's construction:

Lemma 2.2.2 (Britton's Lemma [10]) *Consider the presentation* (2.2). *If w is a word involving the stable letter t which is equivalent to 1, then w contains a subword of the form $t^{-1}ut$ or of the form tvt^{-1}, where u and v are words on $\mathbf{x} \cup \mathbf{x}^{-1}$ representing elements of the subgroups A and B respectively.* ┘

Let $\mathcal{P}_H$ be a finite complete rewriting system for a group H, with words u, v on the alphabet $\mathbf{x}$ such that

$$\{\ldots, u^{-1}, 1, u, u^2, \ldots\} \quad \text{and} \quad \{\ldots, v^{-1}, 1, v, v^2, \ldots\}.$$

are sets of irreducible words. Then these sets are sets of normal forms of the infinite cyclic subgroups generated by $\overline{u}$ and $\overline{v}$ are, respectively.

By associating the cyclic subgroups $\langle\overline{u}\rangle$, $\langle\overline{v}\rangle$ under the isomorphism induced by the map $\overline{u} \mapsto \overline{v}$, we can form the HNN-group

$$G = H *_{\langle\overline{u}\rangle \cong \langle\overline{v}\rangle}.$$

Theorem 2.2.3 *The rewriting system $\mathcal{Q}$ obtained from $\mathcal{P}_H$ by adding new letters t and t^{-1} together with the additional rules*

$$\begin{array}{lll} tt^{-1} = 1, & t^{-1}u = vt^{-1}, & t^{-1}u^{-1} = v^{-1}t^{-1}, \\ t^{-1}t = 1, & tv = ut, & tv^{-1} = u^{-1}t \end{array}$$

is a λ-complete rewriting system for G.

Proof: We first show that the new rewriting system is noetherian. Let $F(\tilde{F})$ be the free monoid on the alphabet

$$\widetilde{F} = \{\widetilde{w} : \text{ w is a word on the alphabet } \mathbf{x}\}.$$

(Note that for two words w_1 and w_2 on $\mathbf{x}$, $(\widetilde{w}_1)(\widetilde{w}_2)$ is *not* equal to $\widetilde{w_1w_2}$.)

We can use the reduction order on $F(\mathbf{x})$ induced by the rules of $\mathcal{P}_H$ to give a noetherian partial order $\triangleright$ on the alphabet $\widetilde{F}\cup\{t,t^{-1}\}$ as follows:

1. $t^{\varepsilon} \triangleright \widetilde{w}$ for $\varepsilon = \pm 1$ and each letter $\widetilde{w} \in \widetilde{F}$;

2. $\widetilde{w}_1 \triangleright \widetilde{w}_2$ if $w_1 \rightarrow_{\mathbf{r}}^{*} w_2$ but $w_1 \neq w_2$.

Any word w on the alphabet $\mathbf{x}\cup\{t,t^{-1}\}$ of $\mathcal{Q}$ is of the form

$$w = w_1 t^{\varepsilon_1} w_2 t^{\varepsilon_2} \ldots w_n t^{\varepsilon_n} w_{n+1}$$

where $\varepsilon_i = \pm 1$ and w_i is a word on $\mathbf{x}$ for $i = 1, 2, \ldots, n$. Let $\widetilde{w}$ be the corresponding word

$$\widetilde{w} = \widetilde{w}_1 t^{\varepsilon_1} \widetilde{w}_2 t^{\varepsilon_2} \ldots \widetilde{w}_n t^{\varepsilon_n} \widetilde{w}_{n+1}$$

on the alphabet $\widetilde{F}\cup\{t,t^{-1}\}$.

For any two words w and z on the alphabet of $\mathcal{Q}$ we then write $w \succ z$ if and only if $\widetilde{w} \succ_{\triangleright} \widetilde{z}$, where $\succ_{\triangleright}$ is the recursive path ordering from the left (Definition 1.1.3) on $F(\widetilde{F}\cup\{t,t^{-1}\})$ induced by the partial order $\triangleright$. It remains to show that the rules of $\mathcal{Q}$ are compatible with this reduction ordering on $F(\mathbf{x}\cup\{t,t^{-1}\})$. This is easily checked. Firstly,

> $r_{+1} \succ r_{-1}$ holds for each rule r in $\mathcal{P}_H$, since $\widetilde{r}_{+1} \triangleright \widetilde{r}_{-1}$ immediately gives $\widetilde{r}_{+1} \succ_{\triangleright} \widetilde{r}_{-1}$.

The additional rules are seen to be compatible as follows.

1. For $\varepsilon = \pm 1$, $t^{\varepsilon} t^{-\varepsilon} \succ 1$ holds because $t^{\varepsilon} t^{-\varepsilon} \succ_{\triangleright} 1$ by definition.

2. The relation $t^{-1} u \succ v t^{-1}$ holds because $t^{-1} \widetilde{u} \succ_{\triangleright} \widetilde{v} t^{-1}$, since $t^{-1} \triangleright \widetilde{v}$ and $t^{-1} \widetilde{u} \succ_{\triangleright} t^{-1}$; the relation $t^{-1} \widetilde{u} \succ_{\triangleright} t^{-1}$ holds because $\widetilde{u} \succ_{\triangleright} 1$.

3. The relation $t^{-1}u^{-1} \succ vt$ holds because $t^{-1}\widetilde{u} \succ_{\triangleright} \widetilde{v}t^{-1}$, since $t^{-1} \triangleright \widetilde{v}$ and $t^{-1}\widetilde{u} \succ_{\triangleright} t^{-1}$; this latter holds because $\widetilde{u} \succ_{\triangleright} 1$;

4. The relation $tv \succ ut$ follows from the fact that $t\widetilde{v} \succ_{\triangleright} \widetilde{u}t$, because $t \triangleright \widetilde{u}$ and $t\widetilde{v} \succ_{\triangleright} t$; this latter holds because $\widetilde{v} \succ_{\triangleright} 1$.

5. The relation $tv^{-1} \succ u^{-1}t$ holds because $\widetilde{tv^{-1}} \succ_{\triangleright} \widetilde{u^{-1}}t$, because $t \triangleright \widetilde{u^{-1}}$ and $\widetilde{tv^{-1}} \succ_{\triangleright} t$; this latter holds because $\widetilde{v^{-1}} \succ_{\triangleright} 1$.

Therefore $\mathcal{Q}$ is indeed noetherian, and it remains to show that the empty word is the unique irreducible in its congruence class.

Suppose that w is an irreducible word which is congruent to the empty word. If w contains the letter t or t^{-1} then by Britton's Lemma it must contain a subword of the form $t^{-1}wt$ or $tw't^{-1}$ where w, w' are words on $\mathbf{x} \cup \mathbf{x}^{-1}$ representing elements of $\langle \overline{u} \rangle$, $\langle \overline{v} \rangle$, respectively, and so it will not be irreducible - a contradiction. If on the other hand w does not contain any instances of the letter t or the letter t^{-1}, then by Lemma 2.2.1 we see that it is the empty word, which is therefore the unique irreducible in its congruence class. ┘

2.2.2 Some examples

Baumslag-Solitar groups

The infinite cyclic group has a finite complete rewriting system consisting of the rules

$$xx^{-1} = 1 \quad \text{and} \quad x^{-1}x = 1$$

on the alphabet x, x^{-1}. For each pair of integers p and q, the normal forms of the subgroups generated by $\overline{x}^p$ and $\overline{x}^q$ are, respectively,

$$\{x^{kp} : k \in \mathbb{Z}\} \quad \text{and} \quad \{x^{kq} : k \in \mathbb{Z}\},$$

and so by Theorem 2.2.3, the HNN-group with the presentation

$$\mathcal{G}_{p,q} = \langle x, t \,;\, tx^p t^{-1} = x^q \rangle \tag{2.3}$$

obtained by associating these subgroups has the following λ-complete rewriting system:

$$\begin{array}{rll} \textbf{alphabet:} & x, x^{-1}, t, t^{-1} & \\ \textbf{rules:} & xx^{-1} = 1, & t^{-1}x^p = x^q t^{-1}, \\ & x^{-1}x = 1, & t^{-1}x^{-p} = x^{-q}t^{-1}, \\ & tt^{-1} = 1, & tx^q = x^p t, \\ & t^{-1}t = 1, & tx^{-q} = x^{-p}t. \end{array}$$

This rewriting system is not complete, since for example a critical pair of the form

$$\begin{array}{c} \qquad\qquad x^q t^{-1} x^{-1} \\ \nearrow \\ \underline{t^{-1}x^p}\,x^{-1} \\ \parallel \\ t^{-1}x^{p-1}\underline{xx^{-1}} \\ \searrow \\ \qquad\qquad t^{-1}x^{p-1} \end{array}$$

will not generally be resolved.

In fact $\mathcal{G}_{(p,q)}$ is the standard presentation of the Baumslag-Solitar group $G_{(p,q)}$, for which finite complete rewriting systems have been exhibited elsewhere [22, §7.4].

A small cancellation group

The free group $\mathcal{F}_3$ with basis $\{a, b, c\}$ has a complete rewriting system of the form (1.1), whose rules consist of all free reductions.

The normal forms of the cyclic subgroups generated by $\overline{c}$ and $\overline{cb^{-1}a^{-1}ba}$ are, respectively, the sets

$$\{c^k : k \in \mathbb{Z}\} \quad \text{and} \quad \{(cb^{-1}a^{-1}ba)^k : k \in \mathbb{Z}\},$$

and so by Theorem 2.2.3, the HNN-group with the presentation

$$\langle a, b, c, d\,;\; a^{-1}b^{-1}abc^{-1}d^{-1}cd\rangle. \tag{2.4}$$

obtained from $\mathcal{F}_3$ by associating these cyclic subgroups has the following λ-complete rewriting system:

$$
\begin{array}{rllc}
\textbf{alphabet:} & a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1} & & \\
\textbf{rules:} & aa^{-1} = 1, & a^{-1}a = 1, & d^{-1}c = cb^{-1}a^{-1}bad^{-1}, \\
& bb^{-1} = 1, & b^{-1}b = 1, & d^{-1}c^{-1} = a^{-1}b^{-1}abc^{-1}d^{-1}, \\
& cc^{-1} = 1, & c^{-1}c = 1, & dcb^{-1}a^{-1}ba = cd, \\
& dd^{-1} = 1, & d^{-1}d = 1, & da^{-1}b^{-1}abc^{-1} = c^{-1}d.
\end{array}
$$

But (2.4) is the standard presentation of the fundamental group of a closed orientable surface of genus 2, which was shown by Dehn himself to be (what we now call) a Dehn presentation (see [62, §6.1.4]). Therefore this group admits a different λ-complete rewriting system of the form (2.1) which is length reducing.

A group whose isoperimetric function grows faster than any simple exponential.

For each pair (p, q) of integers, the Baumslag-Solitar group $G_{p,q}$ with presentation (2.3) has a finite complete rewriting system [22, §7.4] on the alphabet x, x^{-1}, t, t^{-1}, moreover where the normal forms of the cyclic subgroups generated by $\overline{x}$ and $\overline{t}$ are, respectively,

$$\{x^k : k \in \mathbb{Z}\} \qquad \text{and} \qquad \{t^k : k \in \mathbb{Z}\}.$$

Again we can use Theorem 2.2.3 to construct λ-compete rewriting systems for the HNN-groups formed by associating these subgroups. In particular, the group

$$G_{1,2} *_{\langle \overline{x} \rangle \cong \langle \overline{t} \rangle}$$

has a finite λ-complete rewriting system. This group has presentation

$$\langle x, t, s \; ; \; txt^{-1} = x^2, \; t = sxs^{-1} \rangle$$

from which we can remove the generator t by a type 2 Tietze transformation to get the presentation

$$\langle x, s \; ; \; x^{(x^s)} = x^2 \rangle,$$

where we use the notation x^t to denote the word txt^{-1}. This presentation was shown by Gersten [24] to have an *isoperimetric function*[1] growing faster than the function E_r for all $r \geq 0$, where E_r is the function $\mathbb{N} \longrightarrow \mathbb{N}$ defined recursively by writing

$$E_r(n) = 2^{E_{r-1}(n)}; \qquad E_0(n) = n.$$

2.2.3 Remarks concerning Open Question 2.1.8

The Anick-Groves-Squier Theorem (Theorem 1.4.4) tells us that if a group has a finite complete rewriting system then it is of type FP_∞. Therefore, in order to answer Open Question 2.1.8, one might attempt to find a finite λ-complete rewriting system for a non-FP_∞ group.

Proposition 2.13 (b) of Bieri's book [5] states that if a group G is of type FP_n and has isomorphic subgroups of type FP_{n-1}, then an HNN extension $G*_{A \cong B}$ obtained by associating these subgroups is itself of type FP_n. As a consequence, the construction used in Theorem 2.2.3 will only produce λ-complete rewriting systems for groups which are of type FP_∞: the base group is always of type FP_∞, since it is required to have a finite complete rewriting system, as are the cyclic subgroups which are associated (see, for example, [5, Proposition 2.7]).

However, we do not need the associated subgroups to be cyclic, or indeed to have a finite complete rewriting system for the base group; the requirements of Theorem 2.2.3 just allow a general result to be given. We might hope to use the same method, that is, using Britton's Lemma to examine the irreducible words congruent to the empty word, in specific cases of non-FP_∞ groups. The author has tried (but not succeeded) to produce a λ-complete rewriting system for Abels' group in this way (see the appendix, §A.1).

[1] Isoperimetric functions give some idea of the complexity of word problems (see [26]): for λ-complete rewriting systems this function describes, in terms of the length of any word w satisfying $\overline{w} = 1$, the minimum number of applications of rules that one needs to allow in order to be certain that w can be rewritten to the empty word.

Chapter 3

A 3-dimensional complex and its homology

Abstract

We begin by attaching 3-cells to the 2-complex $\mathcal{D}^{\mathbf{p}}$ to form a 3-complex, which we denote $\overline{\mathcal{D}^{\mathbf{p}}}$. We show that the second homology group of this new 3-complex has, like the first homology of $\mathcal{D}$, a natural $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule structure. In §3.3 we exhibit this bimodule as the kernel of a presentation of Π (the short exact sequence (4)), and in §3.4 we give the short exact sequence (5).

3.1 A 3-dimensional complex

Let $\mathcal{D}$ be the Squier complex and let $\mathbf{p}$ be a set of closed paths. Recall from §1.3.1 that $\mathcal{D}^{\mathbf{p}}$ is the 2-complex obtained by attaching 2-cells $[w, p, w']$ ($p \in \mathbf{p}$, $w, w' \in F$) along all closed paths of the form

$$\partial[w, p, w'] = w.p.w'.$$

We now extend $\mathcal{D}^{\mathbf{p}}$ to form the 3-complex $\overline{\mathcal{D}^{\mathbf{p}}}$: the 2-skeleton is just $\mathcal{D}^{\mathbf{p}}$, and the construction is completed by adding 3-cells as follows. For each positive edge f and each 2-cell σ, where $\partial\sigma = e_1^{\varepsilon_1} \ldots e_n^{\varepsilon_n}$ ($e_i \in \mathbf{e}^+$, $\varepsilon_i = \pm 1, i = 1, \ldots, n$), 3-cells $[f, \sigma]$ and $[\sigma, f]$ are attached to the 2-skeleton by mapping their boundaries to, respectively:

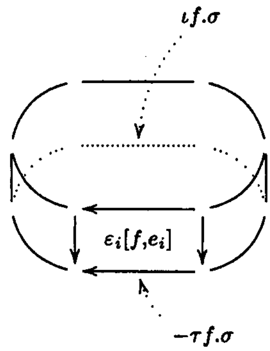

Figure 3.1: Boundary of the 3-cell $[f, \sigma]$

$$\begin{tikzcd} \cdot \arrow[d, "f.\tau e_i"'] \arrow[dr, phantom, "{[f,e_i]}"] & \cdot \arrow[l, "\iota f.e_i"'] \arrow[d, "f.\iota e_i"] \\ \cdot & \cdot \arrow[l, "\tau f.e_i"] \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "\iota f.e_i"] \arrow[d, "f.\iota e_i"'] \arrow[dr, phantom, "{-[f,e_i]}"] & \cdot \arrow[d, "f.\tau e_i"] \\ \cdot \arrow[r, "\tau f.e_i"'] & \cdot \end{tikzcd}$$

$\varepsilon_i = 1$ $\qquad$ $\varepsilon_i = -1$

Figure 3.2: A 2-cell of $\mathcal{D}$

1. the 2-cells $\iota f.\sigma$ and $-\tau f.\sigma$, together with 2-cells $\varepsilon_i[f, e_i]$ for $1 \leq i \leq n$ according to the diagram Figure 3.1, where $\varepsilon_i[f, e_i]$ is the 2-cell shown in Figure 3.2.

2. 2-cells $\sigma.\iota f$ and $-\sigma.\tau f$ together with 2-cells $-\varepsilon_i[e_i, f]$ for $1 \leq i \leq n$.

Remark 3.1.1 Note the similarities to the construction of the 2-complex $\mathcal{D}$, where for each pair of positive edges e and f, 2-cells $[e, f]$ are attached to Γ by mapping their boundaries to the closed paths composed of the edges $e.\iota f$ and $-e.\tau f$ together with the edges $\tau e.f$ and $-\iota e.f$.

Remark 3.1.2 For $e, f, g \in \mathbf{e}^+$ the added 3-cells $[e, [f, g]]$ and $[[e, f], g]$ are cubes, as shown in Figure 3.3.

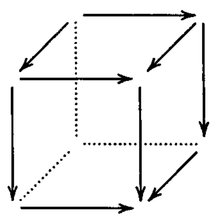

Figure 3.3: A cube

The 2-sided action of F again extends naturally to the 3-cells. For $[f, \sigma]$ and $[\sigma, f]$ ($f \in \mathbf{e}^+$, σ a 2-cell), and $u, v \in F$,

$$u.[f, \sigma].v = [u.f, \sigma.v] \quad \text{and} \quad u.[\sigma, f].v = [u.\sigma, f.v]\,.$$

3.2 The 2-dimensional homology of the 3-complex $\overline{\mathcal{D}^{\mathbf{p}}}$

We now expand the chain complex $\mathbf{C}(\mathcal{D})$ in §1.2.2 to get the chain complex

$$\mathbf{C}(\overline{\mathcal{D}^{\mathbf{p}}}): \qquad 0 \longrightarrow C_3^{\mathbf{p}} \xrightarrow{\partial_3} C_2 \oplus C_2^{\mathbf{p}} \xrightarrow{\tilde{\partial}_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

associated to $\overline{\mathcal{D}^{\mathbf{p}}}$. Here $C_3^{\mathbf{p}}$ is the free abelian group with basis the set of all 3-cells and $C_2^{\mathbf{p}}$ is the free abelian group with basis the set of all 2-cells $[u, p, v]$ ($p \in \mathbf{p}$, $u, v \in F$). The map $\tilde{\partial}_2$ restricted to C_2 is ∂_2, and a 2-cell $[u, p, v]$ (where $u, v \in F$ and $p = f_1^{\delta_1} f_2^{\delta_2} \ldots f_m^{\delta_m}$ ($\delta_i = \pm 1$, $f_i \in \mathbf{e}^+$, $i = 1, \ldots, m$) is a path in $\mathbf{p}$) is mapped to

$$\sum_{i=1}^{m} \delta_i u.f_i.v \in C_1\,.$$

We define ∂_3 as follows. If $f \in \mathbf{e}^+$ and σ is any 2-cell with

$$\tilde{\partial}_2 \sigma = \sum_{i=1}^{n} \varepsilon_i e_i \qquad (\varepsilon_i = \pm 1,\ e_i \in \mathbf{e}^+,\ i = 1, \ldots, n)$$

then

$$\partial_3[f,\sigma] = (\iota f - \tau f).\sigma + \sum_{i=1}^{n} \varepsilon_i [f, e_i] \tag{3.1}$$

and

$$\partial_3[\sigma, f] = \sigma.(\iota f - \tau f) - \sum_{i=1}^{n} \varepsilon_i [e_i, f]. \tag{3.2}$$

Again these chain groups are (free) $(\mathbb{Z}F, \mathbb{Z}F)$-bimodules, where the F-action is inherited from the two-sided action of F on $\overline{\mathcal{D}^{\mathbf{P}}}$, and the boundary maps are $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule homomorphisms. For future reference we note in particular that $C_2^{\mathbf{P}}$ is free with basis

$$\hat{\mathbf{p}} = \{\hat{p} = [1, p, 1] : p \in \mathbf{p}\}. \tag{3.3}$$

We now examine the second homology group

$$H_2(\overline{\mathcal{D}^{\mathbf{P}}}) = \frac{Z_2(\overline{\mathcal{D}^{\mathbf{P}}})}{B_2(\overline{\mathcal{D}^{\mathbf{P}}})}$$

where $Z_2(\overline{\mathcal{D}^{\mathbf{P}}}) = \mathrm{Ker}\tilde{\partial}_2$ and $B_2(\overline{\mathcal{D}^{\mathbf{P}}}) = \mathrm{Im}\partial_3$. Again $H_2(\overline{\mathcal{D}^{\mathbf{P}}})$ is a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule, the F-action induced by the two-sided action of F on the bases of the chain groups. Recall that J denotes the kernel of the ring homomorphism $\mathbb{Z}F \longrightarrow \mathbb{Z}S$.

Lemma 3.2.1 *$J.H_2(\overline{\mathcal{D}^{\mathbf{P}}}) = 0$ and $H_2(\overline{\mathcal{D}^{\mathbf{P}}}).J = 0$.*

Proof: We will only show that $J.H_2(\overline{\mathcal{D}^{\mathbf{P}}}) = 0$; the other equality is obtained similarly. Suppose that ξ is a 2-cycle, say

$$\xi = \sum_{i=1}^{n} \varepsilon_i \sigma_i \in Z_2(\overline{\mathcal{D}^{\mathbf{P}}}) \qquad (\varepsilon_i = \pm 1 \text{ and } \sigma_i \text{ a 2-cell, } i = 1, \ldots, n),$$

where for $i = 1, \ldots, n$

$$\tilde{\partial}_2 \sigma_i = \sum_{j=1}^{k_i} \delta_{i_j} e_{i_j} \qquad (\delta_{i_j} = \pm 1,\ e_{i_j} \in \mathbf{e}^+,\ j = 1, \ldots, k_i).$$

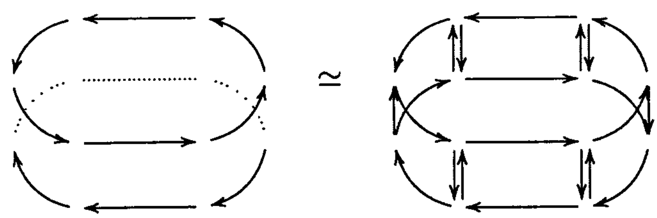

Figure 3.4: Homologous 2-chains

Then

$$0 = \tilde{\partial}_2 \xi = \sum_{i=1}^{n} \varepsilon_i \sum_{j=1}^{k_i} \delta_{i_j} e_{i_j}. \tag{3.4}$$

By (3.1), if $f \in \mathrm{e}^+$ then $(\iota f - \tau f)\sigma_i$ is homologous to the 2-chain

$$-\sum_{j=1}^{k_i} \delta_{i_j} [f, e_{i_j}].$$

(see Figure 3.4). and it follows that

$$(\iota f - \tau f).\xi + B_2(\overline{\mathcal{D}\mathrm{P}}) \quad = \quad -\sum_{i=1}^{n} \varepsilon_i \sum_{j=1}^{k_i} \delta_{i_j} [f, e_{i_j}] + B_2(\overline{\mathcal{D}\mathrm{P}}).$$

But by comparison with (3.4),

$$\sum_{i=1}^{n} \varepsilon_i \sum_{j=1}^{k_i} \delta_{i_j} [f, e_{i_j}] = 0,$$

and so $(\iota f - \tau f).\xi$ is null-homologous. By Lemma 1.1.6, J is generated as an abelian group by elements of the form $\iota f - \tau f$ $(f \in \mathrm{e}^+)$, and it follows immediately that $J.H_2(\overline{\mathcal{D}\mathrm{P}}) = 0$. ⌟

It follows that there is an induced $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule structure on the second homology with the action

$$\bar{u}(\xi + B_2(\overline{\mathcal{D}\mathrm{P}}))\bar{v} = u.\xi.v + B_2(\overline{\mathcal{D}\mathrm{P}}) \qquad (\xi \in Z_2(\overline{\mathcal{D}\mathrm{P}}), u, v \in F).$$

We denote this bimodule by $\Pi_2 = \Pi_2(\mathcal{P}, \mathbf{p})$.

Clearly $B_1(\mathcal{D}) \subseteq B_1(\overline{\mathcal{D}^{\mathbf{p}}}) \subseteq Z_1(\mathcal{D}) = Z_1(\overline{\mathcal{D}^{\mathbf{p}}})$. Note that the quotient

$$\Lambda(\overline{\mathcal{D}^{\mathbf{p}}}) = \frac{B_1(\overline{\mathcal{D}^{\mathbf{p}}})}{B_1(\mathcal{D})}$$

has a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule structure, and in fact as a submodule of $H_1(\mathcal{D})$ we can deduce from (1.4) that

$$J.\Lambda(\overline{\mathcal{D}^{\mathbf{p}}}) = \Lambda(\overline{\mathcal{D}^{\mathbf{p}}}).J = 0\,,$$

so that $\Lambda(\overline{\mathcal{D}^{\mathbf{p}}})$ also has an induced $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule structure. This module is generated by the elements

$$\tilde{\partial}_2[1, p, 1] + B_1(\mathcal{D}) \ (p \in \mathbf{p}),$$

and so we have an epimorphism

$$\nu : \mathbb{Z}S.\mathbf{p}.\mathbb{Z}S \longrightarrow \Lambda(\overline{\mathcal{D}^{\mathbf{p}}}) \qquad p \longmapsto \tilde{\partial}_2[1, p, 1] + B_1(\mathcal{D}).$$

Clearly $\Lambda(\overline{\mathcal{D}^{\mathbf{p}}}) = \Pi$ if and only if $H_1(\overline{\mathcal{D}^{\mathbf{p}}}) = 0$, that is, if and only if $\mathbf{p}$ is a homology trivializer.

3.3 The basic short exact sequence

Theorem 3.3.1 *If the homology classes of the paths in* $\mathbf{p}$ *give rise to a set of bimodule generators of* Π *then there is a short exact sequence*

$$0 \longrightarrow \Pi_2 \xrightarrow{\Phi} \mathbb{Z}S.\mathbf{p}.\mathbb{Z}S \xrightarrow{\nu} \Pi \longrightarrow 0. \tag{3.5}$$

of $(\mathbb{Z}S, \mathbb{Z}S)$*-bimodules.*

The proof we give here was inspired by the methods used by Kobayashi and Otto [40] to give the exact sequence (1) in the introduction.

Let $\mathbb{Z}S.\mathbf{p}.\mathbb{Z}S$ be the free $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule with basis $\mathbf{p}$, which we can consider as a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule via the homomorphism of F onto S. Then since $C_2^{\mathbf{p}}$ is the free $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule with basis the set $\hat{\mathbf{p}}$ of 2-cells attached along paths in $\mathbf{p}$ (3.3), we can define a $(\mathbb{Z}F, \mathbb{Z}F)$-homomorphism

$$\varphi: \; C_2 \oplus C_2^{\mathbf{p}} \longrightarrow \mathbb{Z}S.\mathbf{p}.\mathbb{Z}S \tag{3.6}$$

which maps C_2 to 0 and a 2-cell $u.\hat{p}.v = [u, p, v]$ ($u, v \in F$ and $p \in \mathbf{p}$) is mapped to $\bar{u}.p.\bar{v}$. Let $K^{\mathbf{p}}$ denote the $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule $\mathrm{Ker}\varphi$.

Lemma 3.3.2 *We can write* $K^{\mathbf{p}} = C_2 + J.\hat{\mathbf{p}}.\mathbb{Z}F + \mathbb{Z}F.\hat{\mathbf{p}}.J$.

Proof: This follows immediately from Lemma 1.2.4. ⌟

It is clear from the definition of the boundary map ∂_3 that $B_2(\overline{\mathcal{D}^{\mathbf{p}}}) \subseteq K^{\mathbf{p}}$ (see (3.1) and (3.2)). Also, the restriction of $\tilde{\partial}_2$ to $K^{\mathbf{p}}$ sends $K^{\mathbf{p}}$ onto $B_1(\mathcal{D})$ (if we apply $\tilde{\partial}_2$ to (3.1) where $\sigma = [1, p, w]$ ($w \in F$, $p \in \mathbf{p}$) the left-hand-side is mapped to 0, giving

$$(\iota f - \tau f).\tilde{\partial}_2 \hat{p}.w = -\tilde{\partial}_2 \sum_{i=1}^{n} \varepsilon_i [f, e_i] \in B_1(\mathcal{D})$$

so that $\tilde{\partial}_2(J.\hat{\mathbf{p}}.\mathbb{Z}F) \subseteq B_1(\mathcal{D})$; similarly, $\tilde{\partial}_2(\mathbb{Z}F.\hat{\mathbf{p}}.J) \subseteq B_1(\mathcal{D})$) and, moreover, since $\tilde{\partial}_2\partial_3 = 0$, $B_2(\overline{\mathcal{D}^{\mathbf{p}}})$ is in the kernel of this homomorphism. Therefore we have a complex

$$0 \longrightarrow B_2(\overline{\mathcal{D}^{\mathbf{p}}}) \xrightarrow{incl.} K^{\mathbf{p}} \xrightarrow{\tilde{\partial}_2} B_1(\mathcal{D}) \longrightarrow 0. \tag{3.7}$$

We shall show that if $\mathbf{p}$ is a homology trivializer then $B_2(\overline{\mathcal{D}^{\mathbf{p}}})$ is precisely the kernel of the epimorphism $\tilde{\partial}_2 : K^{\mathbf{p}} \longrightarrow B_1(\mathcal{D})$, so that (3.7) is a short exact sequence. We first prove the following lemma.

Lemma 3.3.3 *Let* $\mathbf{q}$ *be another homology trivializer of* $\mathcal{D}$. *If the sequence*

$$0 \longrightarrow B_2(\overline{\mathcal{D}^{\mathbf{q}}}) \xrightarrow{incl.} K^{\mathbf{q}} \xrightarrow{\tilde{\partial}_2} B_1(\mathcal{D}) \longrightarrow 0$$

is exact, then so is (3.7).

Proof: Let ξ be a 2-cycle in $K^{\mathbf{p}}$. By Lemma 3.3.2 we can write ξ as a sum

$$\xi = \xi' + \sum_{i=1}^{m} \varepsilon_i(\iota e_i - \tau e_i).[1, p_i, w_i] + \sum_{i=m+1}^{m+n} \varepsilon_i[w_i, p_i, 1].(\iota e_i - \tau e_i)$$

where ξ' is a 2-chain in $C_2(\mathcal{D})$ and, for $i = 1, \ldots, m+n$, $\varepsilon_i = \pm 1$, $w_i \in F$, e_i is a positive edge in $\mathcal{D}$ and $p_i \in \mathbf{p}$.

Suppose that, for some $i \in \{1, \ldots, m+n\}$, $\partial p_i = f_1^{\delta_1} \ldots f_k^{\delta_k}$, where for $j = 1, \ldots, k$ $\delta_j = \pm 1$ and f_j is a positive edge in $\mathcal{D}$. If $i \leq m$ then because of the 3-cell $[e_i, [1, p_i, w_i]]$ we have

$$(\iota e_i - \tau e_i).[1, p_i, w_i] = -\sum_{j=1}^{k} \delta_j [e_i, f_j].w_i \qquad (\text{modulo } B_2(\overline{\mathcal{D}^{\mathbf{p}}})),$$

and if $m < i$ then because of the 3-cell $[[w_i, p_i, 1], e_i]$ we have

$$[w_i, p_i, 1].(\iota e_i - \tau e_i) = \sum_{j=1}^{k} \delta_j w_i.[f_j, e_i] \qquad (\text{modulo } B_2(\overline{\mathcal{D}^{\mathbf{p}}})).$$

Thus ξ is homologous to some 2-cycle ζ, say, in $C_2(\mathcal{D})$, and since $C_2(\mathcal{D}) \subseteq K^{\mathbf{q}}$ by hypothesis we can write ζ as the boundary of some 3-chain ω in $C_3(\overline{\mathcal{D}^{\mathbf{q}}})$.

But because $\mathbf{p}$ is also a homology trivializer, for each $q \in \mathbf{q}$ we can choose a 2-chain

$$\sum_{i=1}^{t} \alpha_i \varsigma_i \qquad (\alpha_i = \pm 1 \text{ and } \varsigma_i \text{ a 2-cell, for } i = 1, \ldots, t)$$

in $C_2(\overline{\mathcal{D}^{\mathbf{p}}})$ with boundary the 1-cycle arising from the closed path q, and if in the 3-chain ω we replace each 3-cell of the form $[f, [u, q, v]]$ or $[[u, q, v], f]$ ($u, v \in F$, f a positive edge) with the 3-chain

$$\sum_{i=1}^{t} \alpha_i [f, [u, \varsigma_i, v]] \quad \text{or} \quad \sum_{i=1}^{t} \alpha_i [[u, \varsigma_i, v], f]$$

respectively, then we obtain a 3-chain in $C_3(\overline{\mathcal{D}^{\mathbf{p}}})$ with the same boundary ζ. Therefore $\xi \in B_2(\overline{\mathcal{D}^{\mathbf{p}}})$, and the sequence (3.7) is exact. ⌟

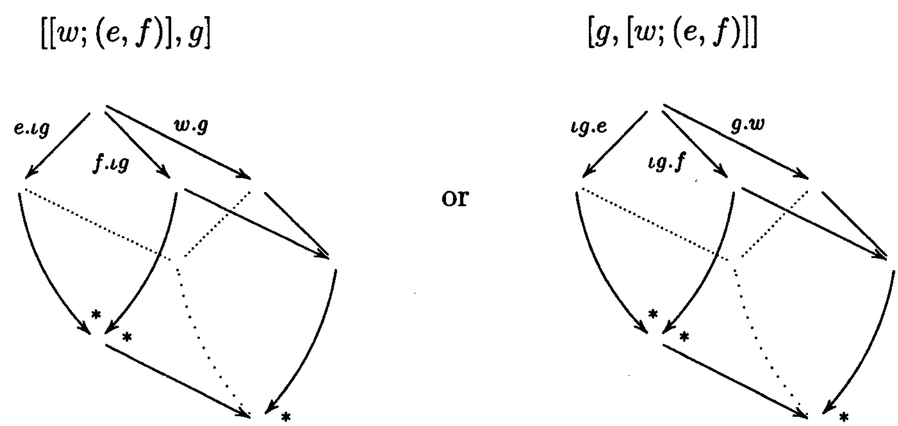

Figure 3.5: Typical 3-cells in $\overline{\mathcal{D}^{\mathbf{p}}}$

Proposition 3.3.4 *If* $\mathbf{p}$ *is a homology trivializer, then the sequence* (3.7) *is exact.*

Special case: We assume that $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ is a *complete* rewriting system such that $\mathbf{r}$ is compatible with some length-lexicographical ordering $<_{llex}$ of F, and $\mathbf{p}$ is a trivializer *obtained by choosing resolutions of the critical pairs of* $\mathbf{r}$ (see Remark 1.4.14).

Suppose ξ is a 2-cycle in $K^{\mathbf{p}}$. We will show that ξ is null-homologous by an inductive argument, for which we first need to order the 2-cells in $\overline{\mathcal{D}^{\mathbf{p}}}$. In what follows we shall use the notation for 2-cells introduced in Remark 1.4.14. We give the 2-cells of $\mathcal{D}^{\mathbf{p}}$ the noetherian total order described by writing $[w; (e, f)] < [w'; (e', f')]$ if:

(i) $w <_{llex} w'$; or

(ii) $w = w'$ and $f <_w f'$; or

(iii) $w = w'$, $f = f'$ and $e <_w e'$.

Remark 3.3.5 Note that each 3-cell in $\overline{\mathcal{D}^{\mathbf{p}}}$ is of one of the two types shown in Figure 3.5, where $w \in F$ and $e, f, g \in \mathbf{e}^+$ with $e, f \in star^+(w)$. The boundary of each 3-cell will therefore consist of three 2-cells with maximal vertex $w.\iota g$ or $\iota g.w$ respectively, together with 2-cells with lesser maximal vertices under the length-lexicographic ordering of F.

We can express ξ uniquely as a sum

$$\xi = \sum_{i=1}^{m} n_i \sigma_i$$

where $m \geq 0$, $\sigma_1, \ldots, \sigma_m$ are distinct 2-cells, and $n_1, \ldots, n_m$ are non-zero integers.

Suppose that ξ is non-zero, that is $m > 0$, and that the 2-cell σ_1 is maximal, so that $\sigma_i < \sigma_1$ for $i = 2, \ldots, m$. Then according to the following procedure we can always replace $n_1\sigma_1$ with a 2-chain composed of lesser 2-cells to obtain a homologous 2-cycle ξ', say. Because $\xi - \xi' \in B_2(\overline{\mathcal{D}^{\mathbf{P}}}) \subseteq K^{\mathbf{P}}$, we have $\xi' = \xi - (\xi - \xi') \in K^{\mathbf{P}}$ also. If we inductively assume that any such 2-cycle (whose maximal 2-cell with non-zero coefficient is beneath σ_1 in our ordering) is null-homologous, then we immediately deduce that ξ itself is null-homologous. The special case of the proposition follows by noetherian induction (Lemma 1.4.2) on the set of all 2-cycles in $K^{\mathbf{P}}$, ordered by their maximal 2-cell with non-zero coefficient.

(I) Suppose first that we can write $\sigma_1 = \iota e.\sigma_1'$ where $e \in \mathbf{e}^+$ and $\sigma_1' = [w; (f, g)]$ for some $w \in F$ and $f, g \in star^+(w)$. Then

$$\partial_3[e, \sigma_1'] = \sigma_1 + [\iota e.w; (e.w, \iota e.f)] - [\iota e.w; (e.w, \iota e.g)] + \zeta$$

where ζ is a 2-chain composed of 2-cells with maximal vertices which are beneath $\iota e.w$ in the length-lexicographical order on F, and so σ_1 is homologous to the 2-chain

$$-[\iota e.w; (e.w, \iota e.f)] + [\iota e.w; (e.w, \iota e.g)] - \zeta$$

composed of lesser 2-cells under the ordering described above. Therefore up to homology we can remove $n_1\sigma_1$ from ξ, replacing it with a 2-chain ξ', say, composed of lesser 2-cells.

(II) If we cannot write σ_1 as in **(I)**, and if $\sigma_1 = [w;(e,f)]$ for some $w \in F$, then there is no $g \in star^+(w)$ satisfying $g <_w e$ and such that g is disjoint from e. Suppose instead that there is an edge $g \in star^+(w)$ satisfying $f <_w g$ with f and g disjoint. In this case σ_1 is homologous to the 2-chain

$$[w;(e,g)] - [w;(f,g)] + \zeta$$

where ζ is a 2-chain composed of 2-cells with lesser maximal vertices. Since we now have $[w;(e,g)], [w;(f,g)] > \sigma_1$ we do not want to simply replace $n_1\sigma_1$ as in **(I)**. However, we know that $\tilde{\partial}_2\xi = 0$, and since σ_1 is maximal it follows that the elementary 1-chains e and $-f$ arising from σ_1 must be cancelled in the boundary map by edges from 2-cells with the same maximal vertex. In fact, part of ξ must consist of a 2-chain

$$\xi' = n_1\sigma_1 + \sum_{i=1}^{k} \varepsilon_i[w;(e_i,f_i)] \qquad (\varepsilon_i = \pm 1, e_i, f_i \in star^+(w), i = 1,\ldots,k)$$

of 2-cells with maximal vertex w, with

$$n_1(e-f) + \sum_{i=1}^{k} \varepsilon_i(e_i - f_i) = 0. \tag{3.8}$$

Again since σ_1 is maximal, we cannot have $f <_w f_i$ for any $i \in \{1,\ldots,k\}$, and so ξ' is homologous to a 2-chain

$$n_1([w;(e,g)] - [w;(f,g)]) + \sum_{i=1}^{k} \varepsilon_i([w;(e_i,g)] - [w;(f_i,g)]) + \zeta'$$

where ζ' is a 2-chain consisting of 2-cells whose maximal vertices are beneath w in the length-lexicographic order. But because of (3.8), we can cancel the terms in the first part of this 2-chain to be left with ζ', and thus we may again (up to homology) replace part of ξ including $n_1\sigma_1$ with a 2-chain composed of lesser 2-cells.

(III) Next, suppose $\sigma_1 = [w;(e,g)]$ and there is an edge $f \in star^+(w)$ disjoint from both e and g and satisfying $e <_w f <_w g$ (so e and g must themselves be disjoint, and σ_1 is a 2-cell from $\mathcal{D}$), but that there is no such edge satisfying $f <_w e$ or $g <_w f$. In this case we have

$$[w;(e,f)] < \sigma_1 < [w;(f,g)].$$

As in part (**II**), because all the edges in $star^+(w)$ have to cancel under the boundary map, part of ξ is a 2-chain

$$\xi' = n_1\sigma_1 + \sum_{i=1}^{k} \varepsilon_i[w;(e_i,g)] \qquad (\varepsilon_i = \pm 1, \text{ and } e_i \in star^+(w), i = 1,\ldots,k)$$

consisting of all 2-cells with maximal vertex w and with their boundary including g, and where $e_i <_w e$ for all $i = 1,\ldots,k$ since σ_1 is maximal. We then have

$$n_1 + \sum_{i=1}^{k} \varepsilon_i = 0,$$

as this sum expresses the coefficient of the edge g in $\tilde{\partial}_2\xi$ (more precisely, in $\tilde{\partial}_2(-\xi)$). Then ξ' is homologous to the 2-chain

$$n_1([w;(e,f)]+[w;(f,g)]) + \sum_{i=1}^{k} \varepsilon_i([w;(e_i,f)]+[w;(f,g)]) + \zeta$$

$$= (n_1 + \sum_{i=1}^{k} \varepsilon_i)[w;(f,g)] + n_1[w;(e,f)] + \sum_{i=1}^{k} \varepsilon_i[w;(e_i,f)] + \zeta$$

$$= n_1[w;(e,f)] + \sum_{i=1}^{k} \varepsilon_i[w;(e_i,f)] + \zeta,$$

where ζ is a 2-chain consisting of 2-cells with maximal vertices beneath w in the length-lexicographical order. Therefore we can again (up to homology) replace a part of ξ including $n_1\sigma_1$ with a 2-chain composed of lesser 2-cells only.

(IV) Next suppose that $\sigma_1 = [w; (f, g)]$ where $f, g \in star^+(w)$ are again disjoint, but that now there are no other edges in $star^+(w)$ which are disjoint from both f and g. The coefficient of g in $\tilde{\partial}_2\xi$ must be 0, and therefore 2-cells of the form $[w; (e, g)]$ must also be represented in ξ, where $e <_w f$ and e and f arise from some critical pair. Then σ_1 is homologous to a 2-chain

$$[w; (e, g)] - [w; (e, f)] + \zeta$$

where ζ is a 2-chain consisting of 2-cells with maximal vertices beneath w in the length-lexicographical order. Therefore we can again (up to homology) replace $n_1\sigma_1$ in ξ with a 2-chain composed of lesser 2-cells only.

(V) The remaining possibility is that the maximal 2-cell σ_1 can be written in the form $u.[w; (e, f)].v$ where $w, u, v \in F$ with u and v irreducible, and (e, f) is a critical pair. Since $\varphi\xi = 0$ it follows that $\xi - n_1\sigma_1$ must include a 2-chain of the form

$$\sum_{j=1}^{k} \varepsilon_j u_j.[w; (e, f)].v_j \qquad (\varepsilon_j = \pm 1 \text{ and } u_j, v_j \in F \text{ for } j = 1, \ldots, k)$$

where $\bar{u}_j = \bar{u}$ and $\bar{v}_j = \bar{v}$ for all $j = 1, \ldots, k$, and such that

$$\sum_{j=1}^{k} \varepsilon_j \bar{u}_j.[w; (e, f)].\bar{v}_j = -n_1\bar{u}.[w; (e, f)].\bar{v}$$

But for each $j = 1, \ldots, k$ we must have either $u_j \neq u$ or $v_j \neq v$, and because $\mathcal{P}$ is complete either $u <_{llex} u_j$ and $v \leq_{llex} v_j$, or $u \leq_{llex} u_j$ and $v <_{llex} v_j$, from which it follows that $\sigma_1 < u_j.[w; (e, f)].v_j$, a contradiction. ⌟

Remark 3.3.6 The above proof does not require the set $\mathbf{r}$ to be finite, since we may in any case assume that the set $star^+(w)$ of any $w \in F$ is well-ordered. This is because w has at most a finite number of subwords, and if there is a pair of positive edges $(u, r, +1, v), (u, r', +1, v)$ (where $u, v \in F$ and $r, r' \in \mathbf{r}$) in $star^+(w)$ which rewrite the same subword then

$$(u, r, +1, v) <_w (u, r', +1, v) \quad \text{if} \quad r_{-1} <_{llex} r'_{-1},$$

so that the ordering of such edges in $star^+(w)$ is well-founded. It follows that the 2-cells of $\overline{\mathcal{D}^{\mathbf{p}}}$ are well-ordered even if the set of rules $\mathbf{r}$ and the homology trivializer $\mathbf{p}$ are both infinite.

General case: Let $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ be a finite rewriting system, with homology trivializer $\mathbf{p}$. We can assume that $\mathbf{r}$ is compatible with some length-lexicographical order on F and we can complete $\mathbf{r}$ using the Knuth-Bendix completion procedure (§1.4.1), obtaining a complete (but possibly infinite) rewriting system $\mathcal{P}^\infty = [\mathbf{x}; \mathbf{r}^\infty]$ with $\mathbf{r} \subseteq \mathbf{r}^\infty$ and where $\mathbf{r}^\infty$ is also compatible with the order on F. Then the 2-complex $\mathcal{D}(\mathcal{P}^\infty)$ has trivializer $\mathbf{p}^\infty$ obtained by choosing resolutions of all the critical pairs of $\mathbf{r}^\infty$, and by the Special Case and Remark 3.3.6 the sequence

$$0 \longrightarrow B_2(\overline{\mathcal{D}(\mathcal{P}^\infty)^{\mathbf{p}^\infty}}) \xrightarrow{incl.} K^\infty \xrightarrow{\tilde{\partial}_2} B_1(\mathcal{D}(\mathcal{P}^\infty)) \longrightarrow 0 \tag{3.9}$$

is exact, where $K^\infty = C_2(\mathcal{D}(\mathcal{P}^\infty)) + J.\hat{\mathbf{p}}^\infty.\mathbb{Z}F + \mathbb{Z}F.\hat{\mathbf{p}}^\infty.J$. We will deduce from this and Lemma 3.3.3 that the corresponding sequence for $\overline{\mathcal{D}(\mathcal{P})^{\mathbf{p}}}$ is also exact.

There is a natural inclusion of $\mathcal{D}(\mathcal{P})$ into $\mathcal{D}(\mathcal{P}^\infty)$, with a retraction ρ of $\mathcal{D}(\mathcal{P}^\infty)$ onto $\mathcal{D}(\mathcal{P})$ given by some choice of a path in $\mathcal{D}(\mathcal{P})$ from r_{+1} to r_{-1} for each rule $r \in (\mathbf{r}^\infty - \mathbf{r})$:

> if $\rho(u, r, +1, v) = e_1^{\varepsilon_1} \ldots e_m^{\varepsilon_m}$ and $\rho(u', r', +1, v') = f_1^{\delta_1} \ldots f_n^{\delta_n}$ for $u, v, u', v' \in F$ and $r, r' \in \mathbf{r}^\infty$ (e_i, f_j positive edges in $\mathcal{D}$ and $\varepsilon_i, \delta_j = \pm 1$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$) then the 2-cell $[(u, r, +1, v), (u', r', +1, v')]$ is mapped by ρ to the subcomplex composed of the 2-cells
>
> $$\{[e_i, f_j] : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

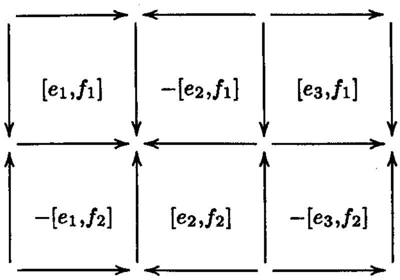

Figure 3.6: The subcomplex in Example 3.3.7

Example 3.3.7 As as illustration, if say $m = 3$ and $n = 2$, with $\varepsilon_1 = \varepsilon_3 = \delta_1 = 1$ and $\varepsilon_2 = \delta_2 = -1$, then the subcomplex will be as shown in Figure 3.6, where as in §3.1 the sign indicates the orientation of the 2-cell. ⌟

Let $\mathbf{q}$ denote the set of closed paths $\{\rho(p) : p \in \mathbf{p}^\infty\}$. The retraction ρ can be extended to a map of $\overline{\mathcal{D}(\mathcal{P}^\infty)^{\mathbf{p}^\infty}}$ onto $\overline{\mathcal{D}(\mathcal{P})^{\mathbf{q}}}$. A 2-cell $[u, p, v]$ $(u, v \in F, p \in \mathbf{p})$ will be mapped to the 2-cell $[u, \rho(p), v]$, and for a positive edge f in $\mathcal{D}(\mathcal{P}^\infty)$ mapped to $\rho f = e_1^{\varepsilon_1} \ldots e_m^{\varepsilon_m}$ ($\varepsilon_i = \pm 1$ and e_i a positive edge in $\mathcal{D}(\mathcal{P})$ for $i = 1, \ldots, m$) the 3-cells $[f, [u, p, v]]$ and $[[u, p, v], f]$ will be mapped to the subcomplexes composed of the 3-cells

$$\{[e_i, [u, \rho(p), v]] : 1 \leq i \leq m\} \text{ and } \{[[u, \rho(p), v], e_i] : 1 \leq i \leq m\}$$

respectively. Similarly, those 3-cells whose boundaries consist only of 2-cells in $\mathcal{D}(\mathcal{P}^\infty)$ are mapped to subcomplexes composed of 3-cells whose boundaries consist only of 2-cells in $\mathcal{D}(\mathcal{P})$. We do not need to describe the map ρ completely as we are mainly concerned with the induced chain map

$$\hat{\rho} : \mathbf{C}(\overline{\mathcal{D}(\mathcal{P}^\infty)^{\mathbf{p}^\infty}}) \longrightarrow \mathbf{C}(\overline{\mathcal{D}(\mathcal{P})^{\mathbf{q}}})$$

which is as follows (where for brevity the chain groups and boundary maps of

$\mathsf{C}(\overline{\mathcal{D}(\mathcal{P}^\infty)^{\mathbf{p}^\infty}})$ are distinguished by the superscript $-^\infty$):

$$\begin{tikzcd}
\mathsf{C}(\overline{\mathcal{D}(\mathcal{P}^\infty)^{\mathbf{p}^\infty}}) : 0 \arrow[r] \arrow[d, "\hat\rho"] & (C_3^\infty)^{\mathbf{p}^\infty} \arrow[r, "\partial_3^\infty"] \arrow[d, "\hat\rho_3"] & C_2^\infty \oplus (C_2^\infty)^{\mathbf{p}^\infty} \arrow[r, "\tilde\partial_2^\infty"] \arrow[d, "\hat\rho_2"] & C_1^\infty \arrow[r, "\partial_1^\infty"] \arrow[d, "\hat\rho_1"] & C_0 \arrow[r] \arrow[d, equal] & 0 \\
\mathsf{C}(\mathcal{D}(\mathcal{P}), \mathbf{q}) : 0 \arrow[r] & C_3^{\mathbf{q}} \arrow[r, "\partial_3"] & C_2 \oplus C_2^{\mathbf{q}} \arrow[r, "\tilde\partial_2"] & C_1 \arrow[r, "\partial_1"] & C_0 \arrow[r] & 0
\end{tikzcd}$$

For a positive edge e in $\mathcal{D}(\mathcal{P}^\infty)$ with $\rho e = e_1^{\varepsilon_1} \ldots e_m^{\varepsilon_m}$ ($\varepsilon_i = \pm 1$ and e_i a positive edge in $\mathcal{D}(\mathcal{P})$ for $i = 1, \ldots, m$)

$$\hat\rho_1 e = \sum_{i=1}^{m} \varepsilon_i e_i,$$

and if f is another positive edge in $\mathcal{D}(\mathcal{P}^\infty)$ with $\rho f = f_1^{\delta_1} \ldots f_n^{\delta_m}$ ($\delta_j = \pm 1$ and f_j a positive edge in $\mathcal{D}(\mathcal{P})$ for $j = 1, \ldots n$) then

$$\hat\rho_2[e, f] = \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_i \delta_j [e_i, f_j].$$

To complete the description of $\hat\rho_2$, a free abelian group generator $[u, p, v]$ of $(C_2^\infty)^{\mathbf{p}^\infty}$ ($u, v \in F$, $p \in \mathbf{p}^\infty$) is mapped to $[u, \rho(p), v]$. Next, in $\overline{\mathcal{D}(\mathcal{P}^\infty)^{\mathbf{p}^\infty}}$ let e be a positive edge and σ a 2-cell with

$$\hat\rho_1 e = \sum_{i=1}^{m} \varepsilon_i e_i \qquad \text{and} \qquad \hat\rho_2 \sigma = \sum_{j=1}^{n} \delta_j \sigma_j$$

where (for $i = 1, \ldots, m$ and $j = 1, \ldots, n$) $\varepsilon_i, \delta_j = \pm 1$, and e_i is a positive edge and σ_j a 2-cell in $\overline{\mathcal{D}(\mathcal{P})^{\mathbf{q}}}$. Then $\hat\rho_3$ is defined by putting

$$\rho_3[e, \sigma] = \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_i \delta_j [e_i, \sigma_j] \qquad \text{and} \qquad \rho_3[\sigma, e] = \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_i \delta_j [\sigma_j, e_i].$$

Since the retraction ρ respects the two-sided action of F on the chain groups, there is an induced $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule epimorphism $\Pi(\mathcal{P}^\infty) \longrightarrow \Pi(\mathcal{P})$. Therefore $\mathbf{q}$ is a homology trivializer of $\mathcal{D}(\mathcal{P})$, and we will use the exactness of (3.9) to show that

$$0 \longrightarrow B_2(\overline{\mathcal{D}(\mathcal{P})^{\mathbf{q}}}) \xrightarrow{incl.} K^{\mathbf{q}} \xrightarrow{\tilde\partial_2} B_1(\mathcal{D}(\mathcal{P})) \longrightarrow 0 \qquad (3.10)$$

is also exact.

Suppose that ξ is a 2-cycle in $K^{\mathbf{q}}$, written as

$$\xi = \xi' + \sum_{i=1}^{m} \varepsilon_i(\iota e_i - \tau e_i).[1, q_i, w_i] + \sum_{i=m+1}^{m+n} \varepsilon_i[w_i, q_i, 1].(\iota e_i - \tau e_i)$$

where $\xi' \in C_2(\mathcal{D}(\mathcal{P}))$ and (for $i = 1, \ldots, m+n$) $\varepsilon_i = \pm 1$, e_i is a positive edge in $\mathcal{D}(\mathcal{P})$, $w_i \in F$ and $q_i \in \mathbf{q}$.

Now for each $q \in \mathbf{q}$ choose some $\tilde{q} \in \mathbf{p}^\infty$ such that $\rho\tilde{q} = q$. For each edge f of $\tilde{q}$, $\rho(f)f^{-1}$ is a closed path in $\mathcal{D}(\mathcal{P}^\infty)$ and so gives rise to the 1-cycle $\hat{\rho}_1(f) - f$. Hence, since $\mathbf{p}^\infty$ is a homology trivializer, there is a 2-chain $c_2(q)$ in $C_2(\overline{\mathcal{D}(\mathcal{P}^\infty)^{\mathbf{p}^\infty}})$ with boundary

$$\tilde{\partial}_2^\infty c_2(q) = \sum_{f \in \tilde{q}} (\hat{\rho}_1 f - f).$$

Note that $\hat{\rho}_2 c_2(q)$ is a 2-cycle, because

$$\tilde{\partial}_2 \hat{\rho}_2 c_2(q) = \hat{\rho}_1 \tilde{\partial}_2^\infty c_2(q) = \sum_{f \in \tilde{q}} (\hat{\rho}_1 f - \hat{\rho}_1 f) = 0.$$

Consider the 2-chain

$$\zeta = \xi' + \sum_{i=1}^{m} \varepsilon_i(\iota e_i - \tau e_i).([1, \tilde{q}_i, 1] + c_2(q_i)).w_i$$

$$+ \sum_{i=m+1}^{m+n} \varepsilon_i w_i.([1, \tilde{q}_i, 1] + c_2(q_i)).(\iota e_i - \tau e_i)$$

in K^∞. Since for any $q \in \mathbf{q}$

$$\tilde{\partial}_2^\infty([1, \tilde{q}, 1] + c_2(q)) = \sum_{f \in \tilde{q}} f + \sum_{f \in \tilde{q}} (\hat{\rho}_1 f - f) = \sum_{f \in \tilde{q}} (\hat{\rho}_1 f) = \tilde{\partial}_2[1, q, 1]$$

we have

$$\tilde{\partial}_2^\infty \zeta = \tilde{\partial}_2 \xi = 0$$

so that $\zeta \in K^{\infty}$ is a 2-cycle.

Moreover,

$$\begin{aligned}
\hat{\rho}_2\zeta &= \xi + \sum_{i=1}^{m} \varepsilon_i(\iota e_i - \tau e_i).\hat{\rho}_2 c_2(q_i).w_i + \sum_{i=m+1}^{m+n} \varepsilon_i w_i.\hat{\rho}_2 c_2(q_i).(\iota e_i - \tau e_i) \\
&= \xi \qquad \text{modulo } (B_2(\overline{\mathcal{D}(\mathcal{P})^{\mathbf{q}}})),
\end{aligned}$$

because, as in the proof of Lemma 3.2.1, if z is any 2-cycle and e a positive edge, then both $(\iota e - \tau e).z$ and $z.(\iota e - \tau e)$ are null-homologous.

As the sequence (3.9) is exact we know that ζ is the boundary of some 3-chain ω, say, in $C_3^{\infty} = C_3(\overline{\mathcal{D}(\mathcal{P}^{\infty})^{\mathbf{p}^{\infty}}})$. But then the 3-chain $\hat{\rho}_3\omega$ in $C_3 = C_3(\overline{\mathcal{D}^{\mathbf{q}}})$ has boundary

$$\partial_3\hat{\rho}_3\omega = \hat{\rho}_2\partial_3^{\infty}\omega = \hat{\rho}_2\zeta$$

which is homologous to ξ, and therefore ξ is a 2-boundary and the sequence (3.10) is exact also. The proposition now follows because of Lemma 3.3.3. ⌟

Proof of Theorem 3.3.1: If $H_1(\mathcal{D}^{\mathbf{p}}) = 0$ then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_2(\overline{\mathcal{D}^{\mathbf{p}}}) & \xrightarrow{incl.} & K^{\mathbf{p}} & \xrightarrow{\tilde{\partial}_2} & B_1(\mathcal{D}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_2(\overline{\mathcal{D}^{\mathbf{p}}}) & \xrightarrow{incl.} & C_2 \oplus C_2^{\mathbf{p}} & \xrightarrow{\tilde{\partial}_2} & Z_1(\mathcal{D}) & \longrightarrow & 0
\end{array}$$

where all vertical maps are inclusions and both rows are exact. From the Snake Lemma (Lemma 1.3.2) we immediately have the short exact sequence (3.5), with

$$\begin{aligned}
\Phi: \ \Pi_2 = \operatorname{coker}(B_2 \to Z_2) \quad &\longrightarrow \quad \operatorname{coker}(K^{\mathbf{p}} \to C_2 \oplus C_2^{\mathbf{p}}) = \mathbb{Z}S.\mathbf{p}.\mathbb{Z}S \\
z_2 + B_2 \quad &\longmapsto \quad \varphi(z_2) \qquad (z_2 \in Z_2(\overline{\mathcal{D}^{\mathbf{p}}})).
\end{aligned}$$

where φ is the homomorphism (3.6), and

$$\nu : \mathbb{Z}S.\mathbf{p}.\mathbb{Z}S = \operatorname{coker}(K^{\mathbf{p}} \to C_2 \oplus C_2^{\mathbf{p}}) \longrightarrow \operatorname{coker}(B_1 \to Z_1) = \Pi$$

is the natural homomorphism taking each $p \in \mathbf{p}$ to the homology class of the corresponding 1-cycle. ⌟

3.4 Killing the left or right action

Let Π_2^l (respectively Π_2^r) denote the left $\mathbb{Z}S$-module $\Pi_2 \otimes_{\mathbb{Z}S} \mathbb{Z}$ (respectively, the right $\mathbb{Z}S$-module $\mathbb{Z} \otimes_{\mathbb{Z}S} \Pi_2$).

Theorem 3.4.1 *If the homology classes of the paths in* $\mathbf{p}$ *give rise to a set of bimodule generators of* Π *then there is a short exact sequence*

$$0 \longrightarrow \Pi_2^l \longrightarrow \mathbb{Z}S.\mathbf{p} \longrightarrow \Pi^l \longrightarrow 0 \tag{3.11}$$

of left $\mathbb{Z}S$*-modules.*

Remark 3.4.2 *A similar sequence of right* $\mathbb{Z}S$*-modules involving* Π_2^r *can also be obtained.*

We first prove several lemmas, and we need the following definition.

Definition 3.4.3 (Kobayashi and Otto [40]) *The* leftmost path $p_l(w)$ *from* w *to the minimal element* $\widehat{w}$ *in the congruence class of* w *with respect to some chosen length-lexicographical ordering of* F *is defined as follows. First for each pair* $(\widehat{w}, x)$ $(w \in F, x \in \mathbf{x})$ *we choose an arbitrary path* $p_l(\widehat{w}x)$ *from* $\widehat{w}x$ *to* $\widehat{\widehat{w}x}$*. The leftmost paths are then defined recursively as follows:*

(i) *if* $w = \widehat{w}$ *then* $p_l(w)$ *is the empty path at* w*; or*

(ii) *if* $w \neq \widehat{w}$ *and* $w = ux$ *for some* $u \in F$ *and* $x \in \mathbf{x}$ *then* $p_l(w)$ *is defined to be* $(p_l(u).x)p_l(\widehat{u}x)$.

Lemma 3.4.4 (Kobayashi and Otto [40]) *The leftmost paths have the property that for all* $w_1, w_2 \in F$,

$$p_l(w_1 w_2) = (p_l(w_1).w_2)p_l(\widehat{w}_1 w_2). \tag{3.12}$$

Proof: This identity is proved using induction on the length n of w_2. Firstly, if $n = 0$ then (3.12) reduces to $p_l(w_1) = p_l(w_1)$. Suppose on the other hand that $n \geq 1$ and that by inductive hypothesis (3.12) holds whenever the length of the second word is less than n. We can write $w_2 = w_2'x$ where $x \in \mathbf{x}$ and $w_2' \in F$. Then

$$\begin{aligned} p_l(w_1w_2) &= (p_l(w_1w_2').x)p_l(\widehat{w_1w_2'}x) \\ &= ((p_l(w_1).w_2')(p_l(\widehat{w_1}w_2')).x)p_l(\widehat{w_1w_2'}x) \\ &= (p_l(w_1).w_2)p_l(\widehat{w_1}w_2). \end{aligned}$$

The lemma follows by induction. ⌟

Example 3.4.5 Consider the following complete rewriting system for the free abelian group of rank 2.

$$\begin{array}{rlll} \textbf{alphabet:} & a, a^{-1}, b, b^{-1} & & \\ \textbf{rules:} & aa^{-1} = 1, & b^{-1}b = 1, & ba^{-1} = a^{-1}b, \\ & a^{-1}a = 1, & ba = ab, & b^{-1}a = ab^{-1}, \\ & bb^{-1} = 1, & b^2a^2 = a^2b^2, & b^{-1}a^{-1} = a^{-1}b^{-1}. \end{array}$$

This rewriting system is compatible with the length-lexicographic ordering induced by the order $b^{-1} \triangleright b \triangleright a^{-1} \triangleright a$, and the normal forms consist of all words of the form $w = a^m b^n$ where m and n are integers. For each such word there are unique paths $p_l(wa^\varepsilon)$ and $p_l(wb^\varepsilon)$ ($\varepsilon = \pm 1$) to the corresponding normal form - these paths do not use the rule $b^2a^2 = a^2b^2$.

The leftmost path $p_l(b^2a^2)$ from the vertex b^2a^2 to the normal form a^2b^2 is the path

$$b\underline{ba}a \to \underline{ba}ba \to ab\underline{ba} \to a\underline{ba}b \to a^2b^2$$

(Since $p_l(b^2a)$ is the path

$$b\underline{ba} \to \underline{ba}b \to ab^2 \qquad).$$

This example shows that the leftmost path is not generally the same as the *path composed of left principal edges* (see § 1.2.1), which in this case is the single edge

$$\underline{b^2a^2} \to a^2b^2.$$

⌟

Lemma 3.4.6 *The left $\mathbb{Z}F$-homomorphism*

$$\hookrightarrow \otimes 1 : Z_2(\overline{\mathcal{D}^{\mathbf{p}}})\otimes_{\mathbb{Z}F}\mathbb{Z} \longrightarrow (C_2 \oplus C_2^{\mathbf{p}})\otimes_{\mathbb{Z}F}\mathbb{Z}$$

is injective, where $\mathbf{p}$ *is a homology trivializer of* $\mathcal{D}$ *and* $\hookrightarrow$ *denotes the inclusion*

$$Z_2(\overline{\mathcal{D}^{\mathbf{p}}}) \longrightarrow (C_2 \oplus C_2^{\mathbf{p}}).$$

Proof: For each edge e there is a 1-cycle $z_1(e)$ corresponding to the closed path $p_l(\iota e)^{-1} e p_l(\tau e)$, and for any $w \in F$ we have

$$z_1(e.w) = z_1(e).w\,.$$

(By Lemma 3.4.4, $p_l(\iota e.w) = (p_l(\iota e).w)p_l(\widehat{\iota e}w)$ and $p_l(\tau e.w) = (p_l(\tau e).w)p_l(\widehat{\tau e}w)$, so that in writing down the 1-cycle $z_1(e.w)$ we can cancel the edges corresponding to the paths $p_l(\widehat{\iota e}w) = p_l(\widehat{\tau e}w)$.)

Next, since $\mathbf{p}$ is a homology trivializer, for each right-reduced positive edge $f = (u, r, +1, 1)$ ($u \in F, r \in \mathbf{r}$) we can choose a 2-chain $c_2(f)$ with boundary $\tilde{\partial}_2 c_2(f) = z_1(f)$, and, for $\varepsilon = \pm 1, v \in F$, we define $c_2(f^{\varepsilon}.v)$ to be $\varepsilon c_2(f).v$. Then to each 2-cell σ of $\overline{\mathcal{D}^{\mathbf{p}}}$ with boundary

$$\tilde{\partial}_2 \sigma = \sum_{i=1}^{n} \delta_i e_i \qquad (e_i \in \mathbf{e}^{+} \text{ and } \delta_i = \pm 1 \text{ for } i = 1, \ldots, n)$$

we can associate a 2-cycle

$$z_2(\sigma) = \sigma - \sum_{i=1}^{n} \delta_i c_2(e_i)$$

with the property that $z_2(\sigma.w) = z_2(\sigma).w$ for all $w \in F$. The homomorphism of abelian groups $\kappa : C_2 \oplus C_2^{\mathbf{p}} \longrightarrow Z_2(\overline{\mathcal{D}^{\mathbf{p}}})$ defined by mapping σ to $z_2(\sigma)$ is therefore a homomorphism of right $\mathbb{Z}F$-modules.

Suppose that

$$\xi = \sum_{i=1}^{n} \varepsilon_i \sigma_i \in Z_2(\overline{\mathcal{D}^{\mathbf{p}}})$$

where (for $i = 1, \ldots, n$) $\varepsilon_i = \pm 1$ and σ_i is a 2-cell with

$$\tilde{\partial}_2 \sigma_i = \sum_{j=1}^{k_i} \delta_{i_j} e_{i_j} \qquad (e_{i_j} \in \mathrm{e}^+ \text{ and } \delta_{i_j} = \pm 1 \text{ for } j = 1, \ldots, k_i).$$

Then since

$$\sum_{i=1}^{n} \sum_{j=1}^{k_i} \varepsilon_i \delta_{i_j} e_{i_j} = \tilde{\partial}_2 \xi = 0$$

we have

$$(\kappa)(\hookrightarrow)\xi = \sum_{i=1}^{n} \varepsilon_i \sigma_i - \sum_{i=1}^{n} \sum_{j=1}^{k_i} \varepsilon_i \delta_{i_j} c_2(e_{i_j}) = \sum_{i=1}^{n} \varepsilon_i \sigma_i = \xi$$

and so κ is a retraction, with $\kappa\hookrightarrow$ the identity map of $Z_2(\overline{\mathcal{D}^{\mathrm{P}}})$. Because κ is a right $\mathbb{Z}F$-homomorphism, we have a group homomorphism

$$\kappa \otimes 1 : (C_2 \oplus C_2^{\mathrm{P}}) \otimes_{\mathbb{Z}F} \mathbb{Z} \longrightarrow Z_2(\overline{\mathcal{D}^{\mathrm{P}}}) \otimes_{\mathbb{Z}F} \mathbb{Z}.$$

But then $(\kappa \otimes 1)((\hookrightarrow) \otimes 1) = (\kappa(\hookrightarrow) \otimes 1)$ is the identity map of $Z_2(\overline{\mathcal{D}^{\mathrm{P}}}) \otimes_{\mathbb{Z}F} \mathbb{Z}$ and therefore $(\hookrightarrow) \otimes 1$ is injective. ⌟

Therefore, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & B_2(\overline{\mathcal{D}^{\mathrm{P}}}) \otimes_{\mathbb{Z}F} \mathbb{Z} & \longrightarrow & K^{\mathrm{P}} \otimes_{\mathbb{Z}F} \mathbb{Z} & \longrightarrow & B_1(\mathcal{D}) \otimes_{\mathbb{Z}F} \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_2(\overline{\mathcal{D}^{\mathrm{P}}}) \otimes_{\mathbb{Z}F} \mathbb{Z} & \longrightarrow & (C_2 \oplus C_2^{\mathrm{P}}) \otimes_{\mathbb{Z}F} \mathbb{Z} & \longrightarrow & Z_1(\mathcal{D}) \otimes_{\mathbb{Z}F} \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Let L denote the kernel of the left $\mathbb{Z}F$-homomorphism

$$\hookrightarrow \otimes 1 : B_1(\mathcal{D}) \otimes_{\mathbb{Z}F} \mathbb{Z} \longrightarrow Z_1(\mathcal{D}) \otimes_{\mathbb{Z}F} \mathbb{Z}$$

where $\hookrightarrow$ denotes the inclusion $B_1(\mathcal{D}) \longrightarrow Z_1(\mathcal{D})$. By the Snake Lemma there is a homomorphism

$$d : L \longrightarrow \Pi_2^l \ (\cong \operatorname{coker}(B_2(\overline{\mathcal{D}^{\mathrm{P}}}) \otimes_{\mathbb{Z}F} \mathbb{Z} \longrightarrow Z_2(\overline{\mathcal{D}^{\mathrm{P}}}) \otimes_{\mathbb{Z}F} \mathbb{Z}))$$

where for $b_1 \in B_1(\mathcal{D})$

$$d(b_1 \otimes 1) = (z_2 + B_2(\overline{\mathcal{D}^{\mathbf{P}}})) \otimes 1$$

where $z_2 \in Z_2(\overline{\mathcal{D}^{\mathbf{P}}})$ is such that $z_2 \otimes 1$ has the same image in $(C_2 \oplus C_2^{\mathbf{P}}) \otimes_{\mathbb{Z}F} \mathbb{Z}$ as some lift of $b_1 \otimes 1$ in $K^{\mathbf{P}} \otimes_{\mathbb{Z}F} \mathbb{Z}$, and there is an exact sequence

$$L \xrightarrow{d} \Pi_2^l(\overline{\mathcal{D}^{\mathbf{P}}}) \longrightarrow \mathbb{Z}S.\mathbf{p} \longrightarrow \Pi^l(\mathcal{D}) \longrightarrow 0. \tag{3.13}$$

Lemma 3.4.7 *L is generated as a left $\mathbb{Z}F$-module by elements of the form*

$$\xi.(\iota f - \tau f) \otimes 1$$

where ξ is a 1-cycle and f is a positive edge.

Proof: First note that $\xi.(\iota f - \tau f)$ is a 1-boundary, for if

$$\xi = \sum_{i=1}^{n} \varepsilon_i e_i \qquad (e_i \in \mathbf{e}^+ \text{ and } \varepsilon_i = \pm 1 \text{ for } i = 1, \ldots n)$$

then

$$\begin{aligned} \partial_2 \sum_{i=1}^{n} \varepsilon_i [e_i, f] &= \sum_{i=1}^{n} \varepsilon_i (e_i.(\iota f - \tau f) - (\iota e_i - \tau e_i).f) \\ &= \xi.(\iota f - \tau f) - \partial_1 \xi.f \\ &= \xi.(\iota f - \tau f). \end{aligned}$$

Also $\xi.(\iota f - \tau f) \otimes 1 \in L$, since in $Z_1(\mathcal{D}) \otimes_{\mathbb{Z}F} \mathbb{Z}$ we can write

$$\begin{aligned} \xi.(\iota f - \tau f) \otimes 1 &= \xi.\iota f \otimes 1 - \xi.\tau f \otimes 1 \\ &= \xi \otimes 1 - \xi \otimes 1 \\ &= 0. \end{aligned}$$

Now suppose that some 1-boundary

$$\zeta = \partial_2 \sum_{i=1}^{n} \varepsilon_i [e_i, f_i] \in B_1(\mathcal{D}) \qquad (e_i, f_i \in \mathbf{e}^+ \text{ and } \varepsilon_i = \pm 1 \text{ for } i = 1, \ldots, n)$$

is such that $0 = \zeta\otimes 1$ holds in $Z_1(\mathcal{D})\otimes_{\mathbb{Z}F}\mathbb{Z}$. In $C_1(\mathcal{D})\otimes_{\mathbb{Z}F}\mathbb{Z}$ we may write

$$\begin{aligned}
0 &= \zeta\otimes 1 \\
&= \sum_{i=1}^{n}\varepsilon_i(e_i.(\iota f_i - \tau f_i) - (\iota e_i - \tau e_i).f_i)\otimes 1 \\
&= \sum_{i=1}^{n}\varepsilon_i(\iota e_i - \tau e_i).f_i'\otimes 1
\end{aligned}$$

where if (for $i = 1,\ldots,n$) f_i is the edge $(u_i, r_i, +1, v_i)$ $(u_i, v_i\in F, r_i\in\mathbf{r})$ then we use f_i' to denote the right-reduced edge $(u_i, r_i, +1, 1)$. Now as a $(\mathbb{Z}F,\mathbb{Z}F)$-bimodule, $C_1(\mathcal{D})$ is freely generated by edges of the form $(1, r, +1, 1)$ $(r\in\mathbf{r})$, and so $C_1(\mathcal{D})\otimes_{\mathbb{Z}F}\mathbb{Z}$ is naturally isomorphic to the free left $\mathbb{Z}F$-module $\mathbb{Z}F.\mathbf{r}$, the isomorphism mapping $\zeta\otimes 1$ to

$$\sum_{i=1}^{n}\varepsilon_i(\iota e_i - \tau e_i)u_i.r_i = 0.$$

It follows that we can partition the indexing set $\{1,\ldots,n\}$ as a disjoint union

$$I_1\cup I_2\cup\ldots\cup I_m,$$

where $i, j\in I_k$ $(k\in\{1,\ldots,m\})$ if and only if $r_i = r_j$, so that for each k we must have

$$\sum_{i\in I_k}\varepsilon_i(\iota e_i - \tau e_i)u_i = 0 \tag{3.14}$$

and

$$\begin{aligned}
0 &= \partial_1\partial_2\sum_{(i\in I_k)}\varepsilon_i[e_i, f_i'] \\
&= \partial_1\sum_{(i\in I_k)}\varepsilon_i(e_i.u_i((r_i)_{+1} - (r_i)_{-1}) - (\iota e_i - \tau e_i).f_i') \\
&= \partial_1\sum_{(i\in I_k)}\varepsilon_i e_i.u_i((r_i)_{+1} - (r_i)_{-1}) \qquad \text{(by (3.14))} \\
&= (\partial_1\sum_{(i\in I_k)}\varepsilon_i e_i.u_i).((s_k)_{+1} - (s_k)_{-1})
\end{aligned}$$

where $s_k = ((s_k)_{+1}, (s_k)_{-1}) = r_i$ for all $i\in I_k$. Since $\mathbb{Z}F$ has no zero-divisors it follows that $\sum_{I_k}\varepsilon_i e_i.u_i$ is a 1-cycle, which we denote by ξ_k.

But consider the 1-boundary

$$\hat{\zeta} = \partial_2 \sum_{i=1}^{n} \varepsilon_i [e_i, f_i'].$$

In $B_1(\mathcal{D}) \otimes_{\mathbb{Z}F} \mathbb{Z}$ we have

$$\hat{\zeta} \otimes 1 = \sum_{i=1}^{n} \partial_2 \varepsilon_i [e_i, f_i'] \otimes 1 = \sum_{i=1}^{n} \partial_2 \varepsilon_i [e_i, f_i'].v_i \otimes 1 = \zeta \otimes 1$$

and since

$$\hat{\zeta} = \sum_{k=1}^{m} \xi_k.((s_k)_{+1} - (s_k)_{-1})$$

we can write

$$\zeta \otimes 1 = \hat{\zeta} \otimes 1 = \sum_{k=1}^{m} \xi_k.((s_k)_{+1} - (s_k)_{-1}) \otimes 1.$$

┘

Proof of Theorem 3.4.1: It suffices to show that the map d in the exact sequence (3.13) applied to a generator $\xi(\iota f - \tau f) \otimes 1$ (ξ a 1-cycle and $f \in \mathbf{e}^+$) of L is zero. Since $\mathbf{p}$ is a homology trivializer of $\mathcal{D}$ there is some 2-chain $\zeta \in C_2 \oplus C_2^{\mathbf{p}}$ such that $\tilde{\partial}_2 \zeta = \xi$. Then $\zeta.(\iota f - \tau f) \in K^{\mathbf{p}}$ is such that $\zeta.(\iota f - \tau f) \otimes 1$ satisfies

$$(\tilde{\partial}_2 \otimes 1)(\zeta.(\iota f - \tau f) \otimes 1) = \xi.(\iota f - \tau f) \otimes 1.$$

But in $(C_2 \oplus C_2^{\mathbf{p}}) \otimes_{\mathbb{Z}F} \mathbb{Z}$

$$\begin{aligned} \zeta.(\iota f - \tau f) \otimes 1 &= \zeta.\iota f \otimes 1 - \zeta.\tau f \otimes 1 \\ &= \zeta \otimes 1 - \zeta \otimes 1 \\ &= 0. \end{aligned}$$

It follows easily from the definition of d that $d(\xi.(\iota f - \tau f) \otimes 1) = 0$ ┘

Chapter 4

Some new finiteness conditions

Abstract

In this chapter we introduce new finiteness conditions FDT_2 and FHT_2, and prove that they are invariants of finite presentations, finite trivializers and of retractions. In §4.4 we show that monoids with finite complete rewriting systems are FDT_2 and FHT_2.

4.1 The properties FDT_2 and FHT_2

Definition 4.1.1 We shall say that the finite rewriting system $\mathcal{P}$ is of *second order finite derivation type* (FDT_2) if:

1. it is of type FDT;
2. for some finite homotopy trivializer $\mathbf{p}$ of $\mathcal{D}$ the 3-complex $\overline{\mathcal{D}^{\mathbf{p}}}$ has a finite set $\mathbf{X}$ of spherical subcomplexes such that attaching 3-cells to the set $F.\mathbf{X}.F$ gives a 3-complex with trivial second homotopy groups.

Definition 4.1.2 We shall say that the finite rewriting system $\mathcal{P}$ is of *second order finite homological type* (FHT_2) if:

1. it is of type FHT;

2. for some finite homology trivializer $\mathbf{p}$ of $\mathcal{D}$ there is a finite set $\mathbf{Y}$ of 2-cycles whose homology classes generate the $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule $\Pi_2 = H_2(\overline{\mathcal{D}^{\mathbf{p}}})$.

It is immediate from the definitions that if a finite rewriting system has the property FDT_2 then it has the property FHT_2 also.

Remark 4.1.3 Suppose that for some rewriting system $\mathcal{P}$ we choose a subset Υ of F containing a unique representative of each congruence class. If $\mathbf{p}$ is some set of closed paths in $\mathcal{D}$ then there is the Hurewicz homomorphism [48]

$$\bigoplus_{w\in\Upsilon}\pi_2(\overline{\mathcal{D}^{\mathbf{p}}}, w)\longrightarrow H_2(\overline{\mathcal{D}^{\mathbf{p}}}) \tag{4.1}$$

which maps the homotopy class of a continuous map of S^2 into $\overline{\mathcal{D}^{\mathbf{p}}}$ based at $w\in\Upsilon$ to the homology class of the corresponding 2-cycle. Furthermore if $\mathbf{p}$ is a homotopy trivializer, so that $\overline{\mathcal{D}^{\mathbf{p}}}$ is simply-connected, then by the Hurewicz Isomorphism Theorem [48] the above homomorphism is in fact an isomorphism.

Thus if $\mathbf{p}$ is a homotopy trivializer of $\mathcal{D}$ and if $\mathbf{X}$ is a set of spherical subcomplexes of $\overline{\mathcal{D}^{\mathbf{p}}}$, then the 3-complex obtained by attaching 3-cells according to the set $F.\mathbf{X}.F$ has trivial second homotopy groups if and only if the homology classes of the 2-cycles arising from the set $\mathbf{X}$ generate the bimodule $\Pi_2 = H_2(\overline{\mathcal{D}^{\mathbf{p}}})$. In particular, we deduce that

> for rewriting systems *which are of type* FDT, the properties FDT_2 and FHT_2 are equivalent.

Consequently,

> *if* the properties FDT and FHT turn out to be equivalent[1], then the properties FDT_2 and FHT_2 are also equivalent.

[1]A recent example of Pride and Otto shows that this is not the case.

4.2 Invariance properties

4.2.1 Invariance of finite presentation and trivializer

We shall say that a finitely presented monoid S has the property FDT_2 (respectively, FHT_2) if some finite rewriting system $\mathcal{P} = [\mathbf{x}; \mathbf{r}]$ of S is of type FDT_2 (respectively, FHT_2). Taking account of Remark 4.1.3, this amounts to saying that for some finite homotopy (respectively, homology) trivializer $\mathbf{p}$ of $\mathcal{D}$, the $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule $\Pi_2(\mathcal{D}, \mathbf{p})$ is finitely generated.

Theorem 4.2.1 *The properties FDT_2 and FHT_2 are monoid invariants, that is, they are independent of the choice of finite rewriting system and finite trivializer.*

Proof: Suppose that $\mathcal{Q} = [\mathbf{y}; \mathbf{s}]$ is another finite rewriting system presenting S, with finite homotopy or homology trivializer $\mathbf{q}$. By a result of Ivanov [32, Proposition 1.7] the $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules

$$\mathcal{M}(\mathcal{P}) \oplus \mathbb{Z}S.\mathbf{y}.\mathbb{Z}S \qquad \text{and} \qquad \mathcal{M}(\mathcal{Q}) \oplus \mathbb{Z}S.\mathbf{x}.\mathbb{Z}S$$

are isomorphic, where $\mathcal{M}$ denotes the relation bimodule (see §1.2.2). By adding free summands to the sequence (1) for $\mathcal{P}$ and then splicing with (4) we obtain an exact sequence:

$$0 \longrightarrow \Pi_2(\mathcal{P}, \mathbf{p}) \longrightarrow \mathbb{Z}S.\mathbf{p}.\mathbb{Z}S \longrightarrow \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S \oplus \mathbb{Z}S.\mathbf{y}.\mathbb{Z}S \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathcal{M}(\mathcal{P}) \oplus \mathbb{Z}S.\mathbf{y}.\mathbb{Z}S \longrightarrow 0$$

A similar sequence can be constructed for the pair $(\mathcal{Q}, \mathbf{q})$. Then applying the generalized Schanuel Lemma (Lemma 1.3.1) to these two sequences, we deduce that

$$\Pi_2(\mathcal{P}, \mathbf{p}) \oplus \mathbb{Z}S.\mathbf{q}.\mathbb{Z}S \oplus \mathbb{Z}S.\mathbf{r}.\mathbb{Z}S \oplus \mathbb{Z}S.\mathbf{y}.\mathbb{Z}S$$

and

$$\Pi_2(\mathcal{Q}, \mathbf{q}) \oplus \mathbb{Z}S.\mathbf{p}.\mathbb{Z}S \oplus \mathbb{Z}S.\mathbf{s}.\mathbb{Z}S \oplus \mathbb{Z}S.\mathbf{x}.\mathbb{Z}S$$

are isomorphic as $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules.Therefore $\Pi_2(\mathcal{P}, \mathbf{p})$ is finitely generated as a $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule if and only if $\Pi_2(\mathcal{Q}, \mathbf{q})$ is. $\lrcorner$

4.2.2 Invariance under retraction

Theorem 4.2.2 *A retract of an FDT_2 (respectively, FHT_2) monoid is also of this type.*

Proof. Suppose that $\pi : S \longrightarrow R$ is a retraction of monoids. If S is finitely presented then we can choose finite rewriting systems $\mathcal{P} = [\mathbf{x};\mathbf{r}]$ and $\mathcal{P}_0 = [\mathbf{x}_0;\mathbf{r}_0]$ for S and R respectively, and such that $\mathbf{x}_0 \subseteq \mathbf{x}$, $\mathbf{r}_0 \subseteq \mathbf{r}$ and there is a homomorphism $\rho : F(\mathbf{x}) \longrightarrow F(\mathbf{x}_0)$ such that $\rho(x) = x$ for all $x \in x_0$ and also $(\rho(r_{+1}), \rho(r_{-1})) \in \mathbf{r}_0$ for each $r \in \mathbf{r}$ (see [66, Theorem 3.3]). We can therefore extend ρ to a retraction

$$\rho : \mathcal{D} = \mathcal{D}(\mathcal{P}) \longrightarrow \mathcal{D}_0 = \mathcal{D}(\mathcal{P}_0)$$

of 2-complexes.

Suppose that $\mathbf{p}$ is a (finite) set of closed paths in $\mathcal{D}$, and let $\mathbf{p}_0 = \rho(\mathbf{p})$. By enlarging $\mathbf{p}$ if necessary, we can assume that $\mathbf{p}_0 \subseteq \mathbf{p}$. By [66, Lemma 3.3], if $\mathbf{p}$ is a homotopy (respectively, homology) trivializer for $\mathcal{D}$, then $\mathbf{p}_0$ is a homotopy (respectively, homology) trivializer for $\mathcal{D}_0$.

We can then extend ρ again to give a retraction

$$\rho : (\overline{\mathcal{D}^{\mathbf{p}}}) \longrightarrow (\overline{\mathcal{D}_0^{\mathbf{p}_0}})$$

which induces a chain map from the chain complex of $\overline{\mathcal{D}^{\mathbf{p}}}$ to the chain complex of $\overline{\mathcal{D}_0^{\mathbf{p}_0}}$, and then induced homomorphisms on homology. In particular, we have a surjective (group) homomorphism

$$\rho_* : \Pi_2(\mathcal{P}, \mathbf{p}) \longrightarrow \Pi_2(\mathcal{P}_0, \mathbf{p}_0)$$

which respects the bimodule structures: for any $s_1, s_2 \in S$ and $[\xi] \in \Pi_2(\mathcal{P}, \mathbf{p})$ we have

$$\rho_*(s_1.[\xi].s_2) = \pi(s_1).\rho_*[\xi].\pi(s_2).$$

It follows that the image of a (finite) set of generators of the $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule $\Pi_2(\mathcal{P}, \mathbf{p})$ will generate $\Pi_2(\mathcal{P}_0, \mathbf{p}_0)$ as a $(\mathbb{Z}R, \mathbb{Z}R)$-bimodule. ⌟

4.3 The properties FDT_2 and FHT_2 for groups

For an FHT monoid S we may splice the exact sequences (5) and (1.8) of left $\mathbb{Z}S$-modules arising from some choice of finite rewriting system and homology trivializer to obtain the partial resolution

$$0 \longrightarrow \Pi_2^l \longrightarrow \mathbb{Z}S.\mathbf{p} \longrightarrow \mathbb{Z}S.\mathbf{r} \longrightarrow \mathbb{Z}S.\mathbf{x} \longrightarrow \mathbb{Z}S \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{4.2}$$

of the trivial left $\mathbb{Z}S$-module $\mathbb{Z}$. As in §1.3.2 we can use the generalized Schanuel's Lemma (Lemma 1.3.1) to deduce that

> *S is of type left FP_4 if and only if Π_2^l is finitely generated as a left $\mathbb{Z}S$-module.*

Theorem 4.3.1 *For finitely presented groups, the properties FDT_2, FHT_2 and FP_4 are equivalent.*

Proof: Since for finitely presented *groups* the properties FDT and FHT are equivalent §1.3.2), by Remark 4.1.3 the properties FDT_2 and FHT_2 are also equivalent.

We now show the equivalence of FHT_2 and FP_4. Let $\mathcal{P}$ be a finite rewriting system defining a group S, and let $\mathcal{D}$ have a finite homotopy trivializer $\mathbf{p}$. As remarked above, S is FP_4 if and only if Π_2^l is finitely generated as a left module. The bimodule Π_2 has a decomposition analogous to (1.5), namely as a direct sum of the second homology groups of each connected component of $\overline{\mathcal{D}^{\mathbf{p}}}$, so as in §1.3.4, we can apply Lemma 1.3.5 to see that Π_2 is finitely generated as a bimodule (that is, S is FHT_2) if and only if Π_2^l is finitely generated as a left module, and the result follows. ⌟

4.4 Monoids with finite complete rewriting systems are FDT_2

This section contains two proofs of the following result:

Theorem 4.4.1 *A monoid with a finite complete rewriting system is of type FDT_2.*

The first uses homology, which was seen in Remark 4.1.3 to be an adequate description of the second homotopy of $\overline{\mathcal{D}^{\mathbf{P}}}$ when $\mathbf{p}$ is a homotopy trivializer. The second proof uses *pictures* to study the homotopy directly.

4.4.1 2-cycles at critical triples

Throughout this section, we assume that $\mathcal{P}$ is a finite complete rewriting system, and that $\mathbf{p}$ is some finite homotopy trivializer arising from a choice of resolution of each critical pair. Because of the choice of $\mathbf{p}$, the boundary of each 2-cell in $\overline{\mathcal{D}^{\mathbf{P}}}$ consists of a pair of positive paths with common initial and terminal vertices, and we shall again use the notation $[w;(e,f)]$ for 2-cells, where $w \in F$ and $e, f \in star^+(w)$, which was introduced in Remark 1.4.14. Recall that $\succ_{\mathbf{r}}$ denotes the reduction ordering induced by the rules of $\mathcal{P}$; thus for $v, w \in F$, we write

$$w \succ_{\mathbf{r}} v \quad \text{if} \quad w \rightarrow_{\mathbf{r}}^* v \quad \text{but} \quad v \neq w.$$

We shall extend this notation to 2-cells and 2-chains, writing $\sigma \succ_{\mathbf{r}} \sigma'$ for 2-cells σ and σ' if their respective maximal vertices w and v, say, satisfy $w \succ_{\mathbf{r}} v$, and $\xi \succ_{\mathbf{r}} \zeta$ for 2-chains ξ and ζ if for each 2-cell σ' represented in ζ there is some σ represented in ξ such that $\sigma \succ_{\mathbf{r}} \sigma'$.

From the construction of $\overline{\mathcal{D}^{\mathbf{P}}}$, for any $w \in F$ and triple of edges $e,f,g \in star^+(w)$ with at least one disjoint from the remaining pair, there is a 3-cell whose boundary gives rise to a 2-cycle $\xi_{(e,f,g)}$ of the form

$$\xi_{(e,f,g)} = [w;(e,f)] + [w;(f,g)] - [w;(e,g)] + \zeta_{(e,f,g)} \tag{4.3}$$

where $\zeta_{(e,f,g)}$ is a 2-chain satisfying $\xi_{(e,f,g)} \succ_{\mathbf{r}} \zeta_{(e,f,g)}$ (see Remark 3.3.5). Moreover, we can suppose that the number of terms in $\zeta_{(e,f,g)}$ is not greater than $k-1$, where $k \geq 4$ is the maximal length of the boundary of any 2-cell in $\overline{\mathcal{D}^{\mathbf{P}}}$.

The critical pairs of a rewriting system are described in Definition 1.4.6; similarly, we may consider the *critical triples*:

Definition 4.4.2 *A triple of positive edges with the same initial vertex form a* critical triple *if either:*

1. *One of the triple is both left- and right-reduced; or*

2. *One of the triple is left-reduced but not right-reduced, one of the remaining pair of edges is right-reduced but not left-reduced, and no single edge is disjoint from the other two.*

Example 4.4.3 The rewriting system given in Example 2.1.5 has a single critical triple (of type *2*):

$$\begin{tikzcd}
 & & & cdcaea^2 \\
 & & \underline{cdca}\,aa \arrow[ur] \arrow[dd, equal] & \\
c^2da^3 & \underline{cdc}\,aaa \arrow[l] \arrow[ur, equal] \arrow[dr, equal] & & \\
 & & cd\underline{caaa} \arrow[dr] & \\
 & & & cda^2
\end{tikzcd}$$

┘

We want to show that there is also a 2-cycle with boundary of the form (4.3) for each critical triple of edges (Definition 4.4.2), but we first need the following technical definition and lemma.

For any $w \in F$ we shall say that $P(w)$ holds if for any pair $p = e_1 \ldots e_m$ and $q = f_1 \ldots f_n$ of positive paths from w to $Irr(w)$ there is a 2-chain whose boundary is the 1-cycle $\sum_{i=1}^{m} e_i - \sum_{j=1}^{n} f_j$ arising from the closed path pq^{-1}, and which can be written as either

1. $[w; (e_1, f_1)] + \zeta$ (if we assume that $e_1 <_w f_1$), or

2. ζ (if $e_1 = f_1$),

where ζ is a 2-chain which, if non-zero, contains only 2-cells with maximum vertices beneath w in the reduction order $\succ_r$.

Lemma 4.4.4 *For all $w \in F$, $P(w)$ holds.*

Proof: The proof is by noetherian induction on F using the reduction order $\succ_{\mathbf{r}}$ induced by the rules $\mathbf{r}$.

Let $w \in F$, and assume inductively that $P(w')$ holds for each $w' \in F$ satisfying $w \succ_{\mathbf{r}} w'$. We want to deduce that P_w holds.

If $w = Irr(w)$ then $P(w)$ holds since we may just take the trivial 2-chain 0.

So suppose that $w \neq Irr(w)$, and that $p = e_1 \ldots e_m$ and $q = f_1 \ldots f_n$ are positive paths from w to $Irr(w)$. Thus $m, n \geq 1$, and so either $e_1 = f_1$, in which case by our inductive hypothesis there is a suitable 2-chain with boundary corresponding to the closed path $e_2 \ldots e_m f_n^{-1} \ldots f_2^{-1}$, or there is a 2-cell $[w; (e_1, f_1)]$ with boundary $e_1 p'(q')^{-1} f_1^{-1}$, say, for some positive paths p', q'. If we choose a positive path r from $\tau p' = \tau q'$ to $Irr(w)$ then by inductive assumption there are 2-chains ζ_{e_1} and ζ_{f_1} whose boundaries are 1-cycles arising from the closed paths $e_2 \ldots e_m r^{-1}(p')^{-1}$ and $f_2 \ldots f_n r^{-1}(q')^{-1}$ respectively, and with $[w; (e_1, f_1)] \succ_{\mathbf{r}} \zeta_{e_1}, \zeta_{f_1}$. (This construction is described by Figure 1.2.)

Then $[w; (e_1, f_1)] + \zeta_{e_1} - \zeta_{f_1}$ is a suitable 2-chain for this pair of paths , and so $P(w)$ holds. The lemma now follows by noetherian induction (Lemma 1.4.2). ⌟

We shall now show that there is also a 2-cycle with boundary of the form (4.3) for each critical triple of edges.

Suppose that (e_1, e_2, e_3) is a critical triple of edges at $w \in F$ with $e_1 <_w e_2 <_w e_3$. Then for $1 \leq i < j \leq 3$ there is a 2-cell $[w; (e_i, e_j)]$ with boundary $e_i p_{i,j} q_{i,j}^{-1} e_j^{-1}$ say, where $p_{i,j}$ and $q_{i,j}$ are positive paths. If for $1 \leq i < j \leq 3$ we also choose positive paths $r_{i,j}$ from $\tau p_{i,j}$ to $Irr(w)$, then by Lemma 4.4.4 there exist 2-chains ζ_1, ζ_2 and ζ_3 whose boundaries are 1-cycles corresponding to the closed paths $p_{1,3} r_{1,3} r_{1,2}^{-1} p_{1,2}^{-1}$, $q_{1,2} r_{1,2} r_{2,3}^{-1} p_{2,3}^{-1}$ and $q_{2,3} r_{2,3} r_{1,3}^{-1} q_{1,3}^{-1}$ respectively, from which we can construct the 2-cycle

$$\xi_{(e_1,e_2,e_3)} = [w; (e_1, e_2)] - [w; (e_1, e_3)] + [w; (e_2, e_3)] + \zeta_1 + \zeta_2 + \zeta_3,$$

which is of the form (4.3), because $\xi_{(e_1,e_2,e_3)} \succ_{\mathbf{r}} \zeta_1 + \zeta_2 + \zeta_3$.

Let $\mathbf{X}$ be a set consisting of such a 2-cycle $\xi_{(e,f,g)}$ for each critical triple (e,f,g) of positive edges. For a pair of 2-cycles z, z' we shall write $z \sim_{\mathbf{X}} z'$ if $z - z'$ is in the sub-bimodule of the $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule $Z_2(\overline{\mathcal{D}_\mathbf{P}})$ generated by $B_2(\overline{\mathcal{D}_\mathbf{P}})$ and $\mathbf{X}$.

Any triple of positive edges at some vertex, where no edge is disjoint from the other pair, is a translate of a critical triple by the two-sided F-action. Therefore, by our choice of the set $\mathbf{X}$ there is now a 2-cycle $\xi_{(e,f,g)}$ of the form (4.3) for *any* $w \in F$ and triple (e,f,g) of positive edges in $star^+(w)$ $(w \in F)$ such that

$$\xi_{(e,f,g)} = [w;(e,f)] + [w;(f,g)] - [w;(e,g)] + \zeta_{(e,f,g)} \sim_{\mathbf{X}} 0,$$

where $\xi_{(e,f,g)} \succ_{\mathbf{r}} \zeta_{(e,f,g)}$. Moreover, there is a constant c (depending on the choice of set $\mathbf{X}$ and the maximal length of the boundary of any 2-cell in $\overline{\mathcal{D}_\mathbf{P}}$) which bounds the number of terms in each such 2-cycle.

4.4.2 Proof of Theorem 4.4.1

We shall show that the homology classes of the set $\mathbf{X}$ generate Π_2.

Let ξ be a 2-cycle. We can write ξ as

$$\xi = \sum_{i=1}^{m} n_i \sigma_i$$

where $m \geq 0$, $\sigma_1, \ldots, \sigma_m$ are distinct 2-cells, and $n_1, \ldots, n_m \in \mathbb{Z} - \{0\}$. Each 2-cell can be written as $\sigma_i = [w_i;(f_i,g_i)]$ for $i = 1, \ldots, m$, where $w_i \in F$ and $f_i, g_i \in star^+(w_i)$.

Denote by $\Omega = \Omega(\xi)$ the set of 2-cells of the form $[u;(e,f)]$ where $u \in F$ is a *descendant* of one of the vertices $w_1, \ldots, w_m$, that is

$$u \in \bigcup_{i=1}^{m} \{v \in F : w_i \to^* v\},$$

and $e, f \in star^+(u)$. Because Γ is locally finite, it follows from Lemma 1.2.2 that Ω is finite. We partially order Ω as follows: For $u_i \in F$ and positive edges $e_i, e_i' \in star^+(u_i)$ $(i = 1,2)$ we write $[u_1;(e_1,e_1')] < [u_2;(e_2,e_2')]$ if

(i) $u_2 \succ_{\mathbf{r}} u_1$; or

(ii) $u_1 = u_2$ and $e'_1 <_{u_1} e'_2$; or

(iii) $u_1 = u_2$, $e'_1 = e'_2$ and $e_1 <_{u_1} e_2$.

Next, choose some bijection

$$\psi : \Omega \longrightarrow \{1, 2, \ldots, |\Omega|\}$$

which is compatible with this ordering, and define

$$\Psi(\xi) = \sum_{i=1}^{m} c^{\psi\sigma_i}.$$

Suppose that ξ is non-zero (that is, $m > 0$), and assume that $\psi(\sigma_1) > \psi(\sigma_i)$ for all $1 < i \leq m$. If the edge f_1 is not the left principal edge e at w_1, then we can replace $n_1\sigma_1$ with a 2-chain

$$n_1([w_1; (e, g_1)] - [w_1; (e, f_1)] - \zeta_{(e,f_1,g_1)})$$

where $\sigma_1 \succ_{\mathbf{r}} \zeta_{(e,f_1,g_1)}$, giving a 2-cycle ξ' with $\xi' \sim_{\mathbf{X}} \xi$ and $\Psi(\xi') < \Psi(\xi)$.

Otherwise f_1 is left principal, and so because σ_1 is maximal it is the only 2-cell represented in ξ with the edge g_1 in its boundary. Since ξ is a 2-cycle it follows that $n_1 = 0$, a contradiction. By induction, $\xi \sim_{\mathbf{X}} 0$. ⌟

4.5 An alternative proof of Theorem 4.4.1 using homotopy

Again let $\mathcal{P}$ be a finite complete rewriting system, and $\mathbf{p}$ a finite trivializer arising from a choice of resolution of each critical pair. To study the second homotopy of $\overline{\mathcal{D}^{\mathbf{p}}}$ we shall use *pictures*:

4.5.1 Pictures

In this subsection we shall only review the material that we shall need; more details and proofs can be found in, for example, [6, 31]. A picture $\mathbb{P}$ over $\overline{\mathcal{D}^{\mathcal{P}}}$ consists of the following:

1. A disc D^2.

2. Disjoint discs $d_1, \ldots, d_m$ in the interior of D^2.

3. A finite number of disjoint arcs $\alpha_1, \ldots, \alpha_n$. Each arc lies in the closure of $D^2 - (d_1 \cup d_2 \cup \cdots \cup d_m)$ and is either a simple closed curve having trivial intersection with the boundaries of the discs $D^2, d_1, \ldots, d_m$, or is a simple non-closed curve whose intersection with the boundaries of these discs consists of its endpoints only.

4. Each arc has a normal orientation indicated by a transverse arrow and is labelled by a positive edge of $\overline{\mathcal{D}^{\mathcal{P}}}$.

5. Travelling around the boundary ∂d_i of each interior disc d_i reading the edges labelling the arcs with endpoints on the boundary will give the boundary of a 2-cell in $\overline{\mathcal{D}^{\mathcal{P}}}$. If we cross an arc labelled by a positive edge e in the direction of its normal orientation then we read e; otherwise we read e^{-1}.

6. The *regions* of $\mathbb{P}$ are the connected components of

$$D^2 - (\partial D^2 \cup \bigcup_{i=1}^{n} d_i \cup \bigcup_{j=1}^{m} \alpha_j)$$

and each region is labelled by an element of F: If we travel across an arc labelled by a positive edge e in the direction of its normal orientation then we move from a region labelled ιe to a region labelled τe.

We define $\partial\mathbb{P}$ to be ∂D^2. By travelling around this boundary we read a closed path in the 1-skeleton Γ of $\overline{\mathcal{D}^{\mathcal{P}}}$. We say that $\mathbb{P}$ is *spherical* if this is the empty path, which is to say that no arc meets $\partial\mathbb{P}$: When a picture is spherical we call the region surrounding $\partial\mathbb{P}$ the *outer region*, and $\partial\mathbb{P}$ is not drawn. The *area* of $\mathbb{P}$ is the number m of interior discs, and the *components* of $\mathbb{P}$ are the connected components of

$$\bigcup_{i=1}^{n} d_i \cup \bigcup_{j=1}^{m} \alpha_j .$$

We say that $\mathbb{P}$ is *connected* if it has at most one component.

Suppose we embed a circle in $\mathbb{P} - \cup_i d_i$ so that it meets the arcs in only finitely many transverse intersections. Then the part of $\mathbb{P}$ enclosed by this circle is said to be a *subpicture* of $\mathbb{P}$. There is a 2-sided action of F on the collection of pictures induced by the action of F on the labels of the regions and arcs.

Each picture describes a continuous map of the disc into the 2-skeleton of $\overline{\mathcal{D}\mathrm{P}}$ in the following way. Each interior disc maps onto the 2-cell whose boundary corresponds to the boundary of the disc, each arc maps onto the mid-point of the edge labelling it, and each region is mapped in some way to the vertex w labelling it and a subset of

$$\{e, e^{-1} : e \in star^{+}(w)\}.$$

If $\mathbb{P}$ is a spherical picture then we may assume that all of $\partial\mathbb{P}$ is mapped to the vertex $w \in F$ which labels the outer region, so that $\mathbb{P}$ describes a continuous map of the 2-sphere S^2 into $\overline{\mathcal{D}\mathrm{P}}$. The *homotopy class* of $\mathbb{P}$ is the corresponding element of the second homotopy group $\pi_2(\overline{\mathcal{D}\mathrm{P}}, w)$, and in fact every element of the second homotopy groups of $\overline{\mathcal{D}\mathrm{P}}$ can be represented in this way by a spherical picture. We shall use $\xi(\mathbb{P})$ to denote the 2-chain obtained by reading the labels on each interior disc of a picture $\mathbb{P}$.

Example 4.5.1 The connected spherical picture in Figure 4.1 describes a map of S^2 onto the boundary of the 3-cell $[e, [f, f']]$ $(e, f, f' \in \mathrm{e}^{+})$, and $\xi(\mathbb{P})$ is the 2-cycle

$$\begin{aligned} & [e, f].\iota f' - \tau e.[f, f'] - [e, \iota f.f'] + [e, \tau f.f'] - [e, f].\tau f' + \iota e.[f, f'] \\ = \;& (\iota e - \tau e).[f, f'] + [e, \tau f.f'] - [e, \iota f.f'] + [e, f].(\iota f' - \tau f'), \end{aligned}$$

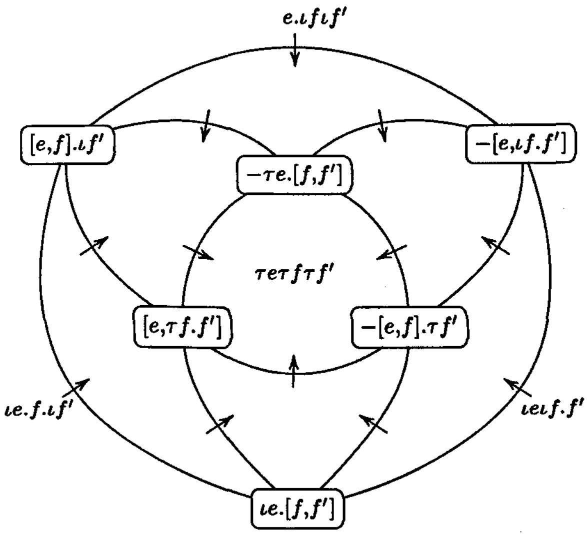

Figure 4.1: A spherical picture

where the coefficient gives the orientation of the 2-cell. The outer region is labelled $\iota e \iota f \iota f'$, and the labelling of the unmarked edges and regions follows from the labelling of the interior discs.

Suppose that Υ is a subset of F containing a unique representative of each congruence class. The Hurewicz homomorphism (4.1) maps the homotopy class of a spherical picture $\mathbb{P}$ with outer region labelled by some $w \in F$ to the homology class of the 2-cycle $\xi(\mathbb{P})$. Since $\mathbf{p}$ is a homotopy trivializer then the above homomorphism is in fact an isomorphism, as noted in §4.1.

4.5.2 3-cells at critical triples

Since a spherical picture over $\overline{\mathcal{D}^{\mathbf{P}}}$ describes a continuous map of S^2 into the 2-skeleton of the augmented Squier complex, we can describe the attaching map of each 3-cell in $\overline{\mathcal{D}^{\mathbf{P}}}$ in this way.

From our choice of trivializer $\mathbf{p}$, and by the construction of $\overline{\mathcal{D}^{\mathbf{P}}}$, for any $w \in F$ and triple of edges $e, f, g \in star^+(w)$ with at least one disjoint from the remaining pair

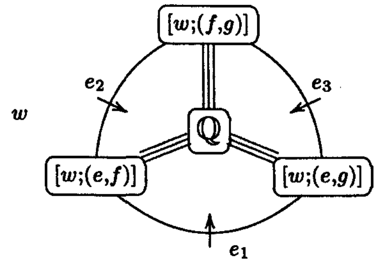

Figure 4.2: Spherical picture at triple (e_1, e_2, e_3)

there is a 3-cell whose attaching map can be described by a spherical picture of the form shown in Figure 4.2 for some subpicture $\mathbb{Q}$, with the outer region labelled w, and all other regions labelled by vertices v_i satisfying $\vartheta v_i < \vartheta w$, where ϑ is the height function (Definition 1.2.3). Furthermore, we can assume that the area of this picture is not greater than $k+2$, where $k \geq 4$ is the maximum length of the boundary of any 2-cell.

We shall now give a set of spherical pictures describing the attaching maps of a set of 3-cells at each critical triple (Definition 4.4.2), and which have the same form as in Figure 4.2. We need the following lemma:

Lemma 4.5.2 *For any pair of positive paths $p = e_1 \ldots e_m$ and $q = f_1 \ldots f_n$ from the same initial vertex $w \in F$ to $Irr(w)$ there is a picture over $\overline{\mathcal{D}^{\mathbf{P}}}$ with boundary pq^{-1} with only one region labelled w and such that any other region is labelled by some $v \in F$ with $\vartheta v < \vartheta w$, where ϑ is the height function.*

Proof: The proof is by noetherian induction on F with the reduction order $\succ_{\mathbf{r}}$ induced by the rules $\mathbf{r}$.

Let $w \in F$ and assume inductively that for any $w' \in F$ satisfying $w \succ_{\mathbf{r}} w'$ and pair of paths r, s from w' to $Irr(w')$ there is a picture over $\overline{\mathcal{D}^{\mathbf{P}}}$ with boundary rs^{-1} with only one region labelled w' and such that any other region is labelled by some $v \in F$ with $\vartheta v < \vartheta w'$. We want to construct a suitable picture for w.

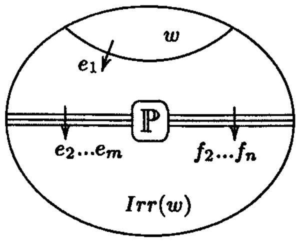

Figure 4.3: Construction used in the proof of Lemma 4.5.2

If $w = Irr(w)$ then p and q are both the empty path at w and we just take the empty picture with a single region labelled w.

So suppose that $w \neq Irr(w)$, and that $p = e_1 \dots e_m$ and $q = f_1 \dots f_n$ are a pair of positive paths from w to $Irr(w)$, with $m, n \geq 1$. If $e_1 = f_1$ then there is by inductive hypothesis a picture $\mathcal{P}$ with boundary $e_2 \dots e_m f_n^{-1} \dots f_2^{-1}$ where a single region is labelled τe_1, and all other regions labelled by some $v \in F$ with $\vartheta v < \vartheta \tau e_1 < \vartheta w$. A suitable picture can be obtained from $\mathcal{P}$ by simply adding an arc with label $e_1 = f_1$ as shown in Figure 4.3.

Otherwise, there is a 2-cell $[w; (e_1, f_1)]$ with boundary $e_1 p' (q')^{-1} f_1^{-1}$ for some pair of positive paths p' and q'. If we choose a positive path r from $\tau p' = \tau q'$ to $Irr(w)$ then by our inductive assumption we have pictures $\mathbb{P}$ and $\mathbb{Q}$ with boundaries $\partial \mathbb{P} = e_2 \dots e_m r^{-1} (p')^{-1}$ and $\partial \mathbb{Q} = q' r f_n^{-1} \dots f_2^{-1}$ giving the picture shown in Figure 4.4.

The lemma follows by noetherian induction (Lemma 1.4.2). ⌟

Proposition 4.5.3 *We can construct a spherical picture of the form shown in Figure 4.2 for each critical triple* (e_1, e_2, e_3) *of positive edges at* $w \in F$.

Proof. For $1 \leq i < j \leq 3$ there is a 2-cell $[w; (e_i, e_j)]$ with boundary $e_i p_{i,j} q_{i,j}^{-1} e_j^{-1}$ say, where $p_{i,j}$ and $q_{i,j}$ are positive paths. If we also choose positive paths $r_{i,j}$ from $\tau p_{i,j}$ to $Irr(w)$ then by using Lemma 4.5.2 we can construct a spherical picture of the form shown in Figure 4.5 from pictures $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$ which have boundaries $p_{1,3} r_{1,3} r_{1,2}^{-1} p_{1,2}^{-1}$,

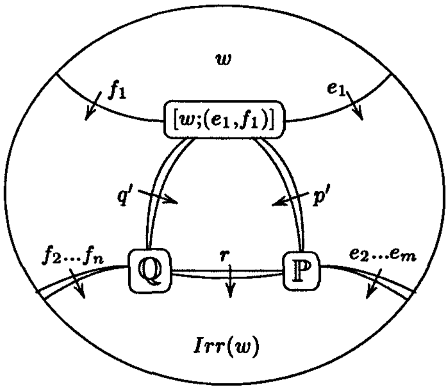

Figure 4.4: Second construction used in the proof of Lemma 4.5.2

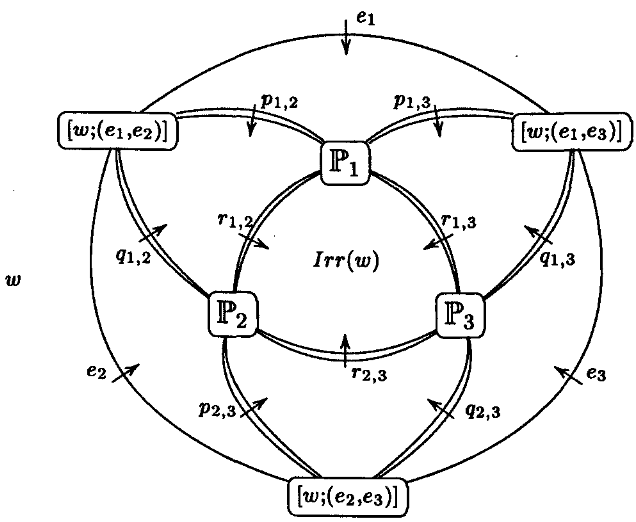

Figure 4.5: Spherical picture at a critical triple.

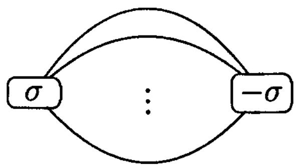

Figure 4.6: Cancelling pair

$q_{1,2}r_{1,2}r_{2,3}^{-1}p_{2,3}^{-1}$ and $q_{2,3}r_{2,3}r_{1,3}^{-1}q_{1,3}^{-1}$ respectively. Then there is only one region labelled w, and every other region is labelled by some $v \in F$ satisfying $\vartheta v < \vartheta w$. ⌟

We choose such a spherical picture for each critical pair, and we claim that 3-cells attached according to this set of pictures together with their translates under the 2-sided action of F trivialize the second homotopy groups of $\overline{\mathcal{D}^{\mathbf{P}}}$, so that S is of type FDT_2. Note that this choice has given us a set $\mathbf{X}$ of spherical pictures of the form shown in Figure 4.2 for every triple of positive edges in $star^+(w)$ for every $w \in F$.

4.5.3 Operations on pictures

Let $\mathbf{X}$ be the chosen set of spherical pictures, one for every triple of positive edges in $star^+(w)$ for some $w \in F$. By an $\mathbf{X}$-*picture* we will mean either a picture $\mathbb{P}$ from $\mathbf{X}$, or $-\mathbb{P}$, the picture obtained by a planar reflection of $\mathbb{P}$.

We allow the following operations on spherical pictures.

(A) Deletion of a closed arc which encircles no arcs or discs (a *floating circle*).

(B) Deletion of a *cancelling pair*, a subpicture of the form shown in Figure 4.6.

(C) *Bridge move*: see Figure 4.7.

(D) *Replace*($\mathbf{X}$), the replacement of an interior disc with label a 2-cell $[w; (f, f')]$ where $w \in F$ and $f, f' \in star^+(w)$ are not left principal edges. If we let e denote the left principal edge at w then this operation is carried out as follows. First

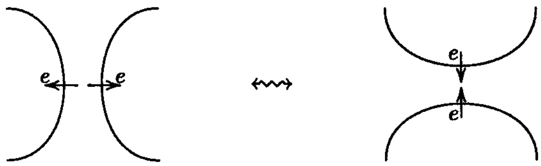

Figure 4.7: Bridge move

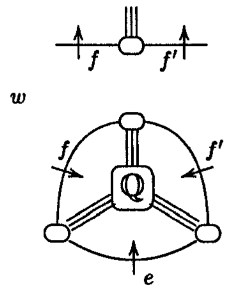

Figure 4.8: Insertion of **X**-picture

we insert the appropriate **X**-picture corresponding to the triple (e, f, f') into the original picture as shown in Figure 4.8. Next, a series of bridge moves can be applied to obtain the picture shown in Figure 4.9, and we then delete the cancelling pair to finish with the picture shown in Figure 4.10.

Operations (A), (B) and (C) do not change the homotopy class of the picture $\mathbb{P}$. This is also true of operation (D) if there are 3-cells whose attaching maps correspond to the **X**-pictures, for then we are essentially just pushing the image of $\mathbb{P}$ across the appropriate 3-cell.

4.5.4 A 2-dimensional Dehn-type algorithm

Let $\mathbb{P}$ be a spherical picture over $\mathcal{D}^{\mathsf{P}}$ whose regions are labelled by a set of vertices $w_1, w_2, \ldots, w_n \in F$. Following [6] we shall use a type of "2-dimensional" Dehn algorithm

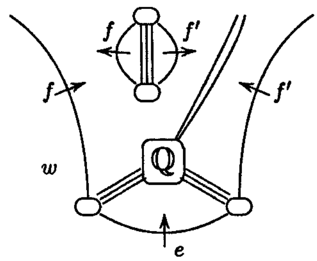

Figure 4.9: Subpicture after performing bridge moves

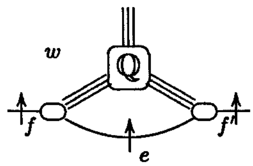

Figure 4.10: Result of replace($\mathbf{X}$)

to show that $\mathbb{P}$ is $\mathbf{X}$-equivalent to the empty picture. The algorithm consists of a series of transformations of the original picture which we now describe.

Firstly we remove any floating circle in $\mathbb{P}$. If there is an interior disc with label a 2-cell $[w;(f,f')]$ for some $w \in F$ and positive edges $f, f' \in star^{+}(w)$ neither of which are left principal then we can perform an operation (D) and we are done.

Otherwise, choose some connected spherical subpicture of $\mathbb{P}$, and in this subpicture choose a region that is labelled by some vertex (w_1, say) that is maximal with respect to the height function ϑ. If this region is bounded by a pair of discs and arcs of the form as in Figure 4.11 where $\sigma = [w_1;(e,f)]$ (for edges $e, f \in star^{+}(w_1)$ with e left principal) has boundary $epq^{-1}f^{-1}$ where p, q are positive paths, then by a series of bridge moves this subpicture can be transformed to a subpicture containing a cancelling pair as shown in Figure 4.12. and we can now remove the cancelling pair, at the same time removing the region with label w_1.

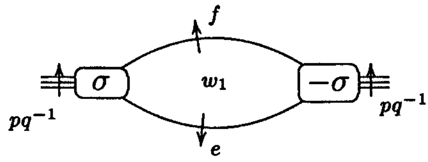

Figure 4.11: Subpicture composed of the discs labelled σ and $-\sigma$

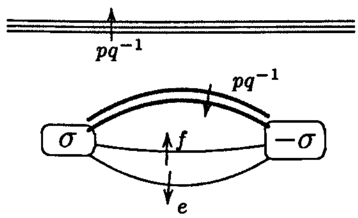

Figure 4.12: New subpicture containing a cancelling pair

On the other hand it is possible that the region with label w_1 is bounded by more than 3 interior discs and arcs as shown in Figure 4.13, where the arcs are labelled alternately by the left principal edge e at w and by edges $f, f', \ldots \in star^+(w_1) - \{e\}$. In this case, we can perform a bridge move between two arcs labelled by e to obtain a subpicture of the form shown in Figure 4.11, which again leads to the removal of a cancelling pair.

It is also possible that the region labelled by w_1 is the outer region of the spherical subpicture. If this subpicture is of the form shown in Figure 4.14 for some subpicture $\mathbb{Q}$, then by applying bridge moves we can obtain the picture shown in Figure 4.15. from which we can remove the cancelling pair. If the subpicture is of the form shown in Figure 4.16 we can similarly apply bridge moves to isolate a cancelling pair which we then remove.

Lemma 4.5.4 *For any spherical picture $\mathbb{P}$ any sequence of transformations will terminate in a finite number of steps.*

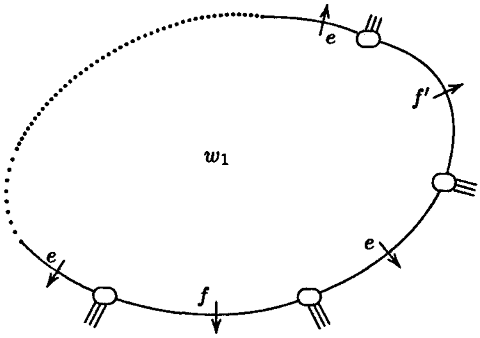

Figure 4.13: The maximal region bounded by more than two discs

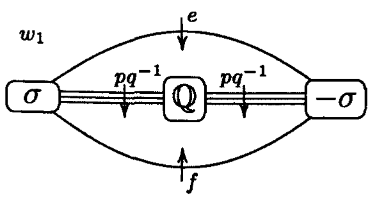

Figure 4.14: The maximal region is the outer region

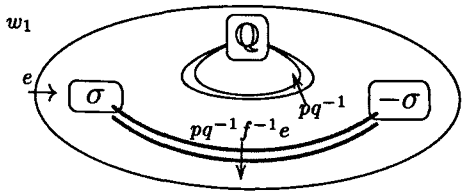

Figure 4.15: After performing bridge moves on Figure 4.14

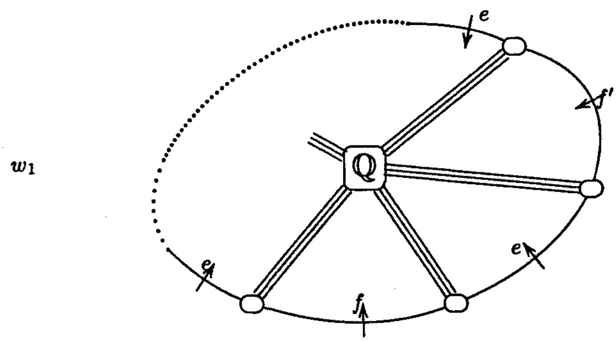

Figure 4.16: The maximal region is the outer region (case 2)

Proof: For any given $u \in F$ let $\Delta(u)$ denote the subgraph of Γ formed by taking the union of all positive paths originating at u. As its vertices $\Delta(u)$ has the set

$$vert(\Delta(u)) = \{v \in F : u \rightarrow^*_{\mathbf{r}} v\}$$

of descendants of u and the edge set is

$$\{e, e^{-1} : e \in star(v), u \rightarrow^*_{\mathbf{r}} v\}.$$

Because Γ is locally finite, it follows from Lemma 1.2.2 that $\Delta(u)$ is finite. We denote by $\Omega = \Omega(\mathbb{P})$ the set of 2-cells of the form $[u; (e, f)]$ with

$$u \in \bigcup_{i=1}^{n} vert(\Delta(w_i)) \subset F \text{ and } e, f \in star^+(u)$$

where $w_1, \ldots, w_n \in F$ are the labels of the regions of $\mathbb{P}$. Again this set is finite.

Note that each single transformation of a picture either removes a cancelling pair, or uses an **X**-picture to replace a subpicture with a single disc, with label a 2-cell σ, say, with a subpicture containing only discs labelled by 2-cells in Ω which are beneath σ in the partial ordering introduced in the proof of the special case of Proposition 3.3.4. We must show that this process terminates in a finite number of steps.

First, we choose some bijective function

$$\psi : \Omega \longrightarrow \{1, 2, \ldots, |\Omega|\}$$

requiring only that for 2-cells $\sigma, \sigma' \in \Omega$ we have $\psi\sigma < \psi\sigma'$ if $\sigma < \sigma'$.

Based on the values assigned by ψ to each 2-cell of Ω we define another function

$$\Psi : \Upsilon \longrightarrow \mathbb{N} \cup \{0\}$$

by extending to the set of interior discs the function

$$\pm\sigma \longmapsto k^{\psi\sigma}$$

where $k+1$ is the maximum area of any **X**-picture.

To prove the lemma, note that applying a single transformation to any picture $\mathbb{Q}$ in Υ will either reduce the value of $\Psi(\mathbb{Q})$ by $2k^i$ for some $1 \leq i \leq |\Omega|$ (if we have removed a cancelling pair), or will reduce the value of $\Psi(\mathbb{Q})$ by k^i while increasing the value by adding less than $k(k^{(i-1)})$. $\lrcorner$

Chapter 5

Alternative viewpoints

5.1 Competing finiteness conditions

5.1.1 An alternative definition of FHT_2

Our definition of FDT_2 seems to allow the possibility of further development of these ideas, for example we could study the third homotopy groups of a 4-complex obtained by attaching 3-cells to the 3-complex $\overline{\mathcal{D}^{\mathbf{p}}}$ in order to trivialize the second homotopy groups, together with certain "obvious" 4-cells.

On the other hand, in our definition of FHT_2 we only require that for some finite *homology* trivializer $\mathbf{p}$ the bimodule $\Pi_2(\mathcal{P}, \mathbf{p})$ is finitely generated. In this situation, where $\mathbf{p}$ may not be a homotopy trivializer, if the Hurewicz homomorphism (4.1) is not surjective then it may be that we are not in fact able to kill the second homology by attaching 3-cells $\overline{\mathcal{D}^{\mathbf{p}}}$.

Without being able to realize this geometrically, it is not clear how we could define a property FHT_3 by studying a 4-complex based on $\overline{\mathcal{D}^{\mathbf{p}}}$. An alternative definition of FHT_2 would require that not only the bimodule Π_2 was finitely generated, but that we could attach a finite set of 3-cells to trivialize the second homology. This problem does not exist with the definition of the property FHT because any 1-cycle arises from a (finite) set of closed paths.

5.1.2 The property bi-FP_n

Following Otto and Kobayashi [41], we say that a monoid S is of type *bi-FP_n* if the monoid ring $\mathbb{Z}S$ (thought of as a $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule by left and right multiplication) has a partial resolution of length n by free $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules of finite rank.

Applying the functor $-\otimes_{\mathbb{Z}S}\mathbb{Z}$ to such a resolution will give a partial resolution of the trivial left $\mathbb{Z}S$-module $\mathbb{Z}$ ($\cong\mathbb{Z}S\otimes_{\mathbb{Z}S}\mathbb{Z}$) by finite rank free left $\mathbb{Z}S$-modules [41]. Thus if a monoid is of type bi-FP_n then it is necessarily of type left-FP_n. Similarly, applying $\mathbb{Z}\otimes_{\mathbb{Z}S}-$ will show that such a monoid is also of type right-FP_n. On the other hand, Kobayashi and Otto [42] have given an example of a monoid which is both left and right FP_∞ but is not of type bi-FP_3.

In [41] it is shown that there is a partial resolution of $\mathbb{Z}S$ by free $(\mathbb{Z}S, \mathbb{Z}S)$-bimodules analogous to (1.8), but constructed using the short exact sequence (3), giving the characterization

FHT monoids are those finitely presented monoids of type bi-FP_3.

By splicing this partial resolution with the short exact sequence (4) arising from some choice of finite homology trivializer, we immediately have the analogous characterization

FHT_2 monoids are those finitely presented monoids of type bi-FP_4.

5.1.3 Homological finite derivation type

Alonso and Hermiller [2] have introduced the alternative finiteness conditions *n-dimensional homological finite derivation type* ($HFDT_n$) for all $n\geq 0$, and have shown that a monoid S of type left and right FP_n is necessarily of type $HFDT_n$. However, the example mentioned above due to Kobayashi and Otto of a monoid which is both left and right FP_∞ but not of type FHT or FDT shows us that (in particular) the property $HFDT_3$ is not equivalent to either FHT or FDT, nor is the property $HFDT_4$ equivalent to FHT_2 or FDT_2.

5.1.4 A finiteness condition for higher dimensional rewriting systems

In order to extend the notion of finite derivation type and finite homological type one dimension higher, we have defined a 3-complex $\overline{\mathcal{D}^{\mathbf{p}}}$ based on $\mathcal{D}$ and studied its second homotopy and homology. We now outline another approach (as suggested in [36, 43]) which is to consider some kind of finite derivation type on some new graph based on the derivations of a higher dimensional "rewriting system" which operates on paths in Γ.

In the next section (§5.2), we describe such a rewriting system on paths, or *homotopy reduction system*, which can be associated to any finite complete rewriting system. The new graph of derivations of this system (introduced in [17]) would have as its vertices the set of all paths in Γ, and there would be edges which correspond to pushing a subpath across a 2-cell in $\overline{\mathcal{D}^{\mathbf{p}}}$.

A closed path in this new graph will correspond to a 2-cycle in $\overline{\mathcal{D}^{\mathbf{p}}}$, and so this method is related to our earlier study of the second homology of this 3-complex, and it is possible that they are equivalent. The advantage of our earlier approach is in obtaining the short exact sequence (4) which allows us to relate the properties FHT_2 and FDT_2 to homological finiteness conditions, and the invariance property (Theorem 4.2.1) can be proved easily using homological algebra together with known results.

5.2 A complete homotopy reduction system for complete rewriting systems

Suppose that $\mathcal{P}$ is a finite complete rewriting system, and let $\mathbf{p}$ be a homotopy trivializer given by resolutions of the critical pairs associated with the set of rules $\mathbf{r}$. As we have already noted (Remark 1.4.14), the boundary of each 2-cell σ consists of two positive paths arising from resolutions of either disjoint or critical pairs, according as to whether σ is in $\mathcal{D}$ or not.

We now describe a *homotopy reduction system* for $\mathcal{D}^{\mathrm{p}}$: a "rewriting system" on paths in $\mathcal{D}^{\mathrm{p}}$, which operates by pushing subpaths across 2-cells. It will turn out that this homotopy reduction system is *complete* in the sense that in a finite number of steps it will rewrite any path to a unique irreducible path sharing the same endpoints (see Theorem 5.2.2).

A single-step homotopy reduction, applied to subpaths of paths in $\mathcal{D}^{\mathrm{p}}$, will be of one of two types:

1. For any $e \in \mathbf{e}^{+}$ we allow reductions of the form
   (a) $ee^{-1} \rightsquigarrow 1_{\iota e}$ (where $1_{\iota e}$ denotes the empty path at ιe); and
   (b) $e^{-1}e \rightsquigarrow 1_{\tau e}$.

2. For any pair $e, f \in star^{+}(w)$ there is a 2-cell in $\mathcal{D}^{\mathrm{p}}$ with boundary of the form

   $$epq^{-1}f^{-1}$$

   where p and q are positive paths ending at some vertex z, say, which resolve the pair (e, f). If $e <_w f$ holds we allow the reductions
   (a) $f \rightsquigarrow epq^{-1}$, and
   (b) $f^{-1} \rightsquigarrow qp^{-1}e^{-1}$,

   illustrated by Figure 5.1.

We use the notation $\rightsquigarrow^*$ to denote the reflexive and transitive closure of the relation $\rightsquigarrow$ on paths in Γ.

Proposition 5.2.1 *The homotopy reduction system just given is* noetherian, *that is, we can only perform a finite number of single-step homotopy reductions on any given path p.*

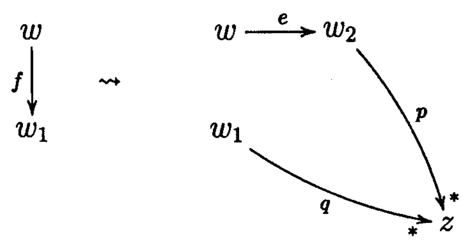

Figure 5.1: Single-step homotopy reduction (type 2)

Proof: First note that each single-step homotopy reduction either removes an edge-pair $e^{\varepsilon}e^{-\varepsilon}$ ($e \in \mathbf{e}^+$, $\varepsilon = \pm 1$) (a Type 1 reduction), or replaces a subpath f^{ε} ($f \in \mathbf{e}^+$, $\varepsilon = \pm 1$) with a subpath consisting of one edge e^{ε} where $e \in star^+(\iota f)$ satisfies $e <_{\iota e} f$, together with edges of the form g^{δ} ($g \in \mathbf{e}^+$, $\delta = \pm 1$) satisfying $\vartheta \iota g < \vartheta \iota f$, where ϑ denotes the height function (Definition 1.2.3) (a Type 2 reduction). We show that this process must terminate in a finite number of steps.

For the given path p, by Lemma 1.2.2 and the fact that Γ is locally finite, the set

$$A = A(p) = \{v \in F : v \prec_{\mathbf{r}} w \text{ for some vertex } w \text{ of } p\}$$

of descendants of vertices in p is finite, and again because Γ is locally finite the set

$$\Omega = \Omega(p) = \{e \in \mathbf{e}^+ : \iota e, \tau e \in A\}$$

of positive edges between vertices in A is also finite. Firstly, we can order A in some arbitrary way as $A = (v_1, v_2, \ldots v_{|A|})$, but requiring that if $\vartheta v_i < \vartheta v_j$ then $i < j$, and then we can order Ω as

$$\begin{aligned}\Omega &= (e_1, e_2, \ldots e_{|\Omega|}) \\ &= (e_{v_1,1}, e_{v_1,2}, \ldots, e_{v_1,k_{v_1}}, e_{v_2,1}, \ldots, e_{v_{|A|},k_{v_{|A|}}})\end{aligned}$$

where the edges indexed by $v_i \in A$ are those edges with initial vertex v_i, and then according to the ordering of edges in $star^+(v_i)$:

if $e_{v_i,j} <_{v_i} e_{v_i,k}$, *then we require that* $j < k$.

Now we consider the set $\Xi = \Xi(p)$ of all finite paths with the same initial and terminal vertices as our original path p and composed of edges e^ε with $e \in \Omega$ and $\varepsilon = \pm 1$, and define a function

$$\psi : \Xi \longrightarrow \mathbb{N}_0 \qquad e_{i_1}^{\varepsilon_k} \ldots e_{i_k}^{\varepsilon_k} \mapsto \sum_{j=1}^{k} n^{i_j} ,$$

where n is the maximum length of the boundary of any 2-cell in $\mathcal{D}^{\mathrm{p}}$.

To prove the proposition, note that applying a Type 1 single-step homotopy reduction to any path q in $\Xi(p)$ will reduce the value of $\psi(q)$ by $2n^i$ for some $i > 0$, and that a Type 2 reduction will reduce the value $\psi(q)$ by n^i for some $i > 1$ while increasing the value by adding less than $n(n^{(i-1)})$; clearly this process must terminate, as we have associated with any sequence of single-step reductions on an arbitrary path a strictly decreasing sequence in $\mathbb{N}_0$. ⌟

Theorem 5.2.2 *The system is complete.*

Proof: The only irreducible paths are those composed of left principal edges and that have no spurs (subpaths of the form $\ldots e^\varepsilon e^{-\varepsilon} \ldots$, where $e \in \mathbf{e}^+$ and $\varepsilon = \pm 1$); these are the unique edge-paths of shortest length between two points in a connected component of the maximal forest of left principal edges in Γ. ⌟

Remark 5.2.3 Homotopy reduction systems were introduced by Kobayashi [36], who considered more general complete homotopy reduction systems. The system that we have associated to a complete rewriting system is a variant of the *Guba-Sapir reduction system* [29, 39], in which the type 2 single step homotopy reductions are restricted to only those reductions which push an edge across a 2-cell from $\mathcal{D}$. The advantage of extending this as we have done is that the overlapping rules of the new homotopy reduction system are easily described (§5.2.1), and then there is a nice partial resolution (5.1) of $\mathbb{Z}$ extending the partial resolution (1.9).

5.2.1 Critical pairs of the homotopy reduction system

Associated to any finite complete rewriting system $\mathcal{P}$ we have described a complete homotopy reduction system based on a natural homotopy trivializer $\mathbf{p}$ of $\mathcal{D}$ given by choosing resolutions of the critical pairs. We shall now examine the *overlapping pairs* of the homotopy reduction system, that is, the pairs of single-step homotopy reductions that can operate on non-disjoint subpaths. The analogue of the result (1.10) that a noetherian rewriting system is complete if and only if it is confluent holds for homotopy reduction systems, as does the Diamond Lemma (Lemma 1.4.3), and therefore each overlapping pair can be resolved by positive homotopy reductions to some common path in Γ.

Overlaps of Type 1 reductions

These overlaps occur on subpaths of the form $ee^{-1}e$ or $e^{-1}ee^{-1}$ ($e \in \mathbf{e}^+$), and are immediately resolved (to e or e^{-1} respectively).

Overlaps between Type 1 and Type 2 reductions

Suppose that we can perform a type 2 reduction

$$f \rightsquigarrow epq^{-1}$$

on a positive edge f, where $e \in \mathbf{e}^+$ and p and q are positive paths. Then, for example, we have overlaps between the reductions

$$ff^{-1} \rightsquigarrow 1_{\iota f} \qquad \text{and} \qquad ff^{-1} \rightsquigarrow epq^{-1}f^{-1}.$$

This overlap is resolved easily as follows:

$$epq^{-1}f^{-1} \rightsquigarrow^* epq^{-1}qp^{-1}e^{-1} \rightsquigarrow^* epp^{-1}e^{-1} \rightsquigarrow^* ee^{-1} \rightsquigarrow^* 1_{\iota f}.$$

Overlaps of Type 2 reductions

These overlapping pairs are of four types (see [37]), and occur when we have edges e, f, g in $star^+(w)$ with $e <_w f <_w g$, so that there is a choice of two different Type

2 reductions at g (and, similarly, at g^{-1}). The first three are *inessential* overlapping pairs and are shown to be easily resolved, and the final type of overlapping pair will be seen to correspond precisely to translates of the critical triples of $\mathcal{P}$.

1. Suppose firstly that e, f and g are all disjoint. Then we can write w as $w = (\iota e')(\iota f')(\iota g')$ for some $e', f', g' \in \mathbf{e}^+$ such that $e = e'.\iota f'.\iota g'$, $f = \iota e'.f'.\iota g'$ and $g = \iota e'.\iota f'.g'$, and the overlapping pair can be resolved along the following pair of positive reductions:

$$\begin{aligned} h_1: \; g &\rightsquigarrow e.(\tau e' \iota f'.g')(e'.\iota f' \tau g')^{-1} \\ &\rightsquigarrow e(\tau e'.f'.\iota g')(\tau e' \tau f'.g')(\tau e'.f'.\tau g')^{-1}(e'.\iota f' \tau g')^{-1} \end{aligned}$$

and

$$\begin{aligned} h_2: \; g &\rightsquigarrow f(\iota e' \tau f'.g')(\iota e'.f'.\tau g')^{-1} \\ &\rightsquigarrow e(\tau e'.f'.\iota g')(e'.\tau f' \iota g')^{-1}(\iota e' \tau f'.g')(\iota e'.f'.\tau g')^{-1} \\ &\rightsquigarrow e(\tau e'.f'.\iota g')(e'.\tau f' \iota g')^{-1}(e'.\tau f' \iota g') \\ &\qquad (\tau e' \tau f'.g')(e'.\tau f' \tau g')^{-1}(\iota e'.f'.\tau g')^{-1} \\ &\rightsquigarrow e(\tau e'.f'.\iota g')(\tau e' \tau f'.g')(e'.\tau f' \tau g')^{-1}(\iota e'.f'.\tau g')^{-1} \\ &\rightsquigarrow e(\tau e'.f'.\iota g')(\tau e' \tau f'.g')(e'.\tau f' \tau g')^{-1} \\ &\qquad (e'.\tau f' \tau g')(\tau e'.f'.\tau g')^{-1}(e'.\iota f' \tau g')^{-1} \\ &\rightsquigarrow e(\tau e'.f'.\iota g')(\tau e' \tau f'.g')(\tau e'.f'.\tau g')^{-1}(e'.\iota f' \tau g')^{-1} \end{aligned}$$

which is illustrated in Figures 5.2 and 5.3 (the reduction h_1 is on the left and h_2 is on the right).

2. Next suppose that e is disjoint from f and g, which arise from some critical pair. We can write w as $w = \iota e' w'$ for some $w' \in F$ and $e' \in \mathbf{e}^+$ such that $e = e'.w'$, and

$$\begin{tikzcd}
\iota e'\iota f'\iota g' \arrow[r] & \iota e'\iota f'\tau g'
\end{tikzcd}
\qquad
\begin{tikzcd}
\iota e'\iota f'\iota g' \arrow[r] & \iota e'\iota f'\tau g'
\end{tikzcd}$$

$$\begin{tikzcd}
& \iota e'\iota f'\iota g' \arrow[dl] \arrow[rr, dash, dotted] & & \iota e'\iota f'\tau g' & \\
\tau e'\iota f'\iota g' \arrow[rrrr] & & & & \tau e'\iota f'\tau g' \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
\iota e'\iota f'\iota g' \arrow[r, dash, dotted] \arrow[d] & \iota e'\iota f'\tau g' \\
\iota e'\tau f'\iota g' \arrow[r] & \iota e'\tau f'\tau g' \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
& \iota e'\iota f'\iota g' \arrow[dl] \arrow[rr, dash, dotted] & & \iota e'\iota f'\tau g' & \\
\tau e'\iota f'\iota g' \arrow[rrrr, dash, dotted] \arrow[d] & & & & \tau e'\iota f'\tau g' \arrow[ul] \\
\tau e'\tau f'\iota g' \arrow[rrrr] & & & & \tau e'\tau f'\tau g' \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \iota e'\iota f'\iota g' \arrow[dl] \arrow[r, dash, dotted] \arrow[dd, dash, dotted] & \iota e'\iota f'\tau g' \\
\tau e'\iota f'\iota g' \arrow[dd] & & \\
& \iota e'\tau f'\iota g' \arrow[r] & \iota e'\tau f'\tau g' \arrow[uu] \\
\tau e'\tau f'\iota g' \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
& \iota e'\iota f'\iota g' \arrow[dl] \arrow[r, dash, dotted] \arrow[dd, dash, dotted] & \iota e'\iota f'\tau g' & \\
\tau e'\iota f'\iota g' \arrow[dd] & & & \\
& \iota e'\tau f'\iota g' \arrow[r, dash, dotted] \arrow[dl, shift left] & \iota e'\tau f'\tau g' \arrow[uu] & \\
\tau e'\tau f'\iota g' \arrow[ur, shift left] \arrow[rrr] & & & \tau e'\tau f'\tau g' \arrow[ul]
\end{tikzcd}$$

Figure 5.2: Resolution of type 1 overlap

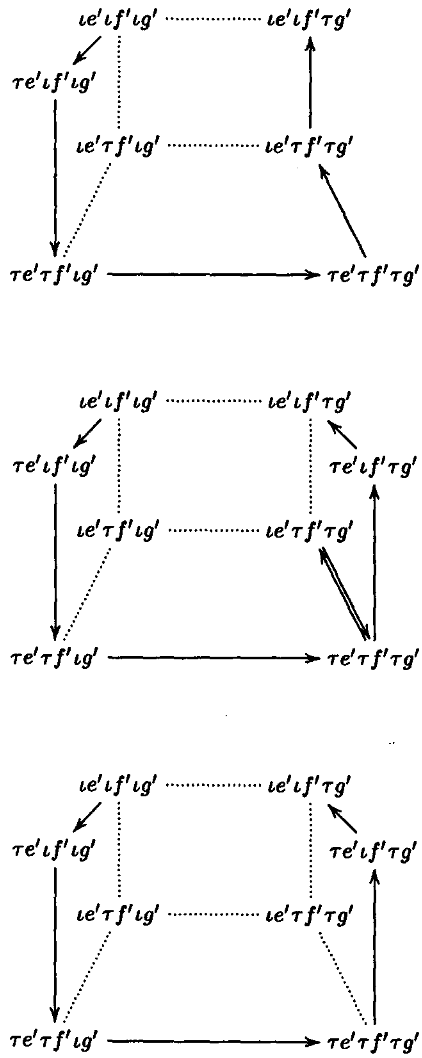

Figure 5.3: Resolution of type 1 overlap (continued)

there are also edges $f', g' \in star^+(w')$ such that $f = \iota e'.f'$ and $g = \iota e'.g'$. By our choice of $\mathbf{p}$, the edges f' and g' are resolved along positive paths $p = f_1 \ldots f_m$ and $q = g_1 \ldots g_n$ on the boundary of a 2-cell in $\mathcal{D}^{\mathbf{P}}$ to some vertex z. We may resolve this overlapping pair along positive homotopy reductions as follows (see figure 5.4):

$$
\begin{aligned}
h_1 : g &\rightsquigarrow e(\tau e'.g')(e'.\tau g')^{-1} \\
&\rightsquigarrow e(\tau e'.f'pq^{-1})(e'.\tau g')^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
h_2 : g &\rightsquigarrow f(\iota e'.pq^{-1}) \\
&\rightsquigarrow e(\tau e'.f')(e'.\tau f')^{-1}(\iota e.f_1 f_2 \ldots f_m g_n^{-1} g_1^{-1}) \\
&\rightsquigarrow e(\tau e'.f')(e'.\tau f')^{-1}(e'.\tau f')(\tau e'.f_1)(e'.\tau f_1)^{-1} \\
&\qquad\qquad (\iota e.f_2 \ldots f_m g_n^{-1} g_1^{-1}) \\
&\rightsquigarrow e(\tau e'.f')(e'.\tau f')^{-1}(e'.\tau f')(\tau e'.f_1)(e'.\tau f_1)^{-1} \\
&\qquad\qquad (e'.\tau f_1)(\tau e'.f_2)(e'.\tau f_2)^{-1}(\iota e.f_3 \ldots f_m g_n^{-1} g_1^{-1}) \\
&\rightsquigarrow^* e(\tau e'.f')(e'.\tau f')^{-1}(e'.\tau f')(\tau e'.f_1)(e'.\tau f_1)^{-1}(e'.\tau f_1) \\
&\qquad\qquad (\tau e'.f_2)(e'.\tau f_2)^{-1} \ldots (e'.\tau g_1)(\tau e'.g_1)^{-1}(e'.\iota g_1)^{-1} \\
&\rightsquigarrow^* e(\tau e'.f')(\tau e'.f_1 f_2 \ldots f_m g_n^{-1} \ldots g_1^{-1})(e'.\iota g_1)^{-1} \\
&= e(\tau e'.(f'pq^{-1}))(e'.\tau g')^{-1}
\end{aligned}
$$

3. Next suppose that g is disjoint from both e and f, which arise from some critical pair. We can write w as $w'\iota g'$ for some $w \in F$ and $g' \in \mathbf{e}^+$ such that $g = w'.g'$, and there are edges $e', f' \in star^+(w')$ such that $e = e'.\iota g'$ and $f = f'.\iota g'$. Again we suppose that the edges e' and f' at w' are resolved along positive paths

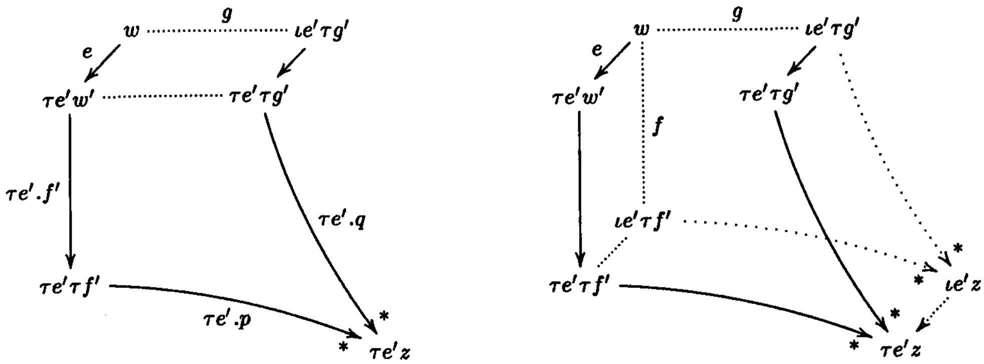

Figure 5.4: Resolution of type 2 overlap

$p = e_1 \ldots e_m$ and $q = f_1 \ldots f_n$ on the boundary of a 2-cell to some vertex z. This overlapping pair can be can be resolved along the following positive homotopy reductions:

$$\begin{aligned} h_1 : \ g \ &\rightsquigarrow \ e(\tau e'.g')(e'.\tau g')^{-1} \\ &\rightsquigarrow \ e(e_1.\iota g')(\tau e_1.g')(e_1.\tau g')^{-1}(e'.\tau g')^{-1} \\ &\rightsquigarrow^* \ e(p.\iota g')(z.g')(p^{-1}.\tau g')(e'.\tau g')^{-1} \end{aligned}$$

and

$$\begin{aligned} h_2 : \ g \ &\rightsquigarrow \ f(\tau f'.g')(f'.\tau g')^{-1} \\ &\rightsquigarrow^* \ f(q.\iota g')(z.g')(q^{-1}.\tau g')(f'.\tau g')^{-1} \\ &\rightsquigarrow^* \ e(pq^{-1}q.\iota g')(z.g')(q^{-1}.\tau g')(e'pq^{-1}.\tau g')^{-1} \\ &\rightsquigarrow^* \ e(p.\iota g')(z.g')(p^{-1}.\tau g')(e'.\tau g')^{-1} \end{aligned}$$

4. **(critical overlap)** Lastly suppose that none of the edges e, f or g are disjoint from the other two. Then we have the pair of homotopy reductions shown

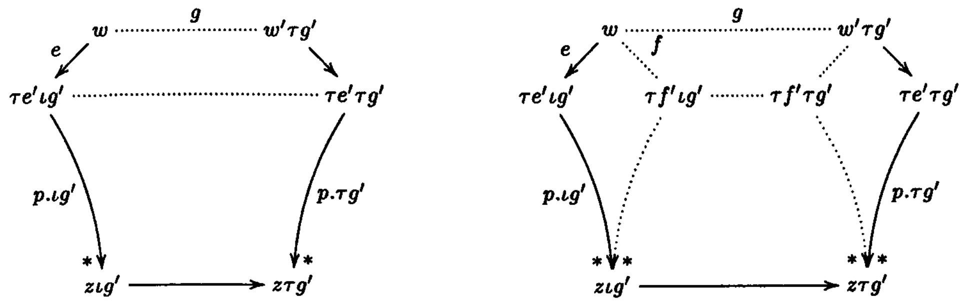

Figure 5.5: Resolution of type 3 overlap

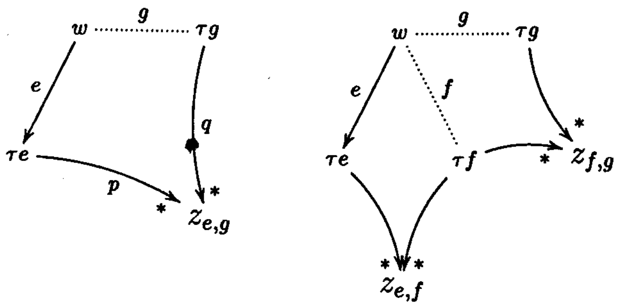

Figure 5.6: Critical overlap

in Figure 5.6 across 2-cells arising from the trivializer $\mathbf{p}$. Since the reduction system is complete this pair can be resolved, and in fact there is a pair of positive homotopy reductions

$$g \rightsquigarrow^* er.$$

where r is the irreducible path which we obtain by performing homotopy reductions on the path pq^{-1}.

Remark 5.2.4 Note that the critical overlaps of this homotopy reduction system correspond precisely to the critical triples (Definition 4.4.2) of the original rewriting system, together with their translates under the two-sided action of F.

5.2.2 Homology

In [37], the critical overlaps which are not translates of others under the two-sided action of F are called the *substantial critical pairs* of the homotopy reduction system. It is shown that in the case when the homotopy reduction system obtained from $\mathbf{p}$ is complete, we can extend the partial resolution (1.9) to dimension 4:

$$\mathbb{Z}S.\mathbf{c} \xrightarrow{\partial_4} \mathbb{Z}S.\mathbf{p} \xrightarrow{\partial_3} \mathbb{Z}S.\mathbf{r} \xrightarrow{\partial_2} \mathbb{Z}S.\mathbf{x} \xrightarrow{\partial_1} \mathbb{Z}S \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0 \tag{5.1}$$

where $\mathbf{c}$ denotes, as above, the set of substantial critical pairs of the homotopy reduction system. ♦

Since the substantial critical pairs correspond to the set of critical triples of the rewriting system, we have the corollary that a monoid with a finite complete rewriting system is of type left FP_4 (and also right FP_4). While this is clearly a weaker result than that given in the Anick-Groves-Squier Theorem (Theorem 1.4.4), it is still interesting in itself: in (5.1) the basis $\mathbf{p}$ corresponds to the critical pairs of the rewriting system, whereas the basis of the free module one dimension higher corresponds to the "critical pairs" of a higher dimensional rewriting system.

We shall not discuss this in detail; we just mention it as an interesting alternative to our method of extending the partial resolution (1.9) to dimension 4 in Chapter 3 by introducing a new 3-complex based on $\mathcal{D}$ and studying its second homology.

Appendix A

Appendix

A.1 Abels' group

Abels' group [1] is a finitely presented group which is not of type FP_3, and so by the Anick-Groves-Squier Theorem (Theorem 1.4.4) it has no finite complete rewriting system. We begin this subsection by showing that one of its subgroups does have a finite complete rewriting system, and we can use this as a base to give a finite λ-complete rewriting system for an HNN-group.

Using the abbreviation $[x, y]$ for the word $x^{-1}y^{-1}xy$,

$$\mathcal{H} = \left\langle \begin{array}{c} a, b, c, d, e \end{array} ; \begin{array}{cccc} [a,e], & b=[a,c], & [a,b], & [c,b] \\ d=[c,e], & [c,d], & [e,d], & [b,d] \end{array} \right\rangle$$

is a presentation for the group of matrices with integer entries and 1's on the diagonal [1]. We can add by a type 1 Tietze transformation the new generator $t = [a, d] = [b, e]$. These generators correspond to matrices with 1's on the diagonal and a 1 in the entry labelled by the corresponding capital letter:

$$\begin{pmatrix} 1 & A & B & T \\ & 1 & C & D \\ & & 1 & E \\ & & & 1 \end{pmatrix}$$

so that, for example, we identify $\bar{b}$ with the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

From this we can derive the rewriting system $\mathcal{P}_{\mathcal{H}}$ for $\mathcal{H}$, consisting of the alphabet

$$a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}, e, e^{-1}, t, t^{-1},$$

together with the following set of rules:

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
b^{\sharp}a^{\flat} & = & a^{\flat}b^{\sharp} & c^{\sharp}b^{\flat} & = & b^{\flat}c^{\sharp} & d^{\sharp}c^{\flat} & = & c^{\flat}d^{\sharp} \\
ca & = & ab^{-1}c & d^{\sharp}b^{\flat} & = & b^{\flat}d^{\sharp} & ec & = & cd^{-1}e \\
c^{-1}a & = & abc^{-1} & eb & = & bt^{-1}e & e^{-1}c & = & cde^{-1} \\
ca^{-1} & = & a^{-1}bc & e^{-1}b & = & bte^{-1} & ec^{-1} & = & c^{-1}de \\
c^{-1}a^{-1} & = & a^{-1}b^{-1}c^{-1} & eb^{-1} & = & b^{-1}te & e^{-1}c^{-1} & = & c^{-1}d^{-1}e^{-1} \\
da & = & at^{-1}d & e^{-1}b^{-1} & = & b^{-1}t^{-1}e^{-1} & t^{\sharp}c^{\flat} & = & c^{\sharp}t^{\flat} \\
d^{-1}a & = & atd^{-1} & t^{\sharp}b^{\flat} & = & b^{\flat}t^{\sharp} & & & \\
da^{-1} & = & a^{-1}td & & & & e^{\sharp}d^{\flat} & = & d^{\flat}e^{\sharp} \\
d^{-1}a^{-1} & = & a^{-1}t^{-1}d^{-1} & & & & d^{\sharp}t^{\flat} & = & t^{\flat}d^{\sharp} \\
e^{\sharp}a^{\flat} & = & a^{\flat}e^{\sharp} & & & & & & \\
t^{\sharp}a^{\flat} & = & a^{\flat}t^{\sharp} & & & & e^{\sharp}t^{\flat} & = & t^{\flat}e^{\sharp}
\end{array}$$

where the superscripts $\sharp$ and $\flat$ may take the values -1 or $+1$ on each side of a rule, together with the set of rules of the form

$$x^{\varepsilon}x^{-\varepsilon} = 1 \qquad (\varepsilon = \pm 1, x \in \{a, b, c, d, e, t\})$$

describing free reductions.

Proposition A.1.1 *The rewriting system $\mathcal{P}_{\mathcal{H}}$ for is complete.*

Proof: The rules have been chosen in order that any word on the generators is rewritten in a finite number of steps to a word of the form

$$a^{\alpha}b^{\beta}c^{\gamma}t^{\tau}d^{\delta}e^{\epsilon} \tag{A.1}$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \tau \in \mathbb{Z}$. To show that it is noetherian we can check that the rules are compatible with recursive path order from the left (Definition 1.1.3) induced by the partial order

$$e^{\pm 1} \rhd d^{\pm 1} \rhd t^{\pm 1} \rhd c^{\pm 1} \rhd b^{\pm 1} \rhd a^{\pm 1}$$

on the alphabet. For example, $ca \succ ab^{-1}c$ holds because $c \rhd a$ and $ca \succ b^{-1}c$, the latter being true because $c \rhd b^{-1}$ and $ca \succ c$, again the latter holding because $a \succ 1$.

The fact that $\mathcal{P}_{\mathcal{H}}$ is complete follows easily by checking that all the critical pairs are resolved. For example caa^{-1} can be rewritten to either c or $ab^{-1}ca^{-1}$, thus forming a critical pair which can be resolved by the path

$$ab^{-1}\underline{ca^{-1}} \to a\underline{b^{-1}a}^{-1}bc \to \underline{aa}^{-1}b^{-1}bc \to \underline{b^{-1}b}c \to c.$$

$\lrcorner$

Remark A.1.2 If we wanted to verify that $\mathcal{P}_{\mathcal{H}}$ is a rewriting system for the group of matrices we are studying we would also need to show that each matrix can be represented by a word of the form (A.1) - this is not difficult to check. The standard presentation of the group of 3×3 upper triangular integral matrices with 1's on the diagonal (the *integral Heisenberg group*) is derived in a similar manner in [33, §5.4].

We want to use $\mathcal{P}_{\mathcal{H}}$ to produce a λ-complete rewriting system by forming an HNN extension.

Lemma A.1.3 *The subgroups $\langle a, b, c^p, e\rangle$ and $\langle a^p, b, c, e\rangle$ of H are isomorphic.*

Proof: Any word on the generators a, b, c^p and e will be rewritten to an irreducible word of the form

$$a^{\alpha}b^{\beta}c^{p\gamma}t^{\tau}d^{p\delta}e^{\epsilon}$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \tau \in \mathbb{Z}$, whereas any word on the generators a^p, b, c and e will be rewritten to an irreducible word of the form

$$a^{p\alpha}b^{\beta}c^{\gamma}t^{\tau}d^{\delta}e^{\epsilon}$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \tau \in \mathbb{Z}$.

It is then straightforward to show that these subgroups are, respectively, the groups of matrices of the form

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & p* & p* \\ & & 1 & * \\ & & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & p* & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}$$

with integers in the entries marked with a $*$. There is an isomorphism

$$\langle a, b, c^p, e\rangle \longrightarrow \langle a^p, b, c, e\rangle$$

given by

$$\begin{pmatrix} 1 & \alpha & \beta & \tau \\ & 1 & p\gamma & p\delta \\ & & 1 & \epsilon \\ & & & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & p\alpha & \beta & \tau \\ & 1 & \gamma & \delta \\ & & 1 & \epsilon \\ & & & 1 \end{pmatrix}$$

again where $\alpha, \beta, \gamma, \delta, \epsilon, \tau \in \mathbb{Z}$. ⌟

We can then form the HNN-extension

$$G = \langle H \mid x\, ;\, x^{-1}ax = a^p,\ [x, b],\ xcx^{-1} = c^p,\ [x, e]\rangle$$

with base group H and stable letter x associating the subgroups $\langle a, b, c^p, e\rangle$ and $\langle a^p, b, c, e\rangle$ according to the isomorphism described in Lemma A.1.3. We can extend $\mathcal{P}_{\mathcal{H}}$ to obtain a rewriting system $\mathcal{P}_G$ for G by adding the letters x, x^{-1}, together with the set of rules

$$\begin{array}{rclrclrcl}
x^{\sharp}x^{-\sharp} & = & 1 & x^{\sharp}b^{\flat} & = & b^{\flat}x^{\sharp} & x^{-1}d^p & = & dx^{-1} \\
 & & & & & & x^{-1}d^{-p} & = & d^{-1}x^{-1} \\
x^{-1}a & = & a^px^{-1} & x^{-1}c^p & = & cx^{-1} & xd & = & d^px \\
x^{-1}a^{-1} & = & a^{-p}x^{-1} & x^{-1}c^{-p} & = & c^{-1}x^{-1} & xd^{-1} & = & d^{-p}x \\
xa^p & = & ax & xc & = & c^px & & & \\
xa^{-p} & = & a^{-1}x & xc^{-1} & = & c^{-p}x & x^{\sharp}e^{\flat} & = & e^{\flat}x^{\sharp} \\
 & & & & & & x^{\sharp}t^{\flat} & = & t^{\flat}x^{\sharp}
\end{array}$$

(again $\sharp$ and $\flat$ may take the values $+1$ or -1).

Proposition A.1.4 *The rewriting system $\mathcal{P}_G$ is λ-complete.*

Proof: First, the rewriting system noetherian, since it is compatible with the recursive path ordering from the left induced by the partial order

$$a^{\pm 1} \lhd b^{\pm 1} \lhd c^{\pm 1} \lhd t^{\pm 1} \lhd d^{\pm 1} \lhd e^{\pm 1} \lhd x^{\pm 1}.$$

Now suppose that w is a non-empty irreducible word such that $\overline{w} = 1$. Since $\mathcal{P}_H$ is complete, we can assume that w contains instances of the letters x, x^{-1}. By Brittons' Lemma (Lemma 2.2.2) w must contain a subword of the form

$$x^{-1} a^{\alpha} b^{\beta} c^{p\gamma} t^{\tau} d^{p\delta} e^{\epsilon} x$$

or of the form

$$x a^{p\alpha} b^{\beta} c^{\gamma} t^{\tau} d^{\delta} e^{\epsilon} x^{-1},$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \tau \in \mathbb{Z}$, and this contradicts the assumption that w is an irreducible word. ⌟

Remark A.1.5 Note that $\mathcal{P}_{\mathcal{G}}$ is not complete. For example, the two single edges

$$\underline{xc}a \to c^p xa \qquad \text{and} \qquad x\underline{ca} \to xab^{-1}c$$

rewrite the word xca to two distinct irreducible words.

Remark A.1.6 Abels [1] has shown that the group with the presentation

$$\mathcal{A} = \langle \mathcal{G} \mid z\,;\; z^{-1}cz = c^p,\; zcz^{-1} = c^p \rangle$$

is not of type FP_3, and therefore by the Anick-Groves-Squier Theorem it has no finite complete rewriting system. On the other hand $\mathcal{A}$ is a presentation of the group of matrices of the form

$$\begin{pmatrix} 1 & * & * & * \\ & * & * & * \\ & & * & * \\ & & & 1 \end{pmatrix}$$

where the entries marked with $*$ take values in the ring of rational numbers with denominator a power of p, and with positive units on the diagonal. As such, the group has solvable word problem.

Exhibiting a finite λ-complete rewriting system for this group would therefore give an example of a group with such a rewriting system but without any finite complete rewriting system, thereby answering Open Question 2.1.8. Unfortunately, $\mathcal{A}$ does not define an HNN-extension with base group the group presented by $\mathcal{G}$, and so we cannot use Britton's Lemma to derive a λ-complete rewriting system for Abels' group using $\mathcal{P}_{\mathcal{G}}$. This statement can be justified by considering the word $w = [a, [ab, z^{-1}cz]]$ which is not equivalent to 1 in $\mathcal{G}$. In $\mathcal{A}$, however, it is equivalent to 1, since because z corresponds to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1/p & 0 \\ & & & 1 \end{pmatrix}$$

w describes the commutation of the matrices

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 1/p & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}.$$

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