Existence and approximation of solutions of nonlinear boundary value problems

by

Rahmat Ali Khan

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Preface

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow.
Almost all of the results of this thesis are the original work of the author with the exception of several results specifically mentioned in the text and attributed there to the authors concerned.
Chapter one contains preliminary material that include some basic definitions, notions and some known results which will be used in this thesis.
A part of chapter two is a joint work with Prof. J. J. Nieto and Angela Torres and is submitted for publication [51]. Much of chapters three and six is a joint work with Prof. J. R. L. Webb. The results of chapter three, section 1, have appeared [53] and the results of chapter three, section 2, is accepted for publication [49]. Much of chapter four is joint work with Rosana Rodriguez Lopez and part of it is accepted for publication [52]. The results of chapter three, section 2, and chapter five are the original work of the author alone. The results of chapter five, section 1, have appeared [45], while the results of section 2, chapter five, is accepted for publication [48]. The results of chapter six section 1, is accepted for publication [54] while section 2 and 3, are submitted for publication [55, 56].

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Summary

This thesis is concerned with the existence, uniqueness, approximation and multiplicity of solutions of nonlinear boundary value problems. In chapter one, we recall some basic concepts and methods which we use throughout the thesis. They include the notions of upper and lower solutions, modification of the nonlinearity, Nagumo condition, the method of quasilinearization, its generalization and the concept of degree theory.

In chapter two, we establish new results for periodic solutions of some second order nonlinear boundary value problems. We develop the upper and lower solutions method to show existence of solutions in the closed set defined by the well ordered lower and upper solutions. We develop the method of quasilinearization to approximate our problem by a sequence of solutions of linear problems that converges to the solution of the original problem quadratically. Finally, to show the applicability of our technique, we apply the theoretical results to a medical problem namely, a biomathematical model of blood flow in an intracranial aneurysm.

In chapter three we study some nonlinear boundary value problems with nonlinear nonlocal three-point boundary conditions. We develop the method of upper and lower solutions to establish existence results. We show that our results hold for a wide range of nonlinear problems. We develop the method of quasilinearization and show that there exist monotone sequences of solutions of linear problems that converges to the unique solution of the nonlinear problems. We show that the sequences converges quadratically to the solutions of the problems in the C^1 norm. We generalize the technique by introducing an auxiliary function to allow weaker hypotheses on the nonlinearity involved in the differential equations.

In chapter four, we extend the results of chapter three to nonlinear problems with linear four point boundary conditions. We generalize previously existence results studied with constant lower and upper solutions. We show by an example that our results are more general. We develop the method of quasilinearization and its generalization for the four point problems which to the best of our knowledge is the first time the method has been applied to such problems.

In chapter five, we extend the results to second order problems with nonlinear integral boundary conditions in two separate cases. In the first case we study the upper and lower

solutions method and the generalized method of quasilinearization for the integral boundary value problem with the nonlinearity independent of the derivative. While in the second case we allow the nonlinearity to depend also on the first derivative.

Finally, in chapter six, we study multiplicity results for three point nonlinear boundary value problems. We use the method of upper and lower solutions and degree arguments to show the existence of at least two solutions for certain range of a parameter r and no solution for other range of the parameter. We show by an example that our results are more general than the results studied previously. We also study existence of at least three solutions in the presence of two lower and two upper solutions for some three-point boundary value problems. In one problem, we employ a condition weaker than the well known Nagumo condition which allows the nonlinearity $f(t, x, x')$ to grow faster than quadratically with respect to x' in some cases.

Introduction

The theory of nonlinear boundary value problems is an important and interesting area of research in differential equations. Due to the entirely different nature of the underlying physical processes, its study is more difficult than that of initial value problems. A variety of techniques are employed in the theory of nonlinear boundary value problems for existence results. One of the most powerful tools for proving existence of solutions is the method of upper and lower solutions. The basic idea is to modify the given problems suitably and then use known existence results of the modified problems together with the theory of differential inequalities, to establish existence of solutions of the given problems. Since explicit analytic solutions of nonlinear boundary value problems in terms of familiar functions of analysis are rarely possible, one needs to exploit various approximate methods. One of the most powerful techniques, which yields a monotone sequence with rapid convergence, is the method of quasilinearization.

The basic idea of the original method of quasilinearization, developed by Bellman and Kalaba [9, 10], is to provide an explicit analytic representation for the solution of a nonlinear differential equations which yields point-wise lower estimates for the solution of the problem whenever the function involved is convex. Recently, Lakshmikantham [61–63] generalized the method of quasilinearization by not demanding the convexity assumption so as to be applicable to a much larger class of nonlinear problems [4,5,46,47,60]. Chapter one of this thesis deals with these and some other basic concepts and definitions.

In chapter two, we establish new results for periodic solutions of some second order nonlinear boundary value problems [51], of the type

$$\begin{aligned} x''(t) &= f(t, x, x'), t \in [0, T], \\ x(0) &= x(T), x'(0) = x'(T),\ T > 0, \end{aligned} \tag{0.0.1}$$

where the nonlinearity f is continuous on $[0, T] \times \mathbb{R}^2$. We prove the validity of the classical upper and lower solutions method and of monotone iterative technique to show existence of at least one solution in the closed ordered interval defined by lower and upper solutions. This provides estimates for the solution and a numerical procedure to approximate the solution of the problem. We assume some monotone conditions on f to show uniqueness of solutions. We develop the method of quasilinearization to approximate our problem as a sequence of solutions of linear problems that converges to a solution of the nonlinear problem quadratically. We improve previously studied results where the nonlinearity did not depend on the derivative [50, 72]. We introduce an auxiliary function to allow weaker hypothesis on the nonlinear function and hence prove results on the generalized quasilinearization method. To show the applicability of our techniques, we apply the theoretical results to a medical problem: a model of blood flow inside an intracranial aneurysm [77–79]

$$\begin{aligned} x''(t) &= px' + ax - bx^2 - cx^3 - F\cos(ht),\ t \in [0,T], \\ x(0) &= x(T), x'(0) = x'(T), \end{aligned}$$

where x represents the velocity of blood flow inside an aneurysm, and p, a, b, c, F, h are medical parameters, varying from patient to patient.
In chapter three, we study some nonlinear boundary value problems of the type

$$x''(t) = f(t,x,x'),\ t \in [0,1] = J, \tag{0.0.2}$$

subject to each of the nonlinear three-point boundary conditions [49, 53]

$$\begin{aligned} &x(0) = a, \quad x(1) = g(x(\eta)), \\ &x(0) = a, \quad x'(1) = g(x(\eta)), \quad 0 < \eta < 1, \end{aligned} \tag{0.0.3}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous and may be nonlinear. We develop the method of upper and lower solutions to establish existence results. Assuming the existence of a lower solution α and an upper solution β, such that $\alpha \leq \beta$ on J, and the nonlinear function f satisfies a Nagumo condition relative to α, β, we prove existence of at least one solution x of the problem such that $\alpha(t) \leq x(t) \leq \beta(t)$ on J. Moreover, we prove that there exists a constant C which depends on α, β and ω (Nagumo function) such that $|x'(t)| < C$ on J. We assume some monotone condition on f to prove uniqueness results. We show that our results hold for a wide range of nonlinear problems, including problems of the form

$$x'' = |x'|^{p-1}x' - f(x), \quad \text{where } 0 < p \leq 2,$$

subject to the well-studied boundary conditions

$$x(0) = a > 0, \quad x(1) = \delta x(\eta), \quad 0 < \delta\eta < 1.$$

For these particular problems we show the existence of a positive solution when f satisfies conditions a little weaker than $0 \leq f(x) \leq \gamma x^p + C$ for some $\gamma < 1$. For the approximation of solution, we develop the method of quasilinearization. We assume f and its partial derivatives up to second order are continuous and the boundary function g is convex. We prove that there is a sequence of solutions of linear problems which converges uniformly and quadratically to the unique solution of the original problems. We control both the function and its first derivative and prove results on quadratic convergence in the C^1 norm. Finally, we introduce an auxiliary function ϕ and generalize the method of quasilinearization, using weaker hypotheses on f.

In chapter four, we extend the results of chapter three to nonlinear problems with linear four point boundary conditions [52], of the type

$$\begin{aligned} &x''(t) = f(t, x, x'),\ t \in I = [a, b], \\ &x(a) = x(c),\ x(b) = x(d), \end{aligned} \tag{0.0.4}$$

where $a < c \leq d < b$. Existence theory for the solution of four point boundary value problems have been given in a number of papers by Rachunkova [82–84]. In theorem 1 of [84], Rachunkova, proved existence of solutions for the four point boundary value problem (0.0.4) under various combinations of sign conditions on the function $f(t, x, x')$. In theorem 4.2.5, we study existence results under more general conditions, assuming existence of upper and lower solutions, which are not necessarily constant, that is, there exist α, $\beta \in C^2[a, b]$ such that

$$f(t, \alpha(t), \alpha'(t)) \leq \alpha''(t), \quad f(t, \beta(t), \beta'(t)) \geq \beta''(t),\ t \in [a, b]. \tag{0.0.5}$$

The conditions (0.0.5) include the corresponding conditions

$$f(t, r_1, 0) \leq 0,\ f(t, r_2, 0) \geq 0, \quad r_1 \leq r_2,\ r_1,\ r_2 \in \mathbb{R},$$

studied in [84] as a special case. We show by an example that our results are more general. The conditions

$$\begin{aligned} &f(t, x, R_1) \leq 0,\ f(t, x, R_2) \geq 0 \text{ for all } x \in [r_1, r_2] \text{ and a.e. } t \in [a, b], \\ &f(t, x, R_3) \geq 0,\ f(t, x, R_4) \leq 0 \text{ for a.e } t \in [d, b] \text{ and } x \in [r_1, r_2], \end{aligned} \tag{0.0.6}$$

where, $R_1 \leq 0 \leq R_2$, $R_3 \leq 0 \leq R_4$, $R_1 \neq R_3$, $R_2 \neq R_4$ studied in [84] are more restrictive than the ones we study, and can never be satisfied in the type of example we give. Moreover, we also study existence results under the following conditions

$$\begin{cases} f(t, x, -R) \leq 0,\ f(t, x, R) \geq 0 \text{ for } x \in [\min \alpha(t), \max \beta(t)],\ t \in [a, d], \\ f(t, x, -R) \geq 0,\ f(t, x, R) \leq 0 \text{ for } x \in [\min \alpha(t), \max \beta(t)],\ t \in (d, b], \end{cases} \tag{0.0.7}$$

where $R \geq \max\{\|\alpha'\|, \|\beta'\|\}$, which are less restrictive than (0.0.6). We develop the method of quasilinearization and the so called generalized method of quasilinearization for the four point problems which to the best of our knowledge is the first time the method has been applied to such problems.

In chapter five, we extend the results to second order problems with nonlinear integral boundary conditions [45, 48]. They include two, three, multipoint and nonlocal boundary value problems as special cases. Existence results for boundary value problems with linear integral boundary conditions are studied in [25,43,44,66]. We discuss two separate cases. In the first case, we study the integral boundary value problem with the nonlinearity independent of the derivative

$$\begin{aligned} &x''(t) = f(t,x),\ t \in J = [0,1], \\ &x(0) - k_1 x'(0) = \int_0^1 h_1(x(s))ds, \quad x(1) + k_2 x'(1) = \int_0^1 h_2(x(s))ds, \end{aligned} \tag{0.0.8}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ and $h_i : \mathbb{R} \to \mathbb{R}\,(i = 1,2)$ are continuous functions and k_i are nonnegative constants. In the second case, we allow the nonlinearity to depend also on the first derivative

$$\begin{aligned} &-x''(t) = f(t,x,x'),\ t \in J = [0,1], \\ &x(0) = a, \quad x(1) = \int_0^1 g(x(s))ds \end{aligned} \tag{0.0.9}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ are continuous and are nonlinear. In each case existence and uniqueness of solutions is shown via the method upper and lower solutions. We approximate our problems by a sequence of linear problems to obtain a monotone sequences of approximants. We show that the sequences of approximants converges quadratically to a solution of the BVPs.

Finally, in chapter six, we study multiplicity results for three point nonlinear boundary value problems with linear boundary conditions [54–56]. We study two type of problems, one with a parameter r and the other without a parameter. For the first problem

$$\begin{aligned} -x'' &= f(t,x,x') - r\phi(t),\ t \in J = (0,1) \\ x(0) &= 0,\ x(1) = \delta x(\eta),\ 0 < \delta\eta < 1,\ 0 < \eta < 1, \end{aligned} \tag{0.0.10}$$

we use the method of upper and lower solutions and degree arguments to show that, for a certain range of the parameter r, there are no solutions while for some other range of values there are at least two solutions of the problem. Existence of at least two solutions for other boundary conditions have been studied in many papers, for example [17,23,27,80,82]. In [82], existence of at least two solutions for (0.0.10) with four-point boundary conditions is studied in the presence of constant lower and upper solutions. In [27], ϕ is taken to be the normalized positive eigenfunction associated with the first eigenvalue $\lambda = 1$ of the linear problem

$$x'' + \lambda x = 0,\ x(0) = 0,\ x(\pi) = 0,$$

existence of at least two solutions for (0.0.10) with the Dirichlet boundary conditions is studied. Both the authors assumed that the set of all solutions is bounded above and the nonlinearity $f(t,x,x')$ is bounded for bounded x. In contrast, here we study the problem (0.0.10) not only with different boundary conditions (three-point boundary conditions) but also assume the existence of lower and upper solutions which are not necessarily constants. Moreover, we do not require the set of solutions to be bounded and allow ϕ to be any positive continuous bounded function without requiring $f(t,x,x')$ to be bounded. In the second case, we study existence of at least three solutions in the presence of two lower and two upper solutions for the three-point boundary value problem

$$-x''(t) = f(t,x,x'),\ t \in J = [0,1], \tag{0.0.11}$$

subject to each of the boundary conditions

$$\begin{aligned} &x(0) = 0, \quad x(1) = \delta x(\eta), \quad 0 < \delta\eta < 1,\ 0 < \eta < 1, \\ &x'(0) = 0, \quad x(1) = \delta x(\eta), \quad 0 < \delta,\ \eta < 1. \end{aligned} \tag{0.0.12}$$

In [35], existence of at least three solutions for (0.0.11) with the Dirichlet conditions is studied. The main assumption in [35] is the existence of two lower solutions α_1, α_2 and two upper solutions β_1, β_2 such that $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$ and that f satisfies a Nagumo condition. The Nagumo condition permits a maximum growth rate of $f(t,x,x')$ with respect to x' which is roughly quadratic. We study different problems and in one problem, we use a condition weaker than the well known Nagumo condition, namely we suppose there are functions ϕ, ψ such that

$$\begin{aligned} &\phi_t(t,x) + \phi_x(t,x)\phi(t,x) + f(t,x,\phi(t,x)) < 0, \\ &\psi_t(t,x) + \psi_x(t,x)\psi(t,x) + f(t,x,\psi(t,x)) > 0, \end{aligned}$$

which conditions were studied in [11], to allow higher growth rate of f with respect to x'. Existence of at least one solution under the above conditions is studied in [87] whereas we obtain at least three solutions. We show by an example that our results are more general than the results studied in [35] and that the results of [35] can never be applied to the type of example we give.

Contents

Chapter 1

Preliminaries

Throughout this work we will be interested in the theory of existence, uniqueness, approximation and multiplicity of solutions of nonlinear boundary value problems. Depending on the nature of a boundary value problem, we will use different tools of nonlinear analysis, for example, lower and upper solutions methods, the method of quasilinearization, the generalized method of quasilinearization and degree theory. In this chapter, we give an introduction to these concepts.

1.1 Upper and lower solutions method

When we study boundary value problems for second order nonlinear differential equations of the type

$$x''(t) = f(t, x, x'),\ t \in I = [a, b] \tag{1.1.1}$$

with certain linear or nonlinear boundary conditions on the compact interval $[a, b] \subset \mathbb{R}$, we often use the properties of lower and upper solutions for (1.1.1) to establish existence of solutions. The basic idea is to modify the given problem suitably and then employ Leray-Schauder theory or known existence results of the modified problem, together with the theory of inequalities, to establish existence results of the given problem. The method of upper and lower solutions for ordinary differential equations has been introduced by E. Picard in 1893 [81], but the method has been further developed by Scorza Dragoni in 1931 [20]. Such methods allow us to ensure the existence of at least one solution of the considered problem lying between a lower solution α and an upper solution β, such that $\alpha \leq \beta$ are in $\mathbb{R}$. The case where the upper and lower solutions are in the reversed ordered $\alpha \geq \beta$ has also received some attention. Recently, the author studied existence results for Neumann problems [46] and periodic problems [47] in the presence of lower and upper solutions in the reversed ordered. Let us first note the definition of lower and upper solutions

Definition 1.1.1. **[Upper and lower solutions]** Let f be continuous on $I \times \mathbb{R}^2$. The functions α, $\beta \in C^2(I)$ are called lower and upper solutions for (1.1.1), if they satisfy

$$\begin{aligned} f(t, \alpha(t), \alpha'(t)) &\leq \alpha''(t), \\ f(t, \beta(t), \beta'(t)) &\geq \beta''(t), \end{aligned} \tag{1.1.2}$$

for all $t \in I$. If the inequalities of (1.1.2) are strict, then α, β are called strict lower and upper solutions. For the estimation at the end points a, b of I, we use certain connections between α, β and the boundary conditions. It is well known that for the classical two-point boundary conditions such a connection has the following form.

- For the Dirichlet boundary conditions ($x(a) = c$, $x(b) = d$), α, β satisfy

$$\alpha(a) \leq c,\ \alpha(b) \leq d,\ \beta(a) \geq c,\ \beta(b) \geq d.$$

- For the periodic boundary conditions: $x(a) = x(b)$, $x'(a) = x'(b)$, we have

$$\alpha(a) = \alpha(b),\ \alpha'(a) \geq \alpha'(b),$$
$$\beta(a) = \beta(b),\ \beta'(a) \leq \beta'(b).$$

  Similarly,

- for the three-point nonlinear boundary conditions: $x(a) = c$, $x(b) = g(x(\eta))$, where g is continuous and $\eta \in (a, b)$, we have

$$\alpha(a) \leq c,\ \alpha(b) \leq g(\alpha(\eta)),$$
$$\beta(a) \geq c,\ \beta(b) \geq g(\beta(\eta)).$$

- For the nonlinear nonlocal three-point conditions: $x(a) = c$, $\quad x'(b) = g(x(\eta))$, $\quad a < \eta < b$. we have

$$\alpha(a) \leq c,\ \alpha'(b) \leq g(\alpha(\eta)),$$
$$\beta(a) \geq c,\ \beta'(b) \geq g(\beta(\eta)).$$

- For the three point boundary conditions: $x'(0) = 0$, $x(1) = \delta x(\eta)$, we have

$$\alpha'(0) \geq 0,\ \alpha(1) \leq \delta(\alpha(\eta)),$$
$$\beta'(0) \leq 0,\ \beta(1) \geq \delta(\beta(\eta)).$$

- For the four-point boundary conditions: $x(a) = x(c)$, $x(d) = x(b)$, where $a < c \leq d < b$, we have

$$\alpha(a) \leq \alpha(c),\ \alpha(d) \leq \alpha(b),$$
$$\beta(a) \geq \beta(c),\ \beta(d) \geq \beta(b).$$

- For the integral boundary conditions:

$$x(a) - k_1 x'(a) = \int_a^b h_1(x(s))d(s),\ x(b) + k_2 x'(b) = \int_a^b h_2(x(s))d(s),$$

where k_1, $k_2 \in \mathbb{R}$ and h_1, $h_2 \in C(I)$ are nonlinear, we have

$$\alpha(a) - k_1\alpha'(a) \leq \int_a^b h_1(x(s))d(s),\ \alpha(b) + k_2\alpha'(b) \leq \int_a^b h_2(x(s))d(s),$$
$$\beta(a) - k_1\beta'(a) \geq \int_a^b h_1(x(s))d(s),\ \beta(b) + k_2\beta'(b) \geq \int_a^b h_2(x(s))d(s).$$

A fundamental result concerning the upper and lower solutions for the problem (1.1.1) with Dirichlet boundary conditions, is the following.

Theorem 1.1.2. *Assume that α, $\beta \in C^2(I)$ are lower and upper solutions of* (1.1.1). *If $f : I\times\mathbb{R}^2 \to \mathbb{R}$ is continuous and $f(t,x,x')$ is strictly increasing in x for each $(t,x') \in I\times\mathbb{R}$, then $\alpha(t) \leq \beta(t)$ on I. In particular, there is at most one solution.*

Proof. Define $w(t) = \alpha(t) - \beta(t)$, $t \in I$, then $w \in C^2(I)$ and

$$w(a) \leq 0,\ w(b) \leq 0. \tag{1.1.3}$$

Suppose $w(t)$ has a positive maximum at some $t_0 \in I$. The boundary conditions (1.1.3) implies that $t_0 \in (a,b)$ and hence

$$w(t_0) > 0,\ w'(t_0) = 0 \text{ and } w''(t_0) \leq 0.$$

However, using the increasing property of f in x, we obtain

$$w''(t_0) = \alpha''(t_0) - \beta''(t_0) \geq f(t_0,\alpha(t_0),\beta'(t_0)) - f(t_0,\beta(t_0),\beta'(t_0)) > 0,$$

a contradiction. Hence $\alpha(t) \leq \beta(t)$ on I. □

We need the following theorem (theorem 1.1.3 [11]) for our later work.

Theorem 1.1.3. *Let $f \in C(I \times \mathbb{R} \times \mathbb{R})$ be bounded on $I \times \mathbb{R} \times \mathbb{R}$. Then the boundary value problem* (1.1.1) *with $x(a) = 0 = x(b)$, has a solution.*

Proof. Let M be the bound of f on $I \times \mathbb{R} \times \mathbb{R}$. Define a mapping $T : E \to E$ by

$$Tx(t) = \int_a^b G(t,s)f(s,x(s),x'(s))ds,$$

where the Banach space $E = C^1(I)$ with the norm

$$\|x\|_E = \max_{t\in I}|x(t)| + \max_{t\in I}|x'(t)|,$$

and $G(t,s)$ is the Green's function defined by

$$G(t,s) = \frac{-1}{b-a}\begin{cases}(b-t)(s-a), & \text{if } a \le s \le t \le b,\\ (b-s)(t-a), & \text{if } a \le t \le s \le b.\end{cases}$$

We note that the derivative $G_t(t,s)$ has a jump discontinuity $G_t(s+,s) - G_t(s-,s) = 1$. It follows that $|G_t(t,s)| \le 1$ on $I \times I$. Letting

$$N = \max_{s,t\in I} |G(t,s)(b-a)|,\ N_1 = \max_{s,t\in I} |G_t(t,s)(b-a)|,$$

then

$$|(Tx)(t)| \le NM,\ |(Tx)'(t)| \le N_1 M.$$

Hence, T maps the closed, bounded, and convex set

$$B_0 = \{x \in E : |x(t)| \le NM,\ |x'(t)| \le N_1 M\}$$

into itself. Moreover, since $|(Tx)''(t)| \le M$, using the relation

$$(Tx)'(t) = (Tx)'(a) + \int_a^t (Tx)''(\tau)d\tau$$

we have

$$|(Tx)'(t) - (Tx)'(s)| = \int_s^t (Tx)''(\tau)d\tau \le M|t-s|,$$

for any $s,\ t \in I\ (s \le t)$. Thus $\{(Tx)\}$ is equicontinuous and hence T is completely continuous by Ascoli's theorem. The Schauder's fixed point theorem then yields the fixed point of T which is a solution of the boundary value problem. □

Employing the notion of upper and lower solutions, let us first define a modification of the nonlinear function f (see for example [11]).

Definition 1.1.4. **[Modified function]** Let $\alpha,\ \beta \in C^2(I)$ be lower and upper solutions of (1.1.1) with $\alpha \le \beta$ on I and let $C > 0$ be such that $C > \max\{|\alpha'(t)|,\ |\beta'(t)| : t \in I\}$. Define

$$F^*(t,x,x') = \begin{cases} f(t,x,C), & \text{for } x' > C,\\ f(t,x,x'), & \text{for } |x'| \le C,\\ f(t,x,-C), & \text{for } x' < -C,\end{cases}$$

and

$$F(t,x,x') = \begin{cases} F^*(t,\beta(t),x') + \frac{x-\beta(t)}{1+x^2}, & \text{for } x > \beta(t),\\ F^*(t,x,x'), & \text{for } \alpha(t) \le x \le \beta(t),\\ F^*(t,\alpha(t),x') + \frac{x-\alpha(t)}{1+x^2}, & \text{for } x < \alpha(t).\end{cases} \quad (1.1.4)$$

The function $F(t,x,x')$ is called a modification of $f(t,x,x')$ associated with the triple $\alpha,\ \beta,\ C$.

It follows from the definition that $F(t,x,x')$ is continuous and bounded on $I \times \mathbb{R}^2$ and that

$$|F(t,x,x')| \leq M \text{ on } I \times \mathbb{R}^2,$$

with $M = M_0 + 1$ where

$$M_0 = \max\{|f(t,x,x')| : t \in I,\ \alpha(t) \leq x \leq \beta(t),\ |x'(t)| \leq C\} + \max_{t\in I}|\alpha(t)| + \max_{t\in I}|\beta(t)|.$$

Here we remark that we will use different modification of f according to different boundary value problems. Now, we consider the modified problem

$$\begin{aligned} x''(t) &= F(t,x,x'),\ t \in I \\ x(a) &= c,\ x(b) = d. \end{aligned} \tag{1.1.5}$$

We show that solutions of the modified problem lie in a region where f is unmodified and hence are solutions of the unmodified problem. Relative to the modified problem (1.1.5), we have the following theorem from [11].

Theorem 1.1.5. *Let α, $\beta \in C^2(I)$ be lower and upper solutions of* (1.1.1) *such that $\alpha \leq \beta$ on I. Then the modified boundary value problem* (1.1.5) *has a solution $x \in C^2(I)$ such that*

$$\alpha(t) \leq x(t) \leq \beta(t) \text{ on } I. \tag{1.1.6}$$

Proof. We write the boundary value problem (1.1.5) as an integral equation

$$x(t) = \frac{1}{b-a}[(bc-ad)+(d-c)t] + \int_a^b G(t,s)F(s,x(s),x'(s))ds, \tag{1.1.7}$$

where $G(t,s)$ is the Green's function and is defined by

$$G(t,s) = \frac{-1}{b-a}\begin{cases} (b-t)(s-a), & \text{if } a \leq s \leq t \leq b, \\ (b-s)(t-a), & \text{if } a \leq t \leq s \leq b. \end{cases}$$

Since F is continuous and bounded on $I \times \mathbb{R}^2$, and also $G(t,s)$ is continuous and bounded on $I \times I$, it follows that the integral equation (1.1.7) has a fixed point. Hence the BVP (1.1.5) has a solution $x \in C^2(I,\mathbb{R})$. Thus we only need to show that (1.1.6) hold. We shall only prove that $x(t) \leq \beta(t)$ on I. The arguments are essentially the same for the case $\alpha(t) \leq x(t)$. Assume, if possible, that $x(t) > \beta(t)$ for some $t \in I$. Then $x(t) - \beta(t)$ has a positive maximum at a point $t_0 \in (a,b)$. Hence it follows that $x'(t_0) = \beta'(t_0)$, $|x'(t_0)| < C$ and

$$\begin{aligned} x''(t_0) &= F(t_0, x(t_0), x'(t_0)) \\ &= f(t_0,\beta(t_0),\beta'(t_0)) + \frac{x(t_0)-\beta(t_0)}{1+x^2(t_0)}. \end{aligned}$$

Since β is an upper solution,

$$\beta''(t_0) \leq f(t_0, \beta(t_0), \beta'(t_0))$$

and therefore, we arrive at

$$x''(t_0) - \beta''(t_0) \geq \frac{x(t_0) - \beta(t_0)}{1 + x^2(t_0)} > 0,$$

which is impossible at a maximum of $x(t) - \beta(t)$. We conclude that $x(t) \leq \beta(t)$ on I. $\square$

We see that existence results for solutions of boundary value problems depend on finding an a priori bounds for the solution and its derivative. Here we present a sufficient conditions for obtaining such bounds.

1.1.1 Nagumo condition

For nonlinear differential equations, it is commonly assumed that the nonlinearity grows not faster than quadratically with respect to the gradient. This is a useful assumption in order to get an a priori bound on the gradient of solution, which is part of most existence proofs for nonlinear second order ordinary differential equations that depend continuously on the first derivative. Consider a quasilinear equation on an interval (a, b)

$$x''(t) = f(t, x, x'), \; t \in I. \tag{1.1.8}$$

The Nagumo condition permits a maximum growth rate of $f(t, x, x')$ with respect to x' which is roughly quadratic. To prove that (1.1.8) together with some boundary conditions has a solution, one needs to estimate x' on I. To this end the following *natural condition* (going back to the work of M. Nagumo [74], and S. Bernstein [12]) is imposed: for all $t \in I$ and $x' \in \mathbb{R}$, we have

$$|f(t, x, x')| \leq c(x)(1 + |x'|^2). \tag{1.1.9}$$

To see that (1.1.9) implies boundness of $|x'|$, assuming boundness of $|x|$. Let t_0 be the extremum point of $x(t)$ and $t_1 \in I$ with $t_0 < t_1$ and $|x(t)| \leq M$, $t \in I$. If $x'(t) \geq 0$ on $[t_0, t_1]$, then in view of (1.1.9), we have

$$\frac{x''(t)x'(t)}{1 + x'^2(t)} \leq c(x(t))x'(t), \; t \in [t_0, t_1],$$

which implies that

$$\frac{d}{dt}\frac{1}{2}\ln(1 + x'^2(t)) - \frac{d}{dt}C(x) \leq 0, \; t \in [t_0, t_1],$$

where $C(x) = \int_0^x c(s)ds$. This means that the function $\frac{\ln(1+x'^2(t))}{2} - C(x)$ is decreasing on $[t_0, t_1]$ and hence

$$\frac{\ln(1 + x'^2)}{2} \leq C(x) - C(x_0), \; t \in [t_0, t_1], \tag{1.1.10}$$

where $x_0 = x(t_0)$. It follows from (1.1.10) that x' is bounded on $[t_0, t_1]$. If $x'(t) \leq 0$ on $[t_2, t_3]$, where $a \leq t_2 \leq t_3 \leq b$. Then, again using (1.1.9), we have

$$x''(t) \geq -c(x(t))(1 + |x'(t)|^2).$$

Hence

$$x''(t)x'(t) \leq -c(x(t))(1 + |x'(t)|^2)x'(t),\ t \in [t_2, t_3].$$

As before, we can show that x' is bounded on $[t_2, t_3]$. Hence x' is bounded on I.
It is important to note that this argument is independent of boundary conditions. If on the other hand, the Nagumo condition is violated, then x' need not be bounded. For example, the problem [58],

$$x'' + (1 + (x')^2)^{\frac{3}{2}} = 0 \text{ on } (0, 2),\ x(0) = 0,\ x(2) = 0$$

has as a solution the upper half of the circle

$$(t-1)^2 + x^2 = 1,$$

with infinite derivatives at $t = 0$ and $t = 2$. Nagumo proved in [75] that the method of upper and lower solutions is not valid for a Dirichlet problem when no Nagumo condition is imposed. This result is extended by Habets and Pouse [33] to the periodic and separated boundary value conditions.
Now, we give a general definition of the Nagumo condition relative to upper and lower solutions.

Definition 1.1.6. Let $f \in C(I \times \mathbb{R}^2, \mathbb{R})$ and $\alpha, \beta \in C(I, \mathbb{R})$ with $\alpha(t) \leq \beta(t)$ on I. Suppose that for $t \in I$, $\alpha(t) \leq x \leq \beta(t)$ and $x' \in \mathbb{R}$,

$$|f(t, x, x')| \leq \omega(|x'|), \tag{1.1.11}$$

where $\omega \in C(\mathbb{R}_+, (0, \infty))$. If

$$\int_\lambda^\infty \frac{s ds}{\omega(s)} = \infty, \tag{1.1.12}$$

where

$$\lambda(b - a) = \max\{|\alpha(a) - \beta(b)|, |\alpha(b) - \beta(a)|\}, \tag{1.1.13}$$

we say that f satisfies a Nagumo condition on I relative to α, β and the function ω is called a Nagumo function [11].

To see that the Nagumo condition implies boundness of the derivative $|x'|$ of a solution x of the differential equation (1.1.8), provided $\alpha(t) \leq x \leq \beta(t)$ on I, we have the following theorem (see [11]).

Theorem 1.1.7. *Assume that f satisfies a Nagumo condition on I with respect to the pair α, β. Then for any solution $x \in C^2(I,\mathbb{R})$ of the differential equation* (1.1.8) *with $\alpha(t) \le x(t) \le \beta(t)$ on I, there exists an $N > 0$ depending only on α, β and ω such that*

$$|x'(t)| \le N \text{ on } I. \tag{1.1.14}$$

Proof. Because of (1.1.12), we can choose an $N > \lambda$ such that

$$\int_\lambda^N \frac{sds}{\omega(s)} > \max_{t\in I}\beta(t) - \min_{t\in I}\alpha(t).$$

If $t_0 \in (a,b)$ is such that $(b-a)x'(t_0) = x(b) - x(a)$, then by (1.1.13), we have $|x'(t_0)| \le \lambda$. Assume that (1.1.14) is not true. Then there exists an interval $[t_1, t_2] \subset I$ such that the following cases hold:

1. $x'(t_1) = \lambda$, $x'(t_2) = N$ and $\lambda < x'(t) < N$ for $t \in (t_1, t_2)$.

2. $x'(t_1) = N$, $x'(t_2) = \lambda$ and $\lambda < x'(t) < N$ for $t \in (t_1, t_2)$.

3. $x'(t_1) = -\lambda$, $x'(t_2) = -N$ and $-N < x'(t) < -\lambda$ for $t \in (t_1, t_2)$.

4. $x'(t_1) = -N$, $x'(t_2) = -\lambda$ and $-N < x'(t) < -\lambda$ for $t \in (t_1, t_2)$.

Let us consider case (1) on $[t_1, t_2]$, using (1.1.11), we obtain

$$|x''(t)|x'(t) = |f(t,x,x')|x'(t) \le \omega(x')x'(t)$$

and as a result

$$\begin{aligned}\left|\int_{t_1}^{t_2} \frac{x''(r)x'(r)dr}{\omega(x'(r))}\right| &\le \int_{t_1}^{t_2} \frac{|x''(r)|x'(r)dr}{\omega(x'(r))}\\ &\le \int_{t_1}^{t_2} x'(r)dr = x(t_2) - x(t_1) < \max_{t\in I}\beta(t) - \min_{t\in I}\alpha(t).\end{aligned}$$

This contradicts the fact that $\int_\lambda^N \frac{sds}{\omega(s)} > \max_{t\in I}\beta(t) - \min_{t\in I}\alpha(t)$.
Now, we consider case 2. Again using (1.1.11), we have

$$\frac{|x''(t)|x'(t)}{h(x'(t))} \le x'(t),\ t \in [t_1, t_2].$$

Hence

$$\left|\int_{t_1}^{t_2} \frac{x''(t)x'(t)}{h(x'(t))}dt\right| \le \int_{t_1}^{t_2} \frac{|x''(t)|x'(t)}{h(x'(t))}dt \le \int_{t_1}^{t_2} x'(t)ds,$$

which implies that

$$\left|\int_N^\lambda \frac{sds}{h(s)}\right| \le x(t_2) - x(t_1) < \max_{t\in I}\beta(t) - \min_{t\in I}\alpha(t),$$

again leads to a contradiction.
Now we consider the case 3. Using (1.1.11), we obtain

$$|x''(t)|x'(t) = |f(t,x,x')|x'(t) \ge \omega(|x'|)x'(t),\ t \in [t_1, t_2],$$

which implies that

$$\int_{-\lambda}^{-N} \frac{|x''(r)|x'(r)dr}{\omega(|x'(r)|)} \geq \int_{t_1}^{t_2} x'(r)dr.$$

The change of variable $x' = -s$, yield

$$\int_{\lambda}^{N} \frac{sds}{\omega(s)} \leq -\int_{t_1}^{t_2} x'(r)dr = x(t_1) - x(t_2) < \max_{t\in I} \beta(t) - \min_{t\in I} \alpha(t),$$

a contradiction. We can deal with the case 4 in a similar way and therefore we conclude that (1.1.14) is valid. □

We note that the above result is independent of boundary conditions. The following results are useful for our later work and for detail see [11].

Corollary 1.1.8. *Let $\omega \in C(\mathbb{R}_+, (0,\infty))$, $f \in C(I \times \mathbb{R}^2, \mathbb{R})$ and for $t \in I$, $|x| \leq M$ and $x' \in \mathbb{R}$, $|f(t,x,x')| \leq \omega(|x'|)$. Assume that*

$$\int_{\lambda}^{\infty} \frac{sds}{\omega(s)} = \infty.$$

Then, for any solution $x \in C^2(I,\mathbb{R})$ with $|x(t)| \leq M$, there exists an $N > 0$ depending only on M, ω, $(b-a)$ such that

$$|x'(t)| \leq N \text{ on } I.$$

Also, $N \to 0$ as $M \to 0$.

Now, we use the method of upper and lower solutions to prove a theorem on existence of solution of the boundary value problem

$$\begin{aligned} x''(t) &= f(t,x,x'),\ t \in I \\ x(a) &= c,\ x(b) = d. \end{aligned} \tag{1.1.15}$$

Theorem 1.1.9. *Let α, $\beta \in C^2(I, \mathbb{R})$ be respectively, lower and upper solutions of the boundary value problem (1.1.15) such that $\alpha(t) \leq \beta(t)$ on I. Suppose further that f satisfies a Nagumo condition on I relative to the pair α, β. Then the boundary value problem (1.1.15) has a solution $x \in C^2(I)$ such that $\alpha(t) \leq x(t) \leq \beta(t)$ and $|x'(t)| \leq N$ on I, where N depends only on α, β and the Nagumo function ω.*

Proof. By Theorem 1.1.7, there is $N > 0$ depending on α, β, ω such that $|x'(t)| \leq N$ on I for any solution x with $\alpha(t) \leq x(t) \leq \beta(t)$ on I. Choose $C_1 > N$ so that

$$|\alpha'(t)| < C_1,\ |\beta'(t)| < C_1 \text{ on } I.$$

Then, by Theorem 1.1.7, the boundary value problem

$$\begin{aligned} x''(t) &= F(t,x,x'),\ t \in (a,b), \\ x(a) &= c,\ x(b) = d. \end{aligned} \tag{1.1.16}$$

has a solution $x \in C^2(I, \mathbb{R})$ such that $\alpha(t) \leq x(t) \leq \beta(t)$ on I, where $F(t,x,x')$ is the modification of $f(t,x,x')$ with respect to α, β and C_1 as given in (1.1.4). By the mean value theorem, there is $t_0 \in (a,b)$ such that

$$(b-a)x'(t_0) = x(b) - x(a),$$

and using (1.1.13) it follows that

$$|x'(t_0)| \leq \lambda < N < C_1.$$

This implies that there is an interval containing t_0, where $x(t)$ is a solution of

$$x''(t) = f(t,x,x').$$

By Theorem 1.1.7, we have $|x'(t)| \leq N < C_1$ on this interval. However, $x(t)$ is a solution of $x''(t) = f(t,x,x')$ as long as $|x'(t)| < C_1$. Since $\alpha(t) \leq x(t) \leq \beta(t)$ on I and f satisfies a Nagumo condition, We conclude by Theorem 1.1.7 that $x(t)$ is a solution (1.1.15) on I. □

The Nagumo condition permits a maximum growth rate of $f(t,x,x')$ with respect to x' which is roughly quadratic. In one problem we will replace the Nagumo condition by more specialist conditions to allow higher growth rate of f with respect to x', see Chapter 6.

We recall some definitions which we use in our later work.

Definition 1.1.10. [Uniform convergence] A sequence of real valued functions $\{f_n\}$ defined on a set $A \subseteq \mathbb{R}$ is said to converge uniformly to a function f defined on A if for each $\epsilon > 0$, there exists a natural number $m(\epsilon)$ such that if $n \geq m$, then

$$|f_n(x) - f(x)| < \epsilon \text{ for all } x \in A.$$

Definition 1.1.11. [Quadratic convergence] Let X be a normed space with the maximum norm $\|.\|$ and $\{f_n\}$ be a sequence in X. Suppose that $\{f_n\}$ converge to $f \in X$. We say that the convergence is quadratic if there exists a natural number m and $k > 0$ ($k \in \mathbb{R}$) such that

$$\|f_{n+1} - f\| \leq k\|f_n - f\|^2 \text{ for } n \geq m.$$

Definition 1.1.12. [Higher order convergence] Let X be a normed space with norm $\|.\|$ and $\{f_n\}$ be a sequence in X. Suppose that $\{f_n\}$ converge to $f \in X$. We say that the convergence is of order $p \in \mathbb{N}$, if there exists a natural number m and $k > 0$ ($k \in \mathbb{R}$) such that

$$\|f_{n+1} - f\| \leq k\|f_n - f\|^p, \text{ for } n \geq m.$$

1.2 Quasilinearization

Now, we study approximation of solutions by the method of quasilinearization. The basic idea of the original method of quasilinearization developed by Bellman and Kalaba [9,10] is to provide an explicit analytic representation for a solution of nonlinear differential equations, which yields point-wise lower estimates for the solution of the problem whenever the function involved is convex. The most important applications of this method has been to obtain a sequence of lower bounds which are solutions of linear differential equations that converge quadratically. As a result, the method has been popular in applied areas. However, the convexity assumption that is demanded by the method of quasilinearization has been a stumbling block for further development of the theory. Recently, this method has been generalized by not demanding the convexity property so as to be applicable to a much larger class of nonlinear problems and make the method more useful in applications (see for example [60]).

To explain the basic idea of the original method of quasilinearization, we consider some two-point boundary value problem of the type [9,10]

$$\begin{aligned} x''(t) &= f(t,x),\ t \in I \\ x(a) &= c,\ x(b) = d. \end{aligned} \tag{1.2.1}$$

Let us begin with the case where we have established by some other means the existence of a unique solution of (1.2.1). We further assume that $f(t,x)$ is convex as a function of x for $t \in I$. Then (1.2.1) may be written

$$x'' = \max_y \left[f(t,y) + f_x(t,y)(x-y)\right],\ t \in I, \tag{1.2.2}$$

hence

$$x'' \geq f(t,y) + f_x(t,y)(x-y),\ t \in I, \tag{1.2.3}$$

for arbitrary y. In particular, we have

$$x'' \geq f(t,x_0) + f_x(t,x_0)(x-x_0),\ t \in I, \tag{1.2.4}$$

where x_0 is an initial approximation for the solution x of (1.2.1), and is specified. For example, x_0 may be taken to be the straight line determined by the two-point boundary condition,

$$x_0 = d + \frac{d-c}{b-a}(t-b).$$

We wish to compare (1.2.4) with the equation for x_1, namely,

$$\begin{aligned} x_1''(t) &= f(t,x_0) + f_x(t,x_0)(x_1-x_0),\ t \in I, \\ x_1(a) &= c,\ x_1(b) = d. \end{aligned} \tag{1.2.5}$$

The comparison hinges upon the properties of the equation

$$w''(t) - f_x(t, x_0)w = g(t),\ t \in I, \qquad (1.2.6)$$
$$w(a) = 0,\ w(b) = 0,$$

since $w = x - x_1$ satisfies (1.2.6) with a positive forcing function. If the associated Green's function is non negative, then

$$x - x_1 > 0.$$

A sufficient condition for the non negativity of the Green's function is that $f_x > 0$ for all $x \in \mathbb{R}$. In general, we require the condition

$$f_x > -\lambda_1,$$

where λ_1 is the smallest characteristic value associated with the Sturm-Liouville equation

$$w''(t) + \lambda w = 0,\ t \in I,$$
$$w(a) = 0,\ w(b) = 0.$$

In this case $\lambda_1 = \frac{\pi^2}{(b-a)^2}$. Returning to (1.2.3), we have

$$x(t) \geq u(t, y),\ t \in I, \qquad (1.2.7)$$

where $u(t, y)$ is a solution of the linear problem

$$x''(t) = f(t, y) + f_x(t, y)(x - y),\ t \in I, \qquad (1.2.8)$$
$$x(a) = c,\ x(b) = d.$$

We note that (1.2.7) is valid for all function $y(t)$ and that the equality holds for the function $y(t) = x(t)$. We can therefore write

$$x(t) = \max_y u(t, y),\ t \in I, \qquad (1.2.9)$$

which provides an explicit analytic representation for the solution of (1.2.1). Moreover, we observe that the function which maximizes (1.2.2), is the solution x itself. This suggests that we employ a method of successive approximation in which $y(t)$ is chosen at each stage to be a lower estimate of the desired solution $x(t)$ of (1.2.1). Continuing in this fashion, we construct the sequence of functions $\{x_n\}$ bounded by $x(t)$ and defined by

$$x''_{n+1}(t) = f(t, x_n) + f_x(t, x_n)(x_{n+1} - x_n),\ t \in I,$$
$$x_{n+1}(a) = c,\ x_{n+1}(b) = d.$$

Let us now establish the fundamental result that the sequence thus generated is monotone increasing,

$$x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n \leq x.$$

We note, using (1.2.3), that $w = x_{n+1} - x_n$, satisfies (1.2.6) with a positive forcing function and hence $x_{n+1} \geq x_n$. It follows from the monotonicity and the uniform boundedness that the sequence $\{x_n(t)\}$ converges for $t \in I$.

Finally, to establish the desired quadratic convergence, we write, using the mean value theorem,

$$x''(t) = f(t, x_n) + f_x(t, x_n)(x - x_n) + f_{xx}(t, \theta)\frac{(x - x_n)^2}{2}, \; t \in I,$$

where θ lies between x and x_n. Hence using the equation for x_{n+1} in terms of x_n, we have

$$(x - x_{n+1})'' = f_x(t, x_n)(x - x_{n+1}) + f_{xx}(t, \theta)\frac{(x - x_n)^2}{2}, \; t \in I.$$

Regarding this as a linear equation for $x - x_{n+1}$ with the forcing term $f_{xx}(t, \theta)\frac{(x-x_n)^2}{2}$, it follows that

$$\max_{t \in I} |x - x_{n+1}| \leq k_1 \max_{t \in I} |x - x_n|^2,$$

where k_1 is a constant.

Thus, we conclude that the quasilinearization technique provides a sequence of function which converges to a solution of a nonlinear differential equation. The essential property of this sequence of solutions is that each of its term satisfies a linear differential equation. Secondly, the sequence of approximate solutions possesses the important properties of monotonicity and quadratic convergence. The most exploited property of the quasilinearization technique is its ability to solve nonlinear differential equations as a sequence of linear ones.

1.2.1 Generalized quasilinearization

In this section, we explain the generalized method of quasilinearization by not demanding $f(t, x)$ to be convex but impose a less restrictive assumption, namely, $f(t, x) + \phi(t, x)$ is convex for some convex function $\phi(t, x)$. In [63], the authors studied the generalized method of quasilinearization and obtained sequences of approximate solutions converging quadratically to a solution of an initial value problem for a first order differential equation of the type

$$x' = f(t, x), \; x(0) = x_0, \; t \in I = [0, T], \tag{1.2.10}$$

where $f \in C(I \times \mathbb{R}, \mathbb{R})$. Let $D = \{(t, x) : \alpha(t) \leq x(t) \leq \beta(t), \; t \in I\}$. The main result is the following theorem.

Theorem 1.2.1. *Assume that*

(A_1) α, $\beta \in C^1(I)$ *are lower and upper solutions of* (1.2.10) *such that* $\alpha(t) \leq \beta(t)$, $t \in I$.

($\mathbf{A_2}$) *$f \in C[D,\mathbb{R}]$, $f_x(t,x)$, $f_{xx}(t,x)$ exist and are continuous and satisfying $f_{xx}(t,x) + \phi_{xx}(t,x) \geq 0$ on D, where $\phi \in C[D,\mathbb{R}]$ and $\phi_x(t,x)$, $\phi_{xx}(t,x)$ exist, are continuous and $\phi_{xx}(t,x) \geq 0$ on D.*

Then, there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ which converge uniformly to the unique solution of the problem and the convergence is quadratic.

(For the proof see [63]).

Recently, J. J. Nieto [76], studied the method of generalized quasilinearization for second order nonlinear boundary value problem of the type

$$\begin{aligned} -x''(t) = f(t,x(t)),\ t \in [0,\pi] = I \\ x(0) = x(\pi) = 0. \end{aligned} \tag{1.2.11}$$

Assuming the existence of a lower solution α and an upper solution β such that $\alpha \leq \beta$ on $[0,\pi]$ and that f satisfies the following conditions

($\mathbf{A_1}$) $\frac{\partial f}{\partial x}(t,x)$, $\frac{\partial^2 f}{\partial x^2}(t,x)$ are continuous for every $(t,x) \in D$,

($\mathbf{A_2}$) $\frac{\partial f}{\partial x}(t,x) < 1$, for every $(t,x) \in D$

($\mathbf{A_3}$) There exists $m > 0$ such that $\frac{\partial^2 f}{\partial x^2}(t,x) \geq -2m$, for every $(t,x) \in D$.

Then there exists a monotone sequence $\{x_n\}$ of approximate solutions which converges uniformly to a solution of (1.2.11) and the convergence is quadratic. Here $x_0 = \alpha$, and the elements x_n of the sequence are solutions of the problems defined by the iterative scheme

$$\begin{aligned} -x''(t) &= f(t,x_{n-1}(t)) + \Big[\frac{\partial f}{\partial x}(t,x_{n-1}(t)) + 2mx_{n-1}(t)\Big](x(t) - x_{n-1}(t)) \\ &\quad - m(x^2(t) - x_{n-1}^2(t)),\ t \in [0,\pi], \\ x(0) &= x(\pi) = 0. \end{aligned}$$

For more details see [76].

1.3 Degree theory

Definition 1.3.1. Let S be a subset of a Banach space. S is compact if every sequence of elements of S has a subsequence convergent to a point of S. S is said to be relatively compact if and only if its closure $\overline{S}$ is compact [39].

Let Ω denote a subset on $\mathbb{R}^n$ and $C(\Omega)$ be the Banach space of continuous functions defined on Ω equipped with the supremum norm.

Definition 1.3.2. A subset S of $C(\Omega)$ is said to be uniformly bounded if there exists a constant $M > 0$ with

$$|f(x)| \leq M \text{ for all } x \in \Omega \text{ and all } f \in S,$$

and equicontinuous if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $x, y \in \Omega$ then $|x - y| < \delta$ implies

$$|f(x) - f(y)| < \epsilon \text{ for all } f \in S.$$

Theorem 1.3.3. *[**The Arzelà-Ascoli Theorem**] Let Ω denote a subset of $\mathbb{R}^n$ and S be a subset of $C(\Omega)$. Then S is relatively compact if and only if it is bounded and equicontinuous [39].*

Definition 1.3.4. [Compact operator] Suppose that S is a subset of a Banach space B. An operator $A : S \to B$ is said to be compact (or Completely continuous) if and only if it is continuous and it maps every bounded subset of S into a relatively compact set [39].

Now, we recall the concept of Brouwer degree for continuous mappings and state some results that we will use later. Let Ω be a bounded open set in $\mathbb{R}^n$. For each continuous map $f : \overline{\Omega} \to \mathbb{R}^n$ and $y \notin f(\partial\Omega)$ we can define an integer $\deg(f, \Omega, y)$ which, roughly speaking, corresponds to the number of solutions $x \in \Omega$ of the equation $f(x) = y$. If f is a smooth function and y is not a critical value for f, the degree is given by the simple formula

$$\deg(f, \Omega, y) = \Sigma_{x \in f^{-1}(y)} J_f(x),$$

where $J_f(x) = \det f'(x)$. When y is a critical value we can define the degree by approximation (see [65] for details). In general for a continuous function the Brouwer degree is constructed via approximation with a smooth function g. Let $g \in C^1(\overline{\Omega})$ be such that

$$\|f(x) - g(x)\| < \operatorname{dist}(y, f(\partial\Omega)),$$

where $\partial\Omega$ denotes the boundary of Ω. We define the degree of f by setting

$$\deg(f, \Omega, y) = \deg(g, \Omega, y).$$

It can be shown that this definition does not depend on the choice of the function g, (again [65] is a good reference).

Definition 1.3.5. [Homotopy] Let ϕ, $\psi \in C(\overline{\Omega})$ and $H : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$. We say that H is a homotopy between ϕ, ψ, if H is continuous on $\overline{\Omega} \times [0, 1]$, $H(x, 0) = \phi(x)$ and $H(x, 1) = \psi(x)$ for every $x \in \overline{\Omega}$.

We are now able to state some properties of the Brouwer degree.

Theorem 1.3.6. *Let Ω be an open bounded set in $\mathbb{R}^n$, $f \in C(\bar{\Omega})$ and $y \notin f(\partial\Omega)$. Then the Brouwer degree has the following properties [18]*

(d_1) *[**Normalization**]* $\deg(I, \Omega, y) = 1$ *for $y \in \Omega$, where I denotes the identity mapping of $\mathbb{R}^n$.*

($\mathbf{d_2}$) *[**Additivity**]* $\deg(f,\Omega,y) = \deg(f,\Omega_1,y) + \deg(f,\Omega_2,y)$, *where* Ω_1 *and* Ω_2 *are disjoint open subsets of* Ω *such that* $y \notin f(\overline{\Omega} \setminus \Omega_1 \cup \Omega_2)$.

($\mathbf{d_3}$) *[**Homotopy**]* $\deg(h(t,.),\Omega,y(t))$ *is independent of* t, *whenever* $h : [0,1] \times \Omega \to \mathbb{R}^n$ *and* $y : [0,1] \to \mathbb{R}^n$ *are continuous and* $y(t) \notin h(t,\partial\Omega)$ *for every* $t \in [0,1]$.

($\mathbf{d_4}$) *[**Existence**]* $\deg(f,\Omega,y) \neq 0$ *implies* $f^{-1}(y) \neq \emptyset$.

($\mathbf{d_5}$) $\deg(.,\Omega,y)$ *is constant on* $\{g \in C(\bar{\Omega}) : \|g - f\| < r\}$, *where* $r = \text{dist}(y, f(\partial\Omega))$.

($\mathbf{d_6}$) $\deg(f,\Omega,.)$ *is constant on every component of* $\mathbb{R}^n \setminus f(\partial\Omega)$.

($\mathbf{d_7}$) *[**Boundary dependence**]* $\deg(f,\Omega,y) = \deg(g,\Omega,y)$ *whenever* $f_{|\partial\Omega} = g_{|\partial\Omega}$.

($\mathbf{d_8}$) *[**Excision Property**]* $\deg(f,\Omega,y) = \deg(f,\Omega_1,y)$ *for every open* $\Omega_1 \subset \Omega$ *such that* $y \notin f(\bar{\Omega} \setminus \Omega_1)$.

1.3.1 The Leray-Schauder degree

We recall that a nonlinear map $f : X \to Y$ is said to be compact if f maps bounded sets into relatively compact sets in Y. The Leray-Schauder degree is an extension of the Brouwer degree to the case of infinite dimensional spaces, in the particular case of maps of the form $T = I - C$, where I is the identity and C is a compact map. The key theorem used in order to define the Leray-Schauder degree is the following [65].

Theorem 1.3.7. *Let* $\Omega \subset X$ *be a bounded open set and* $C : \overline{\Omega} \to Y$ *be compact. Given* $\epsilon > 0$, *there exists a continuous map* $C_\epsilon : \overline{\Omega} \to Y$, *whose range* $C_\epsilon(\overline{\Omega})$ *is finite dimensional such that, for every* $x \in \overline{\Omega}$

$$\|C(x) - C_\epsilon(x)\| < \epsilon.$$

By virtue of Theorem 1.3.7, we can define the Leray-Schauder degree for a map of the type $T = I - C$ by using Brouwer degree. Indeed let $\tilde{T} = I - C_\epsilon$, where C_ϵ is a continuous map on $\overline{\Omega}$ with finite dimension range such that

$$\sup \|C_\epsilon(x) - C(x)\| < \text{dist}(y, T(\partial\Omega)) = \epsilon$$

and $\tilde{\Omega}$ be the finite dimensional subspace of X which contains y and $C_\epsilon(\overline{\Omega})$. Then we can set

$$\deg_{LS}(T,\Omega,y) = \deg(\tilde{T},\tilde{\Omega},y).$$

In [65], it is shown that $\deg_{LS}(T,\Omega,y)$ does not depend on the particular C_ϵ chosen to approximate C. We are ready to state the main properties of the Leray-Schauder degree [18,65].

Theorem 1.3.8. *Let $\Omega \subset X$ be a bounded open set in X. Let $T = I - C : \overline{\Omega} \to X$ be such that $C : \overline{\Omega} \to X$ is compact and $y \notin T(\partial\Omega)$. Then the Leray-Schauder degree $\deg_{LS}(T, \Omega, y)$ is well defined and inherits the properties $(d_1) - (d_8)$ of the Brouwer degree (Theorem 1.3.6).*

Theorem 1.3.9. ***[The Schauder-Fixed point Theorem]*** *Let K be a non-empty closed bounded convex subset of the Banach space B, and suppose that $A : K \to B$ is compact and maps K into itself. Then A has a fixed point in K .*

Proof. Since K is bounded, there is $\rho > 0$ such that $K \subseteq B_\rho(0)$, where $B_\rho(0)$ is ball of radius ρ centered at 0. There exists a continuous retraction $R : B \to K$, with $R(x) = x$ for $x \in K$. Consider $\bar{A} = A \circ R \in C(\overline{B_\rho(0)}, \overline{B_\rho(0)})$. The compact homotopy $H(t,x) = t\bar{A}(x)$, $t \in [0,1]$, shows that

$$\deg(I - \bar{A}, B_\rho(0), 0) = \deg(I, B_\rho(0), 0) = 1.$$

Hence there is a point $x_0 = \bar{A}(x_0) \in K$. Since $\bar{A}(x_0) = A(x_0)$ for $x_0 \in K$. Hence A has a fixed point in K. □

Chapter 2

Periodic nonlinear problems with an application to a nonlinear biomathematical model of blood flow in intracranial aneurysms.

In this chapter, we study a nonlinear second order ordinary differential equation with periodic boundary conditions. We show the validity of the classical upper and lower solution method and of the monotone iterative technique [59] and present a new version related to [91]. This provides estimates for the solution and a numerical procedure to approximate the solution. Then we develop the quasilinearization technique [60] to obtain monotone sequences of approximate solutions converging quadratically to a solution of the nonlinear problems. We improve previous results where the nonlinearity did not depend on the derivative [50,72]. To show the applicability of our techniques we apply the theoretical results to a medical problem: a model of blood flow inside an intracranial aneurysm [77–79], which is a joint work of the author and Prof. J. J. Nieto and Angela Torres and is submitted for publication [51].

2.1 Maximum and antimaximum principles

Consider the nonlinear periodic boundary value problem (PBVP)

$$\begin{aligned} -x''(t) &= f(t,x,x'),\ t \in [0,T], \\ x(0) &= x(T),\, x'(0) = x'(T), \end{aligned} \tag{2.1.1}$$

where $f \in C([0,T] \times \mathbb{R} \times \mathbb{R})$. We know that the linear homogeneous problem

$$-x''(t) + \lambda x(t) = 0,\ t \in [0,T],$$
$$x(0) = x(T),\ x'(0) = x'(T),$$

has only a trivial solution if $\lambda \neq -\frac{4n^2\pi^2}{T^2}$, $n \in \mathbb{Z}$. Consequently, for $\lambda \neq -\frac{4n^2\pi^2}{T^2}$ and any $\sigma \in C[0,T]$, the nonhomogeneous problem

$$\begin{aligned} -x''(t) + \lambda x(t) &= \sigma(t),\ t \in [0,T], \\ x(0) = x(T),&\ x'(0) = x'(T), \end{aligned} \tag{2.1.2}$$

has a unique solution

$$x(t) = \int_0^T G_\lambda(t,s)\sigma(s)ds,$$

where $G_\lambda(t,s)$ is the Green's function, and for $\lambda > 0$,

$$G_\lambda(t,s) = \frac{1}{2\sqrt{\lambda}\sinh\sqrt{\lambda}\frac{T}{2}} \begin{cases} \cosh\sqrt{\lambda}(\frac{T}{2} + (t-s)), & \text{if } 0 \le t < s \le T \\ \cosh\sqrt{\lambda}(\frac{T}{2} + (s-t)), & \text{if } 0 \le s < t \le T, \end{cases}$$

and for $\lambda < 0$,

$$G_\lambda(t,s) = \frac{-1}{2\sqrt{|\lambda|}\sin\sqrt{|\lambda|}\frac{T}{2}} \begin{cases} \cos\sqrt{|\lambda|}(\frac{T}{2} + (t-s)), & \text{if } 0 \le t < s \le T \\ \cos\sqrt{|\lambda|}(\frac{T}{2} + (s-t)), & \text{if } 0 \le s < t \le T. \end{cases}$$

We note that if $\lambda > 0$, then $G_\lambda(t,s) > 0$ and if $\frac{-\pi^2}{T^2} \le \lambda < 0$, then $G_\lambda(t,s) < 0$ on $(0,T) \times (0,T)$. Thus, we have the following maximum and anti-maximum principles.

Maximum principle 2.1.1. If $\lambda > 0$, $\sigma \ge 0$, then the solution x of (2.1.2) is such that $x \ge 0$ on $[0,T]$.

Anti-maximum principle 2.1.2. *If $\frac{-\pi^2}{T^2} \le \lambda < 0$ and $\sigma \ge 0$, then the solution x of (2.1.2) is such that $x \le 0$ on $[0,T]$. On the other hand, if $\sigma \le 0$, then $x \ge 0$ on $[0,T]$.*

2.2 Upper and lower solutions method

In this section, we study existence results of the BVP (2.1.1), using the method of upper and lower solutions. We show that in the presence of lower and upper solutions, there exists a unique solution of the BVP (2.1.1). We recall the concept of lower and upper solution for the PBVP (2.1.1).

Definition 2.2.1. Let $\alpha \in C^2[0,T]$. We say that α is a lower solution of (2.1.1) if

$$-\alpha''(t) \le f(t, \alpha(t), \alpha'(t)), \quad t \in [0,T]$$
$$\alpha(0) = \alpha(T),\ \alpha'(0) \ge \alpha'(T).$$

An upper solution β of the PBVP (2.1.1) is defined similarly by reversing the inequalities.

Now, we study existence results in the form of the following theorems.

Theorem 2.2.2. *Assume that* α, β *are lower and upper solutions of the boundary value problem* (2.1.1). *If* $f \in C([0,T] \times \mathbb{R} \times \mathbb{R})$ *and is strictly decreasing in* x *for each* $(t, x') \in [0,T] \times \mathbb{R}$, *then* $\alpha(t) \le \beta(t)$ *for every* $t \in [0,T]$.

Proof. Define $w(t) = \alpha(t) - \beta(t)$, $t \in [0,T]$. Using the boundary conditions, we obtain

$$w(0) = w(T), \tag{2.2.1}$$

$$w'(0) \ge w'(T). \tag{2.2.2}$$

We claim that $w(t) \le 0$ for every $t \in [0,T]$. If not, then $w(t)$ has a positive maximum at some $t_0 \in [0,T]$. If $t_0 = 0$ or T, then $w(0) = w(T)$ is a positive maximum so that

$$w(0) > 0,\ w'(0) \le 0,\ w(T) > 0,\ w'(T) \ge 0. \tag{2.2.3}$$

The boundary conditions (2.2.2) and (2.2.3) imply that

$$w'(0) = 0,\ w'(T) = 0. \tag{2.2.4}$$

Now, using (2.2.4) and the decreasing property of $f(t, x, x')$ in x, we obtain

$$w''(0) = \alpha''(0) - \beta''(0) \ge -f(0, \alpha(0), \alpha'(0)) + f(0, \beta(0), \alpha'(0)) > 0,$$

which implies that the function w' is strictly increasing in some interval $(0,\ \delta)$ and hence

$$w'(t) > w'(0) = 0,\ 0 < t < \delta.$$

This implies that w is strictly increasing on $(0,\ \delta)$ and hence $w(t) > w(0)$, a contradiction. Hence $t_0 \in (0,T)$. Then, $w(t_0) > 0$, $w'(t_0) = 0$ and $w''(t_0) \le 0$. The definition of upper and lower solutions and the decreasing property of the function f in x gives

$$-w''(t_0) = -\alpha''(t_0) + \beta''(t_0) \le f(t_0, \alpha(t_0), \alpha'(t_0)) - f(t_0, \beta(t_0), \alpha'(t_0)) < 0,$$

a contradiction. □

Corollary 2.2.3. *[Uniqueness] Under the conditions of Theorem 2.2.2, the (PBVP)* (2.1.1) *has at most one solution.*

Theorem 2.2.4. *Assume that* α, $\beta \in C^2[0,T]$ *are lower and upper solutions of* (2.1.1) *respectively such that* $\alpha < \beta$ *on* $[0,T]$. *If* $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and satisfies a Nagumo condition, then there exists a solution* x *of the boundary value problem* (2.1.1) *such that*

$$\alpha(t) \le x(t) \le \beta(t),\ t \in [0,T].$$

Proof. Let $r = \max_{t\in[0,T]}\beta(t) - \min_{t\in[0,T]}\alpha(t)$, then there exists $N > 0$, such that

$$\int_0^N \frac{sds}{\omega(s)} > r.$$

Choose $C \geq \max\{N, \|\alpha'\|, \|\beta'\|\}$ and define $q(y) = \max\{-C, \min\{y, C\}\}$. Then $q(y) = y$ for $|y| \leq C$ and $\text{sgn}(q(y)) = \text{sgn}(y)$. Moreover,

$$\int_0^C \frac{sds}{\omega(q(s))} = \int_0^C \frac{sds}{\omega(s)} \geq \int_0^N \frac{sds}{\omega(s)} > r. \tag{2.2.5}$$

Let $n \in \mathbb{N}$ and consider the modified problem

$$\begin{aligned} -x''(t) &= f_n(t, x, x'),\ t \in [0, T], \\ x(0) &= x(T),\ x'(0) = x'(T), \end{aligned} \tag{2.2.6}$$

where,

$$f_n(t,x,x') = \begin{cases} f(t,\beta(t),\beta'(t)), & \text{if } x \geq \beta(t) + \frac{1}{n}, \\ f(t,\beta(t),q(x')) + \big[f(t,\beta(t),\beta'(t)) - \\ \quad f(t,\beta(t),q(x'))\big]n(x-\beta(t)), & \text{if } \beta(t) < x < \beta(t) + \frac{1}{n}, \\ f(t,x,q(x')), & \text{if } \alpha(t) \leq x \leq \beta(t), \\ f(t,\alpha(t),q(x')) - \big[f(t,\alpha(t),\alpha'(t)) - \\ \quad f(t,\alpha(t),q(x'))\big]n(x-\alpha(t)), & \text{if } \alpha(t) - \frac{1}{n} < x < \alpha(t), \\ f(t,\alpha(t),\alpha'(t)), & \text{if } x \leq \alpha(t) - \frac{1}{n}. \end{cases}$$

We note that $f_n(t, x, x')$ is continuous and bounded on $[0, T] \times \mathbb{R}^2$. Moreover, any solution x of (2.2.6) which satisfies the relations $\alpha(t) \leq x(t) \leq \beta(t)$ and $|x'(t)| \leq C$ on $[0, T]$, is a solution of (2.1.1). For $t \in [0, T]$ and $x \in \mathbb{R}$, define

$$p(\alpha, x, \beta) = \max\{\alpha(t),\ \min\{x, \beta(t)\}\}.$$

Consider the system

$$\begin{aligned} -x''(t) &= sf_n(t, x, x') + (1-s)(\rho_n(t, x) - \lambda x),\ t \in [0, T] \\ x(0) &= x(T),\ x'(0) = x'(T), \end{aligned} \tag{2.2.7}$$

where $s \in [0, 1]$, $\lambda > 0$ and

$$\begin{aligned} \rho_n(t,x) = \frac{1}{\beta(t)-\alpha(t)}\big[&(p(\alpha(t),x,\beta(t)) - \alpha(t))(f(t,\beta(t),\beta'(t)) + \lambda(\beta(t)+1/n)) + \\ &(\beta(t) - p(\alpha(t),x,\beta(t)))(f(t,\alpha(t),\alpha'(t)) + \lambda(\alpha(t)-1/n))\big],\ t \in [0,T]. \end{aligned}$$

For $s = 0$, the system reduces to

$$\begin{aligned} -x''(t) + \lambda x(t) &= \rho_n(t, x),\ t \in [0, T], \\ x(0) &= x(T),\ x'(0) = x'(T), \end{aligned} \tag{2.2.8}$$

and for $s = 1$, it is (2.2.6). That is, (2.2.7) has a solution for $s = 0$. Now, for $s \in [0,1]$, we claim that any solution x_n of (2.2.7) satisfies

$$\alpha(t) - \frac{1}{n} \leq x_n(t) \leq \beta(t) + \frac{1}{n},\ t \in [0,T]. \tag{2.2.9}$$

Once this is shown we can apply Schauder's fixed point theorem to conclude that (2.2.6) has a solution. To verify (2.2.9), we set $v_n(t) = x_n(t) - \beta(t) - 1/n$, $t \in [0,T]$. Then the boundary conditions imply that

$$v_n(0) = v_n(T) \text{ and } v_n'(0) \geq v_n'(T). \tag{2.2.10}$$

Assume that $\max\{v_n(t) : t \in [0,T]\} = v_n(t_0) > 0$. If $t_0 = 0$ or T, then we have

$$v_n(0) > 0,\ v_n'(0) \leq 0 \text{ and } v_n(T) > 0,\ v_n'(T) \geq 0. \tag{2.2.11}$$

From the boundary conditions (2.2.10) and (2.2.11), we obtain $v_n'(0) = 0$ and $v_n'(T) = 0$. There exists $t_1 \in (0,T)$ such that $v_n(t) \geq 0$, $v_n'(t) \leq 0$ on $[0,t_1]$. For every $t \in [0,t_1]$, we have

$$\begin{aligned}-v_n''(t) &= -x_n''(t) + \beta''(t) \leq sf(t,\beta(t),\beta'(t)) + (1-s)\Big[f(t,\beta(t),\beta'(t)) \\ &\quad + \lambda(\beta(t)+1/n) - \lambda x_n(t)\Big] - f(t,\beta(t),\beta'(t)) = -\lambda(1-s)v_n(t) < 0.\end{aligned}$$

This implies that $v_n'(t)$ is strictly increasing on $[0,t_1)$ and hence $v_n'(t) > v_n'(0) = 0$ on $[0,t_1)$, a contradiction. It follows that $t_0 \in (0,T)$ and hence $v_n(t_0) > 0$, $v_n'(t_0) = 0$ and $v_n''(t_0) \leq 0$. However,

$$\begin{aligned}-v_n''(t_0) &= -\,x_n''(t_0) + \beta''(t_0) \leq sf(t_0,\beta(t_0),\beta'(t_0)) + (1-s)\Big[f(t_0,\beta(t_0),\beta'(t_0)) \\ &\quad + \lambda(\beta(t_0)+1/n) - \lambda x_n(t_0)\Big] - f(t_0,\beta(t_0),\beta'(t_0)) = -\lambda(1-s)v_n(t_0) < 0,\end{aligned}$$

again a contradiction. Hence $x_n(t) \leq \beta(t) + \frac{1}{n}$, $t \in [0,T]$. Similarly, we can show that $x_n(t) \geq \alpha(t) - 1/n$, $t \in [0,T]$.

The sequence $\{x_n\}$ of solutions of (2.2.6) is bounded and equicontinuous in $C^1[0,T]$ since f_n are bounded independently of n. Hence the Arzelà-Ascoli theorem guarantees the existence of a subsequence converging in $C^1[0,T]$ to a function $x \in C^1[0,T]$. Since (2.2.9) holds for every $n \in \mathbb{N}$ and every $t \in [0,T]$, it follows that

$$\alpha(t) \leq x(t) \leq \beta(t),\ t \in [0,T].$$

It remains to show that $|x'(t)| \leq C$ on $[0,T]$. The boundary condition, $x(0) = x(T)$ implies that there exists $t^* \in (0,T)$ such that $x'(t^*) = 0$. Suppose that there exists $t_0 \in [0,T]$ such that $x'(t_0) \geq C$. Let $[t^*,t_2] \subset [0,T]$ be the maximal interval containing t_0 such that $x'(t) \geq 0$ on $[t^*,t_2]$. Let $\max\{x'(t) : t \in [t^*,t_2]\} = x'(t^{**}) = \widehat{C}$, then $t^{**} \neq t^*$ and $\widehat{C} \geq C$. It follows that

$$\int_0^{\widehat{C}} \frac{sds}{\omega(q(s))} \geq \int_0^{C} \frac{sds}{\omega(q(s))} > r. \tag{2.2.12}$$

Now, for each $t \in [t^*, t_2]$, since $x \in [\min \alpha(t), \max \beta(t)]$ and $x' \geq 0$, we have

$$|-x''(t)| = |f(t,x,q(x'))| \leq \omega(q(x')).$$

It follows that

$$\frac{x'(t)|x''(t)|}{\omega(q(x'))} \leq x'(t).$$

Integrating from t^* to t^{**}, we obtain

$$\int_0^{\widehat{C}} \frac{sds}{\omega(q(s))} \leq x(t^{**}) - x(t^*) < \max_{t\in[0,T]} \beta(t) - \min_{t\in[0,T]} \alpha(t) < r,$$

a contradiction. Similarly, we can show that $x'(t) > -C$, $t \in [0,T]$.
Hence $|x'(t)| < C$, $t \in [0,T]$. □

2.3 Quasilinearization technique

In this section, we approximate our problem by the method of quasilinearization. We prove that under suitable conditions on the function f, there exists a monotone sequence of solutions of linear problems which converges to a solution of the nonlinear problem (2.1.1) and that the rate of convergence is quadratic.

Theorem 2.3.1. *Assume that*

($\mathbf{A_1}$) *α and $\beta \in C^2[0,T]$ are lower and upper solutions of* (2.1.1) *such that $\alpha < \beta$ on* $[0,T]$.

($\mathbf{A_2}$) *$f \in C^2([0,T] \times \mathbb{R}^2)$ and satisfies $f_x(t,x,x') \leq -\lambda$, for some $\lambda > 0$. Moreover, we assume that $H(f) \geq 0$ on $[0,T] \times [\min \alpha(t), \max \beta(t)] \times [-C, C]$, where*

$$H(f) = (x-y)^2 f_{xx}(t,z_1,z_2) + 2(x-y)(x'-y')f_{xx'}(t,z_1,z_2) + (x'-y')^2 f_{x'x'}(t,z_1,z_2)$$

is the quadratic form of f with z_1 between x, y, and z_2 lies between x' and y'.

($\mathbf{A_3}$) *For $(t,x) \in [0,T] \times [\min \alpha(t), \max \beta(t)]$, $f_{x'}(t,x,x')$ satisfies*

$$|f_{x'}(t,x,y_1) - f_{x'}(t,x,y_2)| \leq L|y_1 - y_2|,\ y_1, y_2 \in \mathbb{R},$$
$$x' f_{x'}(t,x,x') \geq 0 \text{ for } |x'| \geq C,$$

where $L > 0$ and C is as defined in Theorem 2.2.4.

Then, there exists a monotone sequence $\{w_n\}$ of solutions of linear problems converging uniformly and quadratically to a solution of the problem (2.1.1).

Proof. Let

$$S = \{(t,x,x') \in [0,T] \times \mathbb{R}^2 : (t,x,x') \in [0,T] \times [\min \alpha(t), \max \beta(t)] \times \mathbb{R}\}$$

and assume that

$$N = \max \{|f_{xx}(t,x,q(x'))|, |f_{xx'}(t,x,q(x'))|, |f_{x'x'}(t,x,q(x'))| : (t,x,x') \in S\}.$$

Then

$$|H(f)| \le N\|x-y\|_1^2 \text{ on } [0,T] \times [\min \alpha(t), \max \beta(t)] \times [-C,C], \tag{2.3.1}$$

where $\|x-y\|_1 = \|x-y\| + \|(x-y)'\|$ is the usual C^1 norm. Consider the boundary value problem

$$\begin{aligned} -x''(t) &= f(t,x,q(x')),\ t \in [0,T] \\ x(0) &= x(T),\ x'(0) = x'(T). \end{aligned} \tag{2.3.2}$$

We note that any solution $x \in C^2[0,T]$ of (2.3.2) with $\alpha(t) \le x \le \beta(t)$ is such that

$$|x'(t)| \le C \text{ on } [0,T],$$

and hence is a solution of (2.1.1). Therefore it suffices to study (2.3.2). Expanding $f(t,x,q(x'))$ about $(t,y,q(y')) \in S$ by Taylor's theorem and using (A_2), we have

$$f(t,x,q(x')) \ge f(t,y,q(y')) + f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(q(x') - q(y')), \tag{2.3.3}$$

for $(t,x,x') \in S$. Define the function

$$F(t,x,x';y,y') = f(t,y,q(y')) + f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))[q(x') - q(y')], \tag{2.3.4}$$

where (t,x,x'), $(t,y,y') \in [0,T] \times \mathbb{R}^2$. Then F is continuous and bounded on S and therefore satisfies a Nagumo condition on $[0,T]$ relative to the pair α, β. Hence there exists a constant $C_1 > 0$ such that any solution x of the problem

$$\begin{aligned} -x''(t) + \lambda x(t) &= F(t,x,x';y,y') + \lambda p(y,x,\beta),\ t \in [0,T],\ \lambda > 0 \\ x(0) &= x(T),\ x'(0) = x'(T), \end{aligned}$$

with $\alpha(t) \le x \le \beta(t)$ satisfies $|x'(t)| \le C_1$ on $[0,T]$, where

$$p(y,x,\beta) = \max\{y, \min\{x, \beta(t)\}\}.$$

Moreover, $F_x = f_x(t,y,q(y')) \le -\lambda < 0$ and we have the following relations

$$\begin{cases} f(t,x,q(x')) \ge F(t,x,x';y,y') \\ f(t,x,q(x')) = F(t,x,x';x,x'), \end{cases} \tag{2.3.5}$$

for (t,x,x'), $(t,y,y') \in S$.

Now, we set $w_0 = \alpha$ and consider the linear problem

$$\begin{aligned} -x''(t) + \lambda x(t) &= F(t, x, x'; w_0, w_0') + \lambda p(w_0, x, \beta), \ t \in [0, T], \ \lambda > 0 \\ x(0) &= x(T), \ x'(0) = x'(T). \end{aligned} \tag{2.3.6}$$

This is equivalent to the integral equation

$$x(t) = \int_0^1 G_\lambda(t, s)\left[F(s, x, x'; w_0, w_0') + \lambda p(w_0, x, \beta)\right] ds.$$

Since $F(t, x, x'; w_0, w_0') + \lambda p(w_0, x, \beta)$ is continuous and bounded on S, this integral equation has a fixed point (using again Schauder's fixed point theorem). Now, using (A_1) and (2.3.5), we obtain

$$\begin{aligned} -w_0''(t) + \lambda w_0(t) &\leq f(t, w_0(t), w_0'(t)) + \lambda w_0(t) \\ &= F(t, w_0(t), w_0'(t); w_0(t), w_0'(t)) + \lambda p(w_0(t), w_0(t), \beta(t)), \ t \in [0, T], \end{aligned}$$

$$\begin{aligned} -\beta''(t) + \lambda \beta(t) &\geq f(t, \beta(t), \beta'(t)) + \lambda \beta(t) \\ &\geq F(t, \beta(t), \beta'(t); w_0(t), w_0'(t)) + \lambda p(w_0(t), \beta(t), \beta(t)), \ t \in [0, T], \end{aligned}$$

which imply that w_0 and β are lower and upper solution of (2.3.6). Hence, by Theorems 2.2.2, 2.2.4, there exists a unique solution w_1 of (2.3.6) such that $w_0(t) \leq w_1(t) \leq \beta(t)$, $|w_1'(t)| < C_1$, $t \in [0, T]$. In view of (2.3.5) and the fact that w_1 is a solution of (2.3.6), we have

$$-w_1''(t) = F(t, w_1(t), w_1'(t); w_0(t), w_0'(t)) \leq f(t, w_1(t), q(w_1'(t))), \ t \in [0, T], \tag{2.3.7}$$

which implies that w_1 is a lower solution of (2.3.2). Now, consider the problem

$$\begin{aligned} -x''(t) + \lambda x(t) &= F(t, x, x'; w_1, w_1') + \lambda p(w_1, x, \beta), \ t \in [0, T] \\ x(0) &= x(T), \ x'(0) = x'(T). \end{aligned} \tag{2.3.8}$$

In view of (A_1), (2.3.5) and (2.3.7), we can show that w_1 and β are lower and upper solutions of (2.3.8) and hence by Theorems 2.2.2, 2.2.4, there exists a unique solution w_2 of (2.3.8) such that $w_1(t) \leq w_2(t) \leq \beta(t)$, $|w_2'(t)| < C_1$, $t \in [0, T]$. Moreover w_2 is a lower solution of (2.1.1).

Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0 \leq w_1 \leq w_2 \leq w_3 \leq \ldots \leq w_{n-1} \leq w_n \leq \beta, \ t \in [0, T].$$

That is,

$$\alpha(t) \leq w_n(t) \leq \beta(t), \ |w_n'(t)| < C_1, \ n \in \mathbb{N}, \ t \in [0, T], \tag{2.3.9}$$

where w_n is a solution of

$$\begin{aligned} -w_n''(t) &= F(t, w_n, w_n'; w_{n-1}, w_{n-1}'), \ t \in [0, T] \\ w_n(0) &= w_n(T), \ w_n'(0) = w_n'(T). \end{aligned} \tag{2.3.10}$$

Since $F(t, w_n, w_n'; w_{n-1}, w_{n-1}')$ is bounded, there exists $R > 0$ such that

$$|F(t, w_n, w_n'; w_{n-1}, w_{n-1}')| \leq R,\ n \in \mathbb{N},\ t \in [0, T].$$

Using the relation $w_n'(t) = w_n'(0) + \int_0^t w_n''(u)du$, we have

$$|w_n'(t) - w_n'(s)| \leq \int_s^t |F(u, w_n, w_n'; w_{n-1}, w_{n-1}')|du \leq R|t-s|, \tag{2.3.11}$$

for t, $s \in [0, T]$. From (2.3.9) and (2.3.11), it follows that the sequences

$$\{w_n^{(j)}(t)\}, (j = 0, 1),\ n \in \mathbb{N},$$

are uniformly bounded and equicontinuous on $[0, T]$. The Arzelà- Ascoli theorem guarantees the existence of subsequences converging uniformly to $x^{(j)}(j = 0, 1) \in C^1[0, T]$. Consequently, $F(t, w_n, w_n'; w_{n-1}, w_{n-1}') + \lambda p(w_{n-1}, w_n, \beta) \to f(t, x, q(x')) + \lambda x$ on $[0, T]$ as $n \to \infty$ which implies that x is a solution of (2.1.1).

Now, we show that the convergence is quadratic. For this, we set

$$v_n(t) = x(t) - w_n(t),\ t \in [0, T],\ n \in \mathbb{N},$$

where x is a solution of (2.1.1). Then, $v_n \in C^2[0, T]$, $v_n(t) \geq 0$, $n \in \mathbb{N}$, $t \in [0, T]$ and satisfies the boundary conditions

$$v_n(0) = v_n(T),\ v_n'(0) = v_n'(T).$$

The boundary condition $v_n(0) = v_n(T)$ implies the existence of $t_1 \in (0, T)$ such that $v_n'(t_1) = 0$. Now, in view of (2.3.4), we obtain

$$\begin{aligned} -v_n''(t) &= -x''(t) + w_n''(t) = f(t, x, x') - F(t, w_n, w_n'; w_{n-1}, w_{n-1}') \\ &= f_x(t, w_{n-1}, q(w_{n-1}'))v_n + f_{x'}(t, w_{n-1}, q(w_{n-1}')(x' - q(w_n')) + \frac{1}{2}|H(f)|, \end{aligned} \tag{2.3.12}$$

where

$$\begin{aligned} H(f) = (x - w_{n-1})^2 f_{xx}(t, c_1, c_2) + 2(x - w_{n-1})(x' - q(w_{n-1}'))f_{xx'}(t, c_1, c_2) \\ +(x' - q(w_{n-1}'))^2 f_{x'x'}(t, c_1, c_2), \end{aligned}$$

$w_{n-1}(t) \leq c_1 \leq x(t)$ and c_2 lies between $q(w_{n-1}'(t))$ and $x'(t)$. Thus, $v_n(t)$ satisfies the boundary value problem

$$\begin{aligned} -v_n''(t) + \lambda v_n(t) &= [f_x(t, w_{n-1}, q(w_{n-1}')) + \lambda]v_n(t) + f_{x'}(t, w_{n-1}, q(w_{n-1}'))(x' - q(w_n')) \\ &\quad + \frac{1}{2}|H(f)|, \\ v_n(0) &= v_n(T),\ v_n'(0) = v_n'(T). \end{aligned}$$

This is equivalent to

$$0 \le v_n(t) = \int_0^T G_\lambda(t,s)\Big([f_x(s, w_{n-1}, q(w'_{n-1})) + \lambda]v_n(s) + f_{x'}(s, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) + \frac{1}{2}|H(f)|\Big)ds,$$

which in view of $(A_2)(f_x(t,x,x') + \lambda \le 0)$ and (2.3.1) implies that

$$\begin{aligned} v_n(t) &\le \int_0^T G_\lambda(t,s)\big(f_{x'}(s, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) + \frac{N}{2}\|v_{n-1}\|_1^2\big)ds \\ &= \int_0^T G_\lambda(t,s)\big[f_{x'}(s, w_{n-1}, q(w'_{n-1}))v'_n + f_{x'}(s, w_{n-1}, q(w'_{n-1}))(w'_n - q(w'_n)) \\ &+ \frac{N}{2}\|v_{n-1}\|_1^2\big]ds. \end{aligned} \tag{2.3.13}$$

Now, using (2.3.5), we have

$$-v''_n(t) = f(t,x,x') - F(t, w_n, w'_n; w_{n-1}, w'_{n-1}) \ge f(t,x,x') - f(t, w_n, q(w'_n)),\ t \in [0,T]. \tag{2.3.14}$$

The condition $x'f_{x'}(t,x,x') \ge 0$ for $|x'| \ge C$, implies that

$$f(t, w_n(t), q(w'_n(t))) \le f(t, w_n(t), w'_n(t)),\ t \in [0,T],$$

so that (2.3.14) can be rewritten as

$$\begin{aligned} -v''_n(t) &\ge f(t,x,x') - f(t, w_n(t), w'_n(t)) = f_x(t, d_1, d_2)v_n(t) + f_{x'}(t, d_1, d_2)v'_n(t) \\ &\ge f_x(t, d_1, d_2)v_{n-1}(t) + f_{x'}(t, d_1, d_2)v'_n(t),\ t \in [0,T], \end{aligned} \tag{2.3.15}$$

where $w_n(t) \le d_1 \le x(t)$ and d_2 lies between $x'(t)$ and $w'_n(t)$. Let $\mu(t) = e^{\int_0^t f_{x'}(s,d_1,d_2)ds}$ and $-l_1 \le f_{x'}(t, d_1, d_2) \le L_1$ on $[0,T] \times [\min w_0(t), \max \beta(t)] \times [-C_1, C_1]$, where $l_1,\ L_1 > 0$. Then

$$e^{-l_1 t} \le \mu(t) \le e^{L_1 t},\ t \in [0,T]. \tag{2.3.16}$$

Multiplying (2.3.15) by $\mu(t)$ and using (2.3.16), we obtain

$$\big(v'_n(t)\mu(t)\big)' \le \lambda\|v_{n-1}\|\mu(t) \le \lambda\|v_{n-1}\|e^{L_1 t},\ t \in [0,T], \tag{2.3.17}$$

where $\lambda = \max\{|f_x(t,x,x')| : t \in [0,T],\ x \in [\min w_0(t), \max \beta(t)],\ x' \in [-C_1, C_1]\}$. Thus,

$$\big(v'_n(t)\mu(t) - \lambda\|v_{n-1}\|\frac{e^{L_1 t}}{L_1}\big)' \le 0,\ t \in [0,T]. \tag{2.3.18}$$

This implies that the function $\psi(t) = v'_n(t)\mu(t) - \lambda\|v_{n-1}\|\frac{e^{L_1 t}}{L_1}$, is non-increasing in $t \in [0,T]$. Hence $\psi(0) \ge \psi(t_1) \ge \psi(T)$, which yields

$$v'_n(0) - \frac{\lambda}{L_1}\|v_{n-1}\| \ge -\frac{\lambda e^{L_1 t_1}}{L_1}\|v_{n-1}\| \ge v'_n(T)\mu_1(T) - \frac{\lambda e^{L_1 T}}{L_1}\|v_{n-1}\|.$$

Using the boundary conditions $v_n'(0) = v_n'(T)$, we obtain

$$v_n'(0) = v_n'(T) \leq \frac{\lambda}{\mu(T)L_1}(e^{L_1T} - e^{L_1t_1})\|v_{n-1}\| \leq \frac{\lambda}{\mu(T)L_1}(e^{L_1T} - 1)\|v_{n-1}\|, \tag{2.3.19}$$

$$v_n'(T) = v_n'(0) \geq -\frac{\lambda}{L_1}(e^{L_1t_1} - 1)\|v_{n-1}\| \geq -\frac{\lambda}{L_1}(e^{L_1T} - 1)\|v_{n-1}\|. \tag{2.3.20}$$

Now the relation $\psi(0) \geq \psi(t)$, $t \in [0,T]$, together with (2.3.19), implies

$$v_n'(t) \leq \frac{1}{\mu(t)}\Big[\frac{\lambda}{L_1}(\frac{e^{L_1T}-1}{\mu(T)} - 1 + e^{L_1t})\|v_{n-1}\|\Big] \leq q_1\|v_{n-1}\|, \tag{2.3.21}$$

where $q_1 = \max\{\frac{\lambda}{L_1\mu(t)}(\frac{e^{L_1T}-1}{\mu(T)} - 1 + e^{L_1t}) : t \in [0,T]\}$. The relation $\psi(t) \geq \psi(T)$, $t \in [0,T]$, together with (2.3.20), implies

$$v_n'(t) \geq -\frac{\lambda}{\mu(t)L_1}\Big[\mu(T)(e^{L_1T} - 1) + e^{L_1T} - e^{L_1t}\Big]\|v_{n-1}\| \geq -q_2\|v_{n-1}\|, \tag{2.3.22}$$

where $q_2 = \max\{\frac{\lambda}{\mu(t)L_1}[\mu(T)(e^{L_1T} - 1) + e^{L_1T} - e^{L_1t}] : t \in [0,T]\}$. From (2.3.21) and (2.3.22), it follows that

$$|v_n'(t)| \leq Q\|v_{n-1}\|,\ t \in [0,T], \tag{2.3.23}$$

where $Q = \max\{q_1, q_2\}$. We discuss three cases.

1. If for some $t \in [0,T]$, $w_n'(t) > C$, then

$$q(w_n'(t)) = C,\ 0 < w_n'(t) - q(w_n'(t)) \leq w_n'(t) - x'(t)$$

and by (A_3), we obtain

$$\begin{aligned} f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) \leq & f_{x'}(t, w_{n-1}(t), q(w_n'(t))) + L|q(w_{n-1}'(t)) - q(w_n'(t))| \\ & \leq f_{x'}(t, w_{n-1}(t), q(w_n'(t))) + L(|v_n'(t)| + |v_{n-1}'(t)|). \end{aligned}$$

Hence using (2.3.23), we obtain

$$\begin{aligned} (w_n'(t) - q(w_n'(t)))f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) \leq & -v_n'(t)f_{x'}(t, w_{n-1}(t), q(w_n'(t))) + \\ & L|v_n'(t)|(|v_n'(t)| + |v_{n-1}'(t)|) \\ \leq & -v_n'(t)f_{x'}(t, w_{n-1}(t), q(w_n'(t))) + LQ(1+Q)\|v_{n-1}\|_1^2. \end{aligned}$$

2. If for some $t \in [0,T]$, $w_n'(t) < -C$, then

$$q(w_n'(t)) = -C,\ 0 > w_n'(t) - q(w_n'(t)) \geq w_n'(t) - x'(t)$$

and by (A_3), we obtain

$$\begin{aligned} f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) \geq & f_{x'}(t, w_{n-1}(t), q(w_n'(t))) - L|q(w_{n-1}'(t)) - q(w_n'(t))| \\ & \geq f_{x'}(t, w_{n-1}(t), q(w_n'(t))) - L(|v_n'(t)| + |v_{n-1}'(t)|), \end{aligned}$$

hence

$$\begin{aligned} (w_n'(t) - q(w_n'(t)))f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) \leq & -v_n'(t)f_{x'}(t, w_{n-1}(t), q(w_n'(t))) + \\ & LQ(1+Q)\|v_{n-1}\|_1^2. \end{aligned}$$

3. If for some $t \in [0,T]$, $|w_n'(t)| \leq C$, then $q(w_n'(t)) = w_n'(t)$, $w_n'(t) - q(w_n'(t)) = 0$ and by the same process, we can show that

$$(w_n'(t) - q(w_n'(t)))f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) = 0 \leq -v_n'(t)f_{x'}(t, w_{n-1}(t), q(w_n'(t))) + LQ(1+Q)\|v_{n-1}\|_1^2.$$

Thus, for every $t \in [0,T]$, we have

$$f_{x'}(t, w_{n-1}, q(w_{n-1}'))(w_n' - q(w_n')) \leq -v_n'(t)f_{x'}(t, w_{n-1}(t), q(w_n'(t))) + LQ(1+Q)\|v_{n-1}\|_1^2. \tag{2.3.24}$$

Using (2.3.24) in (2.3.13), we obtain

$$\begin{aligned} v_n(t) \leq & \int_0^T G_\lambda(t,s)\big[\big(f_{x'}(s, w_{n-1}(s), q(w_{n-1}'(s))) - f_{x'}(s, w_{n-1}(s), q(w_n'(s)))\big)v_n'(s) \\ & + (LQ(1+Q) + \frac{N}{2})\|v_{n-1}\|_1^2\big]ds \\ = & \int_0^T G_\lambda(t,s)\big[\big(f_{x'}(s, w_{n-1}(s), q(w_{n-1}'(s))) - f_{x'}(s, w_{n-1}(s), q(w_n'(s)))\big)v_n'(s) \\ & + S_1\|v_{n-1}\|_1^2\big]ds, \end{aligned} \tag{2.3.25}$$

where $S_1 = LQ(1+Q) + \frac{N}{2}$.
Again, using (2.3.24) and (2.3.1) in (2.3.12), we obtain

$$-v_n''(t) \leq \big(f_{x'}(t, w_{n-1}, q(w_{n-1}')) - f_{x'}(t, w_{n-1}, q(w_n'))\big)v_n'(t) + S_1\|v_{n-1}\|_1^2, \ t \in [0,T] \tag{2.3.26}$$

which implies that

$$v_n''(t) + \big(f_{x'}(t, w_{n-1}, q(w_{n-1}')) - f_{x'}(t, w_{n-1}, q(w_n'))\big)v_n'(t) \geq -S_1\|v_{n-1}\|_1^2, \ t \in [0,T]. \tag{2.3.27}$$

Since $(t, w_{n-1}, q(w_{n-1}')) \in [0,T] \times [\min w_0(t), \max \beta(t)] \times [-C, C]$, and $f_{x'}$ is continuous, there exist L_2, $l_2 > 0$ such that

$$-l_2 \leq \big(f_{x'}(t, w_{n-1}, q(w_{n-1}')) - f_{x'}(t, w_{n-1}, q(w_n'))\big) \leq L_2, \ t \in [0,T].$$

Then the integrating factor $\mu_1(t) = e^{\int_0^t \left(f_{x'}(s, w_{n-1}, q(w_{n-1}')) - f_{x'}(s, w_{n-1}, q(w_n'))\right)ds}$ satisfies

$$e^{-l_2 t} \leq \mu_1(t) \leq e^{L_2 t}, \ t \in [0,T]. \tag{2.3.28}$$

Thus,

$$\big(v_n'(t)\mu_1(t)\big)' \geq -S_1 e^{L_2 t}\|v_{n-1}\|_1^2. \tag{2.3.29}$$

Integrating (2.3.29) from 0 to t_1, using (2.3.28) and the boundary conditions $v_n'(0) = v_n'(T)$, we obtain

$$v_n'(T) = v_n'(0) \leq \frac{S_1}{L_2}(e^{L_2 t_1} - 1)\|v_{n-1}\|_1^2 \leq \frac{S_1}{L_2}(e^{L_2 T} - 1)\|v_{n-1}\|_1^2. \tag{2.3.30}$$

Integrating (2.3.29) from t to T, using (2.3.28) and (2.3.30), we obtain

$$\begin{aligned} v_n'(t)\mu_1(t) &\leq v_n'(T)\mu_1(T) + \frac{4S_1}{L_2}(e^{L_2T} - e^{L_2t})\|v_{n-1}\|_1^2 \\ &\leq \frac{S_1}{L_2}\Big[\mu_1(T)(e^{L_2T} - 1) + (e^{L_2T} - e^{L_2t})\Big]\|v_{n-1}\|_1^2, \end{aligned}$$

which implies that

$$v_n'(t) \leq \frac{S_1e^{l_2t}}{L_2}\Big[\mu_1(T)(e^{L_2T} - 1) + (e^{L_2T} - e^{L_2t})\Big]\|v_{n-1}\|_1^2 \leq \delta_1\|v_{n-1}\|_1^2,\ t \in [0, T], \tag{2.3.31}$$

where

$$\delta_1 = \max_{[0,T]} \frac{S_1e^{l_2t}}{L_2}\Big[\mu_1(T)(e^{L_2T} - 1) + (e^{L_2T} - e^{L_2t})\Big].$$

Again, integrating (2.3.29) from t_1 to T, using (2.3.28) and the boundary conditions $v_n'(0) = v_n'(T)$, we have

$$v_n'(0) = v_n'(T) \geq \frac{-S_1(e^{L_2T} - 1)}{L_2\mu_1(T)}\|v_{n-1}\|_1^2. \tag{2.3.32}$$

If we integrate (2.3.29) from 0 to t, use (2.3.28) and (2.3.32), we obtain

$$v_n'(t)\mu_1(t) \geq \frac{-S_1}{L_2}\Big[\frac{(e^{L_2T} - 1)}{\mu_1(T)} + (e^{L_2t} - 1)\Big]\|v_{n-1}\|_1^2,\ t \in [0, T]$$

which implies that

$$v_n'(t) \geq \frac{-S_1e^{l_2t}}{L_2}\Big[\frac{(e^{L_2T} - 1)}{\mu_1(T)} + (e^{L_2t} - 1)\Big]\|v_{n-1}\|_1^2 \geq -\delta_2\|v_{n-1}\|_1^2,\ t \in [0, T], \tag{2.3.33}$$

where

$$\delta_2 = \max_{[0,T]} \frac{S_1e^{l_2t}}{L_2\mu_1(t)}\Big(\frac{(e^{L_2T} - 1)}{(\mu_1(T) - 1)} + (e^{L_2t} - 1)\Big).$$

From (2.3.31) and (2.3.33), it follows that

$$\|v_n'\| \leq \delta\|v_{n-1}\|_1^2, \quad \delta = \max\{\delta_1, \delta_2\}. \tag{2.3.34}$$

Now, using (2.3.31) in (2.3.25), we have

$$v_n(t) \leq \int_0^T G_\lambda(t, s)(\sigma\delta_1 + S_1)\|v_{n-1}\|_1^2 ds,$$

which implies that

$$\|v_n\| \leq \int_0^T G_\lambda(t, s)(\sigma\delta_1 + S_1)\|v_{n-1}\|_1^2 ds \leq D\|v_{n-1}\|_1^2, \tag{2.3.35}$$

where $\sigma = \max\{L_2, l_2\}$ and $D \geq (\sigma\delta_1 + S_1)\int_0^T G_\lambda(t, s)$. Let $R = \max\{\delta, D\}$, then (2.3.34) and (2.3.35) gives

$$\|v_n\|_1 \leq R\|v_{n-1}\|_1^2.$$

□

If $f_{x'} \equiv 0$, then it reduces to the case when the nonlinearity f is independent of the derivative x'. In this case the norm $\|.\|_1$ reduces to the norm $\|.\|$ and this case is studied in [50, 72]. Therefore we have extended previous results.

2.4 Generalized quasilinearization technique

Now we introduce an auxiliary function ϕ to allow weaker hypothesis on f.

Theorem 2.4.1. *Assume that*

(B_1) $\alpha,\ \beta \in C^2[0,T]$ *are lower and upper solutions of* (5.1.2) *respectively, such that*

$$\alpha(t) < \beta(t) \text{ on } [0,T].$$

(B_2) $f \in C^2([0,T] \times \mathbb{R} \times \mathbb{R})$ *satisfying* $f_x(t,x,x') \leq -\lambda$ *and* $H(f+\phi) \geq 0$ *on* $[0,T] \times [\min \alpha(t), \max \beta(t)] \times [-C,C]$, *for some function* $\phi \in C^2([0,T] \times \mathbb{R} \times \mathbb{R})$ *with the property that* $H(\phi) \geq 0$ *on* $[0,T] \times [\min \alpha(t), \max \beta(t)] \times [-C,C]$.

(B_3) *For* $(t,x) \in [0,T] \times [\min \alpha(t),\ \max \beta(t)]$, $f_{x'}(t,x,x')$ *satisfies*

$$y f_{x'}(t,x,y) \geq 0 \text{ for } |y| \geq C \text{ and}$$
$$|f_{x'}(t,x,y_1) - f_{x'}(t,x,y_2)| \leq L|y_1 - y_2|,\ y_1,\ y_2 \in \mathbb{R},$$

where $L > 0$.

Then, there exists a monotone sequence $\{w_n\}$ *of solutions of linear problems converging uniformly and quadratically to a solution of the problem* (2.1.1).

Proof. Define $F : [0,T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F(t,x,y) = f(t,x,y) + \phi(t,x,y),\ t \in [0,T].$$

Then in view of (B_2), we have $F(t,x,y) \in C^2([0,T] \times \mathbb{R} \times \mathbb{R})$ and

$$H(F) \geq 0 \text{ on } [0,T] \times [\min \alpha(t), \max \beta(t)] \times [-C,C]. \tag{2.4.1}$$

The condition (2.4.1) implies that

$$\begin{aligned} f(t,x,q(x')) \geq F(t,y,q(y')) + F_x(t,y,q(y'))(x-y) + F_{x'}(t,y,q(y'))(q(x')-q(y')) \\ -\phi(t,x,q(x')), \end{aligned} \tag{2.4.2}$$

for (t,x,x'), $(t,y,y') \in S$. Applying Taylor's theorem to the function ϕ about $(t,y,q(y'))$, we have

$$\phi(t,x,q(x')) = \phi(t,y,q(y')) + \phi_x(t,y,q(y'))(x-y) + \phi_{x'}(t,y,q(y')))(q(x')-q(y')) + \frac{1}{2}H(\phi), \tag{2.4.3}$$

where,

$$\begin{aligned} H(\phi) = (x-y)^2\phi_{xx}(t,c_1,c_2) + 2(x-y)(q(x')-q(y'))\phi_{xx'}(t,c_1,c_2) \\ +(q(x')-q(y'))^2\phi_{x'x'}(t,c_1,c_2), \end{aligned}$$

where c_1 lies between x and y and c_2 lies between $q(x')$ and $q(y')$. Let

$$M = \max\left\{|\phi_{xx}(t,x,x')|, |\phi_{xx'}(t,x,x')|, |\phi_{x'x'}(t,x,x')| : t \in [0,T],\ x \in [\min\alpha(t), \max\beta(t)],\ x' \in [-C,C]\right\}.$$

Then

$$|H(\phi)| \le \frac{M}{2}(|x-y| + |q(x') - q(y')|)^2. \tag{2.4.4}$$

In view of (B_2) and (2.4.4), we obtain the following relations

$$\phi(t,x,q(x')) \ge \phi(t,y,q(y')) + \phi_x(t,y,q(y'))(x-y) + \phi_{x'}(t,y,q(y'))(q(x') - q(y')), \tag{2.4.5}$$

$$\begin{aligned}\phi(t,x,q(x')) \le \phi(t,y,q(y')) &+ \phi_x(t,y,q(y'))(x-y) + \phi_{x'}(t,y,q(y'))(q(x') - q(y')) \\ &+\frac{M}{2}(|x-y| + |q(x') - q(y')|)^2,\end{aligned} \tag{2.4.6}$$

for (t,x,x'), $(t,y,y') \in S$. Using (2.4.6) in (2.4.2), we get

$$\begin{aligned}f(t,x,q(x')) \ge f(t,y,q(y')) &+ f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(q(x') - q(y')) \\ &-\frac{M}{2}(|x-y| + |q(x') - q(y')|)^2,\end{aligned} \tag{2.4.7}$$

for (t,x,x'), $(t,y,y') \in S$. Define

$$\begin{aligned}k^*(t,x,y;x',y') = f(t,y,q(y')) &+ f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(q(x') - q(y')) \\ &-\frac{M}{2}(|x-y| + |q(x') - q(y')|)^2,\end{aligned}$$

for $t \in [0,T]$, x, y, x', $y' \in \mathbb{R}$. Then k^* satisfies the following relations

$$\begin{cases} f(t,x,q(x')) \ge k^*(t,x,x';y,y'), \\ f(t,x,q(x')) = k^*(t,x,x';x,x'), \end{cases} \tag{2.4.8}$$

for (t,x,x'), $(t,y,y') \in S$. Moreover, for (t,x,x'), $(t,y,y') \in S$, k^* is continuous and bounded and therefore satisfies a Nagumo condition on $[0,T]$ relative to α, β. Hence there exists a constant $C_2 > 0$ such that any solution $x \in C^2([0,T])$ of the BVP

$$\begin{aligned}-x''(t) + \lambda x(t) &= k^*(t,x,x';y,y') + \lambda p(y,x,\beta),\ t \in [0,T], \\ x(0) &= x(T), \quad x'(0) = x'(T),\end{aligned}$$

with $\alpha(t) \le x(t) \le \beta(t)$, $t \in [0,T]$ satisfies

$$|x'(t)| \le C_2,\ t \in [0,T].$$

Now, set $w_0 = \alpha$ and consider the boundary value problem

$$\begin{aligned}-x''(t) + \lambda x(t) &= k^*(t,x,x';w_0,w_0') + \lambda p(w_0,x,\beta),\ t \in [0,T], \\ x(0) &= x(T), \quad x'(0) = x'(T).\end{aligned} \tag{2.4.9}$$

Using (B_1) and (2.4.8), we have

$$\begin{aligned}-w_0''(t)+\lambda w_0(t) &\le f(t,w_0(t),w_0'(t))+\lambda w_0(t)\\ &= k^*(t,w_0(t),w_0'(t);w_0(t),w_0'(t))+\lambda p(w_0,x,\beta),\ t\in[0,T],\end{aligned}$$

$$\begin{aligned}-\beta''(t)+\lambda\beta(t) &\ge f(t,\beta(t),\beta'(t))+\lambda\beta(t)\\ &\ge k^*(t,\beta(t),\beta'(t);w_0(t),w_0'(t))+\lambda p(w_0,\beta,\beta),\ t\in[0,T],\end{aligned}$$

which imply that w_0 and β are lower and upper solutions of (2.4.9). Hence, by Theorems 2.2.2, 2.2.4, there exists a unique solution w_1 of (2.4.9) such that

$$w_0(t)\le w_1(t)\le\beta(t) \text{ and } |w_1'(t)|\le C_2,\ t\in[0,T].$$

In view of (2.4.8) and the fact that w_1 is a solution of (2.4.9), we can show that w_1 is a lower solution of (2.1.1). Now, by (B_1) and (2.4.8), we can show that w_1 and β are lower and upper solutions of

$$\begin{aligned}-x''(t)+\lambda x(t) &= k^*(t,x,x';w_1,w_1')+\lambda p(w_1,x,\beta),\ t\in[0,T],\\ x(0)&=x(T),\ x'(0)=x'(T).\end{aligned}\tag{2.4.10}$$

Hence by Theorems 2.2.2, 2.2.4, there exists a unique solution w_2 of (2.4.10) such that $w_1\le w_2\le\beta$ and $|w_2'|\le C_2$ on $[0,T]$. Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0\le w_1\le w_2\le w_3\le\ldots\le w_{n-1}\le w_n\le\beta \text{ on } [0,T].$$

By the standard arguments as in the previous section, we can show that the sequence of solutions of the problems converges to the solution of the original nonlinear problem (2.1.1).

For the quadratic convergence, let $v_n(t)=x(t)-w_n(t)$, $t\in[0,T]$, $n\in\mathbb{N}$. Then,

$$-v_n''(t)=-x''(t)+w_n''(t)=(F(t,x,x')-\phi(t,x,x'))-k^*(t,w_n,w_n';w_{n-1},w_{n-1}')\tag{2.4.11}$$

Applying Taylor's theorem on $F(t,x,x')$ about $(t,w_{n-1},q(w_{n-1}'))$, using (2.4.5), and the definition of k^*, we have

$$\begin{aligned}-v_n''(t) &\le f(t,w_{n-1},q(w_{n-1}'))+f_x(t,w_{n-1},q(w_{n-1}'))(x-w_{n-1})+f_{x'}(t,w_{n-1},q(w_{n-1}'))\times\\ &\quad (x'-q(w_{n-1}'))+\frac{1}{2}|H(F)|-[f(t,w_{n-1},q(w_{n-1}'))+f_x(t,w_{n-1},q(w_{n-1}'))(w_n-w_{n-1})\\ &\quad +f_{x'}(t,w_{n-1},q(w_{n-1}'))(q(w_n')-q(w_{n-1}'))-\frac{M}{2}(|w_n-w_{n-1}|+|q(w_n')-q(w_{n-1}')|)^2]\\ &= f_x(t,w_{n-1},q(w_{n-1}'))(x-w_n)+f_{x'}(t,w_{n-1},q(w_{n-1}'))(x'-q(w_n'))+\frac{1}{2}|H(F)|\\ &\quad +\frac{M}{2}(|w_n-w_{n-1}|+|q(w_n')-q(w_{n-1}')|)^2,\end{aligned}\tag{2.4.12}$$

where,

$$H(F) = v_{n-1}^2 F_{xx}(t,\xi_1,\xi_2) + 2v_{n-1}(x' - q(w'_{n-1}))F_{xx'}(t,\xi_1,\xi_2) + (x' - q(w'_{n-1})^2 F_{x'x'}(t,\xi_1,\xi_2)$$

where $w_{n-1}(t) \le \xi_1 \le x(t)$ and ξ_2 lies between $q(w'_{n-1}(t))$ and $x'(t)$. Let

$$\begin{aligned} R_1 = \max\big\{ &|F_{xx}(t,\xi_1,\xi_2)|, |F_{xx'}(t,\xi_1,\xi_2)|, |F_{x'x'}(t,\xi_1,\xi_2)| : \\ &(t,\xi_1,\xi_2) \in [0,T] \times [\min \alpha(t), \max \beta(t)] \times [-C_2, C_2] \big\}, \end{aligned}$$

then, $|H(F)| \le R_1(|v_{n-1}| + |x' - q(w'_{n-1}|)^2 \le R_1 \|v_{n-1}\|_1^2$. Using this and the assumption $f_x \le -\lambda$ in (2.4.12), we obtain

$$\begin{aligned} -v_n''(t) + \lambda v_n(t) \le f_{x'}(t, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) + \frac{R_1}{2}\|v_{n-1}\|_1^2 \\ + \frac{M}{2}(|w_n - w_{n-1}| + |q(w'_n) - q(w'_{n-1})|)^2. \end{aligned} \tag{2.4.13}$$

Using the relation

$$|q(w'_n) - q(w'_{n-1})| \le |x' - q(w'_{n-1})| + |x' - q(w'_n)| \le |v'_{n-1}| + |v'_n|,$$

we obtain

$$\begin{aligned} -v_n''(t) + \lambda v_n(t) \le f_{x'}(t, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) + \frac{R_1}{2}\|v_{n-1}\|_1^2 \\ + \frac{M}{2}(\|v'_{n-1}\|_1 + |v'_n|)^2, \ t \in [0,T], \end{aligned} \tag{2.4.14}$$

which is equivalent to

$$v_n(t) \le \int_0^T G_\lambda(t,s)\big(f_{x'}(s, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) + \frac{R_1}{2}\|v_{n-1}\|_1^2 + \frac{M}{2}(\|v'_{n-1}\|_1 + |v'_n|)^2\big)ds. \tag{2.4.15}$$

By the same process as in Theorem 2.3.1 [(2.3.23)], we can show that

$$|v'_n| \le Q\|v_{n-1}\| \le Q\|v_{n-1}\|_1 \text{ on } [0,T].$$

Hence, we can rewrite (2.4.15) as

$$\begin{aligned} v_n(t) &\le \int_0^T G_\lambda(t,s)\big(f_{x'}(s, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) + T\|v_{n-1}\|_1^2\big)ds \\ &= \int_0^T G_\lambda(t,s)\big[f_{x'}(s, w_{n-1}, q(w'_{n-1}))v'_n + f_{x'}(s, w_{n-1}, q(w'_{n-1}))(w'_n - q(w'_n)) \\ &+ T\|v_{n-1}\|_1^2\big]ds, \end{aligned} \tag{2.4.16}$$

where $T = \frac{R_1}{2} + \frac{M}{2}(1+Q)$. This is similar to that (2.3.13) of Theorem 2.3.1. Hence following the same procedure as in Theorem 2.3.1, we can show that the sequence converges to the solution of (2.1.1) quadratically. □

2.5 Application to a blood flow model

This section is a joint work with Prof. J. J. Nieto and Angela Torres and has been submitted for publication in [51].

Now, we apply our theoretical results to a medical problem, a biomathematical model of blood flow inside an intracranial aneurysm. An aneurysm is a local enlargement of the arterial lumen caused by congenital, traumatic, atherosclerotic or other factors. The natural history of the development of aneurysms consists of three phases: pathogenesis, enlargement and rupture. Aneurysmal subarachnoid hemorrhage is a major clinical problem in the world. The incidence of subarachnoid hemorrhage (SAH) is stable, at around six cases per 100000 patient a year [26]. The cause of SAH is a ruptured aneurysm in 85 percent of cases and SAH accounts for a quarter of cerebrovascular deaths [90] . The developments of the epidemiology and pathogenesis of intracranial aneurysms, methods of diagnosis, and approaches to treatment have been discussed by several authors [71,85,89]. Different sensitive, but non-invasive, imaging strategies for the diagnosis of intracranial aneurysms are now used [93]. For effective treatment of patients with intact incidental aneurysms, it is important to have adequate models in order to understand the evolution of aneurysms and to propose prognostic criteria upon which to make clinical recommendations. Mathematical models are now more relevant in biomedical practice [37] and several biomathematical models of intracranial aneurysms have been proposed in the literature [34,77,78] .

We consider the biomathematical model of blood flow inside an intracranial aneurysm

$$\begin{aligned} &x'' + px' + ax - bx^2 + cx^3 - F\cos(ht) = 0,\ t \in [0,T] \\ &x(0) = x(T),\ x'(0) = x'(T), \end{aligned} \tag{2.5.1}$$

where x represents the velocity of blood flow inside the aneurysm, and p, a, b, c, F, h are positive medical parameters depending on each patient. For example, F is related to the pulse pressure, h is the inverse of the cardiac frequency, p depends on the elastic properties of the aneurysm wall, ... for details, see [38,77,79].

Assume that $b^2 > \frac{16ac}{3}$, take $\lambda = \frac{a}{6} > 0$ and write the model (2.5.1) as

$$-x'' + \lambda x = f(t,x,x') = px' + \rho(x) - F\cos(ht), \tag{2.5.2}$$

where $\rho(x) = \frac{7a}{6}x - bx^2 + cx^3 = v(x) + \frac{a}{6}x$, $v(x) = ax - bx^2 + cx^3$. We note that the equation $\rho(x) = 0$ has three real roots namely, 0, x_1, x_2 with $(0 < x_1 < x_2)$, where $x_1 = \frac{b-(b^2-\frac{14ac}{3})^{\frac{1}{2}}}{2c}$, $x_2 = \frac{b+(b^2-\frac{14ac}{3})^{\frac{1}{2}}}{2c}$.

Clearly, $\rho(x) \geq 0$, for $x \in [0,x_1]$ and $\rho(x) \leq 0$, for $x \in [x_1,x_2]$. Let $\psi(x) = \rho(x) + \lambda x$, then

$$\psi_{\max} = \max \psi(x) = \psi(x_3) > 0,$$

$$\psi_{\min} = \min \psi(x) = \psi(x_4) < 0,$$

where $x_3 = \frac{b-(b^2-4ac)^{\frac{1}{2}}}{3c} \in (0, x_1)$ and $x_4 = \frac{b+(b^2-4ac)^{\frac{1}{2}}}{3c} \in (x_1, x_2)$. Moreover, $\psi_x(x) \leq 0$, for every $x \in [x_3, x_4]$ which implies that $\rho_x(x) \leq -\lambda$ for every $x \in [x_3, x_4]$.
Similarly,

$$v(x) \geq 0 \text{ on } (0, \frac{3}{2}x_3),$$
$$v(x) \leq 0 \text{ on } (\frac{3}{2}x_3, \frac{3}{2}x_4).$$

Since $0 < x_3 < \frac{b-(b^2-4ac)^{\frac{1}{2}}}{2c}$, it follows that $v(x_3) > 0$ which implies that $\rho(x_3) - \lambda x_3 > 0$. Thus,

$$\psi_{\max} - 2\lambda(\frac{b-(b^2-4ac)^{\frac{1}{2}}}{3c}) > 0.$$

Let

$$F = \min\{\psi_{\max} - 2\lambda(\frac{b-(b^2-4ac)^{\frac{1}{2}}}{3c}), |\psi_{\min}| + \lambda(\frac{b+(b^2-4ac)^{\frac{1}{2}}}{3c})\}.$$

Taking $\alpha = x_3$ and $\beta = x_4$, we have $\alpha < \beta$ and

$$\alpha'' - \lambda\alpha + f(t, \alpha, \alpha') = -\lambda x_3 + \rho(x_3) - F\cos(ht) \geq \psi_{\max} - 2\lambda x_3 - F \geq 0,$$

$$\beta'' - \lambda\beta + f(t, \beta, \beta') = -\lambda x_4 + \rho(x_4) - F\cos(ht) \leq -2\lambda x_4 - |\psi_{\min}| + F \leq 0,$$

which imply that α, β are lower and upper solutions of (2.5.1).

Now, for $x \in [x_3, x_4]$, $t \in [0, T]$, we have

$$|f(t, x, x')| \leq p|x'| + K = \omega(|x'|),$$

where $K = \max\{\rho(x) - F\cos(ht) : t \in [0, T], x \in [x_3, x_4]\}$. Moreover,

$$\int_0^\infty \frac{sds}{\omega(s)} = \int_0^\infty \frac{sds}{ps + K} = \infty.$$

Thus the Nagumo condition is satisfied. Hence by Theorem 2.2.4, there exists a solution of (2.5.1) in $[x_3, x_4]$.

Now we approximate the solution of (2.5.1). The approximation scheme is given by solutions of the linear problems

$$-x'' + \lambda x = k(t, x, x'; y, y'),\ t \in [0, T],$$
$$x(0) = x(T),\ x'(0) = x'(T),$$

where

$$\begin{aligned} k(t, x, x'; y, y') &= f(t, y, y') + \rho_x(y)(x - y) + p(x' - y') \\ &= \rho(y) + \rho_x(y)(x - y) + px' - F\cos(ht). \end{aligned}$$

We rewrite

$$-x'' - px' + (\lambda - \rho_x(y))x = \rho(y) - y\rho_x(y) - F\cos(ht)$$
$$x(0) = x(T),\ x'(0) = x'(T),$$

in the equivalent form

$$x(t) = \int_0^T G_\lambda(t,s)\big[\rho(y) - y\rho_x(y) - F\cos(hs)\big]ds.$$

Since $\lambda - \rho_x(y) > 0$ for $y \in [x_3, x_4]$, it follows that the Green's function $G_\lambda(t,s) > 0$ on $(0,T) \times (0,T)$.

The first approximation to the solution is x_3. Taking $w_0 = x_3$, the second approximation is given by

$$w_1(t) = \int_0^T G_\lambda(t,s)(\psi_{\max} - F\cos(hs))ds.$$

In general, the quasilinearization iteration scheme for the solution of (2.5.1) is given by

$$w_n = \int_0^T G_\lambda(t,s)\big(\rho(w_{n-1}) - w_{n-1}\rho_x(w_{n-1}) - F\cos(hs)\big)ds. \tag{2.5.3}$$

To show the sequence of iterates converges quadratically to the solution of the problem (2.5.1), we set $v_n(t) = x(t) - w_n(t)$. By the mean value theorem, (2.5.2) and (2.5.3), we obtain

$$\begin{aligned}
-v_n''(t)+\lambda v_n(t) &= (-x''(t) + \lambda x(t)) \\
&\quad - (-w_n''(t) + \lambda w_n(t)) \\
&= px' + \rho(x) - [\rho(w_{n-1}) + \rho_x(w_{n-1})(w_n - w_{n-1}) + pw_n'] \\
&= pv_n' + \rho_x(w_{n-1})v_n + \frac{1}{2}\rho_{xx}(\xi)v_{n-1}^2 \\
&\le pv_n' + \rho_x(w_{n-1})v_n + d\|v_{n-1}\|^2,
\end{aligned}$$

where d is a bound for $\frac{1}{2}|\rho_{xx}(x)|$. Thus, it follows that

$$-v_n''(t) - pv_n'(t) + (\lambda - \rho_x(w_{n-1})v_n(t) \le d\|v_{n-1}\|^2,$$

which implies that

$$\|v_n\| \le \delta\|v_{n-1}\|^2,$$

where $d\int_0^T |G_\lambda(t,s)|ds \le \delta$. This shows that the iterates converges quadratically to a solution of the problem.

Chapter 3

Three-point nonlinear boundary value problems

In this chapter, we study existence, uniqueness and approximation of solutions for second order nonlinear boundary value problems with nonlinear nonlocal three-point boundary conditions. Firstly, we develop the method of upper and lower solutions to establish existence results and then the method of quasilinearization to approximate our problems. Multi-point boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics [73,88]. The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [40,41] motivated by the work of Bitsadze [13] on nonlocal linear elliptic boundary value problems. Since then, multipoint boundary value problems have been studied by several authors, for example, [3,21,28–30,32,36,42,57,67,70,92].

3.1 Three point nonlinear boundary value problems (I)

In this section, we study existence and approximation of solutions of nonlinear second order differential equations with nonlinear nonlocal boundary conditions (BCs) of the type

$$\begin{aligned} x''(t) &= f(t,x,x'),\ 0<t<1, \\ x(0) &= a, \quad x(1) = g(x(\eta)), \quad 0<\eta<1, \end{aligned} \tag{3.1.1}$$

where g is continuous and nondecreasing. Simpler versions of these boundary conditions are well studied [24,28–31,67,70,92]. In contrast to previous studies, we deal with a more general boundary condition and we also treat the case where the nonlinearity f depends on x'.

Existence is shown via the method of upper and lower solutions. We show that our results hold for a wide range of problems, including problems of the form

$$x'' = |x'|^{p-1}x' - f(x), \quad \text{where } 0<p\leq 2,$$

subject to the well-studied boundary conditions

$$x(0) = a > 0, \quad x(1) = \delta x(\eta), \quad 0 < \delta\eta < 1.$$

For these particular problems we show the existence of a positive solution when f satisfies conditions a little weaker than $0 \leq f(x) \leq \gamma x^p + C$ for some $\gamma < 1$.

The approximation of solutions uses the method of quasilinearization. We approximate our problem by a sequence of *linear* problems to obtain a monotone sequence of approximants. We show that, under suitable conditions, these converge quadratically to a solution of the original problem. In our new result, we control both the function and its first derivative and prove a result on quadratic convergence in the C^1 norm. This is more delicate than the corresponding results when there is no x' dependence in f, for example [3, 21].

Recently, three-point (and certain more general multipoint) boundary value problems (BVPs) of the following type have received considerable attention.

$$x''(t) = f(t, x, x') + e(t), \ t \in J = [0, 1], \tag{3.1.2}$$

$$x'(0) = 0, \quad x(1) = \delta x(\eta), \quad 0 < \eta < 1, \tag{3.1.3}$$

$$x(0) = 0, \quad x(1) = \delta x(\eta), \quad 0 < \eta < 1. \tag{3.1.4}$$

Existence theory in these cases had been given in a number of papers of Gupta et al. [28,30]. Gupta [29] has also studied the resonance case giving existence and uniqueness results for the problem (3.1.2), (3.1.3) with $\delta = 1$ and for the problem (3.1.2), (3.1.4) with $\delta\eta = 1$. Positivity of solutions of the three point boundary value problems with $0 \leq \delta < 1$ for the BC (3.1.3), and with $0 \leq \delta\eta < 1$ for the BC (3.1.4), when f does not depend on x' [except in a trivial manner] was studied by Webb [92], using fixed point index theory, and by Ma [67] and others.

When f does not depend on x', the nonlinear BVP with the nonlinear boundary condition $u'(0) = 0, \ u(1) + B(u'(1)) = 0$ has been studied by Infante [42], by using fixed point index theory. Here $B : \mathbb{R} \to \mathbb{R}$ is an odd continuous function such that there exists a number $\delta > 0$ such that $0 \leq B(v) \leq \delta v$ for every $v \geq 0$, but without any monotonicity assumptions.

Recently, Eloe and Gao [21], Ahmad, Khan and Eloe [3] have developed the quasilinearization method for a three point boundary value problem involving a nonlinear boundary condition

$$\begin{aligned} x''(t) =& f(t, x), \ t \in J = [0, 1], \\ x(0) =& a, \quad x(1) = g(x(1/2)), \end{aligned} \tag{3.1.5}$$

when the nonlinearity f is independent of x'. Our results allows f to depend on x', which leads to extra difficulties, and they apply to more general boundary conditions.

3.1.1 Upper and lower solutions method

This section is a joint work with Prof. J. R. L. Webb and has been published in [53]. Consider the nonlinear problem

$$\begin{aligned} x''(t) &= f(t, x, x'), \ t \in J = [0, 1], \\ x(0) &= a, \quad x(1) = g(x(\eta)), \quad 0 < \eta < 1, \end{aligned} \tag{3.1.6}$$

where f and g are continuous. Let $k(t, s)$ denote the Green's function for the Dirichlet boundary value problem, $x(0) = x(1) = 0$, so that

$$k(t, s) = \begin{cases} -t(1 - s), & 0 \le t < s \le 1, \\ -s(1 - t), & 0 \le s < t \le 1. \end{cases}$$

We note that $k(t, s) < 0$ on $(0, 1) \times (0, 1)$. We seek a solution x via the integral equation

$$x(t) = a(1 - t) + g(x(\eta))t + \int_0^1 k(t, s) f(s, x(s), x'(s))\, ds. \tag{3.1.7}$$

We recall the concept of lower and upper solutions for the boundary value problem (3.1.6).

Definition 3.1.1. Let α, $\beta \in C^2(J)$. We say that α is a lower solution of the BVP (3.1.6) if

$$\alpha''(t) \ge f(t, \alpha(t), \alpha'(t)), \quad t \in J$$
$$\alpha(0) \le a, \quad \alpha(1) \le g(\alpha(\eta)).$$

An upper solution β of the BVP (3.1.6) is defined similarly by reversing the inequalities.

For $u \in C(J)$ we write $\|u\| = \max_{t \in J} |u(t)|$ and for $v \in C^1(J)$ we write $\|v\|_1 = \sqrt{\|v\|^2 + \|v'\|^2}$. Now, we prove theorems which establish the existence and uniqueness of solutions.

Theorem 3.1.2. *Assume that $\alpha, \beta \in C^2(J)$ are lower and upper solutions of (3.1.6) respectively such that $\alpha \le \beta$ on J. Assume that $f \in C[J \times \mathbb{R} \times \mathbb{R}]$ and satisfies the Nagumo condition in x' relative to α, β. Suppose that g is continuous and is nondecreasing on $\mathbb{R}$. Then there exists a solution $x(t)$ of (3.1.6) such that $\alpha(t) \le x(t) \le \beta(t)$, $t \in J$.*

Proof. There exists a constant N depending on α, β and ω such that

$$\int_\lambda^N \frac{s ds}{\omega(s)} > \max_{t \in J} \beta(t) - \min_{t \in J} \alpha(t),$$

where ω is the Nagumo function.
Let $C > \max\{N, \|\alpha'\|, \|\beta'\|, \|\alpha\|, \|\beta\|\}$ and $q(x') = \max\{-C, \min\{x', C\}\}$. Define modifications of $f(t, x, x')$ and $g(x)$ as follows.

$$F(t, x, x') = \begin{cases} f(t, \beta(t), q(x')) + \dfrac{x - \beta(t)}{1 + |x - \beta(t)|}, & \text{if } x > \beta(t), \\ f(t, x, q(x')), & \text{if } \alpha(t) \le x \le \beta(t), \\ f(t, \alpha(t), q(x')) + \dfrac{x - \alpha(t)}{1 + |x - \alpha(t)|}, & \text{if } x < \alpha(t). \end{cases}$$

and

$$G(x) = \begin{cases} g(\beta(\eta)), & x > \beta(\eta), \\ g(x), & \alpha(\eta) \le x \le \beta(\eta), \\ g(\alpha(\eta)), & x < \alpha(\eta). \end{cases}$$

Consider the modified boundary value problem

$$\begin{aligned} x''(t) &= F(t, x, x'), \ t \in J, \\ x(0) = a, &\quad x(1) = G(x(\eta)). \end{aligned} \tag{3.1.8}$$

This is equivalent to an integral equation,

$$x(t) = a(1-t) + G(x(\eta))t + \int_0^1 k(t,s)F(s, x(s), x'(s))\, ds. \tag{3.1.9}$$

Since $F(t, x, x')$ and $G(x)$ are continuous and bounded, this integral equation has a fixed point by the Schauder fixed point theorem so the BVP (3.1.8) has a solution $x \in C^2(J)$, [21]. Further, we note that

$$\begin{aligned} \alpha''(t) \ge f(t, \alpha(t), \alpha'(t)) &= F(t, \alpha(t), \alpha'(t)), \ t \in J \\ \alpha(0) \le a, \quad \alpha(1) \le g(\alpha(\eta)) &= G(\alpha(\eta)) \end{aligned}$$

and

$$\begin{aligned} \beta''(t) \le f(t, \beta(t), \beta'(t)) &= F(t, \beta(t), \beta'(t)), \ t \in J \\ \beta(0) \ge a, \quad \beta(1) \ge g(\beta(\eta)) &= G(\beta(\eta)), \end{aligned}$$

that is, α, β are lower and upper solutions of (3.1.8). We claim that any solution x, of (3.1.8) with $\alpha(t) \le x(t) \le \beta(t)$, $t \in J$, is also a solution of (3.1.6). For if x is such a solution of (3.1.8), then for $t \in J$, we have

$$|F(t, x, x')| = |f(t, x, q(x'))| \le \tilde{\omega}(|x'|), \tag{3.1.10}$$

where $\tilde{\omega}(s) = \omega(q(s))$ for $s \ge 0$. We note that $q(s) \ge 0$ for $s \ge 0$ and for $s \le C$ we have $q(s) = s$. Now

$$\int_\lambda^C \frac{s}{\tilde{\omega}(s)}\, ds \ge \int_\lambda^N \frac{s}{\tilde{\omega}(s)}\, ds + \int_N^C \frac{s}{\tilde{\omega}(s)}\, ds \ge \int_\lambda^N \frac{s}{\omega(s)}\, ds > \max_{t\in J} \beta - \min_{t \in J} \alpha \tag{3.1.11}$$

and hence as in the proof of Theorem 1.1.7, we conclude that $|x'(t)| \le C$, which implies that $x(t)$ is a solution of (3.1.6).

Now we show that any solution x of (3.1.8) does satisfy $\alpha(t) \le x(t) \le \beta(t)$, $t \in J$. Set $v(t) = \alpha(t) - x(t)$, $t \in J$. Then, $v(t) \in C^2(J)$ and $v(0) \le 0$. We claim that $v(t) \le 0$ on J. If not, v has a positive maximum at some $t_0 \in (0, 1]$. If $t_0 \ne 1$, then

$$v(t_0) > 0, \quad v'(t_0) = 0, \quad v''(t_0) \le 0.$$

However,

$$\begin{aligned} v''(t_0) &= \alpha''(t_0) - x''(t_0) \\ &\geq f(t_0, \alpha(t_0), \alpha'(t_0)) - f(t_0, \alpha(t_0), \alpha'(t_0)) - \frac{x(t_0) - \alpha(t_0)}{1 + |x(t_0) - \alpha(t_0)|} > 0, \end{aligned}$$

a contradiction. If $t_0 = 1$, then $v(1) > 0$, $v'(1) \geq 0$. Using the boundary conditions, we have

$$v(1) = \alpha(1) - x(1) \leq g(\alpha(\eta)) - G(x(\eta)).$$

If $x(\eta) < \alpha(\eta)$, then $g(\alpha(\eta)) = G(x(\eta))$, and hence $v(1) \leq 0$, a contradiction. If $x(\eta) > \beta(\eta)$, then $G(x(\eta)) = g(\beta(\eta)) \geq g(\alpha(\eta))$, which implies $v(1) \leq 0$, a contradiction. Hence $\alpha(\eta) \leq x(\eta) \leq \beta(\eta)$ and $G(x(\eta)) = g(x(\eta))$. Then $v(1) \leq g(\alpha(\eta)) - g(x(\eta)) \leq 0$, by the increasing property of g, another contradiction. Thus $\alpha(t) \leq x(t)$, $t \in J$. Similarly, we can show that $x(t) \leq \beta(t)$, $t \in J$. □

Theorem 3.1.3. *Assume that α, β are lower and upper solutions of the boundary value problem (3.1.6). Suppose $f(t, x, x') \in C[J \times \mathbb{R} \times \mathbb{R}]$ is strictly increasing in x for each $(t, x') \in J \times \mathbb{R}$ and g is continuous and $x - g(x)$ is strictly increasing in x. Then*

$$\alpha(t) \leq \beta(t) \text{ on } J.$$

Hence, under these conditions, solutions are unique.

Proof. Define $w(t) = \alpha(t) - \beta(t)$ on J. Then

$$w \in C^2(J),\ w(0) \leq 0, \text{ and } w(1) \leq g(\alpha(\eta)) - g(\beta(\eta)),$$

Suppose that w has a positive maximum at some $t_0 \in (0, 1]$. If $t_0 \in (0, 1)$ then

$$w(t_0) > 0,\ w'(t_0) = 0,\ w''(t_0) \leq 0.$$

Using the increasing property of $f(t, x, x')$ in x, we obtain

$$w''(t_0) = \alpha''(t_0) - \beta''(t_0) \geq f(t_0, \alpha(t_0), \alpha'(t_0)) - f(t_0, \beta(t_0), \alpha'(t_0)) > 0,$$

a contradiction. Thus $t_0 = 1$, $w(1) > 0$, and $w'(1) \geq 0$. It also follows, by essentially the same argument as above, that w cannot have a negative minimum on $(0, 1)$, hence $w \geq 0$ on J. If $w(\eta) = 0$ then $w(1) \leq 0$ from the boundary condition, a contradiction. So we must have $0 < w(\eta) \leq w(1)$, that is,

$$\alpha(\eta) - \beta(\eta) \leq g(\alpha(\eta)) - g(\beta(\eta)),$$

which implies $\alpha(\eta) \leq \beta(\eta)$ by the increasing property of $I - g$. This contradicts $w(\eta) > 0$. We have shown that $w(t) \leq 0$, $t \in J$. □

Example 3.1.4. (A class of examples) Consider the three-point boundary value problem

$$\begin{aligned} x''(t) =& |x'(t)|^{p-1}x'(t) - f(x(t)),\ t \in (0,1), \quad \text{where } 0 < p \le 2, \\ x(0) =& a > 0,\ x(1) = \delta x(\eta),\ \eta \in (0,1) \text{ with } 0 \le \delta\eta < 1. \end{aligned}$$

We assume that f satisfies

$$f(x) \ge -(a-x)^p \text{ for } 0 \le x \le a, \quad \text{and } f(x) \le \gamma x^p + C \text{ for } x \ge a, \text{ where } \gamma < 1.$$

Firstly we show that there is a lower solution of the form $\alpha(t) = a - bt$, with $b \ge 0$. In fact we take

$$b = \begin{cases} 0, & \text{if } 1 \le \delta < \frac{1}{\eta} \\ \frac{a(1-\delta)}{1-\delta\eta}, & \text{if } \delta < 1. \end{cases}$$

Note that $b < a$ so that $\alpha(t) > 0$ on J. Then $\alpha(0) = a$ and $\alpha(1) \le \delta\alpha(\eta)$ since $a - b \le \delta(a - b\eta) \iff a(1-\delta) \le b(1-\delta\eta)$. Further,

$$\alpha''(t) \ge |\alpha'(t)|^{p-1}\alpha' - f(\alpha(t))$$

if and only if $0 \ge -b^p - f(\alpha(t))$. This is valid because

$$f(a-bt) \ge -[a-(a-bt)]^p = -[bt]^p \ge -b^p.$$

Hence $\alpha(t)$ is a lower solution of the problem.
Secondly we show that $\beta(t) = a + Bt$ is an upper solution, where $B \ge B_0 \ge 0$ is chosen below and where

$$B_0 = \begin{cases} 0 & \text{if } \delta \le 1 \\ \frac{a(\delta-1)}{1-\delta\eta} & \text{if } 1 < \delta \le \frac{1}{\eta}. \end{cases}$$

In fact $\beta(0) = a$ and it is easy to check that $\beta(1) \ge \delta\beta(\eta)$.

Further, $\beta''(t) \le |\beta'|^{p-1}\beta' - f(\beta)$ if and only if $f(\beta(t)) \le B^p$. Now since $\beta \ge a$,

$$f(\beta(t)) \le \gamma(\beta(t))^p + C \le \gamma(a+B)^p + C.$$

Since $\gamma < 1$, for sufficiently large B we have $f(\beta(t)) \le B^p$ and hence $\beta(t) = a + Bt$, is an upper solution of the problem for B sufficiently large.
Finally we show that the Nagumo condition is satisfied. For $\alpha(t) \le x(t) \le \beta(t)$, we have

$$\begin{aligned} \big||x'|^{p-1}x' - f(x)\big| &\le |x'|^p + f(x) \\ &\le |x'|^p + \gamma x^p + C \le |x'|^p + B^p =: \omega(|x'|), \end{aligned}$$

where $\omega(s) = s^p + B^p$, and

$$\int_1^\infty \frac{s}{s^p + B^p}\, ds = \infty \quad \text{for } 0 \le p \le 2.$$

Thus by Theorem 3.1.2, there is a solution of this BVP which lies between α and β, in particular is positive on $[0,1]$. If $\delta < 1$ and f is decreasing then the solution is unique by Theorem 3.1.3.

3.1.2 Quasilinearization technique

We now study approximation of solutions by the quasilinearization method and show that, under suitable conditions on f and g, there are solutions of linear problems that converge quadratically to the solution of the nonlinear problem.

Theorem 3.1.5. *Assume that*

(A_1) *there exist lower and upper solutions of* (3.1.6), $\alpha, \beta \in C^2(J)$, *such that* $\alpha(t) \leq \beta(t)$ *on* J.

(A_2) $f \in C^2(J \times \mathbb{R} \times \mathbb{R})$ *satisfies* $f_x(t, x, x') \geq 0$ *for* $t \in J$, $x \in [\min\alpha, \max\beta]$, *and* $|x'| \leq C$. *Suppose also that* $H(f) \leq 0$ *whenever* $\min\alpha \leq z_1 \leq \max\beta$, $|z_2| \leq C$, $t \in J$, *where*

$$H(f) = (x-y)^2 f_{xx}(t, z_1, z_2) + 2(x-y)(x'-y')f_{xx'}(t, z_1, z_2) + (x'-y')^2 f_{x'x'}(t, z_1, z_2)$$

is the quadratic form of f.

(A_3) *The function* $g \in C^2(\mathbb{R})$ *satisfies*

$$0 \leq g'(x) < 1 \text{ and } g''(x) \geq 0 \text{ for } \min\alpha \leq x \leq \max\beta.$$

Then, there exists a monotone sequence $\{w_n\}$ *of solutions of linear problems converging uniformly and quadratically to the unique solution of the problem* (3.1.6).

Proof. In view of (A_2), (A_3) and Taylor's theorem, we have

$$f(t, x, x') \leq f(t, y, y') + f_x(t, y, y')(x-y) + f_{x'}(t, y, y')(x'-y'), \tag{3.1.12}$$

where x, $y \in [\min\alpha, \max\beta]$, x', $y' \in [-C, C]$ and $t \in J$, and

$$g(x) \geq g(y) + g'(y)(x-y), \quad \text{for } x \in \mathbb{R},\ \min\alpha \leq y \leq \max\beta. \tag{3.1.13}$$

Let $S = \{(t, x, x') : x \in [\min\alpha, \max\beta],\ t \in J,\ x' \in \mathbb{R}\}$ and define on S the function

$$h(t, x, y; x', y') = f(t, y, q(y')) + f_x(t, y, q(y'))(x-y) + f_{x'}(t, y, q(y'))(q(x') - q(y')). \tag{3.1.14}$$

Note that $h(t, x, y; x', y')$ is continuous and bounded so satisfies a Nagumo condition relative to α, β. Also, for x, $y \in [\min\alpha, \max\beta]$, let

$$G(x, y) = g(y) + g'(y)(x-y). \tag{3.1.15}$$

Then, in view of (A_3), $G(x, y)$ is continuous and bounded and satisfies $0 \leq G_x(x, y) < 1$. Moreover, from (3.1.14), it follows that, for $|x'| \leq C$, $|y'| \leq C$,

$$h_x(t, x, y; x', y') = f_x(t, y, y') \geq 0,$$

and

$$h(t,x,y;x',y') = f(t,y,y') + f_x(t,y,y')(x-y) + f_{x'}(t,y,y')(x'-y'). \tag{3.1.16}$$

From (3.1.12), (3.1.14) and (3.1.13), (3.1.15), we have the following relations for $|x'| \le C$, $|y'| \le C$,

$$\begin{cases} f(t,x,x') \le h(t,x,y;x',y') \\ f(t,x,x') = h(t,x,x;x',x'), \end{cases} \tag{3.1.17}$$

and

$$\begin{cases} G(x,y) \le g(x), \\ G(x,x) = g(x). \end{cases} \tag{3.1.18}$$

Now, we set $w_0 = \alpha$ and consider the following linear problem on S

$$\begin{aligned} x''(t) &= h(t,x,w_0;x',w_0'),\ t \in J \\ x(0) &= a,\ \ x(1) = G(x(\eta), w_0(\eta)). \end{aligned} \tag{3.1.19}$$

Using (A_1), (3.1.17) and (3.1.18), we obtain

$$\begin{aligned} &w_0''(t) \ge f(t,w_0(t),w_0'(t)) = h(t,w_0(t),w_0(t);w_0'(t),w_0'(t)),\ t \in J \\ &w_0(0) \le a, \quad w_0(1) \le g(w_0(\eta)) = G(w_0(\eta), w_0(\eta)), \end{aligned}$$

and

$$\begin{aligned} &\beta''(t) \le f(t,\beta(t),\beta'(t)) \le h(t,\beta(t),w_0(t);\beta'(t),w_0'(t)),\ t \in J \\ &\beta(0) \ge a, \quad \beta(1) \ge g(\beta(\eta)) \ge G(\beta(\eta), w_o(\eta)), \end{aligned}$$

which imply that w_0 and β are respectively lower and upper solution of (3.1.19). Hence, by Theorems 3.1.2, 3.1.3, there exists a unique solution w_1 of (3.1.19) such that $w_0 \le w_1 \le \beta$ and $|w_1'| < C$ on J. Using (3.1.17), (3.1.18) and the fact that w_1 is a solution of (3.1.19), we obtain

$$\begin{aligned} w_1''(t) &= h(t,w_1,w_0;w_1',w_0') \ge f(t,w_1(t),w_1'(t)) \\ w_1(0) &= a, \quad w_1(1) = G(w_1(\eta), w_0(\eta)) \le g(w_1(\eta)), \end{aligned} \tag{3.1.20}$$

which implies that w_1 is a lower solution of (3.1.6). Now, consider the problem

$$\begin{aligned} x''(t) &= h(t,x,w_1;x',w_1'),\ t \in J \\ x(0) &= a, \quad x(1) = G(x(\eta), w_1(\eta)). \end{aligned} \tag{3.1.21}$$

In view of (A_1), (3.1.17), (3.1.18) and (3.1.20), we can show that w_1 and β are lower and upper solutions of (3.1.21) respectively. Hence again by Theorems 3.1.2, 3.1.3, there exists a unique solution w_2 of (3.1.21) such that $w_1 \le w_2 \le \beta$ and $|w_2'| < C$ on J. Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0 \le w_1 \le w_2 \le w_3 \le \dots \le w_{n-1} \le w_n \le \beta,\ t \in J,$$

and

$$\alpha \le w_n(t) \le \beta, \text{ and } |w_n'| < C, \; n = 1,2,3,\ldots, t \in J, \tag{3.1.22}$$

where w_n is a solution of the linear problem

$$w_n(t) = a(1-t) + G(w_n(\eta), w_{n-1}(\eta))t + \int_0^1 k(t,s)h(s, w_n, w_{n-1}; w_n', w_{n-1}')\, ds. \tag{3.1.23}$$

As $h(s, w_n, w_{n-1}; w_n', w_{n-1}')$ is continuous and bounded on J, there is $L > 0$ such that $|h(s, w_n, w_{n-1}; w_n', w_{n-1}')| \le L$, $t \in J$. Also, for t, $s \in J$, we have

$$|w_n'(t) - w_n'(s)| \le \int_s^t |h(\sigma, w_n, w_{n-1}; w_n', w_{n-1}')|\, d\sigma \le L|t-s|. \tag{3.1.24}$$

From (3.1.22), (3.1.23) and (3.1.24), it follows that the sequences $\{w_n^{(j)}(t)\}, (j = 0,1)$ are uniformly bounded and equicontinuous on J. The Arzelà-Ascoli theorem guarantees the existence of a subsequence and a function $x \in C^2(J)$ with $w_n^{(j)}$ $(j = 0,1)$ converging uniformly to $x^{(j)}$ on J as $n \to \infty$. Passing to the limit in (3.1.23), we obtain

$$x(t) = a(1-t) + g(x(\eta))t + \int_0^1 k(t,s)f(s,x,x')\, ds,$$

that is, $x(t)$ is a solution of (3.1.6).

Now we show that the convergence is quadratic. For each n, set $v_n = x - w_n$. Then,

$$v_n \in C^2(J), \; v_n(0) = 0 \text{ and } v_n(t) \ge 0, \; t \in J.$$

Using the definition of $G(x,y)$ and Taylor's theorem we have

$$\begin{aligned} v_n(1) &= g(x(\eta)) - G(w_n(\eta), w_{n-1}(\eta)) = g'(w_{n-1}(\eta))v_n(\eta) + \frac{g''(\xi)}{2} v_{n-1}^2(\eta) \\ &= g'(w_{n-1}(\eta))v_n(\eta) + dv_{n-1}^2(\eta), \end{aligned} \tag{3.1.25}$$

where ξ lies between $x(\eta)$ and $w_{n-1}(\eta)$ and $d = \frac{g''(\xi)}{2} \ge 0$. Further, using Taylor's theorem and (A_2), we obtain

$$\begin{aligned} v_{n+1}''(t) &= x''(t) - w_{n+1}''(t) = f(t,x,x') \\ &- [f(t,w_n,w_n') + f_x(t,w_n,w_n')(w_{n+1} - w_n) + f_{x'}(t,w_n,w_n')(w_{n+1}') - w_n')] \\ &= f(t,w_n,w_n') + f_x(t,w_n,w_n')(x - w_n) + f_{x'}(t,w_n,w_n')(x' - w_n') + \frac{1}{2}H(f) \\ &- [f(t,w_n,w_n') + f_x(t,w_n,w_n')(w_{n+1} - w_n) + f_{x'}(t,w_n,w_n')(w_{n+1}' - w_n')] \\ &= f_x(t,w_n,w_n')v_{n+1} + f_{x'}(t,w_n,w_n')v_{n+1}' - \frac{1}{2}|H(f)|, \end{aligned} \tag{3.1.26}$$

where

$$H(f) = v_n^2 f_{xx}(t,c_1,c_2) + 2v_n v_n' f_{xx'}(t,c_1,c_2) + {v_n'}^2 f_{x'x'}(t,c_1,c_2),$$

with $w_n(t) \leq c_1 \leq x(t)$ and c_2 lies between $q(w_n'(t))$ and $x'(t)$. Hence $\alpha \leq c_1 \leq \beta$ and $|c_2| \leq C$. Then

$$M := \max_{t \in J}\{|f_{xx}(t, c_1, c_2)|, |f_{xx'}(t, c_1, c_2)|, |f_{x'x'}(t, c_1, c_2)|\},$$

is independent of w_n and x and we have

$$|H(f)| \leq 2M\|v_n\|_1^2, \tag{3.1.27}$$

where, $\|v_n\|_1^2 = \|v_n\|^2 + \|v_n'\|^2$. Using (3.1.27) and the assumption $f_x \geq 0$, we obtain from (3.1.26)

$$v_{n+1}''(t) \geq f_{x'}(t, w_n, w_n')v_{n+1}' - M\|v_n\|_1^2. \tag{3.1.28}$$

By a comparison argument (the maximum principle), it follows that $0 \leq v_{n+1}(t) \leq r(t)$, where $r(t)$ is the solution of the linear problem

$$\begin{aligned} r''(t) - f_{x'}(t, w_n, w_n')r'(t) &= -M\|v_n\|_1^2 \\ r(0) = 0,\ r(1) &= g'(w_n(\eta))v_{n+1}(\eta) + dv_n^2(\eta). \end{aligned} \tag{3.1.29}$$

Note that $r'(0) \geq 0$ since $r(t) \geq 0$. Let $\mu(t) = e^{-\int_0^t f_{x'}(s, w_n, w_n')\, ds}$ be the integrating factor, then

$$(r'(t)\mu(t))' = -M\mu(t)\|v_n\|_1^2. \tag{3.1.30}$$

Since $\alpha \leq w_n \leq \beta$ and $|w_n'| \leq C$, $f_{x'}(t, w_n, w_n')$ is uniformly bounded say

$$-l \leq f_{x'}(t, w_n, w_n') \leq L, \quad l \geq 0,\ L \geq 0.$$

Then the integrating factor satisfies

$$e^{-Lt} \leq \mu(t) \leq e^{lt}, \quad \frac{1}{\mu(t)} \leq e^{Lt}. \tag{3.1.31}$$

Integrating (3.1.30) from 0 to t, using (3.1.31) and the boundary condition $r'(0) \geq 0$, we obtain

$$r'(t) \geq -\frac{M}{\mu(t)}\|v_n\|_1^2 \int_0^t \mu(s)\, ds \geq -M\|v_n\|_1^2 e^{Lt}(e^{lt} - 1)/l \geq -M\|v_n\|_1^2 e^{(l+L)t}/l,$$

which on integration from t to 1, gives

$$\begin{aligned} r(t) &\leq r(1) + Me^{L+l}\|v_n\|_1^2/(l(l+L)) \\ &\leq g'(w_n(\eta))v_{n+1}(\eta) + dv_n^2(\eta) + C_1\|v_n\|_1^2, \end{aligned}$$

where $C_1 = \dfrac{Me^{l+L}}{l(l+L)}$. Hence, using $0 \leq g'(w_n(\eta)) \leq L_1 < 1$ we have

$$v_{n+1}(t) \leq L_1 v_{n+1}(\eta) + dv_n^2(\eta) + C_1\|v_n\|_1^2.$$

Taking $t = \eta$, solving for $v_{n+1}(\eta)$ then gives

$$v_{n+1}(t) \leq \frac{L_1}{1-L_1}(dv_n^2(\eta) + C_1\|v_n\|_1^2) + C_1\|v_n\|_1^2.$$

Hence

$$v_{n+1}(t) \leq \frac{L_1}{1-L_1}d\|v_n^2\| + C_2\|v_n\|_1^2,$$

where $C_2 = \dfrac{C_1}{1-L_1}$. Taking the maximum over $[0,1]$ gives

$$\|v_{n+1}\| \leq C_3\|v_n\|_1^2,$$

where $C_3 = \dfrac{L_1}{1-L_1}d + C_2$. We have proved that

$$\|v_{n+1}\| \leq C_3\|v_n\|_1^2. \tag{3.1.32}$$

Since $v_n \in C^2(J)$, using the Mean value theorem, we can find $0 < \xi < 1$, such that $v_n(1) - v_n(0) = v_n'(\xi)$. That is, $v_n(1) = v_n'(\xi)$. Using (A_2) and (3.1.32) in (3.1.26), we have

$$\begin{aligned} v_{n+1}''(t) &= f_x(t, w_n, w_n')v_{n+1} + f_{x'}(t, w_n, w_n')v_{n+1}' - \frac{1}{2}|H(f)| \\ &\leq f_{x'}(t, w_n, w_n')v_{n+1}' + Nv_{n+1} \leq f_{x'}(t, w_n, w_n')v_{n+1}' + NC_3\|v_n\|_1^2, \end{aligned} \tag{3.1.33}$$

where $0 \leq f_x(t, w_n, w_n') \leq N$. It follows that

$$(v_{n+1}'(t)\mu(t))' \leq NC_3\|v_n\|_1^2\mu(t) \leq NC_3e^{lt}\|v_n\|_1^2,$$

which implies that

$$(v_{n+1}'(t)\mu(t) - \frac{NC_3e^{lt}}{l}\|v_n\|_1^2)' \leq 0.$$

that is, the function $\phi(t) = v_{n+1}'(t)\mu(t) - \frac{NC_3e^{lt}}{l}\|v_n\|_1^2$ is non-increasing and hence $\phi(\xi) \geq \phi(1)$ which implies that

$$v_{n+1}'(\xi)\mu(\xi) - \frac{NC_3e^{l\xi}}{l}\|v_n\|_1^2 \geq v_{n+1}'(1)\mu(1) - \frac{NC_3e^{l}}{l}\|v_n\|_1^2.$$

That is,

$$\begin{aligned} v_{n+1}'(1)\mu(1) \leq& v_{n+1}'(\xi)\mu(\xi) + \frac{NC_3}{l}\|v_n\|_1^2(e^l - e^{l\xi}) \\ &= v_{n+1}(1)\mu(\xi) + \frac{NC_3}{l}\|v_n\|_1^2(e^l - e^{l\xi}) \\ &= [g'(w_n(\eta))v_{n+1}(\eta) + dv_n^2(\eta)]\mu(\xi) + \frac{NC_3}{l}\|v_n\|_1^2(e^l - e^{l\xi}) \\ &\leq g'(w_n(\eta))v_{n+1}(\eta) + [d\mu(\xi) + \frac{NC_3}{l}(e^l - e^{l\xi})]\|v_n\|_1^2 \\ &\leq [C_3g'(w_n(\eta)) + d\mu(\xi) + \frac{NC_3}{l}(e^l - e^{l\xi})]\|v_n\|_1^2. \end{aligned} \tag{3.1.34}$$

Using (3.1.28), we get

$$(v'_{n+1}(t)\mu(t))' \geq -Ne^{lt}\|v_n\|_1^2, \; t \in J. \tag{3.1.35}$$

Integrating (3.1.35) from t to 1 and using (3.1.34), we obtain

$$\begin{aligned} v'_{n+1}(t)\mu(t) &\leq v'_{n+1}(1)\mu(1) + \frac{N}{l}(e^l - e^{lt})\|v_n\|_1^2 \\ &\leq [C_3 g'(w_n(\eta)) + d\mu(\xi) + \frac{NC_3}{l}(e^l - e^{l\xi}) + \frac{N}{l}(e^l - e^{lt})]\|v_n\|_1^2. \end{aligned} \tag{3.1.36}$$

It follows that

$$v'_{n+1}(t) \leq R_1\|v_n\|_1^2, \tag{3.1.37}$$

where,

$$R_1 = \max_{t\in J}\left\{\frac{1}{\mu(t)}[C_3 g'(w_n(\eta)) + d\mu(\xi) + \frac{NC_3}{l}(e^l - e^{l\xi}) + \frac{N}{l}(e^l - e^{lt})]\right\}.$$

Now, integrating (3.1.35) from 0 to t, we obtain

$$v'_{n+1}(t) \geq -\frac{N(e^{lt}-1)}{l\mu(t)}\|v_n\|_1^2 \geq -R_2\|v_n\|_1^2, \tag{3.1.38}$$

where,

$$R_2 = \max_{t\in J}\left\{\frac{N(e^{lt}-1)}{l\mu(t)}\right\}.$$

Let $R = \max\{R_1, R_2\}$, then from (3.1.37) and (3.1.38), it follows that

$$|v'_{n+1}(t)| \leq R\|v_n\|_1^2. \tag{3.1.39}$$

Combined with (3.1.32) this establishes

$$\|v_{n+1}\|_1 \leq C_4\|v_n\|_1^2,$$

where $C_4 = C_3 + R$, that is, we have quadratic convergence. □

3.1.3 Generalized quasilinearization technique

We now use an auxiliary function ϕ to allow weaker hypotheses on f.

Theorem 3.1.6. *Assume that*

(B_1) *α and $\beta \in C^2(J)$ are lower and upper solutions of* (3.1.6) *respectively, such that $\alpha(t) \leq \beta(t)$ on J.*

(B_2) *$f \in C[J\times\mathbb{R}\times\mathbb{R}]$ satisfies $f_x(t,x,x') > 0$ for $t \in J$, $x \in [\min\alpha, \max\beta]$, and $|x'| \leq C$. Suppose also that $H(f+\phi) \leq 0$ on $J \times [\min\alpha, \max\beta] \times [-C, C]$, for some function $\phi \in C^2[J\times\mathbb{R}\times\mathbb{R}]$ where $H(\phi) \leq 0$ on $J \times [\min\alpha, \max\beta] \times [-C, C]$.*

(B_3) *$g(x) \in C^2(\mathbb{R})$, $0 \leq g'(x) < 1$ and $g''(x) \geq 0$ on $[\min\alpha, \max\beta]$.*

Then, there exists a monotone sequence $\{w_n\}$ of solutions of linear problems converging uniformly and quadratically to the unique solution of the problem.

Proof. Define $F : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(t,x,y) = f(t,x,y) + \phi(t,x,y), \quad t \in J.$$

Then, from (B_2), we have $F(t,x,y) \in C^2[J \times \mathbb{R} \times \mathbb{R}]$ and

$$H(F) \leq 0 \text{ on } J \times [\min\alpha,\ \max\beta] \times [-C,\ C]. \tag{3.1.40}$$

Let $\Omega = \{(t,x,x') \in J \times [\min\alpha,\ \max\beta] \times [-C,\ C]\}$. Using Taylor's theorem and (3.1.40), we have

$$f(t,x,x') \leq F(t,y,y') + F_x(t,y,y')(x-y) + F_{x'}(t,y,y')(x'-y') - \phi(t,x,x'), \tag{3.1.41}$$

where $(t,x,x'),(t,y,y') \in \Omega$. Further, using Taylor's theorem on the function ϕ, we can find $d_1,\ d_2 \in \mathbb{R}$ with $y \leq d_1 \leq x$ and d_2 between x' and y', such that

$$\phi(t,x,x') = \phi(t,y,y') + \phi_x(t,y,y')(x-y) + \phi_{x'}(t,y,y'))(x'-y') + \frac{1}{2}H(\phi), \tag{3.1.42}$$

where

$$\begin{aligned} H(\phi) =&(x-y)^2\phi_{xx}(t,d_1,d_2) + 2(x-y)(x'-y')\phi_{xx'}(t,d_1,d_2) \\ &+ (x'-y')^2\phi_{x'x'}(t,d_1,d_2) \end{aligned}$$

Let

$$N = \max\{|\phi_{xx}(t,d_1,d_2)|, |\phi_{xx'}(t,d_1,d_2)|, |\phi_{x'x'}(t,d_1,d_2) : (t,d_1,d_2) \in \Omega\}.$$

Then

$$|H(\phi)| \leq 4N\|x-y\|_1^2. \tag{3.1.43}$$

In view of (B_2) and (3.1.43), it follows that

$$\phi(t,x,x') \leq \phi(t,y,y') + \phi_x(t,y,y')(x-y) + \phi_{x'}(t,y,y')(x'-y') \tag{3.1.44}$$

and

$$\phi(t,x,x') \geq \phi(t,y,y') + \phi_x(t,y,y')(x-y) + \phi_{x'}(t,y,y'))(x'-y') - 2N\|x-y\|_1^2. \tag{3.1.45}$$

Using (3.1.45) in (3.1.41), we obtain

$$f(t,x,x') \leq f(t,y,y') + f_x(t,y,y')(x-y) + f_{x'}(t,y,y')(x'-y') + 2N\|x-y\|_1^2. \tag{3.1.46}$$

Define the function h^* on S by

$$\begin{aligned} h^*(t,x,y;x',y') = f(t,y,q(y')) &+ f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(q(x')-q(y')) \\ &+ 2N\|x-y\|_1^2. \end{aligned} \tag{3.1.47}$$

We note that h^* is continuous and bounded and therefore satisfies the Nagumo condition. Moreover, $h^*_x(t,x,y;x',y') = f_x(t,y,y') > 0$ and

$$h^*(t,x,y;x',y') = f(t,y,y')+f_x(t,y,y')(x-y)+f_{x'}(t,y,y')(x'-y')+2N\|x-y\|_1^2, \quad (3.1.48)$$

for $|x'| \leq C$, $|y'| \leq C$. From (3.1.46) and (3.1.48), we get the following relations

$$\begin{cases} f(t,x,x') \leq h^*(t,x,y;x',y') \\ f(t,x,x') = h^*(t,x,x;x',x'), \end{cases} \quad (3.1.49)$$

for $|x'| \leq C$, $|y'| \leq C$. As in the previous section, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$\alpha = w_0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq \beta, \; t \in J.$$

That is

$$\alpha \leq w_n(t) \leq \beta \text{ and } |w_n'(t)| < C, \; n = 1,2,3,...,t \in J \quad (3.1.50)$$

where w_n satisfies

$$w_n(t) = a(1-t) + G(w_n(\eta), w_{n-1}(\eta))t + \int_0^1 k(t,s)h^*(s,w_n,w_{n-1};w_n',w_{n-1}')\,ds. \quad (3.1.51)$$

The same arguments as in theorem 3.1.5, shows that the sequence converges to the unique solution of the boundary value problem (3.1.6).
Now we show that the convergence of the sequence of solutions is quadratic. For that, we set $v_n(t) = x(t) - w_n(t)$, $n = 1,2,3...$ and $t \in J$. Then, we note that

$$v_n \in C^2(J), \; v_n(0) = 0, \text{ and } v_n(t) \geq 0, \; t \in J.$$

Using (B_2), (3.1.44) and Taylor's theorem, we get

$$\begin{aligned} v_{n+1}''(t) &= x''(t) - w_{n+1}''(t) = (F(t,x,x') - \phi(t,x,x')) - f(t,w_n,w_n') \\ &\quad - f_x(t,w_n,w_n')(w_{n+1}-w_n) - f_{x'}(t,w_n,w_n')(w_{n+1}'-w_n') - 2N\|w_{n+1}-w_n\|_1^2 \\ &\geq f(t,w_n,w_n') + f_x(t,w_n,w_n')(x-w_n) + f_{x'}(t,w_n,w_n')(x'-w_n') + H(F)/2 \\ &\quad - f(t,w_n,w_n') - f_x(t,w_n,w_n')(w_{n+1}-w_n) - f_{x'}(t,w_n,w_n')(w_{n+1}'-w_n') \\ &\quad - 2N\|w_{n+1}-w_n\|_1^2 \\ &\geq f_x(t,w_n,w_n')v_{n+1} + f_{x'}(t,w_n,w_n')v_{n+1}' - \frac{1}{2}|H(F)| - 2N\|v_n\|_1^2, \end{aligned} \quad (3.1.52)$$

where

$$H(F) = v_n^2 F_{xx}(t,\xi_1,\xi_2) + 2v_n v_n' F_{xx'}(t,\xi_1,\xi_2) + {v_n'}^2 F_{x'x'}(t,\xi_1,\xi_2).$$

with $w_n \leq \xi_1 \leq x$, and ξ_2 lies between w_n and x'. Let

$$\begin{aligned} M = \max_{t\in[0,1]} \{ &|F_{xx}(t,\xi_1,\xi_2)|, |F_{xx'}(t,\xi_1,\xi_2)|, |F_{x'x'}(t,\xi_1,\xi_2)| : t \in J, \\ &\xi_1 \in [\min w_0(t), \max \beta(t)], \; \xi_2 \in [-C,C]\}, \end{aligned}$$

then $|H(F)| \leq 4M\|v_n\|_1^2$ and hence (3.1.52) becomes

$$v''_{n+1}(t) \geq f_{x'}(t, w_n, w'_n)v'_{n+1} - 2C\|v_n\|_1^2, \tag{3.1.53}$$

where $C = N+M$. Here, we used the assumption $f_x > 0$ and the fact that $\|w_{n+1} - w_n\|_1 \leq \|v_n\|_1$. From (3.1.53), it follows that $v_{n+1}(t) \leq r(t)$, where $r(t)$ is the solution of the linear problem

$$\begin{aligned} &r''(t) = f_{x'}(t, w_n, w'_n)r'(t) - 2C\|v_n\|_1^2 \\ &r(0) = 0,\ r(1) = g'(w_n(\eta))v_{n+1}(\eta) + dv_n^2(\eta), \end{aligned} \tag{3.1.54}$$

which is the same equation as (3.1.29). Hence by the same process, quadratic convergence follows as in Theorem 3.1.5. □

3.2 Three-point nonlinear nonlocal boundary value problems (II)

This section is the original work of the author alone and is accepted for publication in [49]. In this section we study a nonlinear second order differential equations with nonlinear nonlocal three-point boundary conditions of the type

$$\begin{aligned} &-x''(t) = f(t, x, x'),\ t \in J, \\ &x(0) = a, \quad x'(1) = g(x(\eta)), \quad 0 < \eta \leq 1, \end{aligned} \tag{3.2.1}$$

where the nonlinearities $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous. We show the validity of the classical upper and lower solutions method to ensure existence of solution of the problem. Then we develop the quasilinearization method to obtain monotone sequence of solutions of *linear* problems which converges uniformly and quadratically to a solution of the original nonlinear problem. In our new results, we control both function and its first derivative and prove a result on quadratic convergence in the C^1 norm. Our boundary conditions includes the two point boundary conditions

$$y(0) = 0,\ y'(1) + \psi(y(1)) = 0,$$

studied in [2], as a special case. Recently, existence theory for the boundary value problem of the type

$$\begin{aligned} &y''(x) = f(x, y),\ x \in (0,1), \\ &\quad y(0) = 0,\ y'(1) = g(y(\xi), y'(\xi)) = 0,\ 0 < \xi < 1, \end{aligned}$$

is studied in [15] with $f(x, y) = a(x, y)(y - b)$.

We write (3.2.1) as an integral equation

$$x(t) = a + g(x(\eta))t + \int_0^1 k(t, s)f(s, x(s), x'(s))ds, \tag{3.2.2}$$

where,

$$k(t,s) = \begin{cases} t, & 0 \le t < s \le 1, \\ s, & 0 \le s < t \le 1, \end{cases}$$

is the Green's function. Clearly, $k(t,s) > 0$ on $(0,1) \times (0,1)$.

Now, we know that the linear homogeneous problem

$$x''(t) + Lx'(t) = 0, \; t \in J,$$
$$x(0) = 0, \; x'(1) = 0,$$

has only a trivial solution for all $L(\neq 0) \in \mathbb{R}$. Consequently, for any $\sigma(t) \in C(J)$, the corresponding nonhomogeneous problem

$$x''(t) + Lx'(t) = \sigma(t), \; t \in J,$$
$$x(0) = a, \; x'(1) = g(x(\eta)), \; 0 < \eta \le 1,$$

has a unique solution

$$x(t) = a + g(x(\eta))t + \int_0^1 k_L(t,s)\sigma(s)ds,$$

where for $L \neq 0$,

$$k_L(t,s) = -\frac{e^{Ls}}{L} \begin{cases} (1 - e^{-Lt}), & 0 \le t < s \le 1 \\ (1 - e^{-Ls}), & 0 \le s < t \le 1, \end{cases}$$

and for $L = 0$,

$$k_0(t,s) = \begin{cases} -t, & 0 \le t < s \le 1 \\ -s, & 0 \le s < t \le 1. \end{cases}$$

We note that $k_L(t,s) < 0$ on $(0,1) \times (0,1)$ for any $L \in \mathbb{R}$.

3.2.1 The method of upper and lower solutions

We recall the concept of upper and lower solutions for the BVP (3.2.1).

Definition 3.2.1. Let $\alpha \in C^2(J)$. We say that α is a lower solution of the BVP (3.2.1), if

$$-\alpha''(t) \le f(t, \alpha(t), \alpha'(t)), \quad t \in J$$
$$\alpha(0) \le a, \; \alpha'(1) \le g(\alpha(\eta)).$$

An upper solution $\beta \in C^2(J)$ of the BVP (3.2.1) is defined similarly by reversing the inequalities.

Now we state and prove the following theorems which establish the existence and uniqueness of solutions.

Theorem 3.2.2. *Assume that $\alpha, \beta \in C^2(J)$ are lower and upper solutions of (3.2.1) respectively such that $\alpha \le \beta$ on J. Assume that $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a Nagumo condition on J relative to α, β. Assume that g is continuous and is increasing on $\mathbb{R}$. Then there exists a solution $x(t)$ of (3.2.1) such that $\alpha(t) \le x(t) \le \beta(t)$, $t \in J$.*

Proof. Let $\max\{|g(\alpha(\eta))|, |g(\beta(\eta))|\} = \lambda$, then there exists a constant $N(> \lambda)$ depends on α, β and ω such that

$$\int_{\lambda}^{N} \frac{sds}{\omega(s)} > \max_{t\in J} \beta(t) - \min_{t \in J} \alpha(t). \tag{3.2.3}$$

Let $C > \max\{N, \|\alpha'\|, \|\beta'\|\}$ and $q(x') = \max\{-C, \min\{x', C\}\}$. Let $\epsilon > 0$ be fixed and define the modifications of $f(t,x,x')$ and $g(x)$ as follows

$$F(t,x,x') = \begin{cases} f\big(t,\beta(t),\beta'(t)\big) + \frac{x-\beta(t)}{1+\left(x-\beta(t)\right)}, & \text{if } x \ge \beta(t)+\epsilon, \\ f\big(t,\beta(t),q(x')\big) + \Big[f\big(t,\beta(t),\beta'(t)\big) & \\ \quad - f\big(t,\beta(t),q(x')\big) + \frac{x-\beta(t)}{1+\left(x-\beta(t)\right)}\Big]\frac{x-\beta(t)}{\epsilon}, & \text{if } \beta(t) \le x < \beta(t)+\epsilon, \\ f\big(t,x,q(x')\big), & \text{if } \alpha(t) \le x \le \beta(t), \\ f\big(t,\alpha(t),q(x')\big) + \Big[f\big(t,\alpha(t),\alpha'(t)\big) & \\ \quad - f\big(t,\alpha(t),q(x')\big) + \frac{\alpha(t)-x}{1+\left(\alpha(t)-x\right)}\Big]\frac{\alpha(t)-x}{\epsilon}, & \text{if } \alpha(t)-\epsilon < x \le \alpha(t) \\ f\big(t,\alpha(t),\alpha'(t)\big) + \frac{\alpha(t)-x}{1+\left(\alpha(t)-x\right)}, & \text{if } x \le \alpha(t)-\epsilon, \end{cases}$$

and

$$G(x) = \begin{cases} g\big(\beta(\eta)\big) + \frac{x-\beta(\eta)}{1+\left(x-\beta(\eta)\right)}, & \text{if } x > \beta(\eta), \\ g(x), & \text{if } \alpha(\eta) \le x \le \beta(\eta), \\ g\big(\alpha(\eta)\big) + \frac{\alpha(\eta)-x}{1+\left(\alpha(\eta)-x\right)}, & \text{if } x < \alpha(\eta). \end{cases}$$

Then $F : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ are continuous and bounded.

Consider the modified problem

$$\begin{aligned} -x''(t) &= F(t,x,x'),\ t \in J, \\ x(0) = a, &\quad x'(1) = G(x(\eta)). \end{aligned} \tag{3.2.4}$$

This is equivalent to the integral equation

$$x(t) = a + G(x(\eta))t + \int_0^1 k(t,s)F\big(s,x(s),x'(s)\big)ds.$$

Since F and G are continuous and bounded, we apply Schauder's fixed point theorem to conclude that the BVP (3.2.4) has a solution. Moreover,

$$\begin{gathered} -\alpha''(t) \le f(t,\alpha(t),\alpha'(t)) = F(t,\alpha(t),\alpha'(t)),\ t \in J \\ \alpha(0) \le a,\ \alpha'(1) \le g(\alpha(\eta)) = G(\alpha(\eta)) \end{gathered}$$

and

$$-\beta''(t) \geq f(t, \beta(t), \beta'(t)) = F(t, \beta(t), \beta'(t)), \ t \in J$$
$$\beta(0) \geq a, \ \beta'(1) \geq g(\beta(\eta)) = G(\beta(\eta)).$$

Thus, α and β are lower and upper solutions of (3.2.4). We claim that any solution x of (3.2.4) satisfies $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in J$. Firstly, we show that $\alpha(t) \leq x(t)$, $t \in J$. For this, set $v(t) = \alpha(t) - x(t)$, $t \in J$. Then, $v \in C^2(J)$ and $v(0) \leq 0$. Assume that

$$\max\{v(t) : t \in (0, 1]\} = v(t_0) > 0.$$

If $t_0 \in (0, 1)$, then $v'(t_0) = 0$ and $v''(t_0) \leq 0$. However,

$$\begin{aligned} v''(t_0) &= \alpha''(t_0) - x''(t_0) \\ &\geq -f(t_0, \alpha(t_0), \alpha'(t_0)) + [f(t_0, \alpha(t_0), \alpha'(t_0)) + \frac{v(t_0)}{1 + v(t_0)}] > 0, \end{aligned}$$

a contradiction. This implies that v has no positive local maximum on J. If $t_0 = 1$, then $v(1) > 0$ and $v'(1) \geq 0$. Using the boundary conditions and the increasing property of the function g, we have

$$v'(1) = \alpha'(1) - x'(1) \leq g(\alpha(\eta)) - G(x(\eta)).$$

If $x(\eta) < \alpha(\eta)$, then

$$G(x(\eta)) = g(\alpha(\eta)) + \frac{\alpha(\eta) - x(\eta)}{1 + (\alpha(\eta) - x(\eta))} > g(\alpha(\eta)),$$

which implies that $v'(1) < 0$, a contradiction. If $x(\eta) > \beta(\eta)$, then

$$G(x(\eta)) = g(\beta(\eta)) + \frac{x(\eta) - \beta(\eta)}{1 + (x(\eta) - \beta(\eta))} > g(\beta(\eta)) > g(\alpha(\eta)),$$

hence $v'(1) < 0$, again a contradiction. Hence $\alpha(\eta) \leq x(\eta) \leq \beta(\eta)$. Then

$$G(x(\eta)) = g(x(\eta)) \geq g(\alpha(\eta)),$$

which implies that $v'(1) \leq 0$. Thus, in this case $v'(1) = 0$. Since $v(1) > 0$, $v(\eta) \leq 0$ and v has no positive local maximum, so there exists $t_1 \in [\eta, 1)$ such that $v(t_1) = 0$ and $v(t) > 0$ on $(t_1, 1]$. Now for each $t \in [t_1, 1]$, we have

$$v''(t) = \alpha''(t) - x''(t) \geq -f(t, \alpha(t), \alpha'(t)) + [f(t, \alpha(t), \alpha'(t)) + \frac{v(t)}{1 + v(t)}] > 0.$$

Integrating from t to 1, using $v'(1) = 0$, we obtain

$$v'(t) \leq 0 \text{ on } [t_1, 1],$$

which implies that v is decreasing on $[t_1, 1]$ and hence $v(1) \leq 0$, a contradiction. Thus, $\alpha(t) \leq x(t)$, $t \in J$. Similarly, we can show that $x(t) \leq \beta(t)$, $t \in J$.

Now, it remains to show that every solution x of (3.2.4) with $\alpha \leq x \leq \beta$ on J satisfies $|x'(t)| < C$, $t \in J$. Since g is increasing, it follows that $g(\alpha(\eta)) \leq g(x(\eta)) \leq g(\beta(\eta))$ which implies that $x'(1) \in [g(\alpha(\eta)), g(\beta(\eta))]$. Since $\max\{|g(\alpha(\eta))|, |g(\beta(\eta))|\} = \lambda$, we have $|x'(1)| \leq \lambda$. If $|x'(t)| \leq \lambda < C$ for every $t \in J$, then we are done. If not, then there exist t_1, $t_2 \in J$(say) with $(t_1 < t_2)$ such that

$$x'(t_1) = C,\ x'(t_2) = \lambda \text{ and } \lambda \leq x'(t) \leq C,\ t \in [t_1, t_2].$$

Since $f(t, x, x')$ satisfies a Nagumo condition relative to α, β, for every $t \in [t_1, t_2]$ and $x \in [\min \alpha(t), \max \beta(t)]$, there exists a Nagumo function ω such that

$$F(t, x, x') \operatorname{sgn}(x') = f(t, x, q(x')) \operatorname{sgn}(x') \leq \omega(q(x')),$$

which implies that

$$\frac{-x'(t)x''(t)}{\omega(q(x'(t)))} \leq x'(t).$$

Integrating from t_1 to t_2, we have

$$\int_\lambda^C \frac{sds}{\omega(s)} \leq x(t_2) - x(t_1) \leq \max_{t\in J} \beta(t) - \min_{t\in J} \alpha(t),$$

which contradicts (3.2.3). Thus, $|x'(t)| < C$ on J, which implies that $x(t)$ is a solution of (3.2.1). □

Theorem 3.2.3. *Assume that α, β are lower and upper solutions of the boundary value problem* (3.2.1) *such that $\alpha(\eta) < \beta(\eta)$. If $f(t, x, x') \in C(J \times \mathbb{R} \times \mathbb{R})$ is decreasing in x for each $(t, x') \in J \times \mathbb{R}$ and $g \in C(\mathbb{R})$ is increasing. Then $\alpha(t) \leq \beta(t)$ on J.*

Proof. Define $w(t) = \alpha(t) - \beta(t)$ on J. Then

$$w \in C^2(J),\ w(0) \leq 0 \text{ and } w(\eta) < 0.$$

Assume that $w(t) \leq 0$, $t \in J$ is not true, then $w(t)$ has a positive maximum at some $t_0 \in (0, 1]$. If $t_0 \neq 1$, then

$$w(t_0) > 0,\ w'(t_0) = 0 \text{ and } w''(t_0) \leq 0.$$

However, using the decreasing property of $f(t, x, x')$ in x, we obtain

$$w''(t_0) = \alpha''(t_0) - \beta''(t_0) \geq -f(t_0, \alpha(t_0), \alpha'(t_0)) + f(t_0, \beta(t_0), \alpha'(t_0)) > 0,$$

a contradiction. If $t_0 = 1$, then $w(1) > 0$ and $w'(1) \geq 0$. Using the boundary condition at $t = 1$ and the increasing property of the function g, we obtain,

$$w'(1) = \alpha'(1) - \beta'(1) \leq g(\alpha(\eta)) - g(\beta(\eta)) < 0,$$

a contradiction. Thus $w(t) \leq 0$, $t \in J$. □

Example 3.2.4. Consider the boundary value problem

$$\begin{aligned} -x'' &= -|x'|^{p-1}x' + f(x),\ 0 \leq p \leq 2,\ t \in [0,1] \\ x(0) &= 0, \quad x'(1) = \delta x(\eta), \end{aligned}$$

where, $0 < \delta\eta \leq 1$ and f is decreasing and satisfies $a^p \geq f(0) \geq 0$ for a fixed $a \geq 0$. Taking $\alpha = 0$, we have

$$\alpha(0) = 0, \quad \alpha'(1) = \delta\alpha(\eta),$$

$$\alpha''(t) - |\alpha'|^{p-1}\alpha' + f(\alpha) = f(0) \geq 0.$$

Thus 0 is a lower solution. Now, take $\beta = at$, we have

$$\beta(0) = 0, \quad \beta'(1) = a \geq a\delta\eta = \delta\beta(\eta),$$

$$\beta''(t) - |\beta'|^{p-1}\beta' + f(\beta) = -a^p + f(at) \leq -a^p + f(0) \leq 0.$$

Thus, at is an upper solution. Moreover, for $t \in [0,1]$ and $x \in [0, at]$, we have

$$|-|x'|^{p-1}x' + f(x)| \leq |x'|^p + M = \omega(|x'|)$$

where $M = \max\{f(x); x \in [0,a]\}$ and

$$\int_0^\infty \frac{sds}{\omega(s)} = \int_0^\infty \frac{sds}{s^p + M} = \infty.$$

Thus, the Nagumo condition is also satisfied. Hence, by Theorem 3.2.2, the problem has a solution in $[0, at]$, in particular a nonnegative solution.

Example 3.2.5. In the above example, if we take $p = 1$ and $f(x) = e^{-x}$, then the problem reduces to

$$\begin{aligned} -x'' &= -x' + e^{-x}, \quad t \in J \\ x(0) &= 0, \quad x'(1) = \delta x(\eta), \end{aligned}$$

where, $0 < \delta\eta \leq 1$. It is easy to see that $\alpha = 0$ is a lower solution and $\beta = t$ is an upper solution of the probem. Moreover, f satisfies a Nagumo condition. Hence, the problem has a solution in $[\alpha, \beta]$.

3.2.2 Quasilinearization technique

Now we study approximation of solutions by the method of quasilinearization and show that under certain conditions on f and g, there exists a monotone sequence of solutions of linear problems that converges quadratically to a solution of the original problem.

Theorem 3.2.6. *Assume that*

(A_1) *α, $\beta \in C^2(J)$ are lower and upper solutions of (3.2.1) such that $\alpha \leq \beta$ on J.*

(A_2) *$f \in C^2(J \times \mathbb{R} \times \mathbb{R})$ satisfies a Nagumo condition on J relative to α, β and is such that $f_{x'}(t,x,y)$ is non increasing in y for $(t,x) \in J \times [\min\alpha(t), \max\beta(t)]$, $f_x \leq 0$ and $H(f) \geq 0$ on $J \times [\min\alpha(t), \max\beta(t)] \times [-C, C]$, where*

$$H(f) = (x-y)^2 f_{xx}(t,z_1,z_2) + 2(x-y)(x'-y')f_{xx'}(t,z_1,z_2) + (x'-y')^2 f_{x'x'}(t,z_1,z_2)$$

is the quadratic form of f, z_1 lies between y and x, z_2 lies between x' and y'.

(A_3) *For $(t,x) \in I \times [\min\alpha(t), \max\beta(t)]$, $f_{x'}(t,x,x')$ satisfies*

$$|f_{x'}(t,x,y_1) - f_{x'}(t,x,y_2)| \leq L|y_1 - y_2|, \; y_1, \; y_2 \in \mathbb{R},$$
$$f_{x'}(t,x,C) \leq -2LC, \; f_{x'}(t,x,-C) \geq 2LC,$$

where $L > 0$.

(A_4) *$g(x) \in C^2(\mathbb{R})$, $0 \leq g'(x) < 1$ and $g''(x) \geq 0$ on $[\min\alpha(t), \max\beta(t)]$.*

Then, there exists a monotone sequence $\{w_n\}$ of solutions of linear problems converging uniformly and quadratically to the unique solution of the problem (3.2.1).

Proof. The conditions (A_1), (A_4) and the Nagumo condition on f assure the existence of a solution of the boundary value problem (3.2.1), (by Theorem 3.2.2). Moreover, there exists a constant $N \geq \lambda$ depending on α, β and ω (a Nagumo function) such that any solution of (3.2.1) satisfying $\alpha(t) \leq x(t) \leq \beta(t)$ on J satisfies $|x'(t)| \leq N$ on J, (by Theorem 1.1.7). Choose $C > \max\{N, \|\alpha'\|, \|\beta'\|\}$ and define $q(x'(t)) = \max\{-C, \min\{x'(t), C\}\}$.
Now consider the boundary value problem

$$\begin{gathered} -x''(t) = f(t,x,q(x')), \; t \in J \\ x(0) = a, \quad x'(1) = g(x(\eta)), \quad 0 < \eta \leq 1. \end{gathered} \tag{3.2.5}$$

We note that any solution $x \in C^2(J)$ of (3.2.5) with $\alpha(t) \leq x \leq \beta(t)$ is such that $|x'(t)| \leq C$, and hence is a solution of (3.2.1). It suffices to study (3.2.5).

Now, in view of (A_2) and Taylor's theorem, we have

$$f(t,x,q(x')) \geq f(t,y,q(y')) + f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(q(x') - q(y')), \tag{3.2.6}$$

for $(t,x,x'), (t,y,y') \in S$. Also, in view of (A_4), we have

$$g(x) \geq g(y) + g'(y)(x-y), \tag{3.2.7}$$

for x, $y \in [\min\alpha(t), \max\beta(t)]$. Define

$$K(t,x,x';y,y') = f(t,y,q(y')) + f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(q(x') - q(y')), \tag{3.2.8}$$

where x, y, x', $y' \in \mathbb{R}$, $t \in J$. Then K is continuous and satisfies the following relations

$$K_x(t, x, x'; y, y') = f_x(t, x, q(x')) \geq 0 \text{ and}$$

$$\begin{cases} f(t, x, q(x')) \geq K(t, x, x'; y, y') \\ f(t, x, q(x')) = K(t, x, x'; x, x') \end{cases} \tag{3.2.9}$$

for $(t, x, x'), (t, y, y') \in S$. Define $G(x, y)$ by

$$G(x, y) = g(y) + g'(y)(x - y), \tag{3.2.10}$$

where x, $y \in \mathbb{R}$. Then G is continuous, bounded and satisfies the relations $0 \leq G_x(x, y) < 1$ and

$$\begin{cases} G(x, y) \leq g(x), \\ G(x, x) = g(x), \end{cases} \tag{3.2.11}$$

for x, $y \in [\min \alpha(t), \max \beta(t)]$. Since $K(t, x, x'; y, y')$ is continuous and bounded on $J \times [\min \alpha(t), \max \beta(t)] \times \mathbb{R}$, therefore satisfies a Nagumo condition on J relative to α, β. Hence there exists a constant $C_1 > \lambda$ such that any solution x of

$$-x''(t) = K(t, x, x'; y, y'),\ t \in J$$
$$x(0) = a,\ x'(1) = G(x(\eta), y(\eta)),$$

with $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in J$ satisfies

$$|x'(t)| \leq C_1,\ t \in J \text{ for } (t, y, y') \in S.$$

Now, set $\alpha = w_0$ and consider the linear problem with the linear boundary conditions

$$\begin{aligned} -x''(t) &= K(t, x, x'; w_0, w_0'),\ t \in J \\ x(0) &= a,\ x'(1) = G(x(\eta), w_0(\eta)). \end{aligned} \tag{3.2.12}$$

Using (A_1), (3.2.9) and (3.2.11), we get

$$-w_0''(t) \leq f(t, w_0(t), w_0'(t)) = K(t, w_0(t), w_0'(t); w_0(t), w_0'(t)),\ t \in J$$
$$w_0(0) \leq a,\ w_0'(1) \leq g(w_0(\eta)) = G(w_0(\eta), w_0(\eta)),$$

and

$$-\beta''(t) \geq f(t, \beta(t), \beta'(t)) \geq K(t, \beta(t), \beta'(t); w_0(t), w_0'(t)),\ t \in J$$
$$\beta(0) \geq a,\ \beta(1) \geq g(\beta(\eta)) \geq G(\beta(\eta), w_0(\eta)),$$

which imply that w_0 and β are lower and upper solutions of (3.2.12) respectively. Hence, by Theorems 3.2.2, 3.2.3, there exists a unique solution w_1 of (3.2.12) such that

$$w_0(t) \leq w_1(t) \leq \beta(t) \text{ on } J.$$

Using (3.2.9), (3.2.11) and the fact that w_1 is a solution of (3.2.12), we obtain

$$\begin{aligned} -w_1''(t) &= K(t, w_1(t), w_1'(t); w_0(t), w_0'(t)) \leq f(t, w_1(t), q(w_1'(t))),\ t \in J \\ w_1(0) &= a,\ w_1(1) = G(w_1(\eta), w_0(\eta)) \leq g(w_1(\eta)), \end{aligned} \tag{3.2.13}$$

which implies that w_1 is a lower solution of (3.2.5). Now, consider the boundary value problem

$$\begin{aligned} -x''(t) &= K(t, x, x'; w_1, w_1'),\ t \in J \\ x(0) &= a,\ x'(1) = G(x(\eta), w_1(\eta)). \end{aligned} \tag{3.2.14}$$

By (A_1), (3.2.13), (3.2.9) and (3.2.11), we can show that w_1 and β are lower and upper solutions of (3.2.14). Hence by Theorems 3.2.2, 3.2.3, there exists a unique solution w_2 of (3.2.14) such that

$$w_1(t) \leq w_2(t) \leq \beta(t) \text{ on } J.$$

Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0 \leq w_1 \leq w_2 \leq w_3 \leq \ldots \leq w_{n-1} \leq w_n \leq \beta,\ t \in J, \tag{3.2.15}$$

where w_n is a solution of the linear problem

$$\begin{aligned} -x''(t) &= K(t, x, x'; w_{n-1}, w_{n-1}'),\ t \in J \\ x(0) &= a,\ x'(1) = G(x(\eta), w_{n-1}(\eta)), \end{aligned}$$

and

$$w_n(t) = a + G(w_n(\eta), w_{n-1}(\eta))t + \int_0^1 k(t,s)K(s, w_n, w_n'; w_{n-1}, w_{n-1}')ds. \tag{3.2.16}$$

Since $(t, w_{n-1}, w_{n-1}'),\ (t, w_n, w_n') \in J \times [\min w_0(t), \max \beta(t)] \times [-C_1, C_1]$, there exists a constant $M_1 > 0$ such that $|K(t, w_n, w_n'; w_{n-1}, w_{n-1}')| \leq M_1$ on J. Thus for any $s, t \in J (s \leq t)$, we have

$$|w_n'(t) - w_n'(s)| \leq \int_s^t |K(u, w_n, w_n'; w_{n-1}, w_{n-1}')|du \leq M_1|t - s|. \tag{3.2.17}$$

From (3.2.15), (3.2.16) and (3.2.17), it follows that the sequences $\{w_n^{(j)}\}(j = 0, 1)$ are uniformly bounded and equicontinuous on J. The Arzelà-Ascoli theorem guarantees the existence of subsequences and a function $x \in C^2(J)$ such that $w_n^{(j)} \to x^{(j)} (j = 0, 1)$ uniformly on J as $n \to \infty$. It follows that $K(t, w_n, w_n'; w_{n-1}, w_{n-1}') \to f(t, x, q(x'))$ as $n \to \infty$. Passing to the limit in (3.2.16), we obtain

$$x(t) = a + g(x(\eta))t + \int_0^1 k(t,s) f(s, x, q(x'))ds,$$

that is, $x(t)$ is a solution of (3.2.5).

To show the quadratic convergence of the sequence, we set $v_n(t) = x(t) - w_n(t)$, $t \in J$, then $v_n(0) = 0$, and $v_n(t) \geq 0$, $t \in J$. Using the mean value theorem and the definition of $G(x, y)$, we have

$$\begin{aligned}
v'_{n+1}(1) &= g(x(\eta)) - G(w_{n+1}(\eta), w_n(\eta)) \\
&= g(w_n(\eta)) + g'(w_n(\eta))\big(x(\eta) - w_n(\eta)\big) + \frac{g''(\xi)}{2!}\big(x(\eta) - w_n(\eta)\big)^2 \\
&- \Big[g(w_n(\eta)) + g'(w_n(\eta))\big(w_{n+1}(\eta) - w_n(\eta)\big)\Big] \\
&= g'(w_n(\eta))v_{n+1}(\eta) + \frac{g''(\xi)}{2}v_n^2(\eta) \\
&\leq g'(w_n(\eta))v_{n+1}(\eta) + d\|v_n\|_1^2,
\end{aligned}$$

where, $0 \leq \frac{g''(\xi)}{2} \leq d$. Moreover, using (3.2.8) and Taylor's theorem, we get

$$\begin{aligned}
-v''_{n+1}(t) &= -x''(t) + w''_{n+1}(t) = f(t, x, x') - K(t, w_{n+1}, w'_{n+1}; w_n, w'_n) \\
&= f(t, w_n, q(w'_n)) + f_x(t, w_n, q(w'_n))(x - w_n) + f_{x'}(t, w_n, q(w'_n))(x' - q(w'_n)) \\
&+ \frac{1}{2}H(f) - [f(t, w_n, q(w'_n)) - f_x(t, w_n, q(w'_n))(w_{n+1} - w_n) \\
&- f_{x'}(t, w_n, q(w'_n))(q(w'_{n+1}) - q(w'_n))] \\
&= f_x(t, w_n, q(w'_n))v_{n+1} + f_{x'}(t, w_n, q(w'_n))(x' - q(w'_{n+1})) + \frac{1}{2}H(f) \\
&= f_x(t, w_n, q(w'_n))v_{n+1} + f_{x'}(t, w_n, q(w'_n))v'_{n+1} \\
&+ f_{x'}(t, w_n, q(w'_n))(w'_{n+1} - q(w'_{n+1})) + \frac{1}{2}H(f), \; t \in J
\end{aligned} \tag{3.2.18}$$

where,

$$\begin{aligned}
H(f) = (x - w_n)^2 f_{xx}(t, \xi_1, \xi_2) + 2(x - w_n)(x' - q(w'_n))f_{xx'}(t, \xi_1, \xi_2) \\
+(x' - q(w'_n))^2 f_{x'x'}(t, \xi_1, \xi_2),
\end{aligned}$$

$w_n(t) \leq \xi_1 \leq x(t)$ and ξ_2 lies between $q(w'_n(t))$ and $x'(t)$. Let

$$\begin{aligned}
R = \max\big\{|f_{xx}(t, \xi_1, \xi_2)|, |f_{xx'}(t, \xi_1, \xi_2)|, |f_{x'x'}(t, \xi_1, \xi_2)| : t \in J,\ \xi_1 \in [\min w_0(t), \max \beta(t)], \\
\xi_2 \in [-C_1, C_1]\big\},
\end{aligned}$$

then

$$|H(f)| \leq R\big(|x - w_n|^2 + 2|x - w_n||x' - q(w'_n)| + |x' - q(w'_n)|^2\big),$$

which in view of the relation $|x' - q(w'_n)| \leq |x' - w'_n| = |v'_n|$, implies that

$$|H(f)| \leq R(|v_n| + |v'_n|)^2 \leq R\|v_n\|_1^2.$$

Using this and the assumption $(A_2)(f_x \leq 0)$, we obtain

$$-v''_{n+1}(t) \leq f_{x'}(t, w_n, q(w'_n))v'_{n+1}(t) + f_{x'}(t, w_n, q(w'_n))(w'_{n+1} - q(w'_{n+1})) + \frac{R}{2}\|v_n\|_1^2. \tag{3.2.19}$$

1. If for some $t \in I$ we have $|w'_{n+1}(t)| \leq C$, then

$$w'_{n+1}(t) - q(w'_{n+1}(t)) = 0,$$

hence

$$f_{x'}(t, w_{n-1}, q(w'_n))(w'_{n+1} - q(w'_{n+1})) = 0.$$

2. If for some $t \in I$ we have $w'_{n+1}(t) > C$, then $w'_{n+1}(t) - q(w'_{n+1}(t)) > 0$, and using (A_3), we obtain

$$\begin{aligned} f_{x'}(t, w_n(t), q(w'_n(t))) &\leq f_{x'}(t, w_n(t), q(w'_{n+1}(t))) + L|q(w'_n(t)) - q(w'_{n+1}(t))| \\ &\leq f_{x'}(t, w_n(t), C) + 2LC \leq 0. \end{aligned}$$

Hence

$$f_{x'}(t, w_n(t), q(w'_n(t)))(w'_{n+1}(t) - q(w'_{n+1}(t))) \leq 0.$$

3. If for some $t \in I$ we have $w'_{n+1}(t) < -C$, then $w'_{n+1}(t) - q(w'_{n+1}(t)) < 0$, and using (A_3), we have

$$\begin{aligned} f_{x'}(t, w_n(t), q(w'_n(t))) &\geq f_{x'}(t, w_n(t), q(w'_{n+1}(t))) - L|q(w'_n(t)) - q(w'_{n+1}(t))| \\ &\geq f_{x'}(t, w_{n-1}(t), -C) - 2LC \geq 0. \end{aligned}$$

Hence

$$f_{x'}(t, w_n(t), q(w'_n(t)))(w'_{n+1}(t) - q(w'_{n+1}(t))) \leq 0.$$

Thus (3.2.19) can be rewritten as

$$-v''_{n+1}(t) \leq f_{x'}(t, w_n, q(w'_n))v'_{n+1} + \frac{R}{2}\|v_n\|_1^2. \tag{3.2.20}$$

By the maximum principle, $v_{n+1}(t) \leq r(t)$ on J, where $r(t)$ is a solution of the linear boundary value problem

$$\begin{aligned} -r''(t) &= f_{x'}(t, w_n, q(w'_n))r'(t) + \frac{R}{2}\|v_n\|_1^2, \ t \in J \\ r(0) &= 0, \ r'(1) = g'(w_n(\eta))v_{n+1}(\eta) + d\|v_n\|_1^2. \end{aligned} \tag{3.2.21}$$

Since $(t, w_n, q(w'_n)) \in J \times [\min w_0(t), \max \beta(t)] \times [-C, C]$ and $f_{x'}$ is continuous on $J \times \mathbb{R}^2$, there exist L, $l_1 > 0$ such that

$$-l_1 \leq f_{x'}(t, w_n, q(w'_n)) \leq L, \ t \in J.$$

Thus, $L - f_{x'}(t, w_n, q(w'_n)) \geq 0$ on J. We rewrite (3.2.21) as

$$\begin{aligned} r''(t) + Lr'(t) &= \big(L - f_{x'}(t, w_n, q(w'_n))\big)r'(t) - \frac{R}{2}\|v_n\|_1^2, \ t \in J \\ r(0) &= 0, \ r'(1) = g'(w_n(\eta))v_{n+1}(\eta) + d\|v_n\|_1^2. \end{aligned}$$

This is equivalent to the integral equation

$$r(t) = \big(g'(w_n(\eta))v_{n+1}(\eta) + d\|v_n\|_1^2\big)t + \int_0^1 k_L(t,s)\Big[\big(L - f_{x'}(s, w_n, q(w_n'))\big)r'(s) - \frac{R}{2}\|v_n\|_1^2\Big]ds. \tag{3.2.22}$$

Let $\mu(t) = e^{\int_0^t f_{x'}(s,w_n(s),w_n'(s))ds}$, then $\mu(t)$ satisfies

$$e^{-l_1 t} < \mu(t) \leq e^{Lt},\ t \in J, \tag{3.2.23}$$

and from (3.2.21) it follows that

$$(r'(t)\mu(t))' = -\frac{R}{2}\|v_n\|_1^2\mu(t) \leq 0.$$

This implies that the function

$$\psi(t) = r'(t)\mu(t),\ t \in J$$

is non-increasing in t. That is, $\psi(t) \geq \psi(1)$, $t \in J$, which implies that

$$r'(t) \geq \frac{r'(1)\mu(1)}{\mu(t)} \geq 0.$$

Thus,

$$k_L(t,s)(L - f_{x'}(t, w_n, q(w_n')))r'(t) \leq 0. \tag{3.2.24}$$

Substituting (3.2.24) in (3.2.22), we obtain

$$\begin{aligned} v_{n+1}(t) \leq r(t) \leq & \big(g'(w_n(\eta))v_{n+1}(\eta) + d\|v_n\|_1^2\big)t + \frac{R}{2}\|v_n\|_1^2\int_0^1 |k_L(t,s)|ds \\ & \leq l v_{n+1}(\eta) + \Big(dt + \frac{R}{2}\int_0^1 |k_L(t,s)|ds\Big)\|v_n\|_1^2, \end{aligned} \tag{3.2.25}$$

where $0 \leq g'(w_n(\eta)) \leq l < 1$. At $t = \eta$, we have

$$v_{n+1}(\eta) \leq \frac{D}{1-l}\|v_n\|_1^2, \tag{3.2.26}$$

where $D = \max\big\{dt + \frac{R}{2}\int_0^1 |k_L(t,s)|ds;\ t \in J\big\}$. Taking the maximum over J and substituting for $v_{n+1}(\eta)$, we obtain

$$\|v_{n+1}\| \leq \frac{D}{1-l}\|v_n\|_1^2. \tag{3.2.27}$$

Using (3.2.23) in (3.2.20), we have

$$\big(v_{n+1}'(t)\mu(t)\big)' \geq -\frac{R}{2}\|v_n\|_1^2\mu(t) \geq -\frac{R}{2}e^{Lt}\|v_n\|_1^2. \tag{3.2.28}$$

Note that $v_{n+1}'(0) \geq 0$, since $v_{n+1}(t) \geq 0$ on J. Integrating (3.2.28) from 0 to t, using the boundary conditions $v_{n+1}'(0) \geq 0$, we obtain

$$v_{n+1}'(t)\mu(t) \geq -\frac{R(e^{Lt}-1)}{2L}\|v_n\|_1^2,$$

which in view of (3.2.23) implies that

$$v'_{n+1}(t) \geq -\frac{R(e^{Lt}-1)}{2L\mu(t)}\|v_n\|_1^2 \geq -\frac{Re^{l_1 t}(e^{Lt}-1)}{2L}\|v_n\|_1^2 \geq -\delta_1\|v_n\|_1^2,\ t \in J \qquad (3.2.29)$$

where $\delta_1 = \max\{\frac{Re^{l_1 t}(e^{Lt}-1)}{2L} : t \in J\}$. Again integrating (3.2.28) from t to 1, using (3.2.23), we have

$$v'_{n+1}(t)\mu(t) \leq v'_{n+1}(1)\mu(1) + \frac{R(e^L - e^{Lt})}{2L}\|v_n\|_1^2,$$

which implies that

$$v'_{n+1}(t) \leq v'_{n+1}(1)\frac{\mu(1)}{\mu(t)} + \frac{R(e^L - e^{Lt})}{2L\mu(t)}\|v_n\|_1^2 \leq v'_{n+1}(1)\frac{\mu(1)}{\mu(t)} + \frac{Re^{l_1 t}(e^L - e^{Lt})}{2L}\|v_n\|_1^2.$$

Using the boundary conditions $v'_{n+1}(1) \leq g'(w_n(\eta))v_{n+1}(\eta) + d\|v_n\|_1^2$ and (3.2.26), we obtain

$$\begin{aligned} v'_{n+1}(t) &\leq \frac{\mu(1)}{\mu(t)}(g'(w_n(\eta))v_{n+1}(\eta) + d\|v_n\|_1^2) + \frac{Re^{l_1 t}(e^L - e^{Lt})}{2L}\|v_n\|_1^2 \\ &\leq \frac{\mu(1)}{\mu(t)}(\frac{lD}{1-l}\|v_n\|_1^2 + d\|v_n\|_1^2) + \frac{Re^{l_1 t}(e^L - e^{Lt})}{2L}\|v_n\|_1^2 \\ &= (\mu(1)\mu(t)(\frac{lD}{1-l} + d) + \frac{Re^{l_1 t}(e^L - e^{Lt})}{2L})\|v_n\|_1^2 \leq \delta_2\|v_n\|_1^2, \end{aligned} \qquad (3.2.30)$$

where, $\delta_2 = \max\{\frac{\mu(1)}{\mu(t)}(\frac{lD}{1-l} + d) + \frac{Re^{l_1 t}(e^L - e^{Lt})}{2L} : t \in J\}$. Let $\tilde{\delta} = \max\{\delta_1, \delta_2\}$, then

$$\|v'_{n+1}\| \leq \tilde{\delta}\|v_n\|_1^2. \qquad (3.2.31)$$

From (3.2.27) and (3.2.31), we get

$$\|v_{n+1}\|_1 = \|v_{n+1}\| + \|v'_{n+1}\| \leq (\frac{D}{1-l} + \tilde{\delta}) \leq Q\|v_n\|_1^2,$$

where $Q = \frac{D}{1-l} + \tilde{\delta}$, which show that the convergence is quadratic. □

Chapter 4

Existence and approximation of solutions of second order nonlinear four point boundary value problems

4.1 Introduction

In this chapter, we study existence and approximation of solutions of second order nonlinear differential equations with four point boundary conditions (BCs) of the type

$$\begin{aligned} &x''(t) = f(t, x, x'), \ t \in I = [a, b], \\ &x(a) = x(c), \ x(b) = x(d), \end{aligned} \tag{4.1.1}$$

where $a < c \leq d < b$. We use the method of lower and upper solutions to establish existence of solutions. Approximation of solutions uses the method of quasilinearization. We approximate our problem by a sequence of linear problems to obtain a monotone sequence of approximants. We show that under suitable conditions, the sequence converges quadratically to a solution of the original problem, which is joint work with Rosana Rodriguez Lopez and part of it is accepted for publication [52].

Existence theory for the solution of four point boundary value problems had been given in a number of papers by Rachunkova [82–84]. In theorem 1 of [84], Rachunkova, proved existence of solutions for the four point boundary value problem (4.1.1) under various combinations of sign conditions on the function $f(t, x, x')$. In Theorem (4.2.3) of this chapter we study the existence of solutions under the more general conditions of the existence of upper and lower solutions, that is, there exist α, $\beta \in C^2(I)$ such that

$$f(t, \alpha(t), \alpha'(t)) \leq \alpha''(t), \quad f(t, \beta(t), \beta'(t)) \geq \beta''(t), \ t \in I. \tag{4.1.2}$$

The conditions (4.1.2) include the corresponding conditions

$$f(t, r_1, 0) \leq 0,\ f(t, r_2, 0) \geq 0, \quad r_1 \leq r_2,\ r_1,\ r_2 \in \mathbb{R},$$

studied in theorem 1 of [84]. Moreover, we also study existence results under the following conditions

$$\begin{cases} f(t, x, -R) \leq 0,\ f(t, x, R) \geq 0 \text{ for } x \in [\min \alpha(t), \max \beta(t)],\ t \in [a, d], \\ f(t, x, -R) \geq 0,\ f(t, x, R) \leq 0 \text{ for } x \in [\min \alpha(t), \max \beta(t)],\ t \in (d, b], \end{cases} \tag{4.1.3}$$

where $R \geq \max\{\|\alpha'|, \|\beta'\|\}$. The conditions (4.1.3) are less restrictive than the corresponding ones in [84], where it is assumed that

$$f(t, x, R_1) \leq 0,\ f(t, x, R_2) \geq 0 \text{ for all } x \in [r_1, r_2] \text{ and a.e. } t \in [a, b], \tag{4.1.4}$$

$$f(t, x, R_3) \geq 0,\ f(t, x, R_4) \leq 0 \text{ for a.e } t \in [d, b] \text{ and } x \in [r_1, r_2],$$

where, $R_1 \leq 0 \leq R_2$, $R_3 \leq 0 \leq R_4$, $R_1 \neq R_3$, $R_2 \neq R_4$.

In sections 3 and 4, we study approximation of solutions. We develop the generalized quasilinearization technique for the problem (4.1.1) to obtain a monotone sequence of approximate solutions of the four point problem which converge quadratically to a solution of the problem. To the best of our knowledge, the quasilinearization technique for four point problems (4.1.1) seems not to have been studied previously.

4.2 Upper and lower solutions

We recall the concept of lower and upper solution for the BVP (4.1.1), [82].

Definition 4.2.1. Let $\alpha \in C^2(I)$. We say that α is a lower solution of (4.1.1) if

$$\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)), \quad t \in (a, b)$$
$$\alpha(a) \leq \alpha(c),\ \alpha(b) \leq \alpha(d).$$

An upper solution β of the BVP (4.1.1) is defined similarly by reversing the inequalities.

Definition 4.2.2. A continuous function $\omega : [0, \infty) \to (0, \infty)$, will be called a Nagumo function if

$$\int_0^\infty \frac{s ds}{\omega(s)} = +\infty.$$

We say that $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Bernstein-Nagumo condition on I relative to α, β, if there exists a Nagumo function ω such that

$$f(t, x, y) \operatorname{sgn}(y) \leq \omega(|y|) \text{ on } I \times [\min \alpha, \max \beta] \times \mathbb{R}, \tag{4.2.1}$$

$$f(t, x, y) \operatorname{sgn}(y) \geq -\omega(|y|) \text{ on } [a, c] \times [\min \alpha, \max \beta] \times \mathbb{R}. \tag{4.2.2}$$

Now, we state and prove theorems which establish the existence and uniqueness of solutions.

Theorem 4.2.3. *Assume that α and β are respectively lower and upper solutions of* (4.1.1) *such that $\alpha(t) \le \beta(t)$, $t \in I$. If $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies the Bernstein-Nagumo condition, then there exists a solution $x(t)$ of the boundary value problem* (4.1.1) *such that*

$$\alpha(t) \le x(t) \le \beta(t),\ t \in I.$$

Moreover, there exists a constant C, depending only on α, β and ω, such that $|x'(t)| < C$ on I.

Proof. Let $r = \max_{t\in I}\beta(t) - \min_{t\in I}\alpha(t)$, then there exists $N > 0$, such that

$$\int_0^N \frac{sds}{\omega(s)} > r.$$

Let $C > \max\{N, \|\alpha'\|, \|\beta'\|\}$ and define $q(y) = \max\{-C, \min\{y, C\}\}$. Since $\int_0^x \frac{sds}{\omega(s)}$ is continuous and increasing in x, it follows that

$$\int_0^C \frac{sds}{\omega(s)} \ge \int_0^N \frac{sds}{\omega(s)} > r. \tag{4.2.3}$$

Consider the modified problem

$$\begin{aligned} &x''(t) = F(t,x,x'),\ t \in I,\\ &x(a) = x(c),\ x(b) = x(d), \end{aligned} \tag{4.2.4}$$

where

$$F(t,x,x') = \begin{cases} f(t,\beta(t),\beta'(t)) + \frac{x-\beta(t)}{1+|x-\beta(t)|}, & \text{if } x \ge \beta(t)+\epsilon,\\ f(t,\beta(t),x') + \big[f(t,\beta(t),\beta'(t)) & \\ \quad - f(t,\beta(t),x') + \frac{x-\beta(t)}{1+|x-\beta(t)|}\big]\frac{x-\beta(t)}{\epsilon}, & \text{if } \beta(t) \le x < \beta(t)+\epsilon,\\ f(t,x,x'), & \text{if } \alpha(t) \le x \le \beta(t),\\ f(t,\alpha(t),x') - \big[f(t,\alpha(t),\alpha'(t)) & \\ \quad - f(t,\alpha(t),x') + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}\big]\frac{x-\alpha(t)}{\epsilon}, & \text{if } \alpha(t)-\epsilon < x \le \alpha(t),\\ f(t,\alpha(t),\alpha'(t)) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}, & \text{if } x \le \alpha(t)-\epsilon, \end{cases}$$

where $\epsilon > 0$ a small fixed number. Note that $F(t,x,x')$ is continuous on $I \times \mathbb{R}^2$. Further, we note that any solution x of (4.2.4) satisfying the relations $\alpha(t) \le x(t) \le \beta(t)$, $t \in I$, is a solution of (4.1.1). For the existence of solution of (4.2.4), we consider the system

$$\begin{aligned} &x'' = \lambda F(t,x,x') + (1-\lambda)(\sigma(t,x,x') + x),\ t \in I\\ &x(a) = x(c),\ x(b) = x(d), \end{aligned} \tag{4.2.5}$$

where $\lambda \in [0,1]$, and

$$\sigma(t,x,x') = f\big(t,p(t,x),q(x')\big) - p(t,x),$$
$$p(t,x) = \max\{\alpha(t), \min\{x,\beta(t)\}\}.$$

For $\lambda = 0$, the system reduces to

$$\begin{aligned} &x''(t) = x(t) + \sigma(t,x,x'),\ t \in I,\\ &x(a) = x(c),\ x(b) = x(d), \end{aligned} \tag{4.2.6}$$

and for $\lambda = 1$, it is (4.2.4).

Since $\sigma(t,x,x')$ is continuous and bounded and the linear problem

$$x''(t) = x(t),\ t \in I,$$
$$x(a) = x(c),\ x(b) = x(d),$$

has only a trivial solution, it follows from ([82], Lemma 1), that the problem (4.2.6) has a solution. That is, (4.2.5) has a solution for $\lambda = 0$. For $\lambda \in [0,1]$, we claim that any solution x of (4.2.5) satisfies the inequality

$$\alpha(t) \le x(t) \le \beta(t),\ t \in I.$$

Set $v(t) = x(t) - \beta(t)$ and suppose that $v(t)$ has a positive maximum at some $t = t_0 \in I$. The boundary conditions imply that $v(a) \le v(c)$, $v(b) \le v(d)$, so we can suppose $t_0 \in (a,b)$. It follows that $v(t_0) > 0$, $v'(t_0) = 0$, $v''(t_0) \le 0$. If $\beta(t_0) \le x(t_0) < \beta(t_0) + \epsilon$, then

$$\begin{aligned} v''(t_0) =& x''(t_0) - \beta''(t_0) \ge \lambda\big[f(t_0,\beta(t_0),\beta'(t_0)) + \frac{x(t_0)-\beta(t_0)}{1+|x(t_0)-\beta(t_0)|}\frac{x(t_0)-\beta(t_0)}{\epsilon}\big]\\ &+ (1-\lambda)\big[f(t_0,\beta(t_0),\beta'(t_0)) - \beta(t_0) + x(t_0)\big] - f(t_0,\beta(t_0),\beta'(t_0))\\ =& \frac{\lambda(v(t_0))^2}{(1+v(t_0))\epsilon} + (1-\lambda)v(t_0) > 0, \end{aligned}$$

a contradiction. If $x(t_0) \ge \beta(t_0) + \epsilon$, then

$$v''(t_0) \ge \lambda\frac{v(t_0)}{(1+v(t_0))} + (1-\lambda)v(t_0) > 0,$$

again a contradiction. It follows that $x(t) \le \beta(t)$ for every $t \in I$. Similarly, we can show that $x(t) \ge \alpha(t)$ for every $t \in I$. Thus, x is a solution of (4.1.1).

Now, we show that $|x'(t)| < C$ on I. The BCs imply the existence of $a_1 \in (a,c)$ and $b_1 \in (d,b)$ such that $x'(a_1) = 0$, $x'(b_1) = 0$. Suppose that there exists $t_0 \in (a_1,b_1)$ such that $x'(t_0) \ge C$. Let $[t_1,t_2] \subset [a_1,b_1]$ be the maximal interval containing t_0 such that $x'(t) > 0$ for $t \in (t_1,t_2)$. Let $\max\{x'(t) : t \in [t_1,t_2]\} = x'(t^*) = C_1$, then $C_1 \ge C$ and as in the last paragraph of the proof of Theorem 2.2.4, we arrive at a contradiction.

Now, suppose $t_0 \in [a,a_1)$ and $[t_1,t_2] \subset [a,a_1]$ be the maximal interval containing t_0 such that $x'(t) > 0$, for $t \in (t_1, t_2)$. Note that t_1, t_2 are such that $t_2 \leq a_1$, $t_1 \geq a$ with $x'(t_2) = 0$ and $x'(t_1) \geq 0$. Let $\max\{x'(t) : t \in [t_1, t_2]\} = x'(t^*) = C_1$, then $C_1 \geq C$ and $t^* < t_2$. Since $x \in [\alpha(t), \beta(t)]$, for $t \in (t_1, t_2)$, using (4.2.2), we obtain

$$x''(t) = f(t, x, x') \geq -\omega(x'), \text{ for } t \in (t_1, t_2),$$

which implies that

$$-x'(t)x''(t)/\omega(x') \leq x'(t), \; t \in (t_1, t_2).$$

Integrating from t^* to t_2, we obtain

$$\int_0^{C_1} \frac{sds}{\omega(s)} \leq x(t_2) - x(t^*) \leq r,$$

a contradiction.

If $t_0 \in (b_1, b]$, let $[t_1, t_2]$ be the maximal interval in $[b_1, b]$ containing t_0 such that

$$x'(t) > 0, \text{ for } t \in (t_1, t_2),$$

where $t_1 \geq b_1$, and $t_2 \leq b$ are such that $x'(t_1) = 0$ and $x'(t_2) \geq 0$. Then $t^* > t_1$, where $t^* \in (t_1, t_2]$ is the point such that $x'(t^*) = \max\{x'(t) : t \in [t_1, t_2]\}$. Since $x \in [\alpha, \beta]$, for $t \in (t_1, t_2)$, using (4.2.1), we obtain

$$\frac{x'(t)x''(t)}{\omega(x')} \leq x'(t), \text{ for } t \in (t_1, t_2).$$

Integrating from t_1 to t^*, we obtain

$$\int_0^{C_1} \frac{sds}{\omega(s)} \leq x(t^*) - x(t_1) \leq r,$$

a contradiction. Hence $x'(t) < C$, $t \in I$.

Now, we show that $x'(t) > -C$, $t \in I$. Assume that there exists $t_0 \in (a_1, b_1)$ such that $x'(t_0) \leq -C$. Let $[t_1, t_2] \subset [a_1, b_1]$ be the maximal interval containing t_0 such that $x'(t) < 0$ for $t \in (t_1, t_2)$. Let $\min\{x'(t) : t \in [t_1, t_2]\} = x'(t^*) = -C_2$, then $C_2 \geq C$ and in view of (4.2.3), we have $\int_0^{C_2} \frac{sds}{\omega(s)} > r$. Now for $t \in (t_1, t_2)$ since $x \in [\alpha(t), \beta(t)]$, we have

$$x''(t) = f(t, x, x') \geq -\omega(|x'|).$$

It follows that

$$\frac{x'(t)x''(t)}{\omega(|x'|)} \leq -x'(t),$$

and hence

$$\int_0^{C_2} \frac{sds}{\omega(s)} \leq x(t_1) - x(t^*) \leq \max_{t \in I} \beta(t) - \min_{t \in I} \alpha(t) = r,$$

a contradiction. If $t_0 \in [a, a_1)$ or $(b_1, b]$, we get a contradiction in the same way as above. Hence $|x'(t)| < C$, $t \in I$. □

Example 4.2.4. Consider the boundary value problem

$$\begin{aligned} x''(t) &= -x'(t) + g(x)\phi(t),\ t \in [0,1], \\ x(0) &= x(\eta),\ x(\delta) = x(1), \end{aligned} \tag{4.2.7}$$

where, $0 < \eta \leq \delta < 1$, $g : \mathbb{R} \to \mathbb{R}$ is continuous and $\phi \in C^2[0,1]$ satisfies the linear problem with constant coefficients

$$\begin{aligned} \phi''(t) + \phi'(t) - \phi(t) &= -\rho(t),\ t \in [0,1], \\ \phi(0) &= \phi(\eta),\ \phi(\delta) = \phi(1), \end{aligned} \tag{4.2.8}$$

where $\rho(t) \in C[0,1]$ and $\rho(t) \geq 0$ on $[0,1]$. Assume that $g(0) \leq 0$, g is increasing and there exists $a \geq 0$, such that $g(a) \geq 0$. Since

$$y''(t) + y'(t) - y(t) = 0, \tag{4.2.9}$$

$$y(0) = y(\eta),\ y(\delta) = y(1) \tag{4.2.10}$$

has only a trivial solution [83], and $\rho(t)$ is continuous and bounded on $[0,1]$, it follows that the BVP (4.2.8), has a solution. We claim that a solution $\phi(t)$ of (4.2.8) satisfies $\phi(t) \geq 0$. If not, then $\phi(t)$ has a negative minimum at some $t_0 \in [0,1]$. The boundary conditions imply that $t_0 \in (0,1)$ and hence

$$\phi(t_0) < 0,\ \phi'(t_0) = 0,\ \phi''(t_0) \geq 0.$$

However,

$$\phi''(t_0) = -\phi'(t_0) + \phi(t_0) - \rho(t_0) < 0$$

a contradiction. Thus, $\phi(t) \geq 0$ on $[0,1]$ and since $[0,1]$ is compact, there exists $L \geq 0$ such that $0 \leq \phi(t) \leq L$.

Now take $\alpha(t) = 0$. Since $g(0) \leq 0$, we have $\alpha''(t) + \alpha'(t) - g(\alpha)\phi(t) = -g(0)\phi(t) \geq 0$, which implies that α is a lower solution of the problem (4.2.7). Take $\beta(t) = a + g(a)\phi(t)$. As $\beta \geq a$ and g is increasing, we have $g(\beta(t)) \geq g(a)$. Moreover,

$$\begin{aligned} \beta''(t) + \beta'(t) - g(\beta)\phi(t) &\leq g(a)\phi''(t) + g(a)\phi'(t) - g(a)\phi(t) \\ &= g(a)[\phi''(t) + \phi'(t) - \phi(t)] \leq 0, \\ \beta(0) &= a + g(a)\phi(0) = a + g(a)\phi(\eta) = \beta(\eta), \\ \beta(\delta) &= a + g(a)\phi(\delta) = a + g(a)\phi(1) = \beta(1). \end{aligned}$$

Thus, β is an upper solution of the problem (4.2.7). Clearly, $\alpha(t) \leq \beta(t)$ on $[0,1]$. Now, for $t \in [0,1]$ and $x \in [\min \alpha(t), \max \beta(t)]$, we have $g(a + g(a)L) \geq g(\beta(t)) \geq g(x)$. Let $C_3 = \max\{|g(0)|,\ g(a + g(a)L)\}$, then for $t \in [0,1]$ and $x \in [\min \alpha(t), \max \beta(t)]$, we have

$$|-x'(t) + g(x)\phi(t)| \leq |x'(t)| + LC_3 = \omega(|x'(t)|),$$

where $\omega(s) = s + LC_3$ for $s \geq 0$. Moreover, $\int_0^\infty \frac{sds}{\omega(s)} = \int_0^\infty \frac{sds}{s+LC_3} = \infty$. Thus, the Nagumo condition is satisfied. Hence by Theorem 4.2.3, there exists a solution x of the BVP (4.2.7), such that $\alpha \leq x \leq \beta$.

Here we remark that for $x \in [0, a + g(a)\phi(t)]$, we have

$$f(t,x,R) = -R + g(x)\phi(t),$$
$$f(t,x,-R) = R + g(x)\phi(t).$$

Thus, under the assumed hypothesis we proved that

$$f(t,0,R) = -R + g(0)\phi(t) \leq 0, \text{ for } R \geq 0$$
$$f(t,a,-R) = R + g(a)\phi(t) \geq 0, \text{ for } -R \leq 0.$$

If $R > 0$, or $g(0) < 0$ and $\phi(t) > 0$ somewhere in $[0,1]$, or $g(a) > 0$ and $\phi(t) > 0$ somewhere in $[0,1]$, then it is not possible to apply theorem 1 of [84].

Theorem 4.2.5. *If in theorem 4.2.3, we replace the Bernstein-Nagumo conditions by the following sign conditions*

$$\begin{cases} f(t,x,-R) \leq 0, \ f(t,x,R) \geq 0 \ \textit{for } t \in [a,d], \\ f(t,x,-R) \geq 0, \ f(t,x,R) \leq 0 \ \textit{for } t \in (d,b], \end{cases} \tag{4.2.11}$$

where $x \in [\min \alpha(t), \max \beta(t)]$, $R \geq \max\{\|\alpha'\|, \|\beta'\|\}$. Then the conclusion of Theorem 4.2.3 is valid, taking $C = R$.

Our method of proof is close to that of theorem 1 of [84].

Proof. Let $n \in \mathbb{N}$ and consider the problem

$$\begin{aligned} &x''(t) = f_n(t,x,x'), \ t \in I = [a,b], \\ &x(a) = x(c), \ x(b) = x(d), \end{aligned} \tag{4.2.12}$$

where

$$f_n(t,x,x') = \begin{cases} f(t,\beta(t),\beta'(t)) + \frac{x-\beta(t)}{1+|x-\beta(t)|}, & \text{if } x \geq \beta(t) + \frac{1}{n}, \\ f(t,\beta(t),q(x')) + [f(t,\beta(t),\beta'(t)) & \\ \quad - f(t,\beta(t),q(x')) + \frac{x-\beta(t)}{1+|x-\beta(t)|}]n(x-\beta(t)), & \text{if } \beta(t) \leq x < \beta(t) + \frac{1}{n}, \\ f(t,x,q(x')), & \text{if } \alpha(t) \leq x \leq \beta(t), \\ f(t,\alpha(t),q(x')) - [f(t,\alpha(t),\alpha'(t)) & \\ \quad - f(t,\alpha(t),q(x')) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}]n(x-\alpha(t)), & \text{if } \alpha(t) - \frac{1}{n} < x \leq \alpha(t), \\ f(t,\alpha(t),\alpha'(t)) + \frac{x-\alpha(t)}{1+|x-\alpha(t)|}, & \text{if } x \leq \alpha(t) - \frac{1}{n}, \end{cases}$$

where $q(x') = \max\{-R, \min\{x', R\}\}$. Further, we note that $f_n(t,x,x')$ is continuous and bounded on $I \times \mathbb{R}^2$ and any solution $x(t)$ of (4.2.12) satisfying the relations $\alpha(t) \leq x(t) \leq$

$\beta(t)$, $|x'(t)| \le R$, $t \in I$, is a solution of (4.1.1). For the existence of solution of (4.2.12), we consider the homotopy

$$\begin{aligned} &x''(t) = \lambda f_n(t,x,x') + (1-\lambda)(\sigma_n(t,x,x') + \frac{x}{n}),\ t \in I \\ &x(a) = x(c),\ x(b) = x(d), \end{aligned} \tag{4.2.13}$$

where $\lambda \in [0,1]$, and

$$\sigma_n(t,x,x') = f\big(t, p(t,x), q(x')\big) - \frac{p(t,x)}{n},$$

where $p(t,x) = \max\{\alpha(t), \min\{x, \beta(t)\}\}$. For $\lambda = 0$, (4.2.13) has a solution. For $\lambda \in [0,1]$, we claim that any solution x_n of (4.2.13) satisfies the inequality

$$\alpha(t) - \frac{1}{n} \le x_n(t) \le \beta(t) + \frac{1}{n},\ t \in I. \tag{4.2.14}$$

Set $v(t) = x_n(t) - \beta(t) - 1/n$, $t \in I$ and suppose that $v(t)$ has a positive maximum at some $t = t_0 \in I$. The boundary conditions imply that $v(a) \le v(c)$, $v(b) \le v(d)$, so we can suppose $t_0 \in (a,b)$. It follows that $v(t_0) > 0$, $v'(t_0) = 0$, $v''(t_0) \le 0$. On the other hand,

$$\begin{aligned} v''(t_0) =& x_n''(t_0) - \beta''(t_0) \\ \ge& \lambda\Big[f(t_0,\beta(t_0),\beta'(t_0)) + \frac{v(t_0)+\frac{1}{n}}{1+v(t_0)+\frac{1}{n}}\Big] + (1-\lambda)\Big[f(t_0,\beta(t_0),\beta'(t_0)) \\ &- \frac{\beta(t_0)}{n} + \frac{x_n(t_0)}{n}\Big] - f(t_0,\beta(t_0),\beta'(t_0)) \\ =& \big(v(t_0) + \frac{1}{n}\big)\big(\frac{\lambda}{1+v(t_0)+\frac{1}{n}} + \frac{1-\lambda}{n}\big) > 0, \end{aligned}$$

a contradiction. It follows that $x_n(t) \le \beta(t) + \frac{1}{n}$ for every $t \in I$. Similarly, we can show that $x_n(t) \ge \alpha(t) - 1/n$ for every $t \in I$.

We obtain a sequence $\{x_n\}$ of solutions of problem (4.2.13) satisfying

$$\alpha(t) - \frac{1}{n} \le x_n(t) \le \beta(t) + \frac{1}{n},\ t \in I. \tag{4.2.15}$$

Moreover, the boundary conditions guarantee the existence of at least one point $t_1 \in (a,b)$ such that $x_n'(t_1) = 0$. Integrating (4.2.13) from t_1 to t, we obtain

$$|x_n'(t)| \le \int_a^b \big|\lambda f_n(t,x_n(t),x_n'(t)) + (1-\lambda)(\sigma_n(t,x_n(t),x_n'(t)) + x_n(t)/n)\big|dt,$$

which implies that $\{x_n'\}$ is uniformly bounded on I. Thus, the sequence $\{x_n\}$ is bounded and equicontinuous in $C^1(I)$ and so, by the Arzelà-Ascoli theorem it is possible to choose a subsequence converging in $C^1(I)$ to a function $x \in C^1(I)$. Since (4.2.15) holds for every $n \in \mathbb{N}$ and every $t \in I$, it follows that

$$\alpha(t) \le x(t) \le \beta(t),\ t \in I$$

and hence x is a solution of the problem

$$\begin{aligned} x''(t) &= f(t, x, q(x')),\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d). \end{aligned} \tag{4.2.16}$$

Now, we show that $|x'(t)| \leq R$, $t \in I$. Firstly, we show that $x'(t) \leq R$, $t \in I$. The boundary conditions $x(a) = x(c)$, $x(b) = x(d)$ imply that there exist $a_1 \in (a, c)$ and $b_1 \in (d, b)$ such that $x'(a_1) = 0$, $x'(b_1) = 0$. Suppose $\max\{x'(t) : t \in [a, b]\} = x'(t_0) \geq R + \frac{1}{m}$. Then, $t_0 \neq a_1, b_1$. If $t_0 \in [a, a_1)$, then there exist $t_1 \geq t_0$ and $t_2 \leq a_1$ with $t_1 < t_2$ such that

$$\begin{aligned} x'(t_1) = R + \frac{1}{m},\ x'(t_2) = R \text{ and} \\ R \leq x'(t) \leq R + \frac{1}{m}, \text{ for } t \in [t_1, t_2]. \end{aligned} \tag{4.2.17}$$

Integrating (4.2.16) from t_1 to t_2 and using (4.2.17) and conditions (4.2.11), we obtain

$$0 > \int_{t_1}^{t_2} x''(t)dt = \int_{t_1}^{t_2} f(t, x(t), R)dt \geq 0,$$

a contradiction. If $t_0 \in (a_1, b_1)$, then we can choose $t_1, t_2 \in (a_1, b_1]$ with $t_0 \leq t_1 < t_2 \leq b_1$ such that

$$\begin{aligned} x'(t_1) = R + \frac{1}{m},\ x'(t_2) = R \text{ and} \\ R \leq x'(t) \leq R + \frac{1}{m},\ t \in [t_1, t_2]. \end{aligned} \tag{4.2.18}$$

Integrating (4.2.16) from t_1 to t_2, using (4.2.18) and conditions (4.2.11), we obtain

$$0 > \int_{t_1}^{t_2} x''(t)dt = \int_{t_1}^{t_2} f(t, x(t), R)dt \geq 0,$$

again a contradiction. Now, if $t_0 \in (b_1, b]$, then there exist $t_1 \geq b_1$ and $t_2 \leq t_0$ with $t_1 < t_2$ such that

$$\begin{aligned} x'(t_1) = R,\ x'(t_2) = R + \frac{1}{m} \text{ and} \\ R \leq x'(t) \leq R + \frac{1}{m},\ t \in [t_1, t_2]. \end{aligned} \tag{4.2.19}$$

If we integrate (4.2.16) from t_1 to t_2, use (4.2.19) and conditions (4.2.11), we obtain

$$0 < \int_{t_1}^{t_2} x''(t)dt = \int_{t_1}^{t_2} f(t, x(t), R)dt \leq 0,$$

a contradiction. Hence, $x'(t) \leq R$, $t \in I$.

Similarly, using the conditions

$$f(t, x, -R) \leq 0 \text{ for } t \in [a, d] \text{ and } f(t, x, -R) \geq 0 \text{ for } t \in (d, b],$$

we can show that $x'(t) \geq -R$, $t \in I$. Consequently, $x(t)$ is a solution of the BVP (4.1.1).

□

Theorem 4.2.6. *Assume that α and β are lower and upper solutions of the boundary value problem* (4.1.1) *respectively. If $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $f(t,x,y)$ is strictly increasing in x for each $(t,y) \in I \times \mathbb{R}$, then*

$$\alpha(t) \leq \beta(t),\ t \in I.$$

Hence under these conditions, solutions are unique.

Proof. Define $w(t) = \alpha(t) - \beta(t)$, $t \in I$, then $w(t) \in C^2(I)$ and

$$w(a) \leq w(c),\ w(b) \leq w(d). \tag{4.2.20}$$

Suppose $w(t)$ has a positive maximum at $t_0 \in I$. The boundary conditions (4.2.20) imply that $t_0 \in (a,b)$ and hence $w(t_0) > 0$, $w'(t_0) = 0$ and $w''(t_0) \leq 0$. On the other hand, using the increasing property of the function $f(t,x,x')$ in x, we obtain

$$f(t_0,\alpha(t_0),\alpha'(t_0)) \leq \alpha''(t_0) \leq \beta''(t_0) \leq f(t_0,\beta(t_0),\beta'(t_0)) < f(t_0,\alpha(t_0),\alpha'(t_0)),$$

a contradiction. Hence

$$\alpha(t) \leq \beta(t), \quad t \in I.$$

□

4.3 Quasilinearization technique

Now, we study approximation of solutions by the method of quasilinearization.

Theorem 4.3.1. *Assume that*

(A_1) *α, $\beta \in C^2(I)$ are respectively lower and upper solutions of* (4.1.1) *such that $\alpha(t) \leq \beta(t)$, $t \in I$.*

(A_2) *$f \in C^2(I \times \mathbb{R}^2)$ satisfies a Bernstein-Nagumo condition on I relative to α, β and is such that $f_x(t,x,x') > 0$ and $H(f) \leq 0$ on $I \times \mathbb{R}^2$, where*

$$H(f) = (x-y)^2 f_{xx}(t,z_1,z_2) + 2(x-y)(x'-y')f_{xx'}(t,z_1,z_2) + (x'-y')^2 f_{x'x'}(t,z_1,z_2)$$

is the quadratic form of f and z_1 is between y, x and z_2 lies between x' and y'.

(A_3) *For $(t,x) \in I \times [\min \alpha(t), \max \beta(t)]$ and $P > \max\{\|\alpha'\|, \|\beta'\|\}$, $f_{x'}$ satisfies*

$$|f_{x'}(t,x,y_1) - f_{x'}(t,x,y_2)| \leq L|y_1 - y_2|,\ y_1, y_2 \in \mathbb{R},$$
$$f_{x'}(t,x,P) \geq 2LP, \quad f_{x'}(t,x,-P) \leq -2LP,$$

where $L > 0$.

Then, there exists a monotone sequence $\{w_n\}$ of solutions converging uniformly to the unique solution of the problem. Moreover, the sequence converges quadratically on I.

Proof. The condition (A_2) implies the existence of a positive continuous function $\omega : [0,\infty) \to (0,\infty)$ such that

$$|f(t,x,x')| \le \omega(|x'|) \text{ for } \alpha(t) \le x \le \beta(t),\ t \in I,\ x' \in \mathbb{R},$$

and

$$\int_0^\infty \frac{sds}{\omega(s)} = \infty.$$

Let $r = \max\{\beta(t) : t \in I\} - \min\{\alpha(t) : t \in I\}$, then there exists a constant $N > 0$ such that

$$\int_0^N \frac{sds}{\omega(s)} \ge r,$$

and hence as in the proof Theorem 1.1.7, any solution x of (4.1.1) with $\alpha(t) \le x \le \beta(t)$, $t \in I$ satisfies $|x'(t)| \le N$, $t \in I$.

Let $C > \max\{\|\alpha'\|, \|\beta'\|, N\}$ and define $q(x') = \max\{-C, \min\{x', C\}\}$. Consider the boundary value problem

$$\begin{aligned} x''(t) &= f(t,x,q(x')), t \in I \\ x(a) &= x(c),\ x(b) = x(d). \end{aligned} \tag{4.3.1}$$

Note that any solution $x \in C^2(I)$ of (4.3.1) such that $|x'(t)| \le C$, is a solution of (4.1.1). As in the proof of Theorem 3.1.2, any solution x of (4.3.1) such that $\alpha(t) \le x \le \beta(t)$, $t \in I$, does satisfy $|x'(t)| \le C$ on I and hence, is a solution of (4.1.1).

Now, in view of (A_2) and Taylor's theorem, we have

$$f(t,x,q(x')) \le f(t,y,q(y')) + f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(q(x') - q(y')),\ t \in I, \tag{4.3.2}$$

where $x, y, x', y' \in \mathbb{R}$. Define

$$h(t,x,x';y,y') = f(t,y,q(y')) + f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(q(x') - q(y')), \tag{4.3.3}$$

then, $h(t,x,x';y,y')$ satisfies the following relations

$$h_x(t,x,x';y,y') = f_x(t,y,q(y')) > 0 \text{ and}$$

$$\begin{cases} f(t,x,q(x')) \le h(t,x,x';y,y'), \\ f(t,x,q(x')) = h(t,x,x';x,x'), \end{cases} \tag{4.3.4}$$

for x, y, x', $y' \in \mathbb{R}$ and $t \in I$.

Moreover, h is continuous and is bounded on

$$I \times [\min \alpha(t), \max \beta(t)] \times \mathbb{R} \times [\min \alpha(t), \max \beta(t)] \times \mathbb{R},$$

and therefore satisfies a Bernstein-Nagumo condition on I relative to α, β. Hence there exists a constant $C_1 > 0$ such that any solution x of

$$x''(t) = h(t, x, x'; y, y'),\ t \in I,$$
$$x(a) = x(c),\ x(b) = x(d),$$

for y fixed, $\alpha(t) \leq y \leq \beta(t)$, $t \in I$ with the property $\alpha(t) \leq x \leq \beta(t)$, $t \in I$ must satisfy $|x'(t)| \leq C_1$ on I.

Now, set $w_0 = \alpha$ and consider the linear four point problem

$$\begin{aligned} x''(t) &= h(t, x, x'; w_0, w_0'),\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d), \end{aligned} \tag{4.3.5}$$

Using (A_1) and (4.3.4), we obtain

$$h(t, w_0, w_0'; w_0, w_0') = f(t, w_0, w_0') \leq w_0''(t),\ t \in I$$
$$w_0(a) \leq w_0(c),\ w_0(b) \leq w_0(d),$$

$$h(t, \beta, \beta'; w_0, w_0') \geq f(t, \beta, \beta') \geq \beta''(t),\ t \in I,$$
$$\beta(a) \geq \beta(c),\ \beta(b) \geq \beta(d),$$

which imply that w_0 and β are respectively lower and upper solutions of (4.3.5). Since $w_0 \leq \beta$ on I, hence by Theorem 4.2.3, there exists a solution w_1 of (4.3.5) such that

$$w_0(t) \leq w_1(t) \leq \beta(t),\ |w_1'(t)| \leq C_1,\ t \in I.$$

In view of (4.3.4) and the fact that w_1 is a solution of (4.3.5), we obtain

$$w_1''(t) = h(t, w_1, w_1'; w_0, w_0') \geq f(t, w_1, q(w_1')), \tag{4.3.6}$$

which implies that w_1 is a lower solution of (4.3.1).

Similarly, we can show that w_1 and β are lower and upper solutions of

$$\begin{aligned} x''(t) &= h(t, x, x'; w_1, w_1'),\ t \in I, \\ x(a) &= x(c),\ x(b) = x(d). \end{aligned} \tag{4.3.7}$$

Hence by Theorem 4.2.3, there exists a solution w_2 of (4.3.7) such that

$$w_1(t) \leq w_2(t) \leq \beta(t),\ |w_2'(t)| \leq C_1,\ t \in I.$$

Continuing this process, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0(t) \leq w_1(t) \leq w_2(t) \leq ...w_n(t) \leq \beta(t),\ |w_n'(t)| \leq C_1,\ t \in I, \tag{4.3.8}$$

where the element w_n of the sequence $\{w_n\}$ satisfies the BVP

$$w_n''(t) = h(t, w_n, w_n'; w_{n-1}, w_{n-1}'),\ t \in I,$$
$$w_n(a) = w_n(c),\ w_n(b) = w_n(d).$$

Since $h(t, w_n, w_n'; w_{n-1}, w_{n-1}')$ is bounded, we can find a constant $A > 0$ such that

$$|h(t, w_n, w_n'; w_{n-1}, w_{n-1}')| \leq A \text{ for every } t \in I.$$

Using the relation

$$x'(t) = x'(a) + \int_a^t x''(s)ds,$$

we obtain

$$|w_n'(t) - w_n'(s)| \leq \int_s^t |h(u, w_n, w_n'; w_{n-1}, w_{n-1}')|du \leq A|t-s|, \tag{4.3.9}$$

for any $s, t \in I$, $(s \leq t)$. The inequalities (4.3.8) and (4.3.9) imply that the sequences $\{w_n^i\}(i = 0,1)$ are uniformly bounded and equi-continuous on I and hence the Arzelà-Ascoli theorem guarantees the existence of subsequences and a function $x \in C^1(I)$ with $w_n^{(j)}$ $(j = 0,1)$ converging uniformly to $x^{(j)}$ on I as $n \to \infty$. Passing to the limit, we obtain $h(t, w_n, w_n'; w_{n-1}, w_{n-1}') \to f(t, x, q(x'))$. Thus, $x(t)$ is a solution of the boundary value problem (4.3.1) and hence is a solution of (4.1.1).

Now, we show that the sequence of solutions converges quadratically to a solution of (4.1.1) on I. For this, set $e_n(t) = x(t) - w_n(t)$, $t \in I$. Note that, $e_n(t) \geq 0$, for $t \in I$ and $e_n(a) = e_n(c)$, $e_n(b) = e_n(d)$. The boundary conditions imply the existence of $a_1 \in (a, c)$ and $b_1 \in (d, b)$ such that

$$e_n'(a_1) = 0,\ e_n'(b_1) = 0. \tag{4.3.10}$$

We choose a_1 the smallest and b_1 the largest zeros of $e_n'(t)$, so that $e_n'(t) \neq 0$ on $(a, a_1) \cup (b_1, b)$. Using Taylor's theorem and (4.3.3), we obtain

$$\begin{aligned}
e_n''(t) &= x''(t) - w_n''(t)\\
&= f(t, x, x') - [f(t, w_{n-1}, q(w_{n-1}')) + f_x(t, w_{n-1}, q(w_{n-1}'))(w_n - w_{n-1})\\
&+ f_{x'}(t, w_{n-1}, q(w_{n-1}'))(q(w_n') - q(w_{n-1}')]\\
&= f(t, w_{n-1}, q(w_{n-1}')) + f_x(t, w_{n-1}, q(w_{n-1}'))(x - w_{n-1})\\
&+ f_{x'}(t, w_{n-1}, q(w_{n-1}'))(x' - q(w_{n-1}')) + \frac{1}{2}H(f)\\
&- [f(t, w_{n-1}, q(w_{n-1}')) + f_x(t, w_{n-1}, q(w_{n-1}'))(w_n - w_{n-1})\\
&+ f_{x'}(t, w_{n-1}, q(w_{n-1}'))(q(w_n') - q(w_{n-1}'))]\\
&= f_x(t, w_{n-1}, q(w_{n-1}'))e_n(t) + f_{x'}(t, w_{n-1}, q(w_{n-1}'))(x' - q(w_n')) + \frac{1}{2}H(f),
\end{aligned} \tag{4.3.11}$$

where,

$$H(f) = (x - w_{n-1})^2 f_{xx}(t, c_1, c_2) + 2(x - w_{n-1})(x' - q(w'_{n-1}))f_{xx'}(t, c_1, c_2)$$
$$+(x' - q(w'_{n-1}))^2 f_{x'x'}(t, c_1, c_2),$$

$w_{n-1}(t) < c_1 < x(t)$, c_2 lies between $x'(t)$ and $q(w'_{n-1}(t))$. Let

$$N_1 = \max\big\{|f_{xx}(t, c_1, c_2)|, |f_{xx}(t, c_1, c_2)|, |f_{xx}(t, c_1, c_2)| : t \in I, c_1 \in [\min w_0(t), \max \beta(t)],$$
$$c_2 \in [-C, C]\big\}.$$

Then,

$$|H(f)| \le N_1\big(|e_{n-1}|^2 + 2|e_{n-1}||x' - q(w'_{n-1})| + |x' - q(w'_{n-1})|^2\big).$$

Using the relation

$$|x'(t) - q(w'_{n-1}(t))| \le |x'(t) - w'_{n-1}(t)| = |e'_{n-1}(t)|,$$

we obtain

$$|H(f)| \le N_1\big(|e_{n-1}| + |e'_{n-1}|\big)^2 \le N_1\|e_{n-1}\|_1^2, \tag{4.3.12}$$

where, $\|e_{n-1}\|_1 = \|e_{n-1}\| + \|e'_{n-1}\|$ is a C^1 norm. Using (A_2) and (4.3.12) in (4.3.11), we obtain

$$e''_n(t) \ge f_{x'}(t, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) - \frac{N_1}{2}\|e_{n-1}\|_1^2,\ t \in I. \tag{4.3.13}$$

We rewrite (4.3.13) as follows

$$e''_n(t) - f_{x'}(t, w_{n-1}, q(w'_{n-1}))e'_n(t) \ge f_{x'}(t, w_{n-1}, q(w'_{n-1}))(w'_n - q(w'_n)) - \frac{N_1}{2}\|e_{n-1}\|_1^2,\ t \in I. \tag{4.3.14}$$

Since $(t, w_{n-1}, q(w'_{n-1})) \in I \times [\min w_0(t), \max \beta(t)] \times [-C, C]$ and $f_{x'}$ is continuous, we can find L_1, $l \ge 0$, such that

$$-l \le f_{x'}(t, w_{n-1}, q(w'_{n-1})) \le L_1,\ t \in I. \tag{4.3.15}$$

Let $M = \max\{l, L_1\}$. We have three cases to consider.

1. If for $t \in I$, $|w'_n(t)| \le C$, then

$$w'_n(t) - q(w'_n(t)) = 0,$$

hence

$$f_{x'}(t, w_{n-1}(t), q(w'_{n-1}(t)))(w'_n(t) - q(w'_n(t))) = 0.$$

2. If for $t \in I$, $w_n'(t) > C$, then $w_n'(t) - q(w_n'(t)) > 0$, and using (A_3), we obtain

$$\begin{aligned} f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) &\geq f_{x'}(t, w_{n-1}(t), q(w_n'(t))) - L|q(w_{n-1}'(t)) - q(w_n'(t))| \\ &\geq f_{x'}(t, w_{n-1}(t), C) - 2LC \geq 0, \end{aligned}$$

and

$$\begin{aligned} f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) &\leq f_{x'}(t, w_{n-1}(t), q(w_n'(t))) + L|q(w_{n-1}'(t)) - q(w_n'(t))| \\ &\leq L_1 + 2LC \leq M + 2LC. \end{aligned}$$

Hence

$$0 \leq f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t)))(w_n'(t) - q(w_n'(t))) \leq (M + 2LC)|w_n'(t) - q(w_n'(t))|.$$

3. If for $t \in I$ with $w_n'(t) < -C$, then $w_n'(t) - q(w_n'(t)) < 0$, and using (A_3), we have

$$\begin{aligned} f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) &\leq f_{x'}(t, w_{n-1}(t), q(w_n'(t))) + L|q(w_{n-1}'(t)) - q(w_n'(t))| \\ &\leq f_{x'}(t, w_{n-1}(t), -C) + 2LC \leq 0, \end{aligned}$$

and

$$\begin{aligned} f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) &\geq f_{x'}(t, w_{n-1}(t), q(w_n'(t))) - L|q(w_{n-1}'(t)) - q(w_n'(t))| \\ &\geq f_{x'}(t, w_{n-1}(t), -C) - 2LC \geq -(M + 2LC). \end{aligned}$$

Hence

$$0 \leq f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t)))(w_n'(t) - q(w_n'(t))) \leq (M+2LC)|q(w_{n-1}'(t)) - q(w_n'(t))|.$$

Thus, we have

$$0 \leq f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t)))(w_n'(t) - q(w_n'(t))) \leq (M + 2LC)|q(w_{n-1}'(t)) - q(w_n'(t))| \tag{4.3.16}$$

for every $t \in I$. Using (4.3.16) in (4.3.14), we have

$$e_n''(t) - f_{x'}(t, w_{n-1}, q(w_{n-1}'))e_n'(t) \geq -\frac{N_1}{2}\|e_{n-1}\|_1^2,\ t \in I. \tag{4.3.17}$$

Firstly, we consider the case $l > 0$ [(4.3.15)]. Let $\mu(t) = e^{-\int_a^t f_{x'}(s, w_{n-1}(s), q(w_{n-1}'(s)))ds}$ be the integrating factor, then $\mu(t)$ satisfies

$$e^{-L_1(t-a)} \leq \mu(t) \leq e^{l(t-a)},\ t \in I. \tag{4.3.18}$$

From (4.3.17), in view of (4.3.18), we have

$$\left(e_n'(t)\mu(t)\right)' \geq -\frac{N_1}{2}\|e_{n-1}\|_1^2 e^{l(t-a)}. \tag{4.3.19}$$

Integrating (4.3.19) from a_1 to $t \geq a_1$, using the condition (4.3.10) ($e_n'(a_1) = 0$), we obtain

$$e_n'(t)\mu(t) \geq -\frac{N_1}{2}\|e_{n-1}\|_1^2 \int_{a_1}^t e^{l(s-a)}ds \geq -\frac{N_1}{2}\|e_{n-1}\|_1^2 \int_a^b e^{l(s-a)}ds,$$

which implies, using (4.3.18) that

$$e_n'(t) \geq -\frac{N_1}{2l}(e^{l(b-a)}-1)\|e_{n-1}\|_1^2/\mu(t) \geq -\frac{N_1}{2l}(e^{l(b-a)}-1)e^{L_1(t-a)}\|e_{n-1}\|_1^2,\ t \geq a_1. \quad (4.3.20)$$

On the other hand, if we integrate (4.3.19) from t to b_1 ($t \leq b_1$) and use the boundary condition (4.3.10) ($e_n'(b_1) = 0$), we obtain

$$-e_n'(t)\mu(t) \geq -\frac{N_1}{2}\|e_{n-1}\|_1^2 \int_t^{b_1} e^{l(s-a)}ds \geq -\frac{N_1}{2}\|e_{n-1}\|_1^2 \int_a^b e^{l(s-a)}ds,$$

which implies, using (4.3.18) that

$$e_n'(t) \leq \frac{N_1}{2l}(e^{l(b-a)} - 1)e^{L_1(t-a)}\|e_{n-1}\|_1^2,\ t \leq b_1. \quad (4.3.21)$$

From (4.3.20) and (4.3.21), we have

$$|e_n'(t)| \leq \frac{N_1}{2l}(e^{l(b-a)} - 1)e^{L_1(t-a)}\|e_{n-1}\|_1^2,\ t \in [a_1, b_1] = I_1. \quad (4.3.22)$$

For the case $l = 0$, repeating the procedure, we obtain

$$|e_n'(t)| \leq \frac{N_1}{2}(b-a)e^{L_1(t-a)}\|e_{n-1}\|_1^2,\ t \in I_1. \quad (4.3.23)$$

Considering both the cases, we have

$$|e_n'(t)| \leq D_1\|e_{n-1}\|_1^2 \text{ on } I_1, \quad (4.3.24)$$

where

$$D_1 = \begin{cases} \max\left\{\frac{N_1}{2l}(e^{l(b-a)} - 1)e^{L_1(t-a)} : t \in I\right\}, & \text{if } l > 0 \\ \max\left\{\frac{N_1}{2}(b-a)e^{L_1(t-a)} : t \in I\right\}, & \text{if } l = 0, \end{cases}$$
$$= \begin{cases} \frac{N_1}{2l}(e^{l(b-a)} - 1)e^{L_1(b-a)}, & \text{if } l > 0 \\ \frac{N_1}{2}(b-a)e^{L_1(b-a)}, & \text{if } l = 0. \end{cases}$$

Now, from (4.3.11), we have

$$e_n''(t) = f_x(t, w_{n-1}, q(w_{n-1}'))e_n(t) + f_{x'}(t, w_{n-1}, q(w_{n-1}'))(x' - q(w_n')) + \frac{1}{2}H(f),\ t \in I.$$

Then multiplying by $\mu(t) = e^{-\int_a^t f_{x'}(s, w_{n-1}, q(w_{n-1}'))ds}$, we obtain

$$e_n''(t)\mu(t) = f_x(t, w_{n-1}, q(w_{n-1}'))e_n(t)\mu(t) + f_{x'}(t, w_{n-1}, q(w_{n-1}'))(x' - q(w_n'))\mu(t)$$
$$+ \frac{1}{2}H(f)\mu(t),\ t \in I.$$

This implies that

$$(e_n'(t)\mu(t))' = f_x(t, w_{n-1}, q(w_{n-1}'))e_n(t)\mu(t) + f_{x'}(t, w_{n-1}, q(w_{n-1}'))(x' - q(w_n'))\mu(t)$$
$$- f_{x'}(t, w_{n-1}, q(w_{n-1}'))(x' - w_n')\mu(t) + \frac{1}{2}H(f)\mu(t)$$
$$= f_x(t, w_{n-1}, q(w_{n-1}'))e_n(t)\mu(t) + f_{x'}(t, w_{n-1}, q(w_{n-1}'))(w_n' - q(w_n'))\mu(t)$$
$$+ \frac{1}{2}H(f)\mu(t),\ t \in I.$$

Using the fact that $f_{x'}(t, w_{n-1}(t), q(w'_{n-1}(t)))(w'_n(t) - q(w'_n(t))) \geq 0$, $t \in I$, we obtain

$$(e'_n(t)\mu(t))' \geq [f_x(t, w_{n-1}, q(w'_{n-1}))e_n(t) + \frac{1}{2}H(f)]\mu(t), \ t \in I.$$

The boundary conditions (4.3.10) imply that the function $\psi(t) = e'_n(t)\mu(t)$ has two zeros on (a, b). Then, there exists a $\bar{t} \in (a_1, b_1)$ such that

$$\psi'(\bar{t}) = (e'_n\mu)'(\bar{t}) = 0.$$

This implies that

$$[f_x(\bar{t}, w_{n-1}(\bar{t}), q(w'_{n-1}(\bar{t})))e_n(\bar{t}) + \frac{1}{2}H(f)]\mu(\bar{t}) \leq 0,$$

and using the positivity of μ, we get

$$f_x(\bar{t}, w_{n-1}(\bar{t}), q(w'_{n-1}(\bar{t})))e_n(\bar{t}) + \frac{1}{2}H(f) \leq 0,$$

thus,

$$f_x(\bar{t}, w_{n-1}(\bar{t}), q(w'_{n-1}(\bar{t})))e_n(\bar{t}) \leq -\frac{1}{2}H(f) = \frac{1}{2}|H(f)|.$$

Let $\bar{r} > 0$ be such that

$$\min\left\{f_x(t, x, x') : t \in I, \ x \in [\min w_0(t), \max \beta(t)], \ x' \in [-C, C]\right\} = \bar{r}.$$

Then

$$0 \leq e_n(\bar{t}) = \frac{|H(f)|}{2f_x(t, w_{n-1}(\bar{t}), q(w'_{n-1}(\bar{t})))} \leq \frac{|H(f)|}{2\bar{r}} \leq \frac{N_1}{2\bar{r}}\|e_{n-1}\|_1^2. \tag{4.3.25}$$

Using (4.3.24) and (4.3.25) in $e_n(t) = e_n(\bar{t}) + \int_{\bar{t}}^t e'_n(s)ds$, we obtain

$$|e_n(t)| \leq |e_n(\bar{t})| + \int_a^b D_1\|e_{n-1}\|_1^2 ds \leq D_2\|e_{n-1}\|_1^2, \ t \in I_1 \tag{4.3.26}$$

where $D_2 = \frac{N_1}{2\bar{r}} + D_1(b - a)$. From (4.3.24) and (4.3.26), it follows that

$$\|e_n\|_1 = \|e_n\| + \|e'_n\| \leq D_2\|e_{n-1}\|_1^2 + D_1\|e_{n-1}\|_1^2 = D\|e_{n-1}\|_1^2 \text{ on } I_1, \tag{4.3.27}$$

where $D = D_1 + D_2$. (Note that the choice of N_1, $\bar{r}$ and D_1 ensure that D is independent of n). Also, we note that I_1 contains $[c, d]$.

Now, we consider the interval $[a, a_1]$. By the choice of a_1, we have

$$e'_n(t) \neq 0 \text{ on } (a, a_1).$$

We have two cases to consider.

Case 1 : $e'_n(t) \geq 0$ on (a, a_1).

Integrating (4.3.19) from t to a_1, using (4.3.18) and the boundary condition ($e_n'(a_1)=0$), we obtain

$$0 \le e_n'(t) \le \frac{N}{2l}\left[e^{(L_1+l)(a_1-a)} - e^{(L_1+l)(t-a)}\right]\|e_{n-1}\|_1^2 \le E_1\|e_{n-1}\|_1^2,\ t\in[a,\ a_1], \tag{4.3.28}$$

where $E_1 = \frac{N}{2l}\max\{e^{(L_1+l)(a_1-a)} - e^{(L_1+l)(t-a)} : t \in [a,\ a_1]\}$. Since $e_n(a) = e_n(c)$ and $c \in I_1$, using (4.3.27), we have

$$e_n(a) \le D_2\|e_{n-1}\|_1^2. \tag{4.3.29}$$

The relation $e_n(t) = e_n(a) + \int_a^t e_n'(s)ds$, yields

$$\begin{aligned} 0 \le e_n(t) \le \Big[D_2 + \frac{N}{2l}\big((a_1-a)e^{(L_1+l)(a_1-a)} - \frac{1}{L_1+l}(e^{(L_1+l)(t-a)}-1)\big)\Big]\|e_{n-1}\|_1^2 \\ \le D_3\|e_{n-1}\|_1^2,\ t\in[a,\ a_1], \end{aligned} \tag{4.3.30}$$

where $D_3 = D_2 + \frac{N}{2l}\max\{(a_1-a)e^{(L_1+l)(a_1-a)} - \frac{1}{L_1+l}(e^{(L_1+l)(t-a)}-1) : t\in[a,\ a_1]\}$. From (4.3.28) and (4.3.30), it follows that

$$\|e_n\|_1 \le E\|e_{n-1}\|_1^2 \text{ on } [a, a_1], \tag{4.3.31}$$

where $E = D_3 + E_1$.

Case 2 : $e_n'(t) \le 0$ on $(a,\ a_1)$. In this case, we have

$$e_n(t) \le e_n(a) \le D_2\|e_{n-1}\|_1^2,\ t\in[a,\ a_1]. \tag{4.3.32}$$

Using (4.3.4), we obtain

$$\begin{aligned} e_n''(t) &\le f(t,x,x') - h(t,w_n,w_n';w_{n-1},w_{n-1}') \le f(t,x,x') - f(t,w_n,q(w_n')) \\ &\le f_x(t,w_n,q(w_n'))e_n(t) + f_{x'}(t,w_n,q(w_n'))(x'-q(w_n')),\ t\in[a,\ a_1]. \end{aligned} \tag{4.3.33}$$

If for some $t\in[a,\ a_1]$, $w_n'(t) > C$, then

$$f_{x'}(t,w_n(t),q(w_n'(t)))(x'(t)-q(w_n'(t))) < 0.$$

If for some $t\in[a,\ a_1]$, $w_n'(t) \le C$, then

$$f_{x'}(t,w_n(t),q(w_n'(t)))(x'(t)-q(w_n'(t))) = f_{x'}(t,w_n(t),w_n'(t))(x'(t)-w_n'(t)) \le 0,$$

if $f_{x'}(t,w_n(t),w_n'(t)) \le 0$, and in view of (4.3.15),

$$f_{x'}(t,w_n(t),q(w_n'(t)))(x'(t)-q(w_n'(t))) \le -le_n'(t),$$

if $f_{x'}(t,w_n(t),w_n'(t)) \ge 0$. Thus, we have

$$f_{x'}(t,w_n(t),q(w_n'(t)))(x'(t)-q(w_n'(t))) \le -le_n'(t),\ t\in[a,\ a_1]. \tag{4.3.34}$$

Using (4.3.32) and (4.3.34) in (4.3.33), we obtain

$$e_n''(t) + le_n'(t) \le MD_2\|e_{n-1}\|_1^2,\ t\in[a,\ a_1], \tag{4.3.35}$$

where $M = \max\{f_x(t,x,x') : t \in [a, a_1], x \in [\min\alpha, \max\beta], x' \in [-C, C]\}$. From (4.3.35), it follows that

$$(e_n'(t)e^{lt})' \leq MD_2e^{lt}\|e_{n-1}\|_1^2, \; t \in [a, a_1]. \tag{4.3.36}$$

Integrating from t to a_1, we obtain

$$e_n'(t) \geq -\frac{MD_2}{l}(e^{l(a-t)} - 1)\|e_{n-1}\|_1^2, \; t \in [a, a_1]. \tag{4.3.37}$$

From (4.3.21) and (4.3.37), it follows that

$$|e_n'(t)| \leq E_2\|e_{n-1}\|_1^2, \; t \in [a, a_1],$$

where $E_2 = \max\{\frac{N}{2l}(e^{l(b-a)} - 1)e^{L_1(t-a)}, \frac{MD_2}{l}(e^{l(a-t)} - 1) : t \in [a, a_1]\}$. Hence

$$\|e_n\|_1 \leq E_3\|e_{n-1}\|_1^2 \text{ on } [a, a_1], \tag{4.3.38}$$

where $E_3 = D_2 + E_2$.
Similarly, we can obtain quadratic convergence on $[b_1, b]$, that is, there exists $E_4 > 0$ such that

$$\|e_n\|_1 \leq E_4\|e_{n-1}\|_1^2 \text{ on } [b_1, b]. \tag{4.3.39}$$

Considering (4.3.27), (4.3.38) and (4.3.39), we have

$$\|e_n\|_1 \leq E_5\|e_{n-1}\|_1^2 \text{ on } I,$$

where $E_5 = \max\{D, E_3, E_4\}$.

Remark 4.3.2. 1. If $c = d$, then our boundary conditions reduces to three-point boundary conditions.
2. If c approach a and d approach b, then our boundary conditions reduces to homogeneous Neumann boundary conditions $x'(a) = 0$, $x'(b) = 0$.
Thus our methods apply to three-point and Neumann boundary value problems.

□

4.4 Generalized quasilinearization technique

Now, we generalize the results by introducing an auxiliary function ϕ to allow weaker hypothesis on f. Also, we replace the condition (A_3) of Theorem 4.3.1, by a weaker condition and prove that the conclusion of the Theorem 4.3.1 remains valid.

Theorem 4.4.1. *Assume that*

(B₁) $\alpha, \beta \in C^2(I)$ *are lower and upper solutions of* (4.1.1) *such that* $\alpha(t) \leq \beta(t)$ *on* I.

($\mathbf{B_2}$) *$f \in C^2(I \times \mathbb{R}^2)$ satisfies a Bernstein-Nagumo condition on I relative to α, β and is such that $f_x(t,x,x') > 0$ and $H(f+\phi) \le 0$ on $I \times \mathbb{R}^2$, for some function $\phi \in C^2(I \times \mathbb{R}^2)$ satisfies $H(\phi) \le 0$.*

($\mathbf{B_3}$) *For $(t,x) \in I \times [\min \alpha(t), \max \beta(t)]$, $f_{x'}(t,x,x')$ satisfies*

$$y f_{x'}(t,x,y) \ge 0,\ y \in \mathbb{R} \text{ and}$$
$$|f_{x'}(t,x,y_1) - f_{x'}(t,x,y_2)| \le L|y_1 - y_2|,\ y_1,\ y_2 \in \mathbb{R},$$

where $L > 0$.

Then, there exists a monotone sequence $\{w_n\}$ of solutions converging uniformly to the unique solution of the problem. Moreover, the sequence converges quadratically on I.

Proof. Define $F : I \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$F(t,x,y) = f(t,x,y) + \phi(t,x,y), \quad t \in I,\ x,\ y \in \mathbb{R}. \tag{4.4.1}$$

Then, in view of (B_2), we have $F \in C^2(I \times \mathbb{R}^2)$ and

$$H(F) \le 0 \text{ on } I \times \mathbb{R}^2. \tag{4.4.2}$$

Using Taylor's theorem and (4.4.2), we obtain

$$\begin{aligned} f(t,x,q(x')) \le F(t,y,q(y')) + F_x(t,y,q(y'))(x-y) + F_{x'}(t,y,q(y'))(q(x') - q(y')) \\ -\phi(t,x,q(x')), \end{aligned} \tag{4.4.3}$$

where $t \in I$, x, y, x', $y' \in \mathbb{R}$. Using Taylor's theorem on the function ϕ, we can find d_1, $d_2 \in \mathbb{R}$ where d_1 lies between $x(t)$, $y(t)$ and d_2 lies between $q(x'(t))$, $q(y'(t))$, such that

$$\begin{aligned} \phi(t,x,q(x')) = \phi(t,y,q(y')) + \phi_x(t,y,q(y'))(x-y) + \phi_{x'}(t,y,q(y'))(q(x') - q(y')) \\ +\frac{1}{2}H(\phi), \end{aligned} \tag{4.4.4}$$

where $t \in I$, x, y, x', $y' \in \mathbb{R}$ and

$$\begin{aligned} H(\phi) =& (x-y)^2 \phi_{xx}(t,d_1,d_2) + 2(x-y)(q(x') - q(y'))\phi_{xx'}(t,d_1,d_2) \\ &+ (q(x') - q(y'))^2 \phi_{x'x'}(t,d_1,d_2), \end{aligned}$$

Let $\Omega = \{(t,x,x') : t \in I,\ x \in [\min \alpha(t), \max \beta(t)],\ x' \in [-C, C]\}$ and

$$M_1 \ge \max\{|\phi_{xx}(t,x,x')|,\ |\phi_{xx'}(t,x,x')|,\ |\phi_{x'x'}(t,x,x')|\} : (t,x,x') \in \Omega\},$$

where C is as defined in Theorem 4.3.1. Then

$$|H(\phi)| \le M_1(|x-y| + |q(x') - q(y')|)^2 \text{ on } \Omega. \tag{4.4.5}$$

Using (4.4.5) and (B_2) in (4.4.4), we obtain

$$\phi(t,x,q(x')) \leq \phi(t,y,q(y')) + \phi_x(t,y,q(y'))(x-y) + \phi_{x'}(t,y,q(y'))(q(x')-q(y')), \quad (4.4.6)$$

for $t \in I$, x, y, x', $y' \in \mathbb{R}$, and

$$\begin{aligned}\phi(t,x,q(x')) \geq \phi(t,y,q(y')) &+ \phi_x(t,y,q(y'))(x-y) + \phi_{x'}(t,y,q(y'))(q(x')-q(y')) \\ &-\frac{M_1}{2}(|x-y| + |q(x')-q(y')|)^2,\end{aligned} \quad (4.4.7)$$

on Ω. Substituting (4.4.7) in (4.4.3) and using (4.4.1), we obtain

$$\begin{aligned}f(t,x,q(x')) \leq f(t,y,q(y')) &+ f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(q(x')-q(y')) \\ &+\frac{M_1}{2}(|x-y| + |q(x')-q(y')|)^2,\end{aligned} \quad (4.4.8)$$

on Ω. Define

$$\begin{aligned}k(t,x,x';y,y') = f(t,y,q(y')) &+ f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(q(x')-q(y')) \\ &+\frac{M_1}{2}(|x-y| + |q(x')-q(y')|)^2,\end{aligned} \quad (4.4.9)$$

for x, y, x', $y' \in \mathbb{R}$ and $t \in I$. Since $f_x \geq 0$, selecting M_1 large enough, so that

$$\begin{aligned}f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(q(x')-q(y')) &+ M_1(x-y)|q(x')-q(y')| \\ &+\frac{M_1}{2}(|x-y|^2 + |q(x')-q(y')|^2) \geq 0,\end{aligned}$$

for $x \geq y$ with x, $y \in [\min\alpha(t), \max\beta(t)]$, x', $y' \in \mathbb{R}$, then,

$$f(t,y,q(y')) \leq k(t,x,x';y,y') \text{ for } x \geq y, \quad (4.4.10)$$

$t \in I$, x', $y' \in \mathbb{R}$. Moreover, k satisfies the following relations

$$\begin{cases} f(t,x,q(x')) \leq k(t,x,x';y,y') \text{ on } \Omega, \\ f(t,x,q(x')) = k(t,x,x';x,x'), \end{cases} \quad (4.4.11)$$

for $t \in I$, x, y, x', $y' \in \mathbb{R}$. Also, we note that k is continuous and bounded on $I \times [\min\alpha(t), \max\beta(t)] \times \mathbb{R} \times [\min\alpha(t), \max\beta(t)] \times \mathbb{R}$, therefore satisfies a Bernstein-Nagumo on I. Hence there exists a constant $C_2 > 0$ such that any solution x of the

$$x''(t) = k(t,x,x';y,y'),\ t \in I,$$
$$x(a) = x(c),\ x(b) = x(d),$$

such that $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in I$ satisfies $|x'| \leq C_2$ on I.

Now, we set $\alpha = w_0$ and consider the linear four point problem

$$\begin{aligned} &x''(t) = k(t,x,x';w_0,w_0'),\ t \in I \\ &x(a) = x(c),\ x(b) = x(d). \end{aligned} \quad (4.4.12)$$

Using (B_1) and (4.4.11), we get

$$k(t, w_0(t), w_0'(t); w_0(t), w_0'(t)) = f(t, w_0(t), w_0'(t)) \leq w_0''(t),\ t \in I$$

$$k(t, \beta(t), \beta'(t); w_0(t), w_0'(t)) \geq f(t, \beta(t), \beta'(t)) \geq \beta''(t),\ t \in I$$

which imply that w_0 and β are lower and upper solution of (4.4.12) respectively. Hence, by Theorem 4.2.3, there exists a solution w_1 of (4.4.12) such that

$$w_0(t) \leq w_1(t) \leq \beta(t),\ |w_1'(t)| \leq C_2,\ t \in I.$$

Using (4.4.11) and the fact that w_1 is a solution of (4.4.12), we obtain

$$w_1''(t) = k(t, w_1, w_1'; w_0, w_0') \geq f(t, w_1(t), q(w_1'(t))),\ t \in I \tag{4.4.13}$$

which implies that w_1 is a lower solution of (4.3.1).
In view of (4.4.11), (4.4.13) and (B_1), we can show that w_1 and β are lower and upper solutions of

$$\begin{aligned} x''(t) &= k(t, x, x'; w_1, w_1'),\ t \in I \\ x(a) &= x(c),\ x(b) = x(d). \end{aligned} \tag{4.4.14}$$

Thus, by Theorem 4.2.3, there exists a solution w_2 of (4.4.14) such that $w_1 \leq w_2 \leq \beta$ on I.
Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0(t) \leq w_1(t) \leq w_2(t) \leq w_3(t) \leq ... \leq w_{n-1}(t) \leq w_n(t) \leq \beta(t),\ t \in I$$

That is,

$$w_0(t) \leq w_n(t) \leq \beta(t) \text{ and } |w_n'(t)| \leq C_2,\ n \in \mathbb{N},\ t \in I, \tag{4.4.15}$$

where w_n is a solution of the problem

$$\begin{aligned} x''(t) &= k(t, x, x'; w_{n-1}, w_{n-1}'),\ t \in I \\ x(a) &= x(c),\ x(b) = x(d). \end{aligned}$$

The same arguments as in Theorem 4.3.1, shows that the sequence converges to a solution of the boundary value problem (4.1.1).
Now we show that the convergence of the sequence of solutions is quadratic. For this, we set $e_n(t) = x(t) - w_n(t)$, $t \in I$. Then, $e_n \in C^2(I)$ and $e_n(t) \geq 0$, $t \in I$. Moreover, the boundary conditions imply that

$$e_n(a) = e_n(c),\ e_n(d) = e_n(b), \tag{4.4.16}$$

which implies that there exist $t_1 \in (a, c)$ and $t_2 \in (d, b)$ such that

$$e_n'(t_1) = 0,\ e_n'(t_2) = 0. \tag{4.4.17}$$

Now, by the Taylor's theorem, the expression (4.4.6) for $\phi(t, x, x')$ [by taking $y = w_{n-1}$] and (4.4.9), we obtain

$$\begin{aligned} e_n''(t) &= x''(t) - w_n''(t) = (F(t,x,x') - \phi(t,x,x')) - k(t, w_n, w_n'; w_{n-1}, w_{n-1}') \\ &\geq f(t, w_{n-1}, q(w_{n-1}')) + f_x(t, w_{n-1}, q(w_{n-1}'))(x - w_{n-1}) + f_{x'}(t, w_{n-1}, q(w_{n-1}'))\times \\ &(x' - q(w_{n-1}')) + \frac{1}{2}H(F) - f(t, w_{n-1}, q(w_{n-1}')) - f_x(t, w_{n-1}, q(w_{n-1}'))(w_n - w_{n-1}) \\ &- f_{x'}(t, w_{n-1}, q(w_{n-1}'))(q(w_n') - q(w_{n-1}')) - \frac{M_1}{2}\left(|w_n - w_{n-1}| + |q(w_n') - q(w_{n-1}')|\right)^2 \\ &= f_x(t, w_{n-1}, q(w_{n-1}'))e_n(t) + f_{x'}(t, w_{n-1}, q(w_{n-1}'))(x' - q(w_n')) - \frac{1}{2}|H(F)| \\ &- \frac{M_1}{2}\left(|w_n - w_{n-1}| + |q(w_n') - q(w_{n-1}')|\right)^2,\ t \in I \end{aligned} \tag{4.4.18}$$

which in view of (B_2) implies that

$$e_n''(t) \geq f_{x'}(t, w_{n-1}, q(w_{n-1}'))(x' - q(w_n')) - \frac{1}{2}|H(F)| - \frac{M_1}{2}\left(|w_n - w_{n-1}| + |q(w_n') - q(w_{n-1}')|\right)^2, \tag{4.4.19}$$

where

$$H(F) = e_{n-1}^2 F_{xx}(t, \xi_1, \xi_2) + 2e_{n-1}(x' - q(w_{n-1}'))F_{xx'}(t, \xi_1, \xi_2) + (x' - q(w_{n-1}'))^2 F_{x'x'}(t, \xi_1, \xi_2),$$

$w_{n-1}(t) \leq \xi_1 \leq x(t)$, ξ_2 lies between $q(w_{n-1}'(t))$ and $x'(t)$. Let

$$P_1 = \max\{|F_{xx}(t, \xi_1, \xi_2)|, |F_{xx'}(t, \xi_1, \xi_2)|, |F_{x'x'}(t, \xi_1, \xi_2)| : (t, \xi_1, \xi_2) \in \Omega\},$$

then

$$|H(F)| \leq P_1(|e_{n-1}| + |x' - q(w_{n-1}')|)^2 \leq P_1(|e_{n-1}| + |e_{n-1}'|)^2 \leq P_1\|e_{n-1}\|_1^2. \tag{4.4.20}$$

Using the relations $w_n - w_{n-1} \leq x - w_{n-1}$, $|x' - q(w_{n-1}')| \leq |x' - w_{n-1}'| = |e_{n-1}'|$ and

$$|q(w_n') - q(w_{n-1}')| \leq |x' - q(w_{n-1}')| + |x' - q(w_n')| \leq |e_{n-1}'| + |e_n'|,$$

we obtain

$$\begin{aligned} e_n''(t) &\geq f_{x'}(t, w_{n-1}, q(w_{n-1}'))(x' - q(w_n')) - \frac{P_1}{2}\|e_{n-1}\|_1^2 - \frac{M_1}{2}\left(|e_{n-1}| + |e_{n-1}'| + |e_n'|\right)^2 \\ &\geq f_{x'}(t, w_{n-1}, q(w_{n-1}'))(x' - q(w_n')) - \frac{P_1}{2}\|e_{n-1}\|_1^2 - \frac{M_1}{2}\left(\|e_{n-1}\|_1 + |e_n'|\right)^2,\ t \in I. \end{aligned} \tag{4.4.21}$$

Now, by (4.4.10), we obtain

$$\begin{aligned} e_n''(t) &= x''(t) - w_n''(t) = f(t, x, x') - k(t, w_n, w_n'; w_{n-1}, w_{n-1}') \\ &\leq f(t, x, x') - f(t, w_{n-1}, q(w_{n-1}')). \end{aligned}$$

Using the mean value theorem, we can find c_1, $c_2 \in \mathbb{R}$ with $w_{n-1}(t) < c_1 < x(t)$, and c_2 lies between $q(w'_{n-1}(t))$ and $x'(t)$ such that

$$\begin{aligned} e''_n(t) &\leq f_x(t, c_1, c_2)e_{n-1}(t) + f_{x'}(t, c_1, c_2)(x' - q(w'_{n-1})) \\ &\leq \lambda|e_{n-1}(t)| + L_1|x'(t) - q(w'_{n-1}(t))| \\ &\leq \lambda|e_{n-1}(t)| + L_1|e'_{n-1}(t)| \leq \lambda_1\|e_{n-1}\|_1,\ t \in I \end{aligned} \tag{4.4.22}$$

where, $\lambda = \max\{f_x(t, x, x') : t \in I, x \in [\min w_0(t), \max \beta(t)], x' \in [-C, C]\}$, $\lambda_1 = \max\{\lambda, L_1\}$, and L_1 is as defined in Theorem 4.3.1. Integrating (4.4.22) from t_1 to $t \geq t_1$, using (4.4.17), we obtain

$$e'_n(t) \leq \lambda_1(t - t_1)\|e_{n-1}\|_1 \leq \lambda_1(b - a)\|e_{n-1}\|_1,\ t \in [t_1, b]. \tag{4.4.23}$$

If we integrate (4.4.22) from t to t_2 ($t \leq t_2$), use (4.4.17), we obtain

$$e'_n(t) \geq -\lambda_1(t_2 - t)\|e_{n-1}\|_1 \geq -\lambda_1(b - a)\|e_{n-1}\|_1,\ t \in [a, t_2]. \tag{4.4.24}$$

From (4.4.23) and (4.4.24), it follows that

$$|e'_n(t)| \leq d\|e_{n-1}\|_1,\ t \in [t_1, t_2], \tag{4.4.25}$$

where $d = \lambda_1(b - a)$. Using (4.4.25) in (4.4.21), we obtain

$$e''_n(t) \geq f_{x'}(t, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) - \hat{N}\|e_{n-1}\|_1^2,\ t \in [t_1, t_2],$$

where $\hat{N} = \frac{P_1 + M_1(1+d)^2}{2}$. Thus, we have

$$\begin{aligned} e''_n(t) - f_{x'}(t, w_{n-1}, q(w'_{n-1}))e'_n(t) \geq f_{x'}(t, w_{n-1}, q(w'_{n-1}))(w'_n - q(w'_n)) \\ -\hat{N}\|e_{n-1}\|_1^2,\ t \in [t_1, t_2]. \end{aligned} \tag{4.4.26}$$

We consider three cases.

(1) If $w'_n(t) > C$ for some $t \in [t_1, t_2]$, then $q(w'_n(t)) = C$, $w'_n(t) - q(w'_n(t)) > 0$, and using (B_3), we obtain

$$\begin{aligned} f_{x'}(t, w_{n-1}(t), q(w'_{n-1}(t))) &\geq f_{x'}(t, w_{n-1}(t), q(w'_n(t))) - L|q(w'_{n-1}(t)) - q(w'_n(t))| \\ &\geq -L|q(w'_{n-1}(t)) - q(w'_n(t))|. \end{aligned}$$

Hence

$$(w'_n(t) - q(w'_n(t)))f_{x'}(t, w_{n-1}(t), q(w'_{n-1}(t))) \geq -L|w'_n(t) - q(w'_n(t))||x'(t) - q(w'_n(t))|.$$

Since

$$\begin{aligned} |q(w'_{n-1}(t)) - q(w'_n(t))| &\leq |x'(t) - q(w'_n(t))| + |x'(t) - q(w'_{n-1}(t))| \\ &\leq |e'_n(t)| + |e'_{n-1}(t)|, \end{aligned}$$

using (4.4.25), we obtain

$$\begin{aligned}|q(w'_{n-1}(t)) - q(w'_n(t))||w'_n(t) - q(w'_n(t))| &\le (|e'_n(t)| + |e'_{n-1}(t)|)|e_n(t)| \\ &\le d(1+d)\|e'_{n-1}\|_1^2.\end{aligned}$$

Thus,

$$(w'_n(t) - q(w'_n(t)))f_{x'}(t, w_{n-1}(t), q(w'_{n-1}(t))) \ge -Ld(1+d)\|e_{n-1}\|_1^2.$$

(2) If $w'_n(t) < -C$ for some $t \in [t_1, t_2]$, then $q(w'_n(t)) = -C$, $w'_n(t) - q(w'_n(t)) < 0$, and using (B_3), we have

$$\begin{aligned}f_{x'}(t, w_{n-1}(t), q(w'_{n-1}(t))) &\le f_{x'}(t, w_{n-1}(t), q(w'_n(t))) + L|q(w'_{n-1}(t)) - q(w'_n(t))| \\ &\le L|q(w'_{n-1}(t)) - q(w'_n(t))|,\end{aligned}$$

and hence

$$\begin{aligned}(w'_n(t) - q(w'_n(t)))f_{x'}(t, w_{n-1}(t), q(w'_{n-1}(t))) &\ge -L|q(w'_{n-1}(t)) - q(w'_n(t))| \times \\ &|w'_n(t) - q(w'_n(t))| \ge -Ld(1+d)\|e_{n-1}\|_1^2.\end{aligned}$$

(3) If $|w'_n(t)| \le C$ for some $t \in [t_1, t_2]$, then $w'_n(t) - q(w'_n(t)) = 0$, and

$$(w'_n(t) - q(w'_n(t)))f_{x'}(t, w_{n-1}(t), q(w'_{n-1}(t))) = 0 > -Ld(1+d)\|e_{n-1}\|_1^2.$$

Thus,

$$(w'_n(t) - q(w'_n(t)))f_{x'}(t, w_{n-1}(t), q(w'_{n-1}(t))) > -Ld(1+d)\|e_{n-1}\|_1^2,\ t \in [t_1, t_2]. \quad (4.4.27)$$

Using (4.4.27) in (4.4.26), we obtain

$$\begin{aligned}e''_n(t) - f_{x'}(t, w_{n-1}, q(w'_{n-1}(t)))e'_n(t) &\ge -Ld(1+d)\|e_{n-1}\|_1^2 - \hat{N}\|e_{n-1}\|_1^2 \\ &= -S\|e_{n-1}\|_1^2,\ t \in [t_1, t_2],\end{aligned} \quad (4.4.28)$$

where, $S = Ld(1+d) + \hat{N}$. The equation (4.4.28) is the same equation as (4.3.17), hence we can obtain the corresponding results following the same procedure of Theorem 4.3.1. □

Chapter 5

Boundary value problems with integral boundary conditions

In this chapter, we study existence and approximations of solutions of second order nonlinear boundary value problems with nonlinear integral boundary conditions. They include two, three, multi-point and nonlocal boundary value problems as special cases. For boundary value problems with integral boundary conditions and comments on their importance, we refer to [25, 44, 66] and the references therein. Moreover, existence results for boundary value problems with integral boundary conditions have been studied by a number of authors, for example [7, 8, 14, 19, 43]. In this chapter, we use the upper and lower solutions method for the existence results and develop the generalized method of quasilinearization to approximate our problems. We study two type of problems. Firstly, the case that the nonlinearity is independent of derivative. Secondly, where the nonlinearity depends also on the first derivative with a little bit simpler boundary conditions than the first one.

5.1 Nonlinearity independent of the derivative

The results of this section have appeared [45]. In this section, we study existence and approximation of solutions of second order differential equations with nonlinear integral boundary conditions of the type

$$\begin{aligned} &x''(t) = f(t,x), \ t \in J = [0,1],\\ &x(0) - k_1 x'(0) = \int_0^1 h_1(x(s))ds, \\ &x(1) + k_2 x'(1) = \int_0^1 h_2(x(s))ds, \end{aligned} \tag{5.1.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ and $h_i : \mathbb{R} \to \mathbb{R} \, (i = 1, 2)$ are continuous functions and k_i are nonnegative constants. Firstly, we study existence results by the method of upper and lower solutions, then we approximate our problem by the method of quasilinearization. We

show that the sequence of approximate solutions converges uniformly and quadratically to a solution of the original problem. We also improve the rate of convergence, and show under some reasonable conditions, that the sequence of approximants converges to the solution with rate of convergence greater than quadratic.
Recently, B. Ahmad, R. A. Khan and S. Sivasundaram [7] and J. Tadeusz [43] studied the quasilinearization method for the solution of first order differential equations with linear integral boundary conditions of the type

$$x'(t) = f(t, x(t)),\ t \in [0, T]$$
$$x(0) = ax(T) + \int_0^T b(s)x(s)ds + k = Bx + k.$$

Here, we study a second order problem and also nonlinear boundary conditions. Further, we improve the rate of convergence of the sequence of solutions.

5.1.1 Existence results

We know that the homogeneous problem

$$x''(t) = 0,\ \ t \in J,$$
$$x(0) - k_1x'(0) = 0, \quad x(1) + k_2x'(1) = 0,$$

has only a trivial solution. Consequently, for any $\sigma(t)$, $\rho_1(t)$, $\rho_2(t) \in C(J)$, the corresponding nonhomogeneous linear problem

$$x''(t) = \sigma(t),\ \ t \in J,$$
$$x(0) - k_1x'(0) = \int_0^1 \rho_1(s)ds, \quad x(1) + k_2x'(1) = \int_0^1 \rho_2(s)ds,$$

has a unique solution $x \in C^2(J)$,

$$x(t) = P(t) + \int_0^1 G(t, s)\sigma(s)ds,$$

where

$$P(t) = \frac{1}{1 + k_1 + k_2}\{(1 - t + k_2)\int_0^1 \rho_1(s)ds + (k_1 + t)\int_0^1 \rho_2(s)ds\}$$

is the unique solution of the problem

$$x''(t) = 0,\ \ t \in J,$$
$$x(0) - k_1x'(0) = \int_0^1 \rho_1(s)ds,$$
$$x(1) + k_2x'(1) = \int_0^1 \rho_2(s)ds,$$

and

$$G(t, s) = \frac{-1}{k_1 + k_2 + 1}\begin{cases}(k_1 + t)(1 - s + k_2), & 0 \le t < s \le 1\\ (k_1 + s)(1 - t + k_2), & 0 \le s \le t \le 1\end{cases}$$

is the Green's function of the problem. We note that $G(t,s) < 0$ on $(0,1)\times(0,1)$. We recall the definition of lower and upper solutions.

Definition 5.1.1. We say that $\alpha \in C^2(J)$ is a lower solution of (5.1.1) if

$$\begin{aligned}
&\alpha''(t) \geq f(t,\alpha(t)), \quad t \in J\\
&\alpha(0) - k_1\alpha'(0) \leq \int_0^1 h_1(\alpha(s))ds,\\
&\alpha(1) + k_2\alpha'(1) \leq \int_0^1 h_2(\alpha(s))ds.
\end{aligned}$$

Similarly, $\beta \in C^2(J)$ is an upper solution of the BVP (5.1.1), if β satisfies similar inequalities in the reverse direction.

5.1.2 Method of lower and upper solutions

In this section, we state and prove theorems on the existence and uniqueness of solutions of the BVP (5.1.1), in an ordered interval defined by lower and upper solutions of the boundary value problem (5.1.1).

Theorem 5.1.2. *Assume that α and β are respectively lower and upper solutions of* (5.1.1) *such that $\alpha(t) \leq \beta(t)$, $t \in J$. If $f : J \times \mathbb{R} \to \mathbb{R}$ and $h_i : \mathbb{R} \to \mathbb{R}\,(i=1,2)$ are continuous and $h_i'(x) \geq 0$, then there exists a solution $x(t)$ of the boundary value problem* (5.1.1) *such that*

$$\alpha(t) \leq x(t) \leq \beta(t),\ t \in J.$$

Proof. Define the following modifications of $f(t,x)$ and $h_i(x)(i=1,2)$

$$F(t,x) = \begin{cases} f(t,\beta(t)) + \dfrac{x-\beta(t)}{1+|x-\beta(t)|}, & \text{if } x > \beta(t),\\ f(t,x), & \text{if } \alpha(t) \leq x \leq \beta(t),\\ f(t,\alpha) + \dfrac{x-\alpha(t)}{1+|x-\alpha(t)|}, & \text{if } x < \alpha(t), \end{cases}$$

and

$$H_i(x) = \begin{cases} h_i(\beta(t)), & \text{if } x > \beta(t),\\ h_i(x), & \text{if } \alpha(t) \leq x \leq \beta(t),\\ h_i(\alpha(t)), & \text{if } x < \alpha(t). \end{cases}$$

Consider the modified problem

$$\begin{aligned}
&x''(t) = F(t,x),\ t \in J,\\
&x(0) - k_1x'(0) = \int_0^1 H_1(x(s))\,ds, \quad x(1) + k_2x'(1) = \int_0^1 H_2(x(s))\,ds.
\end{aligned} \tag{5.1.2}$$

Since $F(t,x) : J \times \mathbb{R} \to \mathbb{R}$ and $H_i : \mathbb{R} \to \mathbb{R}$ are continuous and bounded, it follows that the boundary value problem (5.1.2) has a solution. Further, we note that

$$\begin{aligned}
&\alpha''(t) \geq f(t,\alpha(t)) = F(t,\alpha(t)),\ t \in J\\
&\alpha(0) - k_1\alpha'(0) \leq \int_0^1 h_1(\alpha(s))ds = \int_0^1 H_1(\alpha(s))ds,\\
&\alpha(1) + k_2\alpha'(1) \leq \int_0^1 h_2(\alpha(s))ds = \int_0^1 H_2(\alpha(s))ds,
\end{aligned}$$

and

$$\begin{aligned}
&\beta''(t) \leq f(t,\beta(t)) = F(t,\beta(t)),\ t \in J\\
&\beta(0) - k_1\beta'(0) \geq \int_0^1 h_1(\beta(s))ds = \int_0^1 H_1(\beta(s))ds,\\
&\beta(1) + k_2\beta'(1) \geq \int_0^1 h_2(\beta(s))ds = \int_0^1 H_2(\beta(s))ds,
\end{aligned}$$

which imply that α and β are respectively lower and upper solutions of (5.1.2). Also, we note that any solution x of (5.1.2) which lies between α and β, is a solution of (5.1.1). Thus, we only need to show that any solution x of (5.1.2) is such that $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in J$. Assume that $\alpha(t) \leq x(t)$ is not true on J. Then the function $v = \alpha - x$ defined on J, has a positive maximum at some $t = t_0 \in J$. If $t_0 \in (0,1)$, then

$$v(t_0) > 0,\ v'(t_0) = 0,\ v''(t_0) \leq 0$$

and hence

$$0 \geq v''(t_0) = \alpha''(t_0) - x''(t_0) \geq f(t_0,\alpha(t_0)) - \big(f(t_0,\alpha(t_0)) + \frac{x(t_0) - \alpha(t_0)}{1 + |x(t_0) - \alpha(t_0)|}\big) > 0,$$

a contradiction. If $t_0 = 0$, then $v(0) > 0$ and $v'(0) \leq 0$, but then the boundary conditions and the nondecreasing property of h_i gives

$$\begin{aligned}
v(0) &\leq k_1 v'(0) + \int_0^1 [h_1(\alpha(s)) - H_1(x(s))]ds\\
&\leq \int_0^1 [h_1(\alpha(s)) - H_1(x(s))]ds.
\end{aligned}$$

If $x(s) < \alpha(s)$, then $H_1(x(s)) = h_1(\alpha(s))$. If $x(s) > \beta(s)$, then

$$H_1(x(s)) = h_1(\beta(s)) \geq h_1(\alpha(s)).$$

If $\alpha(s) \leq x(s) \leq \beta(s)$, then

$$H_1(x(s)) = h_1(x(s)) \geq h_1(\alpha(s)).$$

Either case contradicts $v(0) > 0$. Similarly, if $t_0 = 1$, we get a contradiction. Thus $\alpha(t) \leq x(t)$, $t \in J$. Similarly, we can show that $x(t) \leq \beta(t)$, $t \in J$. □

Theorem 5.1.3. *Assume that α and β are lower and upper solutions of the boundary value problem* (5.1.1) *respectively. If $f : J \times \mathbb{R} \to \mathbb{R}$ and $h_i : \mathbb{R} \to \mathbb{R}$ are continuous, f is strictly increasing for each $t \in J$ and $0 \le h_i'(x) < 1$. Then $\alpha(t) \le \beta(t)$, $t \in J$.*

Proof. Define $w(t) = \alpha(t) - \beta(t)$, $t \in J$, then $w \in C^2(J)$ and

$$\begin{aligned} w(0) - k_1 w'(0) &\le \int_0^1 [h_1(\alpha(s)) - h_1(\beta(s))]ds \\ w(1) + k_2 w'(1) &\le \int_0^1 [h_2(\alpha(s)) - h_2(\beta(s))]ds. \end{aligned} \tag{5.1.3}$$

Assume that $w(t) \le 0$ is not true for every $t \in J$. Then $w(t)$ has a positive maximum at some $t_0 \in J$. If $t_0 \in (0,1)$, then $w(t_0) > 0$, $w'(t_0) = 0$ and $w''(t_0) \le 0$. Using the increasing property of the function $f(t,x)$ in x, we obtain

$$f(t_0, \alpha(t_0)) \le \alpha''(t_0) \le \beta''(t_0) \le f(t_0, \beta(t_0)) < f(t_0, \alpha(t_0)),$$

a contradiction. If $t_0 = 0$, then $w(0) > 0$ and $w'(0) \le 0$. On the other hand, using the boundary conditions (5.1.3) and the assumption $0 \le h_1'(x) < 1$, we have

$$\begin{aligned} w(0) \le w(0) - k_1 w'(0) &\le \int_0^1 [h_1(\alpha(s)) - h_1(\beta(s))]ds \le \int_0^1 h_1'(c) w(s)ds \\ &\le h_1'(c) \max_{t \in J} w(t) = h_1'(c) w(0) < w(0), \end{aligned} \tag{5.1.4}$$

a contradiction. If $t_0 = 1$, then $w(1) > 0$ and $w'(1) \ge 0$. But again, the boundary conditions (5.1.3) and the assumption $0 \le h_2'(x) < 1$, gives

$$\begin{aligned} w(1) \le w(1) + k_2 w'(1) &\le \int_0^1 [h_2(\alpha(s)) - h_2(\beta(s))]ds \le \int_0^1 h_2'(c) w(s)ds \\ &\le h_2'(c) \max_{t \in J} w(t) = h_2'(c) w(1) < w(1), \end{aligned} \tag{5.1.5}$$

a contradiction. Hence

$$\alpha(t) \le \beta(t),\ t \in J.$$

□

As a consequence of the Theorem 5.1.3, we have

Corollary 5.1.4. *Assume that α and β are lower and upper solutions of the boundary value problem* (5.1.1) *respectively. If $f : J \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous, f is strictly increasing and $0 \le h'(x) < 1$, $x \in \mathbb{R}$. Then the solution of the boundary value problem* (5.1.1) *is unique.*

5.1.3 Approximation of solutions[generalized quasilinearization technique]

In this section, we develop the generalized quasilinearization technique to study approximation of solutions of the BVP (5.1.1). Assume that

($\mathbf{A_1}$) α and $\beta \in C^2(J)$ are respectively lower and upper solutions of (1.1.1) such that $\alpha(t) \leq \beta(t), t \in J$.

($\mathbf{A_2}$) $f(t,x) \in C^2(J \times \mathbb{R})$ is such that $f_x(t,x) > 0$ and $f_{xx}(t,x) + \phi_{xx}(t,x) \leq 0$ on $J \times [\min \alpha(t), \max \beta(t)]$, where $\phi(t,x) \in C^2(J \times \mathbb{R})$ and $\phi_{xx}(t,x) \leq 0$ on $J \times [\min \alpha(t), \max \beta(t)]$.

($\mathbf{A_3}$) $h_i \in C^2(\mathbb{R})\,(i = 1,2)$ are nondecreasing, $0 \leq h_i'(x) < 1$ and $h_i''(x) \geq 0, x \in [\min \alpha(t), \max \beta(t)]$.

Theorem 5.1.5. *Under assumptions $(A_1) - (A_3)$, there exists a monotone sequence $\{w_n\}$ of solutions converging uniformly and quadratically to the unique solution of the problem.*

Proof. Define $F : J \times \mathbb{R} \to \mathbb{R}$ by $F(t,x) = f(t,x) + \phi(t,x)$. Then in view of (A_2), we note that $F \in C^2(J \times \mathbb{R})$ and

$$F_{xx}(t,x) \leq 0 \text{ on } J \times [\min \alpha(t), \max \beta(t)]. \tag{5.1.6}$$

Using Taylor's theorem, (5.1.6) and (A_3), we have

$$f(t,x) \leq F(t,y) + F_x(t,y)(x-y) - \phi(t,x), \tag{5.1.7}$$

for (t,x), $(t,y) \in J \times [\min \alpha(t), \max \beta(t)]$ and

$$h_i(x) \geq h_i(y) + h_i'(y)(x-y), \tag{5.1.8}$$

for x, $y \in [\min \alpha(t), \max \beta(t)]$. Again applying Taylor's theorem to $\phi(t,x)$, we obtain

$$\phi(t,x) = \phi(t,y) + \phi_x(t,y)(x-y) + \frac{1}{2}\phi_{xx}(t,\xi)(x-y)^2, \tag{5.1.9}$$

where x, $y \in \mathbb{R}$ and ξ is between x and y, which in view of (A_2) implies that

$$\phi(t,x) \leq \phi(t,y) + \phi_x(t,y)(x-y), \tag{5.1.10}$$

and

$$\phi(t,x) \geq \phi(t,y) + \phi_x(t,y)(x-y) - \frac{1}{2}|\phi_{xx}(t,\xi)|\|x-y\|^2, \tag{5.1.11}$$

for (t,x), $(t,y) \in J \times [\min \alpha(t), \max \beta(t)]$, where $\|x-y\| = \max\{|x(t)-y(t)| : t \in J\}$ denotes the supremum norm in the space of continuous functions defined on J. Using (5.1.11) in (5.1.7), we obtain

$$f(t,x) \leq f(t,y) + f_x(t,y)(x-y) + \frac{1}{2}|\phi_{xx}(t,\xi)|\|x-y\|^2,$$

for (t,x), $(t,y) \in J \times [\min\alpha(t), \max\beta(t)]$. Let

$$\Omega_1 = \{(t,x) : t \in J,\, x \in [\min\alpha(t), \max\beta(t)]\} \text{ and } M_2 = \max\{|\phi_{xx}(t,x)| : (t,x) \in \Omega_1\}.$$

Then

$$f(t,x) \le f(t,y) + f_x(t,y)(x-y) + \frac{M_2}{2}\|x-y\|^2, \tag{5.1.12}$$

for (t,x), $(t,y) \in \Omega_1$. Define

$$g(t,x,y) = f(t,y) + f_x(t,y)(x-y) + \frac{M_2}{2}\|x-y\|^2, \tag{5.1.13}$$

where (t,x), $(t,y) \in J \times \mathbb{R}$ and

$$H_i(x,y) = h_i(y) + h_i'(y)(x-y), \tag{5.1.14}$$

where x, $y \in \mathbb{R}$. We note that $g(t,x,y)$ and $H_i(x,y)$ satisfy

$$g_x(t,x,y) = f_x(t,y) > 0 \text{ and } 0 \le \frac{\partial}{\partial x}H_i(x,y) < 1,$$

for x, $y \in [\min\alpha(t), \max\beta(t)]$, $t \in J$. Moreover, $g(t,x,y)$ and $H_i(x,y)$ are continuous and bounded on Ω_1 and in view of (5.1.12), (5.1.13) and (5.1.8), (5.1.14) respectively, satisfy the following relations

$$\begin{cases} f(t,x) \le g(t,x,y), \\ f(t,x) = g(t,x,x), \end{cases} \tag{5.1.15}$$

for (t,x), $(t,y) \in \Omega_1$ and

$$\begin{cases} h_i(x) \ge H_i(x,y) \\ h_i(x) = H_i(x,x), \end{cases} \tag{5.1.16}$$

for x, $y \in [\min\alpha(t), \max\beta(t)]$.

Now, we set $w_0 = \alpha$ and consider the linear boundary value problem

$$\begin{aligned} &x''(t) = g(t,x,w_0),\ t \in J, \\ &x(0) - k_1x'(0) = \int_0^1 H_1(x(s), w_0(s))ds, \\ &x(1) + k_2x'(1) = \int_0^1 H_2(x(s), w_0(s))ds. \end{aligned} \tag{5.1.17}$$

Using (A_1), (5.1.15) and (5.1.16), we obtain

$$\begin{aligned} &w_0''(t) \ge f(t,w_0(t)) = g(t,w_0(t),w_0(t)),\ t \in J, \\ &w_0(0) - k_1w_0'(0) \le \int_0^1 h_1(w_0(s))ds = \int_0^1 H_1(w_0(s),w_0(s))ds, \\ &w_0(1) + k_2w_0'(1) \le \int_0^1 h_2(w_0(s))ds = \int_0^1 H_2(w_0(s),w_0(s))ds, \end{aligned}$$

$$\beta''(t) \le f(t,\beta(t)) \le g(t,\beta(t),w_0(t)),\ t \in J,$$
$$\beta(0) - k_1\beta'(0) \ge \int_0^1 h_1(\beta(s))ds \ge \int_0^1 H_1(\beta(s), w_0(s))ds,$$
$$\beta(1) + k_2\beta'(1) \ge \int_0^1 h_2(\beta(s))ds \ge \int_0^1 H_2(\beta(s), w_0(s))ds,$$

which imply that w_0 and β are respectively lower and upper solutions of (5.1.17). It follows by Theorems 5.1.2, 5.1.3 that there exists a unique solution w_1 of (5.1.17) such that

$$w_0(t) \le w_1(t) \le \beta(t),\ t \in J.$$

In view of (5.1.15), (5.1.16) and the fact that w_1 is a solution of (5.1.17), we note that w_1 is a lower solution of (5.1.1).

Now consider the problem

$$\begin{aligned} x''(t) &= g(t,x,w_1),\ t \in J,\\ x(0) - k_1x'(0) &= \int_0^1 H_1(x(s), w_1(s))ds,\\ x(1) + k_2x'(1) &= \int_0^1 H_2(x(s), w_1(s))ds. \end{aligned} \tag{5.1.18}$$

Again we can show that w_1 and β are lower and upper solutions of (5.1.18) and hence by Theorems 5.1.2, 5.1.3, there exists a unique solution w_2 of (5.1.18) such that

$$w_1(t) \le w_2(t) \le \beta(t),\ t \in J.$$

Continuing this process, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0(t) \le w_1(t) \le w_2(t) \le ...w_n(t) \le \beta(t),\ t \in J,$$

where, the element w_n of the sequence $\{w_n\}$ is a solution of the boundary value problem

$$\begin{aligned} x''(t) &= g(t,x,w_{n-1}),\ t \in J,\\ x(0) - k_1x'(0) &= \int_0^1 H_1(x(s), w_{n-1}(s))ds,\\ x(1) + k_2x'(1) &= \int_0^1 H_2(x(s), w_{n-1}(s))ds, \end{aligned}$$

and

$$w_n(t) = P_n(t) + \int_0^1 G(t,s)g(s, w_n(s), w_{n-1}(s))ds, \tag{5.1.19}$$

where

$$P_n(t) = \frac{1}{1+k_1+k_2}\{(1-t+k_2)\int_0^1 H_1(w_n(s), w_{n-1}(s))\,ds + (k_1+t)\int_0^1 H_2(w_n(s), w_{n-1}(s))\,ds\}. \tag{5.1.20}$$

Employing the standard arguments, as before, it follows that the convergence of the sequence of solutions is uniform. If $x(t)$ is the limit point of the sequence, passing to the limit as $n \to \infty$, (5.1.19) gives

$$x(t) = P(t) + \int_0^1 G(t,s)f(s,x(s))ds,$$

where

$$P(t) = \frac{1}{1+k_1+k_2}\{(1-t+k_2)\int_0^1 h_1(x(s))ds + (k_1+t)\int_0^1 h_2(x(s))ds\};$$

that is, x is a solution of the boundary value problem (5.1.1).

Now, we show that the convergence of the sequence is quadratic. For that, set $e_n(t) = x(t) - w_n(t)$, $t \in J$. Note that, $e_n(t) \geq 0$, $t \in J$, where x is the solution of (5.1.1). Using Taylor's theorem and (5.1.14), we obtain

$$\begin{aligned} e_n(0) - k_1 e_n'(0) &= \int_0^1 [h_1(x(s)) - H_1(w_n(s), w_{n-1}(s))]ds \\ &= \int_0^1 [h_1'(w_{n-1}(s))e_n(s) + \frac{1}{2}h_1''(\xi_1)e_{n-1}^2(s)]ds \end{aligned}$$

and

$$\begin{aligned} e_n(1) + k_2 e_n'(1) &= \int_0^1 [h_2(x(s)) - H_2(w_n(s), w_{n-1}(s))]ds \\ &= \int_0^1 [h_2'(w_{n-1}(s))e_n(s) + \frac{1}{2}h_2''(\xi_2)e_{n-1}^2(s)]ds \end{aligned}$$

where, $w_{n-1}(t) \leq \xi_1, \xi_2 \leq x(t)$. In view of (A_3), there exist $\lambda_i < 1$ and $C_i \geq 0$ such that $h_i'(w_{n-1}(s)) \leq \lambda_i$ and $0 \leq \frac{1}{2}h_i''(\xi_1) \leq C_i$ $(i = 1,2)$. Let $\lambda = \max\{\lambda_1, \lambda_2\} < 1$ and $C = \max\{C_1, C_2\} \geq 0$, then

$$\begin{aligned} e_n(0) - k_1 e_n'(0) &\leq \lambda \int_0^1 e_n(s)ds + C\int_0^1 e_{n-1}^2(s)ds \leq \lambda \int_0^1 e_n(s)ds + C\|e_{n-1}\|^2 \\ e_n(1) + k_2 e_n'(1) &\leq \lambda \int_0^1 e_n(s)ds + C\int_0^1 e_{n-1}^2(s)ds \leq \lambda \int_0^1 e_n(s)ds + C\|e_{n-1}\|^2. \end{aligned} \tag{5.1.21}$$

Further, using Taylor's theorem, (A_2), (5.1.10), (5.1.13) and the fact that $\|w_n - w_{n-1}\| \leq \|e_{n-1}\|$, we obtain

$$\begin{aligned} e_n''(t) &= x''(t) - w_n''(t) = (F(t,x) - \phi(t,x)) \\ &\quad - [f(t,w_{n-1}) + f_x(t,w_{n-1})(w_n - w_{n-1}) + \frac{M_2}{2}\|w_n - w_{n-1}\|^2] \\ &\geq f_x(t,w_{n-1})e_n(t) + F_{xx}(t,\xi_3)\frac{e_{n-1}^2(t)}{2!} - \frac{M_2}{2}\|e_{n-1}\|^2 \\ &\geq -\frac{1}{2}(|F_{xx}(t,\xi_3)| + M_2)\|e_{n-1}\|^2 \\ &\geq -M\|e_{n-1}\|^2, \end{aligned} \tag{5.1.22}$$

where $w_{n-1}(t) \le \xi_3 \le x(t)$, $w_{n-1}(t) \le \xi \le w_n(t)$, $|F_{xx}(t,x)| \le M_1$ on Ω_1 and $2M = M_1 + M_2$. From (5.1.21) and (5.1.22), it follows (using comparison principle) that

$$e_n(t) \le \rho(t) \text{ on } J,$$

where, $\rho(t) \ge 0$ is the unique solution of the boundary value problem

$$\begin{aligned} \rho''(t) &= -M\|e_{n-1}\|^2,\ t \in J \\ \rho(0) - k_1\rho'(0) &= \lambda \int_0^1 e_n(s)ds + C\|e_{n-1}\|^2 \\ \rho(1) + k_2\rho'(1) &= \lambda \int_0^1 e_n(s)ds + C\|e_{n-1}\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} e_n(t) \le \rho(t) &= \frac{1}{1+k_1+k_2}\big[(1-t+k_2)\big(\lambda\int_0^1 e_n(s)ds + C\|e_{n-1}\|^2\big) \\ &+ (t+k_1)\big(\lambda\int_0^1 e_n(s)ds + C\|e_{n-1}\|^2\big)\big] - M\int_0^1 G(t,s)\|e_{n-1}\|^2 ds \\ &\le \frac{1}{1+k_1+k_2}\big(\lambda\{(1-t+k_2)+(t+k_1)\}\|e_n\| + C\{(1-t+k_2)+(t+k_1)\}\|e_{n-1}\|^2\big) \\ &+ M\|e_{n-1}\|^2\int_0^1 |G(t,s)|ds \\ &= \lambda\|e_n\| + C\|e_{n-1}\|^2 + Ml\|e_{n-1}\|^2 = \lambda\|e_n\| + L\|e_{n-1}\|^2, \end{aligned}$$

where l is a bound for $\int_0^1 |G(t,s)|ds$ and $L = C + lM$. Taking the maximum over J, and solving for $\|e_n\|$, we get

$$\|e_n\| \le \delta\|e_{n-1}\|^2,$$

where, $\delta = \frac{L}{1-\lambda}$. □

5.1.4 Rapid convergence

We improve the rate of convergence and show that under some suitable conditions there exists a sequence of approximants which converges to the solution of the problem (5.1.1) rapidly. Assume that

(B₁) α, $\beta \in C^2(J)$ are lower and upper solutions of (5.1.1) respectively such that $\alpha(t) \le \beta(t), t \in J$.

(B₂) There exists $k \ge 2$ such that $f(t,x) \in C^k(J \times \mathbb{R})$ is such that $\frac{\partial^j}{\partial x^j} f(t,x) \ge 0\,(j = 1, 2, 3..., k-1)$, and
$\frac{\partial^k}{\partial x^k}(f(t,x) + \phi(t,x)) \le 0$ on Ω_1, where, $\phi \in C^k(J \times \mathbb{R})$ and $\frac{\partial^k}{\partial x^k}\phi(t,x) \le 0$ on Ω_1.

(B₃) $h_j(x) \in C^k(\mathbb{R})$ is such that $\frac{d^i}{dx^i}h_j(x) \le \frac{M}{(\beta-\alpha)^{i-1}}$ $(i = 1, 2, ..., k-1)$ and $\frac{d^k}{dx^k}h_j(x) \ge 0$ on $[\min\alpha(t), \max\beta(t)]$, where $M < 1/3$ and $j = 1, 2$.

Theorem 5.1.6. *Under assumptions $(B_1) - (B_3)$, there exists a monotone sequence $\{w_n\}$ of solutions converging uniformly to a solution of the problem (5.1.1). Moreover the rate of convergence is of order k.*

Proof. Define, $F : J \times \mathbb{R} \to \mathbb{R}$ by $F(t,x) = f(t,x) + \phi(t,x)$, $t \in J$. Then in view of (B_2), we note that $F \in C^k(J \times \mathbb{R})$ and

$$\frac{\partial^k}{\partial x^k} F(t,x) \le 0 \text{ on } \Omega_1. \tag{5.1.23}$$

Using (B_3), Taylor's theorem and (5.1.23), we have

$$f(t,x) \le \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} F(t,y) \frac{(x-y)^i}{i!} - \phi(t,x), \tag{5.1.24}$$

for (t,x), $(t,y) \in \Omega_1$ and

$$h_j(x) \ge \sum_{i=0}^{k-1} \frac{d^i}{dx^i} h_j(y) \frac{(x-y)^i}{i!}, \tag{5.1.25}$$

for x, $y \in [\min \alpha(t), \max \beta(t)]$. Expanding $\phi(t,x)$ about (t,y) by Taylor's theorem, we obtain

$$\phi(t,x) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} \phi(t,y) \frac{(x-y)^i}{i!} + \frac{\partial^k}{\partial x^k} \phi(t,\xi) \frac{(x-y)^k}{k!}, \tag{5.1.26}$$

where x, $y \in \mathbb{R}$, $t \in J$ and ξ is between x and y, which in view of (B_2) implies that

$$\phi(t,x) \le \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} \phi(t,y) \frac{(x-y)^i}{i!}, \tag{5.1.27}$$

for (t,x), $(t,y) \in \Omega_1$. Using (5.1.26) in (5.1.24), we obtain

$$f(t,x) \le \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t,y) \frac{(x-y)^i}{i!} - \frac{\partial^k}{\partial x^k} \phi(t,\xi) \frac{(x-y)^k}{k!}, \tag{5.1.28}$$

for (t,x), $(t,y) \in \Omega_1$. Define

$$g^*(t,x,y) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t,y) \frac{(x-y)^i}{i!} - \frac{\partial^k}{\partial x^k} \phi(t,\xi) \frac{(x-y)^k}{k!}, \tag{5.1.29}$$

for (t,x), $(t,y) \in J \times \mathbb{R}$ and

$$H_j^*(x,y) = \sum_{i=0}^{k-1} \frac{d^i}{dx^i} h_j(y) \frac{(x-y)^i}{i!}, \tag{5.1.30}$$

for x, $y \in \mathbb{R}$. We note that $g^*(t,x,y)$ and $H_j^*(x,y)$ are continuous and bounded on Ω_1 and are such that

$$g_x^*(t,x,y) = \sum_{i=1}^{k-1} \frac{\partial^i}{\partial x^i} f(t,y) \frac{(x-y)^{i-1}}{(i-1)!} - \frac{\partial^k}{\partial x^k} \phi(t,\xi) \frac{(x-y)^{k-1}}{(k-1)!} \ge 0,$$

for (t,x), $(t,y) \in \Omega_1$, and

$$\begin{aligned}\frac{\partial}{\partial x}H_j^*(x,y) &= \sum_{i=1}^{k-1}\frac{d^i}{dx^i}h_j(y)\frac{(x-y)^{i-1}}{(i-1)!}\\ &\leq \sum_{i=1}^{k-1}\frac{M}{(\beta-\alpha)^{i-1}}\frac{(\beta-\alpha)^{i-1}}{(i-1)!} \leq M(3-\frac{1}{2^{k-2}}) < 1,\end{aligned}$$

for x, $y \in [\min\alpha(t), \max\beta(t)]$. Further, from (5.1.28), (5.1.29) and (5.1.25), (5.1.30) respectively, we have the relations

$$\begin{cases} f(t,x) \leq g^*(t,x,y), \\ f(t,x) = g^*(t,x,x), \end{cases} \tag{5.1.31}$$

for (t,x), $(t,y) \in \Omega_1$, and

$$\begin{cases} h_j(x) \geq H_j^*(x,y), \\ h_j(x) = H_j^*(x,x), \end{cases} \tag{5.1.32}$$

for x, $y \in [\min\alpha(t), \max\beta(t)]$.

Now, set $\alpha = w_0$ and consider the problem

$$\begin{aligned} &x''(t) = g^*(t,x,w_0),\ t \in J,\\ &x(0) - k_1x'(0) = \int_0^1 H_1^*(x(s),w_0(s))ds,\\ &x(1) + k_2x'(1) = \int_0^1 H_2^*(x(s),w_0(s))ds.\end{aligned} \tag{5.1.33}$$

The assumption (B_1) and the expressions (5.1.31), (5.1.32) yield

$$\begin{aligned} &w_0''(t) \geq f(t,w_0(t)) = g^*(t,w_0(t),w_0(t)),\ t \in J,\\ &w_0(0) - k_1w_0'(0) \leq \int_0^1 h_1(w_0(s))ds = \int_0^1 H_1^*(w_0(s),w_0(s))ds,\\ &w_0(1) + k_2w_0'(1) \leq \int_0^1 H_2^*(w_0(s))ds = \int_0^1 H_2^*(w_0(s),w_0(s))ds,\end{aligned}$$

$$\begin{aligned} &\beta''(t) \leq f(t,\beta(t)) \leq g(t,\beta(t),w_0(t)),\ t \in J,\\ &\beta(0) - k_1\beta'(0) \geq \int_0^1 h_1(\beta(s))ds \geq \int_0^1 H_1^*(\beta(s),w_0(s))ds,\\ &\beta(1) + k_2\beta'(1) \geq \int_0^1 h_2(\beta(s))ds \geq \int_0^1 H_2^*(\beta(s),w_0(s))ds,\end{aligned}$$

which imply that w_0 and β are respectively lower and upper solutions of (5.1.33). Hence by Theorems 5.1.2, 5.1.3, there exists a solution w_1 of (5.1.33) such that

$$w_0(t) \leq w_1(t) \leq \beta(t),\ t \in J.$$

Continuing this process, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0(t) \le w_1(t) \le w_2(t) \le ...w_n(t) \le \beta(t),\ t \in J,$$

where the element w_n of the sequence $\{w_n\}$ is a solution of the boundary value problem

$$\begin{aligned} x''(t) &= g^*(t, x, w_{n-1}),\ t \in J, \\ x(0) - k_1 x'(0) &= \int_0^1 H_1^*(x(s), w_{n-1}(s))ds, \\ x(1) + k_2 x'(1) &= \int_0^1 H_2^*(x(s), w_{n-1}(s))ds. \end{aligned}$$

By the same process as in Theorem 5.1.5, we can show that the sequence converges uniformly to the unique solution of (5.1.1).

Now, we show that the convergence of the sequence is of order k as in (B_2). For this, we set

$$e_n(t) = x(t) - w_n(t) \text{ and } a_n(t) = w_{n+1}(t) - w_n(t),\ t \in J.$$

where x is a solution of the boundary value problem (5.1.1). Then,

$$e_n(t) \ge 0,\ a_n(t) \ge 0,\ t \in J.$$

Further, we have

$$e_{n+1}(t) = e_n(t) - a_n(t),\ e_n^i(t) \ge a_n^i(t)\ (i = 1, 2...) \text{ on } J,$$

and from the boundary conditions, we have

$$\begin{aligned} e_n(0) - k_1 e_n'(0) &= \int_0^1 [h_1(x(s)) - H_1^*(w_n(s), w_{n-1}(s)]ds \\ e_n(1) + k_2 e_n'(1) &= \int_0^1 [h_2(x(s)) - H_2^*(w_n(s), w_{n-1}(s)]ds. \end{aligned} \tag{5.1.34}$$

Using Taylor's theorem and (5.1.30), we obtain

$$\begin{aligned} h_j(x(t)) - H_j^*(w_n(t), w_{n-1}(t)) &= \sum_{i=0}^{k-1} \frac{d^i}{dx^i} h_j(w_{n-1}) \frac{(x - w_{n-1})^i}{i!} + \frac{d^k}{dx^k} h_j(c) \frac{(x - w_{n-1})^k}{k!} \\ &- \sum_{i=0}^{k-1} \frac{d^i}{dx^i} h_j(w_{n-1}) \frac{(w_n - w_{n-1})^i}{i!} \\ &= \left[\sum_{i=1}^{k-1} \frac{d^i}{dx^i} h_j(w_{n-1}) \frac{1}{i!} \sum_{l=0}^{i-1} e_{n-1}^{i-1-l} a_{n-1}^l\right] e_n + \frac{d^k}{dx^k} h_j(c) \frac{e_{n-1}^k}{k!} \\ &\le p_j(t) e_n(t) + \frac{M}{r^{k-1}} \frac{e_{n-1}^k}{k!} \le p_j(t) e_n(t) + \frac{M}{r^{k-1}} \frac{\|e_{n-1}^k\|}{k!}, \end{aligned}$$

where

$$p_j(t) = \sum_{i=1}^{k-1} \frac{d^i}{dx^i} h_j(w_{n-1}(t)) \frac{1}{i!} \sum_{l=0}^{i-1} e_{n-1}^{i-1-l}(t) a_{n-1}^l(t)$$

and $r = \max_{t\in J}\beta(t) - \min_{t\in J}\alpha(t)$. In view of (B_3), we have

$$p_j(t) \le \sum_{i=1}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}} \frac{1}{i!} \sum_{l=0}^{i-1} e_{n-1}^{i-1} \le \sum_{i=1}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}} \frac{1}{(i-1)!} (\beta-\alpha)^{i-1} < 1.$$

It follows that we can find $\lambda < 1$ such that $p_j(t) \le \lambda$, $t \in J$, $(j = 1, 2)$ and hence

$$\begin{aligned} e_n(0) - k_1 e_n'(0) &\le \lambda \int_0^1 e_n(s)ds + \frac{M}{r^{k-1}k!}\|e_{n-1}\|^k \\ e_n(1) + k_2 e_n'(1) &\le \lambda \int_0^1 e_n(s)ds + \frac{M}{r^{k-1}k!}\|e_{n-1}\|^k. \end{aligned} \tag{5.1.35}$$

Now, using Taylor's theorem and (5.1.27), we obtain

$$\begin{aligned} e_n''(t) &= x''(t) - w_n''(t) \\ &= [F(t,x) - \phi(t,x)] - \Big[\sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_{n-1}) \frac{a_{n-1}^i}{i!} - \frac{\partial^k}{\partial x^k}\phi(t,\xi)\frac{a_{n-1}^k}{k!}\Big] \\ &= \sum_{i=1}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_{n-1}) \frac{(e_{n-1}^i - a_{n-1}^i)}{i!} + \frac{\partial^k}{\partial x^k} F(t, c_1)\frac{e_{n-1}^k}{k!} + \frac{\partial^k}{\partial x^k}\phi(t,\xi))\frac{a_{n-1}^k}{k!} \\ &\ge \sum_{i=1}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_{n-1}) \frac{\sum_{l=0}^{i-1} e_{n-1}^{i-1-l} a_{n-1}^l}{i!} e_n + \frac{e_{n-1}^k}{k!}(\frac{\partial^k}{\partial x^k}F(t,c_1) + \frac{\partial^k}{\partial x^k}\phi(t,\xi)) \\ &\ge -N\frac{\|e_{n-1}\|^k}{k!}, \end{aligned} \tag{5.1.36}$$

where $-N_1 \le \frac{\partial^k}{\partial x^k}F(t,x) \le 0$, $-N_2 \le \frac{\partial^k}{\partial x^k}\phi(t,x) \le 0$ and $N = \max\{N_1, N_2\}$. From (5.1.35) and (5.1.36), it follows that $0 \le e_n(t) \le \rho(t), t \in J$, where $\rho(t)$ is the unique solution of the problem

$$\begin{aligned} \rho''(t) &= -N\frac{\|e_{n-1}\|^k}{k!},\ t \in J \\ \rho(0) - k_1\rho'(0) &\le \lambda \int_0^1 e_n(s)ds + \frac{M}{r^{k-1}k!}\|e_{n-1}\|^k \\ \rho(1) + k_2 r'(1) &\le \lambda \int_0^1 e_n(s)ds + \frac{M}{r^{k-1}k!}\|e_{n-1}\|^k, \end{aligned}$$

and

$$\begin{aligned} e_n(t) \le \rho(t) &= \frac{1}{1+k_1+k_2}\Big[(1-t+k_2)(\lambda\int_0^1 e_n(s)\,ds + \frac{M}{r^{k-1}k!}\|e_{n-1}\|^k) + \\ &\quad (t+k_1)(\lambda\int_0^1 e_n(s)\,ds + \frac{M}{r^{k-1}k!}\|e_{n-1}\|^k\Big] - N\int_0^1 G(t,s)\frac{\|e_{n-1}\|^k}{k!}\,ds \\ &\le \frac{1}{1+k_1+k_2}\Big[\lambda\{(1-t+k_2) + (t+k_1)\}\|e_n\| \\ &\quad + \{(1-t+k_2)+(t+k_1)\}\frac{M}{r^{k-1}k!}\|e_{n-1}\|^k\Big] + N\|e_{n-1}\|^2\int_0^1 |G(t,s)|\,ds \\ &= \lambda\|e_n\| + C'\|e_{n-1}\|^k, \end{aligned}$$

where L is a bound for $\int_0^1 |G(t,s)|ds$ and $C' = \frac{M}{r^{k-1}k!} + NL$. Taking the maximum over J, we get

$$\|e_n\| \leq \delta_1 \|e_{n-1}\|^k,$$

where $\delta_1 = \frac{C'}{1-\lambda}$. □

5.2 Nonlinearity depending on the derivative

In this section, we study existence, uniqueness and approximation of solutions for a nonlinear boundary value problem with nonlinear integral boundary conditions in the case when the nonlinearity f depends also on the derivative x'. In this case, we control both the function and its first derivative so prove a result on quadratic convergence in the C^1 norm. It is more delicate than the corresponding results when there is no x' dependence in f as in the previous section. We consider boundary value problem of the form

$$\begin{aligned} -x''(t) &= f(t,x,x'),\ t \in J = [0,1], \\ x(0) = a, \quad x(1) &= \int_0^1 h(x(s))ds \end{aligned} \tag{5.2.1}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ are continuous and $a \geq 0$. We use the method of upper and lower solutions and the method of quasilinearization for the existence, uniqueness and approximation of solutions. We approximate our problem by a sequence of *linear* problems to obtain a monotone sequence of approximants. We show that the sequence of approximants converges quadratically to a unique solution of the BVP (5.2.1).

5.2.1 Existence results (upper and lower solutions method)

Definition 5.2.1. Let $\alpha \in C^2(J)$. We say that α is a lower solution of the boundary value problem (5.2.1), if

$$\begin{aligned} -\alpha''(t) &\leq f(t,\alpha(t),\alpha'(t)),\ t \in J \\ \alpha(0) \leq a,\ \alpha(1) &\leq \int_0^1 h(\alpha(s))ds. \end{aligned}$$

Similarly, $\beta \in C^2(J)$ is an upper solution of the boundary value problem (5.2.1), if β satisfies similar inequalities in the reverse direction.

Now we prove some results which establish existence and uniqueness of solutions for the boundary value problem (5.2.1).

Theorem 5.2.2. *Assume that α, $\beta \in C^2(J)$ are lower and upper solutions of (5.2.1) respectively such that $\alpha(t) \leq \beta(t)$ on J. Assume that $f \in C(J \times \mathbb{R} \times \mathbb{R})$ and satisfies the Nagumo conditions on J relative to α, β. Then, there exists a solution $x(t)$ of (5.2.1) such that $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in J$.*

Proof. There exists a constant N depending on α, β and ω such that

$$\int_{\lambda}^{N} \frac{sds}{\omega(s)} > \max_{t\in J} \beta(t) - \min_{t\in J} \alpha(t),$$

where ω is the Nagumo function. Let $C > \max\{N, \|\alpha'\|, \|\beta'\|\}$. Let $n \in \mathbb{N}$ and consider the modified boundary value problem

$$\begin{aligned} &-x''(t) = f_n(t,x,x'),\ t \in J,\\ &x(0) = a,\ x(1) = \int_0^1 H(x(s))ds, \end{aligned} \tag{5.2.2}$$

where

$$f_n(t,x,x') = \begin{cases} f(t,\beta(t),\beta'(t)) + \frac{\beta(t)-x}{1+|\beta(t)-x|}, & \text{if } x \geq \beta(t) + \frac{1}{n},\\ f(t,\beta(t),x') + \big[f(t,\beta(t),\beta'(t)) & \\ -f(t,\beta(t),x') + \frac{\beta(t)-x}{1+|\beta(t)-x|}\big]n(x-\beta(t)), & \text{if } \beta(t) \leq x < \beta(t) + \frac{1}{n},\\ f(t,x,x'), & \text{if } \alpha(t) \leq x \leq \beta(t),\\ f(t,\alpha(t),x') - \big[f(t,\alpha(t),\alpha'(t)) & \\ -f(t,\alpha(t),x') + \frac{\alpha(t)-x}{1+|\alpha(t)-x|}\big]n(x-\alpha(t)), & \text{if } \alpha(t) - \frac{1}{n} < x \leq \alpha(t),\\ f(t,\alpha(t),\alpha'(t)) + \frac{\alpha(t)-x}{1+|\alpha(t)-x|}, & \text{if } x \leq \alpha(t) - \frac{1}{n} \end{cases}$$

and

$$H(x) = \begin{cases} h(\beta(t)), & \text{if } x > \beta(t)\\ h(x), & \text{if } \alpha(t) \leq x \leq \beta(t)\\ h(\alpha(t)), & \text{if } x < \alpha(t). \end{cases}$$

Note that $f_n(t,x,x')$ is continuous on $J \times \mathbb{R}^2$. Also, H is continuous and bounded on $\mathbb{R}$. Moreover, any solution of (5.2.2) with $\alpha(t) \leq x(t) \leq \beta(t)$ on J, is a solution of (5.2.1). Choose $M_1 > 0$ large such that

$$M_1\rho(t) \leq f(t,\alpha(t),\alpha'(t)),\ f(t,\beta(t),\beta'(t)) \leq -M_1\rho(t),\ t \in J, \tag{5.2.3}$$

where

$$\rho(t) = -\frac{\beta(t)-\alpha(t)+2/n}{\beta(t)-\alpha(t)+1} < 0,\ t \in J$$

and define a bounded continuous function

$$\sigma_n(t,x) = \begin{cases} M_1\rho(t) & \text{for } x > \beta(t) + 1/n\\ -M_1\big(\frac{2x-(\beta(t)+\alpha(t))}{\beta(t)-\alpha(t)+1}\big) & \text{for } \alpha(t) - 1/n \leq x \leq \beta(t) + 1/n\\ -M_1\rho(t) & \text{for } x < \alpha(t) - 1/n. \end{cases}$$

Let $\lambda \in [0,1]$ and consider the system of BVPs

$$\begin{aligned} &-x''(t) = \lambda f_n(t,x,x') + (1-\lambda)\sigma_n(t,x), \ t \in J, \\ &x(0) = a, \ x(1) = \int_0^1 H(x(s))ds. \end{aligned} \tag{5.2.4}$$

For $\lambda = 0$, the system reduces to

$$\begin{aligned} &-x''(t) = \sigma_n(t,x), \ t \in J, \\ &x(0) = a, \ x(1) = \int_0^1 H(x(s))ds, \end{aligned} \tag{5.2.5}$$

and for $\lambda = 1$, it is (5.2.2). The problem (5.2.5) is equivalent to an integral equation

$$x(t) = a(1-t) + t\int_0^1 H(x(s))ds + \int_0^1 G(t,s)\sigma_n(s,x(s))ds,$$

where $G(t,s)$ is the Green's function corresponding to the problem

$$-x''(t) = 0, \ t \in J, \quad x(0) = 0, \ x(1) = 0.$$

Since $\sigma_n(t,x)$ and $H(x)$ are continuous and bounded. This integral equation has a fixed point by the Schauder's fixed point theorem, which implies that the boundary value problem (5.2.5) has a solution $x \in C^2(J)$. That is, (5.2.4) has a solution for $\lambda = 0$. For $0 \le \lambda \le 1$, we claim that any solution x_n of (5.2.4) satisfies

$$\alpha(t) - \frac{1}{n} \le x_n(t) \le \beta(t) + \frac{1}{n}, \ t \in J.$$

Set $v(t) = x_n(t) - \beta(t) - \frac{1}{n}$, then the boundary conditions imply that $v(0) < 0$ and

$$v(1) \le \int_0^1 H(x(s))ds - \int_0^1 h(\beta(s))ds - \frac{1}{n} < \int_0^1 H(x(s))ds - \int_0^1 h(\beta(s))ds.$$

Using the nonincreasing property of h and the definition of H, we obtain $v(1) < 0$. We claim that $v(t) \le 0$, for $t \in (0,1)$. If not, then $v(t)$ has a positive maximum at some $t_0 \in (0,1)$. Thus,

$$v(t_0) > 0, \ v'(t_0) = 0, \ v''(t_0) \le 0.$$

However, using the definition of β and (5.2.3), we obtain

$$\begin{aligned} v''(t_0) &= x_n''(t_0) - \beta''(t_0) \ge f(t_0,\beta(t_0),\beta'(t_0)) - \Big[\lambda\big(f(t_0,\beta(t_0),\beta'(t_0)) \\ &\quad + \frac{\beta(t_0) - x_n(t_0)}{1 + |\beta(t_0) - x_n(t_0)|}\big) + (1-\lambda)M_1\rho(t)\Big] \\ &= (1-\lambda)[f(t_0,\beta(t_0),\beta'(t_0)) - M_1\rho(t)] + \lambda\frac{x_n(t_0) - \beta(t_0)}{1 + |\beta(t_0) - x_n(t_0)|} \\ &\ge \lambda\frac{x_n(t_0) - \beta(t_0)}{1 + |\beta(t_0) - x_n(t_0)|} > 0, \end{aligned}$$

a contradiction.

Similarly, we can show that $x_n(t) \geq \alpha(t) - \frac{1}{n}$ for every $t \in J$.

We obtain a sequence $\{x_n\}$ of solutions of problems (5.2.4) satisfying

$$\alpha(t) - \frac{1}{n} \leq x_n(t) \leq \beta(t) + \frac{1}{n},\ t \in J. \tag{5.2.6}$$

The sequence is bounded and equicontinuous in $C^2(J)$ and so, by the Arzelà-Ascoli theorem it is possible to choose a subsequence converging to a function $x \in C^2(J)$. Thus,

$$\alpha(t) \leq x(t) \leq \beta(t),\ t \in J \tag{5.2.7}$$

and hence x is a solution of (5.2.1). Moreover, as in the previous chapters, we can show that $|x'(t)| \leq C$ on J. □

Theorem 5.2.3. *Assume that $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of the boundary value problem* (5.2.1) *respectively. If $f(t, x, x') \in C(J \times \mathbb{R} \times \mathbb{R})$ is decreasing in x for each $(t, x') \in J \times \mathbb{R}$, then $\alpha(t) \leq \beta(t)$ on J. Hence under these conditions solutions are unique.*

Proof. Define $w(t) = \alpha(t) - \beta(t)$ on J. Then

$$w \in C^2(J) \text{ and } w(0) \leq 0.$$

We can show that $w(t) \leq 0$ on $[0, 1)$. Now, using the boundary conditions at $t = 1$ and the nondecreasing property of the function h, we obtain

$$w(1) = \alpha(1) - \beta(1) \leq \int_0^1 \big(h(\alpha(s)) - h(\beta(s))\big)ds \leq 0.$$

Thus $w(t) \leq 0$, $t \in J$. □

We need the following result in the sequel.

Lemma 5.2.4. *Consider the linear homogeneous problem*

$$\begin{aligned} &x''(t) + \lambda x'(t) = 0,\ t \in J, \\ &x(0) = 0,\ x(1) = 0, \end{aligned}$$

If $\lambda \neq 0$, this problem has only a trivial solution. Consequently, the corresponding nonhomogeneous problem

$$\begin{aligned} &x''(t) + \lambda x'(t) = \sigma(t),\ t \in J, \\ &x(0) = a,\ x(1) = \int_0^1 h(x(s))ds, \end{aligned} \tag{5.2.8}$$

has a unique solution given by

$$\begin{aligned} x(t) = \frac{1}{(e^{-\lambda} - 1)}\Big[a(e^{-\lambda} - e^{-\lambda t}) + (e^{-\lambda t} - 1)\int_0^1 h(x(s))ds\Big] \\ + \int_0^1 k(t, s)\sigma(s)ds, \end{aligned} \tag{5.2.9}$$

where,

$$k(t,s) = \frac{1}{\lambda(1-e^{-\lambda})e^{-\lambda s}} \begin{cases} (e^{-\lambda t}-1)(e^{-\lambda s}-e^{-\lambda}), & 0 \le t < s \le 1 \\ (e^{-\lambda s}-1)(e^{-\lambda t}-e^{-\lambda}), & 0 \le s < t \le 1 \end{cases}$$

is the Green's function of the problem. We note that $k(t,s) < 0$ *on* $(0,1)\times(0,1)$ *for any* $\lambda(\neq 0) \in \mathbb{R}$.

5.2.2 Quasilinearization technique

Now we study approximation of solutions by the method of quasilinearization and show that under suitable conditions on f and h, there is a monotone sequence of solutions of linear problems converges quadratically to a solution of the nonlinear original problem. Assume that

(D₁) $\alpha,\ \beta \in C^2(J)$ are lower and upper solutions of (5.1.1) such that $\alpha(t) \le \beta(t)$ on J.

(D₂) $f \in C^2(J\times\mathbb{R}\times\mathbb{R})$ is such that $f_x(t,x,x') \le 0$, $f_{x'}(t,x,x')$ is non-increasing in x' and $H(f) \ge 0$ on $J\times[\min\alpha(t), \max\beta(t)]\times\mathbb{R}$, where

$$H(f) = (x-y)^2 f_{xx}(t,z,z') + 2(x-y)(x'-y')f_{xx'}(t,z,z') + (x'-y')^2 f_{x'x'}(t,z,z')$$

is the quadratic form of f, $y \le z \le x$ and z' lies between x' and y'.

(D₃) $h \in C^2(\mathbb{R})$ is such that $h''(x) \ge 0$ and $0 \le h'(x) < 1$ on $[\min\alpha(t), \max\beta(t)]$.

Theorem 5.2.5. *Under assumptions* $(D_1)-(D_3)$*, there exists a monotone sequence* $\{w_n\}$ *of solutions converging uniformly and quadratically to the unique solution of the problem.*

Proof. Let $\Omega = \{(t,x,x') \in J\times\mathbb{R}^2 : (t,x,x') \in J\times[\min\alpha(t), \max\beta(t)]\times[-C,C]\}$, where C is the same as defined in Theorem 5.2.3. Let

$$M = \max_{(t,x,x')\in\Omega}\{|f_{xx}(t,x,x')|, |f_{xx'}(t,x,x')|, |f_{x'x'}(t,x,x')|\},$$

then

$$|H(f)| \le M\|x-y\|_1^2, \tag{5.2.10}$$

where $\|z\|_1 = \|z\| + \|z'\|$ is the C^1 norm. The conditions $H(f) \ge 0$, and $h'' \ge 0$ imply that

$$f(t,x,x') \ge f(t,y,q(y')) + f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(x'-q(y')), \tag{5.2.11}$$

$$h(x) \ge h(y) + h'(y)(x-y), \tag{5.2.12}$$

for $x,\ y \in [\min\alpha(t), \max\beta(t)]$, $x',\ y' \in \mathbb{R}$, where $q(s) = \max\{-C, \min\{s, C\}\}$. Let

$$S = \{(t,x,x') : t \in J,\ x \in [\min\alpha(t), \max\beta(t)],\ x' \in \mathbb{R}\} \subset J\times\mathbb{R}^2$$

and define the function

$$F(t,x,x';y,y') = f(t,y,q(y')) + f_x(t,y,q(y'))(x-y) + f_{x'}(t,y,q(y'))(x'-q(y')), \quad (5.2.13)$$

where (t,x,x'), $(t,y,y') \in J \times \mathbb{R}^2$. We note that F satisfies

$$F_x(t,x,y;x',y') = f_x(t,y,q(y')) \le 0$$

and

$$f(t,x,x') \ge F(t,x,x';y,y'), \quad (5.2.14)$$

for (t,x,x'), $(t,y,y') \in S$. Also, define

$$H(x,y) = h(y) + h'(y)(x-y), \quad (5.2.15)$$

for x, $y \in \mathbb{R}$. Then H is continuous and bounded on $[\min\alpha, \max\beta]$ and satisfies (5.1.16) of the previous section. Let

$$M_1 \ge \max\{|f(t,x,x')|, \ |f_x(t,x,x')|, \ |f_{x'}(t,x,x')| : (t,x,x') \in \Omega\},$$

then for every (t,x,x'), $(t,y,y') \in S$, we have

$$|h(t,x,x';y,y')| \le M_1 + M_1 r + M_1(|x'-q(y')|) \le M_1(1+r+C) + M_1|x'| = \omega|x'|,$$

where $\omega(s) = M_1(1+r+C) + M_1 s$ for $s \ge 0$. It is easy to see that ω is a Nagumo function. Hence there exists a constant C_1 such that any solution x of

$$-x''(t) = F(t,x,x';y,y'), \ t \in J$$
$$x(0) = a, \ x(1) = \int_0^1 H(x(s),y(s))ds,$$

with $\alpha(t) \le x(t) \le \beta(t)$, $t \in J$ and $(t,y,y') \in S$, satisfies $|x'(t)| \le C_1$, $t \in J$. Now we set $\alpha = w_0$ and consider the linear problem

$$\begin{aligned} -x''(t) &= F(t,x,x';w_0,w_0'), \ t \in J \\ x(0) &= a, \ x(1) = \int_0^1 H(x(s),w_0(s))ds. \end{aligned} \quad (5.2.16)$$

Using (D_1), (5.1.16) and (5.2.14), we have

$$-w_0''(t) \le f(t,w_0(t),w_0'(t)) = F(t,w_0(t),w_0'(t);w_0(t),w_0'(t)), \ t \in J$$
$$w_0(0) \le a, \ w_0(1) \le \int_0^1 h(w_0(s))ds = \int_0^1 H(w_0(s),w_0(s))ds,$$

and

$$-\beta''(t) \ge f(t,\beta(t),\beta'(t)) \ge F(t,\beta(t),\beta'(t);w_0(t),w_0'(t)), \ t \in J$$
$$\beta(0) \ge a, \ \beta(1) \ge \int_0^1 h(\beta(s))ds \ge \int_0^1 H(\beta(s),w_0(s))ds,$$

which imply that w_0 and β are lower and upper solution of (5.2.16) respectively. Hence, by Theorems 5.2.2, 5.2.3, there exists a unique solution w_1 of (5.2.16) such that $w_0 \le w_1 \le \beta$ and $|w_1'| \le C_1$ on J.

Using (5.1.16), (5.2.14) and the fact that w_1 is a solution of (5.2.16), we obtain

$$\begin{aligned} -w_1''(t) &= F(t, w_1, w_1'; w_0, w_0') \le f(t, w_1(t), w_1'(t)),\ t \in J \\ w_1(0) = a,\ w_1(1) &= \int_0^1 H(w_1(s), w_0(s))ds \le \int_0^1 h(w_1(s))ds \end{aligned} \tag{5.2.17}$$

which implies that w_1 is a lower solution of (5.2.1).

Now, we consider the BVP

$$\begin{aligned} -x''(t) &= F(t, x, x'; w_1, w_1'),\ t \in J \\ x(0) = a,\ x(1) &= \int_0^1 H(x(s), w_1(s))ds. \end{aligned} \tag{5.2.18}$$

By (5.2.13) and the mean value theorem, we obtain

$$\begin{aligned} F(t, w_1, w_1'; w_1, w_1') - f(t, w_1, w_1') &= f(t, w_1, q(w_1')) + f_{x'}(t, w_1, q(w_1'))(w_1' - q(w_1')) \\ &\quad - f(t, w_1, w_1') \\ &= (f_{x'}(t, w_1, q(w_1')) - f_{x'}(t, w_1, d))(w_1' - q(w_1')), \end{aligned}$$

where d is between $w_1'(t)$ and $q(w_1'(t))$

If for $t \in J$, $w_1'(t) > C_1$, then using the non-increasing property of $f_{x'}(t, x, x')$ in x', we obtain

$$(f_{x'}(t, w_1(t), q(w_1'(t))) - f_{x'}(t, w_1(t), d))(w_1'(t) - q(w_1'(t))) \ge 0.$$

If for $t \in J$, $|w_1'(t)| \le C_1$, then

$$(f_{x'}(t, w_1(t), q(w_1'(t))) - f_{x'}(t, w_1(t), d))(w_1'(t) - q(w_1'(t))) = 0.$$

If for $t \in J$, $w_1'(t) < -C_1$, then again by the non-increasing property of $f_{x'}(t, x, x')$ in x', we obtain

$$(f_{x'}(t, w_1(t), q(w_1'(t))) - f_{x'}(t, w_1(t), d))(w_1'(t) - q(w_1'(t))) \ge 0.$$

Thus,

$$F(t, w_1, w_1'; w_1, w_1') \ge f(t, w_1, w_1'),\ t \in J. \tag{5.2.19}$$

Now, by (5.2.19), (5.1.16) and (5.2.17), we obtain

$$\begin{aligned} F(t, w_1, w_1'; w_1, w_1') &\ge f(t, w_1, w_1') \ge -w_1''(t),\ t \in J \\ w_1(0) = a,\ w_1(1) &= \int_0^1 H(w_1(s), w_0(s))ds \le \int_0^1 h(w_1(s))ds, \end{aligned}$$

which implies that w_1 is a lower solution of (5.2.18). Now, by (D_1), (5.1.16) and (5.2.14) we can show that β is an upper solution of (5.2.18). Hence by Theorems 5.2.2, 5.2.3, there exists a unique solution w_2 of (5.2.18) such that $w_1 \leq w_2 \leq \beta$ and $|w_2'| \leq C_1$ on J. Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0 \leq w_1 \leq w_2 \leq w_3 \leq \dots \leq w_{n-1} \leq w_n \leq \beta \text{ on } J,$$

that is

$$\alpha(t) \leq w_n(t) \leq \beta(t) \text{ and } |w_n'(t)| < C_1,\ n \in \mathbb{N},\ t \in J, \tag{5.2.20}$$

where w_n is a solution of the linear problem

$$\begin{aligned} -x''(t) &= F(t, x, x'; w_{n-1}, w_{n-1}'),\ t \in J \\ x(0) = a,\ x(1) &= \int_0^1 H(x(s), w_{n-1}(s))ds. \end{aligned}$$

This is equivalent to

$$\begin{aligned} w_n(t) = a(1-t) + t\Big(&\int_0^1 H(w_n(s), w_{n-1}(s))ds\Big) \\ &+ \int_0^1 G(t,s) F(s, w_n, w_n'; w_{n-1}, w_{n-1}')ds, \end{aligned} \tag{5.2.21}$$

where,

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t < s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1 \end{cases}$$

is the Green's function. We note that $G(t,s) > 0$ on $(0,1) \times (0,1)$ and

$$F(t, w_n, w_n'; w_n, w_n') \geq f(t, w_n, w_n'),\ t \in J. \tag{5.2.22}$$

As $F(t, w_n, w_n'; w_{n-1}, w_{n-1}')$ is continuous and bounded, there exists $L_1 \geq 0$ such that $|F(t, w_n, w_n'; w_{n-1}, w_{n-1}')| \leq L_1$ for $t \in J$. Also, for $t,\ s \in J (s \leq t)$, we have

$$|w_n'(t) - w_n'(s)| \leq \int_0^t |F(u, w_n, w_n'; w_{n-1}, w_{n-1}')| du \leq L_1 |t-s|. \tag{5.2.23}$$

The expressions (5.2.20), (5.2.21) and (5.2.23) imply that the sequences $\{w_n^{(j)}(t)\}, (j = 0, 1)$ are uniformly bounded and equicontinuous on J. The Arzelà-Ascoli theorem guarantees the existence of subsequences and a function $x \in C^2(J)$ such that $w_n^{(j)} (j = 0, 1)$ converging uniformly to $x^{(j)}$ on J as $n \to \infty$. Thus, by (5.2.14) and (5.2.22), it follows that $F(t, w_n, w_n'; w_{n-1}, w_{n-1}') \to f(t, x, x')$ as $n \to \infty$. Passing to the limit in (5.2.21), we obtain

$$x(t) = a(1-t) + t\Big(\int_0^1 h(x(s))ds\Big) + \int_0^1 G(t,s) f(s, x, x')ds,$$

that is, $x(t)$ is a solution of (5.2.1).

Now we show that the convergence of the sequence of solutions is quadratic. For that, we set $v_n(t) = x(t) - w_n(t)$, $n \in \mathbb{N}$ and $t \in J$. Then, we note that

$$v_n \in C^2(J),\ v_n(0) = 0,\ v_n(t) \geq 0,\ t \in J,$$

which implies that $v_n'(0) \geq 0$. Further, using the mean value theorem and the definition of $H(w_n(t), w_{n-1}(t))$, we obtain

$$\begin{aligned} v_n(1) &= \int_0^1 h(x(s))ds - \int_0^1 H(w_n(s), w_{n-1}(s))ds \\ &= \int_0^1 [h'(w_{n-1}(s))v_n(s) + \frac{1}{2}h''(\xi(s))v_{n-1}^2(s)]ds \\ &\leq \int_0^1 h'(w_{n-1}(s))v_n(s)ds + C_2\|v_{n-1}\|_1^2, \end{aligned} \tag{5.2.24}$$

where $C_2 = \max\left\{\frac{1}{2}h''(x) : x \in [\min w_0(t), \max \beta(t)]\right\}$. Now, using Taylor's theorem on $f(t, x, x')$ about $(t, w_n, q(w_n'))$, the definition (5.2.13) of F, we have

$$\begin{aligned} -v_{n+1}''(t) &= -x''(t) + w_{n+1}'' = f(t, x, x') - F(t, w_{n+1}, w_{n+1}'; w_n, w_n') \\ &= [f(t, w_n, q(w_n')) + f_x(t, w_n, q(w_n'))(x - w_n) + f_{x'}(t, w_n, q(w_n'))(x' - q(w_n') \\ &+ \frac{1}{2}H(f)] - [f(t, w_n, q(w_n')) + f_x(t, w_n, q(w_n'))(w_{n+1} - w_n) \\ &+ f_{x'}(t, w_n, q(w_n'))(w_{n+1}' - q(w_n'))] \\ &= f_x(t, w_n, q(w_n'))v_{n+1} + f_{x'}(t, w_n, q(w_n'))v_{n+1}' + \frac{1}{2}H(f),\ t \in J \end{aligned} \tag{5.2.25}$$

where,

$$H(f) = v_n^2 f_{xx}(t, \xi_1, \xi_2) + 2v_n(x' - q(w_n'))f_{xx'}(t, \xi_1, \xi_2) + (x' - q(w_n'))^2 f_{x'x'}(t, \xi_1, \xi_2),$$

$w_n(t) \leq \xi_1 \leq x(t)$ and ξ_2 lies between $w_n'(t)$ and $q(x'(t))$. In view of (5.2.10),

$$|H(f)| \leq M(|v_n| + |x' - q(w_n')|)^2 \leq M(|v_n| + |v_n'|)^2 \leq M\|v_n\|_1^2.$$

Using this and the assumption $f_x \leq 0$, we obtain

$$-v_{n+1}''(t) \leq f_{x'}(t, w_n, q(w_n'))v_{n+1}' + \frac{M}{2}\|v_n\|_1^2. \tag{5.2.26}$$

Thus, it follows that $v_{n+1}(t) \leq r(t)$ on J, where $r(t)$ is a solution of the linear boundary value problem

$$\begin{aligned} -r''(t) &= f_{x'}(t, w_n, q(w_n'))r'(t) + \frac{M}{2}\|v_n\|_1^2,\ t \in J \\ r(0) &= 0,\ r(1) = \int_0^1 h'(w_n(s))v_{n+1}(s)ds + C_2\|v_n\|_1^2. \end{aligned} \tag{5.2.27}$$

Since $(t, w_n, q(w_n')) \in \Omega$ and $f_{x'}$ is continuous, there exist constants $L, l > 0$, such that $-l \leq f_{x'}(t, w_n, q(w_n')) \leq L$. Define $\rho(t) = L - f_{x'}(t, w_n, q(w_n'))$, then, $0 \leq \rho(t) \leq L+l$ on J. We rewrite the BVP (5.2.27) in the form

$$r''(t) + Lr'(t) = \rho(t)r'(t) - \frac{M}{2}\|v_n\|_1^2,\ t \in J$$
$$r(0) = 0,\ r(1) = \int_0^1 h'(w_n(s))v_{n+1}ds + C_2\|v_n\|_1^2,$$

which is equivalent to the integral equation

$$\begin{aligned} r(t) = \frac{1-e^{-Lt}}{1-e^{-L}}\Big(\int_0^1 h'(w_n(s))v_{n+1}(s)ds + C_2\|v_n\|_1^2\Big) \\ + \int_0^1 k(t,s)\Big[\rho(s)r'(s) - \frac{M}{2}\|v_n\|_1^2\Big]ds. \end{aligned} \tag{5.2.28}$$

The integrating factor $\mu(t) = e^{\int_0^t f_{x'}(s,w_n,q(w_n'))ds}$ for the differential equation (5.2.27) satisfies

$$e^{-lt} \leq \mu(t) \leq e^{Lt},\ t \in J. \tag{5.2.29}$$

Using (5.2.29) in (5.2.27), we obtain

$$\big(r'(t)\mu(t)\big)' \geq -\frac{M}{2}e^{Lt}\|v_n\|_1^2,\ t \in J.$$

Integrating from 0 to t, using (5.2.29) and the boundary condition $r'(0) \geq 0$, we obtain

$$r'(t) \geq -\frac{Me^{lt}(e^{Lt}-1)}{2L}\|v_n\|_1^2,\ t \in J$$

hence, it follows that

$$r'(t)\rho(t)k(t,s) \leq \frac{Me^{lt}(e^{Lt}-1)(L+l)}{2L}|k(t,s)|\|v_n\|_1^2,\ t \in J. \tag{5.2.30}$$

Substituting (5.2.30) in (5.2.28), we obtain

$$\begin{aligned} v_{n+1}(t) \leq r(t) \leq \frac{1-e^{-Lt}}{1-e^{-L}}\int_0^1 h'(w_n(s))v_{n+1}(s)ds + \Big[C_2\frac{1-e^{-Lt}}{1-e^{-L}} + \\ \frac{M}{2}\int_0^1 |k(t,s)|\big(\frac{L+l}{L}e^{ls}(e^{Ls}-1)+1\big)ds\Big]\|v_n\|_1^2,\ t \in J. \end{aligned} \tag{5.2.31}$$

Taking the maximum over J, using (D_3), we get

$$\|v_{n+1}\| \leq \frac{\delta}{1-\sigma}\|v_n\|_1^2, \tag{5.2.32}$$

where

$$0 \leq \sigma = \max_J\Big\{\frac{1-e^{-Lt}}{1-e^{-L}}\int_0^1 h'(w_n(s))ds\Big\} < 1,$$

and

$$0 \leq \delta = \max_J\Big\{C_2\frac{1-e^{-Lt}}{1-e^{-L}} + \frac{M}{2}\int_0^1 |k(t,s)|\big(\frac{L+l}{L}e^{ls}(e^{Ls}-1)+1\big)ds\Big\}.$$

Now, using (5.2.29) in (5.2.26), we have

$$\left(v'_{n+1}(t)\mu(t)\right)' \geq -\frac{M}{2}e^{Lt}\|v_n\|_1^2,\ t \in J. \tag{5.2.33}$$

Integrating from 0 to t, using (5.2.29) and the boundary condition $v'_{n+1}(0) \geq 0$, we obtain

$$v'_{n+1}(t) \geq -\frac{e^{M}lt(e^{Lt}-1)}{2L}\|v_n\|_1^2,\ t \in J. \tag{5.2.34}$$

Since $v_{n+1} \in C^2(J)$, by the mean value theorem, there exists $\eta \in (0,1)$ such that

$$v'_{n+1}(\eta) = v_{n+1}(1) - v_{n+1}(0) = v_{n+1}(1).$$

It follows from (5.2.32) and (5.2.24) that

$$v'_{n+1}(\eta) \leq \left(\frac{\delta}{1-\sigma}\int_0^1 h'(w_n(s))ds + C_2\right)\|v_n\|_1^2. \tag{5.2.35}$$

As $0 \geq f_x(t, w_n, q(w'_n)) \geq -M_1$, $t \in J$ and $H(f) \geq 0$, it follows from (5.2.25), that

$$-v''_{n+1}(t) \geq -M_1 v_{n+1} + f_{x'}(t, w_n, q(w'_n))v'_{n+1},\ t \in J$$

which implies that

$$\left(v'_{n+1}(t)\mu(t)\right)' \leq M_1 v_{n+1}(t)\mu(t),\ t \in J.$$

Thus, using (5.2.29) and (5.2.32), we obtain

$$\left(v'_{n+1}(t)\mu(t) - \frac{M_1\delta}{L(1-\sigma)}e^{Lt}\|v_n\|_1^2\right)' \leq 0,\ t \in J.$$

Then the function

$$\psi(t) = v'_{n+1}(t)\mu(t) - \frac{M_1\delta}{L(1-\sigma)}e^{Lt}\|v_n\|_1^2,\ t \in J$$

is non-increasing, which implies that $\psi(1) \leq \psi(\eta)$, that is

$$v'_{n+1}(1)\mu(1) - \left(\frac{M_1\delta}{L(1-\sigma)}e^{L}\|v_n\|_1^2\right) \leq v'_{n+1}(\eta)\mu(\eta) - \left(\frac{M_1\delta}{L(1-\sigma)}e^{L\eta}\|v_n\|_1^2\right).$$

Hence,

$$\begin{aligned} v'_{n+1}(1)\mu(1) &\leq v_{n+1}(1)\mu(\eta) + \left(\frac{M_1\delta}{L(1-\sigma)}(e^L - e^{L\eta})\|v_n\|_1^2\right) \\ &\leq \left[\mu(\eta)\left(\frac{\delta}{1-\sigma}\int_0^1 h'(w_n(s))ds + C_2\right) + \frac{M_1\delta}{L(1-\sigma)}(e^L - e^{L\eta})\right]\|v_n\|_1^2 \\ &\leq A\|v_n\|_1^2, \end{aligned} \tag{5.2.36}$$

where

$$A = \max\left\{\mu(\eta)\left(\frac{\delta}{1-\sigma}\int_0^1 h'(w_n(s))ds + C_2\right) + \frac{M_1\delta}{L(1-\sigma)}(e^L - e^{L\eta})\right\}.$$

Now, integrating (5.2.33) from t to 1 and using (5.2.29), we get

$$v'_{n+1}(t)\mu(t) \leq v'_{n+1}(1)\mu(1) + \frac{2N}{L}(e^L - e^{Lt})\|v_n\|_1^2,\ t \in J$$

which in view of (5.2.36) and (5.2.29) implies that

$$v'_{n+1}(t) \leq e^{lt}\big(A + \frac{2N}{L}(e^L - e^{Lt})\big)\|v_n\|_1^2,\ t \in J. \tag{5.2.37}$$

From (5.2.34) and (5.2.37), we have

$$\|v'_{n+1}\| \leq R_1\|v_n\|_1^2, \tag{5.2.38}$$

where $R_1 = \max\big\{e^{lt}(A + \frac{2N}{L}(e^L - e^{Lt})), \frac{2Ne^{lt}(e^{Lt}-1)}{L} : t \in J\big\}$. From (5.2.32) and (5.2.38), it follows that

$$\|v_{n+1}\|_1 = \|v_{n+1}\| + \|v'_{n+1}\| \leq (\frac{\delta}{1-\sigma} + R_1)\|v_n\|_1^2 = D\|v_n\|_1^2,$$

where, $D = \frac{\delta}{1-\sigma} + R_1$, which gives the desired quadratic convergence in the C^1 norm. □

Chapter 6

Existence of multiple solutions for three point problems

In this chapter we study existence of multiple solutions for some three point boundary value problems. We study two types of problems, one with a parameter r and the other without a parameter. For the first type, we use the method of upper and lower solutions and degree arguments to show that, for a certain range of values of the parameter, there are no solution while for other values, there are at least two solutions of the problem. For the second case we study existence of at least three solutions in the presence of two lower solutions and two upper solutions. This chapter is a joint work with Prof. J. R. L. Webb. The results of section 1 is accepted for publication [54] while results of section 2, 3, are submitted for publication [55, 56].

6.1 Existence of at least two solutions of second order nonlinear three point boundary value problems

In this section, we study existence of at least two solutions of a second order nonlinear three point boundary value problem depending on a parameter of the type

$$\begin{aligned} -x''(t) = f(t, x, x') - r\phi(t),\ t \in J = (0, 1) \\ x(0) = 0,\ x(1) = \delta x(\eta),\ 0 < \delta\eta < 1,\ 0 < \eta < 1, \end{aligned} \tag{6.1.1}$$

where $f : J \times \mathbb{R}^2 \to \mathbb{R}$ and $\phi : J \to \mathbb{R}$ are continuous and r is a parameter. We show that for a certain range of the parameter there are no solutions while for some other range of values there are at least two solutions. Our method is based on the use of upper and lower solutions and on topological degree arguments. Such BVPs have been extensively studied in recent years by many authors by applying Leray-Schauder degree theory, for example, [28–30]. The study of (multiple) positive solutions has been done using fixed point theory in cones, mainly when f does not depend on x', see, for example [70, 92].

Existence of at least two solutions for other boundary conditions have been studied in many papers, for example [17, 23, 27, 80, 82]. In [82], existence of at least two solutions for (6.1.1) with $\phi(t) = 1$ subject to the four-point boundary conditions is studied in the presence of constant lower and upper solutions, and it is assumed that the set of all solutions is bounded above. In contrast, here we study the problem (6.1.1) not only with different boundary conditions (three-point boundary conditions) but also assume the existence of lower and upper solutions which are not necessarily constants. Moreover, we do not require the set of solutions to be bounded. In [27], ϕ is taken to be the normalized positive eigenfunction associated with the first eigenvalue $\lambda = 1$ of the linear problem

$$x'' + \lambda x = 0,\ x(0) = 0,\ x(\pi) = 0,$$

existence of at least two solutions for (6.1.1) with the Dirichlet boundary conditions is studied. It is assumed that the set of all solutions is bounded above and the nonlinearity $f(t, x, x')$ is bounded for bounded x. In this paper, we allow ϕ to be any positive continuous bounded function and relax the condition on f.

By different methods, existence of at least three solutions of the two point boundary value problem $x'' + f(t, x, x') = 0$, $x(0) = 0$, $x(1) = 0$, in the presence of two lower and two upper solutions is studied in [35].

The existence of at least one positive solution for some three-point boundary value problems, in the case that the nonlinearity f is a non-negative, is given in [32], by using a fixed point theorem in a cone under different hypotheses.

The purpose of this section is to study how the number of solutions changes as the parameter r varies and to show the existence of no solution and the existence of at least two solutions for certain range of values of r. We give a class of realistic examples to show that our hypotheses can easily be satisfied. Some of the conditions in [32] are more restrictive, in particular $(H_3) - (H_5)$ of [32] can never be satisfied in the type of example we give.

6.1.1 Existence results

Consider the problem

$$\begin{aligned} &-x''(t) = f(t, x, x') - r\phi(t),\ t \in J = [0, 1],\\ &x(0) = 0, \quad x(1) = \delta x(\eta), \quad 0 < \delta\eta < 1,\ \eta < 1. \end{aligned} \tag{6.1.2}$$

We seek a solution x via the integral equation

$$x(t) = \int_0^1 G(t, s)\big[f(s, x(s), x'(s)) - r\phi(s)\big]ds, \tag{6.1.3}$$

where, $G : J \times J \to \mathbb{R}$ defined by

$$G(t, s) = \begin{cases} G_1(t, s), & 0 < \eta \le s \\ G_2(t, s), & 0 < s \le \eta, \end{cases}$$

$$G_1(t,s) = \frac{1}{1-\delta\eta}\begin{cases}(1-s)t, & 0 \le t \le \eta \le s \le 1\\ (1-\delta\eta)s + (\delta\eta - s)t, & \eta \le s \le t \le 1,\end{cases}$$

$$G_2(t,s) = \frac{1}{1-\delta\eta}\begin{cases}[(1-s)+\delta(s-\eta)]t, & 0 \le t \le s \le \eta\\ [(1-t)+\delta(t-\eta)]s, & 0 \le s \le \eta \le t \le 1.\end{cases}$$

We recall the concept of upper and lower solutions for the BVP (6.1.2).

Definition 6.1.1. Let $\alpha \in C^2(J)$. We say that α is a strict lower solution of the BVP (6.1.2), if

$$-\alpha''(t) < f(t,\alpha(t),\alpha'(t)) - r\phi(t), \quad t \in J$$
$$\alpha(0) \le 0,\ \alpha(1) \le \delta\alpha(\eta).$$

A strict upper solution $\beta \in C^2(J)$ of the BVP (6.1.2) is defined similarly by reversing the inequalities.

Let $R(\alpha,\beta) = \{(t,x,x') \in J \times \mathbb{R}^2 : \alpha(t) \le x \le \beta(t)\}$. Now we state and prove the following theorem which establish the existence of solutions.

Theorem 6.1.2. *Assume that*

($\mathbf{a_1}$) *$\bar{r} \in \mathbb{R}$ and $\alpha,\beta \in C^2(J)$ be strict lower and upper solutions of* (6.1.2) *with $r = \bar{r}$, such that $\alpha < \beta$ on* $(0,1]$.

($\mathbf{a_2}$) *$f : J \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, satisfies a Nagumo condition on $R(\alpha,\beta)$ and is such that*

$$f(t,\alpha(t),y) \ge f(t,\alpha(t),\alpha'(t)) \text{ for } y \le \alpha'(t),\ t \in J$$
$$f(t,\beta(t),y) \le f(t,\beta(t),\beta'(t)) \text{ for } y \ge \beta'(t),\ t \in J.$$

($\mathbf{a_3}$) *$\phi : J \to \mathbb{R}$ is continuous, positive and is bounded on J.*

Then there exists a solution $x(t)$ of (6.1.2) *(with $r = \bar{r}$) such that $\alpha(t) < x(t) < \beta(t)$, $t \in J$.*

Proof. Let $\max\{\delta|\alpha(\eta)|, \delta|\beta(\eta)|\} = \lambda$. Since f satisfies a Nagumo condition on $R(\alpha,\beta)$, it follows that for every $x \in [\min\alpha(t), \max\beta(t)]$, there exist a Nagumo function $\omega : [0,\infty) \to (0,\infty)$ and a constant $N(>\lambda)$ depending on α, β and ω such that

$$|f(t,x,x')| \le \omega(|x'|) \text{ on } R(\alpha,\beta) \text{ and}$$

$$\int_\lambda^N \frac{sds}{\omega(s)+M} > \max_{t\in J}\beta(t) - \min_{t\in J}\alpha(t), \tag{6.1.4}$$

where $M = \bar{r}\max\{\phi(t) : t \in J\}$. Let $C > \max\{N, \|\alpha'\|, \|\beta'\|\}$ and define

$$q(y) = \max\{-C, \min\{y, C\}\},$$

a retraction onto $[-C, C]$. Consider the modified problem

$$\begin{aligned} -x'' &= F(t,x,x') - \bar{r}\phi(t),\ t \in J, \\ x(0) &= 0, \quad x(1) = \delta x(\eta),\ 0 < \delta\eta < 1,\ \eta \in (0,1), \end{aligned} \tag{6.1.5}$$

where,

$$F(t,x,x') = \begin{cases} f(t,\beta(t),\beta'(t)), & \text{if } x \ge \beta(t)+\epsilon, \\ f(t,\beta(t),q(x'(t))) + [f(t,\beta(t),\beta'(t)) & \\ \quad - f(t,\beta(t),q(x'(t)))]\frac{x-\beta(t)}{\epsilon}, & \text{if } \beta(t) \le x < \beta(t)+\epsilon, \\ f(t,x,q(x'(t))), & \text{if } \alpha(t) < x < \beta(t), \\ f(t,\alpha(t),q(x'(t))) + [f(t,\alpha(t),\alpha'(t)) & \\ \quad - f(t,\alpha(t),q(x'(t)))]\frac{\alpha(t)-x}{\epsilon}, & \text{if } \alpha(t)-\epsilon < x \le \alpha(t), \\ f(t,\alpha(t),\alpha'(t)), & \text{if } x \le \alpha(t)-\epsilon, \end{cases}$$

and $\epsilon > 0$ is a fixed number. The modified problem (6.1.5) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s)[F(s,x(s),x'(s)) - \bar{r}\phi(s)]ds.$$

Since F, ϕ and G are continuous and bounded, this integral equation has a fixed point by Schauder's fixed point theorem. Hence the BVP (6.1.5) has a solution. Moreover,

$$\begin{aligned} -\alpha''(t) < f(t,\alpha(t),\alpha'(t)) - \bar{r}\phi(t) &= F(t,\alpha(t),\alpha'(t)) - \bar{r}\phi(t),\ t \in J \\ \alpha(0) \le 0,\ \alpha(1) \le \delta\alpha(\eta), \end{aligned}$$

$$\begin{aligned} -\beta''(t) > f(t,\beta(t),\beta'(t)) - \bar{r}\phi(t) &= F(t,\beta(t),\beta'(t)) - \bar{r}\phi(t),\ t \in J \\ \beta(0) \ge 0,\ \beta(1) \ge \delta\beta(\eta), \end{aligned}$$

which imply that α and β are strict lower and upper solutions of (6.1.5).

We show that any solution x of (6.1.5) satisfies $\alpha(t) < x(t) < \beta(t)$, $t \in (0,1]$. For this, set $v(t) = \alpha(t) - x(t)$, $t \in J$, where x is a solution of (6.1.5). Then, $v \in C^2(J)$ and the boundary conditions imply that

$$v(0) \le 0,\ v(1) \le \delta v(\eta). \tag{6.1.6}$$

Let $\max\{v(t) : t \in J\} = v(t_0) \ge 0$. If $t_0 \in (0,1)$, then $v'(t_0) = 0$ and $v''(t_0) \le 0$. However,

$$v''(t_0) = \alpha''(t_0) - x''(t_0) > -F(t_0,\alpha(t_0),\alpha'(t_0)) + \bar{r}\phi(t) + F(t_0,\alpha(t_0),\alpha'(t_0)) - \bar{r}\phi(t) = 0,$$

a contradiction. Thus $v(t)$ has no nonnegative local maximum.

If $t_0 = 1$, then $v(1) \ge 0$ and $v'(1) \ge 0$. If $v(1) = 0$, then the boundary condition $v(1) \le \delta v(\eta)$ imply that $v(\eta) \ge 0$ and hence $v(1)$ cannot be the maximum of $v(t)$. Thus

$v(1) > 0$. Since $v(0) \leq 0$, it follows that there exists $t_1 \in [0, \eta) \subset J$ such that $v(t_1) = 0$ and $v'(t_1) \geq 0$. We claim that

$$v'(t) \geq 0 \text{ on } [t_1, 1].$$

If not, then there exists a $t_2 \in (t_1, 1)$ such that $v'(t_2) < 0$. But then $v(t)$ must have a positive local maximum at some $t \in (t_1, t_2)$, which leads to a contradiction. Hence $v'(t) \geq 0$ on $[t_1, 1]$, which implies that

$$v(t) \geq 0 \text{ on } [t_1, 1].$$

The condition $v'(t) \geq 0$ on $[t_1, 1]$ yields

$$q(x'(t)) \leq x'(t) \leq \alpha'(t),\ t \in [t_1, 1].$$

There exists $t_3 \in (t_1, 1)$ such that $v(t_3) = \epsilon$, and

$$0 \leq v(t) \leq \epsilon,\ t \in [t_1, t_3]$$
$$v(t) \geq \epsilon,\ t \in [t_3, 1].$$

Firstly, we consider the case that $0 \leq v(t) \leq \epsilon$, $t \in [t_1, t_3]$. Using (a_2), we obtain $f(t, \alpha(t), q(x'(t))) \geq f(t, \alpha(t), \alpha'(t))$, $t \in [t_1, t_3]$. Hence

$$\begin{aligned} F(t, x(t), x'(t)) &= f(t, \alpha(t), q(x'(t))) + [f(t, \alpha(t), \alpha'(t)) - f(t, \alpha(t), q(x'(t)))]\frac{v(t)}{\epsilon} \\ &= \frac{v(t)}{\epsilon} f(t, \alpha(t), \alpha'(t)) + (1 - \frac{v(t)}{\epsilon}) f(t, \alpha(t), q(x'(t))) \\ &\geq f(t, \alpha(t), \alpha'(t)),\ t \in [t_1, t_3]. \end{aligned}$$

Also, for every $t \in [t_1, t_3]$, we have

$$\begin{aligned} v''(t) = \alpha''(t) - x''(t) &> -f(t, \alpha(t), \alpha'(t)) + \bar{r}\phi(t) + F(t, x(t), x'(t)) - \bar{r}\phi(t) \\ &\geq -f(t, \alpha(t), \alpha'(t)) + f(t, \alpha(t), \alpha'(t)) = 0. \end{aligned}$$

Secondly, we consider the case that $v(t) \geq \epsilon$, $t \in [t_3, 1]$, then

$$F(t, x, x') = f(t, \alpha(t), \alpha'(t)) \text{ on } [t_3, 1],$$

and for every $t \in [t_3, 1]$, we have

$$v''(t) > -f(t, \alpha(t), \alpha'(t)) + \bar{r}\phi(t) + f(t, \alpha(t), \alpha'(t)) - \bar{r}\phi(t) = 0.$$

This means that the graph of v is convex on $[t_1, 1]$. Hence

$$v(\eta) < v(1)\frac{\eta - t_1}{1 - t_1} < \frac{v(\eta)}{\eta}\frac{(\eta - t_1)}{(1 - t_1)},$$

which leads to $\eta > 1$, a contradiction. Thus the maximum of v occurs at $t_0 = 0$. Since $v(0) \leq 0$ and $v(t)$ has no nonnegative local maximum on $(0, 1]$, it follows that $v(t) < 0$ on $(0, 1]$.

Similarly, we can show that $x(t) < \beta(t)$, $t \in (0,1]$.

Now, it remains to show that every solution x of (6.1.5) with $\alpha(t) < x < \beta(t)$ on J satisfies $|x'(t)| < C$, $t \in J$. For $t \in J$ and $x \in [\min \alpha(t), \max \beta(t)]$,

$$|F(t,x,x') - \bar{r}\phi(t)| = |f(t,x,q(x'(t))) - \bar{r}\phi(t)| \le \omega(|q(x'(t))|) + M = \tilde{\omega}(|x'|) \text{ on } J,$$

where $\tilde{\omega}(s) = \omega(q(s)) + M$ for $s \ge 0$. Since

$$\int_\lambda^\infty \frac{sds}{\tilde{\omega}(s)} = \int_\lambda^C \frac{sds}{\omega(s)+M} + \int_C^\infty \frac{sds}{\omega(C)+M} = \infty,$$

which implies that $F - \bar{r}\phi(t)$ satisfies a Nagumo condition on $R(\alpha,\beta)$. Further, in view of (6.1.4), we have

$$\int_\lambda^C \frac{sds}{\tilde{\omega}(s)} \ge \int_\lambda^N \frac{sds}{\omega(s)+M} > \max_{t\in J} \beta(t) - \min_{t\in J} \alpha(t),$$

and hence, by a standard argument as in the proof of Theorem 1.1.7, we conclude that $|x'(t)| < C$ on J, which implies that $x(t)$ is a solution of (6.1.2). □

6.1.2 Existence of at least two solutions

In this section, we establish multiplicity results for the boundary value problem (6.1.2). Firstly, we introduce some notations and make some observations. Let $k > \max\{\|\alpha\|, \|\beta\|\}$, and assume that

$$f(t,k,0) - \bar{r}\phi(t) < 0 \text{ for every } t \in J. \tag{6.1.7}$$

Let

$$\sigma = \max\{|f(t,k,0)| : t \in J\},$$
$$c = \min\{\phi(t) : t \in J\}, \text{ and } d = \max\{\phi(t) : t \in J\},$$

then σ, c, $d > 0$. Define $r_0 = \frac{-\sigma}{c}$, then $r_0 < \bar{r}$ and

$$f(t,k,0) - r_0\phi(t) \ge 0 \text{ for every } t \in J.$$

For $r \le r_0$, we have

$$f(t,k,0) - r\phi(t) \ge 0 \text{ for every } t \in J. \tag{6.1.8}$$

Let

$$r_1 = \sup\{r \in [r_0,\bar{r}) : f(t,k,0) - r\phi(t) \ge 0 \text{ on } J\}, \tag{6.1.9}$$

then $r_0 \le r_1 < \bar{r}$ and for every $r \in (r_1,\bar{r}]$, we have

$$f(t,k,0) - r\phi(t) < 0 \text{ for every } t \in J. \tag{6.1.10}$$

Let $D_\alpha = \{x \in C^2(J) : \alpha(t) < x < k,\ t \in J\}$. We prove the following theorem.

Theorem 6.1.3. *Assume that* $(a_1) - (a_3)$ *of Theorem 6.1.2 hold and that the assumption in equation* (6.1.7) *holds. Then,* (6.1.2) *has at least two solutions in* D_α *for* $r \in (r_1, \bar{r}]$ *and there exists* r_n *such that* (6.1.2) *has no solution for* $r \leq r_n$.

Proof. In view of (a_1), we have

$$-\alpha''(t) < f(t,\alpha(t),\alpha'(t)) - \bar{r}\phi(t) \leq f(t,\alpha(t),\alpha'(t)) - r\phi(t) \text{ for } r \leq \bar{r}$$

and

$$-\beta''(t) > f(t,\beta(t),\beta'(t)) - \bar{r}\phi(t) \geq f(t,\beta(t),\beta'(t)) - r\phi(t) \text{ for } r \geq \bar{r}.$$

Thus, α is a strict lower solution of (6.1.2) for every $r \leq \bar{r}$ and β is a strict upper solution of (6.1.2) for every $r \geq \bar{r}$. Let

$$R^* = \sup\{r \in \mathbb{R} : -\alpha''(t) < f(t,\alpha(t),\alpha'(t)) - r\phi(t)\} \geq \bar{r},$$

$$r^* = \inf\{r \in \mathbb{R} : -\beta''(t) > f(t,\beta(t),\beta'(t)) - r\phi(t)\} \leq \bar{r}.$$

Then for every $r \in [r^*, R^*]$, α and β are strict lower and upper solutions of (6.1.2) and are such that $\alpha < \beta$ on $(0,1]$. Moreover, $f(t,x,x') - r\phi(t)$ satisfies a Nagumo condition on $R(\alpha, \beta)$ for every $r \in [r^*, R^*]$. Hence by Theorem 6.1.2, there exists a solution $x \in D_\alpha$ of (6.1.2) such that

$$\alpha(t) < x(t) < \beta(t),\ t \in (0,1] \text{ and } |x'(t)| < C \text{ on } J$$

for every $r \in [r^*, R^*]$, where C is a constant.

Define the modification of f as

$$F(t,x,x') = \begin{cases} f(t,k,x'), & \text{if } x \geq k, \\ f(t,x,x') & \text{if } \alpha(t) < x < k, \\ f(t,\alpha(t),x'), & \text{if } x \leq \alpha(t). \end{cases}$$

Consider the modified problem

$$\begin{aligned} -x''(t) &= F(t,x,x') - r\phi(t),\ t \in J, \\ x(0) &= 0, \quad x(1) = \delta x(\eta). \end{aligned} \tag{6.1.11}$$

We note that any solution x of (6.1.11) with $\alpha < x < k$ on J, is a solution of (6.1.2). We will show that for every $r \in (r_1, \bar{r}]$, any solution x of (6.1.11) does satisfy $\alpha < x < k$ on J and hence is a solution of (6.1.2). It will then suffice to show that (6.1.11) has at least two solutions in D_α. Let x be a solution of (6.1.11) for some $r \in (r_1, \bar{r}]$. Firstly, we show that $\alpha < x$ on J. Assume that

$$\max\{v(t) = \alpha(t) - x(t) : t \in J\} = v(t_0) \geq 0.$$

As in the proof of Theorem 6.1.2, we can show that $t_0 \in (0,1)$. Hence $v'(t_0) = 0$, $v''(t_0) \le 0$. However,

$$\begin{aligned}-v''(t_0) &= -\alpha''(t_0) + x''(t_0) \\ &< f(t_0, \alpha(t_0), \alpha'(t_0)) - \bar{r}\phi(t) - f(t_0, \alpha(t_0), \alpha'(t_0)) + r\phi(t) \le 0,\end{aligned}$$

a contradiction. Secondly, we show that $x < k$ on J. Assume that

$$\max\{x(t) : t \in J\} = x(t_0) \ge k,$$

then $x'(t_0) = 0$, $x''(t_0) \le 0$. However, by (6.1.10), we have

$$-x''(t_0) = f(t_0, k, 0) - r\phi(t) < 0,$$

a contradiction. Hence $\alpha < x < k$ on J. On the other hand, we note that for every $r \in (r_1, \bar{r}]$,

$$\begin{aligned}F(t, \alpha(t), \alpha'(t)) - r\phi(t) &= f(t, \alpha(t), \alpha'(t)) - r\phi(t) \\ &\ge f(t, \alpha(t), \alpha'(t)) - \bar{r}\phi(t) > -\alpha''(t),\ t \in J,\end{aligned}$$

and in view of (6.1.10),

$$F(t, k, 0) - r\phi(t) = f(t, k, 0) - r\phi(t) < 0,\ t \in J,$$

which imply that α, k are strict lower and upper solutions of (6.1.11). Moreover, for every $(t, x, y) \in R(\alpha, k)$, we have

$$|F(t, x, x') - r\phi(t)| = |f(t, x, x') - r\phi(t)| \le \hat{\omega}(|x'|),$$

where $\hat{\omega}(s) = \omega(s) + M_r$ for $s \ge 0$, $M_r = \max\{|r\phi(t)| : t \in J,\ r \in [r_1, \bar{r}]\}$. Thus, $F - r\phi$ satisfies a Nagumo condition on $R(\alpha, k)$. Hence by Theorem 6.1.2, the boundary value problem (6.1.11) has a solution $x \in D_\alpha$ for every $r \in (r_1, \bar{r}]$.

Let $r_2 = \inf\{r < \bar{r} : \text{(6.1.11) has a solution in } D_\alpha\}$, then (6.1.11) has no solution in D_α for $r < r_2$. We show that r_2 is finite. Let $x_2 \in D_\alpha$ be a solution of (6.1.11) for $r = r_2$. Then, there exists a constant C depending on k and $\hat{\omega}$ such that $|x_2'| < C$ on J. Let

$$m = \min\{F(t, x, x') : \alpha(t) < x < k,\ |x'(t)| < C,\ t \in J\},$$

then,

$$-x_2''(t) \ge m - r_2\phi(t),\ t \in J,$$

which implies that

$$x_2''(t) \le -m + r_2\phi(t) \le -m + r_2 c,\ t \in J. \tag{6.1.12}$$

Since $x_2 \in C^1(J)$, by the mean value theorem there exists $\xi \in (0,1)$ such that

$$x_2'(\xi) = x_2(1) - x_2(0) = \delta x_2(\eta). \tag{6.1.13}$$

If $\eta \in [\xi, 1)$, then integrating (6.1.12) from ξ to t, using (6.1.13), we obtain

$$x_2'(t) \le x_2'(\xi) - m(t-\xi) + r_2c(t-\xi) < \delta x_2(\eta) - m + r_2c(t-\eta), \text{ for } t \ge \xi. \quad (6.1.14)$$

Integrating (6.1.14) from η to 1, we have

$$x_2(1) - x_2(\eta) < \delta x_2(\eta)(1-\eta) - m(1-\eta) + r_2\frac{c(1-\eta)^2}{2},$$

which in view of the boundary condition $x_2(1) = \delta x_2(\eta)$, implies that

$$-(1-\delta\eta)x_2(\eta) < -m(1-\eta) + r_2\frac{c(1-\eta)^2}{2},$$

which gives

$$r_2 > -\frac{2}{c}\Big[\frac{x_2(\eta)(1-\delta\eta)}{(1-\eta)^2} - \frac{m}{1-\eta}\Big]. \quad (6.1.15)$$

On the other hand, if $\eta \in (0, \xi]$, then integrating (6.1.12) from t to ξ, we obtain

$$x_2'(\xi) - x_2'(t) \le -m(\xi - t) + r_2c(\xi - t) < -m + r_2c(\eta - t),$$

which in view of (6.1.13) implies that

$$-x_2'(t) < -\delta x_2(\eta) - m + r_2c(\eta - t). \quad (6.1.16)$$

Integrating (6.1.16) from 0 to η, we obtain

$$-x_2(\eta) < -\delta\eta x_2(\eta) - m\eta + r_2c\frac{\eta^2}{2},$$

which leads to

$$r_2 > -\frac{2}{c}\Big[\frac{x_2(\eta)(1-\delta\eta)}{\eta^2} - \frac{m}{\eta}\Big]. \quad (6.1.17)$$

Let $\rho = \max\{\eta, 1-\eta\}$, then $\rho > 0$ and from (6.1.15) and (6.1.17), it follows that

$$r_2 > -\frac{2}{c}\Big[\frac{x_2(\eta)(1-\delta\eta)}{\rho^2} - \frac{m}{\rho}\Big] = r_n \text{ (say)}.$$

Hence r_2 is finite.

Now, let $\tilde{r} \in [r_2, \bar{r})$ be such that (6.1.11) has a solution $\tilde{x}$ in D_α. Then for every $r \in (\tilde{r}, \bar{r}]$, we have

$$-\alpha''(t) < F(t, \alpha(t), \alpha'(t)) - \bar{r}\phi(t) \le F(t, \alpha(t), \alpha'(t)) - r\phi(t), \ t \in J$$

$$-\tilde{x}''(t) = F(t, \tilde{x}(t), \tilde{x}'(t)) - \tilde{r}\phi(t) > F(t, \tilde{x}(t), \tilde{x}'(t)) - r\phi(t), \ t \in J.$$

These imply that α and $\tilde{x}$ are strict lower and upper solutions of (6.1.11). Moreover, for $(t, x, y) \in R(\alpha, \tilde{x})$, we have

$$|F(t, x, x') - r\phi(t)| \le \hat{\omega}(|x'|).$$

Hence by Theorem 6.1.2, there exists a solution $x \in D_\alpha$ of (6.1.11) such that

$$\alpha(t) < x(t) < \tilde{x}(t),\ t \in (0,1].$$

Also, there exists a constant $C_1 > 0$ which depends on α, $\tilde{x}$, $\hat{\omega}$ such that

$$|x'(t)| < C_1 \text{ on } J.$$

In case, $r \geq r^*$, the solution x of (6.1.11) satisfies $\alpha(t) < x(t) < \beta(t)$, $t \in (0,1]$. Define the modification of F with respect to α, $\tilde{x}$ as

$$h(t,x,x') = \begin{cases} F(t,\tilde{x}(t),\tilde{x}'(t)), & \text{if } x \geq \tilde{x}(t)+\epsilon, \\ F(t,\tilde{x}(t),x') + [F(t,\tilde{x}(t),\tilde{x}'(t)) & \\ \qquad - F(t,\tilde{x}(t),x')]\frac{x-\tilde{x}(t)}{\epsilon}, & \text{if } \tilde{x}(t) \leq x < \tilde{x}(t)+\epsilon, \\ F(t,x,x'), & \text{if } \alpha(t) < x < \tilde{x}(t), \\ F(t,\alpha(t),x') + [F(t,\alpha(t),\alpha'(t)) & \\ \qquad - F(t,\alpha(t),x')]\frac{\alpha(t)-x}{\epsilon}, & \text{if } \alpha(t)-\epsilon < x \leq \alpha(t), \\ F(t,\alpha(t),\alpha'(t)), & \text{if } x \leq \alpha(t)-\epsilon, \end{cases}$$

and define

$$\psi(t,x) = \frac{(\tilde{x}(t) - p(x,\alpha(t),\tilde{x}))F(t,\alpha(t),\alpha'(t)) + (p(x,\alpha(t),\tilde{x}) - \alpha(t))F(t,\tilde{x}(t),\tilde{x}'(t))}{\tilde{x}(t)-\alpha(t)},\ t \in J$$

where

$$p(x,\alpha(t),\tilde{x}) = \max\big\{\alpha(t),\ \min\{x,\tilde{x}(t)\}\big\},\ x \in \mathbb{R}.$$

Consider the system of equations

$$\begin{aligned} &-x''(t) = \lambda h(t,x,x') + (1-\lambda)\psi(t,x) - r\phi(t),\ t \in J,\ r \in (\tilde{r},\bar{r}] \\ &x(0) = 0, \quad x(1) = \delta x(\eta),\ \lambda \in [0,1]. \end{aligned} \tag{6.1.18}$$

For $\lambda = 0$, the system reduces to

$$\begin{aligned} -x''(t) &= \psi(t,x) - r\phi(t),\ t \in J,\ r \in (\tilde{r},\bar{r}] \\ x(0) &= 0, \quad x(1) = \delta x(\eta). \end{aligned} \tag{6.1.19}$$

This is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s)\big[\psi(s,x) - r\phi(s)\big]ds.$$

Since ψ, ϕ and $G(t,s)$ are continuous and bounded, this integral equation has a fixed point by Schauder's fixed point theorem. Moreover,

$$\begin{aligned} \psi(t,\alpha(t)) - r\phi(t) &= F(t,\alpha(t),\alpha'(t)) - r\phi(t) \\ &\geq F(t,\alpha(t),\alpha'(t)) - \bar{r}\phi(t) > -\alpha''(t), \end{aligned}$$

and

$$\begin{aligned}\psi(t,\tilde{x}(t)) - r\phi(t) =& F(t,\tilde{x}(t),\tilde{x}'(t)) - r\phi(t)\\ &< F(t,\tilde{x}(t),\tilde{x}'(t)) - \tilde{r}\phi(t) = \tilde{x}''(t).\end{aligned}$$

Thus, α, $\tilde{x}$ are strict lower and upper solutions of (6.1.19). Hence any solution of (6.1.19) satisfies $\alpha < x < \tilde{x}$ on $(0,1]$. Since $\psi(t,x) - r\phi(t)$ is continuous and bounded, it satisfies a Nagumo condition on $R(\alpha,\tilde{x})$. Hence there exists $C_1 > 0$ such that any solution x of (6.1.19) with $\alpha < x < \tilde{x}$ satisfies $|x'| < C_1$ on J.

For $\lambda = 1$, (6.1.18) is equivalent to

$$\begin{aligned}&-x''(t) = h(t,x,x') - r\phi(t),\ t\in J,\ r\in(\tilde{r},\bar{r}]\\ &x(0)=0,\quad x(1)=\delta x(\eta).\end{aligned}\tag{6.1.20}$$

Again, α, $\tilde{x}$ are strict lower and upper solutions of (6.1.20) and for every $(t,x)\in J\times[\min\alpha(t),\max\tilde{x}(t)]$ we have

$$|h(t,x,x') - r\phi(t)| = |F(t,x,x') - r\phi(t)| \le \hat{\omega}(|x'|).$$

This implies that h satisfies a Nagumo condition on $R(\alpha,\tilde{x})$. Hence by Theorem 6.1.2, the boundary value problem (6.1.20) has a solution x such that

$$\alpha < x < \tilde{x} \text{ on } (0,1] \text{ and } |x'(t)| < C_2 \text{ on } J.$$

Now, for any $\lambda\in[0,1]$, we note that

$$\begin{aligned}&\lambda h(t,\alpha(t),\alpha'(t)) + (1-\lambda)\psi(t,\alpha(t)) - r\phi(t) = \lambda F(t,\alpha(t),\alpha'(t)) + (1-\lambda)F(t,\alpha(t),\alpha'(t))\\ &- r\phi(t) = F(t,\alpha(t),\alpha'(t)) - r\phi(t) \ge F(t,\alpha(t),\alpha'(t)) - \bar{r}\phi(t) > -\alpha''(t),\ t\in J\end{aligned}$$

and

$$\begin{aligned}&\lambda h(t,\tilde{x}(t),\tilde{x}'(t)) + (1-\lambda)\psi(t,\tilde{x}(t)) - r\phi(t) = \lambda F(t,\tilde{x}(t),\tilde{x}'(t)) + (1-\lambda)F(t,\tilde{x}(t),\tilde{x}'(t))\\ &- r\phi(t) < F(t,\tilde{x}(t),\tilde{x}'(t)) - \tilde{r}\phi(t) = -\tilde{x}''(t),\ t\in J.\end{aligned}$$

These imply that α, $\tilde{x}$ are strict lower and upper solutions of (6.1.18). Further, for $\lambda\in[0,1]$, $(t,x)\in J\times[\min\alpha(t),\max\tilde{x}(t)]$ and $x'\in\mathbb{R}$, we have

$$\begin{aligned}|\lambda h(t,x,x') + (1-\lambda)\psi(t,x) - r\phi(t)| &\le \lambda|F(t,x,x')| + (1-\lambda)\psi|(t,x)| + M_1\\ &\le \omega(|x'|) + M_1 + 2M = \bar{\omega}(|x'|),\end{aligned}$$

where, $\bar{\omega}(s) = \omega(s) + M_2$ if $s\ge 0$, $M_2 = M_1 + 2M$. It is easy to see that $\hat{\omega}$ is a Nagumo function. Thus, $\lambda h(t,x,x') + (1-\lambda)\psi(t,x) - r\phi(t)$ satisfies a Nagumo condition on $R(\alpha,\tilde{x})$. Hence by Theorem 6.1.2, any solution of the boundary value problem (6.1.18) satisfies

$$\alpha < x < \tilde{x} \text{ on } (0,1] \text{ and } |x'(t)| < C_3 \text{ on } J,$$

where $C_3 > 0$ is a constant depending on α, $\tilde{x}$ and $\bar{\omega}$.

To use the Leray-Schauder degree theory, we define the operators

$$F_r : C_b^2(J) \to C_b^2(J), \Psi : C_b^2(J) \to C_b^2(J), \; H_r : C_b^2(J) \to C_b^2(J),$$

by

$$F_r x(t) = \int_0^1 G(t,s)\big[F(s,x(s),x'(s)) - r\phi(s)\big]ds$$

$$H_r x(t) = \int_0^1 G(t,s)\big[h(s,x(s),x'(s)) - r\phi(s)\big]ds$$

$$\Psi x(t) = \int_0^1 G(t,s)\big[\psi(s,x(s)) - r\phi(s)\big]ds,$$

where $C_b^2 = \{x \in C^2(J) : x(0) = 0, \, x(1) = \delta x(\eta)\}$. We write (6.1.11) and (6.1.18) in the following form

$$(I - F_r)x = 0 \tag{6.1.21}$$

$$(I - \lambda H_r - (1-\lambda)\Psi)x = 0 \tag{6.1.22}$$

Let $C > \max\{C_1, C_2, C_3\}$ and consider two open bounded sets in $C^2(J)$

$$\Omega = \{x \in C_b^2(J) : |x| < k, |x'| < C \text{ on } J\},$$

$$\Omega_1 = \{x \in C_b^2(J) : |x'| < C \text{ on } J, \, \alpha < x < \tilde{x} \text{ on } (0,1]\}.$$

For any $\lambda \in [0,1]$, each solution x of (6.1.18), belongs to Ω and so $x \notin \partial\Omega$. For $\lambda = 0$, (6.1.22) is equivalent to

$$(I - \Psi)(x)(t) = 0,$$

and the homotopy invariance of degree implies that

$$d(I - \Psi, \Omega) = d(I, \Omega) = 1.$$

Again by the homotopy property of degree, we have ($\lambda = 1$),

$$d(I - H_r, \Omega) = 1.$$

Since each solution of (6.1.20) belongs to Ω_1, by the excision property of degree, we have

$$d(I - H_r, \Omega_1) = 1.$$

Since $H_r = F_r$ on Ω_1, it follows that

$$d(I - F_r, \Omega_1) = 1.$$

Now, since (6.1.21) has no solution for $r \le r_n$, it follows by the existence property that, for $r \le r_n$

$$d(I - F_r, \Omega) = 0.$$

Also, for $r \le \bar{r}$, any solution x of (6.1.21) belongs to Ω and so $x \notin \partial\Omega$. Letting r vary from r_n to $\bar{r}$, by the homotopy property of degree, we have

$$d(I - F_r, \Omega) = 0, \text{ for } r \le \bar{r}.$$

Using the additivity property of degree, we have

$$d(I - F_r, \Omega \setminus \Omega_1) = d(I - F_r, \Omega) - d(I - F_r, \Omega_1) = 0 - (1) = -1.$$

Therefore, the problem (6.1.21) has at least two solutions in D_α. □

Example 6.1.4. (Class of examples) Consider the following boundary value problem

$$\begin{aligned} -x''(t) = f(t, x, x') - r = -|x'|^{p-1}x' + \psi(x) - r,\ t \in [0, 1], \\ x(0) = 0,\ x(1) = \delta x(\eta),\ 0 < \delta\eta < 1, \end{aligned} \tag{6.1.23}$$

where $0 \le p \le 2$. We take $\psi : \mathbb{R} \to \mathbb{R}$ to be continuous and to satisfy $\psi(0) = 0$ and $\psi(x) > 0$ for $x < 0$ and $-cx^p \le \psi(x) \le 0$ for $x \ge 0$, where $0 < c < 1$. Take $\bar{r} = a^p$ $(a > 0)$, $\alpha = -at$ and $\beta = at$, then α, β satisfy the boundary conditions

$$\alpha(0) = 0,\ \alpha(1) < \delta\alpha(\eta)$$
$$\beta(0) = 0,\ \beta(1) > \delta\beta(\eta).$$

Moreover, for every $t \in (0, 1]$, we have

$$f(t, -at, -a) - \bar{r} = \psi(-at) > 0,$$

and

$$f(t, at, a) - \bar{r} = -2a^p + \psi(at) < 0.$$

Thus, α is a strict lower solution and β is a strict upper solution of (6.1.23). Further, we note that for $t \in J$, $x \in [-a, a]$ and $x' \in \mathbb{R}$, we have

$$|f(t, x, x') - \bar{r}| = \big| - |x'|^{p-1}x' + \psi(x) - a^p\big| \le |x'|^p + a^p(1 + c) = \omega(|x'|),$$

$\omega(s) = s^p + a^p(1 + c)$. Since

$$\int_0^\infty \frac{sds}{\omega(s)} = \infty,$$

it follows that ω is a Nagumo function and hence $f - \bar{r}$ satisfies a Nagumo condition on $R(-a, a)$. Now, let $k \ge a$, then $f(t, k, 0) - \bar{r} = \psi(k) - a^p < 0$, and for every $r \le -k^p$, we have

$$f(t, k, 0) - \bar{r} \ge \psi(k) + k^p \ge k^p(1 - c) > 0 \text{ for every } t \in J.$$

Hence, by Theorem (6.1.3), there exists $r_0 < -k^p$ such that the problem (6.1.23) has no solution in D_α for $r \le r_0$, and has at least two solutions in D_α for $r \in (-k^p, a^p]$, where $D_\alpha = \{x \in C^2 : -at < x < at \text{ on } (0, 1]\}$.

6.2 Existence of at least three solutions of a second order three point boundary value problem

In this section, we study existence of at least three solutions in the presence of two lower and two upper solutions of some second order nonlinear differential equations of the type

$$\begin{aligned} -x''(t) &= f(t,x,x'),\ t \in J = [0,1], \\ x(0) = 0, &\quad x(1) = \delta x(\eta), \end{aligned} \tag{6.2.1}$$

where $0 < \delta\eta < 1$, $0 < \eta < 1$ and $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. The growth of f with respect to x' is allowed to be quadratic. We use some degree theory arguments to get the multiplicity result. We are particularly interested in the case where f depends explicitly on x'. An abstract result giving the existence of three solutions of nonlinear equations has been given by Leggett and Williams [64]. Using the theory of fixed point index in an ordered Banach spaces they developed a fixed point theorem which guaranteed the existence of three fixed points. Recently, using the Leggett and Williams fixed point theorem, existence of three positive solutions is studied [86] when the nonlinearity f is independent of the derivative. There are many papers dealing with existence of solutions of (6.2.1), for example, [3, 21, 24, 28–30, 67–70, 86, 92] and of positive solutions but most of these do not have explicit x' dependence in the nonlinear term. Moreover the methods of these papers are entirely different from ours. In [35], existence of at least three solutions for the two point problem of the type

$$\begin{aligned} x''(t) + f(t,x,x') &= 0,\ t \in [0,1], \\ x(0) = 0 &= x(1), \end{aligned}$$

under a smoothness assumption is studied. The main assumption in [35] is the existence of two lower solutions α_1, α_2 and two upper solutions β_1, β_2 such that $\alpha_1 \le \alpha_2$, $\beta_1 \le \beta_2$ and that f satisfies a Nagumo condition. In this section we study three point boundary value problems (6.2.1). We will assume the existence of upper and lower solutions which are not necessarily constants and we use some topological degree arguments to get multiplicity result. We give a class of examples to show that the hypothesis of our main result can be easily verified.

6.2.1 Existence of at least three solutions

Define $L : C(J) \to C^1(J)$ by

$$L\phi(t) = \int_0^1 G(t,s)\phi(s)ds,$$

for $\phi \in C(J)$, where $G(t,s)$ is the Green's function and is defined in section 6.1.1. From section 6.1.1, we note that

$$G(t,s) \le \frac{\rho}{1-\delta\eta},$$

where $\rho = \max\{\eta, 1-\eta\}$. By standard arguments L is a compact operator. Define $\hat{f}: C^1(J) \to C(J)$ by

$$\hat{f}(\phi)(t) = f(t, \phi(t), \phi'(t)),\ t \in J.$$

Then, x is a solution of (6.2.1) if and only if $x \in C^1(J)$ is a solution of the equation

$$(I - L\hat{f})x = 0, \text{ that is, a fixed point of } L\hat{f}. \tag{6.2.2}$$

We recall the concept of upper and lower solutions for the BVP (6.2.1).

Definition 6.2.1. We say $\alpha \in C^2(J)$ a lower solution for the BVP (6.2.1) if

$$-\alpha''(t) \le f(t, \alpha(t), \alpha'(t)), \quad t \in J$$
$$\alpha(0) \le 0,\ \alpha(1) \le \delta\alpha(\eta).$$

An upper solution $\beta \in C^2(J)$ for the BVP (6.2.1) is defined similarly by reversing the inequalities.

Our main result is the following theorem in which we shall prove existence of at least three solutions of the BVP (6.2.1). We use the method of upper and lower solutions and degree arguments to prove that under suitable conditions, there exist at least three solutions of the BVP (6.2.1). Assume that

(A₁) $\alpha_1, \alpha_2 \in C^2(J)$ are two lower solutions and $\beta_1, \beta_2 \in C^2(J)$ are two upper solutions of (6.2.1) such that

$$\alpha_1 \le \alpha_2 \le \beta_2,\ \alpha_1 \le \beta_1 \le \beta_2 \text{ and } \alpha_2 \not\le \beta_1 \text{ on } J.$$

(A_2) $f: J \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and is nonincreasing in the third variable for every $(t, x) \in J \times \mathbb{R}$.

(A_3) Let $R(\alpha_1, \beta_2) = \{(t, x, x') \in J \times \mathbb{R}^2 : \alpha_1(t) \le x \le \beta_2(t)\}$. Suppose that f is continuous and satisfy a Nagumo condition on $R(\alpha_1, \beta_2)$.

(A_4) α_2 and β_1 are not solutions of (6.2.1).

Theorem 6.2.2. *Under assumptions $(A_1)-(A_4)$, the boundary value problem* (6.2.1) *has at least three solutions x_i $(i = 1, 2, 3)$ such that*

$$\alpha_1 \le x_1 \le \beta_1,\ \alpha_2 \le x_2 \le \beta_2 \text{ and } x_3 \not\le \beta_1 \text{ and } x_3 \not\ge \alpha_2 \text{ on } J.$$

Proof. Let $\max\{\delta|\alpha(\eta)|, \delta|\beta(\eta)|\} = \lambda$. Since f satisfies a Nagumo condition, it follows that for $x \in [\min \alpha_1(t), \max \beta_2(t)]$, there exist a Nagumo function $\omega : [0, \infty) \to (0, \infty)$ and a constant $N(> \lambda)$ depending on α, β and ω such that

$$|f(t, x, x')| \le \omega(|x'|) \text{ for } (t, x) \in J \times [\min \alpha_1(t), \max \beta_2(t)] \text{ and}$$

$$\int_{\lambda}^{N} \frac{sds}{\omega(s)} > \max_{t\in J} \beta(t) - \min_{t\in J} \alpha(t). \tag{6.2.3}$$

Let $C > \max\{N, \|\alpha'\|, \|\beta'\|\}$ and define $q(y) = \max\{-C, \min\{y, C\}\}$, a retraction onto $[-C, C]$.

We modify f with respect to α_1 and β_2 to obtain a second boundary value problem and reformulate the new problem as an integral equation. We show that solutions of the modified problem lie in the region where f is unmodified and hence are solutions of the original problem. Let $\varepsilon > 0$ be fixed and define

$$F(t,x,x') = \begin{cases} f(t,\beta_2(t),\beta_2'(t)) + \frac{x-\beta_2(t)}{1+|x-\beta_2(t)|}, & \text{if } x \geq \beta_2(t)+\varepsilon, \\ f(t,\beta_2(t),q(x')) + \big[f(t,\beta_2(t),\beta_2'(t)) & \\ \quad + \frac{x-\beta_2(t)}{1+|x-\beta_2(t)|} - f(t,\beta_2(t),q(x'))\big]\frac{x-\beta_2(t)}{\varepsilon}, & \text{if } \beta_2(t) \leq x \leq \beta_2(t)+\varepsilon, \\ f(t,x,q(x')), & \text{if } \alpha_1(t) \leq x \leq \beta_2(t), \\ f(t,\alpha_1(t),q(x')) + \big[f(t,\alpha_1(t),\alpha_1'(t)) & \\ \quad + \frac{\alpha_1(t)-x}{1+|\alpha_1(t)-x|} - f(t,\alpha_1(t),q(x'))\big]\frac{\alpha_1(t)-x}{\varepsilon}, & \text{if } \alpha_1(t)-\varepsilon \leq x \leq \alpha_1(t), \\ f(t,\alpha_1(t),\alpha_1'(t)) + \frac{\alpha_1(t)-x}{1+|\alpha_1(t)-x|}, & \text{if } x \leq \alpha_1(t)-\varepsilon. \end{cases} \tag{6.2.4}$$

We note that F is continuous and bounded on $J \times \mathbb{R}^2$ so there exists $M > 0$ such that

$$|F(t,x,x')| \leq M \text{ on } J \times \mathbb{R}^2.$$

We choose M_1 so that $M_1 > \max\{\|\alpha_1\|, \|\beta_2\|, \frac{M\rho}{1-\delta\eta}\}$. Consider the modified problem

$$\begin{aligned} &-x''(t) = F(t,x,x'),\ t \in J, \\ &x(0) = 0, \quad x(1) = \delta x(\eta),\ 0 < \delta\eta < 1,\ \eta \in (0,1). \end{aligned} \tag{6.2.5}$$

Define $\widehat{F} : C^1(J) \to C(J)$ by

$$\widehat{F}(x)(t) = F(t,x(t),x'(t)),\ t \in J.$$

Then x is a solution of (6.2.5) if and only if $x \in C^1(J)$ is a fixed point of $L\widehat{F}$. By definition of F and choice of C, we have

$$F(t,\alpha_1(t),\alpha_1'(t)) = f(t,\alpha_1(t),\alpha_1'(t)) \geq -\alpha_1''(t),\ t \in J$$
$$F(t,\beta_2(t),\beta_2'(t)) = f(t,\beta_2(t),\beta_2'(t)) \leq -\beta_2''(t),\ t \in J$$

so that α_1 and β_2 are lower and upper solutions of (6.2.5). Moreover, for every $(t,x) \in J \times [\min \alpha_1(t), \max \beta_2(t)]$, we have

$$|F(t,x,x')| = |f(t,x,q(x'(t)))| \leq \omega(|q(x'(t))|) = \tilde{\omega}(|x'|),$$

where $\tilde{\omega}(s) = \omega(q(s))$ for $s \geq 0$. Since

$$\int_{\lambda}^{\infty} \frac{sds}{\tilde{\omega}(s)} = \int_{\lambda}^{C} \frac{sds}{\omega(s)} + \int_{C}^{\infty} \frac{sds}{\omega(C)} = \infty,$$

this implies that F satisfies a Nagumo condition. Further, in view of (6.2.3), we have

$$\int_\lambda^C \frac{sds}{\bar{\omega}(s)} \geq \int_\lambda^N \frac{sds}{\omega(s)} > \max_{t\in J}\beta(t) - \min_{t\in J}\alpha(t),$$

and hence, by a standard argument as in the proof of Theorem 1.1.7, we conclude that $|x'(t)| < C$ on J. Thus any solution x of (6.2.5) with $\alpha_1(t) \leq x(t) \leq \beta_2(t)$, $t \in J$ satisfies

$$|x'(t)| < C,\ t \in J$$

and hence is a solution of (6.2.1). We now show that any solution x of (6.2.5) does satisfy

$$\alpha_1(t) \leq x(t) \leq \beta_2(t),\ t \in J.$$

For this, set $v(t) = \alpha_1(t) - x(t)$, $t \in J$. Then, $v \in C^2(J)$ and the boundary conditions imply that

$$v(0) \leq 0,\ v(1) \leq \delta v(\eta). \tag{6.2.6}$$

Suppose that $\alpha_1(t) \not\leq x(t)$ on J, then $v(t) = \alpha_1(t) - x(t)$ has a positive maximum at some $t_0 \in J$. If $t_0 \neq 1$, then $v'(t_0) = 0$ and $v''(t_0) \leq 0$. However, if $0 < v(t_0) < \varepsilon$, then

$$\begin{aligned} v''(t_0) &= \alpha_1''(t_0) - x''(t_0) \\ &\geq -f(t_0, \alpha_1(t_0), \alpha_1'(t_0)) + [f(t_0, \alpha_1(t_0), \alpha_1'(t_0)) + \frac{v^2(t_0)}{\varepsilon(1 + v(t_0))}] \\ &= \frac{v^2(t_0)}{\varepsilon(1 + v(t_0))} > 0, \end{aligned}$$

a contradiction, and if $v(t_0) \geq \varepsilon$, then

$$v''(t_0) \geq -f(t_0, \alpha_1(t_0), \alpha_1'(t_0)) + [f(t_0, \alpha_1(t_0), \alpha_1'(t_0)) + \frac{v(t_0)}{(1 + v(t_0))}] = \frac{v(t_0)}{(1 + v(t_0))} > 0,$$

again a contradiction. Thus $v(t)$ has no positive local maximum.
If $t_0 = 1$, then $v(1) \geq 0$ and $v'(1) \geq 0$. If $v(1) = 0$, then the boundary condition $v(1) \leq \delta v(\eta)$ implies that $v(\eta) \geq 0$ and hence $v(1)$ cannot be the maximum of $v(t)$. Thus $v(1) > 0$. Moreover, if $\delta \leq 1$, then $v(1) \leq v(\eta)$ and hence $v(1)$ can not be the maximum of v. Then $v(t) \leq 0$, $t \in J$ without demanding the assumption (A_2). Now, we discuss the case where $\delta > 1 > \delta\eta$. Since $v(0) \leq 0$, it follows that there exists $t_1 \in [0, \eta) \subset J$ such that $v(t_1) = 0$ and $v'(t_1) \geq 0$. We claim that

$$v'(t) \geq 0 \text{ on } [t_1, 1].$$

If not, then there exists a $t_2 \in (t_1, 1)$ such that $v'(t_2) < 0$. But then $v(t)$ must have a positive local maximum at some $t \in (t_1, t_2)$, which yields a contradiction. Hence $v'(t) \geq 0$ on $[t_1, 1]$, so that

$$v(t) \geq 0 \text{ on } [t_1, 1],$$

and also

$$q(x'(t)) \leq \alpha_1'(t),\ t \in [t_1, 1].$$

Then by (A_2), we have

$$f(t, \alpha_1(t), \alpha_1'(t)) - f(t, \alpha_1(t), q(x'(t)) \leq 0. \tag{6.2.7}$$

There exists $t_3 \in (t_1, 1)$ such that

$$v(t_3) = \varepsilon, \text{ and } 0 \leq v(t) \leq \varepsilon,\ t \in [t_1, t_3],\ \ v(t) \geq \varepsilon,\ t \in [t_3, 1].$$

Firstly, we consider the case that $t \in [t_1, t_3]$ where $0 \leq v(t) \leq \varepsilon$. Then, using (6.2.7), we have

$$\begin{aligned} F(t, x, x') = {} & f(t, \alpha_1(t), q(x'(t))) + \frac{v^2(t)}{(1 + v(t))\varepsilon} \\ & + \left[f(t, \alpha_1(t), \alpha_1'(t)) - f(t, \alpha_1(t), q(x'(t)))\right]\frac{v(t)}{\varepsilon} \\ & \geq f(t, \alpha_1(t), \alpha_1'(t)) + \frac{v^2(t)}{(1 + v(t))\varepsilon} > f(t, \alpha_1(t), \alpha_1'(t)). \end{aligned}$$

Thus, for every $t \in [t_1, t_3]$, we have

$$v''(t) = \alpha''(t) - x''(t) \geq -f(t, \alpha(t), \alpha'(t)) + F(t, x, x') > 0.$$

Secondly, we consider the case $t \in [t_3, 1]$. In this case

$$F(t, x, x') = f(t, \alpha_1(t), \alpha_1'(t)) + \frac{v(t)}{(1 + v(t))} > f(t, \alpha_1(t), \alpha_1'(t)),$$

and for every $t \in [t_3, 1]$,

$$v''(t) = \alpha''(t) - x''(t) > 0.$$

Thus, $v''(t) > 0$, for $t \in [t_1, 1]$, which implies that the graph of v is convex on $[t_1, 1]$. Hence

$$v(\eta) < v(1)\frac{\eta - t_1}{1 - t_1} < \frac{v(\eta)}{\eta}\frac{(\eta - t_1)}{(1 - t_1)},$$

which leads to $\eta > 1$, a contradiction. Thus the maximum of v occurs at $t_0 = 0$. Since $v(0) \leq 0$ it follows that $v(t) \leq 0$ on J.

Similarly, we can show that $x(t) \leq \beta_2(t)$, $t \in J$.

Thus, it is suffices to show that (6.2.5) has at least three solutions x_i such that

$$\alpha_1(t) \leq x_i(t) \leq \beta_2(t),\ t \in J,\ i = 1, 2, 3.$$

Let $\Omega = \{x \in C^1(J) : |x(t)| < M_1, |x'(t)| < C,\ t \in J\}$, then Ω is a bounded open subset of $C^1(J)$ and is convex. Since $L\widehat{F}(\overline{\Omega}) \subset \Omega$ it follows that the degree

$$d(I - L\widehat{F}, \Omega, 0) = 1.$$

Let

$$\Omega_{\alpha_2} = \{x \in \Omega : x > \alpha_2 \text{ on } (0,1)\} \text{ and } \Omega^{\beta_1} = \{x \in \Omega : x < \beta_1 \text{ on } (0,1)\}.$$

Therefore $\overline{\Omega}_{\alpha_2} \cap \overline{\Omega}_1^{\beta} = \emptyset$ and, since $\alpha_2 \not\leq \beta_1$ on J, the set $\Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega_1^{\beta}}$ is not empty. By (A_4) and choice of M_1, C, there are no solutions on $\partial\Omega_{\alpha_2} \cup \partial\Omega^{\beta_1}$. The additivity of degree implies that

$$d(I - L\widehat{F}, \Omega, 0) = d(I - L\widehat{F}, \Omega_{\alpha_2}, 0) + d(I - L\widehat{F}, \Omega^{\beta_1}, 0) + d(I - L\widehat{F}, \Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega_1^{\beta}}, 0). \quad (6.2.8)$$

Now we show that $d(I - L\widehat{F}, \Omega_{\alpha_2}, 0) = d(I - L\widehat{F}, \Omega^{\beta_1}, 0) = 1$. Firstly, we show that

$$d(I - L\widehat{F}, \Omega_{\alpha_2}, 0) = 1.$$

Define $F_2(t, x, x')$ as follows

$$F_2(t,x,x') = \begin{cases} f(t, \beta_2(t), \beta_2'(t)) + \frac{x - \beta_2(t)}{1 + |x - \beta_2(t)|}, & \text{if } x \geq \beta_2(t) + \varepsilon, \\ f(t, \beta_2(t), q(x')) + \big[f(t, \beta_2(t), \beta_2'(t)) & \\ \quad + \frac{x - \beta_2(t)}{1 + |x - \beta_2(t)|} - f(t, \beta_2(t), q(x'))\big] \frac{x - \beta_2(t)}{\varepsilon}, & \text{if } \beta_2(t) \leq x \leq \beta_2(t) + \varepsilon, \\ f(t, x, q(x')), & \text{if } \alpha_2(t) \leq x \leq \beta_2(t), \\ f(t, \alpha_2(t), q(x')) + \big[f(t, \alpha_2(t), \alpha_2'(t)) & \\ \quad + \frac{\alpha_2(t) - x}{1 + |\alpha_2(t) - x|} - f(t, \alpha_2(t), q(x'))\big] \frac{\alpha_2(t) - x}{\varepsilon}, & \text{if } \alpha_2(t) - \varepsilon \leq x \leq \alpha_2(t), \\ f(t, \alpha_2(t), \alpha_2'(t)) + \frac{\alpha_2(t) - x}{1 + |\alpha_2(t) - x|}, & \text{if } x \leq \alpha_2(t) - \varepsilon. \end{cases} \quad (6.2.9)$$

We note that $F_2 = F$ on Ω_{α_2}. Now we consider the problem

$$\begin{aligned} &-x''(t) = F_2(t, x, x'), \ t \in J, \\ &x(0) = 0, \quad x(1) = \delta x(\eta), \ 0 < \delta\eta < 1, \ \eta \in (0,1). \end{aligned} \quad (6.2.10)$$

Then (6.2.10) is equivalent to

$$(I - L\widehat{F}_2)x = 0, \quad (6.2.11)$$

where $\widehat{F}_2(x)(t) = F_2(t, x(t), x'(t))$.

By the same process as we used for the problem (6.2.5), we can show that any solution x of (6.2.10) satisfies $x \geq \alpha_2$ on J, which in view of (A_4) implies that $x \neq \alpha_2$ and hence belongs to Ω_{α_2}. Since $L\widehat{F}_2(\overline{\Omega}) \subseteq \Omega$, we have

$$d(I - L\widehat{F}_2, \Omega, 0) = 1.$$

Now $d(I - L\widehat{F}, \Omega_{\alpha_2}, 0) = d(I - L\widehat{F}_2, \Omega_{\alpha_2}, 0)$. It follows that

$$1 = d(I - L\widehat{F}_2, \Omega, 0) = d(I - L\widehat{F}_2, \Omega_{\alpha_2}, 0) + d(I - L\widehat{F}_2, \Omega \setminus \overline{\Omega}_{\alpha_2}, 0) = d(I - L\widehat{F}_2, \Omega_{\alpha_2}, 0).$$

Similarly, we can show that $d(I - L\widehat{F}, \Omega^{\beta_1}, 0) = 1$. Thus from (6.2.8), we obtain

$$d(I - L\widehat{F}, \Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega_1^{\beta}}, 0) = -1.$$

Hence there are 3 solutions, one in each of the sets Ω_{α_2}, Ω^{β_1} and $\Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega_1^{\beta}}$. □

Remark 6.2.3. As noted in the proof, if $\delta \leq 1$, assumption (A_2) is not needed in Theorem 6.2.2.

We now give a class of examples which illustrate that the assumptions of our theorem can easily be satisfied.

Example 6.2.4. Consider the following boundary value problem

$$-x''(t) = f(t, x, x') = g(x') + \Psi(x),\ t \in (0,1), \tag{6.2.12}$$

$$x(0) = 0,\ x(1) = \delta x(\eta),\ \eta \in (0,1), \tag{6.2.13}$$

where $\delta < 1$ and g, $\Psi : \mathbb{R} \to \mathbb{R}$ are continuous, Ψ is increasing near the origin and

$g(0) = 0$, $\Psi(0) < 0$, $\Psi(-a) \geq 0$, and $\Psi(b) \leq 0$ for some a and b, which may be large ,

while near the origin, they behave as follows

$$\begin{cases} \Psi(x) \geq -c, & \text{for } 0 \leq x \leq \frac{c}{8}, \\ \Psi(x) \geq 2c, & \text{for } \frac{c}{8} \leq x \leq \frac{c}{4}, \end{cases} \tag{6.2.14}$$

and

$$\begin{cases} g(y) \geq 3c, & \text{for } y \in [-c, -\frac{c}{\sqrt{2}}] \cup [\frac{c}{\sqrt{2}}, c], \\ g(y) \geq 0, & \text{for } y \in [-\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}] \end{cases} \tag{6.2.15}$$

for small c. Moreover, we assume $|g(y)| \leq c_1 + c_2|y|^p$, $p \leq 2$, for large $|y|$. Take

$$\alpha_1(t) = -a, \quad \alpha_2(t) = ct(1-t),\ t \in J.$$

Then α_1, $\alpha_2 \in C^2(J)$ satisfy the boundary conditions

$$\alpha_1(0) < 0,\ \ \alpha_1(1) < \delta\alpha_1(\eta)$$
$$\alpha_2(0) = 0,\ \ \alpha_2(1) < \delta\alpha_2(\eta).$$

Moreover, for every $t \in (0, 1]$, we have

$$\alpha_1''(t) + f(t, \alpha_1(t), \alpha_1'(t)) = g(0) + \Psi(-a) \geq 0,$$

and by conditions imposed in (6.2.14) and (6.2.15), we obtain

$$\alpha_2''(t) + f(t, \alpha_2(t), \alpha_2'(t)) = -2c + g(c(1-2t)) + \Psi(ct(1-t)) \geq 0,\ t \in J.$$

Thus, α_1 and α_2 are lower solutions of (6.2.12).

Now, take $\beta_1 = 0$ and $\beta_2 = b$, then β_1, β_2 satisfy the boundary conditions

$$\beta_1(0) = 0,\ \beta_1(1) = \delta\beta_1(\eta),$$
$$\beta_2(0) > 0,\ \beta_2(1) > \delta\beta_2(\eta).$$

Moreover, for every $t \in J$, we have

$$\beta_1''(t) + f(t, \beta_1(t), \beta_1'(t)) = g(0) + \Psi(0) < 0,$$
$$\beta_2''(t) + f(t, \beta_2(t), \beta_2'(t)) = g(0) + \Psi(b) \leq 0.$$

Thus, β_1 and β_2 are upper solutions of (6.2.12). Further, we note that

$$\alpha_1(t) \leq \alpha_2(t) \leq \beta_2(t),\ \text{on } J$$
$$\alpha_1(t) \leq \beta_1(t) \leq \beta_2(t),\ \text{on } J$$
$$\alpha_2(t) \not\leq \beta_1(t),\ \text{on } J.$$

Moreover for every $(t, x) \in J \times [-a, b]$, we have

$$|f(t, x, x'))| \leq |g(x')| + m \leq c_2|x'|^p + m_1 = \omega(|x'|), \tag{6.2.16}$$

where $1 \leq p \leq 2$ and $m = \max\{|\psi(x)| : x \in [-a, b]\}$. Since $\int_0^\infty \frac{sds}{\omega(s)} = \int_0^\infty \frac{sds}{c_2 s^p + m_1} = \infty$, f satisfies a Nagumo condition. Hence all the conditions of Theorem 6.2.2 are satisfied, and so the problem has at least three solutions satisfying

$$\alpha_1(t) \leq x_i(t) \leq \beta_2(t),\ t \in J,\ i = 1, 2, 3.$$

6.3 Existence of at least three solutions of three point boundary value problem with super-quadratic growth

In this section we study existence of at least three solutions of some three point boundary value problem of the type

$$\begin{aligned} -x''(t) &= f(t, x, x'),\ t \in J, \\ x'(0) = 0, \quad x(1) &= \delta x(\eta),\ \eta \in (0, 1), \end{aligned} \tag{6.3.1}$$

where $\delta < 1$. We employ a condition weaker than the well known Nagumo condition which allows the nonlinearity $f(t, x, x')$ to grow faster than quadratically with respect to x' in some cases. Our method uses some degree theory arguments. We replace the Nagumo condition by conditions of the type

$$\phi_t(t, x) + \phi_x(t, x)\phi(t, x) + f(t, x, \phi(t, x)) < 0,$$
$$\psi_t(t, x) + \psi_x(t, x)\psi(t, x) + f(t, x, \psi(t, x)) > 0,$$

studied in [11] , to allow higher growth rate of f with respect to x'. Existence of at least one solution for some nonlinear boundary conditions under the above conditions is studied in [87]. Here we study the existence of at least three solutions of the three-point boundary value problems (6.3.1), by using the method of upper and lower solutions and topological degree. We give a class of examples to show that our theorems can be applied to problems which allow $f(t,x,x')$ to have a growth rate $|x'|^p$, where p is not restricted to be less than 2.

Concerning the existence of solutions, we have the following theorem from [11].

Theorem 6.3.1. *Suppose that*

(1) $\alpha,\ \beta \in C^2(J,\mathbb{R})$ *such that* $\alpha(0) = \beta(0)$, *and* $\alpha(t) \le \beta(t)$ *on* J.

(2) $\phi,\ \psi \in C^1(J \times \mathbb{R}, \mathbb{R})$ *with* $\phi(t,x) \le \psi(t,x)$ *on* $J \times [\min \alpha, \max \beta]$ *such that*

$$\phi_t(t,x) + \phi_x(t,x)\phi(t,x) + f(t,x,\phi(t,x)) < 0,$$
$$\psi_t(t,x) + \psi_x(t,x)\psi(t,x) + f(t,x,\psi(t,x)) > 0,$$

for every $(t,x) \in J \times [\min \alpha, \max \beta]$.

(3) $\phi(0,\alpha(0)) \le \alpha'(0),\ \beta'(0) \le \psi(0,\beta(0))$.

Then for any solution x *of the differential equation* (6.3.1) *such that* $\alpha(t) \le x(t) \le \beta(t)$ *on* J, *we have*

$$\phi(t,x(t)) \le x'(t) \le \psi(t,x(t)),\ t \in J.$$

Proof. Let $x \in C^1(J,\mathbb{R})$ be any solution of (6.3.1) such that $\alpha(t) \le x(t) \le \beta(t)$ on J. Then by (1) and (3), it follows that

$$\phi(0,x(0)) \le x'(0) \le \psi(0,x(0)).$$

Define $v(t) = \phi(t,x(t)) - x'(t)$, $t \in J$. Then $v(0) \le 0$. Suppose there exist $t_0 \in (0,1]$ such that

$$v(t_0) > 0.$$

By continuity we choose $t_1,\ t_2 \in [0,t_0]$ with $t_1 < t_2$ such that

$$v(t_1) = 0,\ v(t) \ge 0 \text{ for } t \in [t_1,t_2].$$

Consequently, $v'(t_1) \ge 0$.

On the other hand, using (2), we obtain

$$\begin{aligned} v'(t_1) = (\phi(t,x(t)) - x'(t))'\big|_{t=t_1} &= \phi_t(t_1,x(t_1)) + \phi_x(t_1,x(t_1))\phi(t_1,x(t_1)) - x''(t_1) \\ &= \phi_t(t_1,x(t_1)) + \phi_x(t_1,x(t_1))\phi(t_1,x(t_1)) + f(t_1,x(t_1),\phi(t_1,x(t_1))) < 0, \end{aligned}$$

a contradiction. Thus, $x'(t) \ge \phi(t,x(t))$ for all $t \in [0,1]$. Similarly,

$$x'(t) \le \psi(t,x(t)) \text{ for all } t \in [0,1].$$

□

It is known [28, 92] that the Green's function $G : J \times J \to \mathbb{R}$ for the BVP (6.3.1) is given by

$$G(t,s) = \begin{cases} G_1(t,s), & 0 < \eta \le s \\ G_2(t,s), & 0 < s \le \eta, \end{cases}$$

where $G_i : J \times J \to \mathbb{R}$ are given by

$$G_1(t,s) = \frac{1}{1-\delta}\begin{cases} (1-s), & 0 \le t \le \eta \le s \le 1 \\ (1-\delta s) + (\delta - 1)t, & \eta \le s \le t \le 1, \end{cases}$$

$$G_2(t,s) = \frac{1}{1-\delta}\begin{cases} [(\delta-1)s + (1-\delta\eta)], & 0 \le t \le s \le \eta \\ [(\delta-1)t + (1-\delta\eta)], & 0 \le s \le \eta \le t \le 1. \end{cases}$$

We note that

$$G(t,s) \le \frac{1-s}{1-\delta} \le \frac{1}{1-\delta}.$$

Our main result is the following theorem in which we shall prove existence of at least three solutions of the boundary value problem (6.3.1).

Theorem 6.3.2. *Assume that*

(A_1) $\alpha_1, \alpha_2 \in C^2(J)$ *are two lower solutions and* $\beta_1, \beta_2 \in C^2(J)$ *are two upper solutions of* (6.3.1) *such that*

$$\alpha_1 \le \alpha_2 \le \beta_2,\ \alpha_1 \le \beta_1 \le \beta_2 \text{ and } \alpha_2 \not\le \beta_1 \text{ on } J.$$

(A_2) $f : J \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and satisfies*

$$f(t,x,x') - f(t,x,y') \ge -L(x'-y') \text{ for } x' \ge y',$$

for $(t,x) \in J \times [\min \alpha_1(t), \max \beta_2(t)]$, *where* $L > 0$.

(A_3) *Suppose that there are* $\phi, \psi \in C^1(J \times \mathbb{R}, \mathbb{R})$ *with*

$$\phi(t,x) \le \psi(t,x) \text{ on } J \times [\min \alpha_1, \max \beta_2],$$

such that

$$\phi(t, x(0)) \le 0 \le \psi(t, x(0)),$$

for any solution x *of* (6.3.1), *and that for* $(t,x) \in J \times [\min \alpha_1, \max \beta_2]$,

$$\phi_t(t,x) + \phi_x(t,x)\phi(t,x) + f(t,x,\phi(t,x)) < 0,$$
$$\psi_t(t,x) + \psi_x(t,x)\psi(t,x) + f(t,x,\psi(t,x)) > 0.$$

(A_4) α_2 *and* β_1 *are not solutions of* (6.3.1).

Then the boundary value problem (6.3.1) *has at least three solutions* x_i $(i = 1,2,3)$ *such that*

$$\alpha_1 \le x_1 \le \beta_1,\ \alpha_2 \le x_2 \le \beta_2 \text{ and } x_3 \nleq \beta_1 \text{ and } x_3 \ngeq \alpha_2 \text{ on } J.$$

Proof. By Theorem 6.3.1 (also see lemma 4 [87]), the assumption (A_3) implies that any solution x of (6.3.1) such that $\alpha_1 \le x \le \beta_2$ on J satisfies

$$\phi(t, x(t)) \le x'(t) \le \psi(t, x(t)),\ t \in J.$$

Let $C > \max\{|\phi(t,x(t))|, |\psi(t,x(t))|, |\alpha_1'(t)|, |\beta_2'(t)| : (t,x) \in J \times [\min\alpha_1, \max\beta_2]\}$, and define

$$q(y) = \max\{-C, \min\{y, C\}\},$$

a retraction on to $[-C, C]$. We modify f with respect to α_1 and β_2 to obtain a second boundary value problem and reformulate the new problem as an integral equation. We show that solutions of the modified problem lie in the region where f is unmodified and hence are solutions of the original problem. Let

$$\lambda = \max\{|f(t,x,x') - f(t,x,y')| : t \in J,\ x \in [\min\alpha_1(t), \max\beta_2(t)],\ x', y' \in [-C, C]\},$$

and $\epsilon > 0$ be fixed. Define the modification of f as follows

$$F(t,x,x') = \begin{cases} f(t,\beta_2(t),\beta_2'(t)) + \lambda, & \text{if } x \ge \beta_2(t) + \epsilon, \\ f(t,\beta_2(t),q(x')) + \big[f(t,\beta_2(t),\beta_2'(t)) & \\ \quad - f(t,\beta_2(t),q(x')) + \lambda\big]\frac{x-\beta_2(t)}{\epsilon}, & \text{if } \beta_2(t) \le x < \beta_2(t) + \epsilon, \\ f(t,x,q(x')), & \text{if } \alpha_1(t) \le x \le \beta_2(t), \\ f(t,\alpha_1(t),q(x')) + \big[f(t,\alpha_1(t),\alpha_1'(t)) & \\ \quad - f(t,\alpha_1(t),q(x')) + \lambda\big]\frac{\alpha_1(t)-x}{\epsilon}, & \text{if } \alpha_1(t) - \epsilon < x \le \alpha_1(t), \\ f(t,\alpha_1(t),\alpha_1'(t)) + \lambda, & \text{if } x \le \alpha_1(t) - \epsilon. \end{cases}$$

We note that F is continuous and bounded on $J \times \mathbb{R}^2$, so there exists $M > 0$ such that

$$|F(t,x,x')| \le M \text{ on } J \times \mathbb{R}^2.$$

We may choose M so that $M \ge \max\{\|\alpha_1\|, \|\beta_2\|\}$. Let $M_1 > \frac{M}{1-\delta}$, and consider the modified problem

$$\begin{aligned} -x''(t) &= F(t,x,x'),\ t \in J, \\ x'(0) &= 0, \quad x(1) = \delta x(\eta), \end{aligned} \tag{6.3.2}$$

where $0 < \delta < 1$, $\eta \in (0,1)$. Define $\widehat{F} : C^1(J) \to C(J)$ by

$$\widehat{F}(x)(t) = F(t, x(t), x'(t)),\ t \in J.$$

Then x is a solution of (6.3.2) if and only if x is a fixed point of $L\widehat{F}$. From (A_1), we have

$$F(t,\alpha_1(t),\alpha_1'(t)) = f(t,\alpha_1(t),\alpha_1'(t)) \geq -\alpha_1''(t), \ \ t \in J$$
$$F(t,\beta_2(t),\beta_2'(t)) = f(t,\beta_2(t),\beta_2'(t)) \leq -\beta_2''(t), \ \ t \in J$$

so that α_1 and β_2 are lower and upper solutions of (6.3.2). Moreover, for every $(t,x) \in J \times [\min \alpha_1(t), \max \beta_2(t)]$, by choice of C and (A_3), we have

$$F(t,x,\phi(t,x)) = f(t,x,q(\phi(t,x))) = f(t,x,\phi(t,x)) < -(\phi_t(t,x) + \phi_x(t,x)\phi(t,x)),$$
$$F(t,x,\psi(t,x)) = f(t,x,q(\psi(t,x))) = f(t,x,\psi(t,x)) > -(\psi_t(t,x) + \psi_x(t,x)\psi(t,x)).$$

Thus any solution x of (6.3.2) with $\alpha_1(t) \leq x(t) \leq \beta_2(t)$, $t \in J$ satisfies

$$\phi(t,x(t)) \leq x'(t) \leq \psi(t,x(t)), \ t \in J$$

and hence is a solution of (6.3.1). We now show that any solution x of (6.3.2) does satisfy

$$\alpha_1(t) \leq x(t) \leq \beta_2(t), \ t \in J.$$

For this, set $v(t) = \alpha_1(t) - x(t)$, $t \in J$. Then, $v \in C^2(J)$ and the boundary conditions imply that

$$v'(0) \geq 0, \ v(1) \leq \delta v(\eta). \tag{6.3.3}$$

Suppose that $\alpha_1(t) \not\leq x(t)$ on J, then $v(t) = \alpha_1(t) - x(t)$ has a positive maximum at some $t = t_0 \in J$. If $t_0 \in (0,1)$, then $v'(t_0) = 0$ and $v''(t_0) \leq 0$. However, for $0 < v(t_0) < \epsilon$,

$$\begin{aligned} -v''(t_0) &= -\alpha_1''(t_0) + x''(t_0) \leq f(t_0,\alpha_1(t_0),\alpha_1'(t_0)) - [f(t_0,\alpha_1(t_0),\alpha_1'(t_0)) + \frac{\lambda v(t_0)}{\epsilon}] \\ &= -\frac{\lambda v(t_0)}{\epsilon} < 0, \end{aligned}$$

a contradiction and for $v(t_0) \geq \epsilon$,

$$-v''(t_0) \leq f(t_0,\alpha_1(t_0),\alpha_1'(t_0)) - [f(t_0,\alpha_1(t_0),\alpha_1'(t_0)) + \lambda] = -\lambda < 0,$$

again a contradiction. Thus $v(t)$ has no positive local maximum.
If $t_0 = 1$, then $v(1) > 0$ and $v'(1) \geq 0$. But the boundary condition $v(1) \leq \delta v(\eta)$ implies that $v(\eta) > 0$ and hence $v(1) < v(\eta)$, so that $v(1)$ cannot be the maximum of $v(t)$.

Hence $t_0 = 0$, which implies that $v(0) \geq 0$ and $v'(0) \leq 0$. This together with the boundary condition $v'(0) \geq 0$, gives $v'(0) = 0$.
Case 1: If $0 < v(0) \leq \varepsilon$, then there exists an interval $[0,t_1] \subseteq J$ such that

$$0 < v(t) \leq \varepsilon, \ v'(t) \leq 0, \ t \in [0,t_1],$$

and also

$$x'(t) \geq q(x'(t)) \geq \alpha_1'(t), \ t \in [0,t_1]. \tag{6.3.4}$$

In view of (A_2) and (6.3.4), we have

$$\begin{aligned} v''(t) &= \alpha_1''(t) - x''(t) \geq -f(t, \alpha_1(t), \alpha_1'(t)) + F(t, x, x') \\ &\geq f(t, \alpha_1(t), q(x')) - f(t, \alpha_1(t), \alpha_1'(t)) \\ &\geq -L(q(x') - \alpha_1'(t)) \geq -L(x' - \alpha_1'(t)) = Lv'(t),\ t \in [0, t_1]. \end{aligned}$$

Thus, for each $t \in [0, t_1]$, $v(t)$ satisfies the differential inequalities

$$v''(t) - Lv'(t) \geq 0,\ v'(t) \leq 0,\ v'(0) = 0,$$

that is, $(v'(t)e^{-Lt})' \geq 0$ on $[0, t_1]$ which on integration gives

$$v'(t)e^{-Lt} \geq 0,\ t \in [0, t_1],$$

a contradiction, unless $v'(t) \equiv 0$ on $[0, t_1]$. If $v'(t) \equiv 0$ on $[0, t_1]$, then $x'(t) = \alpha_1'(t)$ on $[0, t_1]$ and hence

$$F(t, x, x') = f(t, \alpha_1(t), \alpha_1'(t)) + \frac{\lambda v(t)}{\varepsilon} > f(t, \alpha_1(t), \alpha_1'(t)).$$

Consequently, $v''(t) > 0$ on $[0, t_1]$, which implies that $v'(t)$ is strictly increasing on $[0, t_1]$, a contradiction.

Case 2: If $v(0) > \varepsilon$, then by continuity of v there exists an interval $[0, t_2] \subset I$ such that

$$v(t) \geq \varepsilon \text{ and } v'(t) \leq 0,\ t \in [0, t_2].$$

Now, for each $t \in [0, t_2]$, we have

$$F(t, x, x') = f(t, \alpha_1(t), \alpha_1'(t)) + \lambda > f(t, \alpha_1(t), \alpha_1'(t)).$$

Thus

$$v''(t) = \alpha_1''(t) - x''(t) \geq -f(t, \alpha_1(t), \alpha_1'(t)) + F(t, x, x') > 0,\ t \in [0, t_2],$$

which implies that $v'(t)$ is strictly increasing on $[0, t_2]$ and hence $v'(t) > v'(0) = 0$ for $t > 0$, again a contradiction. Thus $v(t) \leq 0$ on J.

Similarly, we can show that $x(t) \leq \beta_2(t)$, $t \in J$.

Thus, it is suffices to show that (6.3.2) has at least three solutions x_i such that

$$\alpha_1(t) \leq x_i(t) \leq \beta_2(t),\ t \in J,\ i = 1, 2, 3.$$

Since $L\widehat{F}(\overline{\Omega}) \subset \Omega$, where $\Omega = \{x \in C^1(J) : |x(t)| < M_1, |x'(t)| < C,\ t \in J\}$ is bounded and convex open subset of $C^1(J)$. It follows that the degree

$$d(I - L\widehat{F}, \Omega, 0) = 1.$$

Let

$$\Omega_{\alpha_2} = \{x \in \Omega : x > \alpha_2 \text{ on } (0,1)\} \text{ and } \Omega^{\beta_1} = \{x \in \Omega : x < \beta_1 \text{ on } (0,1)\}.$$

Therefore $\overline{\Omega}_{\alpha_2} \cap \overline{\Omega}^{\beta_1} = \emptyset$ and, since $\alpha_2 \not\leq \beta_1$ on J, the set $\Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega_1^{\beta}}$ is not empty. By (A_4) and choice of M_1, C, there are no solutions on $\partial\Omega_{\alpha_2} \cup \partial\Omega^{\beta_1}$. The additivity of degree implies that

$$\begin{aligned} d(I - L\widehat{F}, \Omega, 0) &= d(I - L\widehat{F}, \Omega_{\alpha_2}, 0) + d(I - L\widehat{F}, \Omega^{\beta_1}, 0) \\ &\quad + d(I - L\widehat{F}, \Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega_1^{\beta}}, 0). \end{aligned} \tag{6.3.5}$$

Now we show that $d(I - L\widehat{F}, \Omega_{\alpha_2}, 0) = d(I - L\widehat{F}, \Omega^{\beta_1}, 0) = 1$. Firstly, we show that

$$d(I - L\widehat{F}, \Omega_{\alpha_2}, 0) = 1.$$

Define $F_2(t, x, x')$ as follows

$$F_2(t,x,x') = \begin{cases} f(t, \beta_2(t), \beta_2'(t)) + \lambda, & \text{if } x \geq \beta_2(t) + \varepsilon, \\ f(t, \beta_2(t), q(x')) + \big[f(t, \beta_2(t), \beta_2'(t)) & \\ \quad - f(t, \beta_2(t), q(x')) + \lambda\big] \frac{x - \beta_2(t)}{\varepsilon}, & \text{if } \beta_2(t) \leq x \leq \beta_2(t) + \varepsilon, \\ f(t, x, q(x')), & \text{if } \alpha_2(t) \leq x \leq \beta_2(t), \\ f(t, \alpha_2(t), q(x')) + \big[f(t, \alpha_2(t), \alpha_2'(t)) & \\ \quad - f(t, \alpha_2(t), q(x')) + \lambda\big] \frac{\alpha_2(t) - x}{\varepsilon}, & \text{if } \alpha_2(t) - \varepsilon \leq x \leq \alpha_2(t), \\ f(t, \alpha_2(t), \alpha_2'(t)) + \lambda, & \text{if } x \leq \alpha_2(t) - \varepsilon. \end{cases}$$

We note that $F_2 = F$ on Ω_{α_2}. Now we consider the problem

$$\begin{aligned} -x''(t) &= F_2(t, x, x'),\ t \in J, \\ x'(0) = 0, &\quad x(1) = \delta x(\eta), \end{aligned} \tag{6.3.6}$$

where $0 < \delta < 1$, $\eta \in (0, 1)$. Then (6.3.6) is equivalent to

$$(I - L\widehat{F}_2)x = 0, \tag{6.3.7}$$

where $\widehat{F}_2(x)(t) = F_2(t, x(t), x'(t))$, $t \in J$.

By the same process as we did for the problem (6.3.2), we can show that any solution x of (6.3.6) satisfies $x \geq \alpha_2$ on J, which in view of (A_4) implies that $x \neq \alpha_2$ and hence belongs to Ω_{α_2}. Since $L\widehat{F}_2(\overline{\Omega}) \subseteq \Omega$, we have

$$d(I - L\widehat{F}_2, \Omega, 0) = 1.$$

Now $d(I - L\widehat{F}, \Omega_{\alpha_2}, 0) = d(I - L\widehat{F}_2, \Omega_{\alpha_2}, 0)$. It follows that

$$1 = d(I - L\widehat{F}_2, \Omega, 0) = d(I - L\widehat{F}_2, \Omega_{\alpha_2}, 0) + d(I - L\widehat{F}_2, \Omega \setminus \overline{\Omega}_{\alpha_2}, 0) = d(I - L\widehat{F}_2, \Omega_{\alpha_2}, 0).$$

Similarly, we can show that $d(I - L\widehat{F}, \Omega^{\beta_1}, 0) = 1$. Thus from (6.3.5), we obtain

$$d(I - L\widehat{F}, \Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega_1^{\beta}}, 0) = -1.$$

Hence there are 3 solutions, one in each of the sets Ω_{α_2}, Ω^{β_1} and $\Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega_1^{\beta}}$. □

We now give a class of examples which illustrate that we can allow growth in x' larger than quadratic.

Example 6.3.3. Consider the following boundary value problem

$$-x''(t) = f(t,x,x') = g(x') + h(x),\ t \in J, \tag{6.3.8}$$

$$x'(0) = 0,\ x(1) = \delta x(\eta),\ \delta < 1,\ \eta \le \sqrt{3}/2, \tag{6.3.9}$$

where $g,\ h : \mathbb{R} \to \mathbb{R}$ are continuous and are such that $g(0) = 0$, $h(0) < 0$ and

$$h(-a) > 0,\ h(b) < 0,$$
$$g(-a) < -M,\ g(b) > M,$$

where $a > 0$, $b > 0$ and $M = \max\{|h(x)| : x \in [-a,b]\}$. Further for small c, $0 < c < a/8$, we assume that g and h satisfy

$$\begin{aligned} g(y) &\ge 4c \text{ for } -2c \le y \le -c,\\ g(y) &\ge 0 \text{ for } -c \le y < 0\\ h(x) &\ge -2c \text{ for } -c/4 \le x \le c/2\\ h(x) &\ge 2c \text{ for } c/2 \le x \le 3c/4. \end{aligned} \tag{6.3.10}$$

Take

$$\alpha_1(t) = -a, \quad \alpha_2(t) = c(3/4 - t^2).$$

Then $\alpha_1, \alpha_2 \in C^2(J)$ and satisfy the boundary conditions

$$\alpha_1'(0) = 0,\ \ \alpha_1(1) < \delta\alpha_1(\eta)$$
$$\alpha_2'(0) = 0,\ \ \alpha_2(1) < \delta\alpha_2(\eta).$$

Moreover, for every $t \in (0,1]$, we have

$$\alpha_1''(t) + f(t,\alpha_1(t),\alpha_1'(t)) = g(0) + h(-a) > 0,$$

and conditions imposed in (6.3.10), we have

$$\alpha_2''(t) + f(t,\alpha_2(t),\alpha_2'(t)) = -2c + g(-2ct) + h(c(3/4 - t^2)) \ge 0.$$

Thus, α_1 and α_2 are lower solutions of (6.3.8).
Now, take $\beta_1 = 0$ and $\beta_2 = b$, then β_1, $\beta_2 \in C^2(J)$ and satisfy the boundary conditions

$$\beta_1'(0) = 0,\ \beta_1(1) = \delta\beta_1(\eta),$$
$$\beta_2'(0) = 0,\ \beta_2(1) > \delta\beta_2(\eta).$$

Moreover,

$$\beta_1''(t) + f(t,\beta_1(t),\beta_1'(t)) = g(0) + h(0) < 0,$$
$$\beta_2''(t) + f(t,\beta_2(t),\beta_2'(t)) = g(0) + h(b) < 0.$$

Thus, β_1 and β_2 are upper solutions of (6.3.8). Further, we note that

$$\alpha_1(t) \leq \alpha_2(t) \leq \beta_2(t) \text{ on } J$$
$$\alpha_1(t) \leq \beta_1(t) \leq \beta_2(t) \text{ on } J$$
$$\alpha_2(t) \not\leq \beta_1(t) \text{ on } J.$$

Take $\phi(t,x) = -a$, then for $x \in [-a,b]$, we have

$$\phi_t(t,x) + \phi_x(t,x)\phi(t,x) + f(t,x,\phi(t,x)) = g(-a) + h(x) < 0. \tag{6.3.11}$$

Take $\psi(t,x) = b$, then for $x \in [-a,b]$, we have

$$\psi_t(t,x) + \psi_x(t,x)\psi(t,x) + f(t,x,\psi(t,x)) = g(b) + h(x) > 0. \tag{6.3.12}$$

Thus all the conditions of Theorem 6.3.2 are satisfied, and so the problem has at least three solutions satisfying

$$\alpha_1(t) \leq x_i(t) \leq \beta_2(t) \text{ and } -a \leq x_i'(t) \leq b,\ t \in J,\ i = 1,2,3.$$

We note that, $g(x')$ can behave like $|x'|^p$ for $|x'|$ large, so for $p > 2$, f need not satisfy the Nagumo condition, so the results of [35] do not apply to this problem for such function g with $p > 2$.

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