Essays on Real Business Cycle Modelling under Adaptive Learning

Bernardo Xavier Fernandez Telleria

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Abstract

The thesis consists on three chapters aiming to contribute to a growing literature on adaptive learning, a form of bounded rationality that has been attracting increasing interest both in the theoretical and practical fields, as an alternative to the commonly used rational expectations hypothesis on how expectations are formed among economic agents.

The first chapter investigates whether it is possible to improve the ability of the standard real business cycle model to fit the main stylised facts of emerging economies, taking the case of Mexico as an illustration, by assuming that agents are not fully rational and instead form expectations according to an adaptive learning rule. Two well-known rules - recursive least squares and its constant gain variant - are considered for this purpose. The degree of difficulty of the learning process is characterised by different starting values of the algorithms as well as different constant gains.

The simulations show that the model under learning generally outperforms its rational expectations counterpart. Therefore, policymakers should take into account the fact that the expected welfare gains/losses of a particular policy reform, conceived assuming a fully-rational environment, might be significantly different if, in practice, agents behave as learners.

Using a heterogeneous-agent model with three types of agents, namely capitalists, skilled workers and unskilled workers - assuming constant population shares suggesting low social mobility -, and allowing for different degrees of complementarity among these within the productive structure, the second chapter welfare-evaluates tax reforms consistent with a lower long run debt-to-output ratio for the United Kingdom, both under rational expectations and heterogenous learning.

It shows that, relative to the other tax reforms, capital tax cuts lead to the highest aggregate welfare but are skill-biased and can thus increase inequality in the long-run. That is, depending on the elasticity of substitution between capital and unskilled labour, falls in the capital tax can result in higher levels of welfare inequality, even in the absence

of other frictions and increases in other forms of taxation. On the other hand, reductions in labour taxes can hurt the capitalists.

This chapter shows too that including the transition period in the welfare evaluation lowers the inequality effects of capital tax reductions since the complementarity between capital and all labour inputs is higher in the short- than in the long-run. Finally, while heterogeneous learning in the shape of differing initial beliefs after the reform can lead to a form of "irrational exuberance" after a tax cut, it can also exacerbate welfare inequality.

Finally, the third chapter presents an heterogeneous-agent model with two types of agents, capitalists and workers - with constant population shares given the strong evidence on low social mobility -, calibrated to Bolivia's data in order to examine the short and long-run effectiveness and distributional effects of various fiscal rules designed to impose restrictions on the evolution of public debt as a share of output, in response to two different sources of exogenous volatility (i.e. productivity and commodity shocks) and under different ways of forming expectations, namely rational expectations and heterogenous learning.

The results show that under full rationality the fiscal rules generate a trade-off between debt-stabilisation and higher income inequality while, under some conditions, heterogenous learning can help to break such trade-off so that some of the rules can perform well in both fields. However, given the significantly high levels of income inequality and dependence on commodity revenues experienced by Bolivia, finding the best performing rule in response to all the relevant exogenous shocks this economy might face, appears to be a challenging task.

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Preface

While the concept of rational expectations is the standard tool of modelling expectations in macroeconomic models, it has been criticised for its high information requirements. In particular, the rational expectations hypothesis is an equilibrium concept which does not focus on how this equilibrium can be reached in case of a large deviation due, for example, to a policy change or any structural shift in general. Hence, in a context where the transition dynamics after such changes are of interest, the rational expectations hypothesis does not take into account the possibility that in practice agents might have no or very little information on the effects of these changes.

As a consequence, there has been an increasing interest in bounded rationality in the recent years. One increasingly popular form of bounded rationality is the adaptive learning hypothesis. While rational expectations assumes that agents know the structure of the economy, the history of all the endogenous and exogenous variables and all the deep parameter values, the adaptive learning hypothesis reduces these information requirements.

In effect, adaptive learning assumes that agents do not know the structure of the economy and the deep parameter values. However, they are endowed with a model of the economy which they take to estimate the latter with the data they observe using some learning algorithm. Under certain conditions, a plausible learning process will ensure convergence towards rational expectations. Thus, although adaptive learning represents a small deviation from rational expectations, the transition dynamics predicted by both approaches can differ substantially, a result that deserves attention.

In light of the above, this thesis aims to contribute to the growing literature on adaptive learning, looking to pay attention to both theoretical and applied aspects of relevance. It focuses mainly on fiscal policy matters, as these have received relatively less attention in the literature. For this purpose, the thesis has been divided in three main chapters, each of them with different theoretical or policy-oriented motivations, but all sharing the same interest in better understanding the consequences of assuming that agents might no be fully rational and behave as learners instead.

The first chapter investigates if it is possible to improve the ability of the standard real business cycle model to fit the data in the case of emerging economies - taking the case of Mexico as an illustration - by assuming that, instead of being fully rational, agents form expectations according to an adaptive learning rule.

The motivation of this investigation relies in the fact that while it is well-documented that the simple RBC model and its main extensions have been successful in replicating most of the key macroeconomic stylized facts of the US and other developed economies, their performance has been rather poor in the case of emerging economies

The results of this analysis are quite relevant for policy-design purposes in the sense that the expected welfare gains/losses of a particular policy reform conceived assuming a fully rational environment might be significantly different if, in practice, agents behave as learners. This point is also illustrated in this chapter by comparing the effects of two tax reforms.

The second chapter seeks to examine the distributional consequences of a variety of tax reforms in a context of structural and learning heterogeneity among agents, and a productive sector characterised by exhibiting different degrees of complementarity between capital, skilled and unskilled labour. To isolate the effects of changes in each tax rate on all agents, this work considers changes in tax rates that are not revenue neutral. Instead, given its current policy relevance, tax reforms consistent with a lower steady-state debt-to-output ratio are taken into consideration.

For illustrative purposes, the proposed model is calibrated to the UK economy, with the aim of obtaining a realistic assessment of the likely costs and benefits of tax reforms for the different agents.

As said earlier, two types of heterogeneity are considered in this chapter. First, the model's structural heterogeneity in terms of income and savings is generated by means of financial transaction costs which differ substantially between three types of agents - capitalists, skilled workers and unskilled workers. According to relevant evidence suggesting low social mobility in the UK, the model assumes constant population shares with respect to the three types of agents considered.

Second, learning heterogeneity takes the form of differing initial beliefs among those different agents who form expectations and need to learn their equilibrium laws of motion.

This corresponds to an unequal distribution of information right after the tax reform in the economy, an element which has not yet been considered in the tax reform literature.

The results of this study have important implications for fiscal policy design as these help to conclude that tax reforms should be accompanied by a careful evaluation of the production structure in the economy and should also consider the robustness of the results to different plausible ways of forming expectations among agents, in order to clearly identify the groups that are mostly likely to see their returns reduced due to the reform so that appropriate redistributive policies can also be considered.

Finally, the third chapter examines the short and long-run effectiveness and distributional effects of applying various fiscal rules designed to impose restrictions on the evolution of public debt, in response to different sources of exogenous volatility (i.e. productivity and commodity shocks) and under different ways of forming expectations among agents, namely rational expectations and heterogenous learning.

For illustrative purposes, the chapter considers the particular case of Bolivia, an economy that has suffered a long history of severe debt-crisis episodes, triggered both by fiscal and monetary mismanagement, and that currently experiences significantly high levels of income inequality and economic dependence on commodity (natural gas) exports.

For this purpose, a closed-economy stochastic general equilibrium model with structural and learning heterogeneity is considered. As before, to capture key features of wealth and income inequality in the economy, financial intermediation costs which differ substantially between two types of agents, capitalists and workers, are included in a way such that the latter represent a significantly large proportion of the population but have very limited access to the financial markets. As in the previous chapter, strong evidence suggesting very low social mobility in Bolivia and Latin America in general, leads to assume constant population shares in term of the two types of agents considered.

Meanwhile, learning heterogeneity takes the form of differing initial beliefs between agent types whereby agents with limited access to the financial markets tend to exhibit off-equilibrium initial expectations which, in turn, affect their learning process and thus can have important consequences on the performance of the fiscal rules in terms of their debt-stabilisation properties and their effects on income distribution.

The intended contribution of this chapter in terms of fiscal policy design is thus twofold. First, it investigates the impact of debt-stabilising fiscal rules on income distribution to identify possible trade-offs between efficient debt-stabilisation and a higher income inequality. Moreover, it explores the role learning might have with respect to the aforementioned trade-off (i.e. if it helps to break the trade-off or if, conversely, it strengthens it), something that has received little attention in the literature on fiscal policy. Second, with the elements discussed above, the chapter explores the possibilities that might be effectively available to an economy such as Bolivia in its quest for an efficient debt-stabilisation mechanism which does not compromise (or even improves) income distribution, within a context of extreme income and wealth inequality as well as of high dependence on commodity revenues.

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Declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Signature:

Printed name: Bernardo X. Fernandez T.

Chapter 1

Business cycles and learning: the case of emerging economies

1.1 Introduction

Real business cycle (RBC) modelling and its various extensions under the dynamic general equilibrium (DGE) modelling literature are currently the workhorse of modern macroeconomic analysis and its main applications.[1] As many authors point out (see e.g. McGrattan, 1994 and 2006, and Kremer *et al.*, 2006), the current relevance of this approach has much to to thank to the earlier success of simple RBC setups in replicating some of the key macroeconomic stylized facts of the US and other developed economies, while a large number of extensions have helped to further improve these earlier results in several important dimensions, a process that ultimately has promoted the intensive use of DGE modelling in the design and evaluation of fiscal and monetary policy by their relevant economic authorities.

On the other hand, it is widely known that RBC modelling has been much less successful in replicating key stylized facts in emerging economies, as illustrated by Bergoeing and Soto (2002) for Chile, Ellery *et al.* (2002) for Brazil and Garcia-Cicco *et al.* (2006) for Argentina. In particular, RBC models have not been able to replicate key features that are common across these type of economies, such as: a) consumption is more volatile than output; b) investment is significantly (i.e. above four times) more volatile than output;

[1] I would like to thank Charles Nolan for helpful comments to an earlier version of this Chapter.

c) hours worked are more volatile than wages, and d) hours worked and real wages show little contemporaneous correlation. Remarkably, despite their well-documented poor fit to the data, a number of well-known extensions of the RBC model are being adopted by policy-makers in an increasing number of emerging countries (see e.g. Tovar, 2009, and Florian and Montoro, 2009).

In response to this concern, some extensions to the standard RBC model have been proposed over the last decade to improve its overall performance when applied to emerging economies. Most of this research has focused on some particularities often observed in these economies that tend to affect the behavior of their main economic aggregates. For example, Ellis and Fender (2003) suggest that taking account of the costs of corruption in the public sector helps to better explain the transition experiences of several Eastern European economies; Arias and Ardila (2003) show that including military expenditure and the costs of an internal conflict in the model leads to a better description of the economic behavior of Colombia; while Castillo *et al.* (2006) demonstrate that the ability of a DGE model to fit the data for Peru is greatly improved by acknowledging the high levels of dollarization observed in this country.

Looking for more general results on this area, Aguiar and Gopinath (2007) concentrate their attention on the frequent policy regime switches observed in emerging markets and then show that introducing trend shocks along with the typical cycle shocks to total factor productivity (TFP) can account for key features of economic fluctuations in these economies such as the high volatility of consumption and the occurrence of sharp current-account reversals or *sudden stops*. However, Garcia-Cicco, *et al.* (2006) note that for these results to hold a high variability of trend shocks relative to transitory shocks is necessary, a feature that has no robust empirical support when long time series are employed, as demonstrated by them taking the example of Argentina.

More recently, Mendoza (2008) developed a DGE model including frictions in the financial (credit) market in the shape of an endogenous collateral constraint which greatly improved the performance of the model calibrated for Mexico and also helped to generate dynamics consistent with key features of the *sudden stop* episode that hit this country in 1995. Finally, following yet another line of work, Angelopoulos *et al.* (2012) show -

with Mexico as the chosen case study too - that introducing weak property rights derived from major changes in the quality of institutions (e.g. privatization, regulation of the banking system and bankruptcy laws) as an additional source of uncertainty can improve the performance of the standard RBC model.

A common denominator of the literature cited above is the use of the rational expectations (RE) hypothesis which states that, on average, economic agents do not make systemic mistakes in forecasting the future and thus deviations from *perfect foresight* are only random. This, in turn, implies that agents should fully know the structure of the economy, the values of the structural parameters as well as the distribution of any exogenous shocks. As many authors have been increasingly suggesting over the last decade (see e.g. Turnovsky, 2000), such assumption appears to be rather strong, as it ignores the fact that in many cases agents behave in a less rational fashion mostly due to information restrictions of diverse nature that tend to persist for a number of periods and thus might generate temporary deviations from the equilibrium consistent with full rationality.

In light of the above, the aim of this document is to examine whether the overall performance of the standard RBC model can be improved in the case of emerging economics by assuming that, instead of being fully rational, agents in fact behave according to the postulates of the so-called adaptive learning paradigm (see e.g. Marcet and Sargent, 1989, and Evans and Honkapohja, 2001). Adaptive learning (AL) is a form of *bounded rationality* (see Sargent, 1993) which hypotheses that economic agents do not fully know the economic environment in which they interact and that, to overcome this lack of knowledge, they estimate or *learn* a reduced form of this environment in an adaptive and recursive fashion. More specifically, they act as econometricians by recursively running regressions on realized past data to estimate a set of structural parameters required to form expectations about a set of relevant variables.[2]

[2] Note that adaptive learning is only one among several alternative approaches considered in the literature in order to better model expectations. Blume and Easley (2000) postulate an approach called *rational learning,* which suggests that the learning behavior derives from preferences in the sense that if a market participant is an expected utility maximiser, then her beliefs must be revised in light of new information according to Bayes rule. Evans (2001) then proposed a so-called *eductive learning* approach by which learning takes place in mental, not real, time and thus implies more strict conditions for convergence to the fully rational equilibirum than under adaptive learning.

More recently, Adam and Marcet (2011), combining key elements of rational and adaptive learning, proposed a more general approach which assumes agents are internally rational - i.e. they maximize

Even though AL represents a small deviation from RE, as pointed out by Marcet and Nicolini (2003) and also as shown below, the short and medium run dynamics predicted by the model under this assumption can be quite different from their RE counterparts, with the former often showing higher levels of volatility in some key variables such as consumption and labour supply whenever the agents face more severe restrictions in their learning processes and thus remain far away from the RE solution for longer periods of time. An appropriate setup of these especial features might then help the RBC model to improve its performance in terms of better replicating the aforementioned key stylized facts observed in emerging economies.

The version of the RBC model considered in this work is similar to the one proposed by Ireland (2004), with indivisible labour but including a government that has expenditures that are financed by levying taxes on capital and labour income and runs a balanced budget in every period, as in Giannitsarou (2006).

For illustrative purposes, the model is calibrated to Mexican annual data for the period between 1993 and 2005. This choice rests in three key criteria, First, it allows to remain close to most of the relevant literature on this topic (see e.g. Aguiar and Gopinath, 2007, Mendoza, 2008, Boz *et al.*, 2008, and Angelopoulos *et al*, 2012, all of them considering the Mexican case). Second, as a member of the OECD, macroeconomic data for Mexico are available for a reasonably long period of time, something that is not common among most of emerging economies. Third, Mexico's recent economic history, where the debt crisis of the mid-eighties and the so-called "tequila" crisis of 1994 stand out as key episodes in which economic agents probably were not able to behave in a fully-rational fashion, provides an interesting opportunity to consider alternative ways of forming expectations, such as AL, and see how these might affect the performance of the model in terms of fitting the data.

Precisely, in terms of the AL setup, two very well-known algorithms in the AL literature

discounted expected utility under uncertainty given dynamically consistent subjective beliefs about the future -, but might not be externally rational - i.e. they may not know the true stochastic process of relevant variables beyond their control -, which amounts to relaxing the 'prior beliefs' that agents are assumed to hold under full-rationality. Besides providing the adaptive learning literature with more adequate microeconomic foundations, the authors also show that some of the modeling choices in it - e.g. the chosen reduced-form of the model, the assumptions about the learning rules - are less ad-hoc than might initially appear.

are considered: recursive least squares (RLS) and its corresponding constant gain variant (RLS-CG).[3] Moreover, especial emphasis in this document is given to the two key features in any learning setup which ultimately determine how different the predictions of the model under AL are compared to the case when RE is assumed. These are a) the initial or starting values of the learning algorithm and b) the level of the *constant gain,* or relative weight given to the latest forecast error taken into account in the learning algorithm to update the latest estimates of the structural parameters of interest.

The first of these elements is relevant for both learning algorithms and essentially tries to describe how much preliminary information is held by the agents just before they start the actual learning process. The second feature is only relevant when RLS-CG is considered and basically describes the sensitivity of the learning process to the latest forecast error associated with the previous period parameter estimates generated by the algorithm. Hence, if the constant gain is relatively high, the learning algorithm will produce sharp changes in the new parameter estimates even if these had almost converged to their RE values.

Given the particular economic history of Mexico during the period under analysis, the goodness of fit of the RBC model at hand - by means of standard second-moment matching exercises - will be evaluated under two different scenarios that can have a serious impact in the way agents learn about the state of the economy and, as a result, in their consumption and labour supply decisions over time. In the first and most commonly used scenario, the economy is assumed to oscillate around its steady state due to random shocks to technology. In the second, the economy is assumed to face a major recession but the government does not intervene so that the economy must naturally return to its steady state level.

It is important to note that this document follows a similar line of research as Carceles-Poveda and Giannitsarou (2007), Eusepi and Preston (2011) and Huang *et al.* (2009), with the key difference that these authors evaluate the goodness of fit of the RBC under AL for the case of the US only. It is also related to the work of Boz *et al.* (2008), who show

[3] See e.g. Evans and Honkapohja (2001) and Carceles-Poveda and Giannitsarou (2007) for a detailed discussion of these algorithms.

that the performance of the RBC model can be improved for emerging economies if it is assumed that agents know the structure of the economy and the associated structural parameters but they need to learn about the nature of a shock on total factor productivity (TFP). By contrast, in this work it is assumed that agents are interested in learning about the entire structure of the economy but also about the actual impact of the TFP shocks on the economy, given the latter.[4]

The findings of this chapter show that the RBC model under AL generally outperforms its RE counterpart in terms of matching the data for Mexico. Therefore, economic policy design must take into account the fact that the expected welfare gains/losses of a particular policy reform conceived within a RE environment might be significantly different if agents behave as learners in practice. This point is illustrated by examining the effects of two unexpected tax reforms for Mexico both under RE and AL.

1.2 The RBC model

The basic closed-economy RBC model considered in this work is similar to that used by Ireland (2004), but including a simple public sector as in Giannitsarou (2006) so that the impact on welfare of tax reforms can be examined.[5] The main elements of the model are described next.

[4]In this sense, the approach considered here "nests" the one followed by Boz et .al. (2011). Another difference from Boz *et.al.* (2011) is that they assume learning takes place by means of a more general Kalman filter.

[5]This model setup remains quite close to most of the literature discussing the goodness of fit properties of the basic RBC model and its extensions both under rational expectations and adaptive learning. In these, a closed economy is assumed given that the key domestic features of the economies examined are of particular interest while, in the long run, the external sector is assumed to be in equilbrium. See, *inter alia*, King and Rebelo (1999), Ireland (2004), McGrattan (2006) and Angelpoulos et al. (2011) for research in this field assuming rational expectations. Also, see Carceles-Poveda and Giannitsarou (2007) and Huang et al. (2009) for a similar discussion under adaptive learning.

In the particular case of Mexico, due to its significant oil production and exports, trade - defined as exports plus imports as a share of GDP - has behaved in a fairly steady fashion between 1990 to 2005, remaining consistently close to its average for the period, of 58% of GDP, even in the years of economic downturn. Similarly, this country's trade balance deficit during the same period has remained virtually constant at 1% of GDP (source: World Bank Database).

1.2.1 Households

It is first assumed that the single-good economy is composed of a time-invariant number N of identical and infinitely-lived individuals who obtain utility out of consumption and leisure. Hence, every individual j will maximize the following expected utility function:

$$U^j = E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln C_t^j - \gamma H_t^j \right], \tag{1.1}$$

where C_t^j and H_t^j are agent $j's$ consumption and hours worked, respectively. The parameter β is the subjective discount factor. The parameter for labour satisfies $\gamma > 0$ and is consistent with the notion of labour indivisibility proposed by Hansen (1985). That is, $\gamma = -\left[\ln(1 - H_0)\right]/H_0$ where H_0 is the amount of indivisible labour that an agent contracts to provide if she is one of the agents randomly chosen to provide labour. Hence, the ratio H_t^j/H_0 is the probability that a given agent will be providing labour in period t. Also, according to the agreed contract, every agent will always get paid the competitive wage even if she does not work in a given period, an assumption that resembles a type of unemployment insurance which is required here to preserve the desired properties of the utility function (i.e. continuous and differentiable).

Each agent j faces two constraints in every period. The first is the capital accumulation rule:

$$K_t^j = (1 - \delta) K_{t-1}^j + I_t^j, \tag{1.2}$$

where $0 < \delta < 1$ is the depreciation rate, and the second one is the budget constraint given by:

$$W_t H_t^j + R_t K_{t-1}^j + \Pi_t^j - \tau_k \left(R_t - \delta\right) K_{t-1}^j - \tau_h W_t H_t^j = C_t^j + I_t^j, \tag{1.3}$$

where W_t and R_t are the factor rentals determined in their respective competitive markets and thus are taken as given by each individual and Π_t^j represents profits. The tax rate on capital gains is τ_k and the tax rate on labour income is τ_h. This constraint states that each agent's total expenditure can not exceed her income, which is given by the net of taxes remuneration to the inputs she provides plus the profits made by the firms she owns. However, as shown below, it is assumed that firms operate in a perfectly competitive environment and thus no economic profits are made, $\Pi_t^j = 0$.

1.2.2 Firms

To facilitate further manipulations in the following sections, it will be assumed that every firm's output, Y_t^j, depends on the capital stock accumulated until the previous period K_{t-1}^j, the amount of labour H_t^j and a technology shock Z_t, according to a constant-returns to scale Cobb-Douglas function:

$$Y_t^j = Z_t K_{t-1}^{j^\theta} \left(\eta^t H_t^j\right)^{1-\theta}, \tag{1.4}$$

where $\eta > 1$ is the gross rate of labour-augmenting technological progress and $0 < \theta < 1$ is the capital's share of income.[6] The technology shock is available to all firms and follows a first-order autoregressive process of the form:

$$\ln Z_t = (1-\rho)\ln Z + \rho \ln Z_{t-1} + \varepsilon_t, \tag{1.5}$$

where Z is a positive constant, $0 < \rho < 1$ is the autocorrelation coefficient and the random shocks ε_t are normally distributed.

The factor markets are assumed to be perfectly competitive so that both the wage and the capital rental are determined by their respective markets and thus must be taken as given by each individual firm. This means that the profit-maximization behavior of any firm will be given by maximising:

$$\Pi_t^j = Z_t K_{t-1}^{j^\theta} \left(\eta^t H_t^j\right)^{1-\theta} - W_t H_t^j - R_t K_{t-1}^j, \tag{1.6}$$

[6] Most studies define the production in the form $Y_t = f(Z_t, K_t, H_t)$ where K_t represents the capital stock at the the beggining of period t and therefore is taken as given. Here, the function used is of the form $Y_t = f(Z_t, K_{t-1}, H_t)$ where K_{t-1} represents the capital stock at the the end of period $t-1$ so that both specifications are equivalent.

with respect to capital and labour, a procedure that yields the first-order conditions:

$$R_t = \theta Z_t K_{t-1}^{j\,\theta-1} \left(\eta^t H_t^j\right)^{1-\theta}, \tag{1.7}$$

and

$$W_t = (1-\theta) Z_t K_{t-1}^{j\,\theta} \eta^{(1-\theta)t} H_t^{j\,-\theta}, \tag{1.8}$$

which simply state that the optimal amounts of capital and labour to be hired by each firm are such that the marginal products of these inputs equal their respective factor rentals R_t and W_t. Note that, according to (1.4), these two conditions can also be written as:

$$R_t = \theta \frac{Y_t^j}{K_{t-1}^j} \quad \text{and} \quad W_t = (1-\theta) \frac{Y_t^j}{H_t^j} \; .$$

1.2.3 Government

To remain close to the standard RBC model, following e.g. Giannitsarou (2006), it is assumed that the government expenditure takes the form of transfers to private agents, G_t, defined endogenously as they are financed through taxes on labour income and capital gains. That is, G_t is defined according to a balanced-budget rule of the form:

$$G_t = \tau_k \left(R_t - \delta\right) K_{t-1} + \tau_h W_t H_t \tag{1.9}$$

where K_{t-1} is aggregate capital stock (i.e. $K_{t-1} = \sum_{j=1}^{N} K_{t-1}^j$) and H_t is aggregate labour (i.e. $H_t = \sum_{j=1}^{N} H_t^j$). Policy parameters τ_k and τ_h are the tax rates on capital and labour income, respectively, which are assumed to be constant and thus will be set at their average values for the period between 1980 and 2005 in the calibration procedure below.

It is also worth noting that under the model setup considered in this chapter, where G_t is not included in the utility function (see equation (1.1)), this variable is often interpreted as representing public goods or transfers which have not direct impact on the agent's utility but are still necessary for the whole economy to fully operate (see e.g. Marattin and Palestini, 2012).[7]

[7] This model setup regarding on the so-called *wastefulness* of government spending has been criticised

1.2.4 Characterization of the equilibrium allocations

Given the model setup above, the general problem for an agent j at time 0 can be expressed as:

$$\max_{\{C_t^j, K_t^j, H_t^j\}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln C_t^j - \gamma H_t^j\right]. \tag{1.10}$$

subject to the budget constraint - combining (1.2) and (1.3) -:

$$(1-\tau_h) W_t H_t^j + (1-\tau_k)(R_t - \delta) K_{t-1}^j = C_t^j + K_t^j - K_{t-1}^j, \tag{1.11}$$

plus the stochastic process for technology:

$$\ln Z_t = (1-\rho) \ln Z + \rho \ln Z_{t-1} + \varepsilon_t, \tag{1.12}$$

with the aggregate outcomes W_t and R_t and the initial value of capital taken as given.

The Lagrangian representation of this maximization problem is:

$$\begin{aligned} \Lambda \;=\; & E_0 \sum_{t=0}^{\infty} \{\beta^t [\ln C_t^j - \gamma H_t^j + \lambda_t^j ((1-\tau_h) W_t H_t^j + \\ & + (1-\tau_k)(R_t - \delta) K_{t-1}^j - C_t^j - K_t^j + K_{t-1}^j)]\}, \end{aligned} \tag{1.13}$$

and the optimal conditions with respect to C_t^j, K_t^j and H_t^j are:

$$\lambda_t^j = \frac{1}{C_t^j}, \tag{1.14}$$

$$\lambda_t^j = \beta \lambda_{t+1}^j \left[(1-\tau_k)(R_{t+1} - \delta) + 1\right], \tag{1.15}$$

and

$$\gamma = \lambda_t^j (1-\tau_h) W_t, \tag{1.16}$$

for not being able to replicate the values of the fiscal policy multipliers seen in practice (see e.g. Marattin and Marzo, 2010). However, Marattin and Palestini (2012) show that including the government spending into the utility function does not unambiguously solve this issue and thus a much more complex analysis is needed, which is beyond the scope of this work.

respectively. Now, substituting λ_t^j from (1.14) into (1.15) yields the Euler condition:

$$1 = \beta E_t \left[\left(\frac{C_{t+1}^j}{C_t^j} \right)^{-1} [(1-\tau_k)(R_{t+1}-\delta)+1] \right], \tag{1.17}$$

which equates the marginal rate of substitution of consumption between period t and $t+1$ and the corresponding marginal rate of transformation given by the available technology. Likewise, substituting λ_t^j from (1.14) into (1.16) gives:

$$\gamma C_t^j = (1-\tau_h)W_t, \tag{1.18}$$

which states that the utility loss derived from providing a unit of labour must equal the utility gains of an additional unit of consumption purchased with the income earned from providing that extra unit of labour.

1.2.5 Aggregated version of the model

Next, since all the agents are assumed to be identical, it is possible to sum their behavior up and find the aggregate variables of this economy. The aggregate value of any individual variable X^j (with X=$\{C, H, Y, K, I\}$) in any given period will be $\sum_{j=1}^N X^j$, which is assumed to be exogenous and constant. Since all the identical agents will consume exactly the same in each period, it follows that $\sum_{j=1}^N X_t^j = NX_t^j = X_t$. Moreover, since N can be thought as a population index that goes from 0 to 1, then the average value of any variable - i.e. X_t/N - will be simply equal to X_t. Therefore, the aggregate version and the representative-agent version of the model will be equivalent.

Imposing the described aggregation rules into the first-order condition given by (1.17) and using the fact that $R_{t+1} = \theta \frac{Y_{t+1}}{K_t}$ according to the firms' profit maximization problem, gives:

$$1 = \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-1} \left(\left(\theta \frac{Y_{t+1}}{K_t} - \delta \right) (1-\tau_k) + 1 \right) \right]. \tag{1.19}$$

Similarly, applying the aggregation rule into the second first-order condition (1.18) and recalling that $W_t = (1-\theta)\frac{Y_t}{H_t}$, yields:

$$\gamma C_t = (1-\tau_h)\frac{(1-\theta)Y_t}{H_t}. \tag{1.20}$$

The aggregate version of the individual budget constraint (1.11) is:

$$(1-\tau_k)(R_t-\delta)K_{t-1} + (1-\tau_h)W_tH_t = C_t + K_t - K_{t-1},$$

but given that the aggregate production function is homogeneous of degree one so that $Y_t = R_tK_{t-1} + W_tH_t$ and considering the aggregate version of the law of capital accumulation $K_t = (1-\delta)K_{t-1} + I_t$, this constraint can be written as:

$$Y_t = C_t + I_t + \tau_k(R_t-\delta)K_{t-1} + \tau_hW_tH_t. \tag{1.21}$$

where the last two terms in the r.h.s. are equal to G_t according the government's budget constraint, (1.9).

1.2.6 Competitive equilibrium

The competitive equilibrium (CE) is summarised by the following system of five equations:

$$Y_t = Z_tK_{t-1}^{\theta}\left(\eta^tH_t\right)^{1-\theta}, \tag{1.22}$$

$$Y_t = C_t + I_t + \tau_k(R_t-\delta)K_{t-1} + \tau_hW_tH_t, \tag{1.23}$$

$$K_t = (1-\delta)K_{t-1} + I_t, \tag{1.24}$$

$$1 = \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-1}\left(\left[\theta\frac{Y_{t+1}}{K_t}-\delta\right](1-\tau_k)+1\right)\right], \tag{1.25}$$

and

$$\gamma C_t = (1-\tau_h)\frac{(1-\theta)Y_t}{H_t}. \tag{1.26}$$

in the paths of the five variables C_t, H_t , K_t, Y_t, and I_t, given the initial values of capital and TFP, and given the exogenously set stationary AR process for the latter:

$$\ln Z_t = (1-\rho)\ln Z + \rho \ln Z_{t-1} + \varepsilon_t, \tag{1.27}$$

1.2.7 Suitable transformations of the CE

The stationary system

The presence of η^t in the system above implies the existence of a trend in the behavior of the variables, and thus their steady-state values cannot be found. To overcome this, the following stationary variables are defined: $c_t = C_t/\eta^t$, $h_t = H_t$, $k_t = K_t/\eta^t$, $y_t = Y_t/\eta^t$, $i_t = I_t/\eta^t$, $w_t = W_t/\eta^t$, $r_t = R_t$, $g_t = G_t/\eta^t$ and $z_t = Z_t$. Applying these definitions gives the following stationary system:[8]

$$\eta^{\theta} y_t = z_t \eta^{-\theta} k_{t-1}^{\theta} h_t^{1-\theta}, \tag{1.28}$$

$$\ln z_t = (1-\rho)\ln Z + \rho \ln z_{t-1} + \varepsilon_t, \tag{1.29}$$

$$y_t = c_t + i_t + \tau_k \left(r_t - \delta\right) \eta^{-1} k_{t-1} + \tau_h w_t h_t \tag{1.30}$$

$$k_t = (1-\delta)\eta^{-1} k_{t-1} + i_t, \tag{1.31}$$

$$1 = \beta E_t \left[\left(\frac{c_t}{\eta c_{t+1}} \right) \left(\left[\theta \frac{\eta y_{t+1}}{k_t} - \delta \right] (1-\tau_k) + 1 \right) \right], \tag{1.32}$$

and

$$\gamma c_t h_t = (1-\tau_h)(1-\theta) y_t. \tag{1.33}$$

The analytical steady-state solution

In the absence of shocks, this economy converges to a steady state where $x_t = x$ for any $x = \{y, c, i, k, z, h, r, w, g\}$ and for all t. Defining $\chi = \frac{\eta}{\beta} - 1 + \delta(1-\tau_k)$ and $\zeta =$

[8] A detailed derivation of this system is provided in Appendix A.

$(\eta - 1 + \delta)(1 - \tau_k)$ for convenience, the steady state solutions of the variables of the model are:[9]

$$k = \left(\frac{\theta\eta(1-\tau_k)}{\chi}\right) y, \tag{1.34}$$

$$i = \left(\frac{\theta\zeta}{\chi}\right) y \tag{1.35}$$

$$c = \left[1 - \left(\frac{\theta\zeta}{\chi}\right) - \tau_k (r - \delta) \left(\frac{\theta\eta(1-\tau_k)}{\chi}\right) - \tau_h(1 - \theta)\right] y, \tag{1.36}$$

$$h = \frac{(1-\tau_h)(1-\theta)}{\gamma\left[1-\left(\frac{\theta\zeta}{\chi}\right)-\tau_k(r-\delta)\left(\frac{\theta\eta(1-\tau_k)}{\chi}\right)-\tau_h(1-\theta)\right]}, \tag{1.37}$$

and

$$y = Z\frac{^{\frac{1}{1-\theta}}\left(\frac{\theta(1-\tau_k)}{\chi}\right)^{\frac{\theta}{1-\theta}}\frac{(1-\tau_h)(1-\theta)}{\gamma}}{\left[1 - \left(\frac{\theta\zeta}{\chi}\right) - \tau_k (r - \delta) \left(\frac{\theta\eta(1-\tau_k)}{\chi}\right) - \tau_h(1 - \theta)\right]} \tag{1.38}$$

while from (1.29) it follows that the steady state value of z will be equal to the constant parameter Z.

Additionally, the steady-state values of the factor prices are:

$$r = \frac{\chi}{(1 - \tau_k)}, \tag{1.39}$$

and

$$w = (1 - \theta)Z^{\frac{1}{1-\theta}} \left(\frac{\theta(1-\tau_k)}{\chi}\right)^{\frac{\theta}{1-\theta}}, \tag{1.40}$$

while, finally, the steady state value of government expenditure is given by:

$$g = \tau_k (r - \delta) \eta^{-1} k + \tau_h w h. \tag{1.41}$$

The log-linear version of the system

To solve the model, first the first-order Taylor series expansion of the DCE around their steady state values are taken. For any variable $\hat{x} = \{\hat{y}, \hat{c}, \hat{h}, \hat{\imath}, \hat{k}, \hat{z}\}$ the log-linear values

[9] Detailed derivations of these solutions are provided in Appendix A.

are defined as $\hat{x}_t = \ln x_t/x$, and the log-linearised system is:[10]

$$\hat{y}_t = \hat{z}_t + \theta\hat{k}_{t-1} + (1-\theta)\,\hat{h}_t, \tag{1.42}$$

$$\hat{z}_t = \rho\hat{z}_{t-1} + \varepsilon_t, \tag{1.43}$$

$$\kappa\hat{y}_t = \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{\imath}_t - \left(\tau_k\eta^{-1}\tfrac{k}{y}\delta\right)\hat{k}_{t-1}, \tag{1.44}$$

$$\eta\hat{k}_t = (1-\delta)\hat{k}_{t-1} + \varphi\hat{\imath}_t, \tag{1.45}$$

$$\tfrac{\eta}{\beta}E_t\hat{c}_{t+1} - \tfrac{\eta}{\beta}\hat{c}_t = (1-\tau_k)rE_t\hat{y}_{t+1} - (1-\tau_k)r\hat{k}_t, \tag{1.46}$$

and

$$\hat{h}_t + \hat{c}_t = \hat{y}_t. \tag{1.47}$$

where $\kappa = \left(1 - \frac{\tau_k\eta^{-1}kr+\tau_h wh}{y}\right)^{-1}$ and $\varphi = \eta - 1 + \delta$.

1.3 Calibration and steady-state values

1.3.1 Calibration

Table 1.1 summarizes the calibration exercise of the model consistent with annual data for Mexico for the period between 1980 and 2005. It is important to highlight that the main interest of this section is to define parameter values which are in line with typical calibration exercises in the RBC literature, so that the effects of the adaptive learning assumption can be properly isolated.

The subjective discount factor β is assumed to be equal to 0.885, consistent with an annual discount rate of $12-13\%$, which is in line with the average interest rates observed in the Mexican credit market during the period of analysis.[11] It is reasonable to find such high interest rates in emerging economies, since they are often characterised by higher volatility in the economic activity, lower investment and shorter time horizons compared to developed economies.

[10] A detailed derivation of this system is presented in Appendix A.

[11] See http://www.banxico.org.mx/portal_disf/wwwProyectoInternetTasas.jsp

Table 1.1: Parameter values

parameter	value	description
$0 < \beta < 1$	0.885	discount factor
$\delta > 0$	0.06	depreciation rate
$\eta > 1$	1.025	labour augmenting growth rate
$\gamma > 0$	2.06	indivisible labour coefficient
$0 < \theta < 1$	0.40	capital's share of income
$\tau_h \geq 0$	0.13	labour income tax
$\tau_k \geq 0$	0.09	capital income tax
$Z > 0$	2.00	productivity constant
$0 < \rho < 1$	0.914	AR(1) coefficient productivity
$\sigma_\varepsilon > 0$	0.025	std. deviation productivity shock

Following Angelopoulos *et al.* (2012), the growth rate of labour augmenting technology is set equal to 1.025, the gross annual rate of growth of output for the US, for the aforementioned period. This seems a reasonable choice since the US is Mexico's most important trade and financial partner and, accordingly, there is solid empirical evidence suggesting these two countries share a common trend (see e.g. Herrera, 2004). The depreciation rate is set at 6% as in Kehoe and Rhul (2009), while the capital's share in income is set to 40% as computed by Garcia-Verdu (2005) and also in the same line as Kydland and Zarazaga (2002), Bergoeing and Soto (2002) and Lubker (2007).

The effective labour and capital income taxes are taken from the calculations of Anton (2005) following the methodology proposed by Mendoza *et al.* (1994) and thus are set to their 1993 to 2001 averages of 13% and 9%, respectively. Finally, γ has been chosen so that the model can replicate the rather intuitive notion that agents spend one third of their available time working,[12] while the scale parameter Z has been set at a value providing easy-to-read steady-state values. Given the model setup, however, these two last parameters have no effect over the predicted dynamics of the model.

To conclude the calibration procedure, it is worth to note that most of the economic

[12] Angelopoulos *et al.*.(2012) compute an average of 0.467 hours worked for the period between 1991 and 2005 for Mexico, while Kanczuk (2004) obtains an average of 0.33 for Brazil between 1980 and 2001, and Bergoeing and Soto (2002) an average of 0.43 for Chile between 1986 and 2001. Given these different values for different fairly representative emerging economies and, since in the model setup γ has no impact on the dynamics of the model nor in the behavior of the predicted key ratios, it has been chosen such that h is set at its rather most intuitve value of 0.33.

modelling literature for emerging economies agree that TFP in these tends to show lower persistence but higher volatility in response to random exogenous shocks, whereas in developed economies TFP shows the opposite features. In line with this notion, the TFP estimates of Angelopoulos *et al.* (2012) are taken, so that ρ is set equal to 0.941 and σ_ε equal to 0.025.

1.3.2 Steady-state

Given the calibration above, the resulting steady-state values of the variables of interest are exhibited in Table 1.2. The bottom rows of the table show that the chosen parameters produce reasonable values for the key ratios usually considered to examine the feasibility of calibration exercises. With the exception of the c/y ratio which is slightly higher than the $0.7 - 0.65$ historical average, the other three key ratios match almost exactly the average ratios computed for OECD Mexican data between 1980 and 2005, also reported by Angelopoulos *et al.* (2012) and Mendoza (2008).

It is important to note that, comparing these results to similar statistics for developed economies (see e.g. King and Rebelo, 1999), two relevant features are correctly captured in this calibration: the capital - output ratio is general lower in emerging economies while the consumption - output (investment - output) ratio tends to be lower (higher) in developed economies. The former result is consistent with the fact that emerging economies are relatively less capital-intensive, while the latter is in line with historically lower investment rates in these economies, in many cases mostly supported by Foreign Direct Investment given that domestic saving rates tend to be remarkably low (see e.g. Edwards, 1996, and Reinhardt, 2008, for the particular case of Latin-American countries).

Table 1.2: Steady-state values

variable	model	
y	1.513	
c	1.134	
h	0.333	
k	2.652	
i	0.220	
g	0.156	
w	2.725	
r	0.233	
ratios	model	data
k/y	1.75	1.75
c/y	0.75	0.70
i/y	0.15	0.15
g/y	0.11	0.11

1.4 Model Solution

1.4.1 Reduced form of the model

To solve the log-linearised version of the model, note first that it is possible to follow a common approach in the AL literature and substitute all the control variables out from the Euler condition given in (1.46), so that the entire model can be re-written as a second-order difference equation system of the form:[13]

$$\hat{k}_t = a_1 E_t \hat{k}_{t+1} + a_2 \hat{k}_{t-1} + b_1 \hat{z}_t, \tag{1.48}$$

$$\hat{z}_t = \rho \hat{z}_{t-1} + \varepsilon_t, \tag{1.49}$$

which shows that the agents' decisions over the level of capital they wish to hold at any given period, $\hat{k}_t$, can be expressed as a function of the expected value of this variable for the next period, $E_t\hat{k}_{t+1}$, its past value, $\hat{k}_{t-1}$, and the contemporaneous exogenous variable, $\hat{z}_t$, which in turn evolves according to the second equation of the system. Parameters a_1, a_2 and b_1 are constant coefficients that depend only on the structural parameters of the

[13] See, among others using this reduced form, Evans and Hokhaponja (2001), Giannitsarou (2006) and Carceles-Poveda and Giannitsarou (2007 and 2008).

model according to:[14]

$$a_1 = \left[\frac{F\theta\kappa i\eta}{y\varphi D}\right] G^{-1}, \tag{1.50}$$

$$a_2 = \left[\frac{\eta\theta\kappa i(1-\delta)+\eta y\varphi\theta+\tau_k\delta\theta\varphi\kappa k}{\beta y\varphi D}\right] G^{-1}, \tag{1.51}$$

$$b_1 = \left[\frac{(1-\tau_k)r\rho}{\theta} + \frac{\eta}{\beta D} - \frac{F\rho}{D}\right] G^{-1}, \tag{1.52}$$

where:

$$D = \frac{\theta\kappa c+(1-\theta)y}{y}, \quad F = \frac{\eta}{\beta} + \frac{(1-\tau_k)r(1-\theta)}{\theta}$$

and

$$G = \frac{F\left(\theta\kappa i(1-\delta)+y\varphi\theta+\tau_k\delta\eta^{-1}\theta\varphi\kappa k\right)}{y\varphi D} + \frac{\theta\kappa i\eta^2}{\beta y\varphi D}$$

Therefore, by replacing the parameter values as well as the resulting steady-state values found in the calibration procedure above (summarised in Tables 1.1 and 1.2) into equations (1.50) to (1.52), the reduced-form parameters for Mexico are:

$$a_1 = 0.4522, \quad a_2 = 0.5061 \quad \text{and} \quad b = 0.1022, \tag{1.53}$$

Next, the dynamics of each the control variables of the model $q = \{\hat{y}, \hat{c}, \hat{h}, \hat{\imath}, \hat{r}, \hat{w}\}$ can be re-written in terms of the state variables only, according to the following reduced form :

$$q_t = \gamma_{q_1}\hat{k}_t + \gamma_{q_2}\hat{k}_{t-1} + \gamma_{q_3}\hat{z}_t + \gamma_{q_4}E_t\hat{k}_{t+1} \tag{1.54}$$

where, as above, the coefficients γ_{q_j} (for $j = 1, 2, 3, 4$) are convolutions of the structural parameters of the model.[15]

1.4.2 Rational expectations

With the reduced form of the model at hand, it is straightforward to find its solution under RE applying the method of undetermined coefficients proposed by Uhlig (1995).[16]

[14] The derivation of this reduced form is shown in detail in Appendix A.

[15] Note that, for this particular framework, $\gamma_{q_4} = 0$ in all cases. The analytical derivation of this reduced is presented in Appendix A.

[16] While the methods proposed by Blanchard and Khan (1980) and Klien (2002) are often used to solve the linearised versions of dynamic macroeconomic models, in this chapter the undetermined coefficients approach is conveniently chosen instead, as it helps to better visualise the main features of adaptive

First, it is assumed that agents correctly *guess* that the equilibrium law of motion of the state variable has the following linear form:

$$\hat{k}_t = \omega_{kk}\hat{k}_{t-1} + \omega_{kz}\hat{z}_t. \tag{1.55}$$

Substituting (1.49) into the equation above gives:

$$\hat{k}_t = \phi_k \hat{k}_{t-1} + \phi_z \hat{z}_{t-1} + \upsilon_t, \tag{1.56}$$

where $\phi_k = \omega_{kk}$, $\phi_z = \omega_{kz}\rho$ and $\upsilon_t = \frac{\phi_z}{\rho}\varepsilon_t$. The first two (i.e. ϕ_k and ϕ_z) are the two undetermined coefficients of interest. Evaluating this expression at $t+1$ and taking expectations yields:

$$E_t\hat{k}_{t+1} = \phi_k \hat{k}_t + \phi_z \hat{z}_t \tag{1.57}$$

because $E_t\upsilon_{t+1} = (\phi_z/\rho)\, E_t\varepsilon_{t+1} = 0$, given the distribution of ε_t.

If the initial *guess* given by (1.57) is in fact a solution to the model, it must satisfy its reduced form too. Therefore, substituting this equation into (1.48) implies that the system becomes:

$$\hat{k}_t = a_1(\phi_k \hat{k}_t + \phi_z \hat{z}_t) + a_2\hat{k}_{t-1} + b\hat{z}_t, \tag{1.58}$$

plus (1.49). Substituting this last equation into (1.58) yields the unique equation:

$$\hat{k}_t = \frac{a_2}{(1-a_1\phi_k)}\hat{k}_{t-1} + \frac{(a_1\phi_z + b_1)\,\rho}{(1-a_1\phi_k)}\hat{z}_{t-1} + \frac{(a_1\phi_z + b)}{(1-a_1\phi_k)}\varepsilon_t. \tag{1.59}$$

And since this equation must equal the initial guess (1.55) in order to verify that the latter yields a solution to the model, it follows that:

$$\phi_k = \frac{a_2}{(1-a_1\phi_k)}, \tag{1.60}$$

learning within the model and its relationship with the solution under rational expectations presented here.

and

$$\phi_z = \frac{(a_1\phi_z + b)\,\rho}{(1 - a_1\phi_k)}. \tag{1.61}$$

Solving the quadratic equation (1.60) for ϕ_k yields two possible results:

$$\bar{\phi}_{k_1} = \tfrac{1+\sqrt{1-4a_1a_2}}{2a_1} \quad \text{and} \quad \bar{\phi}_{k_2} = \tfrac{1-\sqrt{1-4a_1a_2}}{2a_1}\ , \tag{1.62}$$

and replacing these into (1.61) gives their respective associated results for ϕ_z:

$$\bar{\phi}_{z_1} = \tfrac{b\rho}{1-a_1(\rho+\bar{\phi}_{k_1})} \quad \text{and} \quad \bar{\phi}_{z_2} = \tfrac{b\rho}{1-a_1(\rho+\bar{\phi}_{k_2})}\ . \tag{1.63}$$

Therefore, the RBC model at hand has two solutions under RE which correspond to the so-called minimum state variable (MSV) solutions in the sense that these have been determined by the smallest possible number of state variables including their own lags (see MacCallum, 1983). Furthermore, note that this model is known to be *regular*, meaning that only one of the two solutions in (1.62) is stationary (i.e. has an absolute value less than one) and thus is consistent with an stable equilibrium, while the other solution is ruled out as it implies an explosive path for the state variable.

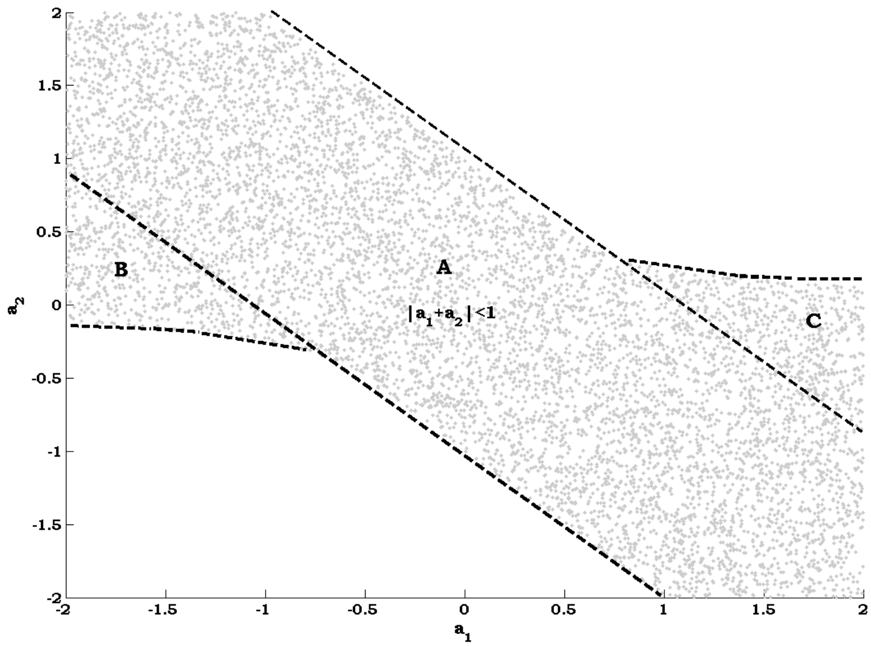
This can be shown more clearly by considering Figure 1.1, which presents a two-dimensional plot of different values for parameters a_1 and a_2 yielding different values for $\bar{\phi}_k$ according to the two solution in (1.62) and satisfying i) $\bar{\phi}_k < 1$ and ii) $\bar{\phi}_k \in \mathbb{R}$.[17]

When the $\bar{\phi}_{k_2}$ solution is considered, the possible combinations of a_1 and a_2 for which the two conditions are satisfied are given by area A - excluding the borders so that it is consistent with condition $|a_1 + a_2| < 1$ - and also by the smaller areas B and C. On the other hand, when the $\bar{\phi}_{k_1}$ solution is considered, the set of combinations satisfying the two conditions is only given by the smaller areas B and C. For any other combination of a_1 and a_2 outside areas A, B and C, both solutions are either non-stationary or non-real.

Therefore, it can be concluded that an RBC-type model will be regular or have a unique stationary solution given by coefficients $\bar{\phi}_{k_2}$ and $\bar{\phi}_{z_2}$, if and only if $|a_1 + a_2| < 1$,

[17] The numerical simulation included 15.000 realizations of the two solutions of the quadratic equation for different values of a_1 and a_2 which were choosen at random from a range between -2 to 2.

Figure 1.1: Numerical simulation of the quadratic equation under conditions $\bar{\phi}_k < 1$ and $\bar{\phi}_k \in \mathbb{R}$.



which graphically means combinations of a_1 and a_2 that always lie inside area A. On this respect, note that the set of parameters presented in (1.53) coming from the calibration of the model for Mexico clearly satisfy the condition $|a_1 + a_2| < 1$, thus verifying that the model at hand has a unique stationary solution associated with coefficients $\bar{\phi}_{k_2}$ and $\bar{\phi}_{z_2}$ in (1.62) and (1.63), respectively.

Given the above considerations, the unique stationary solution of the model is given by coefficients $\bar{\phi}_{k_2}$ and $\bar{\phi}_{z_2}$ which hereafter will be labelled simply $\bar{\boldsymbol{\phi}} = \left[\bar{\phi}_k, \bar{\phi}_z\right]'$ to keep the notation simple. Therefore, the RE equilibrium law of motion of the endogenous state variable is:

$$\hat{k}_t = \bar{\phi}_k \hat{k}_{t-1} + \bar{\phi}_z \hat{z}_{t-1} + \frac{\bar{\phi}_z}{\rho} \varepsilon_t. \tag{1.64}$$

Considering the calibration above, the model solution is given by coefficients:

$$\bar{\phi}_k^{mex} = 0.7839 \quad \text{and} \quad \bar{\phi}_z^{mex} = 0.4020, \tag{1.65}$$

With above results, the analytical policy functions of the control variables of the model can be found by substituting (1.64) into (1.54) for $\hat{k}_t$ and recalling that $E_t \varepsilon_{t+1} = 0$, which yields:

$$\begin{aligned} q_t &= \left[\gamma_{q1}\bar{\phi}_k + \gamma_{q2} + \gamma_{q4}\bar{\phi}_k^2\right]\hat{k}_{t-1} + \\ &+ \left[\gamma_{q1}\frac{\bar{\phi}_z}{\rho} + \gamma_{q3} + \gamma_{q4}\left(\bar{\phi}_z + \bar{\phi}_k\frac{\bar{\phi}_z}{\rho}\right)\right]\hat{z}_t. \end{aligned} \tag{1.66}$$

for $q = \{\hat{y}, \hat{c}, \hat{h}, \hat{\imath}, \hat{r}, \hat{w}\}$, and then plugging the coefficients found in (1.65) into the resulting equations.

1.4.3 Adaptive learning

Motivation

The RE approach to solving the RBC model implies assuming that, on average, agents do not make systemic mistakes in forecasting the future and thus deviations from *perfect foresight* are only random. This in turn implies that they must know the structure of the economy, the values of the deep parameters and the distribution of the random shock. By contrast, the AL approach to modelling expectations formation presented in this work follows a different and perhaps more plausible view of rationality (see e.g. Marcet and Nicolini, 2003) by which it is assumed that the agents face limitations on their knowledge about the economy and, to overcome these, they adopt a learning strategy applying basic estimation techniques on available data.[18]

More specifically, it will be assumed that agents: a) do not know the exact structure of the economy but have a correct *guess* about the specification of the equilibrium policy functions that solve the model, b) do not know the values of the structural parameters that determine the values of the coefficients $\bar{\boldsymbol{\phi}} = \left[\bar{\phi}_k, \bar{\phi}_z\right]'$ in the equilibrium policy functions, but c) they do know the true parameters that characterize the exogenous shock (i.e. ρ and σ_ε). Hence, agents will behave as *econometricians* who run regressions to estimate the coefficients in $\bar{\boldsymbol{\phi}}$ and use these estimates to form their expectations about the behavior of the state variable.

[18] To give a more complete idea of these approaches, a brief description and an illustration of three alternative ways of modelling expectations are provided in Appendix A.

Solution procedure

As discussed above, it will be assumed that since agents are not fully rational they use estimates of the true RE coefficients $\bar{\boldsymbol{\phi}} = [\bar{\phi}_k, \bar{\phi}_z]'$ found in the previous section, which they update in every period by employing a basic econometric technique. In order to avoid a problem of simultaneity in these estimations[19] it will be assumed that agents forecast or form expectations about k_{t+1} using their estimates from the previous period, $\boldsymbol{\phi}_{t-1}$. This means that the agents' expectations formation process derives from what is usually known as their perceived law of motion (PLM), of the form:

$$E_t\hat{k}_{t+1} = \tilde{\phi}_{k,t-1}\hat{k}_t + \tilde{\phi}_{z,t-1}\hat{z}_t. \tag{1.67}$$

where the vector $\tilde{\boldsymbol{\phi}}_t = \left[\tilde{\phi}_k, \tilde{\phi}_z\right]'$ denotes the estimate of $\bar{\boldsymbol{\phi}} = [\bar{\phi}_k, \bar{\phi}_z]'$ for all t. Since the equation above represents the actual forecast of the agents, it can then be plugged into the first equation of reduced-form model given by (1.48) to get:

$$\hat{k}_t = a_1(\tilde{\phi}_{k,t-1}\hat{k}_t + \tilde{\phi}_{z,t-1}\hat{z}_t) + a_2\hat{k}_{t-1} + b\hat{z}_t.$$

Next, using (1.49) and collecting terms yields:

$$\hat{k}_t = \frac{a_2}{(1 - a_1\tilde{\phi}_{k,t-1})}\hat{k}_{t-1} + \frac{\left(a_1\tilde{\phi}_{z,t-1} + b\right)\rho}{(1 - a_1\tilde{\phi}_{k,t-1})}\hat{z}_{t-1} + \frac{\left(a_1\tilde{\phi}_{z,t-1} + b\right)}{(1 - a_1\tilde{\phi}_{k,t-1})}\varepsilon_t. \tag{1.68}$$

or, equivalently:

$$\hat{k}_t = P_1\hat{k}_{t-1} + P_2\hat{z}_{t-1} + V\varepsilon_t, \tag{1.69}$$

where:

$$P_1 = \frac{a_2}{(1 - a_1\tilde{\phi}_{k,t-1})}, \tag{1.70}$$

$$P_2 = \frac{\left(a_1\tilde{\phi}_{z,t-1} + b\right)\rho}{(1 - a_1\tilde{\phi}_{k,t-1})}, \tag{1.71}$$

[19] That is, k_t and ϕ_t would have to be determined at the same time if agents use ϕ_t to form the period-t expectation

$$V = \frac{P_2}{\rho} = \frac{\left(a_1 \tilde{\phi}_{z,t-1} + b\right)}{(1 - a_1 \tilde{\phi}_{k,t-1})}. \tag{1.72}$$

Equation (1.69) is known as the actual law motion (ALM) of the state variable because every new value of $\hat{k}_t$ will be obtained in the model economy but always considering the agents' forecasts according to the PLM. Moreover, and like in the RE case, the ALM of the control variables (denoted by q) can be found by substituting (1.69) for $\hat{k}_t$ and (1.67) for $E_t\hat{k}_{t+1}$ into the reduced form (1.54), to get:

$$\begin{aligned} q_t &= \gamma_{q1}\left(P_1\hat{k}_{t-1} + P_2\hat{z}_{t-1} + V\varepsilon_t\right) + \gamma_{q2}\hat{k}_{t-1} + \gamma_{q3}\hat{z}_t + \\ &\quad + \gamma_{q4}\left(\tilde{\phi}_{k,t-1}\hat{k}_t + \tilde{\phi}_{z,t-1}\hat{z}_t\right), \end{aligned}$$

Next, considering (1.49) means that in the first term of the r.h.s. of the equation above $V\hat{z}_t = P_2\hat{z}_{t-1} + V\varepsilon_t$. Hence:

$$\begin{aligned} q_t &= \left[\gamma_{q1}P_1 + \gamma_{q2} + \gamma_{q4}\tilde{\phi}_{k,t-1}P_1\right]\hat{k}_{t-1} + \\ &+ \left[\gamma_{q1}V + \gamma_{q3} + \gamma_{q4}\left(\tilde{\phi}_{k,t-1}V + \tilde{\phi}_{z,t-1}\right)\right]\hat{z}_t. \end{aligned} \tag{1.73}$$

for $q = \{\hat{c}, \hat{y}, \hat{\imath}, \hat{h}, \hat{r}, \hat{w}\}$.

It is important now to turn the attention on the learning algorithms that agents will be assumed to use in order to find the estimates $\tilde{\boldsymbol{\phi}}_t = \left[\tilde{\phi}_{k,t}, \tilde{\phi}_{z,t}\right]'$. As mentioned earlier, two different algorithms will be considered in the simulations and experiments carried out in this work: recursive least squares and its constant gain variant.

Recursive least squares

Recursive least squares (RLS) is probably the most widely used learning algorithm in the AL literature (see e.g. Marcet and Nicolini, 2003, and Carceles-Poveda and Giannitsarou, 2007). It assumes that the agents behave as econometricians to estimate the coefficients of the model, using a simple ordinary least squares (OLS) regression for this purpose (see Evans and Honkapohja, 2001). That is, at the beginning of each period t, the vector of state variables $\mathbf{x}_t = \left[\hat{k}_t, \hat{z}_t\right]'$ is realized based on the ALM (1.69) and the exogenous process of technology. This means that $\mathbf{x}_t$ is found from:

$$\mathbf{x}_t = \begin{pmatrix} \hat{k}_t \\ \hat{z}_t \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \hat{k}_{t-1} \\ \hat{z}_{t-1} \end{pmatrix} + \begin{pmatrix} V \\ 1 \end{pmatrix} \varepsilon_t. \tag{1.74}$$

Since the realizations of $\mathbf{x}_t$ coming from the matrix system above are observable to the agents, they can now run the regression:

$$\hat{k}_t = \tilde{\boldsymbol{\phi}}' \mathbf{x}_{t-1} + \xi_t, \tag{1.75}$$

to get a new estimate $\tilde{\boldsymbol{\phi}}_t$, where ξ_t is the forecast error. Applying OLS, the estimate $\tilde{\boldsymbol{\phi}}_t$ will be the coefficient vector which minimizes $\sum_{t=1}^{T} \xi_t^2$ and thus is given by:

$$\tilde{\boldsymbol{\phi}}_t = \left(\sum_{i=1}^{t} \mathbf{x}_{i-1} \mathbf{x}_{i-1}^{'} \right)^{-1} \sum_{i=1}^{t} \mathbf{x}_{i-1} \hat{k}_i. \tag{1.76}$$

The equivalent recursive expression of the estimator above is given by the system of two equations:[20]

$$\mathbf{R}_t = \mathbf{R}_{t-1} + g_t \left(\mathbf{x}_{t-1} \mathbf{x}_{t-1}^{'} - \mathbf{R}_{t-1} \right), \tag{1.77}$$

$$\tilde{\boldsymbol{\phi}}_t = \tilde{\boldsymbol{\phi}}_{t-1} + g_t \mathbf{R}_t^{-1} \mathbf{x}_{t-1} \left(\hat{k}_t - \mathbf{x}_{t-1}^{'} \tilde{\boldsymbol{\phi}}_{t-1} \right), \tag{1.78}$$

where $\tilde{\boldsymbol{\phi}}_t$ denotes the coefficient estimates vector and $\mathbf{R}_t = \frac{1}{t} \sum_{i=1}^{t} \mathbf{x}_{i-1} \mathbf{x}_{i-1}^{'}$is the second moment matrix of the variables included in $\mathbf{x}$. In addition, $\left(\hat{k}_t - \mathbf{x}_{t-1}^{'} \tilde{\boldsymbol{\phi}}_{t-1} \right)$ is the latest forecast error which will be used to adjust the current estimates, and the $g_t = 1/t$ is known as the decreasing gain sequence, because it implies that as time goes by, every new forecast error will have a smaller impact in the latest estimation.

Constant gain RLS

Another type of algorithm gaining popularity more recently (see e.g. Orphanides and Williams, 2005, and Milani, 2007 and 2008) is known as the *constant gain* variant of the

[20] The detailed derivation of the recursive version of the OLS regression is provided in Appendix A.

RLS algorithm (hereafter RLS-CG). The main feature of it is that the decreasing gain sequence, g_t, included in the learning rule above is replaced by a constant positive but small number (i.e. less than one), g.

In this case, from equation (1.77), the RLS-CG algorithm is given by:

$$\mathbf{R}_t = \mathbf{R}_{t-1} + g\left(\mathbf{x}_{t-1}\mathbf{x}_{t-1}^{'} - \mathbf{R}_{t-1}\right), \tag{1.79}$$

$$\tilde{\boldsymbol{\phi}}_t = \tilde{\boldsymbol{\phi}}_{t-1} + g\mathbf{R}_t^{-1}\mathbf{x}_{t-1}\left(\hat{k}_t - \mathbf{x}_{t-1}^{'}\tilde{\boldsymbol{\phi}}_{t-1}\right), \tag{1.80}$$

The main difference between this and the decreasing gain algorithm is that under the latter the effect of the latest forecasting error $\left(\hat{k}_t - \mathbf{x}_{t-1}^{'}\tilde{\boldsymbol{\phi}}_{t-1}\right)$ tends to vanish as t increases, while under the constant gain rule every new forecasting error has the same weight as past ones. For this reason, as $t \to \infty$ and under certain conditions (discussed later), a constant gain algorithm ensures convergence to a distribution centered around the RE solution rather than to the true time-invariant values included in $\bar{\boldsymbol{\phi}}$. Because of these properties, this algorithm can be quite useful whenever it is reasonable to assume that agents prefer to use more recent information when forming expectations.

Determinacy, Stationarity, E-stability and Convergence

Evans and Honkapohja (2001) demonstrate that two necessary but not sufficient conditions for local convergence of the adaptive learning solution towards the RE one are: a) the RE solution of the model must be unique (i.e. satisfies the so-called *determinacy* condition) and stationary and b) the RE solution must be expectationally stable or *E-stable.*

Regarding the first condition, in a previous section it was shown that, for the calibration exercise presented the model at hand has a unique solution. Moreover, the solution was stationary since $\left|\bar{\phi}_k^{mex}\right| < 1$ thus ensuring that the policy function does not predict an explosive path for the state variable.

On the other hand, *E-stability* is a condition that determines the stability of the RE solution under a learning rule, in which the estimates $\tilde{\phi}_k$ and $\tilde{\phi}_z$ used in the PLM (1.67) are adjusted slowly in the direction of the implied ALM parameters shown in (1.69). If this adjustment process is completed, feeding the latest estimates $\tilde{\phi}_k$ and $\tilde{\phi}_z$ in the two ALM parameters given by (1.70) and (1.71) should yield exactly the same two estimates again. In such case, these estimates must be equal to the RE solution parameters $\bar{\phi}_k$ and $\bar{\phi}_z$, since it was demonstrated that the model at hand has a unique equilibrium. Intuitively, this implies that the RE solution $\bar{\boldsymbol{\phi}} = [\bar{\phi}_k, \bar{\phi}_z]$ will be E-stable under learning if small deviations from it are returned to $\bar{\boldsymbol{\phi}} = [\bar{\phi}_k, \bar{\phi}_z]$ under the chosen learning rule.

For the RLS algorithm, and recalling that $\mathbf{P} = \begin{bmatrix} P_1 & P_2 \end{bmatrix}'$, Evans and Honkapohja (2001) show that a RE solution of the model is E-stable if we the following 2×2 Jacobian matrix (where $\mathbf{I}$ is the identity matrix):[21]

$$\mathbf{J} = \left.\frac{\partial \mathbf{P}}{\partial \boldsymbol{\phi}}\right|_{\phi=\bar{\phi}} - \mathbf{I} \tag{1.81}$$

is stable, i.e. it has eigenvalues with strictly negative real parts. From (1.70) and (1.71) it follows that the Jacobian evaluated at the RE solution is equal to:

$$\mathbf{J} = \begin{bmatrix} \frac{a_1 a_2}{(1-a_1\bar{\phi}_k)^2} - 1 & 0 \\ \frac{\left(a_1\bar{\phi}_z + b_1\right)a_1\rho}{1-a_1\bar{\phi}_k} & \frac{a_1\rho}{1-a_1\bar{\phi}_k} - 1 \end{bmatrix} \tag{1.82}$$

To ensure that the real parts of the two eigenvalues of this matrix - i.e. the elements in the main diagonal - are negative, the two conditions to be satisfied are:

$$\frac{a_1 a_2}{(1-a_1\bar{\phi}_k)^2} - 1 < 0, \quad \text{and} \quad \frac{a_1\rho}{1-a_1\bar{\phi}_k} - 1 < 0. \tag{1.83}$$

Applying these two conditions to the calibrated model shows that the E-stability condition is satisfied since:

$$0.5491 - 1 < 0, \quad \text{and} \quad 0.6402 - 1 < 0, \tag{1.84}$$

which in turn implies that the model at hand solved assuming RLS locally converges to the RE solution.

In addition, Carceles-Poveda and Giannitsarou (2007) show that for the RLS-CG algorithm the same E-stability condition described above applies plus the condition that $0 < g < 1$. Therefore, the model at hand also shows local convergence to a distribution

[21] Associated with the two coefficients of interest, $\bar{\phi}_k$ and $\bar{\phi}_z$.

centered around the RE solution when this learning rule is used.

To conclude this part, it is important to stress the fact that the conditions described above ensure local but not global convergence of the learning rules towards RE. Hence, if the starting point of the learning process or, if due to major exogenous shocks in the economy, the estimates under the learning rules lie too far away from the area of attraction within the parameter space implied by the E-stability condition, the agents' forecasts might never converge to the RE equilibrium. Therefore, in order to increase the probabilities of local convergence, the learning algorithms will be augmented with a *projection facility* proposed by Marcet and Sargent (1989), according to which any estimate that is considered an *outlier* is ignored and the latest available estimate is repeated instead for forecasting purposes. That is, the learning rule is now defined as:

$$\mathbf{R}_t = \mathbf{R}_{t-1} + g^p \left(\mathbf{x}_{t-1}\mathbf{x}_{t-1}^{\prime} - \mathbf{R}_{t-1} \right), \tag{1.85}$$

$$\tilde{\boldsymbol{\phi}}_t = \begin{cases} \tilde{\boldsymbol{\phi}}_{t-1} + g^p \mathbf{R}_t^{-1} \mathbf{x}_{t-1} \left(\hat{k}_t - \mathbf{x}_{t-1}^{\prime} \tilde{\boldsymbol{\phi}}_{t-1} \right) & \text{if } \tilde{\boldsymbol{\phi}}_t < 1 \\ \tilde{\boldsymbol{\phi}}_{t-1} & \text{if } \tilde{\boldsymbol{\phi}}_t \geqslant 1 \end{cases}, \tag{1.86}$$

where the gain sequence g^p can be set equal to g_t or g, depending on the learning variant chosen.

Initial conditions for learning

An important issue in learning is how to appropriately set the initial values $\tilde{\boldsymbol{\phi}}_0$ and $\mathbf{R}_0$ for the recursions. While it should be expected that the effects of the initial conditions will disappear in the limit, note that these initial values might have an important impact in the short and medium run dynamics of the model. Moreover, as said earlier, if these initial values are too far away from the RE solution, it might be the case that convergence of the estimates to the true values will be very slow or not achievable at all.

Hence, it is sensible to consider some initializing methods which help the learning algorithm start from values that do not compromise convergence. According to type of experiments carried out in this document, two alternative ways of initializing the recursion will be considered.

Initial conditions from randomly generated data (RG) Define t_0 as the time period for which the initial values of the recursion are set (i.e. a pre-estimation time period) which can be rationalised as some preliminary data from which agents can start their recursive estimation procedure. This means that the recursive algorithm itself starts at t_0+1. Hence, the initial values are defined as follows (see Carceles-Poveda and Giannitsarou, 2007):

$$\mathbf{R}_{t_0} = \frac{1}{t_0}\sum_{i=1}^{t_0} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'}, \tag{1.87}$$

$$\tilde{\boldsymbol{\phi}}_{t_0} = \frac{1}{t_0}\mathbf{R}_{t_0}^{-1}\sum_{i=1}^{t_0} \mathbf{x}_{i-1}\hat{k}_i, \tag{1.88}$$

with the initial value $\mathbf{x}_0$ given and usually set equal to 0. A first key element here is to determine t_0. While it is clear that t_0 must be at least equal than the number of regressors (i.e. two), the actual minimum t_0 required must be one that ensures the invertibility of $\mathbf{R}_{t_0}$, implying that t_0 could be in fact larger than two. A second key element is to define how the values $\mathbf{x}_i$ for $i = \{1, ..., t_o\}$ will be (randomly) generated. The RE solution will be used as the benchmark specification for generating the random values. That is, given $\mathbf{x}_0 = 0$, a number of randomly generated values $\mathbf{x}_t$ is obtained from

$$\mathbf{x}_t = \begin{pmatrix} \hat{k}_t \\ \hat{z}_t \end{pmatrix} = \begin{pmatrix} \bar{\phi}_k & \bar{\phi}_z \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \hat{k}_{t-1} \\ \hat{z}_{t-1} \end{pmatrix} + \begin{pmatrix} \bar{\phi}_z/\rho \\ 1 \end{pmatrix} \varepsilon_t. \tag{1.89}$$

for all $t \in \{1, ..., t_o\}$ so that $\mathbf{R}_{t_0}$ and $\tilde{\boldsymbol{\phi}}_{t_0}$ in (1.87) and (1.88), respectively, can be found and subsequently used as the initial values of the recursion given by (1.77) and (1.78) for $t \in I_1\{t_0+1, t_0+2, ..., T\}$. To sum up, the entire recursive algorithm effectively is:

$$\begin{cases} \mathbf{R}_{t_0} = \frac{1}{t_0}\sum_{i=1}^{t_0} \mathbf{x}_{i-1}(\bar{\boldsymbol{\phi}})\mathbf{x}_{i-1}^{'}(\bar{\boldsymbol{\phi}}) \\ \tilde{\boldsymbol{\phi}}_{t_0} = \frac{1}{t_0}\mathbf{R}_{t_0}^{-1}\sum_{i=1}^{t_0} \mathbf{x}_{i-1}(\bar{\boldsymbol{\phi}})\hat{k}_i(\bar{\boldsymbol{\phi}}) \end{cases}, \tag{1.90}$$

$$\begin{cases} \mathbf{R}_t = \mathbf{R}_{t-1} + \frac{1}{t}\left(\mathbf{x}_{t-1}\mathbf{x}_{t-1}^{'} - \mathbf{R}_{t-1}\right) \\ \tilde{\boldsymbol{\phi}}_t = \tilde{\boldsymbol{\phi}}_{t-1} + \frac{1}{t}\mathbf{R}_t^{-1}\mathbf{x}_{t-1}\left(\hat{k}_t - \mathbf{x}_{t-1}^{'}\tilde{\boldsymbol{\phi}}_{t-1}\right) \end{cases} \quad t \in I_1 = \{t_0+1, t_0+2, .., T\} \tag{1.91}$$

where $\mathbf{x}(\bar{\phi})$ represents the regressors that have been generated using the RE solution according to (1.89).

Arbitrary or ad-hoc conditions (AH) Another alternative of initialization at $t_0 = 0$ is to simple choose an arbitrary invertible matrix $\mathbf{R}_0$ and some stationary coefficients $\tilde{\phi}_0$, although in many cases these are assumed to be the ones consistent with the RE solution or, if policy reforms are being considered, the ones associated with the RE solution of the pre-reform economy thus depicting a situation in which the old state of the economy has already been learnt by the agents (see e.g. Giannitsarou, 2006 and Evans *et al.*, 2009).

Given the initial observations $\mathbf{x}_0$, the matrix $\mathbf{R}_0$ is defined simply as $\mathbf{x}_0\mathbf{x}_0^{'}$. This implies that at $t = 1$ the recursion is give by:

$$\mathbf{R}_1 = \mathbf{R}_0 + \mathbf{x}_0\mathbf{x}_0^{'}, \tag{1.92}$$

and:

$$\tilde{\phi}_1 = \tilde{\phi}_0 + \mathbf{R}_t^{-1}\mathbf{x}_0\left(\hat{k}_1 - \mathbf{x}_0^{'}\tilde{\phi}_0\right), \tag{1.93}$$

while from $t = 2$ onwards the recursion again follows the system given by (1.77) and (1.78). Therefore, the entire algorithm is:

$$\begin{cases} \mathbf{R}_1 = \mathbf{R}_0 + \mathbf{x}_0\mathbf{x}_0^{'}, \\ \tilde{\phi}_1 = \tilde{\phi}_0 + \mathbf{R}_t^{-1}\mathbf{x}_0\left(\hat{k}_1 - \mathbf{x}_0^{'}\tilde{\phi}_0\right) \end{cases}, \tag{1.94}$$

$$\begin{cases} \mathbf{R}_t = \mathbf{R}_{t-1} + \frac{1}{t}\left(\mathbf{x}_{t-1}\mathbf{x}_{t-1}^{'} - \mathbf{R}_{t-1}\right) \\ \tilde{\phi}_t = \tilde{\phi}_{t-1} + \frac{1}{t}\mathbf{R}_t^{-1}\mathbf{x}_{t-1}\left(\hat{k}_t - \mathbf{x}_{t-1}^{'}\tilde{\phi}_{t-1}\right) \end{cases} \quad \text{for } t \in \{2, 3, .., T\}, \tag{1.95}$$

with $\mathbf{x}_0$, $\mathbf{R}_0$ and $\tilde{\phi}_0$ given.

1.5 Simulations and results

1.5.1 Predicted dynamics

From the theoretical description of the learning algorithms given above, it is clear that any difference in the predicted dynamics of the model under AL with respect to their RE counterparts can have its origin in only two main elements in the learning setup. The first one is related to how close or, conversely, how far away are the initial values of the learning algorithm from the time-invariant parameters associated with the RE solution. As mentioned earlier, of the former are too far away from the latter, then the learning process might be quite slow and, in some cases, convergence towards the RE equilibrium might not be achieved at all.

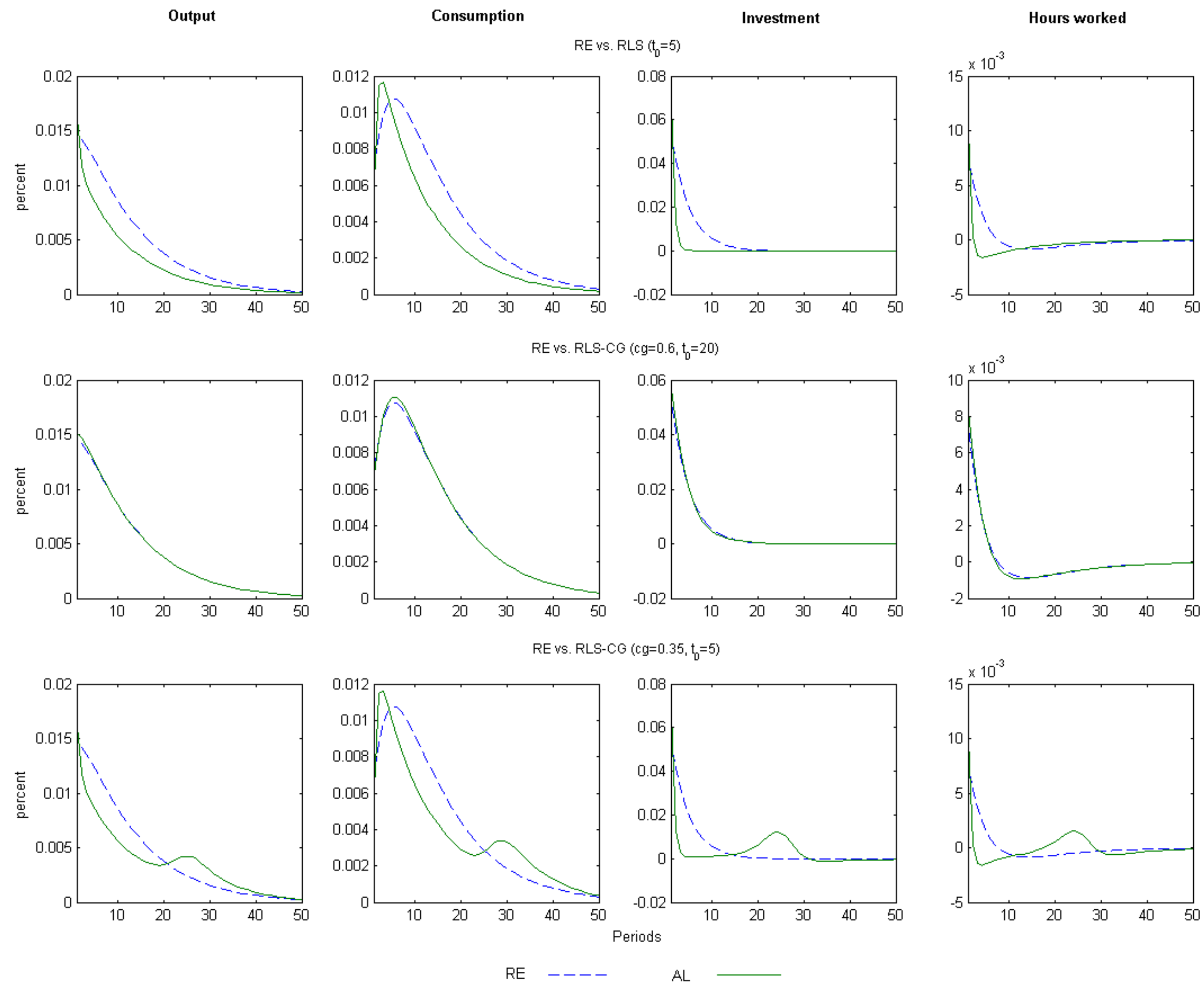
The second element is related to the value of the constant gain sequence of the algorithm and thus is only relevant when RLS-CG is used. As discussed in the previous section, a constant gain sequence (as opposed to a decreasing gain sequence as under RLS) implies that every new forecast error made during the learning process will have the same relative weight as past ones. Hence, if the gain is assumed to be quite high (i.e. closer to one) any small increase in the latest forecast error might cause major corrections in the latest estimation and thus a new temporary departure from the path towards the RE solution occurs.

A useful way to illustrate these differences in the transition paths predicted by the model under RE and AL is by examining the impulse response functions (IRFs) of a 1% shock to technology under these different learning setups.[22] These are presented in Figure 1.2 for output, investment, consumption and hours worked given the relevance of these variables in terms of the stylized facts of interest and also because the latter two have a direct impact on welfare.

The first set graphs in the Figure (first row) show the case where the departing point of the learning process is quite far away from the RE solution. This is depicted by taking RLS as the chosen algorithm and assuming RG as the initialization method but with $t_0 = 5$,

[22] The IRFs of a -1% (negative) shock to technology are symetrically the opposite to the positive shock case, and thus these are not discussed in the document.

Figure 1.2: IRFs of a 1 percent shock to technology under RE and AL



which effectively implies that very few pre-estimation periods were used to compute the initial values $\tilde{\boldsymbol{\phi}}_0$ and $\mathbf{R}_0$ and thus these are far away from their RE counterparts, $\bar{\boldsymbol{\phi}}$ and $\bar{\mathbf{R}}$. As a result, while the direction of the predicted responses under AL is the same as under RE, the reactions seem to be sharper in the first periods (especially in the case of consumption) and then after some periods these tend to decrease quite rapidly (especially in the case of hours worked and investment).

The predicted dynamics seem consistent with some the insights provided by Aguiar and Gopinath (2007) and can be rationalised as follows: if agents in the economy are in fact learners and for some reason their estimates are quite far away from the true RE solution, they might misinterpret a temporary positive shock as a permanent one (i.e. a change in at least one of the structural parameters of the model and implying a new higher steady state), thus leading them to rapidly increase consumption at the expense of investment - implying lower capital accumulation - and to reduce the labour supply due to the wealth effect they *optimistically* perceive. In the following periods, as new data become available and the actual impact of the shock to technology vanishes, agents start recognizing the transient nature of the shock and the responses decrease much faster than when the agents are fully rational.

The second set of graphs in the Figure show the effects of a high constant gain. The learning algorithm is now RLS-CG with a very high constant gain equal to 0.6.[23] To isolate the impact of the chosen gain sequence, the initializing method is RG again but this time with $t_0 = 20$ so that the learning process starts very close to the RE solution. In this case the transition dynamics are quite similar under both RE and AL but there are still some differences in consumption and hours worked that might be of some significance. Moreover, even though these are not shown in the Figure, some periods of extra volatility occur around the 80^{th} period, explained only by the effects of the high constant gain which leads to an overreaction of the adjustment process in the learning algorithm in response to very small differences between the two paths.

[23]In the AL literature the constant gain values often used range between 0.01 to 0.15 (see e.g. Milani, 2007, 2008, and Carceles-Poveda and Giannitsarou, 2007). The gain values chosen in this section are significantly higher but note that this is for illustrative purposes only, and that the condition $0 < cg < 1$ has been not been violated.

The final set of graphs combines the effects of the starting values and the constant gain under the same learning process. As in the previous case, the RLS-CG algorithm is chosen again with a very high constant gain, equal to 0.35. The initialization method is RG but now with $t_0 = 5$ as in the first case. In this learning setup, the starting values are far away from the RE solution but also every new forecast error is adjusted sharply due to the high constant gain associated with it. As a result, the significantly different paths between RE and AL and the remarkable additional volatility of the one under AL can be fully appreciated.

An important result coming from these exercises is that both key elements in the learning setup (i.e. the initial values of the recursion and the constant gain) can be a source of additional volatility in the paths of the relevant variables of the model with respect to their RE counterparts, a feature of AL that might be helpful when trying to improve the ability of the RBC model to match the data in the case of emerging economies.

1.5.2 Second - moment matching results

Given the important insights about the transition dynamics predicted by the model under different learning setups discussed above, this section now pretends to evaluate the goodness of fit properties of the RBC model under AL under different economic scenarios that are likely to occur in practice, especially in emerging economies such as Mexico. For this purpose, the volatility, co-movement and persistence properties of the key variables will be examined, although especial emphasis will be given to the main stylised facts of the Mexican economy as discussed previously.[24]

First, a second-moment matching exercise will be performed assuming (as is standard in the literature) that the economy is oscillating close to its steady-state due to random

[24] As a control, a similar exercise for a calibration of the model using US post-war quarterly data was performed. Following Ireland (2004), the chosen parameter values are $\beta = 0.99$, $\delta = 0.025$, $\alpha = 0.36$, $\eta = 1.0039$, $\rho = 0.95$ and $\sigma_\varepsilon = 0.00712$. Also, following Mendoza et al. (1994) the tax rates are set to $\tau_h = 25\%$ and $\tau_k = 40\%$. The solution of the model under RE (which is unique, stationary and E-stable) is given by coefficients $\phi_k = 0.9516$ and $\phi_z = 0.1349$. The results of the second-moment matching exercise confirm the findings of Eusepi and Preston (2008) and Huang *et al.* (2009), namely, AL - both in the form of RLS or RLS-CG with $cg = 0.02$ - helps to improve the ability of the standard RBC model to fit the US post-war data, only the first scenario of the two described below being necessary (and perhaps reasonable) to obtain these results.

shocks to technology in every period. In a second scenario, it will be assumed that the economy is enduring a major economic depression to which the government does not respond with any explicit policy reform, so that the economy slowly returns to its steady state.

All the simulation exercises performed to compute the relevant statistics presented in this section consider a time horizon of 25 periods (years) in order to match the range of the available data. To get the relevant statistics, logarithms are taken to the simulated data in levels, and then the Hodrick-Prescott filter with a smoothing parameter of 100 is applied. Each experiment is replicated 1000 times and thus the average statistics are reported.[25]

In all cases, the experiments under AL will focus on three key setups: RLS, RLS-CG with $cg = 0.03$ (i.e. a low constant gain) and RLS-CG with $cg = 0.1$ (i.e. a high constant gain). The two constant gains considered are very close, respectively, to the lowest and highest values typically used in the literature (see e.g. Milani, 2007 and 2008).

First scenario: the economy oscillates around the steady-state

In this first experiment, the usual assumption that the economy is fairly close to its steady state and any deviation from this state is due to exogenous shocks to technology is considered. From a learning perspective, one reasonable approach is to assume that in such case the true values of the structural parameter have been already learned by the agents and thus the initial values of the algorithm are equal to their RE counterparts, so that $\tilde{\boldsymbol{\phi}}_0 = \bar{\boldsymbol{\phi}}$ and $\mathbf{R}_0 = \bar{\mathbf{R}}$. In the learning algorithm, this means making use of the AH initialising method and taking the RE solution as the reference. Even in this setup, learning will matter because in each of the following periods, every time there is an exogenous shock to technology, the new parameter estimates $\tilde{\boldsymbol{\phi}}_t$ and $\mathbf{R}_t$ for all $t = 1, 2, ..$ will exhibit deviations from the RE values which will dissipate as long as these deviations do not go beyond the area of attraction implied by the E-stability condition. Moreover,

[25] The *projection facility* was activitated only for the case where the RG initialising method is applied with very few pre-estimation periods (e.g. the minimum possible to ensure the invertibility of $\mathbf{R_0}$). In these cases, on average, the facility was activated in a range of one to two periods out of twenty five, representing less than 4% of the entire simulation horizon.

if a constant gain is assumed instead of a decreasing gain, the response of the estimates to the exogenous shocks will tend to increase with the size of the constant gain. The first three rows of Table 1.3 below show a set of key second moments generated by the model calibrated for Mexico under the different AL setups and assuming that $\tilde{\boldsymbol{\phi}}_0 = \bar{\boldsymbol{\phi}}$ and $\mathbf{R}_0 = \bar{\mathbf{R}}$.

A second approach that could be fairly reasonable for Mexico and other emerging economies - where the time range of available data is relatively short - is to assume that the agents have not fully learned the 'true' structural parameter values $\bar{\boldsymbol{\phi}}$ and $\bar{\mathbf{R}}$ and thus further deviations from the RE solution should be expected in the short-run. This lack of knowledge at the start of the learning process can be fairly depicted by applying the RG initialization method for the learning algorithm and setting very few pre-learning periods, that is t_0 will be set equal to the minimum number of periods required so that $\mathbf{R}_0$ is invertible.[26] The following three rows in Table 1.3, show the resulting moments of this alternative. In addition, the corresponding statistics simulated by the model under RE and those coming from the data are presented in the last two rows to allow for comparisons.[27]

The results in Table 1.3 confirm the widely known poor performance of the standard RBC model under RE in terms of fitting the data, particularly in the cases of the relative volatility of the main variables and virtually all the moments associated with hours worked.[28] In terms of the co-movement and persistence of the variables, the model under RE does fairly well, as shown by the crossed correlation (CCF) and autocorrelation (ACF) functions plus their $\pm$ one standard-deviation intervals in Figure 1.3, but with the exception of the CCF for hours worked. On that matter, Angelopoulos *et al.* (2012) show that acknowledging the fact that property rights in Mexico are not fully protected and can be considered as another source of uncertainty can greatly improve the statistics related to the labour supply.

[26] In all cases, this invertibility criteron meant that t_0 was set to 8 to 10 pre-learning periods.

[27] The data on output, investment and consumption were obtained from the OECD database for the period $1980-2005$. The data on hours worked were obtained from the Instituto Nacional de Estadistica y Geografia (INEGI). All these data were made available by Angelopoulos *et al.* (2012). Finally, due to data limitations on this regard, the two statistics highlighted with (*) are indicative only, according to the stylised facts highlighted by Bergoeing and Soto (2002).

[28] The standard deviation of Mexican output of around 0.03 is reasonably replicated by the model under RE (equal to 0.032) and all the learning setups considered (of around 0.036).

Table 1.3: Second Moments for Mexico - Scenario 1

	$\frac{\sigma(c)}{\sigma(y)}$	$\frac{\sigma(i)}{\sigma(y)}$	$\frac{\sigma(h)}{\sigma(y)}$	$\frac{\sigma(h)}{\sigma(w)}$	$cc(c,y)$	$cc(i,y)$	$cc(h,y)$	$cc(h,w)$
Initialization method: AH $(\tilde{\boldsymbol{\phi}}_0 = \bar{\boldsymbol{\phi}},\ \mathbf{R}_0 = \bar{\mathbf{R}})$								
RLS	0.55	3.70	0.55	1.03	0.91	0.96	0.92	0.68
RLS-CG$_{(0.03)}$	0.54	3.69	0.54	1.00	0.92	0.96	0.92	0.70
RLS-CG$_{(0.1)}$	0.54	3.71	0.53	1.03	0.92	0.96	0.92	0.70
Initialization method: RG $(\tilde{\boldsymbol{\phi}}_0 = \tilde{\boldsymbol{\phi}}_{t_0},\ \mathbf{R}_0 = \mathbf{R}_{t_0},\ t_0 = \min)$								
RLS	0.60	3.54	0.53	0.91	0.89	0.93	0.87	0.58
RLS-CG$_{(0.03)}$	0.61	3.55	0.53	0.87	0.89	0.93	0.86	0.56
RLS-CG$_{(0.1)}$	0.60	3.57	0.53	0.88	0.89	0.93	0.87	0.57
RE	0.54	3.68	0.54	0.99	0.92	0.96	0.92	0.69
Data	1.34	5.86	0.23	>1*	0.92	0.70	-0.2	<0*

On the other hand, the same model under the three learning setups - particularly under RLS - where it is assumed that the structural parameter values were known at the start of the learning process (rows 1 to 3) clearly outperforms its RE counterpart in terms of bringing the predicted statistics closer to the data. While the improvements might not seem very large, it is clear that AL in this case helps to obtain higher relative volatilities for consumption and investment, although still not enough to match the data, and also produces a $\sigma(h)/\sigma(w)$ ratio that is higher than one, thus matching the stylised fact suggested by Bergoeing and Soto (2002). Finally, assuming less initial information available for learning (rows 4 to 6) helps to produce even a higher relative volatility of consumption, at the expense of all the other volatility statistics, but it does help to bring the co-movement statistics closer to the data. The set of graphics in Figure 1.4 show the CCF and ACF of the model under RLS-CG in the sixth row of the Table, as this setup shows the best performance in general. The functions show that the improvements in the relative volatility statistics under learning does not affect negatively the goodness of fit of the model in terms of the predicted persistence or co-movement of the variables of interest. In some cases, very small improvements with respect to their RE counterparts were obtained, although these are difficult to visualize graphically.[29]

Two important elements must be highlighted at this point. First, despite some im-

[29] The ACFs and CCFs of the other learning alternatives given in the Table 1.3 show similar results so these are not reported here.

Figure 1.3: ACFs and CCFs for the RBC model under RE

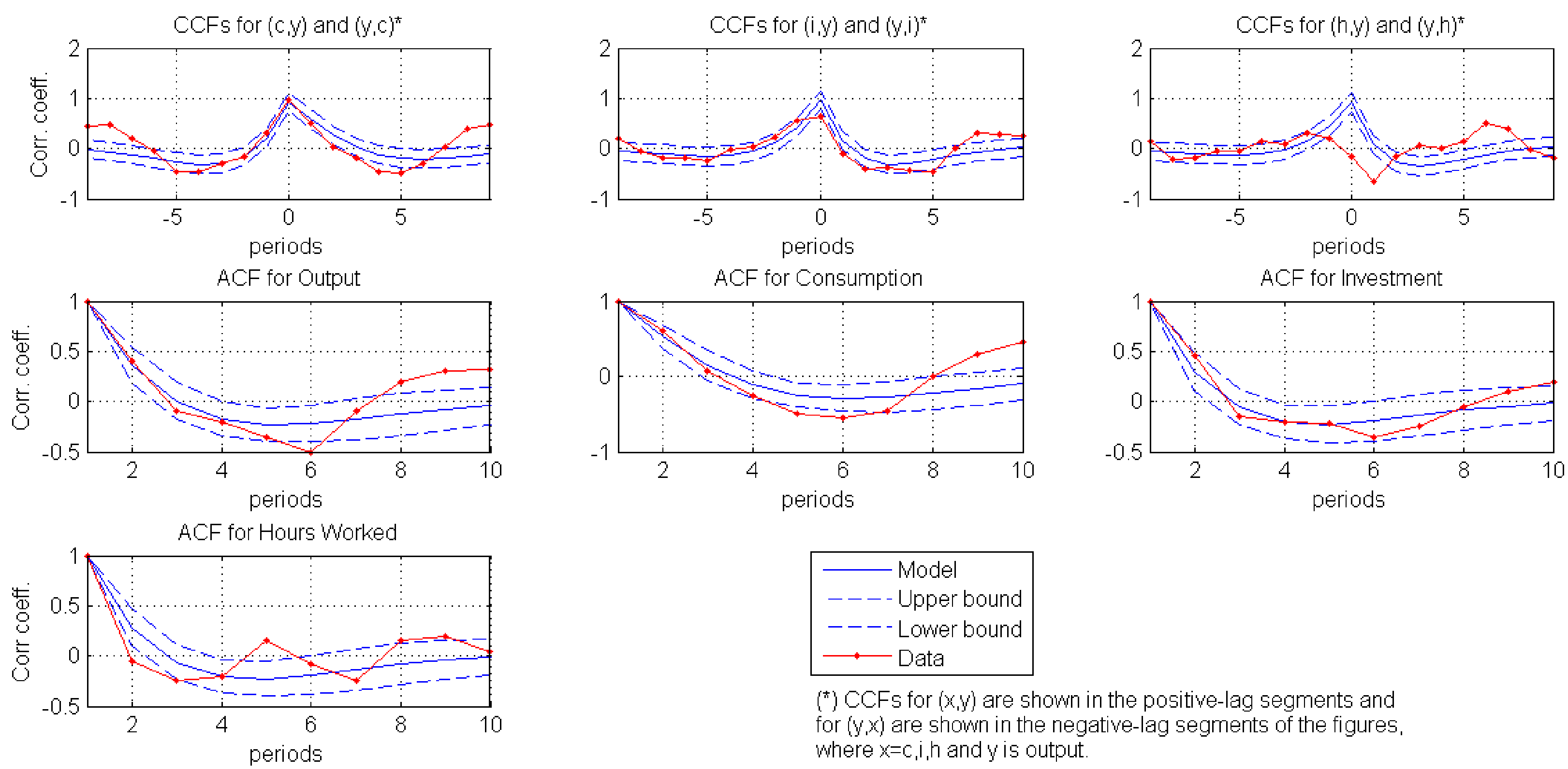
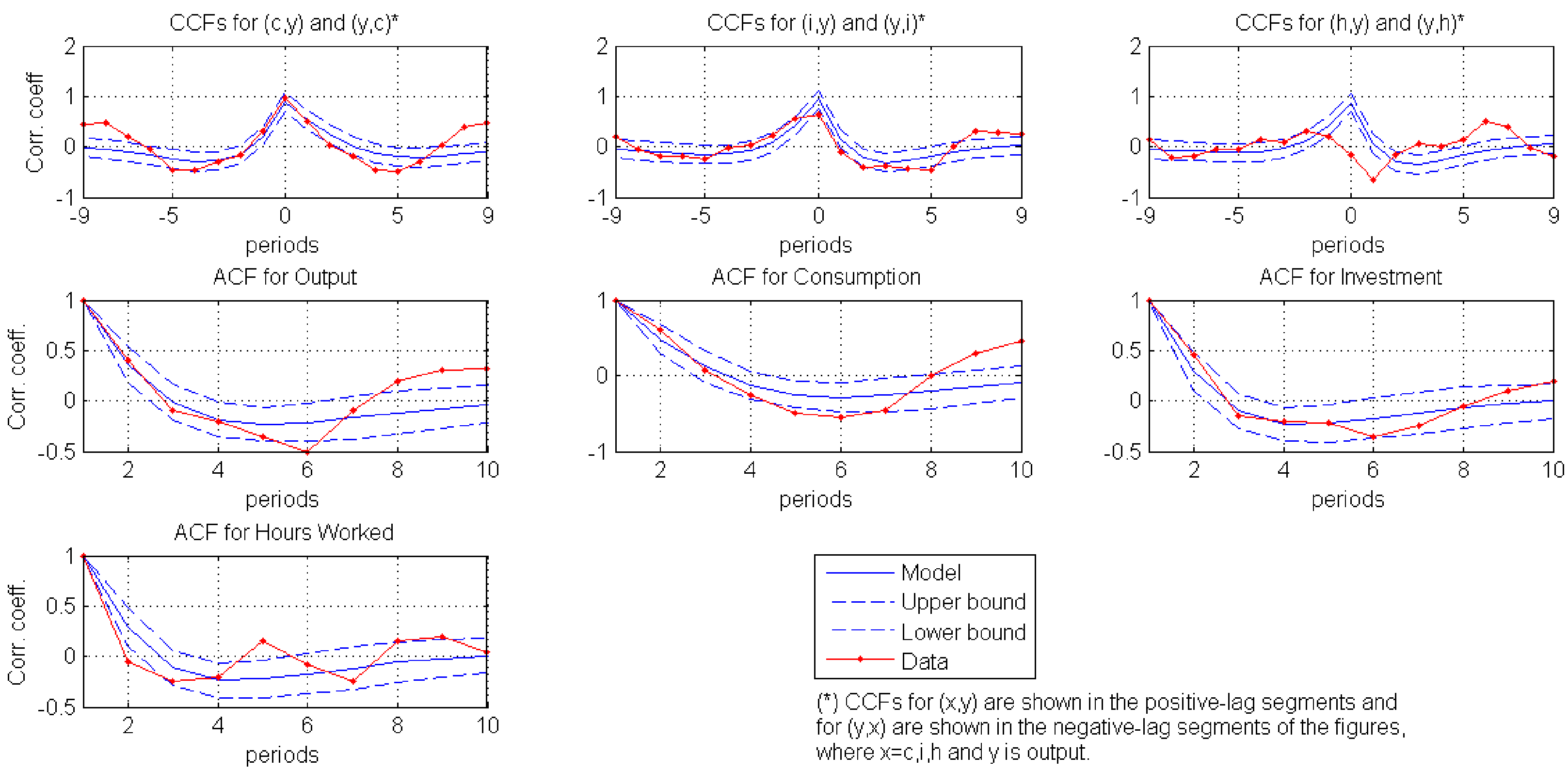


Figure 1.4: ACFs and CCFs for the RBC model under RLS-CG (cg=0.1 and 10 pre-estimation obs.)



provements under AL, the relative volatilities of consumption and investment are still quite far away from the data. Likewise, with the exception of the $\sigma(h)/\sigma(w)$ ratio discussed above, the other predicted statistics involving hours worked clearly fail to match the data. These results thus have motivated a second set of simulation exercises - shown below - which try to verify if these evasive statistics are perhaps more associated with frequent scenarios of economic depression, as suggested by the recent economic history of most emerging economies, precisely such as Mexico.

Second, the model under RLS-CG and $cg = 0.03$ shows the worst performance in terms of fitting the data in this and the following exercises, thus suggesting that a low constant gain in such setup might not be a good representation of the learning dynamics for Mexico. This is potentially a major qualitative difference with respect to developed economies, as most learning exercises carried out for these agree on very low constant gains, between 0.01 and 0.04 (see e.g. Milani, 2007 and 2008, and Gaspar *et al.*, 2006).

Nevertheless, this finding seems quite reasonable since, in one hand, emerging economies in general are known to have been subjected to a number of often economic reforms, structural breaks and other major shocks affecting their institutions (see e.g. Angelopoulos *et al.*, 2012, for Mexico) over the last few decades. Within such context, governments in most of these countries have also endured (or are still enduring) long lasting episodes of very low credibility (see e.g. Calderon *et al.*, 2004). On the other hand, to reflect such particular features of low credibility or expectations of new structural breaks occurring any time soon, recursive algorithms with high constant gains have been often suggested in the AL literature (see e.g. Carceles-Poveda and Giannitasrou, 2006 and 2007). This is because a higher weight can then be assigned to more recent data in the learning algorithm in order to capture the effects of the reforms and/or structural changes right after these have taken place and, in turn, the parameter estimates can be updated faster than under relatively more stable environments.

Second scenario: the economy recovering from a depression

In this section it is now investigated whether by acknowledging the impact of economic depressions on the agents ability to learn about their economic environment can help to improve the above results, particularly in terms of the relative volatility of consumption and investment and the statistics involving hours worked. Studying such scenario seems reasonable since between 1980 and 2005 Mexico experienced two main episodes of severe economic crisis. The first took place during the Latin-American debt crisis between 1982 and 1986, when real GDP fell by 4% in 1983 and 1986, while the second episode known as the "tequila" currency crisis occurred between 1994 and 1995, when real GDP fell by 6%. Such was the negative impact of these episodes, that capital stock in the shape of machinery and equipment actually fell in both periods, the only two occasions where such a severe impact had to be endured by this country in the last 60 years, despite some other (milder) periods of economic downturn taking place over the same period (see e.g. Hoffman, 2000, and Souza *et al.,* 2005).

Therefore, to illustrate such episodes in the model and particularly their impact on the learning dynamics, following Giannitsarou (2006), it will be assumed that the initial value of the only endogenous state variable of the model - capital stock - is placed at a level such that the simulated output starts around 4% below its steady-state value. In other words, instead of setting $\hat{k}_0 = 0$ which means that there are no deviations from the steady-state value of capital, now the simulations will depart from $\hat{k}_0 = 1.84$, level 37% lower than k and consistent with a recession of around 4% in terms of output, from which the economy recovers gradually as no government intervention is assumed. This, in turn, should generate an additional wedge between the RE the AL paths, since the early observations to be used in the estimation process will not be consistent with the steady state of the economy and therefore the early estimates will show a much slower convergence to the RE solution.[30]

As before, the two different assumptions regarding the agents' initial knowledge are considered. The first three rows of Table 1.4 show the case where the structural parameter values are already known by the agents, while the following three rows show the case when these parameter values are still not fully known , for the same selected AL setups in both cases. In the latter case, however, the number of pre-learning periods is set to $t_0 = 10$

[30]Recall that local convergence to the RE solution is ensured given that the E-stability conditions hold, as shown earlier, and that the projection facility can be activated whenever necessary.

Table 1.4: Second Moments for Mexico - Scenario 2

	$\frac{\sigma(c)}{\sigma(y)}$	$\frac{\sigma(i)}{\sigma(y)}$	$\frac{\sigma(h)}{\sigma(y)}$	$\frac{\sigma(h)}{\sigma(w)}$	$cc(c,y)$	$cc(i,y)$	$cc(h,y)$	$cc(h,w)$
Initialization method: AH ($\tilde{\boldsymbol{\phi}}_0 = \bar{\boldsymbol{\phi}}$, $\mathbf{R}_0 = \bar{\mathbf{R}}$)								
RLS	1.11	4.56	0.83	0.75	0.69	0.48	0.27	-0.48
RLS-CG$_{(0.03)}$	1.09	4.55	0.83	0.75	0.68	0.49	0.29	-0.47
RLS-CG$_{(0.1)}$	1.09	4.56	0.83	0.76	0.68	0.49	0.29	-0.47
Initialization method: RG ($\tilde{\boldsymbol{\phi}}_0 = \tilde{\boldsymbol{\phi}}_{t_0}$, $\mathbf{R}_0 = \mathbf{R}_{t_0}$, $t_0 = 10$)								
RLS	1.16	5.11	0.93	0.80	0.63	0.49	0.30	-0.50
RLS-CG$_{(0.03)}$	1.16	5.10	0.93	0.79	0.63	0.49	0.30	-0.50
RLS-CG$_{(0.1)}$	1.16	5.13	0.94	0.81	0.63	0.50	0.31	-0.50
RE	0.54	3.68	0.54	0.99	0.92	0.96	0.92	0.69
Data	1.34	5.86	0.23	>1*	0.92	0.70	-0.20	<0*

rather than the minimum possible as this assumption yields better results.

The results in this scenario are quite encouraging in terms of the relative volatility of consumption and investment but also in terms of the negative correlation between wages and hours worked. In effect, the results of the model under RLS-CG with $cg = 0.1$ in the second approach where the RG initialisation method is used, yielding $\sigma(c)/\sigma(y)$ and $\sigma(i)/\sigma(y)$ ratios that are very close to the data, suggest that such learning setup seems to better depict how expectations are formed among agents in emerging economies, as it shows the best overall results in terms of the stylised facts discussed earlier, considering the different experiments carried out in this section. However, it is also clear that these results are obtained at the expense of the co-movement and persistence statistics which are now quite poor when compared to the results of the first scenario. However, these tend to improve rather fast as more years are gradually allowed into the simulations.[31]

These results suggest that highly volatile consumption and investment plus a negative correlation between hours worked and wages (and as a result also between hours worked and consumption), three key stylised facts often seen in emerging economies - are more consistent with an economy populated by learners, in an scenario of transition towards the steady state after a major deviation - due to a severe economic crisis, for instance - had taken place. Such deviation affects the agent's ability to learn the RE solution faster - particularly in the early periods -, which ultimately leads them to make their consumption and investment choices in a context of large forecast errors and important adjustments

[31] For this reason, these are not reported in the document.

of the estimates in every period, as they find it difficult to recognize whether the current paths of the main variables towards the steady state respond to a temporary positive shock to technology or are simply driven by the deterministic recovery process towards the steady state that follows right after the latest economic depression.

It is worth noting that the above analysis seems consistent with the findings of Aguiar and Gopinath (2007) and Boz et al. (2008) in the sense that agents in emerging economies might find it difficult to recognize between shocks to the trend of the economy - which are quite frequent in these economies - and the more typical transitory fluctuations around a stable trend.

1.5.3 Welfare implications

The results above show that in general AL helps to improve the ability of the RBC model to fit the data for Mexico, specially in terms of the relative volatility and co-movement. Therefore, if AL rather than the mechanical sources of volatility and correlation of the model at hand solved under RE provide a more appropriate representation of this economy, then the outcome of any economic policy carried out by the government might be different from what is expected if it (wrongly) assumes that private agents are fully rational. It might in fact generate unexpected additional welfare gains/costs from misspecifying private expectations formation. A similar point was in fact made by Giannitsarou (2006) for the US case, showing that after a tax reform which brings the capital tax rate down, an economy of learners will be less better off than one of fully-rational agents.[32]

The objective of this final section is thus to verify whether this concern is also relevant for an emerging economy and, if so, to what extent. For this purpose, two unanticipated tax reforms are welfare-evaluated: a) a reduction of the capital tax τ_k from 9% to 0%, inspired by most of the literature on capital taxation suggesting that this tax should be set to zero in the long run (see e.g. Chamley, 1986, and Lucas, 1990); and b) a similar reduction of the labour tax τ_h from 13% to 0%, to complete the analysis given the tax menu available in the model setup.[33]

[32] See also Milani (2007) for another example about this issue related to monetary policy.

[33] A reduction of τ_h to 8.6%, so that the ratio G/Y in both reforms is the same and equal to 8%,

Table 1.5: Tax reforms and steady-states

variable	base.	$\tau_k = 0\%$	$\tau_h = 0\%$
y	1.513	1.550	1.575
c	1.134	1.188	1.304
i	0.220	0.242	0.229
h	0.333	0.326	0.347
h	2.652	2.913	2.761

In each case, following Giannitsarou (2006), the other tax rate will remain constant and G will adjust accordingly in order to satisfy the balanced budget condition.[34] Furthermore, it is worth noting that since in the model setup proposed in this work G is not included in the utility function, the welfare gains resulting from the tax reforms under analysis will be higher than in a case where G is also a determinant of the agent's utility (see e.g. Chapter 3). In effect, the decrease in taxes will naturally result in a fall of government revenues given that no other sources of revenue are assumed. Hence, the positive effects on total utility coming from a less distorted economy - which are discussed below - will be lower because of the relatively lower utility obtained from the consumption of less public goods. Nevertheless, standard model calibration procedures (see e.g. Malley et al., 2009) acknowledge the fact that the relative weight of public good consumption in the utility function tends to be quite small, which implies that while some quantitative differences can be expected between these two setups, the results should still be qualitatively similar.

Table 1.5 reports the post-reform steady-state values consistent with the two tax reforms. In the economy with $\tau_k = 0\%$, output is 2.5% higher than in the pre-reform steady-state, explained mostly by the capital stock which is almost 10% higher as labour is in fact 2% lower. As a result, consumption and investment are 4.8% and 10% higher, respectively. The equilibrium laws of motion under RE in this case are still of the form given by (1.64) with coefficients $\bar{\phi}_k^{post,\tau_k} = 0.7797$ and $\bar{\phi}_z^{post,\tau_k} = 0.4056$.

When $\tau_h = 0\%$, output is 4% higher than its pre-reform value, explained by an increase

was considered too. In this case, the diferences between RE and AL are virtually insignificant. For that reason, the results of a much more dramatic change of this tax are reported.

[34] Giannitsarou (2006) shows that if it is assumed that to compensate a tax cut the other tax rate must increase to a level such that the pre-reform ratio G/Y remains unchanged, the results are qualitatively similar to those presented in the non-revenue-neutral exercise.

of capital stock and labour of 4%. The latter is in clear contrast to the capital tax cut case, implying that agents choose to work more hours in order to benefit from a much higher post-tax labour income and increase their consumption accordingly. In effect, consumption is now 15% higher than in the previous steady-state while investment is only 4% higher. The resulting equilibrium laws of motion are also in the same form as in (1.64), with coefficients $\bar{\phi}_k^{post,\tau_h} = 0.7839$ and $\bar{\phi}_z^{post,\tau_h} = 0.4062$. Finally, it is important to note that the two solutions associated with each tax reform satisfy the conditions of determinacy, stationarity and E-stability discussed earlier, so that both show local convergence to their respective RE solutions under the different AL setups.

In terms of the learning setup, following Evans *et al.* (2009), it will be assumed that the agents have already learned the structural parameters associated with the pre-reform steady-state of the economy, denoted by $\bar{\boldsymbol{\phi}}^{pre} = \left[\bar{\phi}_k^{pre}, \bar{\phi}_z^{pre}\right]$ and $\bar{\mathbf{R}}^{pre}$. However, they do not know the new parameter values associated with the post-reform steady-state after either of the two tax reforms, denoted by $\bar{\boldsymbol{\phi}}^{post,m} = \left[\bar{\phi}_k^{post,m}, \bar{\phi}_z^{post,m}\right]$ and $\bar{\mathbf{R}}^{post,m}$ where $m = \tau_h, \tau_k$. Technically, this implies that the AH initializing method is applied to set $\tilde{\boldsymbol{\phi}}_0 = \bar{\boldsymbol{\phi}}^{pre}$ and $\mathbf{R}_0 = \bar{\mathbf{R}}^{pre}$ in the learning algorithm. Finally, in line with the results in the previous sections, the two learning algorithms with the best performance in terms of matching the data - i.e. RLS and RLS-CG with $cg = 0.1$ - will be considered in these experiments.

Following Lucas (1990) and Giannitsarou (2006), the welfare measure used for this analysis is a compensating consumption supplement or the percentage amount ξ by which consumption should change in all periods in the pre-reform economy so that the agents are equally well off as in the post-reform economy, including the transition period. This will be conditional on the initial random shock to technology the economy is potentially subjected to, which in practice usually tends to be the main factor motivating the tax reforms in the first place (e.g. a tax cut after a recession). The formula of this measure is thus given by:[35]

$$\xi_i^m = \left[e^{(1-\beta)\left(U_{T,m}^{post,i} - U_T^{pre,RE}\right)} - 1\right] \times 100 \;\Big|\; \varepsilon_0 = \varepsilon \tag{1.96}$$

[35]The derivation of this formula is presented in Appendix A.

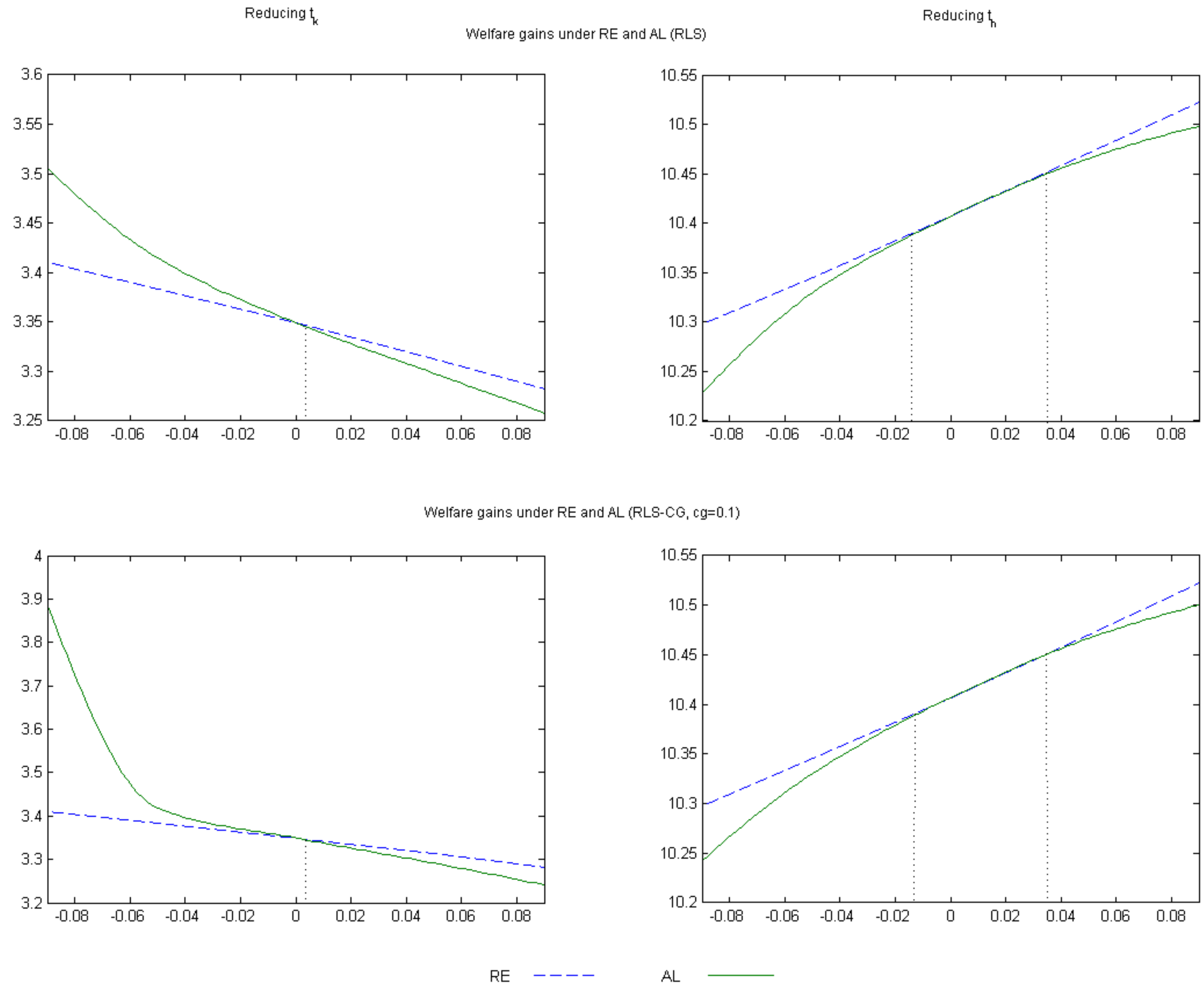
where total discounted utility U_T is computed according to (1.1) with the number of simulated periods set to $T = 1000$ and $m = \tau_h, \tau_k$ and $i = RE,\ AL$. Since it is assumed that the agents have learned the pre-reform steady state, the pre-reform utility is computed using the RE solution of the model, implying that $U_T^{pre,RE}$ (conditional on each exogenous shock) remains unchanged for both tax reform exercises. Finally, ε is a given realization of a shock to technology taking place simultaneously with the tax reform. Given the standard deviation $\sigma_\varepsilon = 0.025$ for TFP in Mexico, shocks in the range $[-0.09, 0.09]$ need to be considered as these cover close to 99% of the associated probability mass. Hence, if $\xi > 0$ there is a (conditional) welfare gain of moving from the pre- to the post-reform steady state.

The conditional welfare computations of the two tax reforms are exhibited in Figure 1.5, where it can be seen that, confirming Giannitsarou's (2006) findings, the differences between AL and RE are more significant when large negative shocks hit the economy. These differences then tend to vanish for relatively small but increasing positive shocks and again become significant but in the opposite direction once large positive shocks are taken into account.

However, unlike Giannitsarou's (2006) main results on welfare, the upper-left plot in the Figure shows that a reduction of the capital tax in response to a negative shock to technology would make learners better off than fully rational agents, while the opposite is true for the same tax reform if a positive shock just above $\varepsilon_0 = 0$ hits the economy. Intuitively, a bad shock coinciding with a tax cut makes learners believe that under the new regime capital is more inelastic than under the old regime (also note that $\bar{\boldsymbol{\phi}}^{pre} > \bar{\boldsymbol{\phi}}^{post}$) and thus they decide to accumulate capital more slowly than under RE. As a result, in the early periods right after the reform and the bad shock the decrease in consumption and increase in hours worked of the learners (motivated by the bad shock) is smaller than in the case of fully rational agents. Then, after some periods, rational agents resume their rapid convergence towards a higher level of consumption and the lower level of hours worked consistent with the post-reform steady state (see Table 1.5 above), while the learners' convergence is slower.

At this point, the discount factor in the utility function, β, enters into play. This

Figure 1.5: Tax reforms and welfare under RE and AL



parameter has been set equal to 0.885 following the relevant literature as a way to depict features often associated with emerging economies such as shorter investment horizons and higher levels of impatience among agents. In terms of welfare calculations, this parameter value implies that every additional future period included in the computation of the present value lifetime utility at $t = 0$ is heavily discounted by β^t for $t = 1, .., \infty$.[36] This element, along with the fact that the capital tax cut has had a relatively small impact in terms of the final level of consumption (i.e. it only increases 4% with respect to its pre-reform level), implies that the early post-reform periods when consumption is higher and labour supply is lower under learning with respect to RE are the most important in relative terms, ultimately leading to the results shown in the left-hand plots of the Figure. Likewise, following the same interpretation as above, if the behavior of learners is in fact better described by the RLS-CG algorithm with $cg = 0.1$ (lower-left plot), then the welfare gains for learners after a large negative shock are even more significant. At the extreme, if a large negative shock $\varepsilon_0 = -0.09$ occurs, then the additional welfare gain under learning is equivalent to 0.5%.

On the other hand, when a (at least slightly above $\varepsilon_0 = 0$) positive shock coincides with the reform, both learners and RE behave virtually in the same way and therefore the differences in welfare are very small. However, when a very large positive shock is realized the welfare gains are higher under RE because in the next few periods that follow the capital stock overshoots in the economy of learners since the positive shock makes them believe that capital is less inelastic than before. As a result, they optimistically decide to work and invest more and thus consume less than fully rational agents. Therefore, the same discussion related to the discount factor is in place once more but now acting in a symmetrically opposite way.

In turn, the right-hand plots in the Figure show that if eliminating τ_h is the chosen tax reform in response to a relatively large shock to technology - i.e. a negative shock larger than $\varepsilon_0 = -0.015$ or a positive shock larger than $\varepsilon_0 = 0.035$ - then learners will be less

[36] In fact, note that if β is modified slightly from 0.885 to 0.89 (*ceteris paribus*) in this calculations, then the results are inverted as the welfare gains under RE become higher than those under learning after a large bad shock while the opposite is true for positive shocks, as suggested by Giannitsarou (2006) considering a calibration of a similar model for the US, and where the parameter β is often set at a much higher level.

better off than rational agents, irrespective of the learning algorithm that better describes the former. In the first case, the same elements related to the behavior of learners in the early periods and the discount factor apply. However, note that in this case the impact of the tax reform on consumption is much more important than in the τ_k case important (i.e. an increase of 15% with respect of the old regime), which implies that, after a few periods of sluggishness, the faster convergence to the higher steady state by fully rational agents generates a series of (discounted) welfare gains which effectively outweight those in favour of learners generated only in the early periods.

For the second case, when a large positive shock is realized, the analysis is analogous to the same case above. That is, the welfare gains are higher under RE because in the economy of learners capital overshoots in the early periods as they work and invest more and consume less than fully rational agents. Finally, if the shock, either positive or negative, is rather small - i.e. inside the $[-0.015, 0.035]$ interval cited earlier - there will be virtually no differences in welfare gains between learners and rational agents.

1.6 Conclusions

While a number of authors have highlighted the relevance of learning to match the behavior of the key aggregates in developed economies, this document tried to verify such potential for emerging economies. This is considered an important contribution to the AL literature since most of the research in this field - with the exception of Marcet and Nicolini (2005) and Boz *et al.* (2008) - has focused exclusively on developed economies.

The simulation exercises under two different learning algorithms showed that in general AL helps to improve the second-moment matching properties of the standard RBC model calibrated for Mexico and perhaps for emerging economies in general as these share a number of key features which have been taken into account in the calibration exercise. However, in order to replicate key stylised facts such as consumption being more volatile than output and labour supply being much more volatile than wages, it has been necessary to assume that the economy is in fact recovering from an economic depression, something that is nevertheless supported by the data.

These results give some indication that AL in the shape of RLS and RLS-CG with a relatively high constant gain, might be a good representation of how agents form their expectations in practice. Therefore, the design of economic policy must take into account the fact that the expected welfare gains/losses of a particular reform conceived assuming a RE environment as it is commonly done might be significantly different if in reality agents tend to behave more as learners. This point has been illustrated by welfare-evaluating the effects of two tax reforms for Mexico, the results of which suggest that, if the government decided to cut taxes in response to a large negative shock to the economy, then a large capital tax cut might be more advisable than a large cut of the labour tax.

Finally, it is worth noting that AL within the framework presented here was not able to improve the performance of the standard RBC model in a way such that all the key stylised facts can be captured using a single model setup with a given set of initial conditions. In effect, some important moments such as the crossed correlation between hours worked and output were couldn't be replicated by the model.

In light of these caveats, the model under AL could be greatly enriched in two important and complementary respects. First, some market frictions which have been proved to be highly relevant for emerging economies (e.g. credit restrictions as in Mendoza, 2008) as well as acknowledging the existence of structural heterogeneity among the economic agents, could help to improve the overall performance of the model, both under RE and AL. Second, as discussed by Giannitsarou (2003) and Honkapohja and Mitra (2006), a better insight of the dynamics predicted by the AL hypothesis could be gained by considering the possibility of expectational heterogeneity, whereby different segments of the economy's population, which might also differ in some of their structural features (and thus structural heterogeneity is also present) or not, follow different learning setups and hence their interaction might bring important dynamic implications to the model. This approach is in fact pursued with more depth in the following two chapters.

Chapter 2

Distributional consequences of tax reforms in the UK under capital-skill complementarity

2.1 Introduction

There now exists a significant and growing literature on tax reforms in dynamic general equilibrium (DGE) models, largely focusing on the aggregate welfare benefits and the distributional consequences of permanent reductions in constant capital tax rates.[1] Studies within the representative agent framework suggest that tax reforms which reduce capital taxation will produce welfare gains for the society, even if the tax burden is concurrently shifted to labour (see e.g. Lucas, 1990, Cooley and Hansen, 1992, Angelopoulos *et al.*, 2012).[2] The aggregate welfare benefits from tax reforms that reduce capital taxation are also confirmed in models with heterogeneous agents (see e.g. Garcia-Mila *et al.,* 2010). However, at the same time, heterogeneous agent models make clear that such reforms can have large redistributive effects that will disadvantage different groups in the society (see e.g. Domeij and Heathcote, 2004, Greulich and Marcet, 2008 and Garcia-Mila *et al.,*

[1]This chapter is an extended version of the paper "The distributional consequences of supply-side reforms" with Jim Malley and Konstantinos Angelopoulos (Discussion Paper 2010-16). I would like to thank Chryssi Giannitsarou, Charles Nolan, Apostolis Philippopoulos and Peter Rosenkranz for helpful comments and suggestions.

[2]At the same time, at the aggregate level, there is also an important literature that examines optimal tax policy. The general message from Ramsey optimal taxation is that the tax rate on capital should be zero in the long-run (see e.g. Chamley, 1986, Chari *et al.,* 1994 and Chari and Kehoe, 1999). This result, however, does not necessarily hold in models incorporating market failures (see e.g. Guo and Lansing, 1999), nor in models under time-consistent optimal taxation (see e.g. Klein *et. al.,* 2008).

2010).[3]

The literature using heterogeneous agent models has also considered different types of market incompleteness and/or agent heterogeneity to demonstrate the distributional effects of tax reforms and, in particular, capital tax cuts. An important dimension in which agents differ, which is central to the analysis of capital tax reforms, is inequality in the distribution of assets or wealth.[4] A common approach to modeling this type of heterogeneity is to assume that some agents do not have access to the capital markets, or more generally, that some agents depend more on labour relative to capital income (see e.g. Judd, 1985, Lansing, 1999, Krusell, 2002, and Garcia-Mila et al. 2010).

In such environments, agents whose capital income is significant can expect to gain after a capital tax cut. However, the total effects of a capital tax cut are not as clear for those agents who depend predominantly on labour income, usually termed as the workers. There are costs to workers from a capital tax cut, if this is accompanied by an increase in labour taxes. Nevertheless, there can also be benefits that take the form of increased labour productivity delivered by the increase in the capital stock. Therefore, to evaluate the distributional effects of capital tax cuts, the productive role of capital and its complementarity with labour need to be explicitly examined.

This complementarity between capital and labour becomes particularly important when the economic structure suggests a distribution in the skill supply of the labour force, in addition to the asset distribution, and even more so when these distributions are positively related. For instance, the PSID data (see e.g. Table 2 in Garcia-Mila *et al.*, 2010) suggest that high wealth is positively related to higher wages, while evidence from the UK, discussed further below, suggests that skill acquisition, in the form of University education, is related to socioeconomic income group.[5]

[3] Studies that take into account the redistributive effects of capital taxation in designing optimal taxation in heterogeneous agent models are fewer. In Judd (1985) and Chamley (1986), Ramsey-type optimal taxation leads to a zero tax on capital in the long-run. However, this result does not necessarily hold when time-consistent taxation is considered (see e.g. Krusell, 2002, and Angelopoulos, Malley, and Philippopoulos., 2011).

[4] See for example, the Panel Study of Income Dynamics (PSID) data for wealth inequality in the USA analysed in Garcia-Mila et al. (2010) and the Family Resources Survey data discussed in more detail below for the UK.

[5] Note that the literature has allowed for joint inequality in asset holdings and labour skill in evaluating capital taxes (see e.g. Conesa et al., 2009, and Garcia-Mila et al., 2010). Our main interest here is the importance of capital-skill complementarity under such joint distributions.

When the production structure exhibits capital-skill complementarities as suggested by e.g. Stokey (1996) and Krusell *et al.* (2000), where skilled labour complements capital more than unskilled, capital-augmenting policies will be skill-biased and will thus increase the wage premium and inequality.[6] In an environment where the unskilled are also those agents who do not own capital stock, the benefits for the unskilled workers from a capital tax cut discussed above are likely to be small and thus the inequality effects of capital tax cuts higher. However, whether a capital tax cut creates benefits for the workers in such an environment depends on the production structure and the joint distribution of asset and skill heterogeneity.

Additionally, the inclusion of the transition period in the welfare evaluation of the reforms is crucial. In existing studies, this is because the benefits associated with the capital tax cut, in the form of higher labour productivity, materialise later in the lifetime of the worker (see e.g. Greulich and Marcet, 2008 and Garcia-Mila et al., 2010). Therefore, capital tax cuts increase inequality more immediately after the reform, compared to the long-run. However, when the production structure exhibits capital-skill complementarities, the timing of the effects on capital tax cuts on unskilled labour will depend on the evolution of the complementarities of capital with the different types of labour over time.

Furthermore, another line of research suggests when studying the transition path after a tax reform, considering alternative expectation generating mechanisms is also useful. For instance, Giannitsarou (2006), has shown that capital tax cuts, which necessitate learning on the part of the agents towards the new equilibrium, can reduce the desirability of such reforms. However, Giannitsarou worked in a representative agent framework and thus did not consider the case where agents might have heterogeneous initial conditions for learning after the reform.

With the above background in mind, this chapter aims to welfare-evaluate changes in income tax rates for different types of agents, in a model that allows for capital-skill complementarity and dynamics that can be influenced by heterogeneity with respect to initial conditions for learning. To isolate the effects of changes in each tax rate on all

[6] See e.g. Hornstein et al. (2005) for a review of the literature and empirical evidence on factor- and sector-specific technologies and inequality.

agents, we consider changes in tax rates that are not revenue neutral. Instead, given its current policy relevance, we consider tax reforms consistent with a lower steady-state debt-to-GDP ratio. Moreover, to focus on the interaction of asset and skill heterogeneity with a production structure that allows for different capital-labour complementarities, we also abstract from other sources of heterogeneity that have already attracted a lot of interest in the literature (i.e. stochastic or unobservable ability).

We calibrate our model to the UK economy, to assess the likely costs and benefits of tax reforms for the different agents. The UK is used to illustrate the quantitative analysis, since the data suggest significant heterogeneities, in both asset holdings and skill in the labour supply which are also generally positively correlated. According to the Family Resources Survey in 2008-2009, 28% of households do not have any savings, 53% have savings up to £ 20,000 and 19% have savings above £ 20,000.[7] Moreover, the Labour Force Survey of the Office for National Statistics[8], suggests that in 2003, 28% of the working population was employed in low-skill, semi-routine and routine occupations, whereas the remaining share worked in supervisory, technical, professional and managerial occupations. There is also support for associating skill with income group. For example, data from the Department for Education and Skills on the participation rates in higher education for different income groups show that the participation ratio was about three times higher in the 1990s for the three highest, relative to the three lowest groups.[9] Finally, the tax structure in the UK stands in stark contrast with other European countries, by having a very high capital to labour income tax ratio.[10]

Our modeling permits us to capture key features of heterogeneity. Following the literature on credit constraints and income inequality (see e.g. Galor and Zeira, 1993, Benabou, 1996 and Aghion and Howitt, 2009), financial intermediation costs allow our model to generate heterogeneity in savings, which is consistent with the UK data. In addition, we use a constant elasticity of substitution (CES) specification for the production function, fol-

[7] The survey is sponsored by the Department for Work and Pensions (see their Table 4.9 for the information reported here).

[8] See http://www.statistics.gov.uk/STATBASE/Expodata/Spreadsheets/D7665.xls.

[9] See www.statistics.gov.uk/STATBASE/Expodata/Spreadsheets/D7308.xls.

[10] See e.g. Martinez-Mongay (2000), for effective tax rates in European countries. More details on tax and other data used for the calibration are provided later in section 3.

lowing e.g. Stokey (1996) and Krusell et al. (2000), which assumes different degrees of complementarity with capital for skilled and unskilled labour. This allows our calibrated model to produce a wage premium that is in line with empirical studies.

We relax the assumption of rational expectations so that we can also consider an adaptive learning environment. This allows us to include an additional source of heterogeneity, in the form of the initial beliefs of the agents who need to learn the equilibrium laws of motion. This corresponds to an unequal distribution of information after the reform in the economy and, as far as we know, has not yet been considered in the tax reform literature.

In this sense, the intended contribution of this Chapter is two-fold. First, while the literature on the welfare effects of changes in income tax has allowed for joint inequality in asset holdings and labour productivity in evaluating capital taxation (see e.g. Domeij and Heathcote, 2004, Conesa *et al.*, 2009, and Garcia-Milà et al., 2010), our main interest is on the importance of the capital-skill complementarity under such joint distributions. This allows us to examine the post-reform evolution of wage inequality that is driven by an endogenous skill premium, and evaluate its contribution in determining the overall inequality effects of a given tax reform. Second, as said above, while the tax reform literature has in general assumed full-rationality of agents (with Giannitsarou, 2006, and Evans *et al.*, 2009, as relevant exceptions, but both assuming a representative-agent environment), we look to assess the additional implications of assuming alternative ways of forming expectations among different types of agents, a feature that is often known as a form of structural- and learning-heterogeneity in the adaptive learning literature, but which has received little attention in the fiscal policy field.[11]

By first focusing on the long-run, we show that tax cuts have sizeable distributional effects even when they are not met by a rise in another tax rate. Instead, these distributional effects work through the structure of the production and, more specifically, the complementarity between labour and capital. In particular, capital tax reductions are skill-biased and thus increase the skill premium and income inequality, consistent with the

[11] See, e.g., Giannitsarou (2003), Honkapohja and Mitra (2006) and Nunes (2009), for earlier discussions about the assumption of both structural and learning heterogeneity in general equilibrium models with applications in the monetary policy field.

results in He and Liu, 2008.[12] In fact, for elasticities of substitution between capital and unskilled labour, within the range of empirical estimates, there are income and welfare losses to unskilled workers in the long-run after the capital tax cut. On the other hand, we find that reductions in labour income taxes also disproportionately favour either skilled or unskilled labour and can reduce the welfare of the capitalists, resulting in reductions in inequality. These results are consistent with the literature on tax reforms discussed above, which suggests that capital taxation increases inequality, while labour taxation decreases it. In our model, the effects of tax policy on inequality are amplified by the skill-premium channel.

Next, by examining the transition to the post-reform steady-state, we find that the skill premium initially falls and then converges to the higher, post- reform levels. This result is driven by the fact that, in general equilibrium, the macroeconomic effects of the complementarity between capital and labour inputs is higher in the short-than in the long-run. In particular, following the tax reform, the relative skill supply increases, as the agents that hold the capital stock increase the skilled labour supply, to increase labour income and thus investment in capital, given the higher returns to capital. The increase in relative skill supply decreases the skill premium, thus providing short-run benefits to the unskilled workers. These benefits are reduced over time, as the capital stock is increased and the relative skill supply decreased.

This dynamic transition of the skill premium implies that the inequality effects of a capital tax cut are lower initially. Based on the findings of the tax reform literature to date, the benefits of capital tax cuts are generally expected to be higher in the long-run for both workers and capitalists. However, allowing capital accumulation to affect the skill premium implies that, in our analysis, the benefits for the workers are lower in the post-reform steady-state since capital and unskilled labour are substitutes. In contrast, capitalists and skilled workers benefit more from the higher capital stock as this is built up over time. Thus, including the initial periods helps to close the lifetime welfare gap between the agents and reduce the inequality effects of capital tax cuts.

[12] Our analysis for the UK also suggests that the combination of high effective capital tax rates with the complementarities in production imply that the tax revenue can be increased in the long-run by a reduction in the capital tax, since this will increase the tax revenue from labour income.

Finally, we show that heterogeneity in learning matters. Consistent with the results in Giannitsarou (2006), under homogeneous initial conditions in learning, the convergence to the new steady-state is slower and this results in welfare costs for the agents after a capital tax cut. However, heterogeneity in initial conditions implies learning dynamics that result in paths for the economic variables that exhibit overshooting relative to the rational expectations solution. In this case, the errors that the learners make in the adjustment process amplify their reaction to the tax reform, so that there is a form of an "irrational exuberance", which is beneficial to all learners. Similarly, heterogeneous learning implies welfare gains for the other tax reforms considered.

2.2 Model

In this section we construct a closed-economy DGE model comprised of a representative capitalist and representative skilled and unskilled workers who all consume output in the product market and supply labour in the factor market in return for labour income. The first two income groups, subject to intermediation costs, allocate savings to physical capital and government bonds in return for capital income whereas unskilled workers do not save. The representative firm is owned by the capitalist who hires (skilled and unskilled) labour services and leases physical capital from the factor market for which it pays the competitive wage and interest rate respectively. Finally, the government taxes economic activity, provides public spending and issues debt to balance its budget.

2.2.1 Population composition

The population size, N, is exogenous and constant. Among N, $N^c < N$ are identical capitalists, $N^s < N$ are identical skilled workers, and the rest, $N^u = N - N^c - N^s$, are identical unskilled workers. Capitalists are indexed by the subscript $c = 1, 2, ..., N^c$, skilled workers by $s = 1, 2, ..., N^s$ and unskilled workers by $u = 1, 2, ..., N^u$. There are also N^f firms, $f = 1, 2, ..., N^f$. We assume that the number of firms equals the number of capitalists, $N^c = N^f$, and that each capitalist owns one firm. It is useful, for what follows, to define $N^c/N = n^c$, $N^s/N = n^s$, $N^u/N = n^u = 1 - n^c - n^s$ and $N^f/N = n^f$.

2.2.2 Firms

Each firm produces a single output, Y_t^f, using physical capital, K_t^f, and two distinct types of labour, unskilled, $h_{u,t}^f$, and skilled, h_t^f, where skilled labour is relatively more complementary to capital than unskilled labour. The production function is given by a constant returns to scale (CRS) technology assumed to take a constant elasticity of substitution (CES) specification following e.g. Krusell *et al.* (2000) and He (2012):[13]

$$Y_t^f = A\left\{\mu\left(h_{u,t}^f\right)^{\alpha} + (1-\mu)\left[\rho\left(K_t^f\right)^{\nu} + (1-\rho)\left(h_t^f\right)^{\nu}\right]^{\frac{\alpha}{\nu}}\right\}^{\frac{1}{\alpha}} \tag{2.1}$$

where $A > 0$ is constant productivity; $0 < \alpha, \nu < 1$, are the parameters determining the factor elasticities, i.e. $1/(1-\alpha)$ is the elasticity of substitution between capital and skilled labour with respect to unskilled labour, whereas $1/(1-\nu)$ is the elasticity of substitution between capital and skilled labor; and $0 < \mu$, $\rho < 1$ are the share parameters. The above CES form allows us to capture the capital-skill complementarity, which is considered to be a main driver of the skill premium and wage inequality (see e.g. Krusell *et al.*, 2000, and Hornstein *et al.*, 2005).

Each firm acts competitively, taking prices and policy variables as given, and maximises profits given by:

$$\Pi_t^f \equiv Y_t^f - r_t^k K_t^f - w_t h_t^f - w_{u,t} h_{u,t}^f \tag{2.2}$$

subject to the technology constraint given by (2.1); where w_t and $w_{u,t}$ are, respectively, the wage rates of skilled and unskilled labour and r_t^k is the interest rate on capital.[14] The different roles in the production function for skilled and unskilled labour imply that there will be a skill premium for the former, in the sense that the ratio of w_t to $w_{u,t}$ will be larger than unity. We will calibrate the production function so that the implied factor input elasticities and the resulting wage premium are in line with empirical studies.

[13] Note that when $\nu = 1$ and $\rho > 1$ capital and skilled labour are perfect substitutes and, when $\mu > 0$ and $\alpha = 1$ and $\rho = 1$ then unskilled labor and capital are perfect substitutes.

[14] Note that, in equilibrium, profits, Π_t^f, are driven to zero due to perfect competition.

2.2.3 Budget constraints of capitalists

The representative capitalist owns one firm and receives its profits. He also receives income from providing skilled labour services, $h_{c,t}$, to the labour market and income from interests on his accumulated stock of financial assets, in the form of capital, $K_{c,t}$, and government bonds, $B_{c,t}$. The interest rate on government bonds is given by r_t^b. All these sources of income are taxed. In particular, financial asset and profit income are taxed at the constant rate τ^k, while labour income is taxed at the constant rate τ^h.

We assume that those agents holding assets need to pay intermediation or transaction premia due to imperfections in capital markets. For instance, these premia can represent the costs of gathering extra information relating to legal issues, asset-specific government regulations, intermediation fees and so on. We follow Persson and Tabellini (1992), and Benigno (2009), and assume a quadratic cost function such that the capitalist incurs a cost of $\varphi_c^k K_{c,t}^2$ for holding physical capital and of $\varphi_c^b B_{c,t}^2$ for holding government bonds, where $\varphi_c^b, \varphi_c^k > 0$ measures the size of the transaction costs. The presence of this capital market imperfection and of the associated transaction costs help the model to capture a feature of realism. However, their main contribution here is that they will allow us, as we shall see below, to capture household heterogeneity in asset holdings.

The capitalist uses his income for consumption, $C_{c,t}$, investment in capital, $I_{c,t}$, and investment in government bonds, $D_{c,t}$. He also receives average (per agent) transfers from the government, $\overline{G}_t \, (= G_t/N)$. Thus, his budget constraint is:

$$\begin{aligned} C_{c,t} + I_{c,t} + D_{c,t} = &\left(1-\tau^k\right)\left(r_t^k K_{c,t} + r_t^b B_{c,t}\right) + \\ &+ \left(1-\tau^k\right)\Pi_t^f + \left(1-\tau^h\right) w_t h_{c,t} + \overline{G_t} - \varphi_c^b B_{c,t}^2 - \varphi_c^k K_{c,t}^2 \end{aligned} \tag{2.3}$$

while the evolution of the stock of capital and government bonds, respectively, are given by:

$$K_{c,t+1} = (1-\delta) K_{c,t} + I_{c,t} \tag{2.4}$$

$$B_{c,t+1} = B_{c,t} + D_{c,t} \tag{2.5}$$

where $0 < \delta < 1$ is a depreciation rate and $K_{c,0}, B_{c,0} > 0$ are given.

2.2.4 Budget constraints of skilled workers

The problem of the skilled worker is similar to the capitalist's, except that he pays different transaction costs, so that the capital market imperfections affect him to a greater extent. We assume that firm ownership gives an insider advantage in financial transactions to the capitalist (due, for instance, to past experience, socioeconomic background, networks, etc.) and thus the size of the transaction costs is lower for the capitalist. The idea that capital market imperfections can explain heterogeneity has been extensively examined in the income inequality literature (see e.g. Galor and Zeira, 1993, Benabou, 1996, and Aghion and Howitt, 2009). Most of these models assume, for simplicity, that the intermediation cost is either infinite for some agents (and thus these agents are effectively excluded from the financial market) or zero. In this paper, we examine the case of non-zero, finite intermediation costs for both capitalists and skilled workers where $\varphi_c^b < \varphi_s^b$, $\varphi_c^k < \varphi_s^k$. We differentiate the skilled worker and capitalist even further by assuming that the former has lower initial holdings of capital and government bonds, i.e. $K_{s,0} < K_{c,0}$, $B_{s,0} < B_{c,0}$.

Accordingly, the budget constraints and the evolution equations for capital and government bonds for the s^{th} skilled worker are:

$$\begin{aligned} C_{s,t} + I_{s,t} + D_{s,t} = \left(1-\tau^k\right)\left(r_t^k K_{s,t} + r_t^b B_{s,t}\right) + \\ + \left(1-\tau^h\right) w_t h_{s,t} + \overline{G}_t - \varphi_s^b B_{s,t}^2 - \varphi_s^k K_{s,t}^2 \end{aligned} \tag{2.6}$$

$$I_{s,t} = K_{s,t+1} - (1-\delta) K_{s,t} \tag{2.7}$$

$$D_{s,t} = B_{s,t+1} - B_{s,t}. \tag{2.8}$$

2.2.5 Budget constraint of unskilled workers

Unskilled workers differ from capitalists and skilled workers in two important respects. First, they start with zero initial holdings of assets and capital market imperfections result in them being excluded from the financial markets as in the models of Benabou (1996) and Aghion and Howitt (2009).[15] Second, we assume that exclusion from capital

[15] See e.g. Aghion *et al.* (1999) for a microeconomic rationalisation of credit constraints that do not allow agents to participate in asset markets.

markets does not allow them to acquire the skills to provide skilled labour services, so that their labour effort differs, in nature, from the labour effort of the other two types of agents. Evidence from the UK, introduced later, suggests that skill acquisition, in the form of University education, is indeed related to socioeconomic income group.

Thus, the budget constraint of the u^{th} unskilled worker is:

$$C_{u,t} = (1 - \tau^u)\, w_{u,t} h_{u,t} + \overline{G}_t \tag{2.9}$$

where $0 \leq \tau^u < 1$ is the tax rate on unskilled labour, $h_{u,t}$ is the labour supply and $C_{u,t}$ is the consumption.

2.2.6 Utility function and optimal choices of agents

Each type of household $i = c, s, u$ maximises:

$$E_0 \sum_{t=0}^{\infty} \beta^t u\,(C_{i,t}, h_{i,t}) \tag{2.10}$$

subject to the relevant budget constraints given above; where E_0 is the conditional expectations operator.

We use the instantaneous utility function:

$$u_{i,t} = (C_{i,t}, h_{j,t}) = \frac{\left[(C_{i,t})^{\mu}\,(1 - h_{i,t})^{1-\mu}\right]^{1-\sigma}}{1 - \sigma} \tag{2.11}$$

where $0 < \mu < 1$ is the consumption weight in utility and $\sigma > 1$ is the coefficient of relative risk aversion.

Therefore, to maximise discounted lifetime utility, the representative capitalist chooses $\{C_{c,t}, h_{c,t}, K_{c,t+1}, B_{c,t+1}\}_{t=0}^{\infty}$ subject to $(2.3 - 2.5)$. Meanwhile, the representative skilled worker chooses $\{C_{s,t}, h_{s,t}, K_{s,t+1},\ B_{s,t+1}\}_{t=0}^{\infty}$ subject to $(2.6 - 2.8)$, and finally the representative unskilled worker chooses $\{C_{u,t}, h_{u,t}\}_{t=0}^{\infty}$ subject to (2.9).

2.2.7 Government budget constraint

Following the literature on tax reforms (see e.g. Lucas, 1990, Cooley and Hansen, 1992, Giannitsarou, 2006, Garcia-Milà *et al.*, 2010, and Angelopoulos, *et al.*, 2012), we do not model government spending. Instead, government expenditure takes the form of transfers to the private agents, G_t. To finance these, it taxes income from labour and financial assets and issues government bonds, B_t. The budget constraint of the government is thus given by:

$$G_t + \left(1 + r_t^b\right) B_t = B_{t+1} + N^c[\tau^k \left(r_t^k K_{c,t} + r_t^b B_{c,t}\right) + \tau^h w_t h_{c,t}] +$$

$$+ N^s[\tau^k \left(r_t^k K_{s,t} + r_t^b B_{s,t}\right) + \tau^h w_t h_{s,t}] + N^u[\tau^u w_{u,t} h_{u,t}]. \tag{2.12}$$

2.2.8 Market-clearing conditions

The market clearing conditions for the capital, bond, skilled and unskilled labour and product markets respectively are:

$$N^f K_t^f = N^c K_{c,t} + N^s K_{s,t} \tag{2.13}$$

$$B_t = N^c B_{c,t} + N^s B_{s,t} \tag{2.14}$$

$$N^f h_t^f = N^c h_{c,t} + N^s h_{s,t} \tag{2.15}$$

$$N^f h_{u,t}^f = N^u h_{u,t} \tag{2.16}$$

$$N^f Y_t^f = N^c C_{c,t} + N^s C_{s,t} + N^u C_{u,t} + N^c \left[K_{c,t+1} - (1-\delta) K_{c,t}\right] + \tag{2.17}$$

$$+ N^s \left[K_{s,t+1} - (1-\delta) K_{s,t}\right] + N^c \left(\varphi_c^b B_{c,t}^2 + \varphi_c^k K_{c,t}^2\right) +$$
$$+ N^s \left(\varphi_s^b B_{s,t}^2 + \varphi_s^k K_{s,t}^2\right)$$

where (2.17) gives the aggregate resource constraint of the economy.

2.2.9 Decentralised competitive equilibrium

The decentralised competitive equilibrium (DCE) is defined when (i) households and firms optimise, taking prices and policy as given; (ii) all constraints are satisfied; and (iii) all markets clear. After the relevant substitutions, we summarise the DCE in the paths of the following variables: $(C_{c,t}, C_{s,t}, C_{u,t}, h_{c,t}, h_{s,t}, h_{u,t}, w_t, w_{u,t}, K_{c,t+1}, K_{s,t+1}, B_{c,t+1}, B_{s,t+1}, r_t^k, r_t^b)$ given the exogenously set stationary processes for technology and fiscal policy instruments which are discussed below.[16] We define the relevant aggregate, economy-wide quantities as, X_t, for $X_t = \{C_t, I_t, K_t, B_t, Y_t\}$.

Given that we wish to analyse the welfare implications of permanent tax regime changes, all tax rates are treated as exogenous constants, $0 \leq \tau^k, \tau^h, \tau^u < 1$. In the policy reforms that we will examine, the economy will start from the steady-state and will be subjected to an exogenous, permanent change in one or more tax instruments, holding the other policy instruments, including G, constant at the pre-reform steady-state values. We examine economic outcomes and welfare in the new steady and during the transition period to the new steady-state.

2.3 Calibration and steady-state

In Table 2.2, we next calibrate the structural parameters of the model so that its steady-state solution, reported in Table 2.1, reflects the main empirical characteristics of the UK economy. The calibration also provides empirical justification for the key modelling decisions made above.

2.3.1 Population shares

We first wish to map out agent heterogeneity and thus distinguish the three types of households by their differing shares in the population, n^i. According to the Family Resources Survey in 2008-2009, 28% of households do not have any savings, 53% have savings up

[16] To save space we have not reported the DCE system here but it is provided in Appendix B.

to £20,000 and 19% have savings above £20,000.[17] In light of this, since we assume that unskilled workers do not have savings, we set n^u equal to 30%. At the other end of the distribution, since we model capitalists as the income group with the highest share of savings and assets, we set n^c to 20% implying that n^s is 50%.

Other data providing an additional dimension by which unskilled workers differ from skilled workers and capitalists is that the former group offers a labour input that is lacking in skills. According to the Labour Force Survey of the Office for National Statistics[18], in 2003, 28% of the working population was employed in semi-routine and routine occupations, whereas the remaining share worked in supervisory, technical, professional and managerial occupations, which require an increasingly higher skilled labour input. Moreover, according to data from the Department for Education and Skills on the participation rates in higher education for different income groups, the participation ratio was about three times higher in the 1990s for the three highest, relative to the three lowest groups.[19] Thus, there appears to be adequate support for associating skill with income group.[20]

2.3.2 Productivity

We next turn to heterogeneity in productivity and returns to labour, which governs the choice of the relevant production parameters. Using the estimates in Krusell *et al.* (2000), we set $\nu = -0.495$ and $\alpha = 0.401$ implying elasticities of substitution between capital and skilled labour and between capital (or skilled labour) and unskilled labour of about 0.67 and 1.67 respectively. As discussed in Krusell *et al.* (2000), and Hornstein *et al.* (2005), these estimates cohere well with the microeconometric evidence reported in the literature.

[17] The survey is sponsored by the Department for Work and Pensions (see their Table 4.9 for the information reported here).

[18] See `http://www.statistics.gov.uk/STATBASE/Expodata/Spreadsheets/D7665.xls`.

[19] See `www.statistics.gov.uk/STATBASE/Expodata/Spreadsheets/D7308.xls`.

[20] Even though the chosen population shares allow the model to succesfully replicate a number of key UK data values, the information considered for this calibration has a couple of caveats that must be acknowledged. First, due to data limitations and availability lags, some of the values considered are quite recent (i.e. 2009) while some other are less so (i.e. 2003). Second, it is reasonable to expect that not all households that hold no savings are in fact unskilled or that not all those with savings above £20.000 are in fact capitalists. In this sense, this calibration should be taken as indicative rather than fully precise. Nevertheless, given the data limitations in this case, the proposed calibration probably yields the best aproximations available to the share parameters of interest.

Table 2.1: Parameter Values

parameter	value	definition
$0 < \frac{1}{1-\alpha} < 1$	1.669	capital/skilled labour to unskilled labour elasticity
$0 < \rho < 1$	0.645	capital weight in composite input share
$0 \leq \delta \leq 1$	0.060	depreciation rate on private capital
$0 < \frac{1}{1-\nu} < 1$	0.669	capital to skilled labour elasticity
$0 < \mu < 1$	0.275	unskilled labour weight
$\varphi_c^k, \varphi_c^b > 0$	0.004	transaction costs, capitalists
$\varphi_s^k, \varphi_s^b > 0$	0.020	transaction costs, skilled workers
$0 < \beta < 1$	0.976	rate of time preference
$0 < \gamma < 1$	0.347	consumption weight in utility
$\sigma > 1$	2.000	coefficient of relative risk aversion
$0 < n^c < 1$	0.200	population share of capitalists
$0 < n^s < 1$	0.500	population share of skilled workers
$0 < G/Y < 1$	0.313	public spending share of output
$0 < \tau^h < 1$	0.300	labour tax rate, skilled
$0 < \tau^k < 1$	0.442	capital tax rate
$0 < \tau^u < 1$	0.200	labour tax rate, unskilled

Our calibration of $\rho = 0.645$, $\mu = 0.275$ and $A_0 = 1.65$ allows us to obtain the labour share of income, the skill premium and the capital to output ratio consistent with the UK data.

2.3.3 Savings

Heterogeneity in savings is controlled for, as explained in the previous section, by the parameters that govern transaction costs in the financial markets. Following the models in e.g. Galor and Zeira (1993), Benabou (1996) and Aghion and Howitt (2009), we set these costs to infinity for the unskilled workers, which implies that these agents do not have any savings. As said above, about 28% of the UK households do not save. Regarding the households with positive savings, data from the Family Resources Survey of 2008-2009 suggest that households in the highest saving bracket have five times higher savings than the other savers, on average. In terms of our model, this difference is applied to the representative capitalist and skilled worker by setting the transaction costs for the latter to be five times greater than the former. For simplicity, we set this cost in capital asset markets to be the same in the bond market. We chose the level of the transaction costs parameter, so that in combination with an annual depreciation rate, δ, of 6%, the total ratio of capital to GDP in the steady-state is about 2 and the transaction costs are about 1% of asset holdings. The latter is broadly consistent with the average difference between the lending and borrowing rates in the UK (see, e.g. World Development Indicators - WDI - database) over the past 30 years.

2.3.4 Effective tax rates

Effective average tax rates for capital and labour income are constructed by following the approach in Conesa *et al.* (2007). We use data from the National Accounts and the Public Sector, Taxation and Market Regulation databases (available from OECD Statistics), to obtain the series for 1970-2005. The average capital tax rate over the time period is $\tau^k = 0.442$, while the average labour income rate is 0.27. Using data from Social Trends 38, Office for National Statistics, we are able to approximate the progressivity of the UK income tax system at about 1.6.[21]

A ratio of $\tau^h/\tau^u = 1.6$, together with the requirement that the weighted average of the two tax rates equal the effective labour income tax rate, would imply that $\tau^h = 0.304$ and $\tau^u = 0.19$. However, the progressivity of income taxation probably overestimates the progressivity of labour income taxation, which is our interest here. This is because, in light of the data discussed, we would expect the higher income brackets to have more capital income compared to lower income brackets. On the other hand, the lower the progressivity ratio, the higher the implied value of τ^u. We thus use a progressivity ratio of $\tau^h/\tau^u = 1.5$ for the calibration, which guarantees that τ^u is equal to the base income tax rate. Accordingly, we approximate the lower tax rate, τ^u, at 20%, and the higher labour income tax rate, τ^h, at 30%.

2.3.5 Parameters common to all agents

We next approximate the rate of time preference, β, so that $1/\beta$ is equal to 1 plus the *ex-post* real interest rate, where we use real interest rate data from OECD Main Economic

[21] This is obtained by calculating the average income tax rate that applies approximately to the lower 30% and the upper 70% of the tax payers. We then add the national insurance contribution rate of 11% and calculate the ratio of these two effective average tax rates.

Indicators, from 1970-2005. This gives a value 0.976 for β. Following Kydland (1995), we set μ, the weight given to consumption relative to leisure in the utility function, equal to the average value of work versus leisure time, which is obtained using data on hours worked from the OECD Economic Outlook database, from 1970-2005.[22] We also use a common value from the literature for the intertemporal elasticity of consumption, $1/\sigma = 0.5$ or $\sigma = 2$.

Given that we will evaluate policies that reduce the debt-to-GDP ratio below, we calibrate the share of government spending in GDP, G/Y to a reasonable value of 31%, to obtain a B/Y ratio of 70% based on official forecasts for 2011-2013 (see e.g. the Pre-Budget Forecast, June 2010, Office for Budget Responsibility)[23].

2.3.6 Steady-state

The steady-state solution of the model is given in Table 2.2 below in terms of the aggregate variables. The figures show that the capitalists consume in total 19.4% of total income (or about 23% of total consumption)[24], skilled workers consume in total 44% of total income (or around 52% of total consumption) and unskilled workers consume in total 22.3% of total income (or approximately 26% of total consumption). In addition, the capitalists in total own about 67% of the capital and government bonds in the economy. As said above, the ratio of savings, I_c/I_s, and assets, K_c/K_s and B_c/B_s, of the representative capitalist to the representative skilled worker, are equal to five. Note also that the net (i.e. after depreciation, tax and transaction costs) interest rates on capital and bonds, are given respectively by:

$$\widetilde{r}_k = r^k(1-\tau^k) - \delta - 2\varphi_c^k\left(\frac{n^c}{n^c+n^s}\right)K_c - 2\varphi_s^k\left(\frac{n^s}{n^c+n^s}\right)K_s \tag{2.18}$$

$$\widetilde{r}_b = r^b(1-\tau^k) - 2\varphi_c^b\left(\frac{n^c}{n^c+n^s}\right)B_c - 2\varphi_s^b\left(\frac{n^s}{n^c+n^s}\right)B_s \tag{2.19}$$

[22] To obtain this we divide total hours worked by total hours available for work or leisure, following Ho and Jorgenson (2001). They assume that there are 14 hours available for work or leisure per day with the remaining 10 hours accounted for by physiological needs. This implies that μ is set to 0.35.

[23] See `http://budgetresponsibility.independent.gov.uk.`

[24] This is calculated as $\frac{(N^c*C_c)/Y}{C/Y} = (N^c*C_c)/C$. The same formula is used below for similar quantities.

Table 2.2: Steady-state (pre-reform)

variable	value	variable	value
$\frac{N^c C_c}{Y}$	0.194	$\frac{w(N^c h_c+N^s h_s)+w_u N^u h_u}{Y}$	0.633
$\frac{N^s C_s}{Y}$	0.444	$\frac{N^c K_c}{Y}$	1.275
$\frac{N^u C_u}{Y}$	0.223	$\frac{N^s K_s}{Y}$	0.637
$\frac{C}{Y}$	0.861	$\frac{K}{Y}$	1.912
$\frac{N^c I_c}{Y}$	0.077	$\frac{N^c B_c}{Y}$	0.467
$\frac{N^s I_s}{Y}$	0.038	$\frac{N^s B_s}{Y}$	0.233
$\frac{I}{Y}$	0.115	$\frac{B}{Y}$	0.700
w	1.375	$\widetilde{r}_k$	0.025
w_u	1.006	$\widetilde{r}_b$	0.025
$\frac{w}{w_u}$	1.366	U_c	-63.02
h_c	0.166	U_s	-68.83
h_s	0.236	U_u	-73.17
h_u	0.235	U_a	-68.97

and are equal in the steady-state. The Table shows that these returns are 2.5%, which compares favourably with the 1970-2005 average UK real interest rate, equal to 2.1% in the WDI database.

It is next worth noting that the ratio of average hours worked by unskilled workers to the average hours worked by skilled labour in the model is 1.1, which is the same as in the UK Labour Force Survey (LFS) data.[25] These work-time allocations imply Frisch (or λ-constant) labour supply elasticities of 3.97 for capitalists, 2.18 for skilled and 2.19 for unskilled workers, which are similar to values calibrated in macro models (see e.g. Browning *et al.* (1999), Chetty *et al.* (2011), and Keane and Rogerson (2012), for a discussion regarding micro and macro elasticities). The value for the capitalists suggests that, as expected, this group is the relatively least dependent on labour income, and is consistent with the research in Low (2005), and Domeij and Floden (2006), which suggests that agents without full access to asset markets prefer increased work hours. Table 2.2 also shows that the labour's share of income in the model, $\frac{w(N^c h_c+N^s h_s)+w_u N^u h_u}{Y} = 0.633$ is close to the value (i.e. 0.601) obtained from the OECD's International Sectoral Database (ISDB) for 1970-2005.

[25] The ratio is calculated as $\frac{h_u}{[(N^c/(N^c+N^s))*h_c+(N^s/(N^c+N^s))*h_s]}$. The data refer to average actual weekly hours of work by industry sector from 1997-2012. Unskilled and skilled hours are obtained respectively by averaging over industries A-I and J-Q reported in the UK LFS.

Turning to the skill premium in the UK, Walker and Zhu (2008), estimate a college premium (defined as the log difference between the wage rate for skilled and unskilled labour) of about 18% for males and 28% for females, while Machin (1996), computes the ratio of wages between non-manual and manual jobs in manufacturing that ranges between 1.3 and 1.5, from 1970 to 1990. For the US, Hornstein *et al.* (2005), report a college premium, in terms of wage ratios, that ranges from about 1.47 in 1982 to 1.79 in 2000. The skill premium predicted by our calibration is $w/w_u = 1.37$ or $ln(w/w_u) = 31\%$, which is thus consistent with the empirical evidence cited above.

Finally, note that in the steady-state, capitalists work considerably less than skilled and unskilled workers, who work more or less the same time (see the $h's$ in Table 2.2) while $C_c = 0.439$, $C_s = 0.402$ and $C_u = 0.331$. Thus in terms of welfare, U, higher consumption and lower work effort make the capitalists better off, followed by the skilled and unskilled workers, respectively. The weighted average measure of aggregate or Benthamite lifetime utility, U_a, is also reported.[26]

2.4 Model Solution

To solve the model, we start by taking the first-order Taylor series expansion of the DCE and exogenous process for productivity around their respective steady-states. For any variable X_t, these values are denoted $\widehat{X}_t = \log X_t - \log X$. We next re-express the model in matrix form as second-order difference equation system:

$$\begin{aligned}
\mathbf{x}_t &= \mathbf{M}_1\mathbf{E}_t\mathbf{x}_{t+1} + \mathbf{M}_2\mathbf{x}_{t-1} + \mathbf{M}_3\mathbf{z}_t \\
\mathbf{y}_t &= \mathbf{N}_1\mathbf{x}_t + \mathbf{N}_2\mathbf{x}_{t-1} + \mathbf{N}_3\mathbf{z}_t + \mathbf{N}_4\mathbf{E}_t\mathbf{x}_{t+1} \\
\mathbf{z}_t &= \boldsymbol{\rho}\mathbf{z}_{t-1} + \mathbf{u}_t.
\end{aligned} \tag{2.20}$$

where $\mathbf{x}_t = \left[\hat{B}_{c,t+1}, \hat{K}_{c,t+1}, \hat{B}_{s,t+1}, \hat{K}_{s,t+1}\right]'$ contains the endogenous state variables; $\mathbf{y}_t = \left[\hat{C}_{c,t}, \hat{C}_{s,t}, \hat{C}_{u,t}, \hat{h}_{c,t}, \hat{h}_{s,t}, \hat{h}_{u,t}, \hat{r}_t^b, \hat{r}_t^k, \hat{w}_t, \hat{w}_{u,t}\right]'$ the endogenous control variables; and $\mathbf{z}_t =$

[26] The lifetime utility of agent i is given by $U_i = \frac{(1-\beta^T)}{1-\beta}\overline{u}_i$, for $i = c, s, u$, where $\overline{u}_i$ is the welfare of i calculated at the steady-state using (2.11) and $T = 1000$. Also note that $U_a = n^c U_c + n^s U_s + n^u U_u$.

$[\hat{a}_{t+1}]$ the exogenous state variables.[27] The various $\mathbf{M}$ and $\mathbf{N}$ matrices contain convolutions of the structural parameters calibrated in Table 2.1. Finally, only for us to be able to examine the behavior of adaptive learning under an stochastic environment, we will consider one exogenous state variable, i.e. total factor productivity (TFP). Hence, $\boldsymbol{\rho} = \rho^a$ is the autoregressive coefficient of the AR(1) process assumed to be followed by this variable, while $\mathbf{u}_t = \varepsilon_{t+1}$ is a normally distributed exogenous shock term.

In Appendix B we use (2.20) to briefly describe how we obtain both the rational expectations (RE) and adaptive learning (AL) solutions of the log-linearised model.

2.5 Tax reforms

In this section, we examine five different tax reforms that meet a debt-to-GDP target of 60% in the steady-state. The latter provides us with a common base for conducting the policy reforms.[28] We start by changing the capital income tax rate, τ^k, holding all other rates constant. Next we examine changes in the labour income tax rates, first on skilled labour, τ^h, and second on unskilled labour, τ^u, each implying that the progressivity of labour income taxation has been altered. We then examine the case where the government changes the effective average labour tax rate, i.e. τ^h and τ^u move proportionately, so that the progressivity in the labour income taxation remains unaffected. Lastly, we evaluate the distributional effects of varying all tax rates proportionately.

For each tax reform considered, we find the steady-state tax rate(s) required to obtain the target debt-to-GDP ratio and welfare-evaluate this tax reform in terms of its aggregate and distributional consequences in the long-run.[29] We also study the transition path by starting the economy at its pre-reform steady-state, implementing the required permanent

[27] Other papers in the literature using this particular reduced form are e.g. Giannitsarou (2006), and Carceles-Poveda and Giannitsarou (2007 and 2008).

[28] Given that we seek to evaluate the distributional effects of tax reforms and not the optimal size of the government or government debt, we take this debt target as given. Hence, we do not evaluate the potential welfare benefits from reducing the debt-to-GDP ratio, in the form of, for instance, lowering the cost of borrowing for the government and reassuring financial markets that there is no risk of default.

[29] Note that a lower level of debt in the steady-state implies that there will also be a reduction in interest payments on debt and thus in total government spending, assuming, as we do here, that the remaining components of government spending do not change. Hence, tax reforms consistent with a lower level of steady-state debt will need to generate a lower level of total tax revenue.

tax reform and then simulating the response of economy until it reaches the new steady-state. This allows us to calculate lifetime welfare under both full-rationality and adaptive learning.

2.5.1 Laffer curves in tax revenue and debt

Prior to undertaking the welfare analysis, it is first useful to demonstrate the long-run general equilibrium effects of tax changes on factor returns and quantities by examining the effect of tax changes on the tax revenue from all tax bases.[30] The relationship between the tax revenue from a particular tax base and the associated tax rate is, in general, given by a Laffer curve (see e.g. Schmitt-Grohé and Uribe, 1997). In our model, changing a tax rate can lead to either increases or decreases in the tax revenue collected from this tax base, depending on whether the economy is on the upward or downward slopping part of the curve, respectively. In the CES production function with capital-skill complementarity that we employ, a tax rate change will have spillover effects to the tax revenue collected from the other tax bases. For instance, an increase in the capital tax rate will decrease the capital supply, but will tend to increase or decrease the supply of unskilled labour, depending on whether the latter substitutes for or complements capital in production. Thus, the tax revenue collected from the tax base of unskilled labour income can either rise or fall after an increase in the capital tax.

As an illustration, we plot the Laffer curves associated with changes in τ^k in Figure 2.1. The B/Y curve (lower-right panel) indicates that the target for the debt to GDP ratio can be obtained by either increasing or decreasing τ^k to to 65.3% or 40.7%, respectively. The relationship between tax revenue from assets and the capital tax rate (upper-right panel) shows that the economy is on the upward slopping part of this Laffer curve. Increasing τ^k increases the tax revenue collected from capital, while falls in τ^k decrease tax revenue from this source. However, the upper-left and upper-middle panels in the Figure suggest that decreases in τ^k crowd-in both skilled and unskilled labour and, accordingly, the tax revenue from these sources increases.

[30] Since in this exercise we are concerned about the long-run effects of tax reforms on total tax revenues, only the RE solution of the model is considered here.

Table 2.3: Tax reforms and steady-state equilibria (B/Y=0.6)

	fall in tax rates			rise in tax rates		
	τ^k	τ^h	τ^u	τ^k	τ^h	τ^u
base ($B/Y = 0.7$)	0.442	0.300	0.200	0.442	0.300	0.200
τ^k	0.407	-	-	0.653	-	-
τ^h	-	0.284	-	-	0.696	-
τ^u	-	-	0.161	-	-	0.817
τ^u, τ^h	-	0.287	0.191	-	0.700	0.467
τ^u, τ^h, τ^k	0.429	0.292	0.194	0.674	0.456	0.304

The UK economy appears to be near the peak of the total tax revenue and debt Laffer curves with respect to τ^k. The results for the tax revenue Laffer curve are very similar to those reported in Trabandt and Uhlig, 2012, for the UK. In particular, both models predict for this economy that the gain in tax revenue by increasing the capital tax to the point where the tax revenue is maximised is only a few percentage points.[31]

The results for the Laffer curves associated with the remaining tax instruments are, in general, similar.[32] They also imply that, consistent with the analysis in Schmitt-Grohé and Uribe (1997), for a given level of debt, when a tax rate is the variable that is chosen to satisfy the government budget constraint, there can be two long-run solutions.[33] In Table 2.3 we summarise the tax changes required to obtain steady-state equilibria that cohere with the target B/Y ratio of 60%. Given the Laffer curves in tax revenue and debt discussed above, this target is consistent with both increases and decreases in tax rates. Table 3 suggests that reductions in each of the taxes individually or jointly are generally smaller than the respective increases.

[31] This is despite the use of different models. Trabandt and Uhlig, 2012, use a representative agent model, with a Cobb-Douglas production function and allow for monopolistic competition in the product market.

[32] These are not presented to save space but are available on request.

[33] A critical condition for this is that a Laffer curve exists with respect to total tax revenue. Further note that Schmitt-Grohé and Uribe (1997), also discuss the parameter range under which some of these *equilibria* can be indeterminate. For our model and the calibrated parameters for the UK, all solutions obtained below are saddle-path stable.

Figure 2.1: Laffer curves for changes in the capital tax

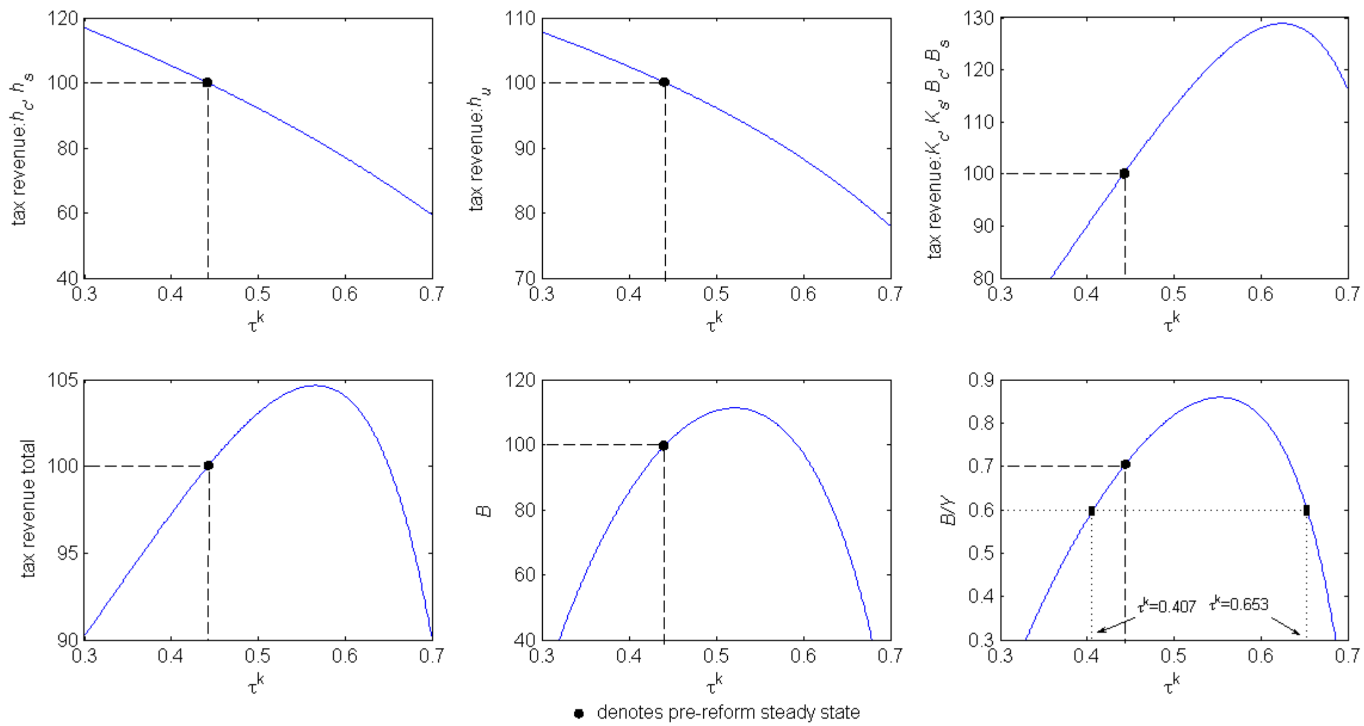


Table 2.4: Steady-state welfare gains/losses relative to pre-reform economy

	fall in tax rates			rise in tax rates		
	Capitalist	Skilled	Unskilled	Capitalist	Skilled	Unskilled
τ^k	0.0237	0.0211	0.0083	-0.2360	-0.1740	-0.0652
τ^h	0.0007	0.0100	0.0032	-0.3041	-0.2997	-0.1070
τ^u	-0.0082	-0.0011	0.0243	-0.1084	-0.0878	-0.2905
τ^u, τ^h	-0.0011	0.0076	0.0079	-0.3261	-0.3130	-0.2170
τ^u, τ^h, τ^k	0.0079	0.0127	0.0082	-0.3318	-0.2722	-0.1470

2.5.2 Evaluation of tax reforms in the long-run

We next calculate the welfare for each agent at the steady-state of these equilibria and present, in Table 2.4, the welfare gains/losses relative to the pre-reform economy.[34] To calculate these welfare changes, we follow Lucas, 1990, and compute the percentage extra consumption that an individual would require so as to be equally well off between the two regimes. This is defined as:

$$\xi_i = \left(\frac{U_{i,ss}^{post}}{U_{i,ss}^{pre}} \right)^{\frac{1}{\mu(1-\sigma)}-1} \tag{2.21}$$

for each agent $i = c, s, u$, where ss denotes welfare calculated in the steady-state.

The first observation regarding the results in Table 2.4 is that, as expected, welfare is always reduced for all agents for increases in tax rates. Therefore, we do not consider these equilibria further in the analysis which follows.

Regarding the fall in tax rates, the results in Table 4 show that there are different welfare effects on the agents. In general, tax cuts imply gains (or, at least, no losses) for all types of agents, with the exception of reductions in τ^u or τ^u, τ^h combined and thus are not Pareto improving. The biggest welfare gains at the aggregate level are obtained for a capital tax cut. However, this is also the tax reform with the largest distributional effects, ranging from sizeable welfare gains for the agents that own capital and supply skilled labour, to near-zero welfare gains for unskilled workers. This trade-off between efficiency and equity is central to the analysis of capital tax reforms and is well-documented in the

[34] Given that in all cases discussed learning converges to the fully-rational post-reform equilibrium, we only report the results under full rationality in this part.

related literature (see e.g. Domeij and Heathcote, 2004, and Garcia-Mila *et al.*, 2010). However, here it is obtained for a capital tax cut that is not followed by a labour tax increase.

The key to interpreting these results lies in the interaction of the asset and skill inequalities with the structure of production. As discussed above when analysing Figure 2.1, a fall in τ^k increases the capital stock and this raises the productivity of both types of labour, so that labour supply and labour income are increased. Therefore, workers also gain by a reduction in the capital tax. This positive productivity spillover effect is an important driver of the zero long-run optimal capital tax results in models that assume a relatively high complementarity between the labour input of the worker and capital stock (e.g., as in models using Cobb-Douglas production functions).

However, consistent with Krusell *et al.*, (2000), a higher capital stock benefits skilled more than unskilled labour, so that the wage premium increases to 32.9% (implying a wage ratio of 1.39) after the reform. Hence, in this model, capital-skill complementarities work to amplify the inequality implications of capital tax cuts. In contrast, reductions in τ^u or τ^u and τ^h result in increases in unskilled labour, which in turn increase skilled labour but crowd out capital, thus leading to lower capital income.[35]

The general message from the above analysis is that the complementarity and/or substitutability between factor inputs is important when assessing the effects of tax reforms. This finding is consistent with related research which has emphasised the importance of different patterns of production and sector- and factor-specific technical changes on inequality (see e.g. Hornstein *et al.*, 2005, for a review). Here, the tax reform plays a similar role to factor-specific technological progress given the way it affects factor returns and productivity (see also e.g. He and Liu, 2008). By reducing τ^k or τ^h the government is effectively introducing a skill-biased change, while reductions in τ^u favour the unskilled.

He and Liu, 2008, also evaluate the effect of capital tax cuts on the skill premium for a model that is calibrated to US data and conclude that the capital tax cuts will lead to

[35] Note that by reducing interest payments in the steady state, the tax cuts considered here imply an additional channel through which they affect the agents differently. Namely, debt in the steady-state represents assets to skilled workers and capitalists. Hence, its reduction implies, *ceteris paribus*, a reduction in an income source for these two agents, but not for unskilled workers. This hurts capitalists and skilled workers, especially when the tax rate on unskilled labour falls.

modest increases in the skill premium. In particular, the elimination of the capital tax and its substitution with labour taxes, results in an increase of the skill premium of about 3.3% in their model.

In our model for the UK, the effects of the capital tax cut on this premium are bigger, since the skill premium rises by 1.8% for a small reduction in the capital tax, by 7.9%. Our model differs in two important ways.[36] First, we allow for agents that differ in both capital ownership and skill supply, whereas He and Liu, 2008, use a representative agent model. The higher concentration of capital that we assume, consistent with the British data, tends to increase the impact of a capital tax cut on the skill premium. In particular, given that the marginal propensity to save increases with income, the increase in the supply of capital after the capital tax cut is expected to be higher in a society characterised by higher concentration of wealth. Second, He and Liu, 2008, allow for endogenous skill formation, so that, in their model a capital tax cut also leads to a larger rise in the relative skill supply, which acts to moderate the skill premium. In light of these findings, our long-run quantitative results can be interpreted as an upper bound on changes in inequality. Nevertheless, for shorter horizons, the composition of skill in the population is more likely to remain unchanged..

2.5.3 Skill premium and inequality during the transition

We next evaluate the aggregate and distributional effects of the above tax reforms over the lifetime of the agents, including the transition period, paying particular attention to how agents form expectations after the reform. First, we evaluate the lifetime welfare of all agents,[37] as they converge to the post-reform steady-state starting from the current economy, assuming rational expectations (RE). In this case, the agents adjust their choices to the new tax rates immediately when the reform takes place.

Second, we evaluate lifetime welfare assuming an adaptive learning (AL) environment

[36] Note also that the policy experiments are different, since He and Liu, 2008, consider a capital tax cut that is met by a labour tax rise, whereas we isolate the effects of the capital tax cut, by allowing the level of debt and interest payments on debt to adjust.

[37] We calculate conditional welfare or discounted lifetime utility using equation (3.6) and a time horizon of 1000 periods.

in which the agents have fully learned the pre-reform rational expectations solution but now must learn the coefficients of their reduced form policy functions associated with the post-reform economy by using a recursive least squares (RLS) learning algorithm, which is widely used in the AL literature. The intuition behind this exercise is that, even though agents might have some preliminary information regarding a policy change to be enacted soon as it is often announced by the government, they are nevertheless quite likely to have no or very little information on the effects of such changes on the aggregate and, in general, on the implications of the new steady-state that can be achieved as a result of the reform, and will thus require to learn about these.[38]

Here, we examine two scenarios which determine the initial conditions for learning. In the first, which serves to contextualise our results relative to the literature, we follow e.g. Giannitsarou (2006) and Evans *et al.* (2009) and assume that the agents start learning using the reduced form coefficients that correspond to the pre-reform economy. In the second, we assume that there is heterogeneity in the initial conditions used for learning, capturing, for instance, unequal information regarding the tax reform, so that a subset of the population - i.e. the capitalists - can make a better initial guess regarding the coefficients in its policy function.[39]

It is worth noting that in the first scenario of AL with homogeneous initial beliefs, the welfare effects of all tax reforms for all agents are effectively the same as under the RE solution, consistent with the results in Giannitsarou (2006).[40] Hence, to save space, we do not discuss results from this solution further and only present results from rational expectations (ξ_{re}) and AL with heterogeneous initial beliefs (ξ_{al}), henceforth heterogenous learning.

[38] See e.g. Giannitsarou (2006) and Evans and Honkapohja (2012), for a similar discussion. For a situation in which agents are assumed to fully anticipate a fiscal reform and a part of its expected macroeconomic effects within a learning environment, see Evans *et al.* (2009).

[39] See Appendix B for the model solution under rational expectations and learning and for details on how the initial conditions for learning are set.

[40] Note it is only when the tax reform was accompanied by a negative shock to TFP that the rational expectations and learning transition paths differed more substantially in Giannitsarou (2006). The results reported below correspond to a non-stochastic case, when there is a zero initial shock to the model at the time of the reform. An "stochastic" transition from the old to the new steady state (obtained by averaging over 2500 simulations) produced transition paths that are very similar to those reported below. For this purpose, an AR(1) process was assumed for TFP, with an autoregressive parameter equal to 0.92 and a standard deviation of the innovations equal to 0.01, according to 1970-2005 data from the Office for National Statistics.

Table 2.5: Lifetime welfare (lower tax rates)

	τ^k =0.407			τ^h =0.284			τ^u =0.161		
i	ξ_{re}	ξ_{al}	ζ	ξ_{re}	ξ_{al}	ζ	ξ_{re}	ξ_{al}	ζ
c	0.0134	0.0139	-0.00049	-0.0011	-0.0011	-5.34e-05	-0.0092	-0.0092	-4.54e-05
s	0.0152	0.0157	-0.00050	0.0090	0.0090	-8.87e-06	-0.0017	-0.0017	-9.65e-06
u	0.0071	0.0073	-0.00022	0.0030	0.0030	-2.22e-05	0.0242	0.0241	-2.15e-05
a	0.0123	0.0127	-0.00041	0.0052	0.0052	-1.73e-06	0.0050	0.0050	-3.30e-06

Table 2.6: Lifetime welfare (combined lower tax rates)

	τ^u=0.191, τ^h=0.287			τ^u =0.194, τ^h =0.292, τ^k =0.429		
i	ξ_{re}	ξ_{al}	ζ	ξ_{re}	ξ_{al}	ζ
c	-0.0028	-0.0028	-5.39e-05	0.0031	0.0033	-0.00019
s	0.0067	0.0067	-9.49e-06	0.0099	0.0101	-0.00009
u	0.0077	0.0077	-2.31e-05	0.0076	0.0077	-9.07e-06
a	0.0053	0.0053	-2.21e-06	0.0080	0.0081	-0.00009

The results for the lifetime welfare gains/losses for each agent under rational expectations and heterogenous learning are shown in Tables 2.5 and 2.6 for each tax reform considered. To quantify the importance of the latter for welfare, we also calculate the cost of the heterogeneous learning, in terms of the consumption supplement, compared to the RE solution. This is defined as ζ in Tables 2.5 and 2.6.

Rational expectations

We first compare lifetime welfare gains/costs in Tables 2.5 and 2.6 to the corresponding steady-state values in Table 2.4 under RE. Consistent with the literature, the results indicate that the larger benefits in terms of aggregate welfare are obtained by capital tax cuts and that these are smaller, compared to the long-run.[41] Moreover, the results show that the inequality effects after the capital tax cuts are also smaller relative to the steady-state. In particular, capital tax cuts result in smaller welfare gains relative to the long-run for capitalists and skilled workers, while the welfare figures are roughly the same for the

[41] Note that the literature on tax reforms (see e.g. Domeij and Heathcote, 2004, and Garcia-Milà *et al.*, 2010), has emphasised that capital tax cuts will lead to welfare losses for those households whose resources depend predominantly on labour income, when the elimination of the capital tax cut is met by a rise in the labour tax to balance the budget. We confirm that this is obtained in this model as well, for a similar tax reform. Results are available upon request but are not shown here, since, to save on space, we focus on the productivity gains after a capital tax cut.

unskilled workers. In other words, the inequality effects are dampened by the inclusion of the transition period.

To further investigate this result we focus again on the capital tax reduction. To this end, in Figure 2.2 we plot the pre-reform steady-state in percent deviations from the post-reform steady-state and the transition paths of capital, labour input and consumption by agent, the relative supply of skilled labour, defined as $\frac{N^c h_c + N^s h_s}{N^u h_u}$, and the skill premium. The paths of consumption and hours are important as these will ultimately determine welfare for each agent.[42]

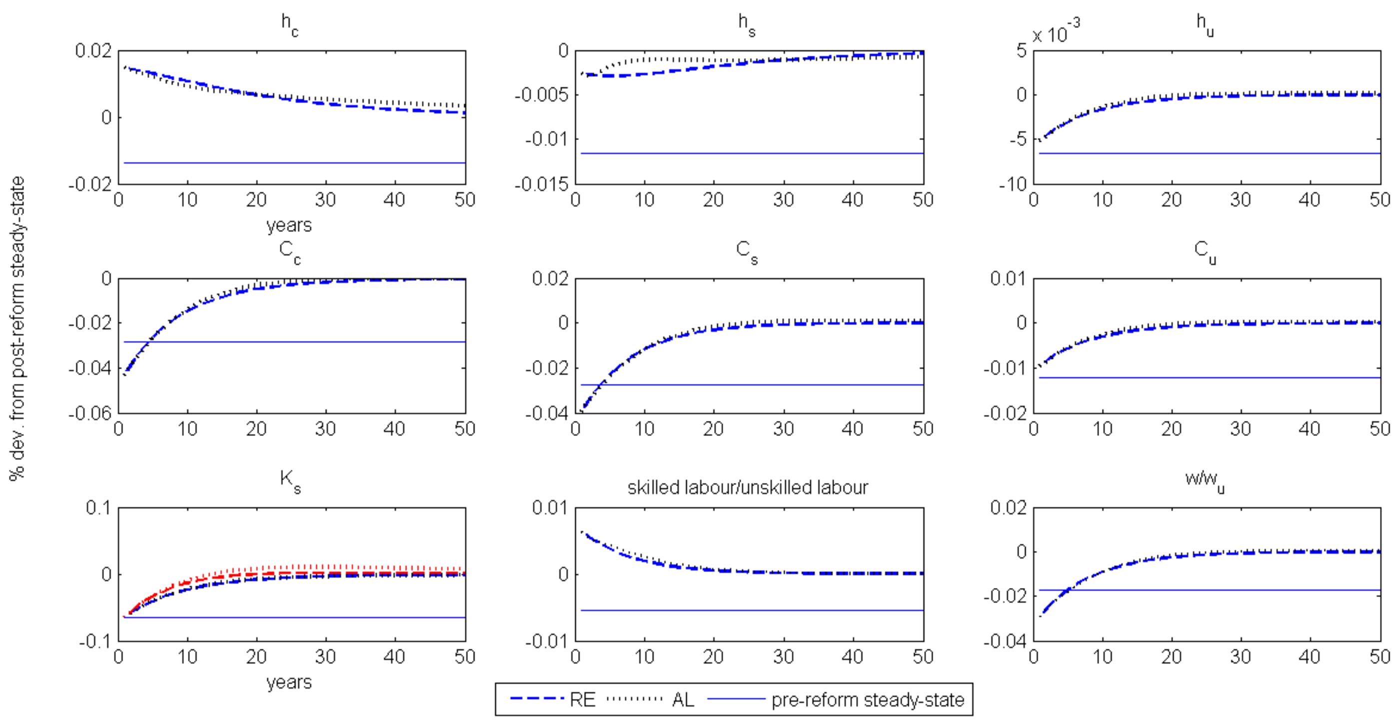
Figure 2.2 shows that a tax reform based on reducing the capital tax implies an increase in the capital stock as the economy gradually converges to the new equilibrium. The capital tax cut has created incentives for those agents who hold capital, i.e. capitalists and skilled workers, to increase their accumulation and thus increase investment. For this to be achieved, capitalists and workers can temporarily decrease consumption, but they also can increase their income by increasing their labour supply. Therefore, in general equilibrium, the increase in the return to capital also increases labour supply for those agents who hold capital. For the capitalists, in particular, the labour supply initially increases above the new steady-state and then converges to it. As they become wealthier over time, given the higher capital stock, they tend to supply less labour as the income effect dominates the substitution effect.

The overshooting in the relative supply of skilled labour in the short-run, driven by the higher returns to capital that will materialise in the long-run, leads to a fall in the skill premium in the short-run, which, in turn, has positive effects for the unskilled workers. However, over time, the relative supply of skilled labour falls and the quantity of capital increases. Both factors lead to a rising skill premium towards the new steady-state. Overall, the dynamic analysis indicates that, in general equilibrium, the complementarity between capital (or skilled labour) and unskilled labour is higher in the short-run, compared with the long-run.

Therefore, our analysis implies that after the capital tax reform, wage inequality

[42] To save space we do not present the Figures associated with the remaining tax reforms reported in Tables 2.5 and 2.6 but these are available on request.

Figure 2.2: Transition path after a capital tax cut



changes initially favour the agents with less wealth, and this works to partially offset the increase in asset income inequality in the short-run.[43] Therefore, in this model of capital-skill complementarity, the biggest relative gains for the poorest segment of the population after the capital tax cut materialise immediately after the reform, when the increase in the capital stock is lower and the relative skill supply overshoots, such that the wage premium moves favourably for the unskilled workers. Over time, the gains for the unskilled worker are diminishing faster than those for the skilled and wealthier groups, since both wage and asset income inequality now move in the same direction, implying that the welfare gap between the agents rises more in the long-run.

Heterogeneous learning

Next, we evaluate the importance of learning under heterogeneous initial beliefs. In particular, the skilled workers initiate their learning by using the coefficients that correspond to RE solution consistent with the old, pre-reform steady state. In contrast, capitalists are able to guess, immediately after the reform, the coefficients that correspond to the RE solution consistent with the new steady state.[44] However, both agents' expectations will be erroneous, because the actual economy, as determined by the interaction of their choices, is neither in the pre- nor in the post-reform RE equilibrium. In Figure 2.2, it can be seen that as the agents revise their errors along the transition path, the general equilibrium response to the tax reform includes an overshooting relative to the rational expectations case. This is different from the case of homogeneous learning, where adaptive learning generally implies a slower convergence to the new equilibrium (see e.g. Giannitsarou, 2006). Therefore, contrary to homogeneous learning, which dampens the reaction to the tax reform, the errors that heterogeneous learners make amplify their reaction.

The intuition for this result is consistent with Giannitsarou's (2006), observation that

[43] In the tax reforms considered in models that do not allow for capital-skill complementarity (e.g. Domeij and Heathcote, 2004, Greulich and Marcet, 2008, and Garcia-Milà *et al.,* 2010), the productivity gains and thus the benefits to the workers from a capital tax cut are stronger in the long run, as the capital stock is built up. However, these models do not allow for an evaluation of the wage inequality following a capital tax cut.

[44] Note that, all the policy experiments performed under AL yield stationary and locally E-stable solutions. In other words, all these reforms are effectively *learnable* (see, e.g. Evans and Honkapohja, 2001, and Honkapohja and Mitra, 2006).

positive shocks that coincide with the reform generate a "boost in optimism" that accelerates convergence to the new steady state and capital and output can overshoot in the early periods under learning as compared to RE. In our case, the overshooting is obtained in the absence of such shocks. Both capitalists and skilled workers are learners but given our assumptions regarding heterogeneity in initial beliefs, the former already know their own post-reform RE coefficients. Hence they realise that returns on investment are higher in the new steady state, so they start immediately investing more and accumulating capital faster. In contrast, the skilled workers are still using the pre-reform coefficients and have no information about the effects of the new regime. Hence, the increased activity they perceive due to the decision of the capitalists appears to them as very high forecast errors, as they keep discovering that total capital is higher than the their latest forecast. These, in effect, act as successive positive shocks which lead them to a faster accumulation of capital than if they were fully rational.

This overshooting from skilled workers implies that capitalists have also made a forecast error, as actual capital is in fact higher than their forecasted capital that would be consistent with the RE path. Hence, they also “correct” their behavior accordingly by investing more, and thus by overshooting themselves. Thus, heterogeneity in initial conditions appears to create forecast errors that effectively act as positive shocks in the early periods, as the actual capital stock is higher than what was expected. This effect is higher for the skilled workers, as can be seen in Figure 2.2.

Therefore, heterogeneous learning leads to a form of "irrational exuberance" which disappears in the long-run as beliefs gradually converge to the RE solution. However, in the particular case of a capital tax cut, this helps to increase the welfare for all agents along the transition path, relative to the case of rational expectations (see, e.g. the relevant ζ figures in Table 2.5). With the exception of the proportional decrease in all taxes, which also shows some sizeable results, the difference between rational expectations and heterogeneous learning is very small (virtually to zero) for the remaining tax reforms, given that the change in tax rates is also very small. However, we report that for bigger tax reforms (e.g. if the tax reforms aim for even lower debt-to-GDP ratios), the quantitative effects of heterogeneous initial beliefs in learning are unambiguously bigger.

2.5.4 Substitutability between capital and unskilled labour

The above results suggest that the elasticity of substitution between capital and unskilled labour is a critical factor in determining the inequality effects of capital tax cuts, irrespective of how expectations are effectively formed. Thus, we next explore the quantitative effects of higher elasticities of substitution. As discussed previously, empirical analyses provide a range of estimates for the critical parameter α in the production function. We consider a set of values of α which are consistent with this range and re-parameterise the model to obtain the same factor shares and B/Y ratio as in the pre-reform economy in Table 2.2.[45] In Table 2.7, we present the results for the welfare gains or losses for the three types of agents post-reform for the steady-state and for all periods according to these alternative calibrations. In each case, different capital tax reductions were applied to reach a debt-to-output ratio of 60% in the new steady-state.

The results in Table 2.7 suggest that over both time horizons considered, the welfare gains from the reduction in the capital tax to capitalists and skilled workers increase when the substitutability between capital (or skilled labour) and unskilled labour is increased. In contrast, the welfare gains to the unskilled workers fall. Therefore, the overall welfare inequality effects of capital tax cuts rise in the presence of higher capital skill complementarity, since reductions in the capital tax are skill-biased and thus raise wage inequality. While these qualitative results are expected, the small quantitative changes obtained for the empirically relevant range of parameters considered, lend support to the robustness of the model predictions in Tables 2.5 and 2.6, both under RE and heteregenous learning.

2.6 Conclusions

Using a heterogeneous agent model allowing for different degrees of complementarity between capital, skilled and unskilled labour, we have evaluated supply-side reforms consistent with a lower public debt-to-GDP ratio. To implement these reforms, we calibrated

[45] See e.g. Cantore and Levine, 2012, on "re-parameterisation" with CES production functions. The re-calibration considered here ensures that the values at the pre-reform steady-state when $\alpha = \{0.42, 0.45, 0.50\}$ are the same as those reported in Table 2 (i.e. when $\alpha = 0.401$) up to the third decimal place. For this purpose, μ took the values $\{0.281, 0.290, 0.303)$ while ρ took the values $\{0.645, 0.645, 0.635\}$, respectively.

Table 2.7: Walfare gains/losses for different alpha's

	Steady-state			Lifetime		
	Capitalist	Skilled	Unskilled	Capitalist	Skilled	Unskilled
	$\alpha = 0.40$, $\frac{\bar{w}}{\bar{w}_u} = 1.366$, $\tau^k = 0.4070$					
RE	0.0237	0.0211	0.0083	0.0134	0.0152	0.0071
AL	-	-	-	0.0139	0.0157	0.0073
	$\alpha = 0.42$, $\frac{\bar{w}}{\bar{w}_u} = 1.370$, $\tau^k = 0.4067$					
RE	0.0240	0.0213	0.0081	0.0136	0.0154	0.0070
AL	-	-	-	0.0141	0.0160	0.0072
	$\alpha = 0.45$, $\frac{\bar{w}}{\bar{w}_u} = 1.373$, $\tau^k = 0.4065$					
RE	0.0245	0.0217	0.0078	0.0139	0.0157	0.0068
AL	-	-	-	0.0144	0.0162	0.0070
	$\alpha = 0.50$, $\frac{\bar{w}}{\bar{w}_u} = 1.386$, $\tau^k = 0.4063$					
RE	0.0245	0.0218	0.0072	0.0139	0.0158	0.0062
AL	-	-	-	0.0144	0.0162	0.0064

the model so that the pre-reform steady-state represented the current state of the UK economy and then simulated different permanent changes in tax rates.

Our results imply that, relative to the other tax reforms, capital tax reductions lead to the highest aggregate welfare but are skill-biased and thus increase inequality in the long-run. Also, including the transition period in the welfare evaluation lowers the inequality effects of reducing the capital tax since the complementarity between capital and all labour inputs is higher in the short- than in the long-run. Finally, our results suggest too that a form of "irrational exuberance" can arise after a tax cut under heterogeneous learning in the initial conditions after the tax reform

Our findings further suggest that it may be appropriate to consider redistributive policies alongside capital tax cuts. While these policies have not been studied here, we expect them to be more effective if they aim to raise the productivity of factor inputs and, in particular, enhance social mobility, rather than simply redistribute income towards the income groups that are not favoured by the reform. A careful evaluation of such policies would be an obvious extension to this work. We leave these issues for future research.

Chapter 3

Debt-targeting fiscal rules and income inequality: the case of Bolivia

3.1 Introduction

Modern macroeconomic policy making has become increasingly constrained by the concern about the long run effects of misguided monetary and fiscal policies. A clear example of this current trend is the result of the high levels of public debt accumulated in most Latin-American economies from the mid-seventies to early-eighties. These, along with weak budgetary institutions, led to severe debt crises as well as costly episodes of hyperinflation and currency depreciation in most of the region between the mid-eighties and early-nineties.

Such negative past experiences, in turn, have brought sustainability and fiscal consolidation to the forefront of economic authorities' concerns in most of these countries. In effect, since year 2000, countries such as Brazil, Peru, Colombia, Argentina and Mexico have all been gradually applying a variety of legislated restrictions or rules on fiscal policy in order to set specific limits on government expenditure, the debt-to-output ratio or some other relevant variables describing the performance of public finances.[1]

Despite the fact that Bolivia was one of the countries that suffered the most during the aforementioned crisis, and while admittedly macroeconomic stability has been reasonably

[1] Brazil started applying its fiscal rule since 2000. Peru and Colombia approved their rules in 2003, while Argentina and Mexico did it in 2004 and 2006, respectively. See Banco de la Republica *et al.* (2010) for a review of the main features of each of these rules.

preserved over the last twenty seven years, the fact that no long-term sustainability issues have been considered by the fiscal authorities, leaves little insurance that such negative events will not take place again in the foreseeable future. Two elements make this concern all the more relevant.

First, although the government revenues have been increasing consistently over the last six years and primary surpluses have been announced in the last five consecutive years, public debt has also been increasing in the last three years, from 37 percent of gross domestic product (GDP) to almost 45 percent in 2011. While the current level is still close to the World Bank's debt sustainability threshold for Bolivia of 40 percent, the increasing trend of the debt-to-GDP ratio seems at odds with the current government's strong fiscal situation and raises questions about the former's sustainability prospects if the country's public finances start to deteriorate in the near future due to an adverse shock.[2]

Second, since the late 80s to the mid-2000s, almost 60 percent of the Bolivian external debt stock had some degree of concessionality.[3] Moreover, an important share of this debt was forgiven between the mid-nineties and the start of the new century thanks a series of multilateral schemes aiming to benefit the poorest and most heavily indebted countries around the world as long as the freed resources are used instead to achieve a number of poverty-reduction targets in terms of health, education, gender equality and others (see e.g. Lopez, 2003). Currently, however, only 30 percent of the Bolivian external debt is concessional while the remaining 70 percent as well as all the domestic debt have been contracted under market conditions, a trend which naturally implies that ensuring its sustainability over time will only tend to be more challenging.[4]

With this in mind, it seems highly relevant to ask whether implementing a fiscal rule that imposes restrictions in the evolution of public debt in Bolivia is an advisable measure or not. Such question is relevant as early research on the impact of fiscal rules (see e.g. Andres and Domenech, 2006a and Gordon and Leeper, 2005) warned against potential welfare costs derived from applying countercyclical fiscal rules when agents behave

[2] Despite reporting a fiscal surplus of around 2% of GDP for 2012 (see e.g. IMF, 2012), the Bolivian government has announced its intention to issue bonds in the international markets for USD 500 million (see www.ft.com (18/3/2012): "Bolivia plans first bond issue since early 1900").

[3] Defined as loans with an original grant element of 25 percent or more.

[4] See http://www.tradingeconomics.com/bolivia/

optimally and have full access to the financial markets.

However, more recent work concludes otherwise. For instance, Andres and Domenech (2006b) show in a New-Keynesian framework that if an important percentage of the population have little or no access to the financial markets, fiscal rules targeting debt consolidation can be highly successful at controlling debt and have little effect on output and consumption. Particularly, using a similar model as above, Garcia *et al.* (2011) and Cordoba and Rojas (2010) find that rules targeting debt or the fiscal deficit will yield the most desirable results in terms of welfare for Chile and Peru, respectively.

These results are also consistent with the more general findings of Kirsanova *et al.* (2009), who use a model with homogeneous agents and nominal frictions within a fiscal and monetary interaction context, and cannot conclude against the so-called current consensus assignment in economic policy, which states that to maximise welfare, fiscal policy should mainly focus on the control of government debt or deficits.

It is worth emphasizing that the study of the Bolivian case allows to investigate and discuss in more detail the impact of three highly relevant aspects on the performance of any given fiscal rule and, consequently, on the criteria that must be considered for the selection of the most adequate one.

The first aspect relates to the significantly high levels of income inequality suffered in Bolivia. In effect, a ranking reported by the World Bank based on periodic calculations of the Gini coefficient places Bolivia among the most unequal economies in the world.[5] While reportedly the Bolivian government has increased its efforts over the last few years in response to this precarious situation (see e.g. Montecino, 2011), more recent figures on inequality have shown little improvement (see e.g. Gasparini *et al.*, 2009). In this sense, it is reasonable to think that any fiscal policy measure or instrument to be considered by the government will aim to reduce income inequality or, at least, ensure that the latter will not be exacerbated. In other words, it is important to investigate whether a given fiscal rule can generate a trade-off between debt-control and inequality, an issue that has not received enough attention in the fiscal policy literature.

[5] The Gini coefficient for Bolivia for 2008 was calculated at 56.3, placing this country at the bottom of the world ranking, only ahead of Colombia, Honduras and South Africa (source: data.worldbank.org).

A second aspect is that Bolivia's economic fate depends strongly on the extractive nature of its productive structure, as is the case of several natural-resource-rich developing countries around the world. In particular, Bolivia holds the second largest reserves of natural gas in South-America, with the revenues originated from exports of this resource representing around 6 percent of GDP and almost 40 percent of the government's total revenues each year.

On this respect, there is now an important literature on natural resource dependence (see e.g. Van der Ploeg and Poelhekke, 2009 and Van der Ploeg, 2011, for a detailed discussion) which shows that developing economies with debilitated public finances and less developed financial markets tend to be more vulnerable to the adverse effects of the potential volatility associated with these resources.[6] Moreover, it is well-documented (see Pindyck, 2004, for a detailed review) that currently almost all energy markets in the world, including the different regional natural gas markets, are suffering from very high levels of price volatility. Therefore, it becomes clear that the role of natural gas revenues as a key source of exogenous volatility in Bolivia cannot be ignored in this work.

A third aspect is related to how expectations are formed among agents in general and particularly in less developed economies like Bolivia. A growing literature has been exploring the notion that in practice agents might not be fully rational and thus must rely on past data in order to forecast the behavior of the economic variables of interest, making use of efficient estimation or learning rules which, under certain conditions, ensure convergence to the fully rational equilibrium.

In particular, a number of authors (see e.g. Carceles-Poveda and Giannitsarou, 2007, Eusepi and Preston, 2011, and Huang *et al.*, 2009) have suggested that if an adaptive learning approach is assumed instead of the commonly used rational expectations hypothesis, the overall goodness-of-fit of standard general equilibrium models tends to improve. Although the particular case of developing countries has received little attention in this field, results in favour of adaptive learning have been also suggested by, for instance, Marcet and Nicolini (2003) and Boz *et al.* (2008).

[6] Notably, such vulnerability has been found to be even higher for landlocked countries with ethnic tensions (see Van der Ploeg and Poelhekke, 2009), two features that also characterise the Bolivian economy.

Accordingly, a related literature has been discussing the consequences of adaptive learning in a large number of policy-oriented matters (see Evans and Honkhapoja, 2011, for a recent detailed review). However, most part of this growing research has paid close attention to monetary policy issues (e.g. determinacy and expectational stability of policy rules and optimal policy design), leaving the fiscal policy field almost unattended.[7] In particular, to the date, the potential impact of adaptive learning on the performance and income-inequality implications of applying different fiscal policy rules in a given economy has been largely overlooked by the learning literature.

In light of the above, it is the main objective of this work to shed more lights on the distributional consequences of applying debt-targeting fiscal rules in Bolivia when the economy is exposed to different sources of volatility and considering different ways of expectations formation among agents. For this purpose, we calibrate a closed-economy stochastic general equilibrium model with heterogenous agents to match Bolivia's main features as described by the data.

The model assumes two types of agents: capitalists, whose main sources of income are their physical and financial assets; and workers, whose main source of income is their labour supply. To capture key features of heterogeneity and wealth and income inequality observed in this economy, we follow the literature on credit constraints and income inequality (see e.g. Galor and Zeira, 1993, Benabou, 1996, and Aghion and Howitt, 2009) and include financial intermediation costs which are significantly higher for workers than for capitalists, implying that the former's participation in the financial markets is very limited.

As said earlier, besides assessing the performance of the fiscal rules as Bolivia is exposed to the effects of standard exogenous technology shocks, natural gas-revenue (or commodity) shocks are also considered as another relevant source of volatility. For this purpose, following Garcia and Restrepo (2007) and Garcia *et al.* (2010), these revenues are modelled exogenously as net revenues, subject to high levels of international price

[7] Few relevant exceptions are Giannitsarou (2006) who discusses the importance of learning on the efectiveness of a capital tax reform after a recession, Evans and Honkapohja (2009) who examine the effects of learning in face of an anticipated fiscal reform and Evans *et al.* (2012) who investigate the effectiveness of fiscal policy under learning.

volatility.

Under this particular model setup we investigate how different ways of forming expectations affect the performance and the distributional impact of each of the proposed fiscal rules. Hence, the results of the model solved assuming rational expectations are presented first. Then, this assumption is relaxed and it is assumed that both types of agents are learners instead.

Specifically, it is assumed that both types of agents follow identical learning rules over time but have different initial beliefs at the start of the learning process - i.e. a type of heterogeneous learning[8]. Here, two well-known rules in the learning literature - recursive least squares and its constant gain variation - are taken into consideration. Also, in order to characterise these differing initial beliefs, three alternatives are examined. In the first one it is assumed that workers do not know the exact impact of the fiscal rule in place while capitalists do. In the second alternative it is assumed that workers do not know the economic implications of the rule and additionally have slightly more pessimistic initial beliefs than capitalists. Finally, in the third alternative it is assumed that workers do not know the implications of the rule but have slightly more optimistic initial beliefs than capitalists.

For every different assumption regarding how expectations are formed, this work aims to examine and rank each of the proposed fiscal rules according to two main criteria: a) the short and long run performance at stabilising debt after an exogenous shock has hit the economy, and b) the associated impact on income inequality.[9]

The main results of this work are as follows. First, under full rationality, the adoption of fiscal rules aiming to control the evolution of public debt in a context of high wealth and income inequality can prove ineffective in response to productivity and, specially, commodity shocks. In effect, in most cases these rules generate a trade-off between debt-stabilisation and higher income-inequality which, as discussed earlier, should not be

[8] Another, stronger, type of heterogeneity consists in assuming different learning algorithms between agents. In this work, however, we want to show that the differences generated by assuming rational expectations or heteregenous learning can be quite significant even if the simplest form of learning heterogeneity is considered.

[9] In addition, and although income inequality is of key interest in this work, the distributional consequences of applying these rules in terms of welfare will also be reported when relevant.

ignored in economies where such aspects are highly sensible from a sociopolitical point of view.

However, if heterogenous learning is assumed, then the transition dynamics are such that in a number of cases the aforementioned trade-off is no longer present and thus some of the rules - i.e. those instrumented via labour taxes - can perform rather well in both fields, specially if TFP shocks are considered.

Second, given the particular features of Bolivia, finding a fiscal rule that shows high debt-stabilising properties without compromising income distribution in response to all relevant sources of exogenous volatility is not an easy task. That is, the fiscal rules instrumented via labour taxes seem the most reasonable candidates, but only in response to productivity shocks. When large negative natural gas revenue shocks are considered, however, the overall performance of the rules is poor, a result that illustrates the high dependence and vulnerability of Bolivia on this source revenue which is subject to high levels of volatility due to sharp changes in its international price.

After this introduction, section 2 describes the model setup and section 3 reports its calibration and steady-state solution. The solution methods of the dynamic model assuming both RE and AL are briefly described in section 4. Then, section 5 presents the main results predicted by the model under the different fiscal rules and alternative ways of forming expectations. Finally, section 6 presents the main conclusions of this work and some final remarks.

3.2 Model setup

In this section we construct a closed-economy DSGE model comprised of two types of agents: a representative capitalist and a representative worker.[10] Both consume output in the product market and supply labour in the factor market in return for labour income. Also, subject to intermediation costs, both types of agents allocate savings to physical capital and government bonds in return for capital income with the main difference that

[10]The names *capitalist* and *worker* were chosen to facilitate their identification along the document. They try to emphasize the fact that *capitalists* have the returns from their assets as the main source of income while *workers* obtain most of their income from their work.

intermediation costs for workers are significantly higher.

The representative firm is owned by the capitalist who hires labour services and leases physical capital from the factor market for which it pays the competitive wage and interest rate respectively. Finally, the government taxes economic activity, provides public spending and issues debt to balance its budget. In such context, as discussed earlier, a set of alternative fiscal rules which focus on the levels of public debt will be considered so that their relative performance and distributional consequences can be examined.

3.2.1 Population composition

The population size, N, is exogenous and constant. Among N, $N^c < N$ are identical capitalists and $N^w = N - N^c$ are identical workers. Capitalists are indexed by the subscript $c = 1, 2, ..., N^c$ and workers by $w = 1, 2, ..., N^w$. There are also N^f firms, $f = 1, 2, ..., N^f$. For simplicity, we assume that the number of firms equals the number of capitalists, $N^k = N^f$, and that each capitalist owns one firm. It is useful, for what follows, to define $N^c/N = n^c$, $N^w/N = n^w = 1 - n^c$ and $N^f/N = n^f$. The shares of each type of agent in the population are constant.

3.2.2 Firms

Each firm produces a single output, Y_t^f, using physical capital, K_t^f, and labour services, H_t^f. The production function is given by a well-known Cobb-Douglas specification (see e.g. Angelopoulos, Jiang and Malley, 2011):

$$Y_t^f = A_t \left(K_t^f\right)^{\alpha} \left(H_t^f\right)^{1-\alpha} \tag{3.1}$$

where A_t is exogenous stochastic productivity whose motion is depicted by a first-order autoregressive - AR(1) - process

$$A_{t+1} = A_0^{(1-\rho^a)} A_t^{\rho^a} e^{\varepsilon_t} \tag{3.2}$$

where $A_0 > 0$ is a constant, $0 < \rho^a < 1$ is the autoregressive parameter, $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ are random shocks to total factor productivity (TFP) and $0 < \alpha_1$,and $(1 - \alpha_1)$, are the productivity of capital and labour, respectively.

Each firm acts competitively, taking prices and policy variables as given, and maximises profits given by:

$$\Pi_t^f \equiv Y_t^f - r_t^k K_t^f - w_t H_t^f \tag{3.3}$$

subject to the technology constraint, where w_t is the wage rate and r_t^k is the interest rate on capital. Note that in equilibrium, profits, Π_t^f, are driven to zero due to perfect competition.

3.2.3 Budget constraints of capitalists

The representative capitalist owns one firm and receives income from providing labour services, $H_{c,t}$, to the labour market and income from interest on his accumulated stock of financial assets, in the form of capital, $K_{c,t}$, and government bonds, $B_{c,t}$. The interest rate on government bonds is given by r_t^b. In line with the Bolivian legal framework, all these sources of income are taxed by the government, with the exception of that coming from the interest earned on government bonds. Hence, financial asset income coming from investing in physical capital is taxed at the rate τ^k, while labour income is taxed at the rate τ^h.

In order to hold assets, capitalists need to pay intermediation or transaction premia due to imperfections in capital markets. For instance, these premia can represent the costs of gathering extra information relating to legal issues, asset-specific government regulations, intermediation fees and so on. Following Persson and Tabellini (1992), we assume a quadratic cost function such that the capitalist incurs a cost of $\varphi_c^k K_{c,t}^2$ for holding physical capital and of $\varphi_c^b B_{c,t}^2$ for holding government bonds, where $\varphi_c^b, \varphi_c^k > 0$ measure the size of the transaction costs. The presence of this capital market imperfection and of the associated transaction costs, helps the model to capture a feature of realism, but also help to define household heterogeneity in the model.

The capitalist uses his income for consumption, $C_{c,t}$, for which he pays a consumption tax, τ^c, investment in capital and government bonds. Thus, his budget constraint is:

$$\begin{aligned}(1+\tau^c)\, C_{c,t} + K_{c,t+1} + B_{c,t+1} = (1-\delta)\, K_{c,t} + B_{c,t} + \left(1-\tau^k\right) r_t^k K_{c,t} + \\ + \left(1-\tau^h\right) w_t H_{c,t} + r_t^b B_{c,t} + \bar{G}_t^{tr} - \varphi_c^b B_{c,t}^2 - \varphi_c^k K_{c,t}^2\end{aligned} \tag{3.4}$$

where $0 < \delta < 1$ is the depreciation rate, $\bar{G}_t^{tr} = G_t^{tr}/N$ are average net transfers from the government and $K_{c,0}, B_{c,0} > 0$ are given.

3.2.4 Budget constraints of workers

The problem of the worker is similar to the problem of the capitalist, in that he provides labour to the labour market, invests the share of his income he does not consume in capital and government bonds, earns interest rate income on his financial stock, receives the same average transfers and pays the same tax rates for these economic activities.

He differs from the capitalist, however, in that he pays different transaction costs, so that the effect of the capital market imperfections affects him to a greater extent.[11] In particular, we assume that firm ownership gives an insider advantage in financial transactions to the capitalist (due, for instance, to past experience, socioeconomic background, networks or geographical issues related to living in less connected rural areas) and thus the size of the transaction costs is lower for the capitalist.

The idea that capital market imperfections can explain heterogeneity has been examined by, *inter alia*, Benabou (1996) and Aghion and Howitt (2009). In this work, we examine the case of non-zero, finite intermediation costs for both capitalists and workers where $\varphi_c^b < \varphi_w^b$, $\varphi_c^k < \varphi_w^k$. This in turn implies that workers have much lower initial holdings of capital and government bonds than capitalists, i.e. $K_{w,0} < K_{c,0}$, $B_{w,0} < B_{c,0}$.[12] Accordingly, the budget constraint of the worker is given by:

$$\begin{aligned}(1+\tau^c)\, C_{w,t} + K_{w,t+1} + B_{w,t+1} = (1-\delta)\, K_{w,t} + B_{w,t} + \left(1-\tau^k\right) r_t^k K_{w,t} + \\ + \left(1-\tau^h\right) w_t H_{w,t} + r_t^b B_{w,t} + \bar{G}_t^{tr} - \varphi_w^b B_{w,t}^2 - \varphi_w^k K_{w,t}^2\end{aligned} \tag{3.5}$$

[11] The worker also differs from the capitalist in that he does not appropriate the profits of the firm. Given that in this model these profits are zero in equilibrium, this difference is trivial.

[12] Note that this notion of heterogeneity among agents is somehow related to other line of research in this field, which assumes that the economy is populated by Ricardian and non-Ricardian agents in a New Keynesian framework (see e.g. Garcia *et al.* 2011 and Garcia and Restrepo, 2006).

3.2.5 Utility function of agents

Each type of household $i = c, w$ maximises:

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(C_{i,t}, H_{i,t}, \overline{G}_t\right) \tag{3.6}$$

subject to the relevant budget constraints given above; where E_0 is the expectations operator; and $\overline{G}_t = G_t/N$ is average (per agent) government services or the total amount of public goods per capita. We use the instantaneous utility function:

$$u_{i,t} = \left(C_{i,t}, H_{i,t}, \overline{G}_{i,t}\right) = \frac{\left[\left(C_{i,t}\right)^{\mu_1}\left(1 - H_{i,t}\right)^{\mu_2}\left(\overline{G}_{i,t}\right)^{1-\mu_1-\mu_2}\right]^{1-\sigma}}{1-\sigma} \tag{3.7}$$

where μ_1, μ_2, and μ_3 are preference parameters while σ is the parameter of risk aversion.

It is worth mentioning that, unlike the previous two chapters, government spending is assumed to enhance utility in this case. This assumption is taken in light of two relevant features of the Bolivian economy. First, from a demand point of view, Bolivia remains as the poorest country of South America and around 60% of its population is still below the national poverty line.[13] In this sense, the relative dependence of this large part of the population on public goods, services and transfers is quite significant with respect to total income.

Second, from a supply point of view, the current government's policy in terms of the outlook of the country's productive structure is to unambiguously favour state-owned activities. Examples of this are the on-going process of nationalisation of firms owned by private foreign investors in so-called "strategic" sectors (e.g. oil, gas and electricity) which started in 2006[14], and the start-up of several state-owned companies which now compete against domestic private companies in relatively more competitive markets (e.g. grains, sugar, almonds, diary, paper, cardboard and cement).[15]

[13] See http://data.un.org/Data.aspx?q=Bolivia+poverty&d=MDG&f=seriesRowID%3A581%3BcountryID%3A68

[14] See http://www.bloomberg.com/news/2012-12-29/spain-s-iberdrola-has-4-units-taken-over-by-bolivian-government.html.

[15] See page 67 of the World Bank's Report No.58674 (Bolivia—Public Financial Management Review, 2011). Available online at: http://www-wds.worldbank.org/external/default/WDSContentServer/WDSP/IB/2011/01/18/000333037_20110118233636/Rendered/PDF/586740ESW0Whit14B01PUBLIC1012120.10.pdf

3.2.6 Government

Budget constraint

The government provides the private agents with utility-enhancing services, in the form of government consumption. In order to finance these expenses, it taxes consumption, income from labour and physical assets and issues government bonds.

In addition, as is the case in a number of emerging/developing economies, the government obtains significant revenues, denoted by R_t, from selling a natural resource often produced within a natural-monopoly structure and thus usually owned entirely (or almost entirely) by the government. Very well-known cases are the cooper industry in Chile and of course the oil industry in several Latin-American, African and Middle-Eastern countries.

In turn, due to a significant increase in exploration activity since the late-90s, Bolivia became known for holding the second largest reserves of natural gas in South-America (after Venezuela). Also, after a major investment to build a cross-border pipeline to Brazil in addition to the existing one to Argentina (built in the early-70s), the revenues originated from exports of this resource to both countries have represented around 6 percent of the country's GDP on average in the last three years.[16]

Following Garcia and Restrepo (2007) and Garcia *et al.* (2010), given the nature of this strategic industry, the additional revenues coming from natural gas exports are modelled here as net revenues.[17] Therefore, the budget constraint of the government is given by:

$$G_t + G_t^{tr} + \left(1 + r_t^b\right) B_t = B_{t+1} + N^c[\tau^c C_{c,t} + \tau^k r_t^k K_{c,t} + \tau^h w_t h_{c,t}] +$$

$$+ N^w[\tau^c C_{w,t} + \tau^k r_t^k K_{w,t} + \tau^h w_t h_{w,t}] + R_t \tag{3.8}$$

where the behavior of R_t will be given by an AR(1) process:

$$R_{t+1} = R_0^{(1-\rho^R)} R_t^{\rho^R} e^{\varepsilon_t^R} \tag{3.9}$$

[16] The 20-year contract with Brazil was agreed in 1996 and a new contract with Argentina (Bolivia also sold gas to Argentina between 1972 to 1999) was signed in 2004.

[17] Note that this a simplifying way of modelling revenues which are net from the operational costs involved in the production/distribution of this commodity, so that more emphasis can be given to the volatile nature of the international prices at which it is sold.

with $R_0 > 0$, $0 < \rho^R < 1$ and $\varepsilon_t^R \sim N(0, \sigma_R^2)$.

Fiscal rules

It has been empirically demonstrated by several authors that fiscal policy in Latin American countries has been historically pro-cyclical (see e.g. Gavin *et al.*, 1996, Gavin and Perotti, 1997 and Talvi and Vegh, 2000), a situation that has led to severe losses in terms of growth and welfare, especially for the poor (see e.g. Perry, 2002).

Bolivia has not been the exception to these empirical findings and, in fact, its lack of fiscal discipline and weak budgetary institutions during the 70s and 80s provoked a severe public debt crisis, with total debt reaching a staggering 229.3 percent of GDP in 1987, which eventually had to be defaulted, as well as unprecedented rates of hyperinflation and currency depreciation.[18]

While admittedly macroeconomic stability has been preserved to the date after that last episode, the fact that no long-term sustainability issues have been considered by the Bolivian fiscal authorities over the last 20 years leaves little insurance that such negative events will not take place again in the foreseeable future. As mentioned earlier, two important elements give raise to this concern.

First, despite the fact that the government revenues increased remarkably (mainly due to increasing tax revenues and natural gas exports), leading to five consecutive years of primary surpluses, public debt has also been increasing, from around 37 percent of GDP in 2008 to a little less than 45 percent of GDP in 2011.

While such levels are below the 10-year (67% of GDP) and 20-year (75% of GDP) averages of this variable and thus are not alarming,[19] its trend seems at odds with the current strong fiscal situation and raises questions about its sustainability if the government's finances start to deteriorate in the near future due to an adverse shock such as a large fall in the price of natural gas or a decrease in tax revenues after an slowdown of economic activity.

Second, unlike ten years ago when more than 60 percent of the Bolivian public debt had

[18] Since that episode and to the date, Bolivia has not taken part in international financial markets again.
[19] Bolivian public debt data source: Reinhart and Roggof (2010).

some degree of concessionality, currently more than 70 percent of the debt stock has been contracted under market conditions, which in turn implies that ensuring its sustainability over time is only becoming more challenging.

In light of the above, the proposed model includes a legislated restriction on fiscal policy in Bolivia that sets specific limits on expenditure and taxation taking the behavior of public debt as the main indicator of fiscal prudence (i.e. a debt-targeting fiscal rule).[20] That is, following Andres and Domenech (2006b), the fiscal rules to be considered are:[21]

$$\frac{G_t}{Y_t} = G_0 \left(\frac{B}{Y} / \frac{B_t}{Y_t} \right)^{\gamma_g} \tag{3.10}$$

$$\tau_t^k = \tau_0^k \left(\frac{B_t}{Y_t} / \frac{B}{Y} \right)^{\gamma_k} \tag{3.11}$$

$$\tau_t^h = \tau_0^h \left(\frac{B_t}{Y_t} / \frac{B}{Y} \right)^{\gamma_h} \tag{3.12}$$

$$\tau_t^c = \tau_0^c \left(\frac{B_t}{Y_t} / \frac{B}{Y} \right)^{\gamma_c} \tag{3.13}$$

where B_t/Y_t is the actual debt-to-GDP ratio and B/Y is its sustainable steady-state level or long-run target, G_0 represents the G/Y ratio in the steady-state while τ_0^k, τ_0^h and τ_0^c are also equal to their respective steady-state values, τ^k, τ^h and τ^c.

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[20] Such rule has been applied in the past by, for example, Canadian provinces in order to contribute to debt consolidation in Canada (see e.g. Fiess, 2002).

[21] Two more general and sophisticated fiscal rules were also tested. First, following Garcia *et al.* (2011), a rule of the form

$$G_t = TR - (r_t + \mu_x)B_t + \alpha_g(TR_t - TR)$$

was considered, where TR_t represents total revenues at time t, α_g is a parameter measuring the acyclicality of the rule, μ_x is an arbitrary parameter which tries to prevent public debt from showing an explosive behavior, and the variables without time subscripts denote steady-state values.

Under this rule, indeterminacy problems started to arise for most of the feasible range of the fiscal policy coefficients, given the calibration to Bolivian data. This indeterminacy issue is in fact not uncommon under this setup and is discussed by Garcia *et al.* (2012) and Garcia and Restrepo (2007).

Second, following Malley et. al (2009), a rule of the form

$$x_t = x_0 \left(\frac{B_t}{Y_t} / \frac{B}{Y} \right)^{\gamma_b} \left(\frac{Y_t}{Y} \right)^{\gamma_y} \left(\frac{R_t}{R} \right)^{\gamma_R}$$

was examined, where the γ's are the coefficients which denote how anticyclical the rule will be, $x_t = G_t/Y_t, \tau_t^k, \tau_t^h, \tau_t^c$, denotes all the possible fiscal instruments at hand and x_0 denotes the steady-state level of these instruments.

The results under this type of rule were in general no better than those under the proposed debt-targeting rules. The inclusion of gas revenues in the rule transmitted the price volatility of this commodity (as warned by *e.g.*, Garcia *et al.*, 2011, Schaechter *et al.*, 2012, and Berganza, 2012), while including the output gap generated no meaningful additional gains with respect to the results reported here.

The reaction parameters γ_i for $i = g, k, h, c$ determine the degree of adjustment implied by the rule in response to a deviation of the actual debt-to-GDP ratio with respect to its steady-state level. Given the form of the rules, positive values of the reaction parameters will ensure that any deviation from the debt target will be gradually corrected. For instance, consider the case where the actual ratio is above the steady-state level. Here, positive values for the $\gamma_i's$ will ensure that the policy instruments will react in order to bring B_t/Y_t back to its target. That is, government spending as a share of GDP will remain below G_0 while taxes will be set above their steady-state levels until the deviation is corrected completely. Conversely, if the B/Y ratio is below its target, then the rules will generate the opposite reaction in these instruments.

Moreover, the larger the parameters, the faster the adjustment towards the B/Y target will be. Naturally, negative values for these parameters will lead the rules to amplify any initial deviation from the target, while zero-values will make them completely irresponsive to any deviation, irrespective of its size. In this sense, this work will only concentrate on the positive ranges of the reaction parameters to ensure that debt-stabilisation is the main goal of the proposed rules. Having said this, note that the positive ranges of these reaction parameters will be further restricted in order to ensure the determinacy and learnability of the model, and issue that is discussed during the calibration procedure.

The above fiscal rules are defined as simpler, more transparent and easy to monitor by agents (see e.g. Berganza, 2012), features that might be perceived as highly valuable in countries such as Bolivia, where debt-sustainability has been a major concern for many decades and governments consistently suffer from low credibility, as discussed by e.g. Calderon *et al.*, (2004), and thus agents might require easy-to-follow measures in order to assess the overall performance of the fiscal sector. On the other hand, it has been shown too that such simple rules have been more effective in helping to strengthen long-term sustainability than in responding to shocks. The next step, thus, seems to be the need for gradual improvement of these rules, such that the sustainability objective can be efficiently combined with greater flexibility to accommodate economic shocks (as suggested by e.g. Schaechter *et al.*, 2012).[22]

[22] This last aspect goes beyond the intended scope of this work.

It is worth noting too that the adoption of the above fiscal rules in the model is fairly consistent with the findings of Kirsanova *et al.* (2009), whose detailed analysis of monetary and fiscal interactions concludes in favour of the current so-called consensus assignment in economic policy, which states that to maximise welfare fiscal policy should focus on the control of government debt or deficits, as its impact on output is very small.. Moreover, the proposed analysis is also in line with the findings of Andres and Domenech (2006b), Cordoba and Rojas (2010) and Garcia *et al.* (2011), which suggest that, if an important part of the population in the economy have little or no access to the financial markets, then fiscal policies aiming for debt-consolidation will tend to be quite successful and thus will also yield the most desirable results in terms of welfare.

3.2.7 Market-clearing conditions

The market clearing conditions for the capital, bond, labour and product markets, respectively, are:

$$N^f K_t^f = N^c K_{c,t} + N^w K_{w,t} \tag{3.14}$$

$$B_t = N^c B_{c,t} + N^w B_{w,t} \tag{3.15}$$

$$N^f H_t^f = N^c H_{c,t} + N^w H_{w,t} \tag{3.16}$$

$$\begin{aligned} N^f Y_t^f = {} & N^c C_{c,t} + N^w C_{w,t} + N^c \left[K_{c,t+1} - (1-\delta) K_{c,t}\right] + \\ & + N^w \left[K_{w,t+1} - (1-\delta) K_{w,t}\right] + G_t + N^c \left(\varphi_c^b B_{c,t}^2 + \varphi_c^k K_{c,t}^2\right) + \\ & + N^w \left(\varphi_w^b B_{w,t}^2 + \varphi_w^k K_{w,t}^2\right) + R_t^{ng} \end{aligned} \tag{3.17}$$

3.2.8 Decentralised competitive equilibrium

The decentralised competitive equilibrium (DCE) is defined when (i) households and firms optimize, taking prices and policy as given; (ii) all constraints are satisfied; and (iii) all markets clear. Each representative capitalist chooses $\{C_{c,t}, H_{c,t}, K_{c,t+1}, B_{c,t+1}\}_{t=0}^{\infty}$ to maximise discounted lifetime utility subject to (3.4) whereas each worker chooses $\{C_{w,t}, H_{w,t}, K_{w,t+1}, B_{w,t+1}\}_{t=0}^{\infty}$ subject to (3.5). Finally, each representative firm chooses $\left\{K_t^f, H_t^f\right\}_{t=0}^{\infty}$ to maximise profits subject to the technology constraint (3.1) resulting in two optimality conditions.

In addition to these, the DCE also includes the production function, one of the two conditions involving the Lagrangian multipliers from the households problems, the government budget constraint, the aggregate resource constraint, the two AR(1) exogenous processes and, according to the case under study, one of the fiscal rules cited above.[23]

3.3 Calibration and Steady-state

The model's structural parameters relating to preferences, production and capital accumulation and the heterogeneity features discussed earlier are next calibrated using annual data for Bolivia from 1990 to 2010. These are reported in Table 3.1 below.

3.3.1 Population shares

In order to map out agent heterogeneity and define the shares in the population of the two types of households described in the model, we consider the Financial Services Access in Latin-America Survey by the Latin American Development Bank (known as CAF) presented in April, 2011. According to this survey, currently 35 percent of the Bolivian population have access to formal (i.e. regulated) financial services, while the rest have to rely in other informal and thus more costly sources of funding. Hence, n^c or the percentage of capitalists in the model will be placed at this level, implying that $n^w = 1 - n^c = 0.65$.

3.3.2 Savings

Heterogeneity in savings and the associated very high levels of concentration observed in the financial markets in Bolivia are controlled for, as explained in the previous section, by the parameters that govern financial transaction costs (see Benabou, 1996, and Aghion and Howitt, 2009), taking a number of elements into account. First, for simplicity we will set this cost in capital asset markets to be the same in the bond market.

Second, although there is no reliable information regarding the size of the informal

[23] To save space the DCE system is not reported here, but it is provided in Appendix C.

financial market Bolivia, some data exists regarding the concentration levels in the government bond market which could be used to characterize both asset markets.

In effect, a new scheme based on retail sales of government bonds was implemented by the Central Bank of Bolivia in October of 2007 (see CBB, 2007) one of its aims being to provide a risk-free saving alternative to agents who, for a number of reasons, do not participate in the formal financial markets. Since then, the stock of bonds sold through this scheme has stabilised at around 3.3 percent of the total stock.[24] Hence, such level could be taken as a fair approximation of the actual holdings of both physical capital and government bonds of those labelled as workers in the model.

Therefore, the transaction costs in the model are set to be fifty five times higher for the workers compared to those for capitalists, so that it can be replicated that 35 percent of the population (i.e. the capitalists) hold around 96.7 percent of the total assets of the economy in the form of both physical capital and securities, while the 65 percent of the population (i.e. the workers) are only responsible for the remaining 3.3 percent.

3.3.3 Parameters common to all agents

We next approximate the subjective time preference, β, so that it is consistent with the average real interest rate in the Bolivian formal credit market from 2004 to 2010,[25] which we calculated to be around 6.2 percent.[26] This gives a parameter value of $\beta = 0.94$. Parameters μ_1, μ_2 and $\mu_3 = 1 - \mu_1 - \mu_2$ in the utility function have been set at usual values in the literature - i.e. leisure is almost twice as important as consumption while the utility provided by public goods is quite small - in order to match the key aggregate ratios observed in the data (i.e. C/Y and G/Y),[27] as well as the notion that agents spend

[24] At the end of 2011, the stock of bonds sold directly to the agents reached Bs310 millions, representing 3.3% of the total internal public debt, excluding the debt to the pension fund system. See http://www.bcb.gob.bo/webdocs/2012/01-enero/semanal/entero20-01.pdf

[25] We do not use previous years in this calculation as the financial market was highly dollarised during that period, and thus the interest rates in domestic currency did not fully reflect the conditions of the markets. Since the mid-2000's, however, the dollarisation levels have decreased significantly (see e.g. CBB, 2010).

[26] Data source: http://www.udape.gob.bo (see statistical dossier section)

[27] National accounts data source: International Financial Statistics (IMF).

Table 3.1: Parameter Values

parameter	value	definition
$0 \leq \delta \leq 1$	0.070	depreciation rate of capital
$\varphi_c^k, \varphi_c^b > 0$	0.001	transaction costs, capitalists
$\varphi_w^k, \varphi_w^b > 0$	0.055	transaction costs, workers
$0 < \beta < 1$	0.940	rate of time preference
$0 < \alpha < 1$	0.373	capital's share of income
$0 < \mu_1 < 1$	0.350	consumption weight in utility
$0 < \mu_2 < 1$	0.600	leisure weight in utility
$0 < n^c < 1$	0.350	population share of capitalists
$0 < G_0 < 1$	0.142	public spending share of output
$0 < \tau_0^h < 1$	0.130	effective labour tax rate
$0 < \tau_0^k < 1$	0.230	effective capital tax rate
$0 < \tau_0^c < 1$	0.130	effective consumption tax rate
$G^{tr} > 0$	0.066	government transfers
$A > 0$	1.000	constant parameter productivity
$0 < \rho^a < 1$	0.602	AR(1) parameter productivity
$\sigma^a > 0$	0.0208	standard deviation productivity
$R_0 > 0$	0.026	const. parameter gas revenues
$0 < \rho^R < 1$	0.720	AR(1) parameter gas revenues
$\sigma^R > 0$	0.120	standard deviation gas revenues

between 15 to 35 percent of their available time working, with the workers at the top of this range. Finally, we also use a common value from the literature for the coefficient of risk aversion, i.e., $\sigma = 2$ (see e.g. Angelopoulos *et al.*, 2011).

The parameter values related to the technology available are somewhat more difficult to define due to the lack of reliable and up-to-date data about the structure of the Bolivian productive sector. To obtain the best possible approximations, first we make use of an important finding by Cole *et al.* (2005), who show that the capital-to-output ratio for a sample of eleven Latin-American economies (in which Bolivia is included) is in fact quite similar (i.e. slightly higher) to that of the US, which in turn is known to have stabilised at around $K/Y = 2$ since the 50s (see e.g. Evans, 2000).

Then, by applying the perpetual inventory method to compute the capital stock for Bolivia using data on gross capital formation for the last sixty years, we find that an annual depreciation rate, δ, of 7 percent is consistent with a capital-to-output ratio that oscillates around a value slightly above 2.[28]

[28] This depreciation rate is similar to the rate used by both Feu (2004) and Feu *et al.* (2007) for the case of Brazil. Also, Gelos and Isgut (2001) considered depreciation rates between 4% and 7% when studying the cases of Mexico and Colombia.

Finally, we turn our attention to the parameter value related to capital's share in income. It has been argued that the capital's share of income tends to be larger in developing economies due to a large supply of unskilled labor that has kept wages at very low levels, thus affecting negatively the share of labor in national income. In fact, very preliminary efforts made by the Bolivian statistics office (INE) in order to obtain some estimates of $1-\alpha$ for the last two decades yielded a value of around 0.48, implying that $\alpha = 0.52$.

However, it must be noted that due to lack of data, most labour's share measurements such as the one presented above tend to underestimate the labor income of the self-employed and family workers who make up a large fraction of the labour force. Moreover, Gollin (2002) and Bergoeing and Soto (2002) demonstrate that when there is enough information to adjust for this mismeasurements, the resulting capital shares lie within a range between 0.2 and 0.37.[29] In fact, for the purposes of this work, we will consider one of Gollin's estimates for Bolivia, situated at the top of the aforementioned range, of $\alpha = 0.373$.

3.3.4 Parameters related to the action of the government

The steady-state effective average tax rate for capital gain is taken from Chen and Mintz (2011) who estimate it to be $\tau^k = 0.23$, quite close to the nominal rate of 25 percent defined by law (Law No.1606, passed in December, 1994). For the case of the labour income tax and the consumption tax, due to the lack of up-to-date estimates and reliable data on their effective counterparts, both are set to the same rate of $\tau^h = \tau^c = 13\%$ as established by law (Tax Law No.843, passed in May, 1986).[30]

Next, the government-to-output ratio at the steady state, G/Y, is set to match the 20-year average ratio according to the data, of 14.2 percent. Likewise, the government transfers, G^{tr}, are set to a value of 0.065 so that the G^{tr}/Y ratio matches its respective 20-year average given by data, of 15 percent of GDP.

[29] With the adequate adjustments, Bergoeing and Soto (2002) managed to correct an official preliminary measurment of α from around 0.59 to a level of 0.36 for Chile.

[30] See http://bolivia.infoleyes.com/shownorm.php?id=367

It is worth noting that these transfers were included in the model only to match the key aggregates observed in the data and no fiscal rules related to them will be considered in this work as these have shown virtually no changes over the last few years, in the form of permanent and periodical bonuses aimed to very sensible social demands (e.g. elder people with no pension income, maternity and lactancy allowances, people with disabilities), and thus are very difficult to modify without either creating a climate of social unrest or risking its financial sustainability over time.

On the other hand, despite the fact that its 10-year average is around 67 percent, the public debt-to-output ratio will be set at the much lower average registered between 2008 to 2010 (with a slight but still worrying increasing trend in the last two years as discussed earlier), of 43 percent.

Since the World Bank's three-year average IDA Resource Allocation Index classifies Bolivia as a medium performer with respect to the overall quality of its macroeconomic policies, implying that the related risk threshold on debt-to-GDP is 40 percent (IMF, 2009), to remain as close to this target as possible seems a reasonable policy in terms of macroeconomic prudence and long-term debt sustainability. Therefore, a ratio of $B/Y = 0.43$ will be taken as the main reference in the fiscal rules to be examined in the following sections.

A word must be said about the reaction or feedback parameters of the fiscal rules given in equations (3.10) to (3.13). While the values of these parameters have no impact whatsoever in the steady-state of the economy, they do have major impact in the transition dynamics as we shall see later. In this sense, recalling that we are interested in the debt-stabilising properties of these fiscal rules, the ranges for each of these parameters which ensure the determinacy and learnability of the model are, respectively, $\gamma_g : [0, 0.19]$, $\gamma_k : [0, 0.39]$, $\gamma_h : [0, 0.39]$, $\gamma_c : [0, 0.33]$.

In order to define the optimal values for the parameters related to each rule within the above ranges, an optimal simple rule (OSR) procedure is performed according to which, for each rule, different values of the feedback parameter are tested. The optimal value of the parameter in each case will be the one that minimises a quadratic objective (or loss) function that takes into account both the standard deviation of the debt-to-GDP ratio and an income inequality ratio, defined as the discounted cumulative total net income earned by a representative capitalist with respect to the cumulative discounted total net income earned by a representative worker after each year. This procedure yields the following optimal parameter values $\gamma_g = 0.19$, $\gamma_k = 0.05$, $\gamma_h = 0.39$ and $\gamma_c = 0.33$, which will be used throughout the rest of this work.[31]

3.3.5 Sources of volatility

To conclude this parameterisation procedure, we now focus on the parameters of the two exogenous sources of volatility considered in the model. First, to estimate the AR(1) relation for the productivity process described by (3.2), TFP data for Bolivia was constructed as in King and Rebelo (1999)[32] using the IFS's National Accounts annual data for Bolivia from 1986 to 2010, in order to avoid the adverse effects in the estimations that arise when earlier periods, characterised by severe episodes of economic crisis (e.g. late-70s and early-80s), are included in the estimation sample.[33] The resulting estimates for ρ^a and σ_a are thus 0.602 and 0.028, respectively; while the constant term A_0 has been set equal to its standard value of one.

Finally, for the process of R_t, we assume that the constant term R_0 is equal to 0.0265, which ensures that at the steady-state the R/Y ratio is equal to 0.06, as observed in the data between 2008 to 2010 (see CBB, 2010). The parameters ρ^R and σ_R are set to 0.72 and 0.12, respectively. These were obtained by estimating an AR(1) process for the evolution of the quarterly average price per British Termal Unit of gas sales to Brazil between 2007 and 2011,[34] because changes in the volumes sold are less likely as these depend on the pipeline's capacity and thus can only be affected by other unexpected events such as

[31] It is worth noting that the rule on capital tax is the only that is kept at very low levels of reaction, mainly because of its significantly high negative impact on income inequality.

[32] The production function in (3.1) written in per-worker terms (considering the economically active population data provided by the Bolivian National Statistics Office - INE) was solved for A_t. By feeding the per-worker time series of output and (also newly built) capital stock, an estimated TFP series was obtained. This was then expressed in logs and detrended, and finally an AR(1) process was fitted to it.

[33] For instance, if the entire sample between 1950 and 2010 is considered, the less realistic resulting estimates are $\rho^a = 0.486$ and $\sigma_a = 0.118$.

[34] Gas sales to Brazil represent 76 percent of the total. The sale prices to both countries are determined by formulas which consider the international market prices of a basket of oil-related products. See http://www.hidrocarburosbolivia.com.

Table 3.2: Steady-state

variable	value	variable	value
$\frac{n^c C_c}{Y}$	0.288	$\frac{n^c K_c}{Y}$	2.001
$\frac{n^w C_w}{Y}$	0.420	$\frac{n^w K_w}{Y}$	0.069
$\frac{C}{Y}$	0.710	$\frac{K}{Y}$	2.070
$\frac{n^c I_c}{Y}$	0.145	$\frac{n^c B_c}{Y}$	0.416
$\frac{n^w I_w}{Y}$	0.005	$\frac{n^w B_w}{Y}$	0.014
$\frac{I}{Y}$	0.150	$\frac{B}{Y}$	0.430
$\frac{G}{Y}$	0.142	H_c	0.168
$\widetilde{r}_k = \widetilde{r}_b$	0.064	H_w	0.347
r_k	0.180	U_c	-30.57
r_b	0.065	U_w	-38.44
$\frac{R}{Y}$	0.06	U_a	-35.67

geopolitical issues, civil unrest in the producing regions, natural disasters, etc. High and increasing price volatility in the natural gas and most energy markets has been widely documented and discussed in the last years, see for instance Pindyck (2004).

3.3.6 Steady-State

The steady-state solution of the model is given in Table 3.2 below in terms of the aggregate variables.[35] First, note that the chosen parametrization allows the model to match the key aggregate ratios observed in Bolivian data between 1990 and 2010. That is, the C/Y and I/Y ratios are just below their observed counterparts of 0.72 and 0.16, while for the remaining aggregate ratios - K/Y, B/Y, G/Y and R/Y - the match is virtually perfect as intended during the parametrization procedure discussed in the previous section.

In addition, note that due to the different transaction costs in the financial markets, the capitalists hold 96.7 percent of the total assets in the economy. In effect, as said above, the ratios of savings, I_c/I_w, and assets, K_c/K_w and B_c/B_w, of the representative capitalist to the representative skilled worker, are equal to fifty five, which gives a clear idea of the high levels of income and wealth inequality in Bolivia.

Second, note that a representative capitalist consumes relatively more than a worker

[35] Variables $\widetilde{r}_b = \widetilde{r}_k$ denote net rates (i.e. after tax, depreciation and transaction costs when applicable), which under the given assumptions have to be equal in both asset markets.

since $C_c = 0.3607$ and $C_w = 0.2834$. This implies that in aggregate terms the capitalists - i.e. representing only 35 percent of the entire population - consume 41 percent of the total consumption in the economy as shown in the Table. On the other hand, the variables H_c and H_w show that, as expected, the capitalists work considerably less than the workers. Thus, in terms of welfare, higher consumption and lower work effort make the capitalists relatively better off as indicated by U_k which is smaller than U_w in absolute terms. The weighted average measure of aggregate or Benthamite lifetime utility (computed for 1000 periods), U_a, is also reported in the Table.

3.4 Model Solution

To solve the model, we start by taking the first-order Taylor series expansion of the DCE and exogenous process for productivity around their respective steady-states. For any variable X_t, these values are denoted $\widehat{X}_t = \log X_t - \log X$. We next re-express the model in matrix form as second-order difference equation system:

$$\begin{aligned} \mathbf{x}_t &= \mathbf{M}_1 \mathbf{E}_t \mathbf{x}_{t+1} + \mathbf{M}_2 \mathbf{x}_{t-1} + \mathbf{M}_3 \mathbf{z}_t \\ \mathbf{y}_t &= \mathbf{N}_1 \mathbf{x}_t + \mathbf{N}_2 \mathbf{x}_{t-1} + \mathbf{N}_3 \mathbf{z}_t + \mathbf{N}_4 \mathbf{E}_t \mathbf{x}_{t+1} \\ \mathbf{z}_t &= \boldsymbol{\rho} \mathbf{z}_{t-1} + \mathbf{u}_t. \end{aligned} \tag{3.18}$$

where $\mathbf{x}_t = \left[\hat{B}_{c,t+1}, \hat{K}_{c,t+1}, \hat{B}_{w,t+1}, \hat{K}_{w,t+1}\right]'$ contains the endogenous state variables; $\mathbf{y}_t = \left[\hat{C}_{c,t}, \hat{C}_{w,t}, \hat{H}_{c,t}, \hat{H}_{w,t}, \hat{r}^b_t, \hat{r}^k_t, \hat{w}_t, \hat{G}_t, \hat{\tau}^k_t, \hat{\tau}^h_t, \hat{\tau}^c_t\right]'$ the endogenous control variables; and $\mathbf{z}_t = \left[\hat{a}_{t+1}, \hat{R}_{t+1}\right]'$ the exogenous state variables.[36] The various $\mathbf{M}$ and $\mathbf{N}$ matrices contain convolutions of the structural parameters. Finally, since we have two exogenous state variables, $\boldsymbol{\rho} = \begin{bmatrix} \rho^a & 0 \\ 0 & \rho^R \end{bmatrix}$ and $\mathbf{u}_t = [\varepsilon^a_{t+1}, \varepsilon^R_{t+1}]'$.

In Appendix C it is briefly described how (3.18) is used to obtain both the rational expectations (RE) and adaptive learning (AL) solutions of the log-linearised model. For the latter, a more detailed description of how the initial conditions for learning have been

[36] For examples of other work in the literature using this particular reduced form, see Evans and Hokhaponja (2001), Giannitsarou (2006) and Carceles Poveda and Giannitsarou (2008).

setup is also provided.

3.5 Results

3.5.1 Setup of the experiments

In this section we examine the performance and distributional implications of each of the fiscal rules cited earlier when the economy faces different sources of volatility and assuming different ways of forming expectations as discussed above. For this purpose, the transition dynamics in response to productivity and gas-revenue or commodity shocks are studied by means of standard impulse-response functions (IRFs).

It is worth noting that, when assessing the impact of productivity shocks, it is common in the literature to examine the behavior of the relevant variables in response to a one percent shock. This seems reasonable as long as the estimated standard deviation σ_ε of the errors in the autoregressive process describing the behavior of the variable of interest is small. For instance, if we consider the well-known standard deviation of 0.00712, estimated for the TFP process in the US (see e.g. Giannitsarou, 2006), shocks in the range $[-0.01, 0.01]$ will cover 90 percent of the probability mass.

Following a similar criterion and taking an standard deviation of $\sigma_\varepsilon = 0.028$ - estimated earlier for TFP in Bolivia - implies that shocks in the range $[-0.045, 0.045]$ also cover 90 percent of the probability mass. In the same line, an estimated standard deviation of $\sigma_R = 0.12$ for the process of the international gas prices faced by the Bolivian government implies that shocks in the range $[-0.2, 0.2]$ are required to cover a similar probability mass. Consequently, TFP shocks and commodity shocks in the order of +/-4.5 percent and +/-20 percent, respectively, are considered in the experiments of this section. However, since the results found when negative shocks are assumed are found to be symmetrically opposite to those found under positive shocks, only the results of the former case will be reported and discussed in detail.

With respect to the learning setup, it is worth noting that when it is assumed that both types of agents are learners but their learning processes, including their initial beliefs, are homogeneous, no additional distributional implications are identified with respect to the results under RE. Hence, we focus instead in a situation where the two types of agents do differ in terms of the initial information they have at the start of their learning processes, which we assume coincides with the date an exogenous shock hits the economy.

To this end, the matrix with the agents' initial beliefs $\tilde{\phi}_0$ is constructed as described by equation $(C.15)$ in Appendix C, denoting the fact that capitalists always know the RE equilibrium parameters of their policy functions while workers do not and instead can have either incomplete initial beliefs (i.e. where the steady-state equilibrium is known but the effects of the fiscal rule are not), or, in addition to the previous case, pessimistic or optimistic initial beliefs associated with off-equilibrium perceptions about the state of the economy.

In effect, the pessimistic beliefs are consistent with an hypothetical lower steady-state with a 0.4 percent lower consumption, virtually the same working hours and an over-estimated debt-to-GDP ratio of 0.47; found by assuming that the three effective taxes in the economy (i.e. capital, labour and consumption) are just one percent higher than under the base calibration in Table 3.1.

This particular way of defining a lower equilibrium was chosen to denote the fact that those agents with less information in the economy might underestimate the actual state of the economy and attribute this misperceived under-performance to the distorting actions of the government. Note too that the changes in the three tax rates are equally proportional to ensure that in the lower state there are no additional distributional implications coming from a different tax schedule.[37]

In similar fashion, the optimistic initial beliefs are consistent with an hypothetical higher steady-state with a 0.4 percent higher consumption, the same number of hours worked and an under-estimated debt-to-GDP ratio equal to 0.34, found by assuming that the effective taxes in the economy are one percent lower than in the true equilibrium.[38]

[37] When it is assumed that workers do not over- or under-estimate the debt-to-GDP ratio, implying that G^{tr} has to be re-calibrated in each of the new steady-states to preserve the above ratio at 43%, learning generates no significant differences with respect to the results under RE. For this reason, this line of research is not taken any further.

[38] When higher-than-one percent differences in the taxes are considered to find the hypothetical equilibria, the model becomes not learnable (see e.g. Evans and Honkapohja, 2001) if the economy is subject to large commodity shocks.

It is important to note too that for those experiments considering the constant gain variant of the RLS algorithm, the gain parameter g is set to its most common value in the literature of 0.03, as confirmed by the survey in Williamson and Orphanides (2005) and by the estimates of Milani (2007).

The performance of each of the rules in terms of stabilising debt is measured by the predicted standard deviation of the debt-to-GDP ratio in levels computed both in the short run (i.e. for the first five years) and over the lifetime.[39]

On the other hand, the distributional impact of the rules is examined by means of the evolution of the ratio between the discounted cumulative total net income (i.e. labour and capital income net of taxes and transaction costs plus average transfers) earned by a representative capitalist and the cumulative discounted total net income earned by a representative worker after each year. Hence, in what follows, a higher income-inequality ratio denotes a worse income distribution in the economy.

In what follows, we first examine the performance of the proposed fiscal rules in terms of their debt-stabilisation properties. Next, the distributional effects derived from applying each of these rules are studied. In both parts, the effects of the different assumptions regarding how expectations are formed are taken into account.[40]

3.5.2 Debt-stabilisation properties of the rules

The upper panel of Table 3.3 reports the standard deviations associated with the evolution of the B/Y ratio under each of the rules after a negative 4.5 percent TFP shock has hit the economy while the lower panel reports similar information in the case of a negative 20% commodity shock.

Consider the performance of the rules under full rationality first (columns denoted with RE in the Table). Recalling that all the rules are parameterised at their optimal values, note that in the short run the fiscal rules on τ^c and τ^k by far offer the best performance

[39] For the lifetime calculations 1000 periods were used.

[40] The results obtained considering the RLS learning algorithm and its constant gain variation show very small differences. For this reason, only the results under the former are reported.

Table 3.3: Standard deviation of B/Y

Rule type	Short Run			Lifetime		
	RE	AL-opt	AL-pes	RE	AL-opt	AL-pes
After a -4.5% TFP shock						
Rule on G/Y	0.0785	0.0788	0.0782	0.0133	0.0137	0.0132
Rule on τ^k	0.0760	0.0773*	0.0749	0.0131*	0.0132*	0.0130
Rule on τ^h	0.0829	0.0914	0.0747*	0.0137	0.0363	0.0114*
Rule on τ^c	0.0691*	0.0790	0.0789	0.0137	0.0132*	0.0140
After a -20% commodity shock						
Rule on G/Y	0.0805	0.0823	0.0757	0.0235	0.0255	0.0205
Rule on τ^k	0.0781*	0.0800*	0.0779	0.0223*	0.0231*	0.0233
Rule on τ^h	0.0889	0.1076	0.0708*	0.0255	0.2353	0.0185*
Rule on τ^c	0.0807	0.0819	0.0749	0.0235	0.0255	0.0198

in response to TFP and commodity shocks, respectively. Increasing these two tax rates in order to rapidly improve the public finances is in fact common practice.[41]

In the long run, the rule on τ^k shows the best debt-stabilising properties after either type of exogenous shock. In the case of TFP shocks, this is expected since the negative productivity shock tends to reduce the fiscal revenues and thus leads to a raise of public debt. Both the increase in debt and the fall of output generate an increase of the B/Y ratio above its long-run target. Hence, according to the rule, the increase of the B/Y ratio immediately motivates a raise in τ^k to increase the government revenues and reduce debt. In a context of high wealth and income inequality, this tax raise affects mostly to capitalists who see their disposable income fall. However, since they can gradually increase their labour supply to compensate for the loss in capital income, aggregate output and income will tend to increase faster under this particular rule, with respect to the other rules.

In the case of commodity shocks, similar mechanisms as those described above take place. However, since shocks of this type are known to be relatively large and since these have a direct impact on the financial markets, the overall performance of the rules - including the rule on τ^k - in a fully-rational environment is worse, as shown by the larger standard deviations of B/Y in the lower panel of the Table.

[41] Examples of this are the increase of the Value Added Tax (VAT) in the UK in 2012, and the gradual increase of the Capital Gain tax in Iceland since 2008, in order to face the aftermath of the 2007-2009 financial crisis.

These last findings illustrate the worrying levels of dependence and vulnerability Bolivia has with respect to its gas exports, which in turn can suffer of high price volatility. As mentioned earlier, such features are quite common among a number of developing countries rich in natural resources and have motivated a more active search of mechanisms aiming to reduce the sensitivity of their public finances to the dynamics of the international markets of these resources.[42]

Next, we turn our attention on how heterogenous learning changes the above results. First, we simply assume that while capitalists know their equilibrium policy functions,including the effects of the fiscal rule applied, workers do not know the latter. In such case, the results are very similar to those found under full rationality above, irrespective of the type of shock or the chosen learning algorithm. These results suggest that the differences between the workers' estimated policy function according to their initial beliefs and their true policy functions under RE are rather small in this case. As a result, they are able to learn the latter very rapidly despite the temporary deviations from equilibrium generated by the exogenous shock.

In light of the above, we further explore the implications of assuming a higher degree of heterogeneity between agents in the form of differing initial beliefs. That is, in addition to the above assumption, we further assume that, while the capitalists' initial beliefs are consistent with the true equilibrium of the economy, off-equilibrium optimistic or pessimistic expectations regarding the situation of the economy govern the initial beliefs among workers.

From these two cases, reported in Table 3.3 as 'AL-opt' and 'AL-pes' respectively, it is clear that the type of initial belief the workers are assumed to hold has an important effect on the evolution of the volatility debt-to-GDP ratio. First, when optimistic initial beliefs prevail among workers, the rule on τ^k stands out as the best performing rule irrespective of the type of exogenous shock. A similar mechanism as under RE above explains this outcome, with the difference that, due to the initial optimism of workers which leads to an increase in the returns on capital, capitalists choose to work more and consume less

[42] Such as stabilisation funds with different characteristics depending on the type of natural resource (see e.g. Sturm *et al.*, 2009).

in order to invest relatively more, thus leading to a faster capital accumulation in the early periods after the shock than under full-rationality. This behavior is then adjusted gradually as they observe the adverse effects of the negative shock.

Second, when these beliefs are pessimistic, the rule taking τ^h as an instrument tops the rank. In this case, workers initially choose to work more to compensate the expected fall in consumption consistent with a lower steady-state, only to gradually realise the levels of consumption are in fact higher than expected. This change in the outlook of the economy leads workers and, later on, capitalists to gradually increase their labour supply in order to fund higher levels of investment. In this sense, given that the labour income tax base increased since the early periods after the shock, the fiscal rule on this tax helps to bring the B/Y ratio back to its target in a more efficient fashion relative the other rules. Having said this, something worth of attention is the fact that this rule provides the most extreme results under adaptive learning, both in the short- and long-run. That is, if the initial beliefs among workers are pessimistic then the rule shows the best performance of all as explained above, but if the beliefs are optimistic instead then it yields the worst possible performance.

3.5.3 Income inequality

The two figures presented in Appendix C show the evolution of the income inequality ratio over time in response to negative TFP and commodity shocks according to each of the proposed rules under both RE and heterogeneous learning. In these, it can be seen that a negative shock of any type generates a sharp temporary fall in the inequality ratio (or a better income distribution) for all cases considered.[43]

Then, the adjustment process back to the equilibrium level of almost 1.6 - denoting that the net income of a representative capitalist is approximately sixty percent higher that than that of a representative worker[44] - can differ in terms of the persistence of the initial deviation from equilibrium, depending on the chosen rule and/or the assumption

[43] Conversely, a positive shock temporarily increases income inequality in all cases.

[44] This result is consistent with a survey carried by UPB(2005), which shows that 77% of the population in Bolivia earn less than 4000 Bolivianos per month, while the remaining 23% earn 7000 Bolivianos per month or more.

Table 3.4: Evolution of the income inequality ratio

Rule type	Short Run			Lifetime		
	RE	AL-opt	AL-pes	RE	AL-opt	AL-pes
After a -4.5% TFP shock						
Rule on G/Y	1.5771*	1.5778*	1.5760	1.5976	1.5977*	1.5975
Rule on τ^k	1.5786	1.5798	1.5777	1.5981	1.5983	1.5978
Rule on τ^h	1.5776	1.5827	1.5736*	1.5975*◊	1.6047	1.5961*◊
Rule on τ^c	1.5772	1.5778*◊	1.5762	1.5976	1.5977*	1.5977
After a -20% commodity shock						
Rule on G/Y	1.5889	1.5876	1.5899*	1.5968	1.5959	1.5976
Rule on τ^k	1.5900	1.5889	1.5908	1.5972	1.5966	1.5974*
Rule on τ^h	1.5886*◊	1.5852*	1.5918	1.5963*◊	1.5775*◊	1.5982
Rule on τ^c	1.5890	1.5878	1.5902	1.5968	1.5959	1.5978

regarding expectations. Table 3.4 reports a summary of these results once more focusing both in the short-(five years) and long-run.

As previously done, let us examine the results under RE first. The rule on τ^h appears to show the best performance overall as it generates the lowest inequality ratios with respect to the other rules, although in the short-run this is only true in the case of the commodity shock, given that the rule on G/Y is the best performer if a negative TFP shock is assumed. The fact that these two rules have a relatively more even impact over workers and capitalists than the capital tax, which tends to favour the latter, explains this result.

Precisely, the rule on τ^k deserves some attention as it generates slightly higher levels of income inequality when measured over the lifetime, thus making it the worst performing rule of all. The unequal wealth and income distribution in the economy implies that any change in τ^k will mainly affect the capitalists. In the early periods after the shock, the initial raise of this tax in response to the increasing debt (leading the B/Y ratio above its long run target) affects negatively their disposable income. However, since they can compensate part of this loss by increasing their labour supply, the short-run impact of this fiscal rule on income inequality is rather small. Then, as the effects of the negative exogenous shock fade over time, consecutive reductions of this tax must follow in line with the rule, which in turn leads to a steady increase of the capitalists' disposable income and, consequently, to higher levels of income inequality over time.

We next focus on the impact of heterogenous learning on these results. As when studying debt-stabilisation, when it is assumed that workers have initial beliefs consistent with the true equilibrium but do not know the potential impact of the rule applied on the economy, the results obtained are very similar to those found assuming RE. For this reason, once again we turn our attention to the cases were optimistic or pessimistic expectations affect the workers' initial beliefs.

Since the results in this part are found to be qualitatively different depending on the type of exogenous shock considered, we first examine the impact of a negative TFP shock. In this case, Table 3.4 shows that if optimistic initial beliefs are assumed among workers then the rules on G/Y and τ^c show the best performance, irrespective of time horizon considered. In contrast, if pessimistic beliefs are assumed then, it is the rule on labour income that one which tend to improve income distribution.

This result is explained by the pessimistic beliefs among workers, which motivate them to work and invest relatively less than under equilibrium, implying that they are hit relatively less than the capitalists by the lower factor prices and the higher tax rates or lower government spending implied by the rules in response to the negative shock, thus naturally leading to lower levels of income inequality.

Finally, consider a negative commodity shock. In this case, the initial fall in revenues due to the bad shock forces the government to increase its demand for debt, which leads to an increase in the returns of public bonds and, given the market clearing conditions, to an increase in the returns of physical capital too. Within this context, optimistic workers invest and work more than if their beliefs where consistent with the true equilibrium and thus benefit from the increase in the asset returns during the early periods after the shock which. These early benefits, in turn, in most cases outweight the future losses in disposable income that follow once taxes (spending) start to increase (decrease) according to the fiscal rule applied aiming to stabilise debt. Under these circumstances, the rule on τ^h shows the best performance.

If, by contrast, pessimistic initial beliefs prevail among workers, they behave in the opposite way, thus losing the chance to benefit from the increasing returns in the financial markets. However, the lower than expected returns on capital generated by this behavior affects capitalists, who see their capital income decrease in the long run. In this sense, the rule on G/Y in the short run, given its more even impact on both types of agents, and the rule τ^k in the long run exhibit the best performances in this case.

Before concluding this part, it is important to note that a welfare-inequality ratio, defined as the ratio between the cumulative utility of the representative capitalist and the cumulative utility of the representative worker, has also been computed for the short run and over the lifetime (see Table C.1 in Appendix C). The ratio shows that, in fifty percent of the cases, whenever a rule generates the best income distribution (cases marked with an '$*$' in the Table), it also leads to the best possible distribution of welfare (cases marked with an '$\Diamond$' in the Table).[45]

3.5.4 Income distribution and debt-stabilisation trade-off

The above results suggest that the relative performance of each of the rules considered in this work can be quite different depending on the criterion chosen to evaluate them but also on the circumstances assumed to prevail in the economy in terms of the type of shock that hits the economy or the way agents form expectations.

Given this diverse set of results available, a reasonable way of better ranking these rules in order to find the most adequate one given the particular features of Bolivia, should consider the performance of each rule in terms of both its debt-stabilisation properties as well as its distributional implications as measured by its impact on the income-inequality ratio.

In this sense, Table 3.5 below presents a summary of the relative performance of each rule according to both criteria. Only those cases in which the rule performs better that the remaining rules in both fields are identified with a '$\checkmark$' mark, while the other cases are assigned an 'x' mark.[46]

[45] Note that, in constrast to the income-inequality ratio, in this case a lower welfare-ratio means higher welfare inequality, because the computed discounted utility values are negative (see Table 3.2).

[46] Note that in the early periods after a TFP shock, the rule on τ^h generates a better income distribution as well as a lower volatility of the debt-to-GDP ratio under heterogeneous learning with pessimistic beliefs among workers. However since it also yields a worse welfare distribution as shown in Table C.1 in Appendix C, it has been marked with an 'x'.

Table 3.5: Performance of the rules according to both criteria

Rule type	Short Run			Lifetime		
	RE	AL-opt	AL-pes	RE	AL-opt	AL-pes
After a -4.5% TFP shock						
Rule on G/Y	x	x	x	x	x	x
Rule on τ^k	x	x	x	x	x	x
Rule on τ^h	x	x	✓	x	x	✓
Rule on τ^c	x	x	x	x	x	x
After a -20% commodity shock						
Rule on G/Y	x	x	x	x	x	x
Rule on τ^k	x	x	x	x	x	x
Rule on τ^h	x	x	x	x	x	x
Rule on τ^c	x	x	x	x	x	x

Consider the case where the economy is populated by fully-rational agents first. Irrespective of the type of shock or the time horizon under analysis, it is clear no rule can satisfy both criteria, a result which denotes a type of trade-off between debt-stabilisation and income inequality for this economy as it faces different types of exogenous shocks.

Next, consider the impact of heterogenous learning on this analysis. The only rule deserving some attention is the one on τ^h because, if pessimistic initial beliefs among workers are assumed[47], it actually breaks the trade-off between debt-stabilisation and income-inequality only if a negative TFP shock is considered.[48] However, the trade-off can not be broken whenever the workers hold optimistic beliefs instead or for any case related to a negative commodity shock.

This last result, as said earlier, can be interpreted as typical symptom suffered by several countries which are rich in a certain commodity and thus become highly dependant on the revenues coming from its exports (see e.g. Van der Ploeg and Poelhekkey, 2009, and Van der Ploeg, 2011) . Moreover, if the price of this commodity tends to be highly volatile - as is the case of natural gas, see e.g. Pyndick (2004) - implies that the country's vulnerability to abrupt price changes becomes a major concern.

To further elaborate on this aspect, it seems important to examine how these results

[47] If positive exogenous shocks are considered instead, then optimistic initial beliefs among workers lead to this and the next results in this section.

[48] Note too that a similar trade-off with respect to welfare-inequality still remains even for this case.

Table 3.6: Performance of the rules after a negative five percent commodity shock

Rule type	Short Run			Lifetime		
	RE	AL-opt	AL-pes	RE	AL-opt	AL-pes
Standard deviation of B/Y						
Rule on G/Y	0.0174	0.0178	0.0167	0.0052	0.0056	0.0048
Rule on τ^k	0.0169	0.0173	0.0169	0.0049	0.0051	0.0051
Rule on τ^h	0.0191	0.0223	0.0158*	0.0056	0.0289	0.0042*
Rule on τ^c	0.0175	0.0177	0.0166	0.0052	0.0056	0.0046
Income inequality ratio						
Rule on G/Y	1.5950	1.5947	1.5952	1.5993	1.5990	1.5994
Rule on τ^k	1.5961	1.5958	1.5963	1.5996	1.5994	1.5996
Rule on τ^h	1.5950	1.5942	1.5952*	1.5992	1.5959	1.5994*
Rule on τ^c	1.5950	1.5947	1.5953	1.5993	1.5991	1.5994
Performance of the rules according to both criteria						
Rule on G/Y	x	x	x	x	x	x
Rule on τ^k	x	x	x	x	x	x
Rule on τ^h	x	x	✓	x	x	✓
Rule on τ^c	x	x	x	x	x	x

change if some smoothing mechanism - e.g. a stabilisation fund, is designed to isolate the economy from the high volatility of these resources.[49] As an illustration, consider a case where, thanks to such smoothing mechanism, a commodity shock in this economy is now quantitatively more similar to a TFP shock, so that, for instance, a +/-5 percent shock also covers 90 percent of the relevant probability mass.

Table 3.6 summarises the results of assuming this smaller commodity shock. In this case, the trade-off is preserved under RE as above, but now assuming heterogeneous learning with pessimistic initial beliefs among workers can help to overcome the debt-stabilisation versus income inequality trade-off, only when the fiscal rule on τ^h is implemented.

3.6 Conclusions

This document presents an heterogeneous-agent DSGE model with two types of agents (namely, capitalists and workers) and no social mobility, calibrated to match Bolivia's key aggregate data, particularly its high levels of wealth and income inequality, with the aim

[49] Funds of this type have been implemented in *e.g.* Norway, Russia and Botswana (see e.g. Sturm *et al.*, 2009).

of examining the short and long-run effectiveness and distributional effects of four fiscal rules designed to impose restrictions on the evolution of public debt. A long history of severe debt-crisis episodes, triggered both by fiscal and monetary mismanagement and the resulting need to apply for a number of debt-relief programs over the last twenty five years justifies this interest.

Taking this economy's particular case as a useful illustration, the contribution of this work is twofold. First, from a theoretical point of view, it shows that the adoption of fiscal rules aiming to control the evolution of public debt in a context of full rationality and high wealth and income inequality can prove challenging in response to productivity and, specially, large commodity shocks.

The reason for this result is that, in general, the proposed rules generate a trade-off between debt-stabilisation and income inequality which should not be ignored, especially in countries like Bolivia, currently ranked among the worst in the world in terms of income distribution, where such aspects clearly stand out as highly sensible from a sociopolitical point of view.

However, if agents are assumed to behave as learners but with initial beliefs between capitalists and workers that differ at the start of their learning processes - which is assumed to coincide with the time an exogenous shock hits the economy - the transition dynamics are such that in a number of cases the aforementioned trade-off is no longer present and hence some of the rules - i.e. those instrumented via the labour income tax - perform rather well in both fields.

Second, from a policy-oriented point of view, this work suggests that, given the particular features of Bolivia, finding a fiscal rule which can show effective debt-stabilising properties without compromising income distribution in response to all relevant types of exogenous shocks is not an easy task. In effect, the fiscal rule on labour income tax (in that order) seem the most reasonable candidates, particularly if heterogeneous learning is assumed to prevail in the economy in terms of how expectations are formed among agents.

When large commodity shocks are considered, however, the overall performance of the rules is much worse, a result that illustrates the high dependence of this economy on the revenues coming from natural gas exports and the significant degree of vulnerability with respect to this commodity's international price, known to be highly volatile.

In light of the above, it could be argued that the implementation of a smoothing mechanism or natural gas stabilisation fund aiming to isolate the economy from this high volatility identified in commodity revenues, could prove useful at increasing the overall effectiveness of the chosen fiscal rule and thus help to preserve a sound fiscal policy in Bolivia over time. We leave this issue for future research.

Appendix A

Appendix to Chapter 1

A.1 The stationary version of the model

Recall the definitions $c_t = C_t/\eta^t$, $h_t = H_t$, $k_t = K_t/\eta^t$, $y_t = Y_t/\eta^t$, $i_t = I_t/\eta^t$, $w_t = W_t/\eta^t$, $r_t = R_t$ and $z_t = Z_t$. Hence, the transformation for the production function is:

$$\begin{aligned}\frac{Y_t}{\eta^t} &= \frac{Z_t K_{t-1}^{\theta}\left(\eta^t H_t\right)^{1-\theta}}{\eta^t}, \\ y_t &= z_t \eta^{-\theta} k_{t-1}^{\theta} h_t^{1-\theta}, \end{aligned} \tag{A1}$$

while the technology process can be written simply as:

$$\begin{aligned}\ln Z_t &= (1-\rho)\ln Z + \rho \ln Z_{t-1} + \varepsilon_t, \\ \ln z_t &= (1-\rho)\ln Z + \rho \ln z_{t-1} + \varepsilon_t. \end{aligned} \tag{A2}$$

Meanwhile, the budget constraint is:

$$\begin{aligned}\frac{Y_t}{\eta^t} &= \frac{C_t + I_t + \tau_k(R_t-\delta)K_{t-1} + \tau_h W_t H_t}{\eta^t}, \\ y_t &= c_t + i_t + \tau_k(r_t-\delta)\eta^{-1}k_{t-1} + \tau_h w_t h_t \end{aligned} \tag{A3}$$

while the capital accumulation law can be transformed as:

$$\begin{aligned}\frac{K_t}{\eta^t} &= \frac{(1-\delta)K_{t-1}+I_t}{\eta^t}, \\ k_t &= \eta^{-1}(1-\delta)k_{t-1}+i_t. \end{aligned} \tag{A4}$$

The Euler equation will be transformed according to:

$$\begin{aligned} 1 &= \beta E_t\left[\left(\frac{C_t}{C_{t+1}}\frac{\eta^t}{\eta^t}\right)\left(\left(\theta\frac{Y_{t+1}}{K_t}\frac{\eta^t}{\eta^t}-\delta\right)(1-\tau_k)+1\right)\right], \\ &= \beta E_t\left[\left(\frac{c_t}{\eta c_{t+1}}\right)\left(\left(\theta\frac{\eta y_{t+1}}{k_t}-\delta\right)(1-\tau_k)+1\right)\right] \end{aligned} \tag{A5}$$

and, finally, the first order condition related to labour becomes:

$$\begin{aligned}\frac{\gamma C_t H_t}{\eta^t} &= \frac{(1-\tau_h)(1-\theta)Y_t}{\eta^t}, \\ h_t\gamma c_t &= (1-\tau_h)(1-\theta)y_t, \end{aligned} \tag{A6}$$

Next, defining $r_t = R_t$ and $w_t = W_t/\eta^t$as the stationary values of the factor rentals, the equilibrium conditions in the factor markets become:

$$\begin{aligned} R_{t+1} &= \theta\frac{Y_{t+1}}{K_t}\frac{\eta^t}{\eta^t}, \\ r_{t+1} &= \theta\frac{\frac{\eta Y_{t+1}}{\eta^{t+1}}}{k_t} = \theta\frac{\eta y_{t+1}}{k_t}, \end{aligned} \tag{A7}$$

and:

$$\begin{aligned}\frac{W_t}{\eta^t}H_t &= (1-\theta)\frac{Y_t}{\eta^t}, \\ w_t h_t &= (1-\theta)y_t. \end{aligned} \tag{A8}$$

Finally, the stationary definition of the government expenditure is:

$$\begin{aligned}\frac{G_t}{\eta^t} &= \tau_k\left(r_t-\delta\right)\frac{K_{t-1}}{\eta^t}+\frac{\tau_h W_t H_t}{\eta^t} \\ g_t &= \tau_k\left(r_t-\delta\right)\eta^{-1}k_{t-1}+\tau_h w_t h_t. \end{aligned} \tag{A9}$$

A.2 The steady-state values

Assuming that $x_t = x$ with $x = \{y, c, i, k, z, h, r, w, g\}$, first the steady-state value of capital can be found from the Euler condition as follows:

$$\begin{aligned}
1 &= \beta\left[\left(\frac{c}{\eta c}\right)\left(\left(\theta\frac{\eta y}{k}-\delta\right)(1-\tau_k)+1\right)\right], \\
\frac{\eta}{\beta} &= \left(\theta\frac{\eta y}{k}-\delta\right)(1-\tau_k)+1, \\
\left(\theta\frac{\eta y}{k}-\delta\right)(1-\tau_k) &= \frac{\eta}{\beta}-1, \\
\theta\frac{\eta y}{k} &= \frac{\eta/\beta-1}{(1-\tau_k)}+\delta, \\
k &= \frac{\theta\eta y}{\frac{\eta/\beta-1+\delta(1-\tau_k)}{(1-\tau_k)}} \\
k &= \left(\frac{\theta\eta(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}\right)y, \qquad \text{(A10)}
\end{aligned}$$

while the value for investment can be found by replacing the above result into the law of capital accumulation and rearranging terms:

$$\begin{aligned}
k &= \eta^{-1}(1-\delta)k+i, \\
i &= \left[1-\eta^{-1}(1-\delta)\right]k, \\
&= \left[1-\eta^{-1}(1-\delta)\right]\left(\frac{\theta\eta(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}\right)y, \\
&= \left(\frac{\theta(\eta-1+\delta)(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}\right)y. \qquad \text{(A11)}
\end{aligned}$$

Similarly, the steady state value of consumption can be found from the resource constraint, by substituting the previous two results for k and i and the equilibrium condition in the labour market for w:

$$\begin{aligned}
c &= y-i-\tau_k\eta^{-1}(r-\delta)k-\tau_h wh, \\
&= y-\left(\frac{\theta(\eta-1+\delta)(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}\right)y-\tau_k\eta^{-1}(r-\delta)\left(\frac{\theta\eta(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}\right)y \\
&\quad -\tau_h(1-\theta)y, \\
&= y\left[1-\frac{\theta(\eta-1+\delta)(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}-\frac{\tau_k(r-\delta)\theta(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}-\tau_h(1-\theta)\right], \qquad \text{(A12)}
\end{aligned}$$

while the value for hours worked comes from the Euler condition for labour and the previous result for consumption:

$$
\begin{aligned}
h &= \frac{(1-\tau_h)(1-\theta)}{\gamma}\frac{y}{c}, \\
&= \frac{(1-\tau_h)(1-\theta)}{\gamma} \frac{y}{y\left[1-\left(\frac{\theta(\eta-1+\delta)(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}\right)-\tau_k(r-\delta)\left(\frac{\theta(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}\right)-\tau_h(1-\theta)\right]}, \\
&= \frac{(1-\tau_h)(1-\theta)}{\gamma\left[1-\left(\frac{\theta(\eta-1+\delta)(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}\right)-\tau_k(r-\delta)\left(\frac{\theta(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}\right)-\tau_h(1-\theta)\right]}.
\end{aligned} \tag{A13}
$$

With the results for capital and labour it is now possible to take the production function and find the steady-state value of output (with $z = Z$ as shown later). For convenience, first define $\chi = \frac{\eta}{\beta} - 1 + \delta(1-\tau_k)$ and $\zeta = (\eta - 1 + \delta)(1-\tau_k)$ to get:

$$
\begin{aligned}
y &= Z\eta^{-\theta}k^{\theta}h^{1-\theta}, \\
&= Z\eta^{-\theta}\left(\left(\frac{\theta\eta(1-\tau_k)}{\chi}\right)y\right)^{\theta} \times \\
&\times \left(\frac{(1-\tau_h)(1-\theta)}{\gamma}\left[1-\left(\frac{\theta\zeta}{\chi}\right)-\tau_k\,(r-\delta)\left(\frac{\theta(1-\tau_k)}{\chi}\right)-\tau_h(1-\theta)\right]^{-1}\right)^{1-\theta},
\end{aligned}
$$

which becomes:

$$
\begin{aligned}
y^{1-\theta} &= Z\left(\frac{\theta(1-\tau_k)}{\chi}\right)^{\theta} \times \\
&\left(\frac{(1-\tau_h)(1-\theta)}{\gamma}\left[1-\left(\frac{\theta\zeta}{\chi}\right)-\tau_k\,(r-\delta)\left(\frac{\theta(1-\tau_k)}{\chi}\right)-\tau_h(1-\theta)\right]^{-1}\right)^{1-\theta},
\end{aligned}
$$

and raising both sides of the equation to the power of $1/(1-\theta)$ finally gives:

$$
\begin{aligned}
y &= Z^{\frac{1}{1-\theta}}\left(\frac{\theta(1-\tau_k)}{\chi}\right)^{\frac{\theta}{1-\theta}}\frac{(1-\tau_h)(1-\theta)}{\gamma} \times \\
&\left[1-\left(\frac{\theta\zeta}{\chi}\right)-\tau_k\,(r-\delta)\left(\frac{\theta(1-\tau_k)}{\chi}\right)-\tau_h(1-\theta)\right]^{-1}.
\end{aligned} \tag{A14}
$$

Note that here the following result obtained for the technology process was also used above:

$$\begin{aligned}
\ln z &= (1-\rho)\ln Z + \rho \ln z, \\
\ln z - \rho \ln z &= (1-\rho)\ln Z, \\
\ln z &= \ln Z, \\
z &= Z. \qquad \text{(A15)}
\end{aligned}$$

In addition, the steady-state values of the factor prices are:

$$\begin{aligned}
r &= \theta \frac{\eta y}{k}, \\
&= \theta \frac{\eta y}{\left(\frac{\theta\eta(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}\right) y}, \\
&= \frac{\theta\eta}{\left(\frac{\theta\eta(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}\right)}, \\
&= \frac{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}{(1-\tau_k)}. \qquad \text{(A16)}
\end{aligned}$$

and:

$$w = \frac{(1-\theta)y}{h},$$

which, after substituting $(A13)$ and $(A14)$ and simplifying, becomes:

$$w = (1-\theta)Z^{\frac{1}{1-\theta}} \left(\frac{\theta(1-\tau_k)}{\frac{\eta}{\beta}-1+\delta(1-\tau_k)}\right)^{\frac{\theta}{1-\theta}}, \qquad \text{(A17)}$$

Finally, the steady-state value for government spending is given by:

$$g = \tau_k \left(r - \delta\right) \eta^{-1} k + \tau_h w h \qquad \text{(A18)}$$

A.3 The log-linear version of the model

The log-linearization has been done defining $\hat{x}_t = \ln x_t/x$ with $x = \{y, c, h, i, k, z\}$. First, the log-linear version of the production function is:

$$
\begin{aligned}
y_t &= z_t \eta^{-\theta} k_{t-1}^{\theta} h_t^{1-\theta}, \\
\eta^{\theta} y e^{\hat{y}_t} &= Z e^{\hat{z}_t} \left(k e^{\hat{k}_{t-1}}\right)^{\theta} \left(h e^{\hat{h}_t}\right)^{1-\theta}, \\
\eta^{\theta} y e^{\hat{y}_t} &= Z e^{\hat{z}_t} k^{\theta} e^{\theta \hat{k}_{t-1}} h^{1-\theta} e^{(1-\theta)\hat{h}_t},
\end{aligned}
$$

but since in the steady state $\eta^{\theta} y = Z k^{\theta} h^{1-\theta}$:

$$
\begin{aligned}
e^{\hat{y}_t} &= e^{\hat{z}_t} e^{\theta \hat{k}_{t-1}} e^{(1-\theta)\hat{h}_t}, \\
\hat{y}_t &= \hat{z}_t + \theta \hat{k}_{t-1} + (1-\theta)\,\hat{h}_t. \qquad \text{(A19)}
\end{aligned}
$$

Next, the technology process can be written as:

$$
\begin{aligned}
\ln z_t &= (1-\rho)\ln Z + \rho \ln z_{t-1} + \varepsilon_t, \\
\ln Z e^{\hat{z}_t} &= (1-\rho)\ln Z + \rho \ln Z e^{\hat{z}_{t-1}} + \varepsilon_t, \\
\ln e^{\hat{z}_t} &= \rho \ln e^{\hat{z}_{t-1}} + \varepsilon_t, \\
\hat{z}_t &= \rho \hat{z}_{t-1} + \varepsilon_t. \qquad \text{(A20)}
\end{aligned}
$$

The log-linear version of the resource constraint is:

$$
\begin{aligned}
y_t &= c_t + i_t + \tau_k (r_t - \delta)\,\eta^{-1} k_{t-1} + \tau_h w_t h_t, \\
y e^{\hat{y}_t} &= c e^{\hat{c}_t} + i e^{\hat{\imath}_t} + \tau_k \left(r e^{\hat{r}_t} - \delta\right) \eta^{-1} k e^{\hat{k}_{t-1}} + \tau_h w e^{\hat{w}_t} h e^{\hat{h}_t}, \\
y(1+\hat{y}_t) &= c(1+\hat{c}_t) + i(1+\hat{\imath}_t) + \tau_k \eta^{-1} k r \left(1 + \hat{r}_t + \hat{k}_{t-1}\right) - \\
&\quad -\tau_k \delta \eta^{-1} k \left(1 + \hat{k}_{t-1}\right) + \tau_h w h \left(1 + \hat{w}_t + \hat{h}_t\right),
\end{aligned}
$$

Recalling that $y = c + i + \tau_k (r - \delta)\,\eta^{-1} k + \tau_h w h$ in the steady state and reordering terms gives:

$$
\begin{aligned}
y\hat{y}_t &= c\hat{c}_t + i\hat{\imath}_t + \tau_k\eta^{-1}kr\hat{r}_t + \tau_k\eta^{-1}k\left(r-\delta\right)\hat{k}_{t-1} + \\
&\quad + \tau_h wh\hat{w}_t + \tau_h wh\hat{h}_t,
\end{aligned}
$$

and by substituting the log-linear versions of the equilibrium conditions in the factor markets discussed below, this expression can be simplified as follows:

$$
\begin{aligned}
y\hat{y}_t &= c\hat{c}_t + i\hat{\imath}_t + \left(\tau_k\eta^{-1}kr + \tau_h wh\right)\hat{y}_t - \left(\tau_k\eta^{-1}k\delta\right)\hat{k}_{t-1} \\
\hat{y}_t &= \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{\imath}_t + \frac{\left(\tau_k\eta^{-1}kr + \tau_h wh\right)}{y}\hat{y}_t - \frac{\left(\tau_k\eta^{-1}k\delta\right)}{y}\hat{k}_{t-1}
\end{aligned}
$$

and reordering terms gives:

$$
\left(1 - \frac{\tau_k\eta^{-1}kr + \tau_h wh}{y}\right)\hat{y}_t = \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{\imath}_t - \left(\tau_k\eta^{-1}\frac{k}{y}\delta\right)\hat{k}_{t-1} \tag{A21}
$$

Next, the law of capital accumulation has the following log-linear version:

$$
\begin{aligned}
k_t &= \eta^{-1}(1-\delta)k_{t-1} + i_t \\
ke^{\hat{k}_t} &= \eta^{-1}(1-\delta)ke^{\hat{k}_{t-1}} + ie^{\hat{\imath}_t} \\
k(1+\hat{k}_t) &= \eta^{-1}(1-\delta)k(1+\hat{k}_{t-1}) + i(1+\hat{\imath}_t) \\
k + k\hat{k}_t &= \eta^{-1}(1-\delta)k + \eta^{-1}(1-\delta)k\hat{k}_{t-1} + i + i\hat{\imath}_t
\end{aligned}
$$

but since at the steady state $k = \eta^{-1}(1-\delta)k + i$ this equation becomes:

$$
\begin{aligned}
k\hat{k}_t &= \eta^{-1}(1-\delta)k\hat{k}_{t-1} + i\hat{\imath}_t, \\
\hat{k}_t &= \frac{(1-\delta)}{\eta}\hat{k}_{t-1} + \frac{(\eta-1+\delta)}{\eta}\hat{\imath}_t, \\
\eta\hat{k}_t &= (1-\delta)\hat{k}_{t-1} + (\eta-1+\delta)\hat{\imath}_t. \qquad \text{(A22)}
\end{aligned}
$$

The FOC for labour has the following log-linear version:

$$\begin{aligned} h_t \gamma c_t &= (1-\tau_h)(1-\theta) y_t, \\ h e^{\hat{h}_t} \gamma c e^{\hat{c}_t} &= (1-\tau_h)(1-\theta) y e^{\hat{y}_t}, \end{aligned}$$

but since at the steady state $h\gamma c = (1-\tau_h)(1-\theta)y$, the equation becomes:

$$\begin{aligned} e^{\hat{h}_t} e^{\hat{c}_t} &= e^{\hat{y}_t}, \\ \hat{h}_t + \hat{c}_t &= \hat{y}_t. \end{aligned} \tag{A23}$$

The Euler condition for consumption has the following log-linear version:

$$\begin{aligned} 1 &= \beta E_t \left[\left(\frac{c_t}{\eta c_{t+1}} \right) \left((r_{t+1} - \delta)(1-\tau_k) + 1 \right) \right], \\ \frac{\eta c e^{E_t \hat{c}_{t+1}}}{\beta c e^{c_t}} &= \left(r e^{\hat{r}_{t+1}} - \delta \right) (1-\tau_k) + 1, \\ \frac{\eta}{\beta} e^{(E_t \hat{c}_{t+1} - \hat{c}_t)} &= (1-\tau_k) \left(r e^{E_t \hat{r}_{t+1}} - \delta \right) + 1, \\ \frac{\eta}{\beta} (1 + E_t \hat{c}_{t+1} - \hat{c}_t) &= (1-\tau_k) r e^{E_t \hat{r}_{t+1}} - (1-\tau_k)\delta + 1, \\ \frac{\eta}{\beta} + \frac{\eta}{\beta} E_t \hat{c}_{t+1} - \frac{\eta}{\beta} \hat{c}_t &= (1-\tau_k)(r-\delta) + (1-\tau_k) r E_t \hat{r}_{t+1} + 1, \end{aligned}$$

and according to $(A16)$ $\frac{\eta}{\beta} = (r-\delta)(1-\tau_k)+1$ at the steady state and thus the expression above becomes:

$$\frac{\eta}{\beta} E_t \hat{c}_{t+1} - \frac{\eta}{\beta} \hat{c}_t = (1-\tau_k) r E_t \hat{r}_{t+1}.$$

or, by considering the log-linear version of the equilibrium condition in the capital market discussed below:

$$\frac{\eta}{\beta} E_t \hat{c}_{t+1} - \frac{\eta}{\beta} \hat{c}_t = (1-\tau_k) r E_t \hat{y}_{t+1} - (1-\tau_k) r \hat{k}_t, \tag{A24}$$

In addition, the log-linear version of the equilibrium condition in the capital market is:

$$\begin{aligned} r_{t+1} &= \theta\frac{\eta y_{t+1}}{k_t}, \\ re^{\hat{r}_{t+1}} &= \theta\frac{\eta y e^{\hat{y}_{t+1}}}{ke^{\hat{k}_t}}, \end{aligned}$$

but given that at the steady state $r = \theta\frac{\eta y}{k}$:

$$\begin{aligned} e^{\hat{r}_{t+1}} &= e^{\left(\hat{y}_{t+1}-\hat{k}_t\right)}, \\ \hat{r}_{t+1} &= \hat{y}_{t+1} - \hat{k}_t. \end{aligned} \tag{A25}$$

And for the equilibrium condition in the labour market is:

$$\begin{aligned} w_t h_t &= (1-\theta)y_t, \\ we^{\hat{w}_t}he^{\hat{h}_t} &= (1-\theta)ye^{\hat{y}_t}, \end{aligned}$$

but using the fact that at the steady state $wh = (1-\theta)y$:

$$\begin{aligned} e^{\left(\hat{w}_t+\hat{h}_t\right)} &= e^{\hat{y}_t}, \\ \hat{w}_t + \hat{h}_t &= \hat{y}_t. \end{aligned} \tag{A26}$$

Finally, the log-linear version of the government budget constraint is:

$$\begin{aligned} ge^{\hat{g}_t} &= \tau_k\left(re^{\hat{r}_t} - \delta\right)\eta^{-1}ke^{\hat{k}_{t-1}} + \tau_h we^{\hat{w}_t}he^{\hat{h}_t} \\ &= \tau_k\eta^{-1}kr\left(1+\hat{r}_t+\hat{k}_{t-1}\right) - \tau_k\delta\eta^{-1}k\left(1+\hat{k}_{t-1}\right) + \\ &\quad +\tau_h wh\left(1+\hat{w}_t+\hat{h}_t\right). \end{aligned}$$

Recalling that $g = \tau_k(r-\delta)\eta^{-1}k + \tau_h wh$ in the steady state and collecting and reordering terms gives:

$$
\begin{aligned}
g\hat{g}_t &= \left(\tau_k \eta^{-1} kr\right) \hat{r}_t + \left[(r-\delta)\, k\eta^{-1}\tau_k\right] \hat{k}_{t-1} + \\
&\quad + (\tau_h wh)\, \hat{w}_t + (\tau_h wh)\, \hat{h}_t,
\end{aligned}
$$

and substituting $(A25)$ and $(A26)$ into the above expression and cancelling terms yields:

$$
g\hat{g}_t = \left(\tau_k \eta^{-1} kr + \tau_h wh\right) \hat{y}_t - \left(\tau_k \eta^{-1} k\delta\right) \hat{k}_{t-1}. \tag{A27}
$$

A.4 The reduced form of the RBC model

First, substitute $(A23)$ into $(A19)$ for $\hat{h}_t$ to get:

$$
\begin{aligned}
\hat{y}_t &= \hat{z}_t + \theta \hat{k}_{t-1} + (1-\theta)\,(\hat{y}_t - \hat{c}_t), \\
\theta \hat{y}_t &= \hat{z}_t + \theta \hat{k}_{t-1} - (1-\theta)\, \hat{c}_t, \qquad \text{(A28)}
\end{aligned}
$$

evaluating this at $t+1$ and recalling that $E_t \hat{z}_{t+1} = \rho \hat{z}_t$ gives:

$$
E_t \hat{y}_{t+1} = \frac{\rho}{\theta} \hat{z}_t + \hat{k}_t - \frac{(1-\theta)}{\theta} E_t \hat{c}_{t+1}. \tag{A29}
$$

Next, substituting $(A29)$ into $(A24)$ gives:

$$
\begin{aligned}
\frac{\eta}{\beta} E_t \hat{c}_{t+1} - \frac{\eta}{\beta} \hat{c}_t &= (1-\tau_k) r \left[\frac{\rho}{\theta} \hat{z}_t + \hat{k}_t - \frac{(1-\theta)}{\theta} E_t \hat{c}_{t+1}\right] \\
&\quad - (1-\tau_k) r \hat{k}_t, \\
\left(\frac{\eta}{\beta} + \tfrac{(1-\tau_k) r (1-\theta)}{\theta}\right) E_t \hat{c}_{t+1} &= \frac{\eta}{\beta} c_t + \frac{(1-\tau_k) r \rho}{\theta} \hat{z}_t. \qquad \text{(A30)}
\end{aligned}
$$

Meanwhile, to find consumption in terms of the state variables, take equation $(A21)$ and define $\kappa = \left(1 - \frac{\tau_k \eta^{-1} kr + \tau_h wh}{y}\right)^{-1}$ to find:

$$
\hat{y}_t = \frac{c}{y} \kappa \hat{c}_t + \frac{i}{y} \kappa \hat{\imath}_t - \left(\tau_k \eta^{-1} \frac{k}{y} \delta\right) \kappa \hat{k}_{t-1},
$$

and substitute this expression into ($A28$) to obtain:

$$\begin{aligned}
\theta\left[\frac{c}{y}\kappa\hat{c}_t + \frac{i}{y}\kappa\hat{\imath}_t - \left(\tau_k\delta\eta^{-1}\frac{k}{y}\right)\kappa\hat{k}_{t-1}\right] &= \hat{z}_t + \theta\hat{k}_{t-1} - (1-\theta)\,\hat{c}_t, \\
\frac{\theta}{y}c\kappa\hat{c}_t + (1-\theta)\,\hat{c}_t + \frac{\theta}{y}\kappa i\hat{\imath}_t &= \hat{z}_t + \theta\hat{k}_{t-1} + \left(\tau_k\delta\eta^{-1}\frac{k}{y}\right)\theta\kappa\hat{k}_{t-1}, \\
\left[\tfrac{\theta c\kappa + (1-\theta)y}{y}\right]\hat{c}_t + \frac{\theta\kappa i}{y}\hat{\imath}_t &= \hat{z}_t + \left[\theta + \left(\tau_k\delta\eta^{-1}\frac{k}{y}\right)\theta\kappa\right]\hat{k}_{t-1}.
\end{aligned}$$

Defining $D = \frac{\theta\kappa c + (1-\theta)y}{y}$ and substituting ($A22$) for investment into the equation above gives:

$$D\hat{c}_t + \frac{\theta\kappa i}{y}\left(\tfrac{\eta\hat{k}_t - (1-\delta)\hat{k}_{t-1}}{(\eta-1+\delta)}\right) = \hat{z}_t + \left[\theta + \left(\tau_k\delta\eta^{-1}\frac{k}{y}\right)\theta\kappa\right]\hat{k}_{t-1}.$$

Next, collecting terms and defining $\varphi = \eta - 1 + \delta$ yields:

$$\begin{aligned}
D\hat{c}_t + \tfrac{\theta\kappa i\eta}{y\varphi}\hat{k}_t - \tfrac{\theta\kappa i(1-\delta)}{y\varphi}\hat{k}_{t-1} - \left[\theta + \left(\tau_k\delta\eta^{-1}\frac{k}{y}\right)\theta\kappa\right]\hat{k}_{t-1} &= \hat{z}_t, \\
D\hat{c}_t + \tfrac{\theta\kappa i\eta}{y\varphi}\hat{k}_t - \left[\tfrac{\theta\kappa i(1-\delta)}{y\varphi} + \theta + \tfrac{\tau_k\delta\eta^{-1}k\theta\kappa}{y}\right]\hat{k}_{t-1} &= \hat{z}_t,
\end{aligned}$$

which can be written as:

$$\begin{aligned}
D\hat{c}_t &= -\tfrac{\theta\kappa i\eta}{y\varphi}\hat{k}_t + \left[\tfrac{\theta\kappa i(1-\delta) + y\varphi\theta + \tau_k\delta\eta^{-1}\theta\varphi\kappa k}{y\varphi}\right]\hat{k}_{t-1} + \hat{z}_t, \\
\hat{c}_t &= -\tfrac{\theta\kappa i\eta}{y\varphi D}\hat{k}_t + \left[\tfrac{\theta\kappa i(1-\delta) + y\varphi\theta + \tau_k\delta\eta^{-1}\theta\varphi\kappa k}{y\varphi D}\right]\hat{k}_{t-1} + \frac{1}{D}\hat{z}_t,
\end{aligned} \tag{A31}$$

and evaluated at $t+1$ (with $E_t\hat{z}_{t+1} = \rho\hat{z}_t$) becomes:

$$E_t\hat{c}_{t+1} = -\tfrac{\theta\kappa i\eta}{y\varphi D}E_t\hat{k}_{t+1} + \left[\tfrac{\theta\kappa i(1-\delta) + y\varphi\theta + \tau_k\delta\eta^{-1}\theta\varphi\kappa k}{y\varphi D}\right]\hat{k}_t + \frac{\rho}{D}\hat{z}_t. \tag{A32}$$

Now, these last two equations ($A31$) and ($A32$) can be substituted for period t and period $t+1$ consumption into the main expectational equation ($A30$), in which for convenience the definition $F = \left(\frac{\eta}{\beta} + \frac{(1-\tau_k)r(1-\theta)}{\theta}\right)$ was also used. Hence, the resulting equation is:

$$
\begin{aligned}
& F\left(-\tfrac{\theta\kappa i\eta}{y\varphi D}E_t\hat{k}_{t+1}+\left[\tfrac{\theta\kappa i(1-\delta)+y\varphi\theta+\tau_k\delta\eta^{-1}\theta\varphi\kappa k}{y\varphi D}\right]\hat{k}_t+\frac{\rho}{D}\hat{z}_t\right) \\
= & \frac{\eta}{\beta}\left(-\tfrac{\theta\kappa i\eta}{y\varphi D}\hat{k}_t+\left[\tfrac{\theta\kappa i(1-\delta)+y\varphi\theta+\tau_k\delta\eta^{-1}\theta\varphi\kappa k}{y\varphi D}\right]\hat{k}_{t-1}+\frac{1}{D}\hat{z}_t\right) \\
& +\tfrac{(1-\tau_k)r\rho}{\theta}\hat{z}_t,
\end{aligned}
$$

and collecting terms:

$$
\begin{aligned}
& -\tfrac{F\theta\kappa i\eta}{y\varphi D}E_t\hat{k}_{t+1}+\left[\tfrac{F\left(\theta\kappa i(1-\delta)+y\varphi\theta+\tau_k\delta\eta^{-1}\theta\varphi\kappa k\right)}{y\varphi D}\right]\hat{k}_t+\frac{F\rho}{D}\hat{z}_t \\
= & -\tfrac{\eta\theta\kappa i\eta}{\beta y\varphi D}\hat{k}_t+\left[\tfrac{\eta\left(\theta\kappa i(1-\delta)+y\varphi\theta+\tau_k\delta\eta^{-1}\theta\varphi\kappa k\right)}{\beta y\varphi D}\right]\hat{k}_{t-1}+\frac{\eta}{\beta D}\hat{z}_t \\
& +\frac{(1-\tau_k)r\rho}{\theta}\hat{z}_t,
\end{aligned}
$$

$$
\begin{aligned}
& \left[\tfrac{F\left(\theta\kappa i(1-\delta)+y\varphi\theta+\tau_k\delta\eta^{-1}\theta\varphi\kappa k\right)}{y\varphi D}+\frac{\eta\theta\kappa i\eta}{\beta y\varphi D}\right]\hat{k}_t \\
= & \left[\tfrac{F\theta\kappa i\eta}{y\varphi D}\right]E_t\hat{k}_{t+1}+\left[\tfrac{\eta\theta\kappa i(1-\delta)+\eta y\varphi\theta+\tau_k\delta\theta\varphi\kappa k}{\beta y\varphi D}\right]\hat{k}_{t-1} \\
& +\left[\tfrac{(1-\tau_k)r\rho}{\theta}+\frac{\eta}{\beta D}-\frac{F\rho}{D}\right]\hat{z}_t,
\end{aligned}
$$

and, finally the reduced form of the model is:

$$
\begin{aligned}
\hat{k}_t = & \; G^{-1}\left[\tfrac{F\theta\kappa i\eta}{y\varphi D}\right]E_t\hat{k}_{t+1}+G^{-1}\left[\tfrac{\eta\theta\kappa i(1-\delta)+\eta y\varphi\theta+\tau_k\delta\theta\varphi\kappa k}{\beta y\varphi D}\right]\hat{k}_{t-1} \\
& +G^{-1}\left[\tfrac{(1-\tau_k)r\rho}{\theta}+\frac{\eta}{\beta D}-\frac{F\rho}{D}\right]\hat{z}_t,
\end{aligned}
\tag{A33}
$$

where:

$$
G=\left[\tfrac{F\left(\theta\kappa i(1-\delta)+y\varphi\theta+\tau_k\delta\eta^{-1}\theta\varphi\kappa k\right)}{y\varphi D}+\tfrac{\eta\theta\kappa i\eta}{\beta y\varphi D}\right].
$$

A.5 The reduced form of the jump variables of the model

Here, the rest of the variables of the model are expressed in terms of the state variables. First, note that the reduced form for consumption was already found in equation $(A31)$, but it is written again below for convenience:

$$\hat{c}_t = -\tfrac{\theta\kappa i\eta}{y\varphi D}\hat{k}_t + \left[\tfrac{\theta\kappa i(1-\delta)+y\varphi\theta+\tau_k\delta\eta^{-1}\theta\varphi\kappa k}{y\varphi D}\right]\hat{k}_{t-1} + \frac{1}{D}\hat{z}_t.$$

Meanwhile, the reduced form for investment can be found directly from the law of capital accumulation:

$$\hat{\imath}_t = \frac{\eta}{(\eta-1+\delta)}\hat{k}_t - \frac{(1-\delta)}{(\eta-1+\delta)}\hat{k}_{t-1}. \tag{A34}$$

For output, replace consumption from $(A31)$ into $(A28)$ to get:

$$\begin{aligned}
\hat{y}_t &= \frac{1}{\theta}\hat{z}_t - \frac{(1-\theta)}{\theta}\left(-\tfrac{\theta\kappa i\eta}{y\varphi D}\hat{k}_t + \left[\tfrac{\theta\kappa i(1-\delta)+y\varphi\theta+\tau_k\delta\eta^{-1}\theta\varphi\kappa k}{y\varphi D}\right]\hat{k}_{t-1} + \frac{1}{D}\hat{z}_t\right) + \hat{k}_{t-1},\\
&= \frac{1}{\theta}\hat{z}_t + \hat{k}_{t-1} + \tfrac{(1-\theta)\kappa i\eta}{y\varphi D}\hat{k}_t - \left[\tfrac{(1-\theta)\left(\kappa i(1-\delta)+y\varphi+\tau_k\delta\eta^{-1}\varphi\kappa k\right)}{y\varphi D}\right]\hat{k}_{t-1} - \tfrac{(1-\theta)}{\theta D}\hat{z}_t,\\
&= \tfrac{(1-\theta)\kappa i\eta}{y\varphi D}\hat{k}_t + \left[1 - \tfrac{(1-\theta)\left(\kappa i(1-\delta)+y\varphi+\tau_k\delta\eta^{-1}\varphi\kappa k\right)}{y\varphi D}\right]\hat{k}_{t-1} + \left[\frac{1}{\theta} - \tfrac{(1-\theta)}{\theta D}\right]\hat{z}_t,
\end{aligned}$$

and after rearranging terms this becomes:

$$\hat{y}_t = \left[\tfrac{(1-\theta)\kappa i\eta}{y\varphi D}\right]\hat{k}_t + \left[\tfrac{y\varphi D-(1-\theta)\left[\kappa i(1-\delta)+y\varphi+\tau_k\delta\eta^{-1}\varphi\kappa k\right]}{y\varphi D}\right]\hat{k}_{t-1} + \left[\tfrac{D-1+\theta}{\theta D}\right]\hat{z}_t. \tag{A35}$$

Likewise, for the case of hours worked, replace the above result into the production function $(A19)$ and then solve for $\hat{h}_t$, as follows:

$$\hat{h}_t = \frac{1}{(1-\theta)}\hat{y}_t - \frac{1}{(1-\theta)}\hat{z}_t - \frac{\theta}{(1-\theta)}\hat{k}_{t-1}$$

$$\hat{h}_t = \left(\left[\tfrac{(1-\theta)\kappa i\eta}{y\varphi D}\right]\hat{k}_t + \left[\tfrac{y\varphi D-(1-\theta)\left[\kappa i(1-\delta)+y\varphi+\tau_k\delta\eta^{-1}\varphi\kappa k\right]}{y\varphi D}\right]\hat{k}_{t-1} + \left[\tfrac{D-1+\theta}{\theta D}\right]\hat{z}_t\right)\frac{1}{(1-\theta)} - \frac{1}{(1-\theta)}\hat{z}_t - \frac{\theta}{(1-\theta)}\hat{k}_{t-1},$$

and collecting terms gives:

$$\hat{h}_t = \left[\tfrac{\kappa i\eta}{y\varphi D}\right]\hat{k}_t + \left[\tfrac{y\varphi D-(1-\theta)\left[\kappa i(1-\delta)+y\varphi+\tau_k\delta\eta^{-1}\varphi\kappa k\right]}{y\varphi D(1-\theta)} - \frac{\theta}{(1-\theta)}\right]\hat{k}_{t-1} + \left[\frac{D-1+\theta}{\theta D\,(1-\theta)} - \frac{1}{(1-\theta)}\right]\hat{z}_t,$$

and this can be further simplified to:

$$\hat{h}_t = \left[\tfrac{\kappa i\eta}{y\varphi D}\right]\hat{k}_t + \left[\tfrac{y\varphi D-(1-\theta)\left[\kappa i(1-\delta)+y\varphi+\tau_k\delta\eta^{-1}\varphi\kappa k\right]-\theta y\varphi D}{y\varphi D(1-\theta)}\right]\hat{k}_{t-1} + \left[\tfrac{D-1+\theta-\theta D}{\theta D(1-\theta)}\right]\hat{z}_t,$$

and, by further simplifying the last two terms of the r.h.s., the expression finally turns into:

$$\hat{h}_t = \left[\tfrac{\kappa i\eta}{y\varphi D}\right]\hat{k}_t + \left[\tfrac{y\varphi D-\left[\kappa i(1-\delta)+y\varphi+\tau_k\delta\eta^{-1}\varphi\kappa k\right]}{y\varphi D}\right]\hat{k}_{t-1} + \left[\tfrac{D-1}{\theta D}\right]\hat{z}_t. \tag{A36}$$

The reduced form for the interest rate is found by replacing $(A35)$ into $(A25)$ evaluated at period t:

$$\hat{r}_t = \hat{y}_t - \hat{k}_{t-1}$$

$$\hat{r}_t = \left[\tfrac{(1-\theta)\kappa i\eta}{y\varphi D}\right]\hat{k}_t + \left[\tfrac{y\varphi D-(1-\theta)\left[\kappa i(1-\delta)+y\varphi+\tau_k\delta\eta^{-1}\varphi\kappa k\right]}{y\varphi D}\right]\hat{k}_{t-1} + \left[\tfrac{D-1+\theta}{\theta D}\right]\hat{z}_t - \hat{k}_{t-1},$$

and collecting terms:

$$\hat{r}_t = \left[\tfrac{(1-\theta)\kappa i\eta}{y\varphi D}\right]\hat{k}_t - \left[\tfrac{(1-\theta)\left[\kappa i(1-\delta)+y\varphi+\tau_k\delta\eta^{-1}\varphi\kappa k\right]}{y\varphi D}\right]\hat{k}_{t-1} + \left[\tfrac{D-1+\theta}{\theta D}\right]\hat{z}_t. \tag{A37}$$

Next, to find the reduced form for the competitive wage note that by combining equations $(A23)$ and $(A26)$, it follows that:

$$\hat{w}_t = \hat{y}_t - \hat{h}_t = \hat{c}_t,$$

and this implies that the reduced form for $\hat{w}_t$ has exactly the same form as the one for consumption. That is:

$$\hat{w}_t = \left[-\frac{\theta\kappa i\eta}{y\varphi D}\right]\hat{k}_t + \left[\frac{\theta\kappa i(1-\delta)+y\varphi\theta+\tau_k\delta\eta^{-1}\theta\varphi\kappa k}{y\varphi D}\right]\hat{k}_{t-1} + \frac{1}{D}\hat{z}_t. \tag{A38}$$

Finally, the reduced form for government spending can be found by substituting $(A35)$ into $(A27)$, to get:

$$\begin{aligned}
\hat{g}_t &= \frac{\left(\tau_k\eta^{-1}kr+\tau_h wh\right)}{g}\hat{y}_t - \frac{\left(\tau_k\eta^{-1}k\delta\right)}{g}\hat{k}_{t-1} \\
\hat{g}_t &= \frac{\left(\tau_k r\frac{k}{\eta}+\tau_h wh\right)}{g}\left[\frac{(1-\theta)\kappa i\eta}{y\varphi D}\hat{k}_t + \frac{y\varphi D-(1-\theta)\left[\kappa i(1-\delta)+y\varphi+\tau_k\delta\varphi\kappa\frac{k}{\eta}\right]}{y\varphi D}\hat{k}_{t-1} + \frac{D-1+\theta}{\theta D}\hat{z}_t\right] - \\
&\quad -\frac{\left(\tau_k\delta\frac{k}{\eta}\right)}{g}\hat{k}_{t-1}
\end{aligned}$$

and, by collecting terms:

$$\begin{aligned}
\hat{g}_t = {} & \left[\frac{\left(\tau_k\eta^{-1}kr+\tau_h wh\right)(1-\theta)\kappa i\eta}{gy\varphi D}\right]\hat{k}_t + \left[\frac{\left(\tau_k\eta^{-1}kr+\tau_h wh\right)(D-1+\theta)}{g\theta D}\right]\hat{z}_t + \\
& \left[\frac{\left(\tau_k\frac{k}{\eta}r+\tau_h wh\right)\left(y\varphi D-(1-\theta)\left[\kappa i(1-\delta)+y\varphi+\tau_k\delta\frac{k}{\eta}\varphi\kappa\right]\right)-\left(\tau_k\frac{k}{\eta}\delta\right)y\varphi D}{gy\varphi D}\right]\hat{k}_{t-1}
\end{aligned} \tag{A39}$$

A.6 A brief review of different approaches about the formation of expectations

A useful way to discuss the different approaches to modelling expectations is by means of an example. Consider again the reduced form of a log-linearized (around the steady state) stochastic model:

$$\begin{aligned}
\hat{k}_t &= a_1 E_t\hat{k}_{t+1} + a_2\hat{k}_{t-1} + b_1\hat{z}_t, \\
\hat{z}_t &= \rho\hat{z}_{t-1} + \varepsilon_t,
\end{aligned} \tag{A40}$$

where $\hat{k}$ is the endogenous state variable (e.g. capital stock), $\hat{z}$ is the exogenous state variable (e.g. technology) and $\varepsilon_t \sim iid(0,\sigma_\epsilon^2)$. Also, note that E_t represents expectations, which might not be necessarily rational. This is the same expression as equation (1.55) in the document. As stressed earlier, many authors (e.g. Evans and Honkapohja, 2001, Carceles-Poveda and Giannitsarou, 2007, 2008, and Giannitsarou, 2006) show that a number of economic models - such as the RBC model and many of its extensions - can be expressed in this convenient form.[1]

From $(A40)$ it is clear that the solution of the model will depend on the assumption made about how agents form expectations. This section first takes a look at the adaptive expectations hypothesis; then the RE solution is reviewed and finally the AL approach is considered compared to the first two. Once the model is solved under these three approaches, the parameterization of the RBC model presented in the document (for the US) is borrowed in order to perform some simulations of the evolution of $\hat{k}_t$ so that the different results coming from these formulations can be appreciated.

A.6.1 Adaptive Expectations

Assume that agents form their expectations about the future level of capital according to the general formulation:

$$E_t\hat{k}_{t+1} = E_{t-1}\hat{k}_t + \vartheta(\hat{k}_t - E_{t-1}\hat{k}_t), \tag{A41}$$

meaning that the previous expectation $E_{t-1}\hat{k}_t$ is adjusted by a fraction $0 < \vartheta < 1$ of the forecast error represented by $\hat{k}_t - E_{t-1}\hat{k}_t$. This formulation is usually known as the adaptive expectations approach to expectations formation. It naturally depends on the value the correction factor ϑ takes: the higher (smaller) it is the faster (slower) the convergence of any deviation of $\hat{k}_t$ to its steady state will be.

To see the impact of this formulation in the model at hand, substitute, $(A41)$ into $(A40)$ to get:

$$\begin{aligned}
\hat{k}_t &= a_1\left(E_{t-1}\hat{k}_t + \vartheta(k_t - E_{t-1}\hat{k}_t)\right) + a_2\hat{k}_{t-1} + b_1\hat{z}_t, \\
&= a_1(1-\vartheta)E_{t-1}\hat{k}_t + a_1\vartheta\hat{k}_t + a_2\hat{k}_{t-1} + b_1\hat{z}_t,
\end{aligned}$$

[1] Also, it was shown earlier that it is always possible to find the solutions for the jump variables of the model. For this reason, this section will focus on solving the model and simulating the evolution of the state variable k_t only.

and rearrange terms to obtain:

$$\begin{aligned}\hat{k}_t - a_1\vartheta\hat{k}_t &= a_1(1-\vartheta)E_{t-1}\hat{k}_t + a_2\hat{k}_{t-1} + b_1\hat{z}_t, \\ \hat{k}_t &= \frac{a_1(1-\vartheta)}{(1-a_1\vartheta)}E_{t-1}\hat{k}_t + \frac{a_2}{(1-a_1\vartheta)}\hat{k}_{t-1}\frac{b_1}{(1-a_1\vartheta)}\hat{z}_t, \end{aligned} \tag{A42}$$

but note that since $\hat{k}_{t-1}$ is the initial capital stock, applying $(A41)$ to find $E_{t-1}\hat{k}_t$ yields:

$$E_{t-1}\hat{k}_t = \hat{k}_{t-1} + \vartheta(\hat{k}_{t-1} - \hat{k}_{t-1}) = \hat{k}_{t-1},$$

and thus collecting terms and recalling that $z_t = \rho z_{t-1} + \varepsilon_t$ means that $(A42)$ finally turns into:

$$\hat{k}_t = \frac{a_1(1-\vartheta)+a_2}{(1-a_1\vartheta)}\hat{k}_{t-1} + \frac{b_1\rho}{(1-a_1\vartheta)}\hat{z}_{t-1} + \frac{b_1}{(1-a_1\vartheta)}\varepsilon_t, \tag{A43}$$

plus the initial values k_{t-1} and z_{t-1}.

The adaptive expectations approach played a significant role in economic modelling in the 1960s and the 1970s, but became under heavy criticism in the late 1970s and eventually was replaced by the RE hypothesis as the main paradigm under which economic theory is constructed. To see why, note that if $\hat{k}_t$ is increasing, assuming adaptive expectations in $(A43)$ will systematically underestimate it and if $\hat{k}_t$ is decreasing then the formulation will consistently overestimate it. This inherent weakness of the updating rule may provide poor forecasts and, as a result, it is reasonable to believe that agents will look for better forecast rules. Having said this, it is worth to note that more insights about this formulation will be given with the intended simulations.

A.6.2 Rational Expectations

Given the discussion above, assume now that agents form rational expectations (RE) which implies that there can be no systematic component in the forecast error which agents could correct. This means that agents will form expectations according to:

$$E_t\hat{k}_{t+1} = E_t^*\hat{k}_{t+1}, \tag{A44}$$

where the $(*)$ superscript denotes rational expectations. Thus the model becomes:

$$\begin{aligned}\hat{k}_t &= a_1 E_t^* \hat{k}_{t+1} + a_2 \hat{k}_{t-1} + b_1 \hat{z}_t, \\ \hat{z}_t &= \rho \hat{z}_{t-1} + \varepsilon_t.\end{aligned} \tag{A45}$$

The method of undetermined coefficients will be used here to find the solution. First, it is assumed that rational agents have a (correct) *guess* about the form of the solution of the endogenous state variable:

$$\hat{k}_t = \omega_{kk} \hat{k}_{t-1} + \omega_{kz} \hat{z}_t. \tag{A46}$$

Substituting the second equation of $(A45)$ into the expression above gives:

$$\hat{k}_t = \phi_k \hat{k}_{t-1} + \phi_z \hat{z}_{t-1} + \frac{\phi_z}{\rho} \varepsilon_t, \tag{A47}$$

where $\phi_k = \omega_{kk}$ and $\phi_z = \omega_{kz}\rho$. These two are the undetermined coefficients of interest. Leading this expression one period and taking expectations yields:

$$\begin{aligned}E_t^* \hat{k}_{t+1} &= \phi_k E_t^* \hat{k}_t + \phi_z E_t^* \hat{z}_t + \frac{\phi_z}{\rho} E_t^* \varepsilon_{t+1}, \\ &= \phi_k \hat{k}_t + \phi_z \hat{z}_t,\end{aligned} \tag{A48}$$

because $E_t^* \varepsilon_{t+1} = 0$. Substitute this initial *guess* into the reduced form model to obtain:

$$\begin{aligned}\hat{k}_t &= a_1 (\phi_k \hat{k}_t + \phi_z \hat{z}_t) + a_2 \hat{k}_{t-1} + b_1 \hat{z}_t, \\ \hat{z}_t &= \rho \hat{z}_{t-1} + \varepsilon_t,\end{aligned} \tag{A49}$$

which can be written as a unique equation:

$$\hat{k}_t = \frac{a_2}{(1 - a_1 \phi_k)} \hat{k}_{t-1} + \frac{(a_1 \phi_z + b_1)\, \rho}{(1 - a_1 \phi_k)} \hat{z}_{t-1} + \frac{(a_1 \phi_z + b_1)}{(1 - a_1 \phi_k)} \varepsilon_t.$$

Finally, since this last equation must equal the initial guess $(A47)$ formed by agents, it follows that:

$$\phi_k = \frac{a_2}{(1-a_1\phi_k)}, \quad \phi_z = \frac{(a_1\phi_z+b_1)\rho}{(1-a_1\phi_k)},$$

and solving the quadratic equation for ϕ_k yields two results, only one of which (i.e. the one with negative sign) is consistent with a stable equilibrium (i.e. it has an absolute value less than one) and is the following:

$$\bar{\phi}_{k_-} = \frac{1-\sqrt{1-4a_1a_2}}{2a_1} \quad \text{and} \quad \bar{\phi}_{z_-} = \frac{b_1\rho}{1-a_1(\rho+\bar{\phi}_{k2})}, \tag{A50}$$

while the other solution (the one with a positive sign in the quadratic equation) is ruled out as it implies an explosive path for the state variable. Therefore, the solution under RE is given by:

$$\hat{k}_t = \frac{1-\sqrt{1-4a_1a_2}}{2a_1}\hat{k}_{t-1} + \frac{b_1\rho}{1-a_1(\rho+\bar{\phi}_{k\_})}\hat{z}_{t-1} + \frac{b_1}{1-a_1(\rho+\bar{\phi}_{k\_})}\varepsilon_t. \tag{A51}$$

plus the initial values $\hat{k}_{t-1}$ and $\hat{z}_{t-1}$.

The example provided above shows clearly that to find the solution it is necessary to assume that agents have full knowledge about the model that generates the data as well as the values of the structural parameters. In that respect, many authors recently have argued that, while such assumption helps to find very neat mathematical results for further analysis and that it might make more sense from a long-run perspective, it is quite unrealistic when the short and medium-run dynamics observed in practice are considered.

A.6.3 Adaptive Learning

Perhaps a more plausible view of rationality is given by the AL approach, in which agents are assumed to behave as econometricians when forecasting. As shown in the main document, two major points of this approach are: a) it makes the forecast functions and the estimation of their parameters fully explicit, and b) expectations and forecast functions influence future data realizations.

Considering the model given in $(A40)$ again, under AL it is assumed that agents believe that the process for $\hat{k}_t$ is of the form (i.e. the perceived law of motion):

$$E_t\hat{k}_{t+1} = \tilde{\phi}_{k,t-1}\hat{k}_t + \tilde{\phi}_{z,t-1}\hat{z}_t, \tag{A52}$$

where $\tilde{\boldsymbol{\phi}}_{t-1}$ denotes estimate of $\bar{\boldsymbol{\phi}}$ (i.e. the true coefficient under RE) with information up to $t-1$. Since this equation represents the behavior of the agents, it can be replaced into the reduced form model $(A41)$ to get:

$$\begin{aligned}
\hat{k}_t &= a_1(\tilde{\phi}_{k,t-1}\hat{k}_t + \tilde{\phi}_{z,t-1}\hat{z}_t) + a_2\hat{k}_{t-1} + b_1\hat{z}_t, \\
\hat{z}_t &= \rho\hat{z}_{t-1} + \varepsilon_t,
\end{aligned}$$

which can be written as a unique equation:

$$\hat{k}_t = \frac{a_2}{(1-a_1\tilde{\phi}_{k,t-1})}\hat{k}_{t-1} + \frac{\left(a_1\tilde{\phi}_{z,t-1} + b_1\right)\rho}{(1-a_1\tilde{\phi}_{k,t-1})}\hat{z}_{t-1} + \frac{\left(a_1\tilde{\phi}_{z,t-1} + b_1\right)}{(1-a_1\tilde{\phi}_{k,t-1})}\varepsilon_t. \tag{A53}$$

Next, it is further assumed that every period agents run an OLS regression to estimate the vector of coefficients $\tilde{\boldsymbol{\phi}} = [\tilde{\phi}_k, \tilde{\phi}_z]'$. It was shown in the document that the OLS formulation can be expressed recursively as:

$$\begin{aligned}
\tilde{\boldsymbol{\phi}}_t &= \tilde{\boldsymbol{\phi}}_{t-1} + t^{-1}\mathbf{R}_t^{-1}\mathbf{x}_{t-1}\left(\hat{k}_t - \mathbf{x}'_{t-1}\tilde{\boldsymbol{\phi}}_{t-1}\right), \\
\mathbf{R}_t &= \mathbf{R}_{t-1} + \frac{1}{t}\left(\mathbf{x}_{t-1}\mathbf{x}'_{t-1} - \mathbf{R}_{t-1}\right),
\end{aligned}$$

where $\mathbf{x}_t = [\hat{k}_t, \hat{z}_t]'$ and $\mathbf{R}_t$ is the 2×2 second moment matrix of $\mathbf{x}_t$. Therefore, the complete model under AL in the form of RLS is given by the system:

$$\begin{aligned}
&\hat{k}_t = \tfrac{a_2}{(1-a_1\check{\phi}_{k,t-1})}\hat{k}_{t-1} + \tfrac{\left(a_1\check{\phi}_{z,t-1}+b_1\right)\rho}{(1-a_1\check{\phi}_{k,t-1})}\hat{z}_{t-1} + \tfrac{\left(a_1\check{\phi}_{z,t-1}+b_1\right)}{(1-a_1\check{\phi}_{k,t-1})}\varepsilon_t, \\
&\tilde{\boldsymbol{\phi}}_t = \tilde{\boldsymbol{\phi}}_{t-1} + t^{-1}\mathbf{R}_t^{-1}\mathbf{x}_{t-1}\left(\hat{k}_t - \mathbf{x}'_{t-1}\tilde{\boldsymbol{\phi}}_{t-1}\right), \\
&\mathbf{R}_t = \mathbf{R}_{t-1} + \tfrac{1}{t}\left(\mathbf{x}_{t-1}\mathbf{x}'_{t-1} - \mathbf{R}_{t-1}\right),
\end{aligned} \tag{A54}$$

plus the initial values $\hat{k}_{t-1}$ and $\hat{z}_{t-1}$ as well as the initial conditions $\tilde{\boldsymbol{\phi}}_{t-1}$ and $\mathbf{R}_{t-1}$.

A.6.4 A simulation of the results

In order to see the different results obtained for a model solved under these three assumptions, the RBC model considered in this document is taken again and calibrated for US post-war quarterly data as in Ireland (2004). Hence the parameter values are $\beta = 0.99$, $\alpha = 0.36$, $\eta = 1.0039$, $\delta = 0.025$, $\gamma = 2.59$, $\rho = 0.95$, $\sigma_\varepsilon = 0.00712$, $Z = 2$, and finally $\tau_k = \tau_h = 0$. Accordingly, the coefficients of the first equation of the reduced-form model in $(A40)$ are $a_1 = 0.4962$, $a_2 = 0.5012$ and $b_1 = 0.0107$, while the second equation has coefficients $\rho = 0.95$ and $\sigma_\varepsilon = 0.00712$.

Before going any further, two major points must be stressed in this exercise. First, to isolate the effects of the different assumptions about the agents' expectations, it is assumed that $\hat{z}_{t-1} = 0$ (i.e. the exogenous state variable is at its steady state value) and that $\varepsilon_t = 0$ so that the environment in this particular exercise is deterministic. Second, to see how different the speeds of convergence towards the steady state are under each expectations formulation, two cases for the initial value of the endogenous state variable are considered: a) when $\hat{k}_{t-1} = -0.1$ or 10% below the steady state value - represented by the *x-axis* as it is equivalent to $k = 0$ - and b) when $\hat{k}_{t-1} = 0.1$ or 10% above the steady state value $k = 0$.

Figure A1 plots the different trajectories of $\hat{k}_t$ under each case of interest. For the model under adaptive expectations it is assumed that the correction factor is very high, equal to 0.9 implying a relatively fast return to equilibrium under this process. This parameter plus the values of the coefficients a_1, a_2 and $b_1$imply that $(A43)$ becomes $\hat{k}_t = 0.9953\hat{k}_{t-1}$. This law of motion is plotted in the Figure using dashed curves: one for the case when $\hat{k}_{t-1} = -0.1$ (i.e. below the 0 straight line) and another one for the case when $\hat{k}_{t-1} = 0.1$.

When RE is assumed, equation $(A51)$ becomes relevant and, after replacing the parameter values, the resulting law of motion is $\hat{k}_t = 0.9349\hat{k}_{t-1}$. The two dotted curves in the Figure exhibit the predicted trajectory of $\hat{k}_t$ under this assumption for the two alternative levels of $\hat{k}_{t-1}$. Finally, when AL is assumed, the solution is given by the system $(A54)$, which implies that only the RLS algorithm has been considered for this exercise as an illustration of the main properties of this approach. In order to find $\tilde{\boldsymbol{\phi}}_{t-1}$ and R_{t-1}, the

RGD initializing method - described in detail in the document - was assumed with $t_0 = 20$ the number of preliminary observations. Given the first estimates $\tilde{\boldsymbol{\phi}}_t$, the law of motion in the first period is $\hat{k}_t = 0.9507\hat{k}_{t-1}$ but, as more information becomes available over time, it converges towards the one associated to RE.

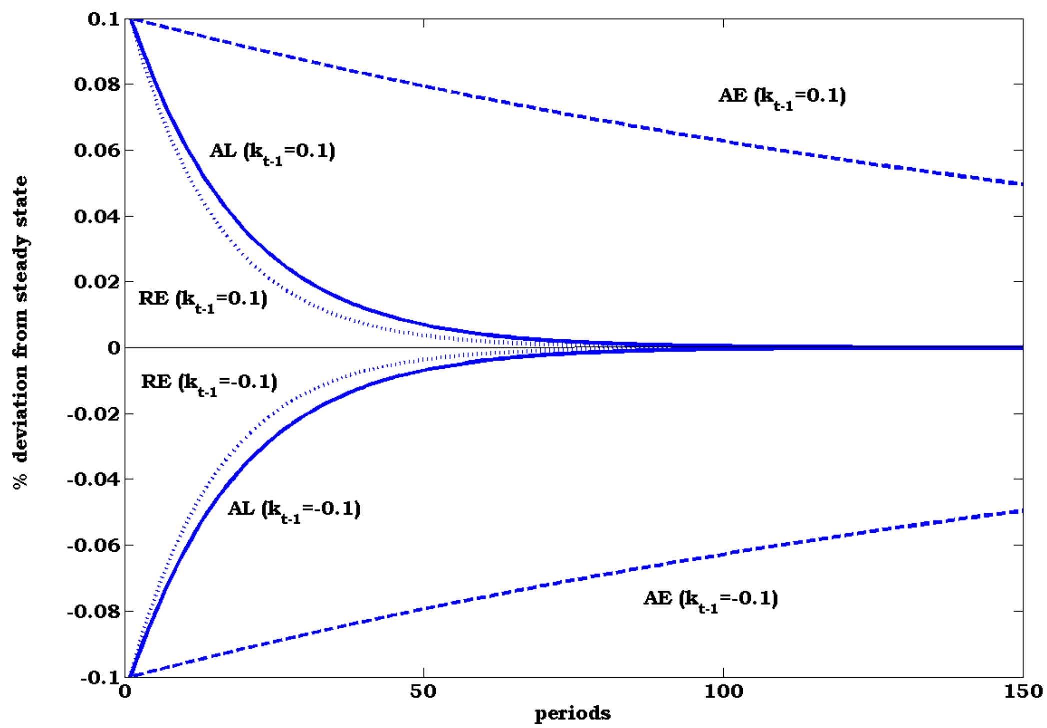
In the three cases at hand the results are found to be symmetric with respect to the two different initial values for $\hat{k}$. However, from the Figure it is clear that the RE hypothesis generates the fastest convergence trajectory to the steady state level of zero. In fact, after 80 periods approximately $\hat{k}_t$ virtually returns to its steady state level.

On the other hand, even with a high correction factor (i.e. 0.9), the adaptive expectations formulation shows a much slower convergence towards equilibrium, a result that reinforces the notion that $(A41)$ systematically underestimates $\hat{k}_t$ if it is increasing (as shown by the the low convergence of the dotted curve below the *x-axis* when compared to the dashed curve) and that systematically overestimates $\hat{k}_t$ if it is decreasing. Moreover, the return to equilibrium under this assumption only occurs after several periods (in this exercise, this is reasonably achieved after 1500 periods, not shown in the Figure), a strong degree of inherent inertia that seems difficult to reconcile with what is seen in practice.

The simulation of the path of $\hat{k}_t$ under AL provides a good illustration of the versatility of this approach. As discussed in the main document, the initial values taken by the RLS learning algorithm to estimate the coefficient $\boldsymbol{\phi}_t$ become quite critical as these define how close to the RE solution the law of motion under learning will be in the short and medium run (i.e. in the long run convergence is ensured). In this example, given the initial values $\tilde{\boldsymbol{\phi}}_{t-1}$ and R_{t-1} (found using 20 preliminary observations), AL generated a slower return to equilibrium compared to RE. However, if more preliminary periods are allowed so that the initial values are closer to the coefficients coming from the RE solution, one should expect the predicted path of $\hat{k}_t$ to be much closer to the one under RE.

In the same line, if the initial values are far away from the RE solution - showing a poor initial ability to learn - then the predicted path will probably start closer to the one under adaptive expectations but then the estimates will unambiguously improve in the following periods and hence these will eventually get much closer to the RE path thus exhibiting a much faster return to equilibrium. These properties show why AL is usually

Figure A.1: Trajectory of capital under different assumptions about expectations



defined as a small deviation from RE which tries to depict a learning process by the agents when these are equipped with adequate techniques that help them to do so as more data become available.

A.7 Ordinary Least Squares in recursive form

As explained in the document, it is assumed that every period agents run an OLS regression:

$$\hat{k}_t = \mathbf{x}'_{t-1}\tilde{\boldsymbol{\phi}}_t + \upsilon_t,$$

with υ_t the forecast error, in order to obtain estimates $\tilde{\boldsymbol{\phi}}_t$, but using data up to $t-1$. The OLS formula in this case is:

$$\tilde{\boldsymbol{\phi}}_t = \left(\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'}\right)^{-1} \sum_{i=1}^{t} \mathbf{x}_{i-1}\hat{k}_i. \tag{A55}$$

The recursive representation of $(A48)$ is found as follows. First, note that it can also be written as:

$$\tilde{\boldsymbol{\phi}}_t = \left(\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'}\right)^{-1} \left(\sum_{i=1}^{t-1} \mathbf{x}_{i-1}\hat{k}_i + \mathbf{x}_{t-1}\hat{k}_t\right). \tag{A56}$$

Moreover, note that at the beginning of the recursion, $\hat{k}_{t-1}$ in the second term of the r.h.s. can be expressed as:

$$\hat{k}_{t-1} = \mathbf{x}'_{t-2}\tilde{\boldsymbol{\phi}}_{t-1},$$

where $\tilde{\boldsymbol{\phi}}_{t-1}$ is the first estimate found one period before (following a specific procedure such as RGD or AHC, which are described in the document) and thus it is given for the current period t. Therefore, replacing this into the main equation $(A55)$ gives:

$$\tilde{\boldsymbol{\phi}}_t = \left(\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'}\right)^{-1} \left(\sum_{i=1}^{t-1} \mathbf{x}_{i-1}\mathbf{x}'_{i-1}\tilde{\boldsymbol{\phi}}_{t-1} + \mathbf{x}_{t-1}\hat{k}_t\right),$$

but expanding the first term in the second parenthesis of the r.h.s gives the equivalent expression:

$$\tilde{\phi}_t = \left(\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'}\right)^{-1} \left(\left[\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{\prime} - \mathbf{x}_{t-1}\mathbf{x}_{t-1}^{\prime}\right] \tilde{\phi}_{t-1} + \mathbf{x}_{t-1}\hat{k}_t\right),$$

which can be written as:

$$\tilde{\phi}_t = \left(\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'}\right)^{-1} \left(\left[\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{\prime}\right] \tilde{\phi}_{t-1} - \mathbf{x}_{t-1}\mathbf{x}_{t-1}^{\prime}\tilde{\phi}_{t-1} + \mathbf{x}_{t-1}\hat{k}_t\right),$$

or, by expanding the whole expression:

$$\begin{aligned} \tilde{\phi}_t &= \left(\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'}\right)^{-1} \left(\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{\prime}\right) \tilde{\phi}_{t-1} - \left(\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'}\right)^{-1} \mathbf{x}_{t-1}\mathbf{x}_{t-1}^{\prime}\tilde{\phi}_{t-1} \\ &\quad + \left(\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'}\right)^{-1} \mathbf{x}_{t-1}\hat{k}_t, \end{aligned}$$

and this can simplified to:

$$\begin{aligned} \phi_t &= \tilde{\phi}_{t-1} - \left(\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'}\right)^{-1} \mathbf{x}_{t-1}\mathbf{x}_{t-1}^{\prime}\tilde{\phi}_{t-1} + \left(\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'}\right)^{-1} \mathbf{x}_{t-1}\hat{k}_t \\ &= \tilde{\phi}_{t-1} + \left(\sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'}\right)^{-1} \mathbf{x}_{t-1}\left[\hat{k}_t - \mathbf{x}_{t-1}^{\prime}\tilde{\phi}_{t-1}\right]. \end{aligned}$$

Next, define:

$$\mathbf{S}_t = \sum_{i=1}^{t} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'},$$

so that the above equation becomes:

$$\tilde{\phi}_t = \tilde{\phi}_{t-1} + \mathbf{S}_t^{-1}\mathbf{x}_{t-1}\left[\hat{k}_t - \mathbf{x}_{t-1}^{\prime}\tilde{\phi}_{t-1}\right], \tag{A57}$$

while the behavior of S can be described by:

$$\begin{aligned} \mathbf{S}_t &= \sum_{i=1}^{t-1} \mathbf{x}_{i-1}\mathbf{x}_{i-1}^{'} + \mathbf{x}_{t-1}\mathbf{x}_{t-1}^{'}, \\ &= \mathbf{S}_{t-1} + \mathbf{x}_{t-1}\mathbf{x}_{t-1}^{'}. \end{aligned} \tag{A58}$$

Therefore, the entire recursive representation of OLS is given by the system:

$$\begin{aligned}
\tilde{\boldsymbol{\phi}}_t &= \tilde{\boldsymbol{\phi}}_{t-1} + \mathbf{S}_t^{-1}\mathbf{x}_{t-1}\left[\hat{k}_t - \mathbf{x}_{t-1}'\tilde{\boldsymbol{\phi}}_{t-1}\right], \\
\mathbf{S}_t &= \mathbf{S}_{t-1} + \mathbf{x}_{t-1}\mathbf{x}_{t-1}^{'}.
\end{aligned}$$

However, note that this recursion can also be written in a different (but perhaps more well-known) way by defining $\mathbf{R}_t = \mathbf{S}_t/\mathbf{t}$. In such case, the second part of the recursion as in $(A57)$ is:

$$\begin{aligned}
\frac{\mathbf{S}_t}{t} &= \frac{1}{t}\mathbf{S}_{t-1} + \frac{1}{t}\mathbf{x}_{t-1}\mathbf{x}_{t-1}^{'}, \\
\mathbf{R}_t &= \frac{1}{t}\mathbf{S}_{t-1} + \frac{1}{t}\mathbf{x}_{t-1}\mathbf{x}_{t-1}^{'},
\end{aligned}$$

but multiplying the first term of the r.h.s. side by $(t-1)/(t-1)$ gives:

$$\begin{aligned}
\mathbf{R}_t &= \frac{1}{t}\frac{(t-1)}{(t-1)}\mathbf{S}_{t-1} + \frac{1}{t}\mathbf{x}_{t-1}\mathbf{x}_{t-1}^{'}, \\
&= \left[\frac{1}{t-1} - \frac{1}{t(t-1)}\right]\mathbf{S}_{t-1} + \frac{1}{t}\mathbf{x}_{t-1}\mathbf{x}_{t-1}^{'}, \\
&= \mathbf{R}_{t-1} + \frac{1}{t}\left(\mathbf{x}_{t-1}\mathbf{x}_{t-1}^{'} - \mathbf{R}_{t-1}\right). \qquad \text{(A59)}
\end{aligned}$$

Meanwhile, the first part of the recursion is:

$$\begin{aligned}
\tilde{\boldsymbol{\phi}}_t &= \tilde{\boldsymbol{\phi}}_{t-1} + \frac{t^{-1}}{t^{-1}}\mathbf{S}_t^{-1}\mathbf{x}_{t-1}\left[\hat{k}_t - \mathbf{x}_{t-1}'\tilde{\boldsymbol{\phi}}_{t-1}\right], \\
&= \tilde{\boldsymbol{\phi}}_{t-1} + \frac{1}{t}\mathbf{R}_t^{-1}\mathbf{x}_{t-1}\left[\hat{k}_t - \mathbf{x}_{t-1}'\tilde{\boldsymbol{\phi}}_{t-1}\right]. \qquad \text{(A60)}
\end{aligned}$$

So, finally, the recursion is given by the system:

$$\begin{aligned}
&\mathbf{R}_t = \mathbf{R}_{t-1} + \tfrac{1}{t}\left(\mathbf{x}_{t-1}\mathbf{x}_{t-1}^{'} - \mathbf{R}_{t-1}\right). \\
&\tilde{\boldsymbol{\phi}}_t = \tilde{\boldsymbol{\phi}}_{t-1} + t^{-1}\mathbf{R}_t^{-1}\mathbf{x}_{t-1}\left(\hat{k}_t - \mathbf{x}_{t-1}'\tilde{\boldsymbol{\phi}}_{t-1}\right),
\end{aligned} \qquad \text{(A61)}$$

and these are the two expressions described and used when applying the RLS learning algorithm in the document.

A.8 Compensating consumption supplement

Following Lucas (1990) and Giannitsarou (2006), the welfare measure considered is the percentage amount, ξ, by which consumption should change in all periods in the pre-reform economy so that agents are equally well off as in the post-reform economy (including the transition periods), but conditional on the initial random shock to technology. Therefore, considering the aggregate version of (1.1), this means:

$$\begin{aligned} E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln C_t^{post} - \gamma H_t^{post}\right] &= \\ E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln\left(C_t^{pre}\left(1+\frac{\xi}{100}\right)\right) - \gamma H_t^{pre}\right] & \quad | \quad \varepsilon_0 = \varepsilon, \end{aligned} \tag{A62}$$

which can be written as:

$$U_T^{post} = \sum_{t=0}^{\infty} \beta^t \ln\left(1+\frac{\xi}{100}\right) + E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln C_t^{pre} - \gamma H_t^{pre}\right] \quad | \, \varepsilon_0 = \varepsilon,$$

for $T = 1, 2, ..., \infty$, or:

$$U_T^{post} = \sum_{t=0}^{\infty} \beta^t \ln\left(1+\frac{\xi}{100}\right) + U_T^{pre} \quad | \, \varepsilon_0 = \varepsilon.$$

Since β is less than unity it follows that $\sum_{t=0}^{\infty} \beta^t \to 1/(1-\beta)$. Hence, solving for ξ gives:

$$\ln\left(1+\frac{\xi}{100}\right) = (1-\beta)\left[U_T^{post} - U_T^{pre}\right] \quad | \, \varepsilon_0 = \varepsilon$$

or, finally:

$$\xi = \left[e^{\left\{(1-\beta)\left[U_T^{post} - U_T^{pre}\right]\right\}} - 1\right] \times 100 \quad | \, \varepsilon_0 = \varepsilon. \tag{A63}$$

Appendix B

Appendix to Chapter 2

B.1 The descentralised competitive equilibrium version of the model

$$\lambda_{c,t} = -\frac{\gamma {C_{c,t}}^{\gamma-1}(1-h_{c,t})^{1-\gamma}}{({C_{c,t}}^{\gamma-1}(1-h_{c,t})^{1-\gamma})^{\sigma}} \tag{B.1}$$

$$\lambda^c_t w_t(\tau^h - 1) = -\frac{C^{\gamma}_{c,t}(\gamma-1)}{(C^{\gamma}_{c,t}(1-h_{c,t})^{1-\gamma})^{\sigma}(1-h_{c,t})^{\gamma}} \tag{B.2}$$

$$\lambda^c_t = -\lambda^c_{t+1}(\delta + 2K_{c,t+1}\phi^k_c + r_{k,t+1}(\tau^k - 1) - 1)\beta \tag{B.3}$$

$$\lambda^c_t = -\lambda^c_{t+1}(2B_{c,t+1}\phi^b_c + r_{b,t+1}(\tau^k - 1) - 1)\beta \tag{B.4}$$

$$\begin{aligned} B_{c,t+1} - B_{c,t} - G_t + K_{c,t+1} \quad = \quad & -K_{c,t}(\delta - 1) - B^2_{c,t}\phi^b_c \\ & -C_{c,t} - K^2_{c,t}\phi^k_c - h_c w(\tau^h - 1) \\ & -(B_{c,t}r_{b,t} + K_{c,t}r_{k,t})(\tau^k - 1) \end{aligned} \tag{B.5}$$

$$\lambda^s_t = -\frac{\gamma {C_{s,t}}^{\gamma-1}(1-h_{s,t})^{1-\gamma}}{({C_{s,t}}^{\gamma-1}(1-h_{s,t})^{1-\gamma})^{\sigma}} \tag{B.6}$$

$$\lambda^s_t w_t(\tau^h - 1) = -\frac{C^{\gamma}_{s,t}(\gamma-1)}{(C^{\gamma}_{s,t}(1-h_{s,t})^{1-\gamma})^{\sigma}(1-h_{s,t})^{\gamma}} \tag{B.7}$$

$$\lambda^s_t = -\lambda^s_{t+1}(\delta + 2K_{s,t+1}\phi^k_s + r_{k,t+1}(\tau^k - 1) - 1)\beta \tag{B.8}$$

$$\lambda^s_t = -\lambda^s_{t+1}(2B_{s,t+1}\phi^b_s + r_{b,t+1}(\tau^k - 1) - 1)\beta \tag{B.9}$$

$$\lambda_t^u = -\frac{\gamma {C_{u,t}}^{\gamma-1}(1-h_{u,t})^{1-\gamma}}{({C_{u,t}}^{\gamma-1}(1-h_{u,t})^{1-\gamma})^{\sigma}} \tag{B.10}$$

$$\lambda_t^u w_{u,t}(\tau^u - 1) = -\frac{C_{u,t}^{\gamma}(\gamma-1)}{(C_{u,t}^{\gamma}(1-h_{u,t})^{1-\gamma})^{\sigma}(1-h_{u,t})^{\gamma}} \tag{B.11}$$

$$C_{u,t} = h_{u,t}w_{u,t}(1-\tau^u) + G_t \tag{B.12}$$

$$\begin{aligned} r_{k,t} \quad = \quad & \frac{1}{\alpha}A(\mu\left(n^u h_{u,t}\right)^{\alpha} + (1-\mu)\left(\rho\left(n^c K_{c,t} + n^s K_{s,t}\right)^{\nu} + \right. \\ & \left.(1-\rho)\left(n^c h_{c,t} + n^s h_{s,t}\right)^{\nu}\right)^{\frac{\alpha}{\nu}})^{\frac{1}{\alpha}-1} \times \\ & (1-\mu)\frac{\alpha}{\nu}(\rho\left(n^c K_{c,t} + n^s K_{s,t}\right)^{\nu} + \\ & (1-\rho)\left(n^c h_{c,t} + n^s h_{s,t}\right)^{\nu})^{\frac{\alpha}{\nu}-1} \times \\ & \rho\nu\left(n^c K_{c,t} + n^s K_s\right)^{\nu-1} \end{aligned} \tag{B.13}$$

$$\begin{aligned} w_t \quad = \quad & \frac{1}{\alpha}A(\mu\left(n^u h_{u,t}\right)^{\alpha} + (1-\mu)\left(\rho\left(n^c K_{c,t} + n^s K_{s,t}\right)^{\nu} + \right. \\ & \left.(1-\rho)\left(n^c h_{c,t} + n^s h_{s,t}\right)^{\nu}\right)^{\frac{\alpha}{\nu}})^{\frac{1}{\alpha}-1} \times \\ & (1-\mu)\frac{\alpha}{\nu}(\rho\left(n^c K_{c,t} + n^s K_{s,t}\right)^{\nu} + \\ & (1-\rho)\left(n^c h_{c,t} + n^s h_{s,t}\right)^{\nu})^{\frac{\alpha}{\nu}-1} \times \\ & (1-p)\nu\left(n^c h_{c,t} + n^s h_{s,t}\right)^{\nu-1} \end{aligned} \tag{B.14}$$

$$\begin{aligned} w_{u,t} \quad = \quad & \frac{1}{\alpha}A(\mu\left(n^u h_{u,t}\right)^{\alpha} + (1-\mu)\left(\rho\left(n^c K_{c,t} + n^s K_{s,t}\right)^{\nu} + \right. \\ & \left.(1-\rho)\left(n^c h_{c,t} + n^s h_{s,t}\right)^{\nu}\right)^{\frac{\alpha}{\nu}})^{\frac{1}{\alpha}-1}\mu\alpha\left(n^u h_{u,t}\right)^{\alpha-1} \end{aligned} \tag{B.15}$$

$$\begin{aligned} G_t + (1+r_{b,t})(n^c B_{c,t} + n^s B_{s,t}) \quad = \quad & n^c B_{c,t+1} + n^s B_{s,t+1} + \tau^k r_{k,t} n^c K_{c,t} \\ & +\tau^k r_{b,t} n^c B_{c,t} + \tau^h w_t n^c h_{c,t} \\ & +\tau^k r_{k,t} n^s K_{s,t} + \tau^k r_{b,t} n^s B_{s,t} \\ & +\tau^h w_t n^s h_{s,t} + \tau^u w_{u,t} n^u h_{u,t} \end{aligned} \tag{B.16}$$

$$
\begin{aligned}
Y_t &= n^c C_{c,t} + n^s C_{s,t} + n^u C_{u,t} + n^c K_{c,t+1} - (1-\delta) n^c K_{c,t} \\
&\quad + n^s K_{s,t+1} - (1-\delta) n^s K_{s,t+1} + \phi_c^b n^c B_{c,t}^2 \\
&\quad + \phi_c^k n^c K_{c,t}^2 + \phi_s^b n^s B_{s,t}^2 + \phi_s^k n^s K_{s,t}^2
\end{aligned} \tag{B.17}
$$

where:

$$
Y_t = A\left\{\mu\left(n^u h_{u,t}\right)^{\alpha} + (1-\mu)\left[\rho\left(n^c K_{c,t} + n^s K_{s,t}\right)^{\nu} + (1-\rho)\left(n^c h_{c,t} + n^s h_{s,t}\right)^{\nu}\right]^{\frac{\alpha}{\nu}}\right\}^{\frac{1}{\alpha}}
$$

Note too that λ_t^c, λ_t^s and λ_t^u are the Lagrangian multipliers associated to the budget constraints of the three types of agents considered in the model. The descentralised competitive equilibrium is completed with the AR(1) process describing the exogenous behavior of productivity.

B.2 Model solution under rational expectations

Consider the system of equations given in (2.20). Employing the undetermined coefficients method, agents first guess that the equilibrium laws of motion for the state variables under RE have the following linear form:

$$
\mathbf{x}_t = \boldsymbol{\gamma}_{\mathbf{x}} \mathbf{x}_{t-1} + \boldsymbol{\gamma}_{\mathbf{z}} \mathbf{z}_t \tag{B.18}
$$

where $\boldsymbol{\gamma}_{\mathbf{x}}$ and $\boldsymbol{\gamma}_{\mathbf{z}}$ are coefficient matrices. Substituting for $\mathbf{z}_t$ using the last equation in (2.20) gives:

$$
\mathbf{x}_t = \boldsymbol{\phi}_{\mathbf{x}} \mathbf{x}_{t-1} + \boldsymbol{\phi}_{\mathbf{z}} \mathbf{z}_{t-1} + \boldsymbol{\phi}_{\mathbf{z}} \boldsymbol{\rho}^{-1} \mathbf{u}_t \tag{B.19}
$$

where $\boldsymbol{\phi}_{\mathbf{x}} = \boldsymbol{\gamma}_{\mathbf{x}}$ and $\boldsymbol{\phi}_{\mathbf{z}} = \boldsymbol{\gamma}_{\mathbf{z}} \boldsymbol{\rho}$. Leading $(B.19)$ by one-period and taking expectations of both sides yields:

$$
\mathbf{E}_t \mathbf{x}_{t+1} = \boldsymbol{\phi}_{\mathbf{x}} \mathbf{x}_t + \boldsymbol{\phi}_{\mathbf{z}} \boldsymbol{\rho} \mathbf{z}_t \tag{B.20}
$$

since $\boldsymbol{\phi}_{\mathbf{z}}\boldsymbol{\rho}^{-1}\mathbf{E}_t[\mathbf{u}_{t+1}] = 0$. Substituting $(B.20)$ and $(B.19)$ into the first equation of (2.20) gives:

$$\begin{aligned}\mathbf{x}_t &= \left[(\mathbf{I}-\mathbf{M}_1\boldsymbol{\phi}_{\mathbf{x}})^{-1}\mathbf{M}_{\mathbf{2}}\right]\mathbf{x}_{t-1}+ \\ &+\left[(\mathbf{I}-\mathbf{M}_1\boldsymbol{\phi}_{\mathbf{x}})^{-1}(\mathbf{M}_1\boldsymbol{\phi}_{\mathbf{z}}+\mathbf{M}_3)\right]\left(\boldsymbol{\rho}\mathbf{z}_{t-1}+\mathbf{u}_t\right).\end{aligned} \tag{B.21}$$

Comparing $(B.21)$ with $(B.19)$ implies that the unique RE solution of the reduced-form model is given by the two parameter matrices, hereafter denoted by $\bar{\boldsymbol{\phi}}_{\mathbf{x}}$ and $\bar{\boldsymbol{\phi}}_{\mathbf{z}}$, that satisfy the following two equations:

$$\begin{aligned}\boldsymbol{\phi}_{\mathbf{x}} &= (\mathbf{I}-\mathbf{M}_1\boldsymbol{\phi}_{\mathbf{x}})^{-1}\mathbf{M}_{\mathbf{2}} \\ \boldsymbol{\phi}_{\mathbf{z}} &= (\mathbf{I}-\mathbf{M}_1\boldsymbol{\phi}_{\mathbf{x}})^{-1}(\mathbf{M}_1\boldsymbol{\phi}_{\mathbf{z}}+\mathbf{M}_3)\,\boldsymbol{\rho}.\end{aligned} \tag{B.22}$$

Assuming $\bar{\boldsymbol{\phi}}_{\mathbf{x}}$ and $\bar{\boldsymbol{\phi}}_{\mathbf{z}}$ exist, the solution for the model's state variables under RE is:[1]

$$\mathbf{x}_t = \bar{\boldsymbol{\phi}}_{\mathbf{x}}\mathbf{x}_{t-1}+\boldsymbol{\rho}^{-1}\bar{\boldsymbol{\phi}}_{\mathbf{z}}\mathbf{z}_t. \tag{B.23}$$

Substituting $(B.23)$ and the expected value of its lead into the second equation of (2.20) gives the RE solution for the model's control variables:

$$\begin{aligned}\mathbf{y}_t &= \left[\mathbf{N}_1\bar{\boldsymbol{\phi}}_{\mathbf{x}}+\mathbf{N}_2+\mathbf{N}_4\bar{\boldsymbol{\phi}}_{\mathbf{x}}^2\right]\mathbf{x}_{t-1}+ \\ &+\left[\mathbf{N}_1\boldsymbol{\rho}^{-1}\bar{\boldsymbol{\phi}}_{\mathbf{z}}+\mathbf{N}_3+\mathbf{N}_4\left(\bar{\boldsymbol{\phi}}_{\mathbf{z}}+\bar{\boldsymbol{\phi}}_{\mathbf{x}}\boldsymbol{\rho}^{-1}\bar{\boldsymbol{\phi}}_{\mathbf{z}}\right)\right]\mathbf{z}_t.\end{aligned} \tag{B.24}$$

B.3 Model solution under adaptive learning

Under the AL hypothesis, it is also assumed that private agents can correctly guess the form of the equilibrium policy functions of the state variables given by $(B.18)$. However, in contrast to the RE solution, it is assumed that they do not know the time-invariant parameter values given by $\bar{\boldsymbol{\phi}}_{\mathbf{x}}$ and $\bar{\boldsymbol{\phi}}_{\mathbf{z}}$, which ultimately govern the dynamics of the economy.[2] Therefore, they must rely on past data and a recursive learning algorithm to estimate these

[1] The two solution matrices $\bar{\boldsymbol{\phi}}_x$ and $\bar{\boldsymbol{\phi}}_z$, were obtained applying the method proposed by Klein (2000).
[2] See Evans and Honkapohja (2001), for further details.

parameters to produce forecasts of the endogenous state variables for the next period. As new data become available in each period, they revise their parameter estimates so that their forecasting errors are corrected gradually.

More formally, agents' expectations are assumed to follow a so-called perceived law of motion (PLM) of the form:

$$\mathbf{E}_t^* \mathbf{x}_{t+1} = \tilde{\boldsymbol{\phi}}_{\mathbf{x},t-1} \mathbf{x}_t + \tilde{\boldsymbol{\phi}}_{\mathbf{z},t-1} \mathbf{z}_t \tag{B.25}$$

where parameters $\tilde{\boldsymbol{\phi}}_{\mathbf{x}}$ and $\tilde{\boldsymbol{\phi}}_{\mathbf{z}}$ are the estimates of $\bar{\boldsymbol{\phi}}_{\mathbf{x}}$ and $\bar{\boldsymbol{\phi}}_{\mathbf{z}}$ coming from a recursive least-squares regression and $\mathbf{E}^*$ denotes that expectations do not follow the RE hypothesis.[3]

Following a similar procedure as under RE, we substitute $(B.25)$ into the first equation of (2.20) to obtain:

$$\mathbf{x}_t = \mathbf{P}_1 \mathbf{x}_{t-1} + \boldsymbol{\rho}^{-1} \mathbf{P}_2 \mathbf{z}_t \tag{B.26}$$

where

$$\begin{aligned} \mathbf{P}_1 &= (\mathbf{I} - \mathbf{M}_1 \tilde{\boldsymbol{\phi}}_{\mathbf{x},t-1})^{-1} \mathbf{M}_2 \\ \mathbf{P}_2 &= (\mathbf{I} - \mathbf{M}_1 \tilde{\boldsymbol{\phi}}_{\mathbf{x},t-1})^{-1} \left(\mathbf{M}_1 \tilde{\boldsymbol{\phi}}_{\mathbf{z},t-1} + \mathbf{M}_3 \right) \boldsymbol{\rho}. \end{aligned} \tag{B.27}$$

Equation $(B.26)$ is referred to as the actual law of motion (ALM) since every new observed value of $\mathbf{x}_t$ depends on the deep parameters of the model economy but also on the agents' forecasts given by the PLM $(B.25)$.

The actual laws of motion for the control variables under learning are found by substituting $(B.26)$ for $\mathbf{x}_t$ and $(B.25)$ for $\mathbf{E}_t \mathbf{x}_{t+1}$ in the second equation of (2.20) giving:

$$\begin{aligned} \mathbf{y}_t &= \left[\mathbf{N}_1 \mathbf{P}_1 + \mathbf{N}_2 + \mathbf{N}_4 \tilde{\boldsymbol{\phi}}_{\mathbf{x},t-1} \mathbf{P}_1 \right] \mathbf{x}_{t-1} + \\ &\quad + \left[\mathbf{N}_1 \boldsymbol{\rho}^{-1} \mathbf{P}_2 + \mathbf{N}_3 + \mathbf{N}_4 \left(\tilde{\boldsymbol{\phi}}_{\mathbf{x},t-1} \boldsymbol{\rho}^{-1} \mathbf{P}_2 + \tilde{\boldsymbol{\phi}}_{\mathbf{z},t-1} \right) \right] \mathbf{z}_t. \end{aligned} \tag{B.28}$$

To estimate $\tilde{\boldsymbol{\phi}}_{\mathbf{x}}$ and $\tilde{\boldsymbol{\phi}}_{\mathbf{z}}$ in $(B.25)$ we first define the matrix $\mathbf{w}_t = [B_{c,t},\ K_{c,t},\ B_{s,t},\ K_{s,t},\ a_t]'$ and then use the recursive least-squares (RLS) learning algorithm which can be written

[3] Note, we follow the common assumption (see, e.g. Evans and Honkaphoja, 2001, and Carceles-Poveda and Giannitsarou, 2007) that at period t agents form expectations for $\mathbf{x}_{t+1}$ using their estimates from the previous period, $\tilde{\boldsymbol{\phi}}_{\mathbf{x},t-1}$ and $\tilde{\boldsymbol{\phi}}_{\mathbf{z},t-1}$, which allows us to avoid a problem of simultaneity in the learning process.

for $t = 1, 2, 3...$, as follows:

$$\begin{aligned} \tilde{\boldsymbol{\phi}}_t &= \tilde{\boldsymbol{\phi}}_{t-1} + g_t \mathbf{R}_t^{-1} \mathbf{w}_{t-1} (\mathbf{x_t} - \tilde{\boldsymbol{\phi}}'_{\mathbf{t-1}} \mathbf{w}_{t-1})' \\ \mathbf{R}_t &= \mathbf{R}_{t-1} + g_t \left(\mathbf{w}_{t-1} \mathbf{w}'_{t-1} - \mathbf{R}_{t-1} \right) \end{aligned} \tag{B.29}$$

where $\mathbf{R}_t$ is a matrix with the second moments of the regressors included in $\mathbf{w}_t$; $(\mathbf{x_t} - \tilde{\boldsymbol{\phi}}'_{\mathbf{t-1}} \mathbf{w}_{t-1})$ is the latest forecast error that will be used to correct the current estimates; and $g_t = 1/t$ is a decreasing gain sequence implying that, as t increases, every new forecast error will have a lower relative importance in the updating process.[4]

B.3.1 Initial conditions for learning

To represent the importance of initial beliefs for the solution of the model under learning, define $\bar{\boldsymbol{\phi}}_{pre} = [\bar{\boldsymbol{\phi}}_{\mathbf{x},pre}, \bar{\boldsymbol{\phi}}_{\mathbf{z},pre}]'$ and $\bar{\boldsymbol{\phi}}_{post} = [\bar{\boldsymbol{\phi}}_{\mathbf{x},post}, \bar{\boldsymbol{\phi}}_{\mathbf{z},post}]'$ as the RE solution matrices for the pre-reform and post-reform economies, respectively, and $\tilde{\boldsymbol{\phi}}_0 = \left[\tilde{\boldsymbol{\phi}}_{\mathbf{x},0}, \tilde{\boldsymbol{\phi}}_{\mathbf{z},0}\right]'$ as the matrix containing the starting values of the learning algorithm. To obtain the rational expectations solution, we assume that:

$$\tilde{\boldsymbol{\phi}}_0 = [\bar{\boldsymbol{\phi}}_{\mathbf{x},post}, \bar{\boldsymbol{\phi}}_{\mathbf{z},post}]' \tag{B.30}$$

where $\mathbf{R}_0$ is the covariance matrix associated with the values of the endogenous state variables as predicted by their corresponding policy functions under the post-reform RE solution $\bar{\boldsymbol{\phi}}_{post}$.[5]

For the case of homogeneous learning, we assume, as in Giannitsarou (2006), that:

$$\tilde{\boldsymbol{\phi}}_0 = [\bar{\boldsymbol{\phi}}_{\mathbf{x},pre}, \bar{\boldsymbol{\phi}}_{\mathbf{z},pre}]' \tag{B.31}$$

[4] See, e.g. Evans and Honkapohja (2001), and Honkapohja and Mitra (2006), for more details on stability conditions under learning. We make use of Matlab functions made available by Carceles-Poveda and Giannitsarou (2007), to solve the model under learning.

[5] To obtain $\mathbf{R}_0$ we make use of a numerical approximation involving the following steps: (i) simulate a series of $N(0, \sigma_a)$ random shocks for the exogenous state variable a_t, for $T_{num} = 100,000$ periods; (ii) using (i), simulate the values for the endogenous state variables as predicted by their corresponding policy functions under the post reform RE solution ($\bar{\boldsymbol{\phi}}_{post}$) for T_{num}; (iii) construct $\mathbf{w}_{(5\times T_{num})}$ including the time series of the simulated values for the five states ($B_{c,t}$, $K_{c,t}$, $B_{s,t}$, $K_{s,t}$, a_t); and (iv) compute the covariance matrix in a recursive fashion according to the second equation of ($B.29$), where the starting values $\mathbf{R}_0$ and $\mathbf{w}_0$ are given by two zero matrices.

where the covariance matrix $\mathbf{R}_0$ is computed as described above, using $(B.31)$ instead of $(B.30)$.

For the case of heterogeneous learning, we assume that the skilled workers "guess" that the coefficients remain the same and thus use the coefficients that correspond to the pre-reform economy in their policy functions for the initial period. In contrast, we assume that the capitalists are able to predict the post-reform RE steady-state and their optimal reduced form coefficients for their policy functions in this equilibrium, so that their "guess" for their initial coefficients correspond to the post-reform RE solution.

This heterogeneity in beliefs implies that the initial guesses for both agents are effectively incorrect, as the actual economy, as determined by the interaction of their choices, is neither in the pre- nor in the post-reform RE equilibrium. Given the gap between the expected and actual outcomes, both agents use thereafter recursive least-squares to learn the coefficients.

Formally, let $\bar{\boldsymbol{\phi}}^{c}_{\mathbf{x},post}$ and $\bar{\boldsymbol{\phi}}^{c}_{\mathbf{z},post}$ be a (4×2) and (1×2) sub-matrices of $\bar{\boldsymbol{\phi}}_{\mathbf{x},post}$ and $\bar{\boldsymbol{\phi}}_{\mathbf{z},post}$, respectively, containing the two columns of $\bar{\boldsymbol{\phi}}_{\mathbf{x},post}$ and $\bar{\boldsymbol{\phi}}_{\mathbf{z},post}$ that correspond to the policy functions of the capitalists. Similarly, let $\bar{\boldsymbol{\phi}}^{s}_{\mathbf{x},pre}$ and $\bar{\boldsymbol{\phi}}^{s}_{\mathbf{z},pre}$ be a (4×2) and (1×2) sub-matrices of $\bar{\boldsymbol{\phi}}_{\mathbf{x},pre}$ and $\bar{\boldsymbol{\phi}}_{\mathbf{z},pre}$, respectively, containing the two columns of $\bar{\boldsymbol{\phi}}_{\mathbf{x},pre}$ and $\bar{\boldsymbol{\phi}}_{\mathbf{z},pre}$ that correspond to the policy functions of the skilled workers. Hence, $\tilde{\boldsymbol{\phi}}_0$ is constructed as:

$$\tilde{\boldsymbol{\phi}}_0 = \begin{bmatrix} \left[\bar{\boldsymbol{\phi}}^{c}_{\mathbf{x},post}\right]_{4\times 2} & \left[\bar{\boldsymbol{\phi}}^{s}_{\mathbf{x},pre}\right]_{4\times 2} \\ \left[\bar{\boldsymbol{\phi}}^{c}_{\mathbf{z},post}\right]_{1\times 2} & \left[\bar{\boldsymbol{\phi}}^{s}_{\mathbf{z},pre}\right]_{1\times 2} \end{bmatrix} \tag{B.32}$$

while, for consistency, $\mathbf{R}_0$ is now computed as above but using $(B.32)$ instead.

Note that for all the post-reform scenarios considered, $\tilde{\boldsymbol{\phi}}_0$ always satisfies the stationarity condition that the real parts of all the eigenvalues of $\tilde{\boldsymbol{\phi}}_{\mathbf{x},0}$ must lie inside the unit circle, while $\mathbf{R}_0$ is always an invertible matrix. These two conditions ensure the algorithm is adequately initialised, see, e.g. Carceles-Poveda and Giannitsarou, 2007.

B.3.2 E-stability and convergence

An important issue is whether the RLS learning algorithm chosen in this work will converge to the RE solution. To verify this, we first consider the so-called expectational stability or E-stability of the model under learning. E-stability determines the stability of the RE solution under a learning rule such as RLS, in which the estimates $\tilde{\boldsymbol{\phi}}_{\mathbf{x}}$ and $\tilde{\boldsymbol{\phi}}_{\mathbf{z}}$ used in the PLM ($B.25$) are adjusted slowly in the direction of the implied ALM parameters shown in ($B.27$).

In fact, if this adjustment process is completed, feeding the latest estimates $\tilde{\boldsymbol{\phi}}_{\mathbf{x}}$ and $\tilde{\boldsymbol{\phi}}_{\mathbf{z}}$ in the two ALM parameters in ($B.27$) should yield exactly the same two estimates for $\tilde{\boldsymbol{\phi}}_{\mathbf{x}}$ and $\tilde{\boldsymbol{\phi}}_{\mathbf{z}}$. In such a case, these estimates must be equal to the RE solution parameters $\bar{\boldsymbol{\phi}}_{\mathbf{x}}$ and $\bar{\boldsymbol{\phi}}_{\mathbf{z}}$, since the model at hand has a unique equilibrium. Evans and Honkapohja (2001) demonstrate that such condition can be verified by computing the following two matrices (associated to $\bar{\boldsymbol{\phi}}_{\mathbf{x}}$ and $\bar{\boldsymbol{\phi}}_{\mathbf{z}}$, respectively):

$$\begin{aligned} \mathbf{Q_x} &= \left[(\mathbf{I} - \mathbf{M}_1\bar{\boldsymbol{\phi}}_{\mathbf{x}})^{-1}\mathbf{M2}\right]' \otimes \left[(\mathbf{I} - \mathbf{M}_1\bar{\boldsymbol{\phi}}_{\mathbf{x}})^{-1}\mathbf{M}_1\right], \\ \mathbf{Q_z} &= \boldsymbol{\rho}' \otimes \left[(\mathbf{I} - \mathbf{M}_1\bar{\boldsymbol{\phi}}_{\mathbf{x}})^{-1}\mathbf{M}_1\right] \end{aligned} \tag{B.33}$$

and then testing if all their corresponding eigenvalues have real parts less than one. For our model, the E-stability condition is met for the base calibration of the model as well as for all the tax reforms considered. This is also true for the additional calibration included when discussing the importance of the degree of substitutability between capital and unskilled labour.

A second condition for convergence is the stationarity of the RE solution. This requires that the eigenvalues of $\bar{\boldsymbol{\phi}}_{\mathbf{x}}$ have real parts less than one, ensuring that the part of the RE solution associated with the lags of the state variables do not have an explosive path. The stationarity condition is also met for the all the experiments considered in this work. Evans and Honkapohja (2001) show that if the E-stability and stationarity conditions are satisfied, then the RLS algorithm converges locally to $\bar{\boldsymbol{\phi}}_{\mathbf{x}}$ and $\bar{\boldsymbol{\phi}}_{\mathbf{z}}$ and thus the model at hand is *learnable*.

Appendix C

Appendix to Chapter 3

C.1 Descentralised competitive equilibrium

$$\frac{(C_{c,t}^{\mu_1}(1-h_{c,t})^{\mu_2}G_{c,t}^{1-\mu_1-\mu_2})^{1-\sigma}\mu_1}{C_{k,t}} = -\lambda_{c,t}(1+\tau_{c,t}) \tag{C.1}$$

$$\frac{-(C_{c,t}^{\mu_1}(1-h_{c,t})^{\mu_2}G_{c,t}^{1-\mu_1-\mu_2})^{1-\sigma}\mu_2}{1-h_{k,t}} = \lambda_{c,t}(1-\tau_{h,t})w_t \tag{C.2}$$

$$-\lambda_{c,t} = \beta\lambda_{c,t+1}(-1+\delta-(1-\tau_{h,t+1})r_{k,t+1}+2\phi_c^k K_{c,t+1}) \tag{C.3}$$

$$-\lambda_{c,t} = \beta\lambda_{c,t+1}(-1-r_{b,t+1}+2\phi_c^b B_{c,t+1}) \tag{C.4}$$

$$\begin{aligned}(1+\tau_{c,t})C_{c,t}+K_{c,t+1}-(1-\delta)K_{c,t} \quad &= \quad -B_{c,t+1}+(1+r_{b,t})B_{c,t} \\ &\quad +(1-\tau_{k,t})r_{k,t}K_{c,t}+G^{tr} \\ &\quad +(1-\tau_{h,t})w_t h_{c,t}+\phi_c^b B_{c,t}^2+\phi_c^k K_{c,t}^2\end{aligned} \tag{C.5}$$

$$\frac{(C_{w,t}^{\mu_1}(1-h_{w,t})^{\mu_2}G_{c,t}^{1-\mu_1-\mu_2})^{1-\sigma}\mu_1}{C_{w,t}} = -\lambda_{w,t}(1+\tau_{c,t}) \tag{C.6}$$

$$\frac{-(C_{w,t}^{\mu_1}(1-h_{w,t})^{\mu_2}G_{c,t}^{1-\mu_1-\mu_2})^{1-\sigma}\mu_2}{1-h_{w,t}} = \lambda_{w,t}(1-\tau_{h,t})w_t \tag{C.7}$$

$$-\lambda_{w,t} = \beta\lambda_{w,t+1}(-1+\delta-(1-\tau_{h,t+1})r_{k,t+1}+2\phi_w^k K_{w,t+1}) \tag{C.8}$$

$$-\lambda_{w,t} = \beta\lambda_{w,t+1}(-1-r_{b,t+1}+2\phi_w^b B_{w,t+1}) \tag{C.9}$$

$$r_{k,t} = \alpha A_t(n_c K_{c,t}+n_w K_{w,t})^{\alpha-1}(n_c h_{c,t}+n_w h_{w,t})^{1-\alpha} \tag{C.10}$$

$$w_t = (1-\alpha)A_t(n_cK_{c,t} + n_wK_{w,t})^{\alpha}(n_ch_{c,t} + n_wh_{w,t})^{-\alpha} \tag{C.11}$$

$$\begin{aligned} G_{c,t} + G^{tr} \;\; = \;\; & -(1+r_{b,t})(n_cB_{c,t} + n_wB_{w,t}) + n_cB_{c,t+1} \\ & + n_wB_{w,t+1} + \tau_{c,t}n_cC_{c,t} + \tau_{k,t}r_{k,t}n_cK_{c,t} \\ & + \tau_{h,t}w_tn_ch_{c,t} + \tau_{c,t}n_wC_{w,t} + \tau_{k,t}r_{k,t}n_wK_{w,t} \\ & + \tau_{h,t}w_tn_wh_{w,t} + R_t^{ng} \end{aligned} \tag{C.12}$$

$$\begin{aligned} Y_t \;\; = \;\; & n_cC_{c,t} + n_wC_{w,t} + n_cK_{c,t+1} \\ & -(1-\delta)n_cK_{c,t} + n_wK_{w,t+1} \\ & -(1-\delta)n_wK_{w,t} + G_{c,t} + \phi_c^b n_cB_{c,t}^2 \\ & + \phi_c^k n_cK_{c,t}^2 + \phi_w^b n_wB_{w,t}^2 + \phi_w^k n_wK_{w,t}^2 + R_t^{ng} \end{aligned} \tag{C.13}$$

where:

$$Y_t = A_t(n_cK_{c,t} + n_wK_{w,t})^{\alpha}(n_ch_{c,t} + n_wh_{w,t})^{1-\alpha}$$

Note that λ_c and λ_w are the Lagrange multipliers associated to the budget constraints of the two types of agents in the model. The descentralised competitive equilibrium is completed with the two AR(1) processes that describe the behavior of productivity and gas revenues, plus one of the debt-targeting rules proposed in the main text.

C.2 Model solution

The solution of the model assuming both RE and AL is qualitatively similar to the description given in parts B.1 and B.2 of Appendix B. In the case of learning, however, the initial conditions are different in this chapter and thus are described next.

C.2.1 Initial conditions for learning

When considering the case of homogeneous learning, we should assume that:

$$\tilde{\phi}_0 = \begin{bmatrix} [\bar{\phi}^c_{\mathbf{x},i}]_{4\times 2} & [\bar{\phi}^s_{\mathbf{x},i}]_{4\times 2} \\ [\bar{\phi}^c_{\mathbf{z},i}]_{1\times 2} & [\bar{\phi}^s_{\mathbf{z},i}]_{1\times 2} \end{bmatrix} \tag{C.14}$$

where i represents any possible set of initial beliefs (e.g. pessimistic or optimistic) which are common to both types of agents but that are away from those consistent with the RE equilibrium.[1] However, as expected, since in this case both capitalists and workers will learn in exactly the same fashion (i.e. due to the same initial beliefs and same learning algorithm), no additional relevant distributional effects coming from this assumption can be found and thus we do not pursue this approach any further.

Hence, for the case of heterogeneous learning, we explore the possibility that the capitalists have already learned the parameters of their policy functions and are aware of the implications of the fiscal rule applied by the authorities 'as if' they were fully rational. In contrast, the workers find themselves with limited information and, as a result, their initial beliefs are slightly off their true parameters according to the RE equilibrium.

Here, we consider three possibilities: a) they do not know the impact the fiscal rules applied might have on the economy, b) they do not know the impact of the rule but also have pessimistic initial beliefs (i.e. consistent with a worse-off equilibrium) regarding the situation of the economy and c) they do not know the impact of the rule and have optimistic initial beliefs instead.

Therefore, if an exogenous shock hits the economy, this heterogeneity in beliefs implies that the initial guesses for both agents in their attempt to return to the steady-state will be effectively incorrect. This is the case because the capitalists do not know that the workers are not fully rational and their initial beliefs are in fact incorrect. On the other hand, the

[1] To obtain $\mathbf{R}_0$ we make use of a numerical approximation involving the following steps: (i) simulate a series of $N(0,\sigma_a)$ random shocks for the exogenous state variables $\hat{a}_t$ or $\hat{R}_t$, depending on the shock under analysis, for $T_{num} = 100,000$ periods; (ii) using (i), simulate the values for the endogenous state variables as predicted by their corresponding policy functions under $\bar{\phi}_0$ for T_{num}; (iii) construct $\mathbf{w}_{(6\times T_{num})}$ including the time series of the simulated values for the six states ($\hat{B}_{c,t}$, $\hat{K}_{c,t}$, $\hat{B}_{w,t}$, $\hat{K}_{w,t}$, $\hat{a}_t$, $\hat{R}_t$); and (iv) compute the covariance matrix in a recursive fashion according to the second equation of $(B.29)$, where the starting values $\mathbf{R}_0$ and $\mathbf{w}_0$ are given by two zero matrices.

workers are not aware of the original error in their initial estimates and also do not know that the capitalists' beliefs are much closer to the RE equilibrium. The interaction between both types of agents will take place in these circumstances, thus affecting their choices over time as unexpected larger forecast errors might take place during their estimation procedures.

More formally, the learning process after any exogenous shock has taken place will start with an heterogeneity in beliefs described by:

$$\tilde{\boldsymbol{\phi}}_0 = \begin{bmatrix} \left[\bar{\boldsymbol{\phi}}^c_{\mathbf{x},re}\right]_{4\times 2} & \left[\bar{\boldsymbol{\phi}}^s_{\mathbf{x},j}\right]_{4\times 2} \\ \left[\bar{\boldsymbol{\phi}}^c_{\mathbf{z},re}\right]_{1\times 2} & \left[\bar{\boldsymbol{\phi}}^s_{\mathbf{z},j}\right]_{1\times 2} \end{bmatrix} \tag{C.15}$$

where j represents the three alternative initial beliefs of the workers.[2] From then on, capitalists and workers will interact using the same learning algorithm as they try to eventually learn their true coefficients, associated to the unique RE equilibrium. Finally, note that for all the scenarios considered, $\tilde{\boldsymbol{\phi}}_0$ always satisfies the stationarity condition that the real parts of all the eigenvalues of $\tilde{\boldsymbol{\phi}}_{\mathbf{x},0}$ must lie inside the unit circle, while $\mathbf{R}_0$ is always an invertible matrix, two conditions that will ensure the learning algorithm is adequately initialised.

C.2.2 E-stability and convergence

Following a similar discussion to that presented in section B.2.2 of Appendix B, the first condition for convergence of the learning process towards the RE solution, i.e. the E-stability condition, is met for the base calibration of the model under all the alternative fiscal rules considered. This is also true for the additional calibrations included in order to obtain the hypothetical lower and higher steady-states used to characterise different initial beliefs of the workers. The second condition for convergence, which consists in the stationarity of the RE solution is also met for all the experiments considered in this Chapter.

[2] For consistency, $\mathbf{R}_0$ is now computed as above (see previous footnote) but using $(C.15)$ instead

Figure C.1: Income inequality ratio for each rule after a negative TFP shock

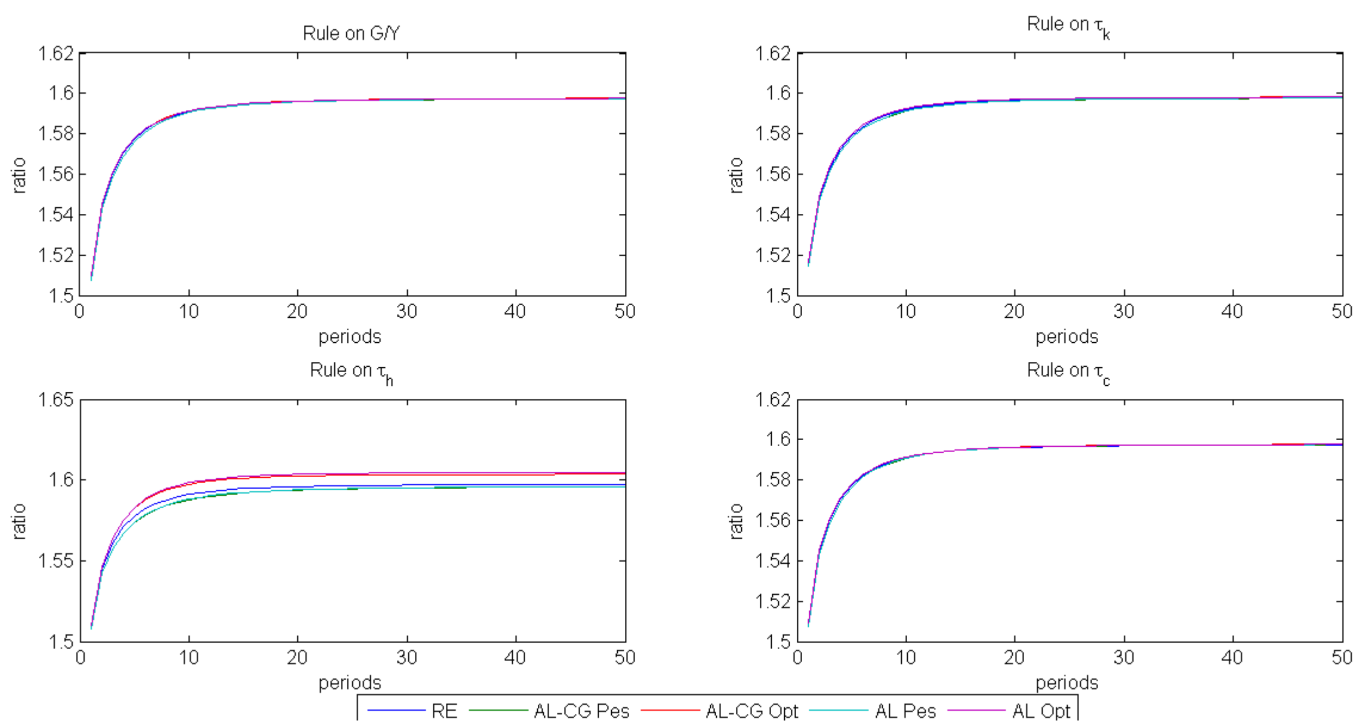


Figure C.2: Income inequality ratio for each rule after a negative commodity shock

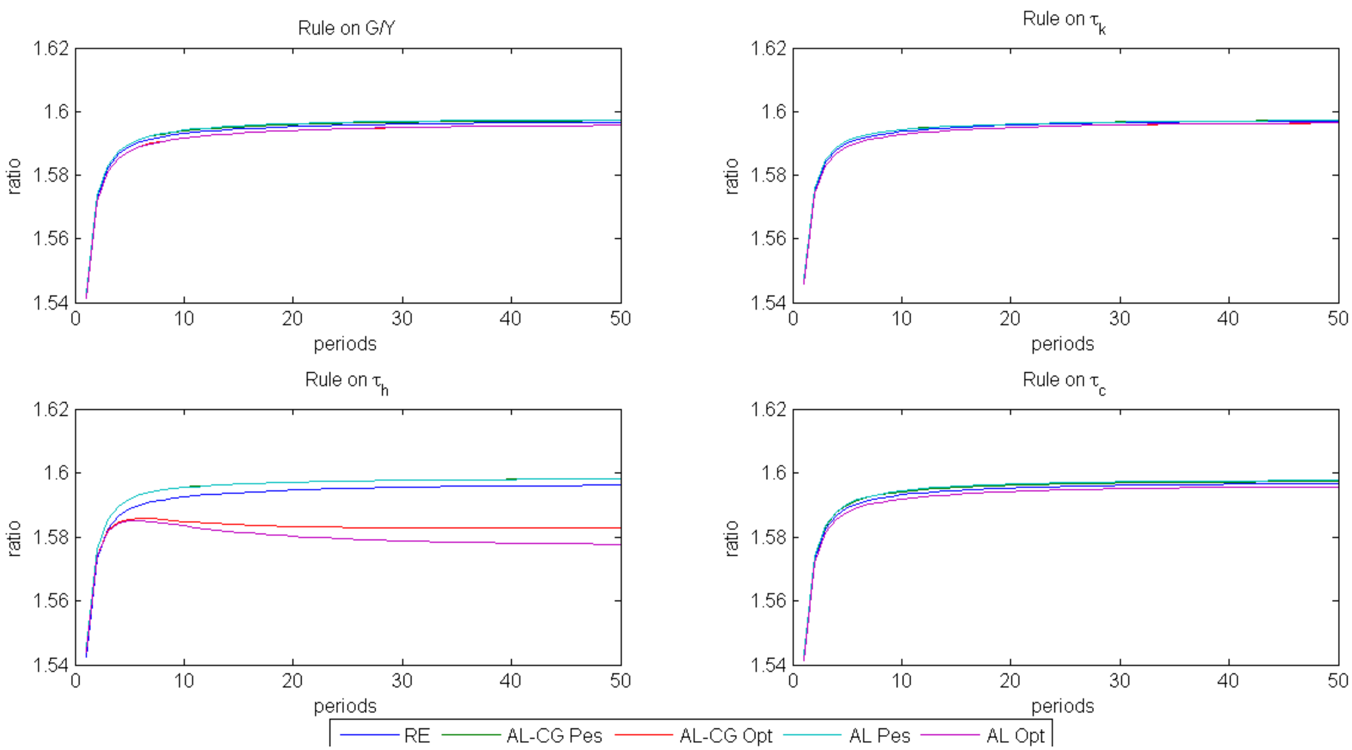


Table C.1: Evolution of the welfare inequality ratio

Rule type	Short Run			Lifetime		
	RE	AL-opt	AL-pes	RE	AL-opt	AL-pes
After a -4.5% TFP shock						
Rule on G/Y	0.7935	0.7936	0.7934	0.7947	0.7946	0.7945
Rule on τ^k	0.7932	0.7934	0.7930	0.7944	0.7946	0.7945
Rule on τ^h	0.7936	0.7954	0.7926	0.7948	0.7947	0.7946
Rule on τ^c	0.7935	0.7936	0.7935	0.7947	0.7946	0.7945
After a -20% commodity shock						
Rule on G/Y	0.8008	0.8005	0.8011	0.7985	0.7985	0.7985
Rule on τ^k	0.8003	0.8000	0.8004	0.7982	0.7981	0.7982
Rule on τ^h	0.8011	0.7997	0.8020	0.7987	0.7986	0.7987
Rule on τ^c	0.8008	0.8005	0.8011	0.7985	0.7985	0.7985

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