Analysis of the Soliton Solutions of a 3-Level Maxwell-Bloch System with Rotational Symmetry.

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A thesis submitted in September 2004 to the Faculty of Engineering at the University of Glasgow for the degree of Ph.D.

Abstract

The dynamics of soliton pulses for use in nonlinear optical devices is mathematically modelled by Maxwell-Bloch systems of equations for the interaction of light with a uniform distribution of quantum-mechanical atoms. We study the Reduced Maxwell-Bloch (RMB) equations occuring when an ensemble of rotationally symmetric 3-level atoms is assumed. The model applies for on and off-resonance conditions and is completely integrable using Inverse Scattering theory, since it arises as the compatibility condition of a 3×3 AKNS-system. Furthermore this integrability remains valid for all timescales of the optical field because only the "one-way wave approximation" is required during the derivation. Solutions are constructed in two ways: 1. Darboux-Bäcklund transforms are applied, generating single soliton pulses of ultrashort (< 1ps) duration, and families of elliptically polarised 2-solitons not possible in lower dimensional problems. 2. A general Inverse Scattering scheme is developed and tested. The Direct Scattering Problem is dealt with first to obtain a complete set of scattering data. Subsequently the Inverse Problem is solved both formally and then in explicit closed form for the special case that the reflection coefficients vanish for real values of the spectral parameter. In this case the main result is a determined system of n linear algebraic equations which yield the n-soliton solution of our RMB-system. It is confirmed that the 1-solitons found by means of Darboux transforms are precisely the same as those given by the full mechanism of Inverse Scattering. Finally we calculate the invariants of the motion for the RMB-equations, and derive an evolution equation giving the variation with propagation distance of the invariant functionals when the original RMB-system is modified by an arbitrary perturbing term. As an application dissipative effects on 1-solitons are considered.

Contents

Chapter 1

Introduction.

1.1 Solitary Waves and Solitons.

As an essential preliminary we must first define the notion of a soliton. Referring to Ablowitz & Clarkson [1] p. 13, pp. 18-19 we make the definitions below:

A *solitary wave* solution of a nonlinear partial differential equation

$$N\left(x, t, u(x, t)\right) = 0,$$

where $x \in \mathbf{R}$, $t \in \mathbf{R}$ are spatial and temporal variables and $u \in \mathbf{R}$ is the dependent variable, is a travelling wave solution of the form

$$u(x, t) = f(x - ct) = f(\xi)$$

whose transition is from one constant asymptotic state as $\xi \to -\infty$, to (possibly) another constant asymptotic state as $\xi \to +\infty$.

A *soliton* is a localised solitary wave solution of a nonlinear equation (or system of equations) which asymptotically preserves its shape and velocity upon nonlinear interaction with other solitary waves, or more generally with another (arbitrary) localised disturbance.

Later (cf. p. 14) the more formal concept of an "n-soliton" will be introduced in the context of the Inverse Scattering Transform method (which is thoroughly described in Chapter 2).

1.2 Solitons in Nonlinear Optical Media.

A significant motivating factor for research into the solitary wave solutions of Maxwell-Bloch (MB) systems of differential equations was the original suggestion by Hasegawa and Tappert (1973) [2] that optical solitons could be used as bits in high capacity digital nonlinear communication systems. Basically Hasegawa and Tappert's idea was that given sufficiently high optical power from a laser source, short stable soliton pulses suitable for data transmission could be produced in a silica glass fibre, as a result of the balance arising between the effects of dispersion and the induced Kerr nonlinear interaction. Now the chief mathematical model representing this process is governed by the Nonlinear Schrödinger equation. However, this equation is just one reduction of the Maxwell-Bloch system for the interaction of an electromagnetic plane wave with a uniform distribution of quantum-mechanical atoms through dipole polarisation. In fact, if the atoms are assumed simply to have two levels, a ground state and a single upper energy state, then there are three cases:

1. The incident carrier wave is off-resonance with the atomic transition, and a Slowly Varying Envelope or Rotating Wave approximation is made.

2. The incident carrier wave is on-resonance, and an envelope approximation is made.

3. The incident carrier wave can have any frequency, and the atomic ensemble is supposed to have low density, allowing use of the "one-way wave approximation" (neglect of backscattered waves).

Cases 1 and 2 lead to the Nonlinear Schrödinger equation and the sine-Gordon equation respectively, which are famous nonlinear partial differential equations for complex wave envelopes. Both of these equations are integrable (i.e., analytically solvable) by the Inverse Scattering Transform [3] [4] and have well documented soliton solutions. In particular, because of its relevance to fibre optic telecommunications, the Nonlinear Schrödinger equation has been exhaustively studied. We have included a brief discussion of soliton communication systems for background interest: see Section 1.4. Case 3 leads to the so-called 2-level Reduced Maxwell-Bloch (RMB) equations which have importance modelling Self Induced Transparency effects, where powerful ultrashort optical pulses are propagated without energy loss through a resonant dielectric medium. The Inverse Scattering scheme and solitons of this system were found by Bullough, Caudrey, Eilbeck and Gibbon (1973) [5] [6] [7].

A substantially more complicated and less well researched problem is to integrate the Maxwell-Bloch equations occuring when the constituent atoms of the ensemble have three distinct energy levels. We shall be investigating the soliton solutions of a recently discovered [8] exactly integrable 3-level Maxwell-Bloch system: the system in question is essentially an extension of the 2-level RMB-equations to allow circularly and elliptically polarised solutions, and is derived by assuming each of the individual atoms is rotationally symmetric with a zero level and two separate orthogonal upper levels. As a general model it answers the need for an analytically solvable basis with which to study near resonant interactions in non-insulating optical media, such as doped optical fibres, semi-conductors or vapours. Indeed its most pertinent application is probably in the description of pulse formation within laser cavities. We note that whilst there exist other examples of integrable 3-level MB-systems [9] [10] [11], their formulation invariably necessitates envelope approximations. On the other hand, as will be seen in Chapter 3, the 3-level RMB-equations with rotational symmetry retain their integrability right down to the carrier timescale, thus allowing physical interpretation of solutions at timescales as short as one optical cycle. Furthermore, in contrast to the

2-level model where by assumption there is no opportunity for the existence of orthogonal dipoles, soliton solutions of the 3-level equations may be circularly or elliptically polarised.

1.3 Objectives and Layout.

The main purposes of this thesis may be summarised as follows:

- To devise a general integration scheme for the 3-level RMB-equations with rotational symmetry using the Inverse Scattering Transform method and obtaining formulae for the n-soliton solution ($\mathrm{N} \ni n \geq 1$).
- To find a way of evaluating the effects of non-Hamiltonian perturbations on the soliton parameters.

These two problems are addressed in Chapters 5 and 7 respectively, after the requisite groundwork has been covered.

In Chapter 4 we employ Darboux-Bäcklund transforms to calculate expressions for the fully polarised 2-solitons of the RMB-system. Darboux transforms give a useful and relatively slick method of aquiring pure multi-soliton solutions of integrable nonlinear equations. Unfortunately though, this is their only function, whereas the techniques of Inverse Scattering provide a general solution to the initial value problem with rapidly decreasing boundary conditions defined in Chapter 3.

The contents of the remaining chapters are clear enough from their titles, so we do not comment further.

In order to make clear the breakdown of original work contained in our thesis, we state that the RMB-equations were derived and proved to be completely integrable by Arnold in 2001 [8]. The basic linearly polarised 1 and 2-soliton solutions were also calculated via Darboux Transform methods at this time. Much of Chapters 3 and 4 therefore amounts to a verification of Arnold's results. However, the calculation to find elliptically polarised solitons is our own, and these formulae are previously unpublished. All the theory in Chapters 5, 6 and 7 is our own work, although as we readily acknowledge, it generalises (in a nontrivial way) mathematical theory due to Faddeev & Takhtajan [12] (Chapter 5), Ablowitz & Clarkson [1] (Chapter 5), and Elgin [13] (Chapter 7).

Regarding equation numbering, we mention that at the beginning of each new chapter, the numbering starts again at (1). If in one chapter an equation from another chapter is referred to, then a page number is included with the reference.

1.4 Optical Soliton Communication Systems.

Following the first observation of an optical fibre soliton by Mollenauer *et al.* (1980) [14], the practical science and theory of soliton communication systems has developed swiftly. In fact, from a wholly scientific perspective, there are no real obstacles to manufacturing multichannel soliton links with each channel offering error-free data rates of more than $10\mathrm{Gbs}^{-1}$ over transatlantic distances: see for example [15]. Nevertheless, commercially speaking these links are currently unviable due simply to lack of demand on the capacity of the fibre optic network already in place. Presumably as demand overrides capacity during the next five-ten years, commercial interest will be reignited. Today there exists just one "dispersion-managed" (see below) soliton transmission system in regular use, connecting Perth and Adelaide in Australia.

By comparison, because of various relaxation effects, absorption and dissipation, the creation of true solitons in near resonant media is much more difficult, and there has been little experimental success. Hence a data transmission system using the characteristic ultrashort soliton pulses predicted by the RMB-equations continues to be only a theoretical possibility.

The efficiency of a typical soliton link is limited by a number of factors; the most important being the effects of dispersion and optical power loss on the information carrying pulse together with any accompanying background radiation. Since fibre losses eventually cause the nonlinear interaction creating the solitons to become negligible, damaging the integrity of a transmitted pulse stream, optical amplifiers are inserted at intervals of roughly 50km. Erbium Doped Fibre Amplifiers (EDFAs) are chosen to do the job, and for the most part they work very well. However, in 1986 Gordon & Haus [16] showed that spontaneous emission of light from the amplifiers created a timing jitter in the output pulse, resulting in serious errors at the receiver over long propagation distances. One way of dealing with this problem is to place optical filters after the amplifiers to eliminate wide deviations in the frequency content of each soliton pulse [17]. However the high cost of these filters is preclusive. A better and more recent solution has been the introduction of dispersion-managed systems, where the fibre link consists of alternate lengths of normally and anomalously dispersive fibre. Multiple channels are accomodated through the technique of Wavelengh-Division-Multiplexing (WDM) [18] [19] [20]. Modulated soliton pulse streams, whose carrier wavelength separation is chosen to minimize the timing jitters caused by soliton collisions and "four-wave mixing", are combined and transmitted down the fibre. Then at the receiving end filters or gratings split the wavelengths up again so that each carrier falls on a separate detector.

Chapter 2

Mathematical Methods for Solving Nonlinear Evolution Equations.

Here we shall describe in some detail the two main techniques which will be used later to find solutions of the initial value-boundary value problem for the RMB-system.

2.1 The Inverse Scattering Transform (IST).

2.1.1 Introduction.

In 1967 Gardner, Greene, Kruskal and Miura [21] published a now famous paper describing an exact method of solving the Korteweg-de Vries (KdV) equation. This caused considerable excitement as the KdV equation is a nonlinear PDE and is therefore unsusceptible to solution via standard methods for linear PDEs, such as the application of Fourier transforms. Intense research over the following years, notably influenced by Lax (1968) [22], resulted in the so-called *Inverse Scattering Transform*, a sophisticated procedure for solving initial value problems for a large number of physically important nonlinear partial differential equations. In particular, amongst many others, the Nonlinear Schrödinger, sine-Gordon, Boussinesq, Kadomtsev-Petviashvili, Three-wave interaction, Davey-Stewartson, and Self-dual Yang-Mills equations are all solvable using variations of the IST method.

Several comprehensive textbooks are available dealing with all aspects of the theory of the transform, its historical development and applications. Especial use will be made of the treatments written by Faddeev and Takhtajan (abbreviated to F&T. subsequently) (1986) [12], and by Ablowitz and Clarkson (A&C.) (1991) [1].

We now begin with some basic concepts and definitions.

A given system of r ($\mathbb{N} \ni r \geq 1$) nonlinear PDEs is said to be *completely integrable* or simply *integrable* if it is exactly solvable by the Inverse Scattering Transform. Suppose our equations are for an unknown vector function $\mathbf{q} = (q_1, ..., q_r)$ of one spatial variable x, and one temporal variable t. Then Ablowitz, Kaup, Newell and Segur (AKNS) (1973b, 1974) [23] and Ablowitz and Haberman (1975b) [24] proved that a wide class of completely integrable systems could be represented as the compatibility condition of the following overdetermined pair of linear equations:

$$
\begin{aligned}
\partial_x F &= UF, \qquad (1)\\
\partial_t F &= VF, \qquad (2)
\end{aligned}
$$

where $U = U(x,t,\zeta)$, $V = V(x,t,\zeta)$ are $N \times N$ complex matrices dependent on q_j, $j = 1, ..., r$ and their derivatives in a way to be defined, $F = F(x,t) = (F_1, ..., F_N)^t$ is an $N \times 1$ complex column vector, and ζ is a complex spectral parameter. Equating $F_{xt} = F_{tx}$ implies that for all ζ

$$
U_t - V_x + [U, V] = 0, \qquad (3)
$$

which is often called the *zero curvature compatibility condition* (cf. F&T. pp. 21-22). Here $[U, V]$ denotes the matrix commutator $UV - VU$. The left-hand side is, in general, an analytic function of ζ with poles, having coefficients depending on the components of $\mathbf{q}(x,t)$. It vanishes identically as long as the original nonlinear PDEs are satisfied.

If $N = 2$, then a typical form for U (AKNS [23]) is

$$U(x,t,\zeta) = \begin{pmatrix} -\iota\zeta & q_1(x,t) \\ q_2(x,t) & \iota\zeta \end{pmatrix}, \qquad (4)$$

where $\iota := \sqrt{-1}$. The elements of V are functions of q_1, q_2 and ζ, independent of the components F_1, F_2 of F.

In the general $N \times N$ case ([24])

$$U(x,t,\zeta) = \iota\zeta J + Q(x,t), \qquad (5)$$

where $J = \mathrm{diag}(J_1, ..., J_N)$, with $J_k \neq J_l$ for $k \neq l$, $k,l = 1,...,N$ and Q is off-diagonal and dependent on the components of $\mathbf{q}$. (As for the 2×2 case the elements of V are functions of the components of $\mathbf{q}$ and ζ, independent of $F_1, ..., F_N$).

Notice that the above forms for U are not all-encompassing: for instance a more complicated ζ-dependence might be chosen, leading to a different class of integrable equations. They are however quite sufficient for our study of the RMB-system.

2.1.2 Inverse Scattering Transform for 2×2 AKNS-Systems.

Suppose that we wish to use the IST to solve the Cauchy problem for a given determined system of nonlinear PDEs, which is known to be equivalent to the compatibility condition of the AKNS pair of differential equations

$$\partial_x F = UF = \begin{pmatrix} -\iota\zeta & q(x,t) \\ r(x,t) & \iota\zeta \end{pmatrix} F, \ \partial_t F = VF. \qquad (6,7)$$

Fixing $t = 0$ in the first equation (6), we have a linear ordinary differential equation containing the spectral parameter ζ linearly.

The first step in applying the IST is to solve this 2×2 scattering problem (which may also be referred to as the *auxiliary linear problem*). That is, knowing $q(x, 0)$, $r(x, 0)$ and assuming q and r are absolutely integrable functions of rapidly decreasing type at infinity, we find the eigenfunctions of equation (6). If certain asymptotic conditions are satisfied at infinity, then these eigenfunctions are called *Jost solutions*. Their behaviour as $|x| \to \infty$ determines scattering data consisting of two components $s(\zeta)$ and s_n corresponding to continuous and discrete spectra, with n standing for a number of discrete eigenvalues. Since the operator U depends on q and r we can define a map

$$\{q(x,0), r(x,0)\} \longmapsto \{s(\zeta,0), s_n(0)\}.$$

Secondly, temporal evolution of q and r according to the original system of nonlinear PDEs, generates evolution of the scattering data through equation (7), the time-part of the AKNS pair. In general the time dependence of the scattering data has the following trivial form

$$\partial_t s(\zeta,t) = \iota\omega(\zeta) s(\zeta,t), \quad \frac{d}{dt} s_n(t) = \omega_n s_n(t),$$

where some of the ω_n's or $\omega(\zeta)$ may be equal to zero.

Finally the most difficult step is to solve the *inverse scattering problem*, which means reconstructing the potentials $q(x,t), r(x,t)$ from the set of scattering data $\{s(\zeta,t), s_n(t)\}$ at an arbitrary time t. As we shall see, solving the inverse scattering problem is equivalent to solving a Riemann-Hilbert boundary value problem in scattering space, and is achieved by the application of a certain projection operator.

We point out that in every case the discrete spectrum scattering data determines solitons, whereas the continuous spectrum scattering data determines a background radiation field. Potentials $q(x), r(x)$ to which there corresponds non-trivial discrete spectrum data $s_n(t)$, but for which the continuous spectrum data implies vanishing of the background radiation are termed *reflectionless*. They evolve into pure "n-soliton solutions" through the mechanism of the IST.

Next we look more closely at the calculations involved in the two principle stages of the transform, i.e., the direct and inverse scattering problems. (Finding the time dependence of the scattering data is straightforward by comparison). Our presentation here amounts to a summary of theory extracted from A&C., Chapter 3, pp. 105-9, and from F&T., Chapter 1, pp. 11-55. Since results from both solution methods will be applied during Chapter 5, a remark comparing the strongly contrasting notation schemes has been included.

2.1.3 The 2×2 Direct Scattering Problem.

Consider once again the auxiliary linear equation

$$\frac{dF}{dx} = U(x, t = 0, \zeta)F. \qquad (8)$$

Typical assumptions on the potential functions q and r are that they are infinitely differentiable, absolutely integrable and satisfy

$$\int_{-\infty}^{\infty} |x|^n |q(x)|\, dx, \int_{-\infty}^{\infty} |x|^n |r(x)|\, dx < \infty, \ \forall n \in \mathbb{N}.$$

Now let γ be the line segment $[y, x]$ $(y \leq x)$ on the x-axis, and partition γ into a large number of tiny adjacent segments $\gamma_1, ..., \gamma_R$, say. Then the *transition matrix* is defined to be the matrix of parallel transport (cf. F&T. p. 22, p. 26) from y to x along the x-axis:

$$T(x, y, \zeta) := \overleftarrow{\exp} \int_y^x U(z, \zeta) dz := \lim_{R \to \infty} \left\{ \left(I + \int_{\gamma_R} U dz \right) ... \left(I + \int_{\gamma_1} U dz \right) \right\}.$$

We also note that $T_L(\zeta) := T(L, -L, \zeta)$ is called the *monodromy matrix*.

The transition matrix is fundamentally useful for solving the direct scattering problem because it satisfies the auxiliary differential equation (8):

$$\partial_x T(x, y, \zeta) = U(x, \zeta) T(x, y, \zeta), \qquad (9)$$

with initial condition

$$[T(x, y, \zeta)]_{x=y} = I. \qquad (10)$$

When $q(x) = 0 = r(x)$, the solution of $(9, 10)$ is simply

$$E(x - y, \zeta) := \exp\left[\iota(x - y)\zeta \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

By virtue of the assumptions on q and r, the Jost solution matrices $T_\pm$ exist and are defined as the limit

$$T_\pm(x, \zeta) = \lim_{y \to \pm\infty} \{T(x, y, \zeta) E(y, \zeta)\}, \text{ for } \zeta \in \mathbf{R}.$$

(cf. F&T. pp. 39-42). Furthermore we have that

$$\frac{d}{dx} T_\pm = U(x, \zeta) T_\pm(x, \zeta), \qquad (11)$$

with

$$T_\pm(x, \zeta) = E(x, \zeta) + o(1), \qquad (12)$$

as $x \to \pm\infty$.

In other words the columns of $T_\pm(x, \zeta) = (T_\pm^{(1)}, T_\pm^{(2)})(x, \zeta)$ say, are eigenfunctions of our 2×2 scattering problem. Since $\mathrm{Tr}\,[U(x, \zeta)] = 0$ (where Tr denotes the matrix trace function), it follows (F&T. p. 27) that

$$\det T(x,y,\zeta) = 1 = \det T_{\pm}(x,\zeta),$$

and so consequently $\left\{T_{\pm}^{(1)}, T_{\pm}^{(2)}\right\}$ are linearly independent sets.

Finally, for $\zeta \in \mathbf{R}$ the following limit exists (F&T. p. 44) and is called the *reduced monodromy matrix*:

$$T(\zeta) = \lim_{x\to\infty,\, y\to-\infty} \{E(-x,\zeta)T(x,y,\zeta)E(y,\zeta)\}.$$

$T(\zeta)$ satisfies $T(\zeta) = T_{+}^{-1}(x,\zeta)T_{-}(x,\zeta)$, i.e.,

$$T_{-}(x,\zeta) = T_{+}(x,\zeta)T(\zeta), \qquad (13)$$

which symbolises the completeness relationship between the columns of the Jost solutions. Elements $T^{ij}(\zeta)$ of the reduced monodromy matrix are called *transition coefficients*. Observe that it is crucial to find all symmetries of the matrix U which respect the definition of the transition matrix. These symmetries are referred to as *involution relations* (F&T. p. 27) and they extend through the whole analysis of the direct scattering problem, minimising the number of independent transition coefficients.

This is a good point at which to see through the notational differences mentioned between the solution methods in F&T. and A&C.. Comparing the asymptotic behaviour of $T_{\pm}$ given by (12) with A&C. (3.1.3) p. 105, i.e.,

$$(\widetilde{\psi},\psi) \sim \begin{pmatrix} e^{-\iota\zeta x} & 0 \\ 0 & e^{\iota\zeta x} \end{pmatrix}, \text{ as } x \to +\infty,$$
$$(\phi,\widetilde{\phi}) \sim \begin{pmatrix} e^{-\iota\zeta x} & 0 \\ 0 & -e^{\iota\zeta x} \end{pmatrix}, \text{ as } x \to -\infty, \qquad (14)$$

and comparing the relation (13) with A&C. (3.1.4) p. 105, i.e.,

$$(\phi,\tilde{\phi}) = (\tilde{\psi},\psi)\begin{pmatrix} a & \tilde{b} \\ b & -\tilde{a} \end{pmatrix}, \qquad (15)$$

it is evident that Ablowitz & Clarkson's eigenfunctions ψ, $\tilde{\psi}$ and ϕ, $\tilde{\phi}$ of $d_xF = UF$ simply correspond to the columns of the Jost solution matrices T_+ and T_- (albeit the normalisation differs trivially). Note that the use of tildas rather than bars in (14) and (15) avoids confusion with the complex conjugate.

Now that the eigenfunctions required have been characterised, they can be explicitly found at $t = 0$ by re-expressing the auxiliary differential equation (8) as a linear integral equation. In fact for our case $U = \begin{pmatrix} -\iota\zeta & q \\ r & \iota\zeta \end{pmatrix}$, the transition matrix $T(x, y, \zeta)$ may be represented as the solution of a Volterra integral equation, whose iterations are of course absolutely convergent. Appropriate limits are then taken in order to obtain integral representations for $T_\pm(x,\zeta)$ and $T(\zeta)$ (cf. F&T. p. 30, pp. 39-41, pp. 46-48).

Lastly, analytic properties of the Jost solutions and the transition coefficients, deduced from their integral representations, in conjunction with the completeness relation (13) imply the discrete and continuous spectrum scattering data.

To be more explicit, consider for example the Nonlinear Schrödinger equation dealt with in Chapter 1 of Faddeev & Takhtajan's book ([12]), where the matrix U of the auxiliary linear equation has the form

$$U(x,\lambda) = \begin{pmatrix} -\iota\lambda/2 & \iota\sqrt{|\chi|}\overline{\psi}(x) \\ \iota\sqrt{|\chi|}\psi(x) & \iota\lambda/2 \end{pmatrix},$$

and the reduced monodromy matrix may be written

$$T(\lambda) = \begin{pmatrix} a(\lambda) & -\overline{b}(\lambda) \\ b(\lambda) & \overline{a}(\lambda) \end{pmatrix}.$$

Here $\psi(x) = \psi(x, t = 0)$ is a known complex-valued function, λ is the arbitrary complex parameter, and the bars denote complex conjugation.

In this (generic) case the discrete spectrum data (λ complex) comprises:

- A set of discrete eigenvalues $\{\lambda_1, ..., \lambda_n\}$ which are the zeros of the transition coefficient $a(\lambda)$ in the upper half λ-plane, together with their complex conjugates $\left\{\overline{\lambda_1}, ..., \overline{\lambda_n}\right\}$.
- $2n$ complex coefficients $\{\gamma_j, \overline{\gamma_j}\,|j = 1, ..., n\}$ called transition coefficients for the discrete spectrum. They are proportionality coefficients existing between the columns of the Jost solution matrices $T_{\pm}(x, \lambda)$ at $\lambda = \lambda_j$ and $\lambda = \overline{\lambda_j}, j = 1, ..., n$.

Whilst the continuous spectrum data just consists of the set $\left\{b(\lambda), \bar{b}(\lambda)\,|\lambda \in \mathbf{R}\right\}$. ($a(\lambda)$, $\lambda \in \mathbf{R}$ can be expressed in terms of its zeros $\lambda_1, ..., \lambda_n$ via a "dispersion relation": cf. F&T. pp. 50-51).

2.1.4 The 2×2 Inverse Scattering Problem.

By taking into account the known analytic properties of $T_{\pm}(x, \zeta)$ and $T(\zeta)$, the completeness relation $T_{-}(x, \zeta) = T_{+}(x, \zeta)T(\zeta)$ will be rewritten and reinterpreted as a vector Riemann-Hilbert problem in scattering space, which can then be formally solved at time t using projections.

Following A&C. pp. 105-109 we suppose the Jost solutions now have the asymptotic behaviour (14) and satisfy the completeness relation (15). Let N, $\widetilde{N}$, M, $\widetilde{M}$ be the modified eigenfunctions with constant boundary conditions defined by

$$\begin{aligned}(\widetilde{N}(x,\zeta), N(x,\zeta)) &= (\widetilde{\psi}(x,\zeta), \psi(x,\zeta)) \begin{pmatrix} e^{\iota\zeta x} & 0 \\ 0 & e^{-\iota\zeta x} \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ as } x \to +\infty,\end{aligned}$$

$$
\begin{aligned}
(M(x,\zeta),\widetilde{M}(x,\zeta)) &= (\phi(x,\zeta),\tilde{\phi}(x,\zeta))\begin{pmatrix} e^{\iota\zeta x} & 0 \\ 0 & e^{-\iota\zeta x}\end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ as } x \to -\infty.
\end{aligned}
$$

Then (15) becomes

$$
(M,\widetilde{M}) = (\widetilde{N},N)\begin{pmatrix} a & \tilde{b}e^{-2\iota\zeta x} \\ be^{2\iota\zeta x} & -\tilde{a}\end{pmatrix},
$$

or componentwise

$$
\frac{M(x,\zeta)}{a(\zeta)} = \widetilde{N}(x,\zeta) + \rho(\zeta)e^{2\iota\zeta x}N(x,\zeta), \quad (16)
$$

$$
\frac{\widetilde{M}(x,\zeta)}{\tilde{a}(\zeta)} = \tilde{\rho}(\zeta)e^{-2\iota\zeta x}\widetilde{N}(x,\zeta) - N(x,\zeta), \quad (17)
$$

where

$$
\rho(\zeta) = b(\zeta)/a(\zeta),\ \tilde{\rho}(\zeta) = \tilde{b}(\zeta)/\tilde{a}(\zeta).
$$

Volterra integral equations for N, $\widetilde{N}$, M, $\widetilde{M}$ and integral representations for a, $\tilde{a}$, b, $\tilde{b}$ are given by equations (3.1.7) p. 106 and (3.1.16) p. 107 of A&C.. They imply that $M(x,\zeta)$, $N(x,\zeta)$, $a(\zeta)$ are analytic in the upper half ζ-plane Π^+, whilst $\widetilde{M}(x,\zeta)$, $\widetilde{N}(x,\zeta)$ and $\tilde{a}(\zeta)$ are analytic in the lower half ζ-plane Π^-. ($b(\zeta)$, $\tilde{b}(\zeta)$ cannot in general be extended away from the real line). Therefore equations (16, 17) are equivalent to the Riemann-Hilbert boundary value problem

$$
(m_+ - m_-)(x,\zeta) = m_-(x,\zeta)V(x,\zeta), \quad (18)
$$

where

$$V(x,\zeta)=\begin{pmatrix}\rho(\zeta)\tilde{\rho}(\zeta) & \tilde{\rho}(\zeta)e^{-2\iota\zeta x}\\ \rho(\zeta)e^{2\iota\zeta x} & 0\end{pmatrix},$$

$$m_{+}(x,\zeta)=\left(\frac{M(x,\zeta)}{a(\zeta)},N(x,\zeta)\right),\ m_{-}(x,\zeta)=\left(\widetilde{N}(x,\zeta),-\frac{\widetilde{M}(x,\zeta)}{\tilde{a}(\zeta)}\right),$$

and

$$m_{\pm}(x,\zeta)\to I,\ \text{as}\ |\zeta|\to\infty.$$

Suppose that given $q(x,0)$, $r(x,0)$ we have solved the linear integral equations for N, $\widetilde{N}$, M, $\widetilde{M}$ and hence obtained $m_{\pm}(x,0,\zeta)$, the discrete and continuous spectrum scattering data and $V(x,0,\zeta)$ using (18). Suppose further that we have found the time dependence of the scattering data from the V-part of the AKNS-pair. Then the inverse problem requires us to solve the vector Riemann-Hilbert problem on $\text{Im}\zeta=0$

$$\begin{aligned}(m_{+}-m_{-})(x,t,\zeta) &= m_{-}(x,t,\zeta)V(x,t,\zeta),\\ m_{\pm}(x,t,\zeta) &\to I,\ \text{as}\ |\zeta|\to\infty,\end{aligned}\qquad(19)$$

finding $m_{\pm}(x,t,\zeta)$ when the matrix function $V(x,t,\zeta)$ is known.

For the sake of brevity in this section we assume $a(\zeta)$, $\tilde{a}(\zeta)$ have no zeros, meaning that $m_{\pm}$ are analytic rather than meromorphic in their respective half-planes. In Chapter 5 we will derive a Riemann-Hilbert problem of the same type as (19) in order to solve the inverse scattering problem for the RMB-system, and we shall explicitly carry out the extra calculations demanded by allowing meromorphic $m_{\pm}$.

Define the projection operator $P^{\pm}$ by

$$\left(P^{\pm}f\right)(\zeta)=\frac{1}{2\pi\iota}\int_{-\infty}^{\infty}\frac{f(u)}{u-(\zeta\pm\iota 0)}du.\qquad(20)$$

If $f_\pm(\zeta)$ are analytic in $\Pi^\pm$, and $f_\pm(\zeta) \to 0$ as $|\zeta| \to \infty$, then it is immediate using Cauchy's Integral Formula from Complex Analysis that

$$\left(P^\pm f_\mp\right)(\zeta) = 0, \ \left(P^\pm f_\pm\right)(\zeta) = \pm f_\pm(\zeta).$$

Let us apply P^- to equation (19). We have

$$P^-\{(m_+ - I) - (m_- - I) - m_- V\} = 0.$$

Therefore

$$\begin{aligned} m_-(x,t,\zeta) &= I + \frac{1}{2\pi\iota}\int_{-\infty}^{\infty} \frac{m_-(x,t,u)V(x,t,u)du}{u-(\zeta-\iota 0)} \\ &\sim I - \frac{1}{2\pi\iota\zeta}\int_{-\infty}^{\infty}\left(\rho(t,u)e^{2\iota ux}N(x,t,u),\ \tilde{\rho}(t,u)e^{-2\iota ux}\widetilde{N}(x,t,u)\right)du \end{aligned}$$

as $|\zeta| \to \infty$.

Alternatively, another asymptotic expansion for m_- in inverse powers of ζ is gained simply by using integation by parts, the rapidly decreasing boundary conditions for q and r at infinity, and the integral representations for $\widetilde{N}$, $\widetilde{M}$ and $\tilde{a}$. In fact

$$m_-(x,t,\zeta) \sim \begin{pmatrix} 1 + \frac{1}{2\iota\zeta}\int_x^\infty q(\xi,t)r(\xi,t)d\xi & \frac{q(x,t)}{2\iota\zeta} \\ -\frac{r(x,t)}{2\iota\zeta} & 1 + \frac{1}{2\iota\zeta}\int_{-\infty}^x q(\xi,t)r(\xi,t)d\xi \end{pmatrix}$$

as $|\zeta| \to \infty$, which upon matching coefficients of ζ^{-1} in the $(1,2)$ and $(2,1)$ positions gives

$$\begin{aligned} q(x,t) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \tilde{\rho}(t,u)e^{-2\iota ux}\widetilde{N_1}(x,t,u)du, \\ r(x,t) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \rho(t,u)e^{2\iota ux}N_2(x,t,u)du, \end{aligned}$$

where $\widetilde{N_1}$ is the $(1,1)$ element of $\widetilde{N}$, and N_2 is the $(2,1)$ element of N.

Evidently these formulae describe q and r at a general time t in terms of the continuous spectrum scattering data (i.e., $\rho(t,\zeta)$, $\widetilde{\rho}(t,\zeta)$ which are known) and the solutions $m_{\pm}$ of (19) (i.e., $N(x,t,\zeta)$, $\widetilde{N}(x,t,\zeta)$).

2.2 Bäcklund and Darboux-Bäcklund Transformations.

In this section we review two allied methods for calculating pure multi-soliton solutions to integrable nonlinear PDEs.

2.2.1 Bäcklund Transformations.

A *Bäcklund transformation* is a system of equations which implicitly defines a mapping between local solutions of either two different partial differential equations or one single partial differential equation.

More precisely, given two uncoupled partial differential equations $P(u) = 0$, $Q(v) = 0$ for $u = u(x,t)$, $v = v(x,t)$, a Bäcklund transformation may be defined [25] as a pair of relations

$$R_i(u, v, u_x, v_x, u_t, v_t, ...; x, t) = 0, \ \ i = 1, 2$$

satisfying the following property: if u is chosen to be a solution of $P(u) = 0$, then $R_i = 0$ are integrable for v and the resulting v is a solution of $Q(v) = 0$, and *vice versa*. If the operators P, Q are the same, then $R_i = 0$, $i = 1, 2$ are called an *auto-Bäcklund transformation.*

It is intriguing that every evolution equation solvable by the IST has its own auto-Bäcklund transformation. Moreover, the AKNS-pair corresponding to a particular integrable equation itself represents a Bäcklund transformation: cf. Ablowitz and Segur [26] pp. 156-157.

Suppose now that $P(u) = 0$ is an integrable PDE with AKNS-pair

$$\begin{aligned} \partial_x F &= U(u)F, \\ \partial_t F &= VF. \end{aligned}$$

Then applying the auto-Bäcklund transformation for $P(u) = 0$ to a given (suitably behaved) solution $u_0(x,t)$, $P(u_0) = 0$, yields a new solution $u_1(x,t)$, $P(u_1) = 0$ characterised by the fact that the spectrum of $\partial_x F = U(u_1)F$ differs from the spectrum of $\partial_x F = U(u_0)F$ by exactly one discrete eigenvalue. Consequently, if u_0 is an n-soliton solution, then u_1 will be an $(n \pm 1)$-soliton solution. A standard trick for obtaining 1-solitons is therefore to apply the auto-Bäcklund transformation to the trivial zero solution of $P = 0$ (providing of course $P = 0$ possesses this solution). Higher order solitons are then found either by further applications of the Bäcklund transformation, which unfortunately requires repeated integrations, or preferably by algebraic means alone if the Bäcklund transformation satisfies a *permutability theorem.* We shall demonstrate this method with a typical example for finding a 2-soliton solution of the sine-Gordon equation

$$u_{xt} = \sin u.$$

An auto-Bäcklund transformation for the sine-Gordon equation is given by

$$\begin{aligned} (u - v)_x &= 2a \sin[(u+v)/2], \\ (u + v)_t &= 2a^{-1} \sin[(u-v)/2], \end{aligned}$$

where $a \neq 0$ is an arbitrary constant. (As can be verified by cross-differentiation). If we choose

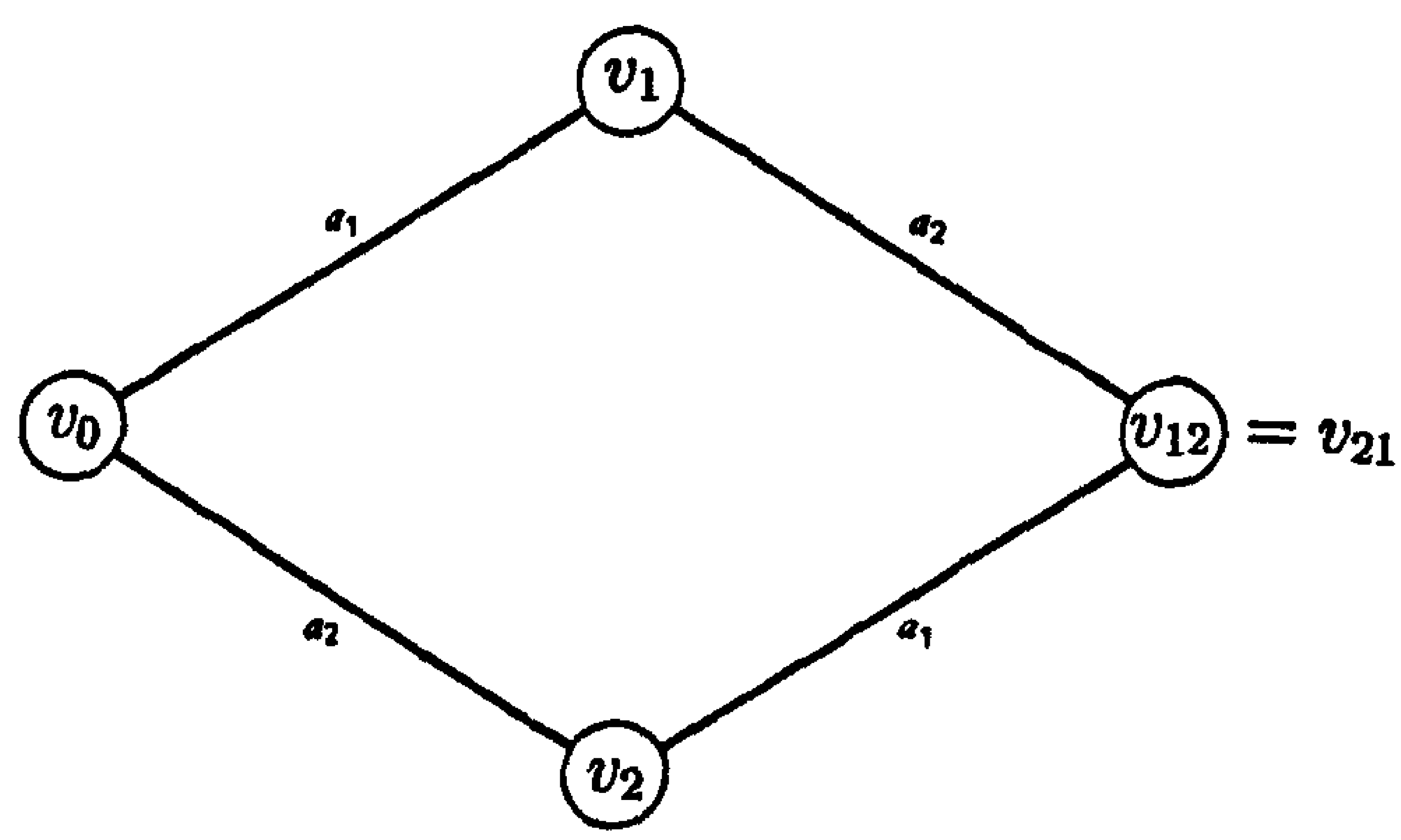
$$v = v_0 \text{ with } (v_0)_{xt} = \sin(v_0),$$

and if we successively take $a = a_i$ $(i = 1, 2)$ in the Bäcklund transformation, then by definition we obtain two solutions $u = v_i$ (say) of the sine-Gordon equation.

In fact if $v_0 = 0$, then

$$v_i = 4 \arctan\left[\exp\left(a_i x + t/a_i\right)\right],$$

each of which represents a 1-soliton solution. Now if we take $v = v_1$, $a = a_2$ then we obtain a solution $u = v_{12}$ (say), and similarly a solution $u = v_{21}$ from $v = v_2$, $a = a_1$.



Thus we have four pairs of equations:

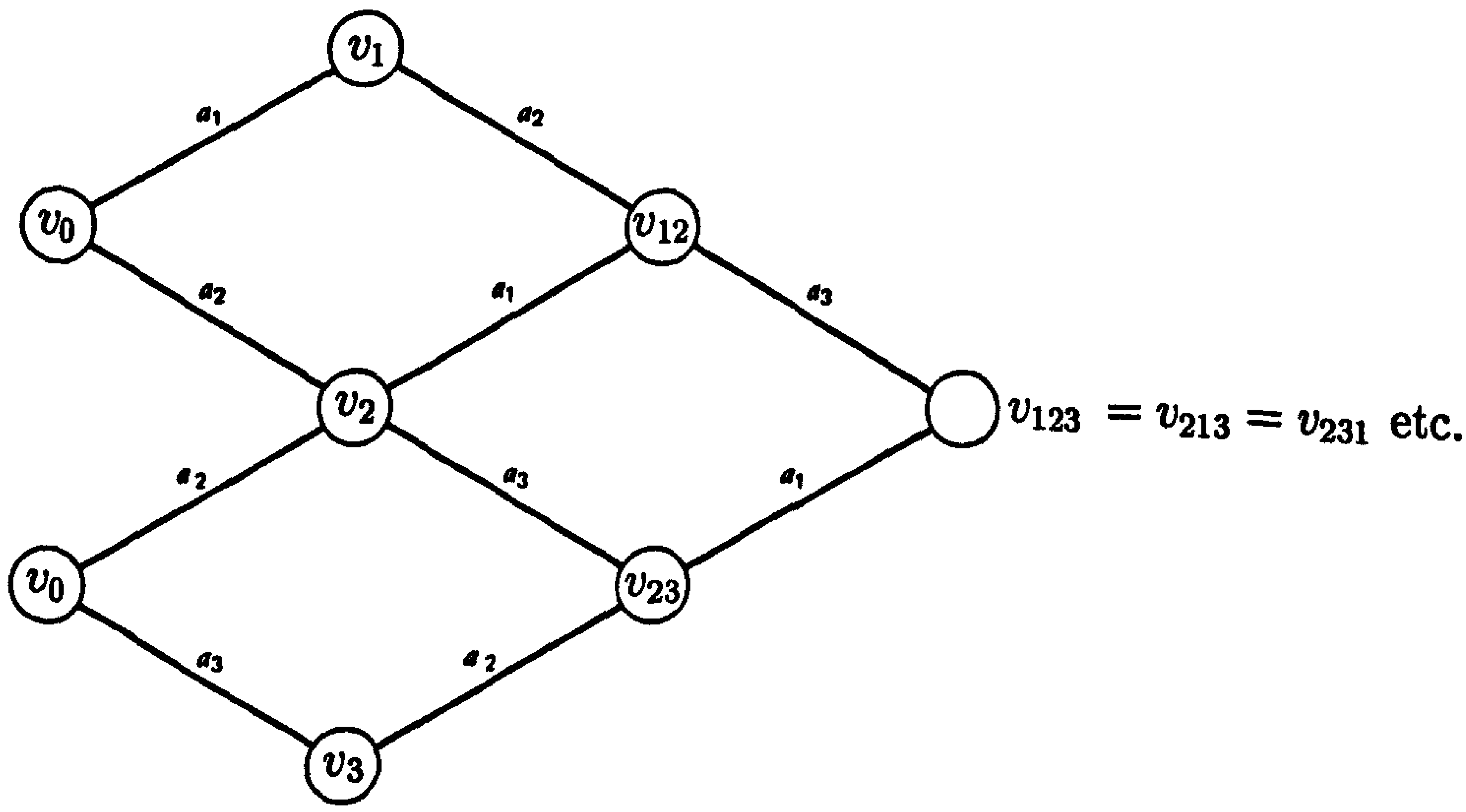
$$\begin{aligned}
(v_1 - v_0)_x &= 2a_1 \sin[(v_1 + v_0)/2] \quad , \quad (v_1 + v_0)_t = 2a_1^{-1} \sin[(v_1 - v_0)/2],\\
(v_2 - v_0)_x &= 2a_2 \sin[(v_2 + v_0)/2] \quad , \quad (v_2 + v_0)_t = 2a_2^{-1} \sin[(v_2 - v_0)/2],\\
(v_{12} - v_1)_x &= 2a_2 \sin[(v_{12} + v_1)/2] \quad , \quad (v_{12} + v_1)_t = 2a_2^{-1} \sin[(v_{12} - v_1)/2],\\
(v_{21} - v_2)_x &= 2a_1 \sin[(v_{21} + v_2)/2] \quad , \quad (v_{21} + v_2)_t = 2a_1^{-1} \sin[(v_{21} - v_2)/2].
\end{aligned}$$

Since Bianchi's Theorem of Permutability (1902) (proved for instance in [27]) demands that $v_{12} = v_{21}$ we can eliminate the differential terms and finally obtain

$$\tan[(v_0 - v_{12})/4] = \frac{(a_2 + a_1)}{(a_2 - a_1)} \tan[(v_1 - v_2)/4].$$

This *nonlinear superposition formula* allows us to calculate algebraically the solution v_{12} providing we can solve for v_1, v_2 given v_0. The 2-soliton solution of the sine-Gordon equation results from the choice $v_0 = 0$, $v_i = 4\arctan[\exp(a_i x + t/a_i)]$.

It is straightforward in principle to continue this process, although the calculations become very involved for large n. The sequence of Bäcklund transformations leading to a 3-soliton solution v_{123} is illustrated below:



2.2.2 Darboux-Bäcklund Transforms.

A Darboux-Bäcklund transform is a special type of gauge transformation which can be used to effect auto-Bäcklund transformations between n and $(n+1)$-soliton solutions of integrable PDEs.

Let $M(\mathrm{q}) = 0$ be an integrable nonlinear system for an unknown vector function $\mathrm{q}(x,t) = (q_1, ..., q_r)(x,t)$, $\mathrm{N} \ni r \geq 1$, and let

$$F_x = U(\mathrm{q}, \zeta)F, \;\; F_t = V(\mathrm{q}, \zeta)F$$

be the associated AKNS $N \times N$ linear system.

Observe that the zero-curvature condition $U_t - V_x + [U, V] = 0$ is invariant under gauge transformations, which are (F&T. p. 22) changes of frame defined by

$$\widehat{F} = GF, \ \widehat{U} = GUG^{-1} + G_x G^{-1} \ \ (20), \ \widehat{V} = GVG^{-1} + G_t G^{-1} \ \ (21).$$

Therefore a gauge transformation $G(\mathrm{q}, \widehat{\mathrm{q}}, \zeta)$ acting on U, V to produce $\widehat{U} = U(\widehat{\mathrm{q}}, \zeta)$, $\widehat{V} = V(\widehat{\mathrm{q}}, \zeta)$ implicitly transforms between two different solutions q, $\widehat{\mathrm{q}}$ of the same nonlinear system $M = 0$.

The form of U fixes the ζ-dependence of G necessary for G to cause transformations between n and $(n+1)$-solitons: cf. Kundu (1987) [28]. In order to make the procedure clearer we work through a routine 2×2 example.

Consider the AKNS-pair

$$F_x = UF = \begin{pmatrix} -\iota\zeta & \frac{1}{2}\theta_x \\ \frac{1}{2}\theta_x & \iota\zeta \end{pmatrix} F, \ F_t = \frac{\iota}{4\zeta} \begin{pmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{pmatrix} F$$

for the sinh-Gordon equation $\theta_{xt} = \sinh\theta$, and let

$$U = \zeta \begin{pmatrix} -\iota & 0 \\ 0 & \iota \end{pmatrix} + \frac{1}{2}\theta_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} := \zeta(\iota A) + \frac{1}{2}\theta_x B,$$

$$\widehat{U} = \zeta(\iota A) + \frac{1}{2}\widehat{\theta}_x B,$$

where $\theta = \theta(x,t)$, $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We shall show that if the gauge transformation G has linear dependence on ζ given by $G = \zeta G_1 + G_0$, then the defining transformation equations (20), (21) constrain G in such a way that θ and $\widehat{\theta}$ are related by the auto-Bäcklund transformation for the sinh-Gordon equation. It is necessary however to make some choices about the form of the matrices G_1 and G_0.

Substituting for G, U and $\widehat{U}$ in (20) we find that the quadratic polynomial

$$\begin{aligned} 0 &= \zeta^2 [G_1, \iota A] + \zeta \left\{ G_{1x} + \frac{1}{2}\theta_x G_1 B - \frac{1}{2}\widehat{\theta}_x B G_1 + [G_0, \iota A] \right\} + \\ & G_{0x} + \frac{1}{2}\theta_x G_o B - \frac{1}{2}\widehat{\theta}_x B G_0 \end{aligned}$$

must be satisfied identically in ζ.

A suitable choice for G_1 is $G_1 = \iota A$, ($G_1 \propto I$ also works), and if we write $G_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then by equating the coefficients of ζ and ζ^0 to zero we obtain

$$b = c = \frac{1}{4}(\widehat{\theta} + \theta)_x \qquad (22)$$

and

$$\begin{aligned} a &= d, \\ a_x &= \frac{1}{2} b(\widehat{\theta} - \theta)_x, \qquad (23) \\ b_x &= \frac{1}{2} a(\widehat{\theta} - \theta)_x, \qquad (24) \end{aligned}$$

respectively.

Now suppose that $a = a\left(\frac{\widehat{\theta}-\theta}{2}\right)$, and define $f = f\left(\frac{\widehat{\theta}-\theta}{2}\right)$ with $f' = a$. Then from equations (23, 22) we have that

$$a' = b = \frac{1}{4}(\widehat{\theta} + \theta)_x = f'',$$

whilst equations (24, 22) give $\frac{1}{4}(\widehat{\theta} + \theta)_{xx} = f_x$, i.e.,

$$\frac{1}{4}(\widehat{\theta} + \theta)_x = f,$$

(taking the integration constant to be zero). Therefore $f = f''$, which has a solution

$$\begin{aligned} f &= \kappa \sinh\left(\frac{\widehat{\theta}-\theta}{2}\right) \\ &= b = c = \frac{1}{4}\left(\widehat{\theta}+\theta\right)_x, \end{aligned}$$

with

$$f' = a = d = \kappa \cosh\left(\frac{\widehat{\theta}-\theta}{2}\right),$$

where κ is an arbitrary constant parameter. In particular, upon rescaling $\phi = \theta/2$, $\widehat{\phi} = \widehat{\theta}/2$, we have the x-part of the auto-Bäcklund transformation

$$(\widehat{\phi}+\phi)_x = 2\kappa \sinh(\widehat{\phi}-\phi).$$

The t-part of the Bäcklund transformation, namely

$$(\widehat{\phi}-\phi)_t = \frac{1}{2\kappa}\sinh(\widehat{\phi}+\phi),$$

ensues by substituting for V and G in equation (21).

Our example has shown how a Darboux transform acts in an equivalent way to an auto-Bäcklund transformation. Multi-soliton solutions are found just as for Bäcklund transformations: the Darboux transform is applied to the trivial zero solution of the nonlinear system in question, then higher order solitons are calculated algebraically where possible using a permutability theorem.

Chapter 3

Formulation and Integrability of the 3-Level Maxwell-Bloch Equations.

Consider the system consisting of an electromagnetic plane wave interacting with a uniform dielectric of 3-level quantum-mechanical atoms. It is assumed that the atoms are rotationally symmetric in the transverse plane. The equations modelling this system can be written:

$$\begin{aligned} \iota\hbar\partial_t\Gamma &= [H,\Gamma], \\ \partial_z^2\mathbf{A} - c^{-2}\partial_t^2\mathbf{A} &= -c^{-2}\varepsilon_0^{-1}\partial_t\mathbf{P}. \end{aligned}$$

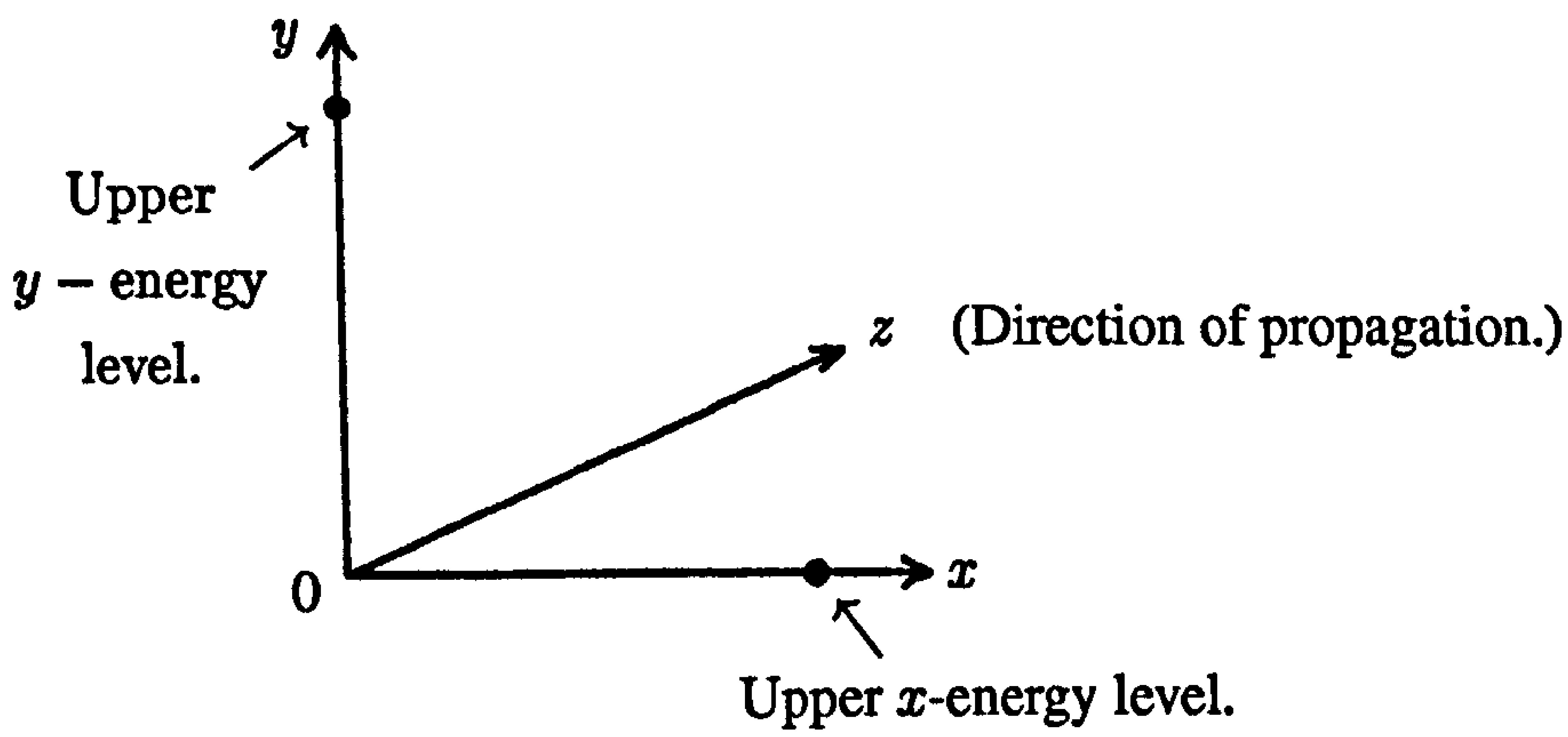
The first equation is the Liouville equation of motion for the 3×3 quantum density matrix Γ. The second is Maxwell's equation, where $\mathbf{A}(z,t)$ is the vector potential field of the plane wave, $\mathbf{P}(z,t)$ is the polarisation density induced in the atomic ensemble, and c is the speed of light in vacuum.

By definition the density matrix Γ is hermitian and has the property $\mathrm{Tr}\Gamma = 1$. However the form of the Liouville equation permits us to define a new traceless density matrix ρ through the relation

$$\Gamma = \frac{1}{3}I + \frac{1}{2}\rho,$$

where $I = \text{diag}\{1, 1, 1\}$. Clearly ρ also satisfies the Liouville equation

$$\iota\hbar\partial_t\rho = [H, \rho].$$



If we suppose that the atom-field coupling is minimal-replacement [29], then the Hamiltonian matrix H takes the form

$$H = \begin{pmatrix} -2\hbar\Omega_0/3 & -\iota e p_x A_x/m & -\iota e p_y A_y/m \\ \iota e p_x A_x/m & \hbar\Omega_0/3 & 0 \\ \iota e p_y A_y/m & 0 & \hbar\Omega_0/3 \end{pmatrix},$$

where $\hbar\Omega_0$ is the energy level of the orthogonal x and y states, the momentum operator is given by

$$\mathbf{p} = (p_1, p_2, p_3) = (p_x \begin{pmatrix} 0 & -\iota & 0 \\ \iota & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, p_y \begin{pmatrix} 0 & 0 & -\iota \\ 0 & 0 & 0 \\ \iota & 0 & 0 \end{pmatrix}, 0),$$

and $-e$, m are the charge and mass of the electron respectively.

The polarisation density is

$$\mathbf{P}(z,t) = -Ne\mathrm{Tr}\left\{\Gamma \mathbf{q}\right\} = -\frac{1}{2}Ne\mathrm{Tr}\left\{\rho \mathbf{q}\right\},$$

where N is the number density of atoms, and $\mathbf{q}$ is the dipole displacement operator

$$\mathbf{q} = (q_1, q_2, q_3) = (q_0 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, q_0 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, 0).$$

Since $\mathbf{p}$, $\mathbf{q}$ are related by the equation

$$\mathbf{p} = -\frac{\iota m}{\hbar}\left[\mathbf{q}, H_0\right],$$

where H_0 is the Hamiltonian in the absence of an applied field, we have $p_x = p_y = mq_0\Omega_0 = p_0$ say, due to the rotational symmetry.

If the number density of atoms is small (for example N less than about 10^{18} atoms/cm^3), then reflected waves may be neglected and use of the "one-way wave approximation" [30] reduces Maxwell's equation to $(\partial_z + c^{-1}\partial_t)\,\mathbf{A} = (2c\varepsilon_0)^{-1}\,\mathbf{P}$.

Finally, by substituting the travelling wave coordinate $\tau = t - z/c$ our system of equations becomes

$$\begin{aligned} \iota\hbar\partial_\tau \rho &= [H, \rho], \qquad (1) \\ \partial_z \mathbf{A} &= \frac{1}{2c\varepsilon_0}\mathbf{P}, \qquad (2) \end{aligned}$$

which are called the reduced 3-level Maxwell-Bloch (RMB) equations.

In order to show that this system is completely integrable it is necessary to use some results from the theory of the Lie algebra su(3) consisting of anti-hermitian 3×3 traceless matrices. (Please see the Appendix p. 135 for the definition of a Lie algebra and a brief review of the terminology used in this section). A basis for su(3) is the set $\{-\iota\lambda_1/2, ..., -\iota\lambda_8/2\}$ such that

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & -\iota & 0 \\ \iota & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\iota \\ 0 & \iota & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & -\iota \\ 0 & 0 & 0 \\ \iota & 0 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad (3-10)$$

with structure constants

$$f_{147} = 1,$$

$$f_{135} = f_{126} = f_{432} = f_{465} = f_{376} = f_{572} = -\frac{1}{2},$$

$$f_{368} = f_{258} = \frac{\sqrt{3}}{2},$$

satisfying

$$2\iota \sum_{j=1}^{8} f_{jkl}\lambda_j = [\lambda_k, \lambda_l]. \quad (11)$$

ρ and H, being hermitian traceless matrices, can therefore be written

$$\rho = \sum_{j=1}^{8} \rho_j \lambda_j,$$

$$H = \sum_{j=1}^{8} H_j \lambda_j = \frac{ep_0}{m}(A_x\lambda_4 + A_y\lambda_6) - \frac{\hbar\Omega_0}{2}\left(\lambda_7 + \frac{1}{\sqrt{3}}\lambda_8\right).$$

Furthermore, since there exists an *outer automorphism* Θ : su(3) $\rightarrow$ su(3) defined by

$$\Theta(g) = -g^t, \ \forall g \in \mathrm{su}(3),$$

it is possible to express ρ and H in terms of their anti-symmetric and symmetric parts as follows

$$\begin{aligned} \rho(\zeta) &= \rho_4\lambda_4 + \rho_5\lambda_5 + \rho_6\lambda_6 + \zeta\left(\rho_1\lambda_1 + \rho_2\lambda_2 + \rho_3\lambda_3 + \rho_7\lambda_7 + \rho_8\lambda_8\right) \\ &= \rho^+ + \zeta\rho^-, \qquad (12) \\ H(\zeta) &= \frac{ep_0}{m}\left(A_x\lambda_4 + A_y\lambda_6\right) - \zeta.\frac{1}{2}\hbar\Omega_0\left(\lambda_7 + \frac{1}{\sqrt{3}}\lambda_8\right) \\ &= H^+ + \zeta H^-, \qquad (13) \end{aligned}$$

where ζ is an arbitrary, generally complex, parameter. Observe that because the automorphism Θ changes the sign of the real matrices λ_1, λ_2, λ_3, λ_7, λ_8 whilst the complex matrices λ_4, λ_5, λ_6 are left unchanged, the same sets of real numbers $\{\rho_j | j = 1, .., 8\}$, $\{H_j | j = 1, ..., 8\}$ are solutions of Liouville's equation at both $\zeta = +1$ and $\zeta = -1$.

We now claim our 3-level RMB-equations $(1, 2)$ are equivalent to the compatibility condition of the AKNS-pair of linear differential equations

$$\begin{aligned} \partial_\tau F &= UF, \qquad (14) \\ \partial_z F &= VF, \qquad (15) \end{aligned}$$

where

$$\begin{aligned} U(\tau, z, \zeta) &= -\frac{\iota}{\hbar}\left(H^+ + \zeta H^-\right), \\ V(\tau, z, \zeta) &= \frac{\iota K}{(\zeta^2 - 1)}\left(\rho^+ + \zeta\rho^-\right), \end{aligned}$$

$K = Ne^2 q_0^2 / 2\hbar c\varepsilon_0$ is a constant normalising factor, and $F(\tau, z) \in \mathbf{C}^3$.

In fact, forming the zero-curvature condition (cf. equation (3), p. 11)

$$\partial_z U - \partial_\tau V + [U, V] = 0,$$

and multiplying throughout by $\hbar K^{-1}(\zeta^2 - 1)$ gives

$$\begin{aligned} 0 &= \zeta^2\left(-\frac{\iota}{K}\partial_z H^+ + \left[H^-, \rho^-\right]\right) + \zeta\left(-\iota\hbar\partial_\tau\rho^- + \left[H^+, \rho^-\right] + \left[H^-, \rho^+\right]\right) + \\ &\quad \left(-\iota\hbar\partial_\tau\rho^+ + \frac{\iota}{K}\partial_z H^+ + \left[H^+, \rho^+\right]\right). \end{aligned}$$

The Fundamental Theorem of Algebra then implies that this quadratic polynomial in ζ will be satisfied identically for all $\zeta \in \mathbf{R}$ if it is satisfied at three distinct values of ζ, which are taken to be $\zeta = \infty$ and $\zeta = \pm 1$. At these values we find

$$\partial_z H^+ = -\iota K\left[H^-, \rho^-\right], \qquad (16)$$

$$\iota\hbar\partial_\tau\left(\rho^+ \pm \rho^-\right) = \left[H^+ \pm H^-, \rho^+ \pm \rho^-\right]. \qquad (17)$$

Equation (17) is evidently just the same as Liouville's equation (1). Moreover calculating the commutator $[H^-, \rho^-]$ using $(3-10)$, $(12, 13)$, and then matching coefficients of λ_4 and λ_6 in equation (16), leads to

$$\begin{aligned} \partial_z A_x &= -\frac{K\hbar}{eq_0}\rho_1 = -\frac{Neq_0}{2c\varepsilon_0}\rho_1, \\ \partial_z A_y &= -\frac{Neq_0}{2c\varepsilon_0}\rho_3, \end{aligned}$$

which are exactly the components of the Maxwell equation (2).

Comparing the AKNS-pair (14, 15) with the generic AKNS-pair (1, 2) on p. 11 we see there is a simple yet important difference. Namely, for the 3-level RMB-system it is the propagation distance z, rather than time t, which is to be thought of as the evolution variable, and the transverse variable is the shifted time $\tau = t - z/c$, as opposed to x. This is a common occurence in optical applications where the initial input pulse at $z = 0$ is known as a function of time, and then the output pulse is detected at some $z > 0$ and again determined as a function of time.

We conclude the present chapter by specifying the initial value-boundary value problem for the RMB-equations. Componentwise the RMB-system $(1,2)$ becomes

$$
\begin{aligned}
\partial_\tau \rho_1 &= \Omega_0 \left[\frac{eq_0}{\hbar}\left(2A_x\rho_7 - A_y\rho_2\right) + \rho_4\right], \\
\partial_\tau \rho_2 &= \frac{\Omega_0 eq_0}{\hbar}\left(A_x\rho_3 + A_y\rho_1\right), \\
\partial_\tau \rho_3 &= \Omega_0 \left\{\frac{eq_0}{\hbar}\left[-A_x\rho_2 + A_y\left(\rho_7 + \sqrt{3}\rho_8\right)\right] + \rho_6\right\}, \\
\partial_\tau \rho_4 &= \Omega_0 \left(\frac{eq_0}{\hbar}A_y\rho_5 - \rho_1\right), \\
\partial_\tau \rho_5 &= \frac{\Omega_0 eq_0}{\hbar}\left(A_x\rho_6 - A_y\rho_4\right), \\
\partial_\tau \rho_6 &= -\Omega_0 \left(\frac{eq_0}{\hbar}A_x\rho_5 + \rho_3\right), \\
\partial_\tau \rho_7 &= -\frac{\Omega_0 eq_0}{\hbar}\left(2A_x\rho_1 + A_y\rho_3\right), \\
\partial_\tau \rho_8 &= -\frac{\sqrt{3}\Omega_0 eq_0}{\hbar}A_y\rho_3, \qquad (18-25) \\
\partial_z A_x &= -\frac{Neq_0}{2c\varepsilon_0}\rho_1, \; \partial_z A_y = -\frac{Neq_0}{2c\varepsilon_0}\rho_3. \qquad (26,27)
\end{aligned}
$$

i.e., ten first order nonlinear PDEs for the ten unknown functions A_x, A_y, ρ_1,..., ρ_8 of two variables τ, $z \in (-\infty, \infty)$.

We prescribe A_x, A_y on $z = 0$, subject to boundary conditions of rapidly decreasing type. If we choose $A_x(\tau,0) = A_1(\tau)$, $A_y(\tau,0) = A_2(\tau)$ to be known functions of τ, then equations $(18-25)$ on $z = 0$ represent a system of ODEs which can be solved uniquely for $\rho_1, ..., \rho_8(\tau, 0)$ as functions of A_1, $A_2(\tau)$, assuming that $\rho_1, ..., \rho_6 \to 0$, $\rho_7 \to 1$, $\rho_8 \to 1/\sqrt{3}$, as $|\tau| \to \infty$. A_x, A_y are supposed to be infinitely differentiable and together with all their derivatives to decay faster than any power of $|\tau|^{-1}$ as $|\tau| \to \infty$.

It is worth emphasizing that whereas we have used the vector potential $\mathbf{A}$ to describe the incident electromagnetic wave, all previous models (such as the 2-level RMB-equations due to Eilbeck, Gibbon, Caudrey & Bullough [5]) have been formulated using the electric field $\mathbf{E}$, where $\mathbf{E} = -\partial_\tau \mathbf{A}$. To our knowledge it is not possible to construct an integrable 3-level system with appropriate rotational symmetry if the atoms are coupled to the field by electric dipole coupling. In particular, since we have assumed rapidly decreasing boundary conditions, this means that soliton solutions of the 3-level RMB-equations always have zero *time area*:

$$\int_{-\infty}^{\infty} \mathbf{E}(z,\tau)d\tau = -\int_{-\infty}^{\infty} \partial_\tau \{\mathbf{A}\}\, d\tau = 0.$$

By contrast solitons of the 2-level RMB-equations have characteristic pulse areas of $2k\pi$, ($\mathbf{N} \ni k \geq 0$):

$$\int_{-\infty}^{\infty} \mathtt{E}(x,t)dt \propto 2k\pi,$$

where $\mathtt{E}$ is the magnitude of the incident plane polarised electromagnetic field [5] [7].

Chapter 4

Use of Darboux-Bäcklund Transformations to find Fully Polarised Soliton Solutions.

Chapter 4 is divided into two sections. In the first section we calculate the fundamental single soliton expressions for A_x, A_y and additionally for the density matrix components $\rho_1, ..., \rho_8$. Taking into account our previous discussion of Darboux transforms (pp. 25-28) a gauge transformation $G = \zeta I + X$, linear in the parameter ζ, will be used to generate these solutions. However, determination of the correct form for the matrix X is a difficult problem as we are now dealing with a 3-level system. Following Arnold [8], and motivated by the work of Park & Shin [9] [10], we shall see that defining X in terms of a hermitian projector matrix $\widehat{G}$ proves successful. In Section 4.2 sequences of two one-parameter Darboux transforms are applied in conjunction with a nonlinear superposition formula to obtain both linearly polarised and elliptically polarised 2-solitons of the RMB-equations. Our formulae specifying the elliptically polarised solitons are the primary results here, they are previously unpublished and represent a natural vector generalisation of the modulated 0π-pulse solutions of the 2-level RMB-equations derived by Bullough, Caudrey, Eilbeck & Gibbon [5] [7].

4.1 Linearly Polarised 1-Soliton Solutions to the RMB-System.

The zero-curvature compatibility condition $\partial_z U - \partial_\tau V + [U, V] = 0$ is left invariant under gauge transformations defined by

$$F' = GF,\ U' = GUG^{-1} + \partial_\tau GG^{-1}\ \ (1),\ V' = GVG^{-1} + \partial_z GG^{-1}.$$

Here we know

$$U(\tau, \zeta) = -\frac{\iota}{\hbar}\left(\zeta H^- + H^+(\tau)\right),$$

$$H^+ = \iota\Omega_0 e q_0 \begin{pmatrix} 0 & -A_x & -A_y \\ A_x & 0 & 0 \\ A_y & 0 & 0 \end{pmatrix},\ H^- = \frac{1}{3}\hbar\Omega_0 \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Choice of the form of G is key to the procedure and a suitable choice is $G = \zeta I + X$, where

$$X = -\iota\eta\widehat{X} = -\iota\eta\left(2\widehat{G} - I\right)$$

($\eta \in \mathbf{R}$ is an arbitrary constant), and

$$\widehat{G}^2 = \widehat{G},\ \widehat{G}^{*t} = \widehat{G}^\dagger = \widehat{G}$$

($\widehat{G}$ is a hermitian projector). Notice that these conditions mean that for $\zeta \in \mathbf{R}$ $GG^\dagger = (\zeta^2 + \eta^2) I$. i.e., G is proportional to a unitary matrix.

Substituting into (1) and demanding that G transforms the matrix U into one of similar form $U' = (\iota\hbar)^{-1}(\zeta H^- + H'^+)$, leads, upon equating powers of ζ, to the equations

$$\begin{aligned} H'^{+} - H^{+} &= \left[X, H^{-}\right], \\ \iota\hbar X_{\tau} &= H'^{+}X - XH^{+}, \end{aligned}$$

where $H'^{+} = \iota\Omega_0 e q_0 \begin{pmatrix} 0 & -A'_x & -A'_y \\ A'_x & 0 & 0 \\ A'_y & 0 & 0 \end{pmatrix}$.

The simplest 1-solitons are generated by our Darboux transform acting on the zero field solution to the RMB-system. Therefore we take $H^{+} = 0$, giving

$$\begin{aligned} H'^{+} &= \left[X, H^{-}\right], \quad (2) \\ \iota\hbar X_{\tau} &= H'^{+}X. \quad (3) \end{aligned}$$

Using equation (2) and the condition $\widehat{G}^2 = \widehat{G}$, we find $\widehat{G}$ may be written as follows

$$\widehat{G} = \frac{1}{g} s s^t = \frac{s s^t}{s^t s}, \quad s = \begin{pmatrix} g \\ C A'_x / 2\eta \\ C A'_y / 2\eta \end{pmatrix},$$

where $g^2 - g + \frac{1}{4} C^2 \eta^{-2} \left(A'^2_x + A'^2_y\right) = 0$, and $C := e q_0 / \hbar$. (Providing s is real valued, $\widehat{G}$ satisfies $\widehat{G}^{\dagger} = \widehat{G}$).

Equation (3) can now be integrated yielding g and the components A'_x, A'_y of the transformed field. In fact (3) implies

$$\begin{aligned} g_{\tau} &= 2\Omega_0 \eta g (g-1), \\ A'_{x\tau} &= \Omega_0 \eta (2g-1) A'_x, \\ A'_{y\tau} &= \Omega_0 \eta (2g-1) A'_y, \end{aligned}$$

with $g(g-1) \geq 0$ and $g(g-1) \to 0$, as $|\tau| \to \infty$, since A'_x, $A'_y \to 0$, as $|\tau| \to \infty$.

Hence

$$g = \frac{1}{e^{2\theta}+1},\ A'_x = \frac{\eta a}{C}\mathrm{sech}\theta,\ A'_y = \frac{\eta b}{C}\mathrm{sech}\theta,$$

where $\theta = \Omega_0\eta\,(\tau - \tau_0)$, and the constants a, b must satisfy $a^2 + b^2 = 1$ for consistency.

A'_x, A'_y are real provided that the arbitrary constants $a := \cos\phi$, $b := \sin\phi$, and τ_0 are also real numbers.

Summarising: linearly polarised 1-soliton solutions

$$\mathbf{A}(\tau, z) = \left(\frac{\eta a}{C}\mathrm{sech}\Omega_0\eta\,(\tau - \tau_0\,(z))\,,\ \frac{\eta b}{C}\mathrm{sech}\Omega_0\eta\,(\tau - \tau_0\,(z))\right) \qquad (4)$$

of the RMB-system, whose polarisation direction lies at an angle ϕ to the x-axis, are generated from the zero solution by the Darboux transform $G = \zeta I - \iota\eta\widehat{X} = (\zeta + \iota\eta) - 2\iota\eta\widehat{G}$, where the real symmetric projection matrix $\widehat{G}$ is given by

$$\widehat{G} = \frac{ss^t}{s^ts},\ s = \frac{1}{e^{2\theta}+1}\begin{pmatrix} 1 \\ ae^{\theta} \\ be^{\theta} \end{pmatrix},$$

$$a^2 + b^2 = \cos^2\phi + \sin^2\phi = 1.$$

The z-dependence of the parameter $\tau_0 = \tau_0(z)$ will be found presently.

$$\widehat{G} = \frac{1}{e^{2\theta}+1}\begin{pmatrix} 1 & ae^{\theta} & be^{\theta} \\ ae^{\theta} & a^2e^{2\theta} & abe^{2\theta} \\ be^{\theta} & abe^{2\theta} & b^2e^{2\theta} \end{pmatrix},$$

$$\widehat{X} = \frac{1}{e^{2\theta}+1}\begin{pmatrix} 1 - e^{2\theta} & 2ae^{\theta} & 2be^{\theta} \\ 2ae^{\theta} & (2a^2-1)e^{2\theta} - 1 & 2abe^{2\theta} \\ 2be^{\theta} & 2abe^{2\theta} & (2b^2-1)e^{2\theta} - 1 \end{pmatrix},$$

$$X = -\iota\eta \begin{pmatrix} -\tanh\theta & a\text{sech}\theta & b\text{sech}\theta \\ a\text{sech}\theta & a^2 e^\theta \text{sech}\theta - 1 & abe^\theta \text{sech}\theta \\ b\text{sech}\theta & abe^\theta\text{sech}\theta & b^2 e^\theta \text{sech}\theta - 1 \end{pmatrix}.$$

Now G acts on the matrix V according to the equation $V'G = GV + \partial_z G$. Substituting for G will determine how ρ is transformed and fix τ_0 as a function of z.

In the absence of a driving field

$$\begin{aligned} V = \frac{\iota K}{\zeta^2 - 1}\left(\rho^+ + \zeta\rho^-\right) &= \frac{\iota N e^2 q_0^2}{3\hbar c\varepsilon_0}\left(\frac{\zeta}{\zeta^2-1}\right)\text{diag}\{2,-1,-1\} \\ &= \frac{\iota K_0 \zeta}{\zeta^2 - 1}\text{diag}\{2,-1,-1\}, \end{aligned}$$

say, where

$$\mathbf{R} \ni K_0 = \frac{2K}{3} > 0.$$

If we write $V' = \frac{\zeta V_1' + V_0'}{\zeta^2 - 1}$, then substituting as described and equating powers of ζ gives

$$\begin{aligned} V_1' &= \iota K_0 D + X_z, & (5) \\ V_1' X + V_0' &= \iota K_0 X D, & (6) \\ V_0' &= \frac{1}{\eta^2} X_z X. & (7) \end{aligned}$$

Here $D = \text{diag}\{2,-1,-1\}$ and (7) follows from $X^2 = -\eta^2 I$.

Using (5) and (7) in equation (6) implies

$$X_z = \frac{-\iota K_0}{(\eta^2+1)}[X, D]X,$$

which is a consistency condition holding for $\tau_0 = \tau_0(z)$ only if $\frac{d}{dz}\tau_0(z) = \frac{3K_0}{\Omega_0(\eta^2+1)}$.

i.e.,

$$\tau_0(z) = \frac{3K_0 z}{\Omega_0\left(\eta^2+1\right)} = \frac{2Kz}{\Omega_0\left(\eta^2+1\right)}.$$

V_0' and V_1' may now be computed from (7) and (5).

We obtain

$$\begin{aligned} V' &= \frac{1}{\zeta^2-1}\left(V_0'+\zeta V_1'\right) = \frac{\iota K}{\zeta^2-1}\left(\rho'^{+}+\zeta\rho'^{-}\right) \\ &= \frac{2\iota K}{\zeta^2-1}\left\{-\frac{\eta \operatorname{sech}\theta}{\eta^2+1}\begin{pmatrix} 0 & -\iota a & -\iota b \\ \iota a & 0 & 0 \\ \iota b & 0 & 0 \end{pmatrix}+\right. \\ &\zeta\left[\frac{\eta^2}{\eta^2+1}\begin{pmatrix} -\operatorname{sech}^2\theta & -a\operatorname{sech}\theta\tanh\theta & -b\operatorname{sech}\theta\tanh\theta \\ -a\operatorname{sech}\theta\tanh\theta & a^2\operatorname{sech}^2\theta & ab\operatorname{sech}^2\theta \\ -b\operatorname{sech}\theta\tanh\theta & ab\operatorname{sech}^2\theta & b^2\operatorname{sech}^2\theta \end{pmatrix}+\right. \\ &\left.\left.\frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right]\right\}. \end{aligned}$$

Hence

$$\rho_1' = -\frac{2\eta^2}{\eta^2+1}a\operatorname{sech}\theta\tanh\theta,\ \rho_2' = \frac{2\eta^2}{\eta^2+1}ab\operatorname{sech}^2\theta,$$

$$\rho_3' = -\frac{2\eta^2}{\eta^2+1}b\operatorname{sech}\theta\tanh\theta,\ \rho_4' = -\frac{2\eta}{\eta^2+1}a\operatorname{sech}\theta,$$

$$\rho_5' = 0,\ \rho_6' = -\frac{2\eta}{\eta^2+1}b\operatorname{sech}\theta,$$

$$\rho_7' = 1-\frac{\eta^2\left(a^2+1\right)}{\eta^2+1}\operatorname{sech}^2\theta,\ \rho_8' = \frac{1}{\sqrt{3}}\left[1-\frac{3\eta^2 b^2}{\eta^2+1}\operatorname{sech}^2\theta\right].\ (8-15)$$

4.2 2-Soliton Solutions of the RMB-System.

We begin as before with the gauge transformation equation for the matrix U,

$$U'G = GU + G_\tau. \quad (16)$$

Let G be a product of two basic Darboux transforms with distinct parameters

$$G = F_2G_1 = F_1G_2,$$

where $G_j = \zeta I + X_j$, $F_j = \zeta I + Y_j$, $j = 1, 2$. Then (16) becomes

$$\left(\zeta H^- + H'^+\right)\left[\zeta^2 I + \zeta\left(X_1 + Y_2\right) + Y_2X_1\right] =$$

$$\left[\zeta^2 I + \zeta\left(X_1 + Y_2\right) + Y_2X_1\right]\left(\zeta H^- + H^+\right) + \iota\hbar\left[\zeta\left(X_1 + Y_2\right)_\tau + \left(Y_2X_1\right)_\tau\right].$$

In particular, equating the coefficients of ζ^2 implies the following formula for the transformed field in terms of the 3×3 matrices X_1 and Y_2,

$$H'^+ = \iota\Omega_0 e q_0 \begin{pmatrix} 0 & -A'_x & -A'_y \\ A'_x & 0 & 0 \\ A'_y & 0 & 0 \end{pmatrix} = \left[\left(X_1 + Y_2\right), H^-\right] + H^+.$$

If we again take $H^+ = 0$, then we have that

$$A'_x = \frac{\iota}{C}\left(X_1 + Y_2\right)_{(1,2)}, \quad A'_y = \frac{\iota}{C}\left(X_1 + Y_2\right)_{(1,3)}, \quad (17)$$

where $M_{(i,j)}$ denotes the (i, j)-th element of a matrix M. This makes it possible to read off the fields A'_x, A'_y once the Darboux matrices X_1 and Y_2 are found.

Now suppose the parameters associated with the matrices X_j are defined to be η_j, $a_j = \cos\phi_j$, $b_j = \sin\phi_j$, $\theta_j = \Omega_0\eta_j\left(\tau - \tau_j\right)$, $j = 1, 2$. From the equality

$F_2G_1 = F_1G_2$, which is a consequence of Bianchi's Permutability Theorem, we find

$$\begin{aligned} X_1 + Y_2 &= X_2 + (X_1 - X_2) X_1 (X_1 - X_2)^{-1} \\ &= \left(X_1^2 - X_2^2\right)(X_1 - X_2)^{-1} \\ \Rightarrow X_1 + Y_2 &= \left(\eta_2^2 - \eta_1^2\right)(X_1 - X_2)^{-1}, \end{aligned} \qquad (18)$$

since $X_j^2 = -\eta_j^2 I, j = 1, 2$.

Depending on the nature of the parameters chosen, equations (17) and (18) can be exploited to generate families of linearly, circularly and elliptically polarised 2-soliton solutions from the vacuum. We begin by considering the case $\phi_1 = \phi_2 = 0$, for which

$$X_j = -\iota\eta_j \begin{pmatrix} -\tanh\theta_j & \text{sech}\theta_j & 0 \\ \text{sech}\theta_j & \tanh\theta_j & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad j = 1, 2.$$

Substituting into (17) and (18) leads to

$$\begin{aligned} A'_x &= \frac{\iota}{C}\left(\eta_2^2 - \eta_1^2\right)(X_1 - X_2)^{-1}_{(1,2)} \\ &= \frac{\left(\eta_1^2 - \eta_2^2\right)(\eta_1 \text{sech}\theta_1 - \eta_2 \text{sech}\theta_2)}{C\left[\eta_1^2 + \eta_2^2 - 2\eta_1\eta_2(\tanh\theta_1 \tanh\theta_2 + \text{sech}\theta_1 \text{sech}\theta_2)\right]}, \\ A'_y &= 0. \end{aligned} \qquad (19)$$

The next step is to allow the various parameters to be complex valued. This is valid providing the choice of parameters always results in real field components A'_x, A'_y. If in equation (19) we take

$$\eta_1 = \eta_2^\star = \eta = \frac{1}{\Omega_0}(p + \iota q),$$

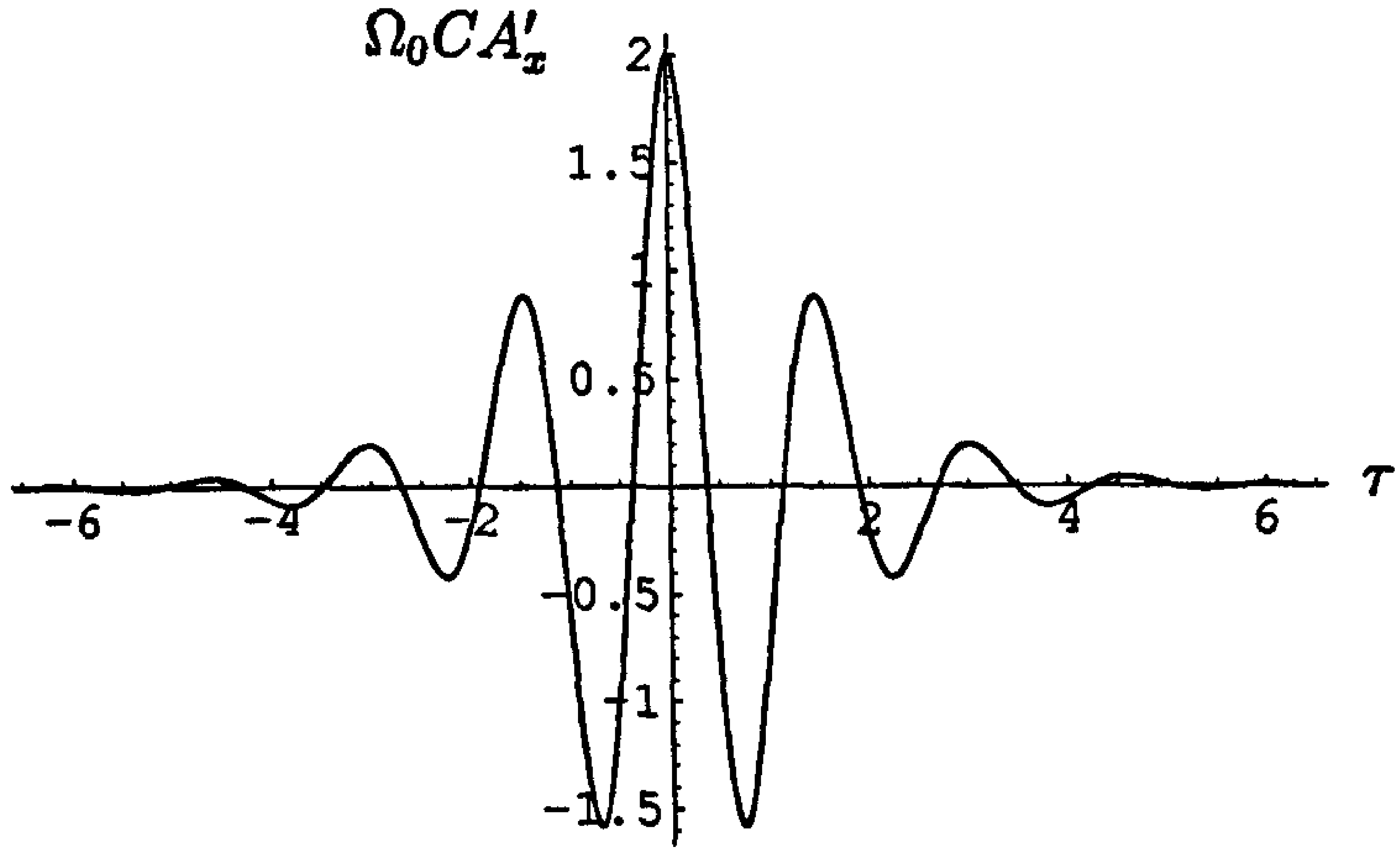
and

$$\begin{aligned}\theta_1 = \theta_2^\star &= \Omega_0 \eta (\tau - \tau_1) \\ &= p(\tau - \varphi) + \iota q (\tau - \tilde{\varphi}), \text{ (say)}\end{aligned}$$

then at $\varphi = \tilde{\varphi} = 0$ we have

$$\begin{aligned}A_x' &= \frac{2p\text{sech}p\tau \left[\cos q\tau - (pq^{-1}) \sin q\tau \tanh p\tau\right]}{\Omega_0 C\left[1 + (p^2 q^{-2}) \sin^2 q\tau \text{sech}^2 p\tau\right]}, \qquad (20) \\ A_y' &= 0.\end{aligned}$$

(Replacing $p\tau$ by $p(\tau - \varphi)$ and $q\tau$ by $q(\tau - \tilde{\varphi})$ gives the correct formula when $\varphi, \tilde{\varphi} \neq 0$). Below there is a plot of $\Omega_0 C A_x'$ as a function of τ when $p = 1$ and $q = 4$:



Expressions (19) and (20) both represent linearly polarised "breather" solitons having zero *time area*. They may be rotated to any given direction of polarisation by applying a τ-independent gauge transformation R so that $G \mapsto RGR^{-1}$, where

$$R := \exp(\iota\phi\lambda_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}, \quad \phi \in \mathbf{R}.$$

Let us move on to the problem of constructing elliptically polarised solitons. In order to guarantee the reality of A'_x, A'_y we require

$$\eta_1 = \eta_2^\star = \eta, \text{ and } \widehat{G_1} = \widehat{G_2}^\star = \widehat{G}.$$

Then

$$\eta_2^2 - \eta_1^2 = -\frac{4\iota pq}{\Omega_0^2},$$

which is pure imaginary, and

$$\begin{aligned} X_1 - X_2 &= -2\iota\left(\eta_1\widehat{G_1} - \eta_2\widehat{G_2}\right) + \iota\left(\eta_1 - \eta_2\right) I \\ &= -2\iota\left(\eta\widehat{G} - \eta^\star\widehat{G}^\star\right) + \iota\left(\eta - \eta^\star\right) I \\ &= \frac{1}{\Omega_0}\left\{-2\iota\left[(p+\iota q)\,\widehat{G} - (p-\iota q)\,\widehat{G}^\star\right] - 2qI\right\} \in \mathbf{R}. \qquad (21) \end{aligned}$$

Hence

$$A'_x = \frac{4pq}{\Omega_0^2 C}\left(X_1 - X_2\right)^{-1}_{(1,2)} \in \mathbf{R},\ A'_y = \frac{4pq}{\Omega_0^2 C}\left(X_1 - X_2\right)^{-1}_{(1,3)} \in \mathbf{R}.$$

It remains to find the inverse of $X_1 - X_2$, only this time we will permit complex parameters a and b. Our task is simplified by first refining the form of $\widehat{G}$.

In the previous section we showed that

$$\widehat{G} = \frac{ss^t}{s^ts},\ s = \frac{1}{e^{2\theta}+1}\begin{pmatrix} 1 \\ ae^\theta \\ be^\theta \end{pmatrix}, \text{ where } a^2 + b^2 = 1.$$

Equivalently, dividing the original s by $\frac{1}{2}\text{sech}\theta$, $\widehat{G}$ is generated by $\begin{pmatrix} e^{-\theta} \\ a \\ b \end{pmatrix}$, which in turn can be rewritten as $s = \begin{pmatrix} e^{-\Omega_0 \eta \tau} \\ ka \\ kb \end{pmatrix}$, $\mathbf{C} \ni k$ constant, by rescaling.

In fact

$$\begin{pmatrix} e^{-\Omega_0 \eta \tau} \\ ka \\ kb \end{pmatrix} = k \begin{pmatrix} \frac{1}{k} e^{-\Omega_0 \eta \tau} \\ a \\ b \end{pmatrix} = k \begin{pmatrix} e^{-\Omega_0 \eta (\tau - \tilde{\tau})} \\ a \\ b \end{pmatrix}$$

$$\mapsto k \begin{pmatrix} e^{-\Omega_0 \eta \tau} \\ a \\ b \end{pmatrix}, \text{ scaling out } \tilde{\tau} = -\frac{1}{\Omega_0 \eta} \ln k.$$

Suppose we choose the case $a = \cosh\phi$, $b = \iota \sinh\phi$, $k = \text{sech}\phi$, $\phi \in \mathbf{R}$. Then $ka = 1$, $kb = \iota\Upsilon$, where $\Upsilon := \tanh\phi$, $\tilde{\tau} = -(2\Omega_0\eta)^{-1} \ln(1 - \Upsilon^2)$,

$$s = \begin{pmatrix} e^{-(p+\iota q)\tau} \\ 1 \\ \iota\Upsilon \end{pmatrix},$$

and

$$\widehat{G} = \frac{1}{\left[e^{-2(p+\iota q)\tau} + 1 - \Upsilon^2\right]} \begin{pmatrix} e^{-2(p+\iota q)\tau} & e^{-(p+\iota q)\tau} & \iota\Upsilon e^{-(p+\iota q)\tau} \\ e^{-(p+\iota q)\tau} & 1 & \iota\Upsilon \\ \iota\Upsilon e^{-(p+\iota q)\tau} & \iota\Upsilon & -\Upsilon^2 \end{pmatrix}.$$

Finally, substituting for $\widehat{G}$ and $\widehat{G}^*$ in equation (21) and inverting the resulting expression for $X_1 - X_2$ gives

$$A'_x = \frac{4pe^{-p\tau}\left\{2\Upsilon^2 p^2 \cos q\tau + pq \sin q\tau \left[e^{-2p\tau} - (1 - \Upsilon^2)\right] + q^2 \cos q\tau \left[e^{-2p\tau} + (1 + \Upsilon^2)\right]\right\}}{\Omega_0 C \left\{q^2 \left[e^{-2p\tau} + (1 + \Upsilon^2)\right]^2 + 4p^2 \left[e^{-2p\tau}\left(\sin^2 q\tau + \Upsilon^2 \cos^2 q\tau\right) + \Upsilon^2\right]\right\}},$$

$$A'_y = \frac{4p\Upsilon e^{-p\tau}\left\{-2p^2 \sin q\tau + pq \cos q\tau \left[e^{-2p\tau} + (1 - \Upsilon^2)\right] - q^2 \sin q\tau \left[e^{-2p\tau} + (1 + \Upsilon^2)\right]\right\}}{\Omega_0 C \left\{q^2 \left[e^{-2p\tau} + (1 + \Upsilon^2)\right]^2 + 4p^2 \left[e^{-2p\tau}\left(\sin^2 q\tau + \Upsilon^2 \cos^2 q\tau\right) + \Upsilon^2\right]\right\}}.$$

Circularly polarised solitons correspond to the limiting case $\Upsilon \to 1$ (or $\Upsilon \to -1$), when we have

$$A'_x = \frac{4pe^{-p\tau}\left[2\left(p^2+q^2\right)\cos q\tau + qe^{-2p\tau}\left(q\cos q\tau + p\sin q\tau\right)\right]}{\Omega_0 C\left[q^2e^{-4p\tau}+4\left(p^2+q^2\right)\left(e^{-2p\tau}+1\right)\right]},$$

$$A'_y = \frac{4pe^{-p\tau}\left[-2\left(p^2+q^2\right)\sin q\tau + qe^{-2p\tau}\left(p\cos q\tau - q\sin q\tau\right)\right]}{\Omega_0 C\left[q^2e^{-4p\tau}+4\left(p^2+q^2\right)\left(e^{-2p\tau}+1\right)\right]}.$$

Using partial fractions and the identity

$$\frac{\sqrt{z}}{z+\mathtt{A}} = \frac{1}{2\sqrt{\mathtt{A}}}\operatorname{sech}\left(p\tau + \ln\sqrt{\mathtt{A}}\right),\ \ z = e^{-2p\tau},$$

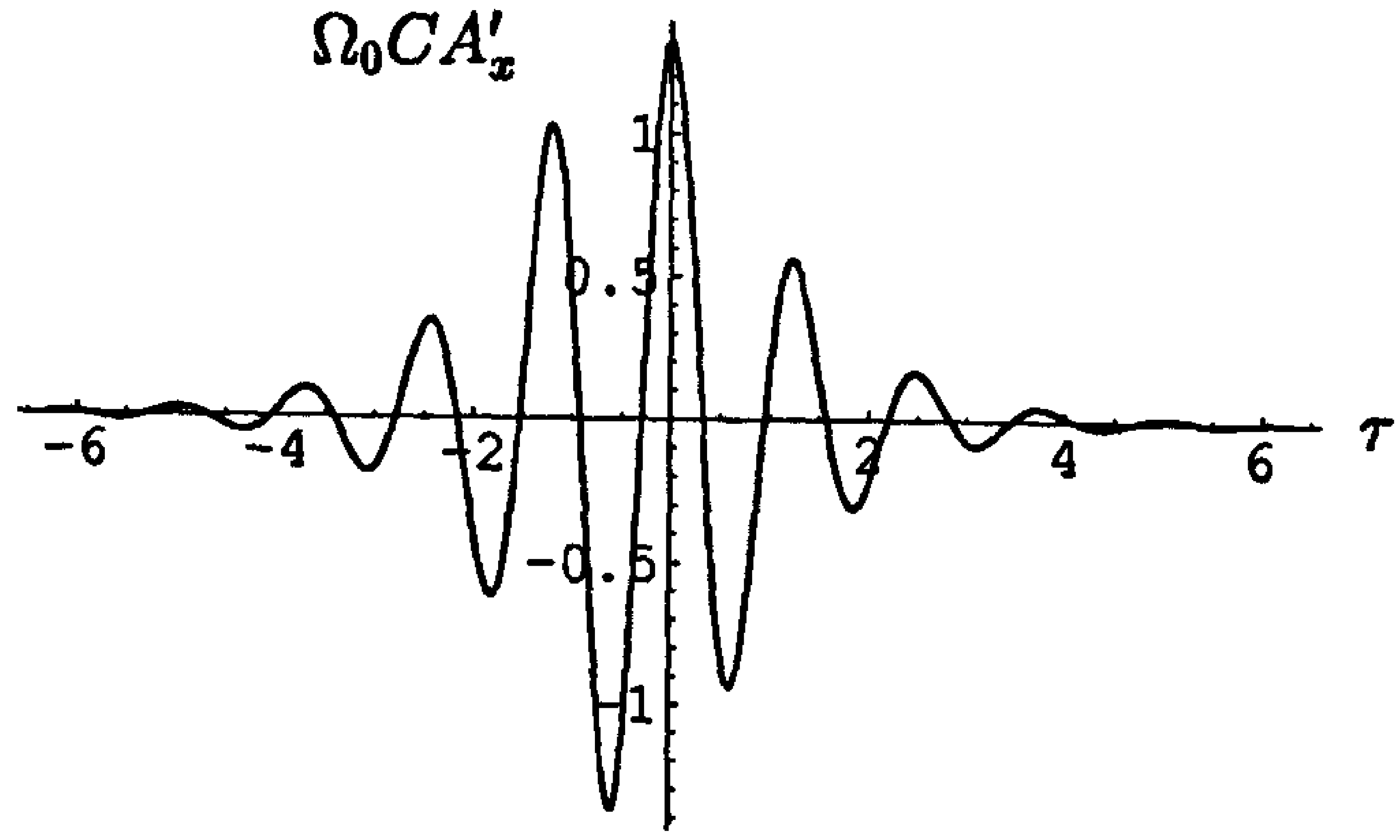
leads to

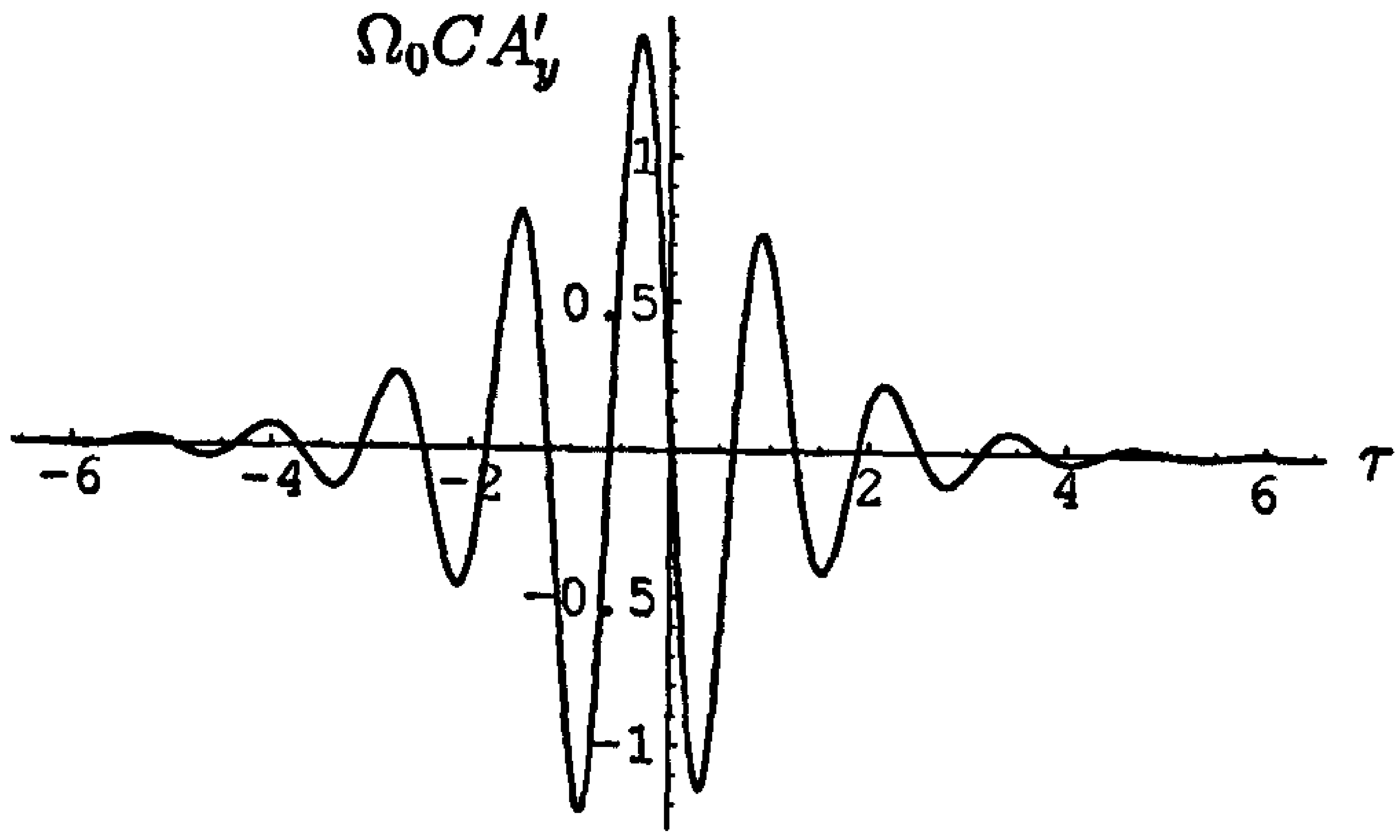
$$\begin{aligned} A'_x &= \frac{p}{\Omega_0 C}\left\{\frac{\left[q\cos q\tau + \left(\Omega_0\left|\eta\right| + p\right)\sin q\tau\right]}{\sqrt{2\Omega_0\left|\eta\right|\left(\Omega_0\left|\eta\right|+p\right)}}\operatorname{sech}\left[p\tau + \ln\frac{\sqrt{2\Omega_0\left|\eta\right|\left(\Omega_0\left|\eta\right|+p\right)}}{q}\right]\right. \\ &+ \left.\frac{\left[q\cos q\tau - \left(\Omega_0\left|\eta\right| - p\right)\sin q\tau\right]}{\sqrt{2\Omega_0\left|\eta\right|\left(\Omega_0\left|\eta\right|-p\right)}}\operatorname{sech}\left[p\tau + \ln\frac{\sqrt{2\Omega_0\left|\eta\right|\left(\Omega_0\left|\eta\right|-p\right)}}{q}\right]\right\}, \end{aligned}$$

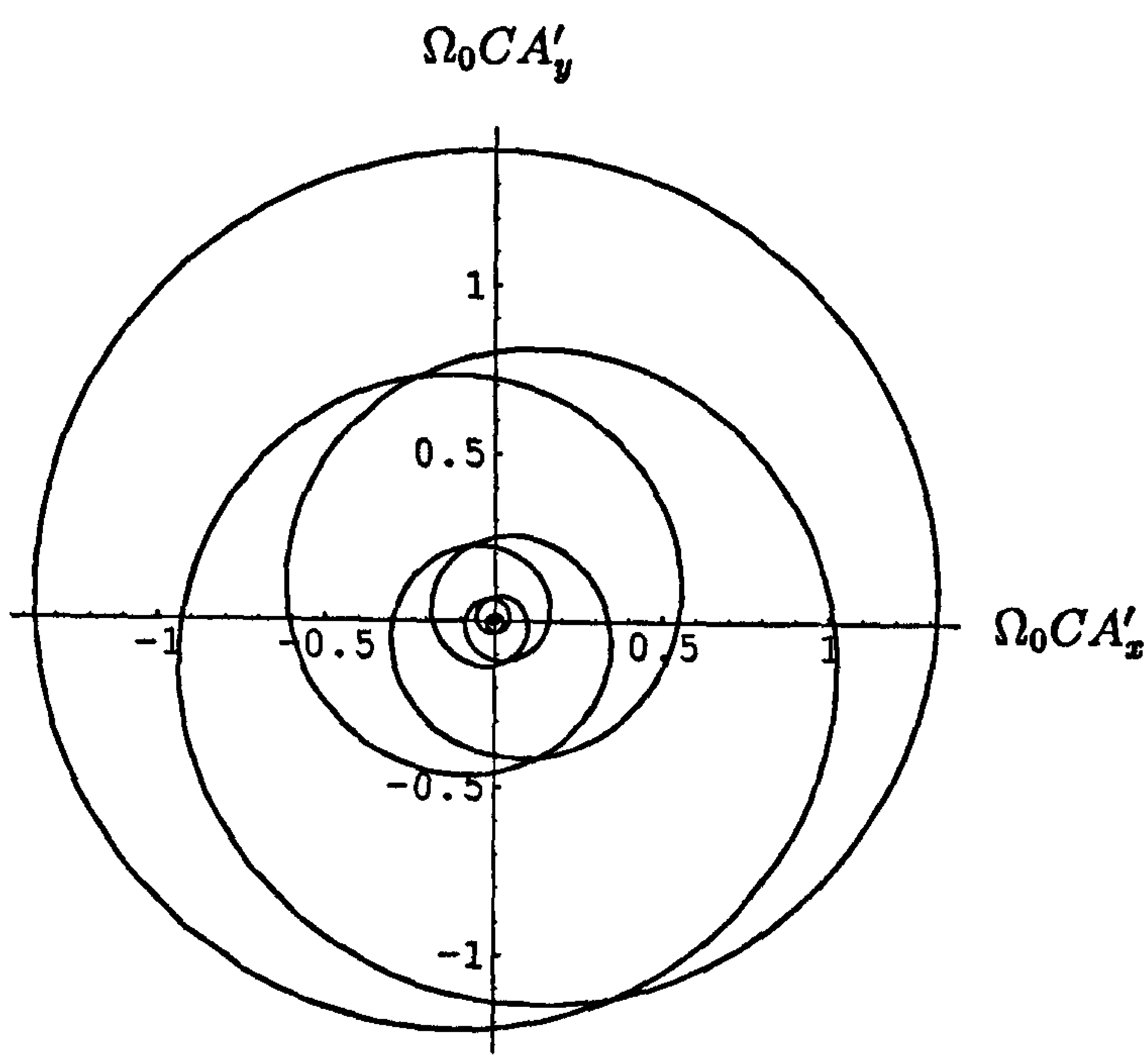
$$\begin{aligned} A'_y &= \frac{p}{\Omega_0 C}\left\{\frac{\left[-q\sin q\tau + \left(\Omega_0\left|\eta\right| + p\right)\cos q\tau\right]}{\sqrt{2\Omega_0\left|\eta\right|\left(\Omega_0\left|\eta\right|+p\right)}}\operatorname{sech}\left[p\tau + \ln\frac{\sqrt{2\Omega_0\left|\eta\right|\left(\Omega_0\left|\eta\right|+p\right)}}{q}\right]\right. \\ &- \left.\frac{\left[q\sin q\tau + \left(\Omega_0\left|\eta\right| - p\right)\cos q\tau\right]}{\sqrt{2\Omega_0\left|\eta\right|\left(\Omega_0\left|\eta\right|-p\right)}}\operatorname{sech}\left[p\tau + \ln\frac{\sqrt{2\Omega_0\left|\eta\right|\left(\Omega_0\left|\eta\right|-p\right)}}{q}\right]\right\}. \end{aligned}$$

There follow some illustrative graphs of the transformed field components in the case $p = 1$, $q = 5$:

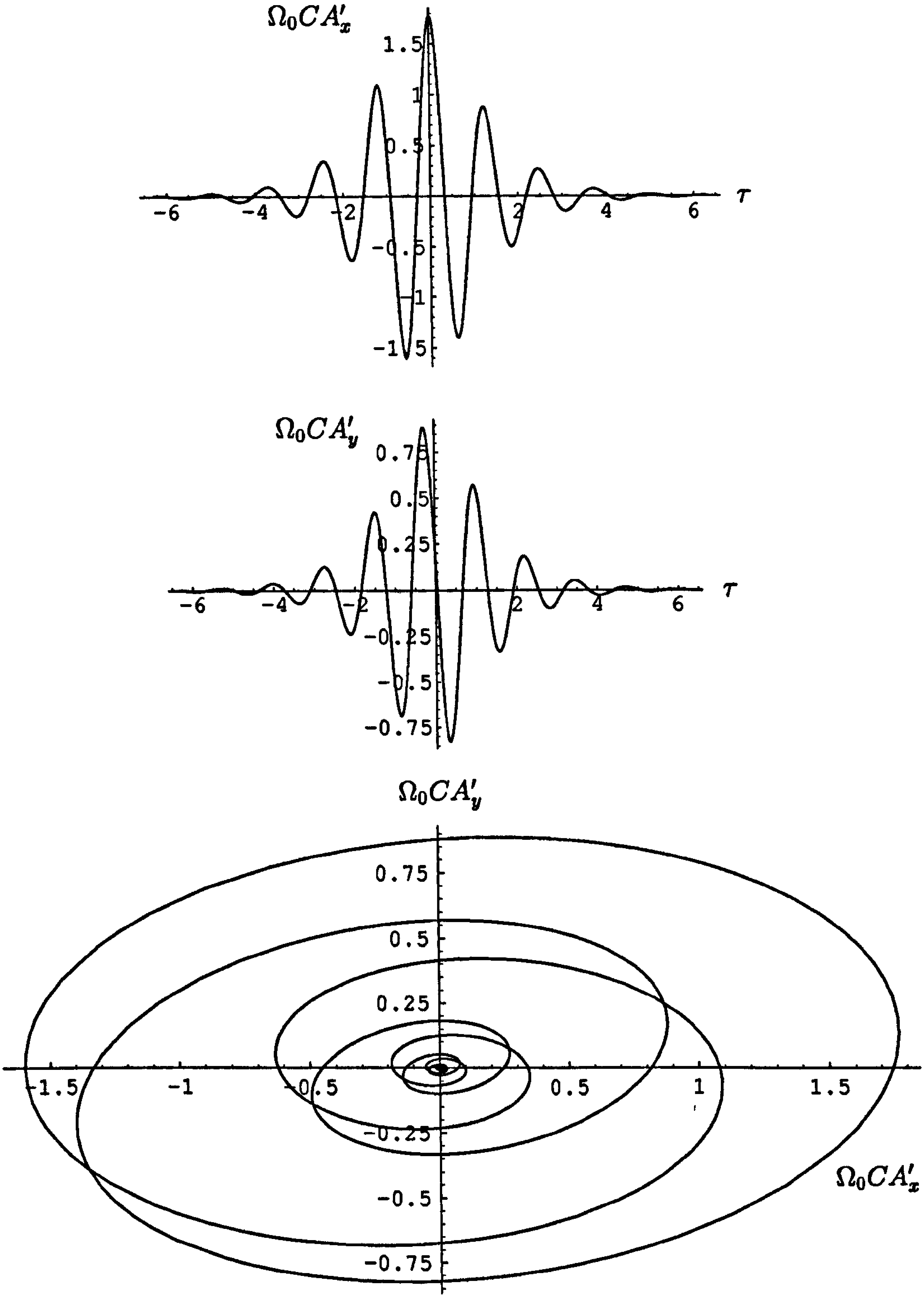
Circularly polarised solitons ($\Upsilon \to 1$).







Elliptically polarised solitons for the case $\Upsilon = 1/2$.



Chapter 5

Inverse Scattering for the RMB-System.

We proceed to solve the initial value-boundary value problem (stated on p. 35) for our RMB-equations by means of the inverse scattering transform method.

Consider the 3×3 AKNS-pair corresponding to the RMB-system which from p. 33 is known to be

$$\partial_\tau F = UF, \ \partial_z F = VF, \qquad (1,2)$$

where

$$U(\tau, z, \zeta) = \frac{-\iota}{\hbar}(\zeta H^- + H^+) \qquad (3)$$

$$= \frac{-\iota}{\hbar}\left[\frac{\zeta\hbar\Omega_0}{3}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \iota\Omega_0 e q_0 \begin{pmatrix} 0 & -A_x & -A_y \\ A_x & 0 & 0 \\ A_y & 0 & 0 \end{pmatrix}(\tau, z)\right],$$

$$V(\tau, z, \zeta) = \frac{\iota K}{(\zeta^2 - 1)}\left[\rho^+(\tau, z) + \zeta\rho^-(\tau, z)\right]. \qquad (4)$$

(H^+, H^- and ρ^+, ρ^- are the anti-symmetric and symmetric parts of H and ρ respectively, and $K := \frac{Ne^2 q_0^2}{2\hbar c \varepsilon_0}$).

Observe that equation (3) is a special case of equation (5) on p. 12, with $N = 3$ and $J_2 = J_3$. This simple degenerate property of the matrix U has far reaching consequences which were originally noticed and exploited by Manakov [31] in the separate context of Schrödinger equations. Manakov developed an inverse scattering scheme for a two component vector Nonlinear Schrödinger equation whose AKNS-pair is similar to the pair for our 3-level system. Unfortunately Manakov's scheme was never published in a particularly detailed form. Very recently ([32]) we discovered that Ablowitz, Prinari and Trubatch (2004) [33] have rectified this situation by giving a rigorous account of the inverse scattering transform for a generalised matrix Nonlinear Schrödinger equation. Nevertheless, since matrix Nonlinear Schrödinger equations do not possess the same (sine-Gordon-type) symmetries as the 3-level RMB-system, the inverse scattering scheme presented below remains the first of its kind. Specifically the degenerate structure of the matrix U (defined by (3)) means that we can find integral representations of Volterra rather than Fredholm type for the transition matrix, allowing us to use a generalised version of Faddeev and Takhtajan's method to solve the direct scattering problem. There is no need to apply the theory of, for example, Caudrey [34] or Beals and Coifman [35] [36], who studied and gave solution methods for general $N \times N$ ($N \geq 3$) direct and inverse scattering problems (cf. A&C., Section 3.1.2, p. 111). We have already described in Chapter 2 the procedural steps involved in using the IST to solve nonlinear PDEs associated to 2×2 AKNS-systems, and the strategy is no different for our 3×3 case. However, there are certain extra intrinsic difficulties which stem from the basic fact that the inverse of a 3×3 matrix is a much more algebraically complicated object than the inverse of a 2×2 matrix. We overcome these difficulties by employing symmetry relations and a crucial formula for the inverse of the transition matrix.

5.1 The Direct Scattering Problem.

5.1.1 Properties of the Transition Matrix.

The auxiliary linear equation and the transition matrix $T(\tau, \tau_0, \zeta)$ for the RMB-system are given by

$$\frac{dF}{d\tau} = U(\tau, z = 0, \zeta)F, \qquad (5)$$

and

$$T(\tau, \tau_0, \zeta) = \overleftarrow{\exp} \int_{\tau_0}^{\tau} U(s, \zeta)ds := \lim_{R\to\infty} \left\{ \left(I + \int_{\gamma_R} U ds \right) \ldots \left(I + \int_{\gamma_1} U ds \right) \right\} \quad (6)$$

respectively, where $\gamma_1, \ldots, \gamma_R$ partition the line segment $[\tau_0, \tau]$ on the τ-axis (cf. p. 14).

The transition matrix has a number of elementary properties (cf. F&T. pp. 26-27):

- It satisfies the auxiliary linear equation

  $$T_\tau(\tau, \tau_0, \zeta) = U(\tau, \zeta)T(\tau, \tau_0, \zeta) \qquad (7)$$

  with initial condition

  $$[T(\tau, \tau_0, \zeta)]_{\tau=\tau_0} = I. \qquad (8)$$

  If $A_x(\tau) = 0 = A_y(\tau)$, then $U = \frac{1}{3}\iota\Omega_0\zeta \mathrm{diag}\{2, -1, -1\}$ and the solution of the differential equation $(7, 8)$ is

  $$E(\tau - \tau_0, \zeta) := \exp \left[\zeta (\tau - \tau_0) \frac{\iota\Omega_0}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right]. \qquad (9)$$

- The superposition property

$$T(\tau,\tau_1,\zeta)T(\tau_1,\tau_0,\zeta) = T(\tau,\tau_0,\zeta), \text{ for } \tau_0 \le \tau_1 \le \tau,$$

which implies

$$T(\tau,\tau_0,\zeta) = T^{-1}(\tau_0,\tau,\zeta). \qquad (10)$$

- The differential equation for $T(\tau,\tau_0,\zeta)$ with respect to τ_0 is

$$T_{\tau_0}(\tau,\tau_0,\zeta) = -T(\tau,\tau_0,\zeta)U(\tau_0,\zeta), \qquad (11)$$

again with

$$[T(\tau,\tau_0,\zeta)]_{\tau_0=\tau} = I. \qquad (12)$$

- $\det T(\tau,\tau_0,\zeta) = 1$. *Proof:* It is easy if laborious to check directly using (7) that

$$\frac{\partial}{\partial\tau}\det T(\tau,\tau_0,\zeta) = \mathrm{TrU}(\tau,\zeta)\det T(\tau,\tau_0,\zeta).$$

Then since $\mathrm{Tr}U = 0$ and $\det\left\{[T(\tau,\tau_0,\zeta)]_{\tau=\tau_0}\right\} = 1$, it follows that

$$\det T(\tau,\tau_0,\zeta) = 1.$$

We now seek to find those symmetries of U which respect the definition of the transition matrix (6). Unfortunately, for our matrix U given by (3), there are no non-trivial involution relations of the kind (3.13) on p. 27 of F&T., namely relations of the form $\widetilde{U} = \sigma U\sigma$, where $\sigma^2 = I$ and $\widetilde{U}$ is obtained from U by some simple operation (such as complex conjugation) which respects the definition (6). (In fact this is the main reason why Faddeev and Takhtajan's method of solving the inverse problem fails to work for the RMB-system).

U does however satisfy

$$\overline{U}(\zeta) = U(-\overline{\zeta}) \qquad (13)$$

and

$$U^t(\zeta) = -\overline{U}(\overline{\zeta}), \qquad (14)$$

where the bars denote complex conjugation and the τ-dependence of U has been suppressed for clarity of expression.

It is immediate from equation (13) that

$$\overline{T}(\tau, \tau_0, \zeta) = T(\tau, \tau_0, -\overline{\zeta}), \qquad (15)$$

which incidentally means T is real when ζ is pure imaginary. Furthermore we claim that the following important relation holds

$$T^{-1}(\tau, \tau_0, \zeta) = \overline{T}^t(\tau, \tau_0, \overline{\zeta}).$$

Proof: Define

$$\Omega_R = \left(I + \int_{\gamma_R} U d\tau\right) \dots \left(I + \int_{\gamma_1} U d\tau\right) = \prod_{i=R}^{1} \left(I + \int_{\gamma_i} U d\tau\right),$$

and let $\Gamma_i = -\gamma_i$, $i = 1, ..., R$. (i.e., Γ_i is the line segment γ_i with negative orientation). Then

$$\begin{aligned} \overline{\Omega_R}^t(\tau, \tau_0, \zeta) &= \left(I + \int_{\gamma_1} \overline{U}^t(\zeta) d\tau\right) \dots \left(I + \int_{\gamma_R} \overline{U}^t(\zeta) d\tau\right) \\ &= \left(I - \int_{\gamma_1} U(\overline{\zeta}) d\tau\right) \dots \left(I - \int_{\gamma_R} U(\overline{\zeta}) d\tau\right) \end{aligned}$$

by (14). On the other hand, since

$$\left(I + \int_{\Gamma_1} U d\tau\right) \dots \left(I + \int_{\Gamma_R} U d\tau\right) \left(I + \int_{\gamma_R} U d\tau\right) \dots \left(I + \int_{\gamma_1} U d\tau\right) = I,$$

we have that

$$\Omega_R^{-1}(\tau, \tau_0, \zeta) = \left(I + \int_{\gamma_1} U(\zeta) d\tau\right)^{-1} \dots \left(I + \int_{\gamma_R} U(\zeta) d\tau\right)^{-1}$$

$$
\begin{aligned}
&= \left(I + \int_{\Gamma_1} U(\zeta)d\tau\right) \dots \left(I + \int_{\Gamma_R} U(\zeta)d\tau\right) \\
&= \left(I - \int_{\gamma_1} U(\zeta)d\tau\right) \dots \left(I - \int_{\gamma_R} U(\zeta)d\tau\right) \\
&= \overline{\Omega_R}^t(\tau, \tau_0, \overline{\zeta}).
\end{aligned}
$$

Therefore as $R \to \infty$ we obtain the stated identity.

Encapsulating the above using (10) and (15) gives

$$T(\tau_0, \tau, \zeta) = T^{-1}(\tau, \tau_0, \zeta) = \overline{T}^t(\tau, \tau_0, \overline{\zeta}) = T^t(\tau, \tau_0, -\zeta). \qquad (16)$$

Next we introduce notation for the elements of $T(\tau, \tau_0, \zeta)$:

$$T(\tau, \tau_0, \zeta) = \begin{pmatrix} \alpha & \beta & \gamma \\ \eta & \theta & \mu \\ \nu & \rho & \chi \end{pmatrix} (\tau, \tau_0, \zeta).$$

Since $\det T(\tau, \tau_0, \zeta) = 1$ and $T^{-1}(\tau, \tau_0, \zeta) = T^t(\tau, \tau_0, -\zeta)$, it is straightforward to show that the transition matrix can be written (τ, τ_0-dependence suppressed for clarity) as

$$T(\zeta) = \begin{pmatrix} (\theta\chi - \rho\mu)(-\zeta) & (\nu\mu - \eta\chi)(-\zeta) & (\eta\rho - \nu\theta)(-\zeta) \\ \eta(\zeta) & \theta(\zeta) & \mu(\zeta) \\ \nu(\zeta) & \rho(\zeta) & \chi(\zeta) \end{pmatrix} \qquad (17)$$

together with the *normalisation* conditions

$$\eta(\zeta)\eta(-\zeta) + \theta(\zeta)\theta(-\zeta) + \mu(\zeta)\mu(-\zeta) = 1, \qquad (18)$$

$$\nu(\zeta)\nu(-\zeta) + \rho(\zeta)\rho(-\zeta) + \chi(\zeta)\chi(-\zeta) = 1, \qquad (19)$$

and the *orthogonality* condition

$$\eta(\zeta)\nu(-\zeta)+\theta(\zeta)\rho(-\zeta)+\mu(\zeta)\chi(-\zeta)=0. \qquad (20)$$

In other words α, β and γ are to be thought of as dependent functions defined by

$$\begin{aligned}\alpha(\tau,\tau_0,\zeta) &= (\theta\chi-\rho\mu)(\tau,\tau_0,-\zeta),\\ \beta(\tau,\tau_0,\zeta) &= (\nu\mu-\eta\chi)(\tau,\tau_0,-\zeta),\\ \gamma(\tau,\tau_0,\zeta) &= (\eta\rho-\nu\theta)(\tau,\tau_0,-\zeta).\end{aligned}$$

We note that the conditions

$$\alpha(\zeta)\alpha(-\zeta)+\beta(\zeta)\beta(-\zeta)+\gamma(\zeta)\gamma(-\zeta)=1,$$

$$\alpha(\zeta)\eta(-\zeta)+\beta(\zeta)\theta(-\zeta)+\gamma(\zeta)\mu(-\zeta)=0,$$

$$\alpha(\zeta)\nu(-\zeta)+\beta(\zeta)\rho(-\zeta)+\gamma(\zeta)\chi(-\zeta)=0,$$

are implicit from equations $(18-20)$.

5.1.2 Integral Representations for the Transition Matrix.

Suppose we write

$$U(\tau,\zeta)=U_0\left(A_x(\tau),A_y(\tau)\right)+\zeta U_1.$$

Then the integral equations corresponding to the differential problems $(7,8)$ and $(11,12)$ are

$$T(\tau,\tau_0,\zeta)=E(\tau-\tau_0,\zeta)+\int_{\tau_0}^{\tau}E(\tau-s,\zeta)U_0(s)T(s,\tau_0,\zeta)ds \qquad (21)$$

and

$$T(\tau,\tau_0,\zeta) = E(\tau-\tau_0,\zeta) + \int_{\tau_0}^{\tau} T(\tau,s,\zeta)U_0(s)E(s-\tau_0,\zeta)ds \qquad (22)$$

respectively. We shall prove (22): the proof of (21) is similar.

Firstly the equality $E(\tau-\tau_0,\zeta) = \exp\left[\zeta\,(\tau-\tau_0)\,U_1\right]$ implies

$$\frac{\partial}{\partial s}E(s-\tau_0,\zeta) = \zeta U_1 E(s-\tau_0,\zeta), \qquad (23)$$

whilst integrating equation (11) between τ_0 and τ gives

$$\int_{\tau_0}^{\tau}\frac{\partial T}{\partial s}(\tau,s,\zeta)E(s-\tau_0,\zeta)ds = -\int_{\tau_0}^{\tau} T(\tau,s,\zeta)U(s,\zeta)E(s-\tau_0,\zeta)ds. \qquad (24)$$

Hence using (23) and integrating by parts we find

$$\int_{\tau_0}^{\tau} T(\tau,s,\zeta)\left[U\,(s,\zeta)-\zeta U_1\right]E(s-\tau_0,\zeta)ds =$$

$$\begin{aligned}\int_{\tau_0}^{\tau} T(\tau,s,\zeta)U(s,\zeta)E(s-\tau_0,\zeta)ds \quad &- \quad E(\tau-\tau_0,\zeta) + \\ \int_{\tau_0}^{\tau}\frac{\partial T}{\partial s}(\tau,s,\zeta)E(s-\tau_0,\zeta)ds \quad &+ \quad T(\tau,\tau_0,\zeta).\end{aligned}$$

By (24) the first and third terms on the right-hand side cancel, and the remaining equation is (22).

We now assert that iterative analysis of the Volterra equations (21) and (22) leads to the following integral representations for $T(\tau,\tau_0,\zeta)$:

$$T(\tau,\tau_0,\zeta) = E(\tau-\tau_0,\zeta)\left[I + \int_{\tau_0}^{\tau} F(s-\tau_0,\zeta)\tilde{\Gamma}(\tau,\tau_0,s)ds\right], \qquad (25)$$

and

$$T(\tau,\tau_0,\zeta)=\left[I+\int_{\tau_0}^{\tau}\Gamma(\tau,\tau_0,s)F(\tau-s,\zeta)ds\right]E(\tau-\tau_0,\zeta), \quad (26)$$

where

$$\tilde{\Gamma}(\tau,\tau_0,s)=U_0(s)+\int_{s}^{\tau}U_0(r)\tilde{\Gamma}(r,\tau_0,r-s+\tau_0)dr, \quad (27)$$

and

$$\Gamma(\tau,\tau_0,s)=U_0(s)+\int_{\tau_0}^{s}\Gamma(\tau,r,r-s+\tau)U_0(r)dr. \quad (28)$$

Here $\tau_0\le s\le\tau$ and $F(\tau,\zeta):=\exp\left[-\iota\Omega_0\zeta\tau\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\right]$.

Equations $(25-28)$ are proved by using the symmetry relations

$$\begin{aligned}U_0E(\tau,\zeta)&=E(\tau,\zeta)F(\tau,\zeta)U_0,\\ U_0F(\tau,\zeta)&=F(-\tau,\zeta)U_0,\end{aligned} \quad (29)$$

which together also give

$$E(\tau,\zeta)U_0=U_0F(\tau,\zeta)E(\tau,\zeta). \quad (30)$$

Let us study the iterations of equation (22). Being a Volterra integral equation and taking into account the postulated behaviour of $A_x(\tau)$, $A_y(\tau)$, we are assured that the Neumann series must converge absolutely. For the first iteration we write

$$\begin{aligned}T_1(\tau,\tau_0,\zeta)&=E(\tau-\tau_0,\zeta)+\int_{\tau_0}^{\tau}E(\tau-s,\zeta)U_0(s)E(s-\tau_0,\zeta)ds\\ &=\left[I+\int_{\tau_0}^{\tau}U_0(s)F(\tau-s,\zeta)ds\right]E(\tau-\tau_0,\zeta)\end{aligned}$$

by virtue of the symmetry relation (30). Comparing this with equation (26) we see that $\Gamma_1(\tau,\tau_0,s) = U_0(s)$. Secondly, making use of (30), we find

$$(T_2 - T_1)(\tau,\tau_0,\zeta) = \left[\int_{\tau_0}^{\tau}\int_{s}^{\tau} U_0(r)U_0(s)F(r-s,\zeta)drds\right] E(\tau-\tau_0,\zeta).$$

If we substitute for $u = s - r + \tau$, change the order of integration and relabel the dummy variables, then this becomes

$$(T_2 - T_1)(\tau,\tau_0,\zeta) = \left[\int_{\tau_0}^{\tau}\int_{\tau_0}^{s} U_0(r-s+\tau)U_0(r)drF(\tau-s,\zeta)ds\right] E(\tau-\tau_0,\zeta),$$

from which we deduce

$$\Gamma_2(\tau,\tau_0,s) = U_0(s) + \int_{\tau_0}^{s} U_0(r-s+\tau)U_0(r)dr.$$

Continuing in the same fashion yields

$$(T_3 - T_2)(\tau,\tau_0,\zeta)E(\tau_0-\tau,\zeta) =$$

$$\int_{\tau_0}^{\tau}\int_{\tau_0}^{s}\left[\int_{r}^{r-s+\tau} U_0(u+s-r)U_0(u)du\right]U_0(r)drF(\tau-s,\zeta)ds,$$

and

$$\Gamma_3(\tau,\tau_0,s) = U_0(s) + \int_{\tau_0}^{s} \Gamma_2(\tau,r,r-s+\tau)U_0(r)dr.$$

The pattern is now apparent, and so as the number of iterations tends to infinity equations (26) and (28) result. Of course the representation (25) with $\tilde{\Gamma}$ satisfying (27) is derived similarly using symmetry relation (29).

Lastly we remark that Γ and $\tilde{\Gamma}$ are connected by the formulae

$$\Gamma^R(\tau,\tau_0,s) = \tilde{\Gamma}^R(\tau,\tau_0,s)F(2s-\tau-\tau_0,\zeta), \qquad (31)$$

$$\Gamma^{OR}(\tau,\tau_0,s) = \tilde{\Gamma}^{OR}(\tau,\tau_0,s), \qquad (32)$$

where the R and OR-parts of an arbitrary 3×3 matrix M refer to the decomposition

$$\begin{aligned} M &= M^R + M^{OR} \\ &= \begin{pmatrix} M^{11} & 0 & 0 \\ 0 & M^{22} & M^{23} \\ 0 & M^{32} & M^{33} \end{pmatrix} + \begin{pmatrix} 0 & M^{12} & M^{13} \\ M^{21} & 0 & 0 \\ M^{31} & 0 & 0 \end{pmatrix}. \end{aligned}$$

The identities (31, 32) follow directly from the integral representations (25, 26) and the commutator relations

$$\left[E, T^R\right] = 0, \ \left[F, \tilde{\Gamma}^R\right] = 0 = \left[F, \Gamma^R\right].$$

5.1.3 Integral Representations for the Jost Solutions.

It is first necessary to introduce some notation. Let $L^1(\mathbf{R})$ be the vector space of absolutely integrable, measurable functions on $\mathbf{R}$ with the standard norm, and suppose $L^1_{3\times3}(\mathbf{R})$ is the normed space of 3×3 matrix functions M such that $M^{ij} \in L^1(\mathbf{R}), i, j = 1, 2, 3$, with norm

$$\|M\| := \int_{-\infty}^{\infty} \sum_{i,j=1,2,3} \left|M^{ij}\right|(s)ds < \infty.$$

Now the Jost solution matrices are defined to be the limits

$$T_{\pm}(\tau, \zeta) = \lim_{\tau_0 \to \pm\infty} \{T(\tau, \tau_0, \zeta)E(\tau_0, \zeta)\},$$

for $\zeta \in \mathbf{R}$, (cf. p. 15). The proof that these limits exist providing $A_x(\tau)$, $A_y(\tau) \in L^1(\mathbf{R})$ exactly parallels the existence proof for the 2×2 solutions $T_{\pm}(x, \lambda)$ in F&T. pp. 39-40, and we choose to omit it here.

From equations (26) and (28) we obtain

$$\begin{aligned}\lim_{\tau_0\to-\infty}\{T(\tau,\tau_0,\zeta)E(\tau_0,\zeta)\} &:= T_-(\tau,\zeta)\\ &= \left[I+\int_{-\infty}^{\tau}\Gamma_-(\tau,s)F(\tau-s,\zeta)ds\right]E(\tau,\zeta), \quad (33)\end{aligned}$$

where

$$\Gamma_-(\tau,s) := \lim_{\tau_0\to-\infty}\Gamma(\tau,\tau_0,s) = U_0(s)+\int_{-\infty}^{s}\Gamma(\tau,r,r-s+\tau)U_0(r)dr, \quad (34)$$

and for each fixed τ, $\Gamma_-(\tau,s)\in L^1_{3\times3}(-\infty,\tau)$. [Since one can show

$$\int_{-\infty}^{\tau}\|\Gamma_-(\tau,s)\|\,ds \le \left[\int_{-\infty}^{\tau}\|U_0(s)\|\,ds\right]\exp\left(\int_{-\infty}^{\tau}\|U_0(s)\|\,ds\right),$$

where by assumption $\int_{-\infty}^{\infty}\|U_0(s)\|\,ds<\infty$.]

Next consider

$$\lim_{\tau_0\to+\infty}\{T(\tau,\tau_0,\zeta)E(\tau_0,\zeta)\}.$$

We know $[T(\tau,\tau_0,\zeta)E(\tau_0,\zeta)]^{-1}=E(-\tau_0,\zeta)T(\tau_0,\tau,\zeta)$ and equation (25) says

$$T(\tau_0,\tau,\zeta)=E(\tau_0-\tau,\zeta)\left[I+\int_{\tau}^{\tau_0}F(s-\tau,\zeta)\tilde{\Gamma}(\tau_0,\tau,s)ds\right].$$

Therefore

$$\begin{aligned}T_+^{-1}(\tau,\zeta) &= \lim_{\tau_0\to+\infty}\{E(-\tau_0,\zeta)T(\tau_0,\tau,\zeta)\}\\ &= E(-\tau,\zeta)\left[I+\int_{\tau}^{\infty}F(s-\tau,\zeta)\tilde{\Gamma}_+(\tau,s)ds\right],\end{aligned}$$

where

$$\tilde{\Gamma}_+(\tau,s)=\lim_{\tau_0\to+\infty}\tilde{\Gamma}(\tau_0,\tau,s)=U_0(s)+\int_{s}^{\infty}U_0(r)\tilde{\Gamma}(r,\tau,r-s+\tau)dr, \quad (35)$$

and for each fixed τ, $\tilde{\Gamma}_+(\tau,s)\in L^1_{3\times3}(\tau,\infty)$. On the other hand it is immediate from equation (16) and the definitions of $T_\pm(\tau,\zeta)$ that

$$T_\pm^{-1}(\tau,\zeta) = T_\pm^t(\tau,-\zeta) \left(= \overline{T_\pm}^t(\tau,\zeta), \text{ when } \zeta \in \mathbf{R}\right). \qquad (36)$$

Hence

$$\begin{aligned} T_+(\tau,\zeta) &:= \lim_{\tau_0\to+\infty}\{T(\tau,\tau_0,\zeta)E(\tau_0,\zeta)\} \\ &= \left[I + \int_\tau^\infty \tilde{\Gamma}_+^t(\tau,s)F(\tau-s,\zeta)ds\right]E(\tau,\zeta). \qquad (37)\end{aligned}$$

Notice $T_\pm$ satisfy the auxiliary differential equation

$$\frac{dT_\pm}{d\tau} = U(\tau,\zeta)T_\pm,$$

with

$$T_\pm(\tau,\zeta) = E(\tau,\zeta) + o(1), \text{ as } \tau \to \pm\infty.$$

5.1.4 The Reduced Monodromy Matrix $T(\zeta)$.

For $\zeta \in \mathbf{R}$,

$$T(\zeta) := \lim_{\tau\to+\infty,\tau_0\to-\infty}\{E(-\tau,\zeta)T(\tau,\tau_0,\zeta)E(\tau_0,\zeta)\}, \qquad (38)$$

and so by (16) it follows that

$$T^{-1}(\zeta) = T^t(-\zeta) \left(= \overline{T}^t(\zeta) \text{ when } \zeta \in \mathbf{R}\right). \qquad (39)$$

Furthermore since $\det T(\tau,\tau_0,\zeta) = 1$, we also have

$$\det T_\pm(\tau,\zeta) = 1 = \det T(\zeta). \qquad (40)$$

Therefore we conclude that the Jost solution matrices and the reduced monodromy matrix are unitary with determinant 1. They are defined for $\zeta \in \mathbf{R}$, but their columns and coefficients respectively may be extended into either the upper half ζ-plane Π^+, or the lower half ζ-plane Π^-, according to the fundamental integral representations (33, 37) of $T_\pm(\tau,\zeta)$. In fact, writing

$$T_\pm(\tau,\zeta) = \left(T_\pm^{(1)}, T_\pm^{(2)}, T_\pm^{(3)}\right)(\tau,\zeta),$$

we deduce from (33, 37) that the columns $T_+^{(1)}(\tau,\zeta)$, $T_-^{(2)}(\tau,\zeta)$, $T_-^{(3)}(\tau,\zeta)$ extend analytically into Π^+, whereas $T_-^{(1)}(\tau,\zeta)$, $T_+^{(2)}(\tau,\zeta)$, $T_+^{(3)}(\tau,\zeta)$ extend analytically into Π^-. Moreover, because $\Gamma_-(\tau,s)$, $\tilde{\Gamma}_+(\tau,s)$ are absolutely integrable functions of s for each fixed τ, the Riemann-Lebesgue Lemma implies for $\mathrm{Im}\zeta \geq 0$, $|\zeta| \to \infty$,

$$\begin{aligned}
T_+^{(1)}(\tau,\zeta)\exp\left(-\frac{2}{3}\iota\Omega_0\zeta\tau\right) &= \begin{pmatrix}1\\0\\0\end{pmatrix} + o(1),\\
T_-^{(2)}(\tau,\zeta)\exp\left(\frac{1}{3}\iota\Omega_0\zeta\tau\right) &= \begin{pmatrix}0\\1\\0\end{pmatrix} + o(1),\\
T_-^{(3)}(\tau,\zeta)\exp\left(\frac{1}{3}\iota\Omega_0\zeta\tau\right) &= \begin{pmatrix}0\\0\\1\end{pmatrix} + o(1), \qquad (41-43)
\end{aligned}$$

whereas for $\mathrm{Im}\zeta \leq 0$, $|\zeta| \to \infty$,

$$T_-^{(1)}(\tau,\zeta)\exp\left(-\frac{2}{3}\iota\Omega_0\zeta\tau\right) = \begin{pmatrix}1\\0\\0\end{pmatrix} + o(1),$$

$$T_+^{(2)}(\tau,\zeta)\exp\left(\frac{1}{3}\iota\Omega_0\zeta\tau\right) = \begin{pmatrix}0\\1\\0\end{pmatrix} + o(1),$$

$$T_+^{(3)}(\tau,\zeta)\exp\left(\frac{1}{3}\iota\Omega_0\zeta\tau\right) = \begin{pmatrix}0\\0\\1\end{pmatrix} + o(1). \qquad (44-46)$$

Now (cf. (13) p.16)

$$\begin{aligned} T(\zeta) &= T_+^{-1}(\tau,\zeta)T_-(\tau,\zeta) \\ &= T_+^t(\tau,-\zeta)T_-(\tau,\zeta), \qquad (47) \end{aligned}$$

which gives

$$\begin{pmatrix}\alpha & \beta & \gamma\\ \eta & \theta & \mu\\ \nu & \rho & \chi\end{pmatrix}(\zeta) =$$

$$\begin{pmatrix} T_+^{(1)}(-\zeta)\cdot T_-^{(1)}(\zeta), & T_+^{(1)}(-\zeta)\cdot T_-^{(2)}(\zeta), & T_+^{(1)}(-\zeta)\cdot T_-^{(3)}(\zeta) \\ T_+^{(2)}(-\zeta)\cdot T_-^{(1)}(\zeta), & T_+^{(2)}(-\zeta)\cdot T_-^{(2)}(\zeta), & T_+^{(2)}(-\zeta)\cdot T_-^{(3)}(\zeta) \\ T_+^{(3)}(-\zeta)\cdot T_-^{(1)}(\zeta), & T_+^{(3)}(-\zeta)\cdot T_-^{(2)}(\zeta), & T_+^{(3)}(-\zeta)\cdot T_-^{(3)}(\zeta) \end{pmatrix}, \qquad (48)$$

where $T_+^{(i)}(-\zeta)\cdot T_-^{(j)}(\zeta) := \sum_{k=1}^{3} T_+^{ki}(-\zeta)T_-^{kj}(\zeta)$, $i,j = 1,2,3$, and we have retained the same Greek letters to denote the transition coefficients that were used for corresponding elements of the transition matrix: so for instance $T^{11}(\zeta) := \alpha(\zeta) = (\theta\chi-\rho\mu)(-\zeta)$ whilst $T^{11}(\tau,\tau_0,\zeta) := \alpha(\tau,\tau_0,\zeta) = (\theta\chi-\rho\mu)(\tau,\tau_0,-\zeta)$.

"Substituting" the analytic and asymptotic behaviour $(41-46)$ of $T_\pm(\tau,\zeta)$ into the right-hand side of (48) establishes the following:

- $\alpha(\zeta) = (\theta\chi-\rho\mu)(-\zeta)$ may be analytically extended into Π^-, with $\alpha(\zeta) = 1+o(1)$, for $\mathrm{Im}\zeta \le 0$, as $|\zeta| \to \infty$.

- $\theta(\zeta)$, $\mu(\zeta)$, $\rho(\zeta)$, $\chi(\zeta)$ may be analytically extended into Π^+, with $\theta(\zeta) =$

$1 + o(1)$, $\mu(\zeta) = o(1)$, $\rho(\zeta) = o(1)$, $\chi(\zeta) = 1 + o(1)$, for $\text{Im}\zeta \geq 0$, as $|\zeta| \to \infty$.

- The coefficients $\beta(\zeta) = (\nu\mu - \eta\chi)(-\zeta)$, $\gamma(\zeta) = (\eta\rho - \nu\theta)(-\zeta)$, $\eta(\zeta)$, and $\nu(\zeta)$ have in general no analytic continuation away from the real line.

In addition there are normalisation and orthogonality conditions for the transition coefficients given for real ζ by (cf. (18 – 20))

$$\eta(\zeta)\eta(-\zeta) + \theta(\zeta)\theta(-\zeta) + \mu(\zeta)\mu(-\zeta) = |\eta(\zeta)|^2 + |\theta(\zeta)|^2 + |\mu(\zeta)|^2 = 1, \quad (49)$$

$$\nu(\zeta)\nu(-\zeta) + \rho(\zeta)\rho(-\zeta) + \chi(\zeta)\chi(-\zeta) = |\nu(\zeta)|^2 + |\rho(\zeta)|^2 + |\chi(\zeta)|^2 = 1, \quad (50)$$

and

$$0 = \eta(\zeta)\nu(-\zeta) + \theta(\zeta)\rho(-\zeta) + \mu(\zeta)\chi(-\zeta) =$$
$$\eta(\zeta)\overline{\nu}(\zeta) + \theta(\zeta)\overline{\rho}(\zeta) + \mu(\zeta)\overline{\chi}(\zeta). \quad (51)$$

5.1.5 Integral Representations for the Transition Coefficients.

We will need the results here in connection with the existence of solutions to the inverse problem.

Let R_c be the ring consisting of functions of the form

$$F(\zeta) = \mathrm{F}\left[c\delta(s) + f(s)\right] = c + \int_{-\infty}^{\infty} f(s)e^{\iota\zeta s}ds,$$

where $f(s) \in L^1(\mathbf{R})$, and F denotes the Fourier transform operator. $R_c^{\pm}$ is the subring of R_c with elements of the form

$$F_{\pm}(\zeta) = c + \int_0^{\infty} f_{\pm}(s)e^{\pm\iota\zeta s}ds,$$

where $f_{\pm}(s) \in L^1(0,\infty)$. Functions in $R_c^{\pm}$ extend analytically into $\Pi^{\pm}$.

We claim $\alpha(\zeta) \in R_1^-$, $\theta(\zeta)$, $\chi(\zeta) \in R_1^+$, $\mu(\zeta)$, $\rho(\zeta) \in R_0^+$ and

$$\eta(\zeta),\ \nu(\zeta),\ \beta(\zeta),\ \gamma(\zeta) \in R_0.$$

Proof. Let us check for example that $\alpha(\zeta) \in R_1^-$, $\eta(\zeta) \in R_0$. Consider the definition (38) of the reduced monodromy matrix. If we let τ, $-\tau_0$ tend to $+\infty$ at the same rate, then (38) can be rewritten

$$T(\zeta) = \lim_{L\to\infty} \{E(-L,\zeta)T(L,-L,\zeta)E(-L,\zeta)\}, \text{ for } \zeta \in \mathbf{R},$$

so that in particular we have

$$\alpha(\zeta) = \lim_{L\to\infty} \left\{e^{-\frac{4}{3}\iota L\Omega_0\zeta}\alpha_L(\zeta)\right\},\ \eta(\zeta) = \lim_{L\to\infty} \left\{e^{-\frac{1}{3}\iota L\Omega_0\zeta}\eta_L(\zeta)\right\}.$$

Now according to equation (26) the monodromy matrix $T(L,-L,\zeta)$ has the integral representation

$$T(L,-L,\zeta) = \left[I + \int_{-L}^{L} \Gamma(L,-L,s)F(L-s,\zeta)ds\right] E(2L,\zeta).$$

Writing out the $(1,1)$ and $(2,1)$-elements of this matrix equation gives

$$\alpha_L(\zeta)e^{-\frac{4}{3}\iota L\Omega_0\zeta} = 1 + \int_{-2L}^{0} \Gamma^{11}(L,-L,s+L)e^{\iota\Omega_0\zeta s}ds, \qquad (52)$$

$$\eta_L(\zeta)e^{-\frac{1}{3}\iota L\Omega_0\zeta} = e^{\iota L\Omega_0\zeta}\int_{-2L}^{0} \Gamma^{21}(L,-L,s+L)e^{\iota\Omega_0\zeta s}ds. \qquad (53)$$

Using equations (28), (31, 32), and the fact that $U_0(\tau) = \Omega_0 C \begin{pmatrix} 0 & -A_x & -A_y \\ A_x & 0 & 0 \\ A_y & 0 & 0 \end{pmatrix}(\tau)$, we find

$$\Gamma^{11}(L,-L,s+L) = \Omega_0 C\int_{-L}^{s+L} \tilde{\Gamma}^{12}(L,r,r-s)A_x(r)+\tilde{\Gamma}^{13}(L,r,r-s)A_y(r)dr, \quad (54)$$

$$
\begin{aligned}
\frac{e^{\iota L\Omega_0\zeta}}{\Omega_0 C}\Gamma^{21}(L,-L,s+L) &= e^{\iota L\Omega_0\zeta}A_x(s+L)+ \\
&\int_{-L}^{s+L}\left[\tilde{\Gamma}^{22}(L,r,r-s)\,A_x(r)+\right. \\
&\left.\tilde{\Gamma}^{23}(L,r,r-s)\,A_y(r)\right]e^{\iota\Omega_0\zeta(r-2s)}d\tau. \quad (55)
\end{aligned}
$$

Since $L_1^{3\times 3}(\tau,\infty) \ni \tilde{\Gamma}_+(\tau,s) := \lim_{\tau_0\to\infty}\tilde{\Gamma}(\tau_0,\tau,s)$ for each fixed $\tau \le s$, it is valid to substitute (54) in (52) and let $L \to \infty$, obtaining

$$
\alpha(\zeta) = 1+\int_{-\infty}^{0}\left[\Omega_0 C\int_{-\infty}^{\infty}\tilde{\Gamma}_+^{12}(r,r-s)A_x(r)+\tilde{\Gamma}_+^{13}(r,r-s)A_y(r)dr\right]e^{\iota\Omega_0\zeta s}ds,
$$

which may be written as

$$
\alpha(\zeta) = 1+\int_0^{\infty}\hat{\alpha}(s)e^{-\iota\zeta s}ds,
$$

with $\hat{\alpha}(s) \in L^1(0,\infty)$, after a trivial rescaling of the integration variable s. Similarly if we substitute (55) into (53), change the order of integration in the double integral, and let $L \to \infty$, then we find

$$
\begin{aligned}
\eta(\zeta) &= \int_{-\infty}^{\infty}\Omega_0 C\left\{A_x(s)+\int_{-\infty}^{0}\left[\tilde{\Gamma}_+^{22}(s,s-r)A_x(s)+\right.\right. \\
&\left.\left.\tilde{\Gamma}_+^{23}(s,s-r)A_y(s)\right]e^{-\iota\Omega_0\zeta r}dr\right\}e^{\iota\Omega_0\zeta s}ds,
\end{aligned}
$$

which may be written

$$
\eta(\zeta) = \int_{-\infty}^{\infty}\hat{\eta}(s)e^{\iota\zeta s}ds,
$$

with $\hat{\eta}(s) \in L^1(\mathbf{R})$, after rescaling s.

5.1.6 Temporal Evolution of the Transition Coefficients.

The evolution equation for the transition matrix is given by (cf. F&T. pp. 28-29)

$$
\frac{\partial}{\partial z}T(\tau,\tau_0,z,\zeta) = V(\tau,z,\zeta)T(\tau,\tau_0,z,\zeta)-T(\tau,\tau_0,z,\zeta)V(\tau_0,z,\zeta).
$$

$$\text{i.e.,}\quad T_z(\tau,\tau_0,\zeta) = V(\tau,\zeta)T(\tau,\tau_0,\zeta) - T(\tau,\tau_0,\zeta)V(\tau_0,\zeta), \qquad (56)$$

suppressing the z argument.

We know that $T_{\pm}(\tau,\zeta) := \lim_{\tau_0\to\pm\infty}\{T(\tau,\tau_0,\zeta)E(\tau_0,\zeta)\}$, and also (cf. p. 41)

$$\lim_{\tau\to\pm\infty}\{V(\tau,z,\zeta)\} = \frac{2\iota K\zeta}{3(\zeta^2-1)}\operatorname{diag}\{2,-1,-1\},$$

where $\mathbf{R} \ni K = \frac{Ne^2q_0^2}{2\hbar c\varepsilon_0} > 0$. Therefore, if we multiply (56) on the right by $E(\tau_0,\zeta)$, then examination of the limit

$$\begin{aligned}\lim_{\tau_0\to\pm\infty}\{T_z(\tau,\tau_0,\zeta)E(\tau_0,\zeta)\} &= \lim_{\tau_0\to\pm\infty}\{V(\tau,\zeta)T(\tau,\tau_0,\zeta)E(\tau_0,\zeta)- \\ &\quad T(\tau,\tau_0,\zeta)V(\tau_0,\zeta)E(\tau_0,\zeta)\}\end{aligned}$$

obviously leads to

$$\frac{\partial}{\partial z}T_{\pm}(\tau,\zeta) = V(\tau,\zeta)T_{\pm}(\tau,\zeta) - \frac{2\iota K\zeta}{3(\zeta^2-1)}T_{\pm}(\tau,\zeta)\begin{pmatrix}2&0&0\\0&-1&0\\0&0&-1\end{pmatrix}. \qquad (57)$$

In a similar way, bearing in mind

$$T(\zeta) := \lim_{\tau\to+\infty,\tau_0\to-\infty}\{E(-\tau,\zeta)T(\tau,\tau_0,\zeta)E(\tau_0,\zeta)\},$$

we obtain

$$T_z(\zeta) = \frac{2\iota K\zeta}{3(\zeta^2-1)}\left[\begin{pmatrix}2&0&0\\0&-1&0\\0&0&-1\end{pmatrix}, T(\zeta)\right]. \qquad (58)$$

As we shall see, the differential equations (57), (58) completely determine remarkably simple temporal evolution of the scattering data for our RMB-system.

From equation (58) it is clear that the transition coefficients θ, μ, ρ, χ and α are all independent of z:

$$\begin{aligned}\theta(z,\zeta)=\theta(0,\zeta) \quad &, \quad \mu(z,\zeta)=\mu(0,\zeta),\\ \rho(z,\zeta)=\rho(0,\zeta) \quad &, \quad \chi(z,\zeta)=\chi(0,\zeta),\\ \text{and } \alpha(z,\zeta)=\alpha(0,\zeta) \quad &= \quad (\theta\chi-\rho\mu)(0,-\zeta), \qquad (59-63)\end{aligned}$$

whilst for $\zeta\in\mathbf{R}$, η, ν, β and γ evolve in accordance with the trivial equations

$$\eta(z,\zeta)=\eta(0,\zeta)\exp\left(\frac{-2\iota K\zeta z}{\zeta^2-1}\right),\ \nu(z,\zeta)=\nu(0,\zeta)\exp\left(\frac{-2\iota K\zeta z}{\zeta^2-1}\right), \quad (64,65)$$

$$\beta(z,\zeta)=(\nu\mu-\eta\chi)(z,-\zeta)=[(\nu\mu-\eta\chi)(0,-\zeta)]\exp\left(\frac{2\iota K\zeta z}{\zeta^2-1}\right),$$

$$\text{and } \gamma(z,\zeta)=(\eta\rho-\nu\theta)(z,-\zeta)=[(\eta\rho-\nu\theta)(0,-\zeta)]\exp\left(\frac{2\iota K\zeta z}{\zeta^2-1}\right).$$

5.1.7 The Scattering Matrix for the Direct Scattering Problem.

Through a simple transformation of the completeness relation (47)

$$T_-(\tau,\zeta)=T_+(\tau,\zeta)T(\zeta),$$

we gain a new equation which not only helps identify the scattering data, but is also fundamental to the inverse problem.

Suppose we define the matrices

$$S_+(\tau,\zeta)=\left(T_+^{(1)},T_-^{(2)},T_-^{(3)}\right)(\tau,\zeta),\ S_-(\tau,\zeta)=\left(T_-^{(1)},T_+^{(2)},T_+^{(3)}\right)(\tau,\zeta). \quad (66,67)$$

Then from pp. 64-65 we know that $S_\pm$ are analytic in $\Pi^\pm$ with the asymptotic behaviour as $|\zeta|\to\infty$

$$S_+(\tau,\zeta)E^{-1}(\tau,\zeta) \quad = \quad I+o(1),\ \text{for}\,\mathrm{Im}\zeta\geq 0, \qquad (68)$$

$$S_-(\tau,\zeta)E^{-1}(\tau,\zeta) \quad = \quad I + o(1), \ \text{for} \, \text{Im}\zeta \le 0. \qquad (69)$$

Furthermore, by writing out columnwise the two forms $T_-(\tau,\zeta) = T_+(\tau,\zeta)T(\zeta)$ and $T_+(\tau,\zeta) = T_-(\tau,\zeta)T^t(-\zeta)$ of equation (47), it is easy to show that

$$S_+(\tau,\zeta) = \left(T_+^{(1)}, T_+^{(2)}, T_+^{(3)}\right)(\tau,\zeta) \begin{pmatrix} 1 & \beta & \gamma \\ 0 & \theta & \mu \\ 0 & \rho & \chi \end{pmatrix} (\zeta), \qquad (70)$$

$$S_-(\tau,\zeta) = \left(T_+^{(1)}, T_+^{(2)}, T_+^{(3)}\right)(\tau,\zeta) \begin{pmatrix} \alpha & 0 & 0 \\ \eta & 1 & 0 \\ \nu & 0 & 1 \end{pmatrix} (\zeta), \qquad (71)$$

and

$$S_+(\tau,\zeta) = \left(T_-^{(1)}, T_-^{(2)}, T_-^{(3)}\right)(\tau,\zeta) \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 1 & 0 \\ \gamma & 0 & 1 \end{pmatrix} (-\zeta), \qquad (72)$$

$$S_-(\tau,\zeta) = \left(T_-^{(1)}, T_-^{(2)}, T_-^{(3)}\right)(\tau,\zeta) \begin{pmatrix} 1 & \eta & \nu \\ 0 & \theta & \rho \\ 0 & \mu & \chi \end{pmatrix} (-\zeta). \qquad (73)$$

The *scattering matrix* $S(\zeta)$ is defined via the key relation

$$S_-(\tau,\zeta) = S_+(\tau,\zeta)S(\zeta), \qquad (74)$$

where using $(70-73)$ we have

$$S(\zeta) \quad = \quad \begin{pmatrix} 1 & \beta & \gamma \\ 0 & \theta & \mu \\ 0 & \rho & \chi \end{pmatrix}^{-1} (\zeta) \begin{pmatrix} \alpha & 0 & 0 \\ \eta & 1 & 0 \\ \nu & 0 & 1 \end{pmatrix} (\zeta)$$

$$= \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 1 & 0 \\ \gamma & 0 & 1 \end{pmatrix}^{-1} (-\zeta) \begin{pmatrix} 1 & \eta & \nu \\ 0 & \theta & \rho \\ 0 & \mu & \chi \end{pmatrix} (-\zeta).$$

Calculating either product leads, courtesy of the normalisation and orthogonality conditions $(49-51)$, to

$$S(\zeta) = \frac{1}{\alpha(-\zeta)} \begin{pmatrix} 1 & \eta(-\zeta) & \nu(-\zeta) \\ -\beta(-\zeta) & \chi(\zeta) & -\mu(\zeta) \\ -\gamma(-\zeta) & -\rho(\zeta) & \theta(\zeta) \end{pmatrix} \qquad (75)$$

Clearly, since $\det T_{\pm}(\tau,\zeta) = 1$, equations (71), (70) imply

$$\det S_{-}(\tau,\zeta) = \det\left[\left(T_{-}^{(1)}, T_{+}^{(2)}, T_{+}^{(3)}\right)(\tau,\zeta)\right] = \alpha(\zeta), \qquad (76)$$
$$\det S_{+}(\tau,\zeta) = \det\left[\left(T_{+}^{(1)}, T_{-}^{(2)}, T_{-}^{(3)}\right)(\tau,\zeta)\right] = \alpha(-\zeta), \qquad (77)$$

and consequently

$$\det S(\zeta) = \frac{\alpha(\zeta)}{\alpha(-\zeta)}. \qquad (78)$$

Note also that for the inverse of the scattering matrix we have the formula

$$S^{-1}(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} S^{t}(-\zeta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

After careful inspection of the structure of $S(\zeta)$ we draw the following conclusions:

1. $\alpha(-\zeta)$ is the crucial transition coefficient whose simple zeros in Π^{+}, when they exist, are bound state eigenvalues of the direct scattering problem. Such zeros are independent of z and correspond to solitons through the action of the inverse scattering transform.

2. The functions $\frac{\eta(-\zeta)}{\alpha(-\zeta)}$ and $\frac{\nu(-\zeta)}{\alpha(-\zeta)}$ are reflection coefficients for the direct scattering problem. Pure soliton solutions are obtained by setting

$$\eta(-\zeta) = 0 = \nu(-\zeta)$$

as will be seen later.

5.1.8 Transition Coefficients for the Discrete Spectrum.

In order to make the analysis more straightforward we assume that the zeros of $\alpha(-\zeta)$ in Π^+ are simple and do not occur on the real line. Since $\alpha(-\zeta) = 1+o(1)$, as $|\zeta| \to \infty$, for $\mathrm{Im}\zeta \geq 0$ (cf. p. 65), there can be no cluster points of zeros in Π^+, and so $\alpha(-\zeta)$ must have only a finite number of zeros.

If $\alpha(-\zeta)$ does possess zeros, then it is trivial to prove they occur either in pairs symmetric about the imaginary axis, or as single pure imaginary zeros. Indeed we know $T(-\zeta) = \overline{T}(\overline{\zeta})$, which means that if $\alpha(-\zeta_\alpha) = 0$, then also $\alpha(\overline{\zeta_\alpha}) = 0$, where $\zeta_\alpha \in \Pi^+$ is some zero of $\alpha(-\zeta)$.

Evidently $\alpha(-\zeta) = (\theta\chi - \rho\mu)(\zeta)$ may have zeros at points where for instance zeros of $\theta(\zeta), \rho(\zeta)$, or $\theta(\zeta), \mu(\zeta)$, or $\chi(\zeta), \rho(\zeta)$, or $\chi(\zeta), \mu(\zeta)$ coincide, as well as zeros at points for which none of $\theta(\zeta), \chi(\zeta), \rho(\zeta), \mu(\zeta)$ are zero.

Let the set of zeros of $\alpha(-\zeta)$ be given by

$$Z\left[\alpha\left(-\zeta\right)\right] = \left\{\zeta_r^+ \in \Pi^+ \middle| r = 1, ..., n = \left(n_1 + 2n_2\right)\left(\alpha, \theta, \chi, \rho, \mu\right)\right\},$$

where n_1 is the number of pure imaginary zeros, and n_2 is the number of symmetric pairs of zeros. Then

$$Z\left[\alpha\left(\zeta\right)\right] = \left\{\zeta_r^- \in \Pi^- \middle| r = 1, ...n(\alpha, \theta, \chi, \rho, \mu)\right\},$$

where $\zeta_r^- = -\zeta_r^+$, for each r.

Now evaluating equation (77) at $\zeta = \zeta_r^+$, and equation (76) at $\zeta = \zeta_r^-$ gives respectively

$$\begin{aligned}
\det\left(T_+^{(1)}, T_-^{(2)}, T_-^{(3)}\right)(\tau, \zeta_r^+) &= 0, \\
\text{and } \det\left(T_-^{(1)}, T_+^{(2)}, T_+^{(3)}\right)(\tau, \zeta_r^-) &= 0.
\end{aligned}$$

Hence there exist $2n$ coefficients $\mathbf{C} \ni p_{jr}^+(z), j = 1, 2, r = 1, ..., n$ such that

$$T_+^{(1)}(\tau, \zeta_r^+) = p_{1r}^+ T_-^{(2)}(\tau, \zeta_r^+) + p_{2r}^+ T_-^{(3)}(\tau, \zeta_r^+) \qquad (79)$$

and also $2n$ coefficients $\mathbf{C} \ni p_{jr}^-(z), j = 1, 2, r = 1, ..., n$ such that

$$T_-^{(1)}(\tau, \zeta_r^-) = p_{1r}^- T_+^{(2)}(\tau, \zeta_r^-) + p_{2r}^- T_+^{(3)}(\tau, \zeta_r^-). \qquad (80)$$

We can determine the z-dependence of $p_{jr}^{\pm}$ from equation (57). In particular the evolution equations for the columns $T_+^{(1)}, T_-^{(2)}, T_-^{(3)}$ are

$$\begin{aligned}
\partial_z T_+^{(1)}(\tau, \zeta) &= \left[V(\tau, \zeta) - \frac{4\iota K\zeta}{3(\zeta^2 - 1)}\right] T_+^{(1)}(\tau, \zeta), \\
\partial_z T_-^{(2)}(\tau, \zeta) &= \left[V(\tau, \zeta) + \frac{2\iota K\zeta}{3(\zeta^2 - 1)}\right] T_-^{(2)}(\tau, \zeta), \\
\partial_z T_-^{(3)}(\tau, \zeta) &= \left[V(\tau, \zeta) + \frac{2\iota K\zeta}{3(\zeta^2 - 1)}\right] T_-^{(3)}(\tau, \zeta).
\end{aligned}$$

Suppose $\zeta = \zeta_r^+$. Then equating expressions for $\partial_z T_+^{(1)}(\tau, \zeta_r^+)$ using (79) yields (at $\zeta = \zeta_r^+$)

$$\begin{aligned}
(p_{1r}^+)_z T_-^{(2)} + (p_{2r}^+)_z T_-^{(3)} &+ p_{1r}^+ \left[V + \frac{2\iota K\zeta}{3(\zeta^2 - 1)}\right] T_-^{(2)} + p_{2r}^+ \left[V + \frac{2\iota K\zeta}{3(\zeta^2 - 1)}\right] T_-^{(3)} \\
&= \left[V - \frac{4\iota K\zeta}{3(\zeta^2 - 1)}\right] \left[p_{1r}^+ T_-^{(2)} + p_{2r}^+ T_-^{(3)}\right]
\end{aligned}$$

$$\Rightarrow \frac{d}{dz}p_{jr}^{+} = -\frac{2\iota K\zeta_r^{+}}{(\zeta_r^{+})^2-1}p_{jr}^{+}, \quad j=1,2,$$

$$\Rightarrow p_{jr}^{+}(z) = p_{jr}^{+}(0)\exp\left[-\frac{2\iota K\zeta_r^{+}z}{(\zeta_r^{+})^2-1}\right], \quad j=1,2, \ r=1,...,n. \qquad (81)$$

Similarly, equating expressions for $\partial_z T_{-}^{(1)}(\tau,\zeta_r^{-})$ we have

$$p_{jr}^{-}(z) = p_{jr}^{-}(0)\exp\left[-\frac{2\iota K\zeta_r^{-}z}{(\zeta_r^{-})^2-1}\right], \quad j=1,2, \ r=1,...,n. \qquad (82)$$

Therefore, because $\zeta_r^{-} = -\zeta_r^{+}$ for each r, equations $(81,82)$ imply

$$p_{jr}^{+}(z)p_{kr}^{-}(z) = p_{jr}^{+}(0)p_{kr}^{-}(0),$$

where $j,k \in \{1,2\}$, $r=1,...,n$.

Consider next equation (48) from which we know $\alpha(\zeta) = T_{+}^{(1)}(-\zeta)\cdot T_{-}^{(1)}(\zeta)$. Let $\zeta=\zeta_r^{-}$. Then using $(79,80)$ gives

$$\begin{aligned} 0 &= p_{1r}^{+}p_{1r}^{-}T_{-}^{(2)}(\zeta_r^{+})\cdot T_{+}^{(2)}(\zeta_r^{-}) + p_{1r}^{+}p_{2r}^{-}T_{-}^{(2)}(\zeta_r^{+})\cdot T_{+}^{(3)}(\zeta_r^{-}) + \\ &\quad p_{2r}^{+}p_{1r}^{-}T_{-}^{(3)}(\zeta_r^{+})\cdot T_{+}^{(2)}(\zeta_r^{-}) + p_{2r}^{+}p_{2r}^{-}T_{-}^{(3)}(\zeta_r^{+})\cdot T_{+}^{(3)}(\zeta_r^{-}) \end{aligned}$$

$$\Rightarrow 0 = p_{1r}^{+}p_{1r}^{-}\theta(\zeta_r^{+}) + p_{1r}^{+}p_{2r}^{-}\rho(\zeta_r^{+}) + p_{2r}^{+}p_{1r}^{-}\mu(\zeta_r^{+}) + p_{2r}^{+}p_{2r}^{-}\chi(\zeta_r^{+}). \qquad (83)$$

Condition (83) holds true if we take

$$p_{1r}^{+}p_{1r}^{-} = -\chi(\zeta_r^{+}),\ p_{1r}^{+}p_{2r}^{-} = \mu(\zeta_r^{+}),\ p_{2r}^{+}p_{1r}^{-} = \rho(\zeta_r^{+}),\ p_{2r}^{+}p_{2r}^{-} = -\theta(\zeta_r^{+}). \quad (84-87)$$

(These choices are consistent with later theory in Section 5.2, cf. p. 89).

In conclusion, the $2n$ functions $p_{jr}^{+}(z)$, $j=1,2$, $r=1,...n$, will be called the *transition coefficients for the discrete spectrum*, they are part of the scattering data which characterises the auxiliary linear problem. Observe that if ζ_r^{+} is pure imaginary, then p_{1r}^+, p_{2r}^+ are real, and if ζ_r^+, ζ_s^+ are a symmetric pair, then

$$p_{1s}^+ = \overline{p_{1r}^+}, \; p_{2s}^+ = \overline{p_{2r}^+},$$

as can easily be checked using (79) and the relation $\overline{T_\pm}(\zeta) = T_\pm(-\overline{\zeta})$. The coefficients $p_{jr}^-(z)$ are dependent on $p_{jr}^+(z)$ through equations (84 – 87).

5.1.9 Trace Formulae.

The results of this section allow us to minimise the number of parameters comprising the scattering data, and will also be needed in Chapter 6.

Let the set of zeros of $\alpha(\zeta)$, $\theta(\zeta)$, and $\chi(\zeta)$ be written respectively as

$$\begin{aligned} Z[\alpha(\zeta)] &= \left\{-\iota\zeta_{\alpha_j} \middle| \mathbf{R} \ni \zeta_{\alpha_j} > 0, j = 1, ..., n_1\right\} \\ &\cup \left\{-\zeta_{\alpha_k}, \overline{\zeta_{\alpha_k}} \in \Pi^- \middle| k = n_1 + 1, ..., n_1 + n_2\right\}, \end{aligned}$$

$$\begin{aligned} Z[\theta(\zeta)] &= \{\iota\zeta_{\theta_l} | \mathbf{R} \ni \zeta_{\theta_l} > 0, l = 1, ..., n_{1\theta}\} \\ &\cup \left\{\zeta_{\theta_m}, -\overline{\zeta_{\theta_m}} \in \Pi^+ \middle| m = n_{1\theta} + 1, ..., n_{1\theta} + n_{2\theta}\right\}, \end{aligned}$$

$$\begin{aligned} Z[\chi(\zeta)] &= \left\{\iota\zeta_{\chi_p} \middle| \mathbf{R} \ni \zeta_{\chi_p} > 0, p = 1, ..., n_{1\chi}\right\} \\ &\cup \left\{\zeta_{\chi_q}, -\overline{\zeta_{\chi_q}} \in \Pi^+ \middle| q = n_{1\chi} + 1, ..., n_{1\chi} + n_{2\chi}\right\}. \end{aligned}$$

Then we claim that for $\zeta \in \Pi^-$,

$$\begin{aligned} \alpha(\zeta) &= \exp\left\{\frac{\iota}{2\pi}\int_{-\infty}^{\infty} \frac{\log\left[1 - \left(|\eta(s)|^2 + |\nu(s)|^2\right)\right]}{s - \zeta} ds\right\} \times \\ &\prod_{j=1}^{n_1} \left(\frac{\zeta + \iota\zeta_{\alpha_j}}{\zeta - \iota\zeta_{\alpha_j}}\right) \prod_{k=n_1+1}^{n_1+n_2} \frac{(\zeta + \zeta_{\alpha_k})\left(\zeta - \overline{\zeta_{\alpha_k}}\right)}{\left(\zeta + \overline{\zeta_{\alpha_k}}\right)(\zeta - \zeta_{\alpha_k})}, \end{aligned} \qquad (88)$$

whilst for $\zeta \in \Pi^+$,

$$\theta(\zeta) = \exp\left\{-\frac{\iota}{2\pi}\int_{-\infty}^{\infty}\frac{\log\left[1-\left(|\eta(s)|^2+|\mu(s)|^2\right)\right]}{s-\zeta}ds\right\}\times$$
$$\prod_{l=1}^{n_{1\theta}}\left(\frac{\zeta-\iota\zeta_{\theta_l}}{\zeta+\iota\zeta_{\theta_l}}\right)\prod_{m=n_{1\theta}+1}^{n_{1\theta}+n_{2\theta}}\frac{\left(\zeta-\zeta_{\theta_m}\right)\left(\zeta+\overline{\zeta_{\theta_m}}\right)}{\left(\zeta-\overline{\zeta_{\theta_m}}\right)\left(\zeta+\zeta_{\theta_m}\right)}, \quad (89)$$

$$\chi(\zeta) = \exp\left\{-\frac{\iota}{2\pi}\int_{-\infty}^{\infty}\frac{\log\left[1-\left(|\nu(s)|^2+|\rho(s)|^2\right)\right]}{s-\zeta}ds\right\}\times$$
$$\prod_{p=1}^{n_{1\chi}}\left(\frac{\zeta-\iota\zeta_{\chi_p}}{\zeta+\iota\zeta_{\chi_p}}\right)\prod_{q=n_{1\chi}+1}^{n_{1\chi}+n_{2\chi}}\frac{\left(\zeta-\zeta_{\chi_q}\right)\left(\zeta+\overline{\zeta_{\chi_q}}\right)}{\left(\zeta-\overline{\zeta_{\chi_q}}\right)\left(\zeta+\zeta_{\chi_q}\right)}. \quad (90)$$

Formulae (88 – 90) can be proved by the same method used in F&T. pp. 50-51 for deriving the "trace formula" in the 2×2 case (following Ablowitz *et al.* [33], equations (94, 95) below will be referred to as *trace formulae*). However, for completeness we prove the most important relation (88):

We know that $\alpha(\zeta) = (\theta\chi - \rho\mu)(-\zeta)$ is analytic in Π^- and for $\zeta \in \mathbf{R}$ we have the normalisation condition

$$|\alpha(\zeta)|^2 = 1 - \left(|\eta(\zeta)|^2 + |\nu(\zeta)|^2\right).$$

Introduce the modified function

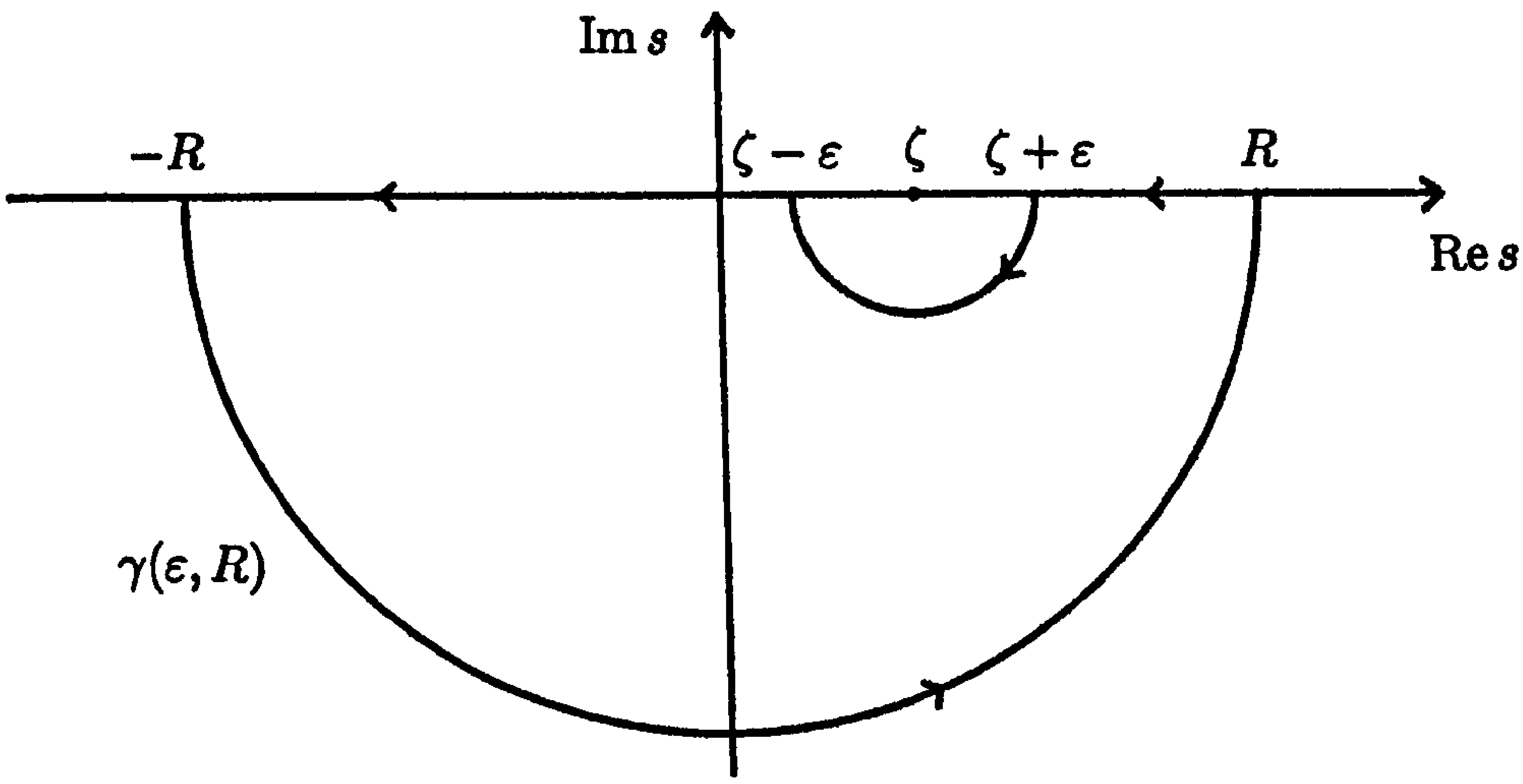
$$\tilde{\alpha}(\zeta) := \alpha(\zeta)\prod_{j=1}^{n_1}\left(\frac{\zeta-\iota\zeta_{\alpha_j}}{\zeta+\iota\zeta_{\alpha_j}}\right)\prod_{k=n_1+1}^{n_1+n_2}\frac{\left(\zeta+\overline{\zeta_{\alpha_k}}\right)\left(\zeta-\zeta_{\alpha_k}\right)}{\left(\zeta+\zeta_{\alpha_k}\right)\left(\zeta-\overline{\zeta_{\alpha_k}}\right)}. \quad (91)$$

Then for $\zeta \in \Pi^-$, $\tilde{\alpha}(\zeta)$ is analytic, has no zeros, and $\tilde{\alpha}(\zeta) = 1 + o(1)$ as $|\zeta| \to \infty$. Furthermore, the function $\log\tilde{\alpha}(\zeta)$ is analytic for $\mathrm{Im}\zeta < 0$, continuous up to the real line (since $\alpha(\zeta)$ is assumed to have no zeros when $\zeta \in \mathbf{R}$), and $\log\tilde{\alpha}(\zeta) \to 0$ as $|\zeta| \to \infty$.

Suppose we define

$$f(s) = \frac{\log \tilde{\alpha}(s)}{s - \zeta}, \quad \zeta \in \mathbf{R},$$

and integrate f around the contour γ shown below:



We have that

$$\int_\gamma f(s)ds = \left(\int_{\Gamma(0,R)} - \int_{\Gamma(\zeta,\varepsilon)} - \int_{[-R,\zeta-\varepsilon]} - \int_{[\zeta+\varepsilon,R]}\right) f(s)ds,$$

where $\Gamma(0, R)$ denotes the semicircular arc centred at 0, radius R, and $\Gamma(\zeta, \varepsilon)$ is the arc centred at ζ, radius ε. Therefore, as

$$\lim_{\varepsilon\to 0} \int_{\Gamma(\zeta,\varepsilon)} f(s)ds = \iota(2\pi - \pi)\log\tilde{\alpha}(\zeta),$$

Cauchy's Theorem yields

$$0 = \lim_{R\to\infty,\varepsilon\to 0} \int_{\gamma(\varepsilon,R)} \frac{\log\tilde{\alpha}(s)}{s-\zeta} ds = -\iota\pi\log\tilde{\alpha}(\zeta) - \mathrm{PV}\int_{-\infty}^{\infty} \frac{\log\tilde{\alpha}(s)}{s-\zeta} ds$$

$$\Rightarrow \mathrm{Im}\{\log\tilde{\alpha}(\zeta)\} = \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty} \frac{\mathrm{Re}\{\log\tilde{\alpha}(s)\}}{s-\zeta} ds. \qquad (92)$$

Now applying the Plemelj Formula (cf. equation (6.24), p. 50, F&T.) to $\mathrm{Re}\{\log\tilde{\alpha}(s)\}$ gives for $\zeta \in \mathbf{R}$

$$\int_{-\infty}^{\infty} \frac{\mathrm{Re}\{\log\tilde{\alpha}(s)\}}{s-\zeta+\iota 0}ds = \mathrm{PV}\int_{-\infty}^{\infty} \frac{\mathrm{Re}\{\log\tilde{\alpha}(s)\}}{s-\zeta}ds - \iota\pi\int_{-\infty}^{\infty} \mathrm{Re}\{\log\tilde{\alpha}(s)\}\,\delta(s-\zeta)ds,$$

and so by (92)

$$\begin{aligned}\log\tilde{\alpha}(\zeta) &= \frac{\iota}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Re}\{\log\tilde{\alpha}(s)\}}{s-\zeta}ds \\ &= \frac{\iota}{2\pi}\int_{-\infty}^{\infty} \frac{\log\left[1-\left(|\eta(s)|^2+|\nu(s)|^2\right)\right]}{s-\zeta}ds, \qquad (93)\end{aligned}$$

since $\mathrm{Re}\{\log\tilde{\alpha}(s)\} = \log|\tilde{\alpha}(s)| = \log|\alpha(s)|$. Equation (88) follows directly from (91) and (93).

Finally, if we take logarithms of (88, 89), and expand the denominator of the integrands using the binomial theorem together with the fact that $|\eta(s)|^2$, $|\nu(s)|^2$, $|\mu(s)|^2$ are even functions, then we obtain

$$\begin{aligned}\log\alpha(\zeta) &= \iota\sum_{r=1}^{\infty}\frac{1}{\zeta^{2r-1}}\left\{-\frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left[1-\left(|\eta(s)|^2+|\nu(s)|^2\right)\right]s^{2r-2}ds\right. \\ &\left.-\frac{2\iota}{(2r-1)}\left[\sum_{j=1}^{n_1}\left(\iota\zeta_{\alpha_j}\right)^{2r-1}+\sum_{k=n_1+1}^{n_1+n_2}\left(\zeta_{\alpha_k}^{2r-1}-\overline{\zeta_{\alpha_k}}^{2r-1}\right)\right]\right\}, \qquad (94)\end{aligned}$$

$$\begin{aligned}\log\theta(\zeta) &= -\iota\sum_{r=1}^{\infty}\frac{1}{\zeta^{2r-1}}\left\{-\frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left[1-\left(|\eta(s)|^2+|\mu(s)|^2\right)\right]s^{2r-2}ds\right. \\ &\left.-\frac{2\iota}{(2r-1)}\left[\sum_{l=1}^{n_{1\theta}}\left(\iota\zeta_{\theta_l}\right)^{2r-1}+\sum_{m=n_{1\theta}+1}^{n_{1\theta}+n_{2\theta}}\left(\zeta_{\theta_m}^{2r-1}-\overline{\zeta_{\theta_m}}^{2r-1}\right)\right]\right\}. \qquad (95)\end{aligned}$$

There is obviously an analagous expression for $\log\chi(\zeta)$.

5.1.10 Statement of the Scattering Data.

The discrete spectrum scattering data is completely determined by the following sets of parameters:

$$\left\{\zeta_{\alpha_j}\in\mathbf{R}\middle|j=1,\dots,n_1\right\}\cup\left\{\zeta_{\alpha_k},\overline{\zeta_{\alpha_k}}\middle|k=n_1+1,\dots,n_1+n_2\right\}$$

and

$$\left\{p_{1j}^+,p_{2j}^+\in\mathbf{R}\middle|j=1,\dots,n_1\right\}\cup\left\{p_{1k}^+,\overline{p_{1k}^+},p_{2k}^+,\overline{p_{2k}^+}\middle|k=n_1+1,\dots,n_1+n_2\right\},$$

where p_{1k}^+, p_{2k}^+ are the transition coefficients for the discrete spectrum associated to ζ_{α_k}, and $\overline{p_{1k}^+}$, $\overline{p_{2k}^+}$ are the coefficients associated to $-\overline{\zeta_{\alpha_k}}$.

The continuous spectrum scattering data is given, for $\zeta\in\mathbf{R}$, by

$$\{\eta(0,\zeta),\,\overline{\eta}(0,\zeta),\,\nu(0,\zeta),\,\overline{\nu}(0,\zeta),\,\mu(0,\zeta),\,\overline{\mu}(0,\zeta),\,\rho(0,\zeta),\,\overline{\rho}(0,\zeta)\},$$

plus either

$$\left\{\theta(0,\zeta),\,\overline{\theta}(0,\zeta),\,\chi(0,\zeta),\,\overline{\chi}(0,\zeta)\right\},$$

or alternatively

$$\left\{\zeta_{\theta_l},\zeta_{\theta_m},\overline{\zeta_{\theta_m}},\zeta_{\chi_p},\zeta_{\chi_q},\overline{\zeta_{\chi_q}}\right\},$$

where

$$l=1,\dots,n_{1\theta},\;\; m=n_{1\theta}+1,\dots,n_{1\theta}+n_{2\theta},$$

$$p=1,\dots,n_{1\chi},\;\; q=n_{1\chi}+1,\dots,n_{1\chi}+n_{2\chi},$$

in view of relations (89, 90).

We already know the z-dependence of the transition coefficients for the discrete spectrum and of η and ν: see equations (81) and (64, 65). The remaining data are constants of the motion.

5.2 The Inverse Scattering Problem.

5.2.1 Derivation of the Riemann-Hilbert Problem for the RMB-System.

Our starting point is the defining equation (74) of the scattering matrix $S(\zeta)$:

$$\left(T_-^{(1)}, T_+^{(2)}, T_+^{(3)}\right)(\tau,\zeta) = \left(T_+^{(1)}, T_-^{(2)}, T_-^{(3)}\right)(\tau,\zeta)S(\zeta) \qquad (96)$$

Suppose we normalise the Jost solution matrices by defining

$$\begin{aligned} M(\tau,\zeta) &= T_-(\tau,\zeta)E^{-1}(\tau,\zeta), \\ N(\tau,\zeta) &= T_+(\tau,\zeta)E^{-1}(\tau,\zeta). \end{aligned}$$

Then (cf. p. 64) the columns $N^{(1)}(\tau,\zeta)$, $M^{(2)}(\tau,\zeta)$, $M^{(3)}(\tau,\zeta)$ are analytic in Π^+, whilst $M^{(1)}(\tau,\zeta)$, $N^{(2)}(\tau,\zeta)$, $N^{(3)}(\tau,\zeta)$ are analytic in Π^-, and directly from equations $(41-46)$ we have for each fixed τ the asymptotic behaviour:

$$\begin{aligned} N^{(1)}(\tau,\zeta) &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + o(1), \\ M^{(2)}(\tau,\zeta) &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + o(1), \qquad \text{for } \mathrm{Im}\zeta \geq 0, \text{ as } |\zeta| \to \infty. \\ M^{(3)}(\tau,\zeta) &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + o(1), \end{aligned}$$

$$M^{(1)}(\tau,\zeta) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + o(1),$$

$$N^{(2)}(\tau,\zeta) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + o(1), \qquad \text{for } \mathrm{Im}\zeta \leq 0, \text{ as } |\zeta| \to \infty.$$

$$N^{(3)}(\tau,\zeta) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + o(1),$$

Moreover equation (96) becomes

$$\left(M^{(1)}, N^{(2)}, N^{(3)}\right)(\tau,\zeta) = \left(N^{(1)}, M^{(2)}, M^{(3)}\right)(\tau,\zeta)E(\tau,\zeta)S(\zeta)E^{-1}(\tau,\zeta) =$$

$$\left(N^{(1)}, M^{(2)}, M^{(3)}\right)(\tau,\zeta)\frac{1}{\alpha(-\zeta)}\begin{pmatrix} 1 & e^{\iota\Omega_0\zeta\tau}\eta(-\zeta) & e^{\iota\Omega\zeta\tau}\nu(-\zeta) \\ -e^{-\iota\Omega_0\zeta\tau}\beta(-\zeta) & \chi(\zeta) & -\mu(\zeta) \\ -e^{-\iota\Omega_0\zeta\tau}\gamma(-\zeta) & -\rho(\zeta) & \theta(\zeta) \end{pmatrix}.$$

Columnwise this reads

$$\begin{aligned} M^{(1)}(\tau,\zeta) &= \frac{N^{(1)}(\tau,\zeta)}{\alpha(-\zeta)} - e^{-\iota\Omega_0\zeta\tau}\frac{\beta(-\zeta)}{\alpha(-\zeta)}M^{(2)}(\tau,\zeta) - e^{-\iota\Omega_0\zeta\tau}\frac{\gamma(-\zeta)}{\alpha(-\zeta)}M^{(3)}(\tau,\zeta), \\ N^{(2)}(\tau,\zeta) &= e^{\iota\Omega_0\zeta\tau}\frac{\eta(-\zeta)}{\alpha(-\zeta)}N^{(1)}(\tau,\zeta) + \frac{\chi(\zeta)}{\alpha(-\zeta)}M^{(2)}(\tau,\zeta) - \frac{\rho(\zeta)}{\alpha(-\zeta)}M^{(3)}(\tau,\zeta), \\ N^{(3)}(\tau,\zeta) &= e^{\iota\Omega_0\zeta\tau}\frac{\nu(-\zeta)}{\alpha(-\zeta)}N^{(1)}(\tau,\zeta) - \frac{\mu(\zeta)}{\alpha(-\zeta)}M^{(2)}(\tau,\zeta) + \frac{\theta(\zeta)}{\alpha(-\zeta)}M^{(3)}(\tau,\zeta), \end{aligned}$$

$$(97-99)$$

Equation (97) can be rearranged to give an expression for $\frac{N^{(1)}(\tau,\zeta)}{\alpha(-\zeta)}$. If we substitute this expression into (98) and (99) and apply the normalisation and orthogonality conditions (49 − 51), then we find

$$\frac{N^{(2)}(\tau,\zeta)}{\alpha(\zeta)} = e^{\iota\Omega_0\zeta\tau}\frac{\eta(-\zeta)}{\alpha(\zeta)}M^{(1)}(\tau,\zeta)+\frac{\theta(-\zeta)}{\alpha(\zeta)}M^{(2)}(\tau,\zeta)+\frac{\mu(-\zeta)}{\alpha(\zeta)}M^{(3)}(\tau,\zeta),$$

$$\frac{N^{(3)}(\tau,\zeta)}{\alpha(\zeta)} = e^{\iota\Omega_0\zeta\tau}\frac{\nu(-\zeta)}{\alpha(\zeta)}M^{(1)}(\tau,\zeta)+\frac{\rho(-\zeta)}{\alpha(\zeta)}M^{(2)}(\tau,\zeta)+\frac{\chi(-\zeta)}{\alpha(\zeta)}M^{(3)}(\tau,\zeta).$$

(100, 101)

Rewriting the system of equations (97, 100, 101) in vector notation gives

$$\frac{N^{(1)}}{\alpha^-} = M^{(1)}+e^{-\iota\Omega_0\zeta\tau}\left(\frac{M^{(2)}}{\alpha^-},\frac{M^{(3)}}{\alpha^-}\right)\begin{pmatrix}\beta^-\\ \gamma^-\end{pmatrix}, \qquad (102)$$

$$\left(\frac{N^{(2)}}{\alpha},\frac{N^{(3)}}{\alpha}\right) = e^{\iota\Omega_0\zeta\tau}\frac{M^{(1)}}{\alpha}\left(\eta^-,\nu^-\right)+\left(M^{(2)},M^{(3)}\right)\frac{1}{\alpha}\begin{pmatrix}\theta^- & \rho^-\\ \mu^- & \chi^-\end{pmatrix}, \qquad (103)$$

where for clarity we have suppressed the arguments, and for example α^- denotes $\alpha(-\zeta)$. Lastly, multiplying equation (103) on the right by $\begin{pmatrix}\chi^- & -\rho^-\\ -\mu^- & \theta^-\end{pmatrix}$ yields

$$\left(\frac{N^{(2)}}{\alpha},\frac{N^{(3)}}{\alpha}\right)\begin{pmatrix}\chi^- & -\rho^-\\ -\mu^- & \theta^-\end{pmatrix} = -e^{\iota\Omega_0\zeta\tau}\frac{M^{(1)}}{\alpha}(\beta,\gamma)+\left(M^{(2)},M^{(3)}\right). \qquad (104)$$

The system of equations (102, 104) may be recast as a vector Riemann-Hilbert problem having the characteristic matrix form (cf. equations (3.1.19), p. 108, A&C.)

$$m_+(\tau,\zeta)-m_-(\tau,\zeta)=m_-(\tau,\zeta)V(\tau,\zeta), \qquad (105)$$

on the contour $\text{Im}\zeta=0$, where

$$m_+(\tau,\zeta) := \left(\frac{N^{(1)}(\tau,\zeta)}{\alpha(-\zeta)},M^{(2)}(\tau,\zeta),M^{(3)}(\tau,\zeta)\right),$$

$$
\begin{aligned}
m_-(\tau,\zeta) \;:=\; & \Big(M^{(1)}(\tau,\zeta), \frac{\chi(-\zeta)}{\alpha(\zeta)}N^{(2)}(\tau,\zeta) - \frac{\mu(-\zeta)}{\alpha(\zeta)}N^{(3)}(\tau,\zeta), \\
& \frac{\theta(-\zeta)}{\alpha(\zeta)}N^{(3)}(\tau,\zeta) - \frac{\rho(-\zeta)}{\alpha(\zeta)}N^{(2)}(\tau,\zeta)\Big),
\end{aligned}
$$

$$
V(\tau,\zeta) := \begin{pmatrix} \frac{1-\alpha(\zeta)\alpha(-\zeta)}{\alpha(\zeta)\alpha(-\zeta)}, & e^{\iota\Omega_0\zeta\tau}\frac{\beta(\zeta)}{\alpha(\zeta)}, & e^{\iota\Omega_0\zeta\tau}\frac{\gamma(\zeta)}{\alpha(\zeta)} \\ e^{-\iota\Omega_0\zeta\tau}\frac{\beta(-\zeta)}{\alpha(-\zeta)}, & 0 & 0 \\ e^{-\iota\Omega_0\zeta\tau}\frac{\gamma(-\zeta)}{\alpha(-\zeta)}, & 0 & 0 \end{pmatrix},
$$

and $m_\pm(\tau,\zeta) \to I$, as $|\zeta| \to \infty$.

For future reference we note

$$
\begin{aligned}
m_-V &= m_+ - m_- \\
&= \left(e^{-\iota\Omega_0\zeta\tau}\left[\frac{\beta^-}{\alpha^-}M^{(2)} + \frac{\gamma^-}{\alpha^-}M^{(3)}\right], e^{\iota\Omega_0\zeta\tau}\frac{\beta}{\alpha}M^{(1)}, e^{\iota\Omega_0\zeta\tau}\frac{\gamma}{\alpha}M^{(1)}\right). \quad (106)
\end{aligned}
$$

5.2.2 Formal Solutions to the Inverse Problem.

Let us first deal with the case when α has no zeros. Having already established the Riemann-Hilbert problem, we may proceed just as in Chapter 2, Section 1.4.

It is necessary to find the asymptotic behaviour of $m_-(\tau,\zeta)$ for large $|\zeta|$. We know from equations (33) and (37) on pp. 62-63 that

$$
\begin{aligned}
M(\tau,\zeta) &= I + \int_{-\infty}^{\tau} \Gamma_-(\tau,s)F(\tau-s,\zeta)ds, \\
N(\tau,\zeta) &= I + \int_{\tau}^{\infty} \tilde{\Gamma}_+^t(\tau,s)F(\tau-s,\zeta)ds,
\end{aligned}
$$

where $F(\tau,\zeta) = \exp\left[-\iota\Omega_0\zeta\tau\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{pmatrix}\right]$. Looking for instance at column one of $m_-(\tau,\zeta)$ we have

$$\begin{aligned}
m_-^{(1)}(\tau,\zeta) &= M^{(1)}(\tau,\zeta) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^{-\iota\Omega_0\zeta\tau}\int_{-\infty}^{\tau}\Gamma_-^{(1)}(\tau,s)e^{\iota\Omega_0\zeta s}ds \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\iota\Omega_0\zeta}\Gamma_-^{(1)}(\tau,\tau) + O(\zeta^{-2}) \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\iota\zeta}\begin{pmatrix} -\Omega_0 C^2 \int_{-\infty}^{\tau} |A|^2(r)dr \\ CA_x(\tau) \\ CA_y(\tau) \end{pmatrix} + O(\zeta^{-2}),
\end{aligned}$$

which follows from the definitions (34) and (28) of Γ_- and Γ, since

$$U_0(\tau) = \Omega_0 C \begin{pmatrix} 0 & -A_x & -A_y \\ A_x & 0 & 0 \\ A_y & 0 & 0 \end{pmatrix}(\tau).$$

Computing the behaviour of $m_-^{(2)}$, $m_-^{(3)}$ in the same way leads to

$$\begin{aligned}
m_-(\tau,\zeta) &= I \\
&+ \frac{C}{\iota\zeta}\begin{pmatrix} -\Omega_0 C\int_{-\infty}^{\tau}|A|^2 dr, & A_x(\tau) & A_y(\tau) \\ A_x(\tau) & -\Omega_0 C\int_{\tau}^{\infty} A_x^2 dr, & -\Omega_0 C\int_{\tau}^{\infty} A_x A_y dr \\ A_y(\tau) & -\Omega_0 C\int_{\tau}^{\infty} A_x A_y dr, & -\Omega_0 C\int_{\tau}^{\infty} A_y^2 dr \end{pmatrix} \\
&+ O\left(\zeta^{-2}\right), \qquad (107)
\end{aligned}$$

as $|\zeta| \to \infty$, where $|A|^2(\tau) := A_x^2(\tau) + A_y^2(\tau)$.

For later use we record the corresponding result for m_+:

$$m_+(\tau,\zeta) = I +$$

$$
+\ \frac{C}{\iota\zeta}\begin{pmatrix} \Omega_0 C\int_\tau^\infty |A|^2\,dr, & A_x(\tau) & A_y(\tau) \\ A_x(\tau) & \Omega_0 C\int_{-\infty}^\tau A_x^2 dr & \Omega_0 C\int_{-\infty}^\tau A_x A_y dr \\ A_y(\tau) & \Omega_0 C\int_{-\infty}^\tau A_x A_y dr & \Omega_0 C\int_{-\infty}^\tau A_y^2 dr \end{pmatrix}
$$

$$
+\ O\left(\zeta^{-2}\right), \qquad (108)
$$

as $|\zeta| \to \infty$.

On the other hand, applying the projection operator P^- (cf. equation (20), p. 20) to equation (105) we find

$$
P^-\left\{m_+(\tau,\zeta) - I - [m_-(\tau,\zeta) - I] - m_-(\tau,\zeta)V(\tau,\zeta)\right\} = 0
$$

$$
\Rightarrow m_-(\tau,\zeta) = I + \frac{1}{2\pi\iota}\int_{-\infty}^{\infty}\frac{m_-(\tau,u)V(\tau,u)}{u-(\zeta-\iota 0)}du.
$$

Hence, for large $|\zeta|$, using equation (106) gives

$$
\begin{aligned}
m_-(\tau,\zeta) &= I - \frac{1}{2\pi\iota\zeta}\int_{-\infty}^{\infty}\left(e^{-\iota\Omega_0 u\tau}\left(\frac{\beta(-u)}{\alpha(-u)}M^{(2)}(\tau,u) + \frac{\gamma(-u)}{\alpha(-u)}M^{(3)}(\tau,u)\right),\right. \\
&\left. e^{\iota\Omega_0 u\tau}\frac{\beta(u)}{\alpha(u)}M^{(1)}(\tau,u), e^{\iota\Omega_0 u\tau}\frac{\gamma(u)}{\alpha(u)}M^{(1)}(\tau,u)\right) du + O(\zeta^{-2}). \quad (109)
\end{aligned}
$$

Now suppose that given $V(\tau,z,\zeta)$ we have solved the inverse problem defined by equation (105), obtaining $m_\pm(\tau,z,\zeta)$. Then comparing (107) and (109) we deduce in particular that

$$
\begin{aligned}
A_x(\tau,z) &= -\frac{1}{2\pi C}\int_{-\infty}^{\infty} e^{\iota\Omega_0 u\tau}\frac{\beta(z,u)}{\alpha(0,u)}M^{11}(\tau,z,u)du, \\
A_y(\tau,z) &= -\frac{1}{2\pi C}\int_{-\infty}^{\infty} e^{\iota\Omega_0 u\tau}\frac{\gamma(z,u)}{\alpha(0,u)}M^{11}(\tau,z,u)du.
\end{aligned}
$$

Secondly, let us study the case when $\alpha(-\zeta)$ has a set of simple zeros

$$
Z[\alpha(-\zeta)] = \left\{\zeta_r^+ \in \Pi^+ \middle| r = 1, ..., n = n_1 + 2n_2\right\}.
$$

Consider equation (102): we have that $\frac{N^{(1)}(\tau,\zeta)}{\alpha(-\zeta)}$ is meromorphic in Π^+ with n simple poles. If we choose to set

$$\frac{N^{(1)}(\tau,\zeta)}{\alpha(-\zeta)} = N^+(\tau,\zeta) + \sum_{r=1}^{n} \frac{A_r(\tau)}{\zeta - \zeta_r^+},$$

where the column vector $N^+(\tau,\zeta)$ is analytic in Π^+, then (102) reads

$$\begin{aligned} N^+(\tau,\zeta) + \sum_{r=1}^{n} \frac{A_r(\tau)}{\zeta - \zeta_r^+} = \frac{N^{(1)}(\tau,\zeta)}{\alpha(-\zeta)} &= M^{(1)}(\tau,\zeta) + \\ & e^{-\iota\Omega_0\zeta\tau}\left[\frac{\beta(-\zeta)}{\alpha(-\zeta)}M^{(2)}(\tau,\zeta) + \right. \\ & \left. \frac{\gamma(-\zeta)}{\alpha(-\zeta)}M^{(3)}(\tau,\zeta)\right]. \end{aligned} \tag{110}$$

We also know that

$$e^{\iota\Omega_0\zeta_r^+\tau}N^{(1)}(\tau,\zeta_r^+) = p_{1r}^+ M^{(2)}(\tau,\zeta_r^+) + p_{2r}^+ M^{(3)}(\tau,\zeta_r^+), \tag{111}$$
$$e^{\iota\Omega_0\zeta_r^-\tau}M^{(1)}(\tau,\zeta_r^-) = p_{1r}^- N^{(2)}(\tau,\zeta_r^-) + p_{2r}^- N^{(3)}(\tau,\zeta_r^-), \tag{112}$$

which are simply equations (79, 80) written in terms of M and N. Therefore, integrating equation (110) around a suitable contour with ζ_r^+ contained inside, Cauchy's Residue Theorem implies

$$A_r(\tau) = -\frac{N^{(1)}(\tau,\zeta_r^+)}{\alpha'(-\zeta_r^+)} = -\frac{e^{-\iota\Omega_0\zeta_r^+\tau}}{\alpha'(-\zeta_r^+)}\left[p_{1r}^+ M^{(2)}(\tau,\zeta_r^+) + p_{2r}^+ M^{(3)}(\tau,\zeta_r^+)\right],$$

for each $r = 1, \ldots, n$.

Next we let P^- operate on equation (110), giving

$$\begin{aligned} 0 = P^-\Big\{ & N^+(\tau,\zeta) + \sum_{r=1}^{n} \frac{A_r(\tau)}{\zeta - \zeta_r^+} - I^{(1)} - \\ & \left[M^{(1)}(\tau,\zeta) - I^{(1)}\right] - [m_-(\tau,\zeta)V(\tau,\zeta)]^{(1)}\Big\}. \end{aligned}$$

Since the function $\frac{1}{(\zeta-\zeta_r^+)}$ is clearly analytic in Π^-,

$$P^-\left\{\frac{1}{\zeta-\zeta_r^+}\right\}=\frac{1}{\zeta_r^+-\zeta}$$

for each r, and so

$$\begin{aligned}
M^{(1)}(\tau,\zeta) &= I^{(1)}+\frac{1}{2\pi\iota}\int_{-\infty}^{\infty}\frac{[m_-V]^{(1)}(\tau,u)}{u-(\zeta-\iota 0)}du- \\
&\sum_{r=1}^{n}\frac{e^{-\iota\Omega_0\zeta_r^+\tau}}{\zeta-\zeta_r^+}\left[\frac{p_{1r}^+}{\alpha'(-\zeta_r^+)}M^{(2)}(\tau,\zeta_r^+)+\right. \\
&\left.\frac{p_{2r}^+}{\alpha'(-\zeta_r^+)}M^{(3)}(\tau,\zeta_r^+)\right] \qquad (113) \\
&= I^{(1)}-\frac{1}{2\pi\iota\zeta}\int_{-\infty}^{\infty}e^{-\iota\Omega_0 u\tau}\left[\frac{\beta(-u)}{\alpha(-u)}M^{(2)}(\tau,u)+\right. \\
&\left.\frac{\gamma(-u)}{\alpha(-u)}M^{(3)}(\tau,u)\right]du- \\
&\frac{1}{\zeta}\sum_{r=1}^{n}\frac{e^{-\iota\Omega_0\zeta_r^+\tau}}{\alpha'(-\zeta_r^+)}\left[p_{1r}^+M^{(2)}(\tau,\zeta_r^+)+p_{2r}^+M^{(3)}(\tau,\zeta_r^+)\right]+O(\zeta^{-2}),
\end{aligned}$$

as $|\zeta|\to\infty$. Alternatively, equation (107) tells us

$$M^{(1)}(\tau,\zeta)=m_-^{(1)}(\tau,\zeta)=I^{(1)}+\frac{C}{\iota\zeta}\begin{pmatrix}-\Omega_0 C\int_{-\infty}^{\tau}|A|^2(r)dr\\ A_x(\tau)\\ A_y(\tau)\end{pmatrix}+O(\zeta^{-2}),$$

as $|\zeta|\to\infty$. Thus, equating coefficients of ζ^{-1}, and assuming once again that given $V(\tau,z,\zeta)$ we have solved the inverse problem for $m_\pm(\tau,z,\zeta)$, we obtain the formal solution

$$A_x(\tau,z) = -\frac{\iota}{C}\sum_{r=1}^{n}\frac{e^{-\iota\Omega_0\zeta_r^+\tau}}{\alpha'(0,-\zeta_r^+)}\left[p_{1r}^+(z)M^{22}(\tau,z,\zeta_r^+)+p_{2r}^+(z)M^{23}(\tau,z,\zeta_r^+)\right]-$$

$$\frac{1}{2\pi C}\int_{-\infty}^{\infty} e^{-\iota\Omega_0 u\tau}\left[\frac{\beta(z,-u)}{\alpha(0,-u)}M^{22}(\tau,z,u)+\right.$$
$$\left.\frac{\gamma(z,-u)}{\alpha(0,-u)}M^{23}(\tau,z,u)\right]du,$$

with a corresponding expression for $A_y(\tau,z)$. However, this is just one representation of the formal solution. In order to acquire the other we choose to write the system of equations (104) as follows

$$\begin{aligned} N_\beta^-(\tau,\zeta)+\sum_{r=1}^{n}\frac{B_r(\tau)}{\zeta-\zeta_r^-} &= -\frac{\chi(-\zeta)}{\alpha(\zeta)}N^{(2)}(\tau,\zeta)+\frac{\mu(-\zeta)}{\alpha(\zeta)}N^{(3)}(\tau,\zeta) \\ &= e^{\iota\Omega_0\zeta\tau}\frac{\beta(\zeta)}{\alpha(\zeta)}M^{(1)}(\tau,\zeta)-M^{(2)}(\tau,\zeta), \quad (114) \end{aligned}$$

$$\begin{aligned} N_\gamma^-(\tau,\zeta)+\sum_{r=1}^{n}\frac{C_r(\tau)}{\zeta-\zeta_r^-} &= \frac{\rho(-\zeta)}{\alpha(\zeta)}N^{(2)}(\tau,\zeta)-\frac{\theta(-\zeta)}{\alpha(\zeta)}N^{(3)}(\tau,\zeta) \\ &= e^{\iota\Omega_0\zeta\tau}\frac{\gamma(\zeta)}{\alpha(\zeta)}M^{(1)}(\tau,\zeta)-M^{(3)}(\tau,\zeta), \quad (115) \end{aligned}$$

where $N_\beta^-(\tau,\zeta)$ and $N_\gamma^-(\tau,\zeta)$ are column vectors analytic in Π^-. Matching residues at poles in the first equation, bearing in mind relation (112), gives

$$B_r(\tau)=\frac{p_{1r}^+}{\alpha'(\zeta_r^-)}\left[p_{1r}^-N^{(2)}(\tau,\zeta_r^-)+p_{2r}^-N^{(3)}(\tau,\zeta_r^-)\right]=\frac{e^{\iota\Omega_0\zeta_r^-\tau}p_{1r}^+}{\alpha'(\zeta_r^-)}M^{(1)}(\tau,\zeta_r^-),$$

for each $r=1,..,n$. Furthermore, recalling the definition of the operator P^+ from (20) p. 20, we have

$$P^+\left\{N_\beta^-+\sum_{r=1}^{n}\frac{B_r(\tau)}{\zeta-\zeta_r^-}+M^{(2)}-[m_-V]^{(2)}\right\}=0$$

$$\Rightarrow M^{(2)}-I^{(2)}+\sum_{r=1}^{n}\frac{B_r(\tau)}{\zeta-\zeta_r^-}-P^+\left\{[m_-V]^{(2)}\right\}=0.$$

Hence

$$
\begin{aligned}
M^{(2)}(\tau,\zeta) &= I^{(2)} + \frac{1}{2\pi\iota}\int_{-\infty}^{\infty}\frac{[m_{-}V]^{(2)}(\tau,u)}{u-(\zeta+\iota 0)}du - \\
&\quad \sum_{r=1}^{n}\frac{e^{\iota\Omega_0\zeta_r^{-}\tau}p_{1r}^{+}}{(\zeta-\zeta_r^{-})\,\alpha'(\zeta_r^{-})}M^{(1)}(\tau,\zeta_r^{-}) \qquad (116)\\
&= I^{(2)} - \frac{1}{2\pi\iota\zeta}\int_{-\infty}^{\infty}e^{\iota\Omega_0 u\tau}\frac{\beta(u)}{\alpha(u)}M^{(1)}(\tau,u)du - \\
&\quad \frac{1}{\zeta}\sum_{r=1}^{n}e^{\iota\Omega_0\zeta_r^{-}\tau}\frac{p_{1r}^{+}}{\alpha'(\zeta_r^{-})}M^{(1)}(\tau,\zeta_r^{-}) + O(\zeta^{-2}), \qquad (117)
\end{aligned}
$$

as $|\zeta| \to \infty$. By a parallel argument applied to equation (115) we also find

$$
\begin{aligned}
M^{(3)}(\tau,\zeta) &= I^{(3)} + \frac{1}{2\pi\iota}\int_{-\infty}^{\infty}\frac{[m_{-}V]^{(3)}(\tau,u)}{u-(\zeta+\iota 0)}du - \\
&\quad \sum_{r=1}^{n}\frac{e^{\iota\Omega_0\zeta_r^{-}\tau}p_{2r}^{+}}{(\zeta-\zeta_r^{-})\,\alpha'(\zeta_r^{-})}M^{(1)}(\tau,\zeta_r^{-}) \qquad (118)\\
&= I^{(3)} - \frac{1}{2\pi\iota\zeta}\int_{-\infty}^{\infty}e^{\iota\Omega_0 u\tau}\frac{\gamma(u)}{\alpha(u)}M^{(1)}(\tau,u)du - \\
&\quad \frac{1}{\zeta}\sum_{r=1}^{n}e^{\iota\Omega_0\zeta_r^{-}\tau}\frac{p_{2r}^{+}}{\alpha'(\zeta_r^{-})}M^{(1)}(\tau,\zeta_r^{-}) + O(\zeta^{-2}), \qquad (119)
\end{aligned}
$$

as $|\zeta| \to \infty$. Therefore, comparing asymptotic expansions for $M^{(2)}$ and $M^{(3)}$ given by (117), (119), and columns two and three of equation (108), results in the second formal solution

$$
\begin{aligned}
A_x(\tau,z) &= -\frac{\iota}{C}\sum_{r=1}^{n}e^{\iota\Omega_0\zeta_r^{-}\tau}\frac{p_{1r}^{+}(z)}{\alpha'(0,\zeta_r^{-})}M^{11}(\tau,z,\zeta_r^{-}) - \\
&\quad \frac{1}{2\pi C}\int_{-\infty}^{\infty}e^{\iota\Omega_0 u\tau}\frac{\beta(z,u)}{\alpha(0,u)}M^{11}(\tau,z,u)du, \qquad (120)\\
A_y(\tau,z) &= -\frac{\iota}{C}\sum_{r=1}^{n}e^{\iota\Omega_0\zeta_r^{-}\tau}\frac{p_{2r}^{+}(z)}{\alpha'(0,\zeta_r^{-})}M^{11}(\tau,z,\zeta_r^{-}) - \\
&\quad \frac{1}{2\pi C}\int_{-\infty}^{\infty}e^{\iota\Omega_0 u\tau}\frac{\gamma(z,u)}{\alpha(0,u)}M^{11}(\tau,z,u)du. \qquad (121)
\end{aligned}
$$

5.2.3 Soliton Solutions of the RMB-System.

If the transition coefficients $\eta(\zeta)$, $\nu(\zeta)$ are zero when ζ is real, then the Riemann-Hilbert problem (105) may be solved in closed form. Indeed setting $\eta(\zeta) = 0 = \nu(\zeta)$, for $\zeta \in \mathbf{R}$, so that the reflection coefficients η/α, ν/α, β/α and γ/α are zero as well, clearly causes the integral terms to vanish in each of the equations (113) and (116 – 121). Specifically from (113), (116) and (118) we have

$$\begin{aligned} M^{11}(\tau,\zeta) &= 1-\sum_{r=1}^{n}\frac{e^{-\iota\Omega_0\zeta_r^+\tau}}{\alpha'(-\zeta_r^+)(\zeta-\zeta_r^+)}\left[p_{1r}^+M^{12}(\tau,\zeta_r^+)+p_{2r}^+M^{13}(\tau,\zeta_r^+)\right], \\ M^{12}(\tau,\zeta) &= -\sum_{s=1}^{n}\frac{e^{-\iota\Omega_0\zeta_s^+\tau}}{\alpha'(-\zeta_s^+)(\zeta+\zeta_s^+)}p_{1s}^+M^{11}(\tau,-\zeta_s^+), \\ M^{13}(\tau,\zeta) &= -\sum_{s=1}^{n}\frac{e^{-\iota\Omega_0\zeta_s^+\tau}}{\alpha'(-\zeta_s^+)(\zeta+\zeta_s^+)}p_{2s}^+M^{11}(\tau,-\zeta_s^+). \qquad (122-124) \end{aligned}$$

Reintroducing the z-dependence, it follows that for $s = 1, ..., n$

$$\begin{aligned} M^{11}(\tau,z,-\zeta_s^+) &= 1-\sum_{r=1}^{n}\frac{e^{-\iota\Omega_0\zeta_r^+\tau}}{\alpha'(-\zeta_r^+)(\zeta_r^++\zeta_s^+)}\times \qquad (125) \\ &\left\{p_{1r}^+(z)\sum_{s=1}^{n}\left[\frac{e^{-\iota\Omega_0\zeta_s^+\tau}}{\alpha'(-\zeta_s^+)(\zeta_r^++\zeta_s^+)}p_{1s}^+(z)M^{11}(\tau,z,-\zeta_s^+)\right]\right. \\ &\left.+p_{2r}^+(z)\sum_{s=1}^{n}\left[\frac{e^{-\iota\Omega_0\zeta_s^+\tau}}{\alpha'(-\zeta_s^+)(\zeta_r^++\zeta_s^+)}p_{2s}^+(z)M^{11}(\tau,z,-\zeta_s^+)\right]\right\}. \end{aligned}$$

This is a system of linear algebraic equations for $M^{11}(\tau,z,-\zeta_s^+)$, $s = 1, ..., n$, which in conjunction with equations (120, 121) evaluated when $\eta = 0 = \nu$, $\zeta \in \mathbf{R}$, i.e.,

$$\begin{aligned} A_x(\tau,z) &= -\frac{\iota}{C}\sum_{r=1}^{n}e^{-\iota\Omega_0\zeta_r^+\tau}\frac{p_{1r}^+(z)}{\alpha'(-\zeta_r^+)}M^{11}(\tau,z,-\zeta_r^+), \qquad (126) \\ A_y(\tau,z) &= -\frac{\iota}{C}\sum_{r=1}^{n}e^{-\iota\Omega_0\zeta_r^+\tau}\frac{p_{2r}^+(z)}{\alpha'(-\zeta_r^+)}M^{11}(\tau,z,-\zeta_r^+), \qquad (127) \end{aligned}$$

allow us to calculate pure n-soliton solutions of our RMB-system.

Consider for example the special case $n = 1$. We know (cf. Section 5.1.8) there must just be one pure imaginary zero of $\alpha(-\zeta)$ at $\zeta = \iota\xi$ (say), $\mathbf{R} \ni \xi > 0$, with associated transition coefficients for the discrete spectrum denoted by $p_{1+}(z)$, $p_{2+}(z) \in \mathbf{R}$. To obtain real potentials $A_x(\tau, z)$, $A_y(\tau, z)$, we assume that the reflection coefficients β/α and γ/α take the form

$$\frac{\beta(\zeta)}{\alpha(\zeta)} = \begin{cases} 0, & \zeta \in \mathbf{R}, \\ \frac{\iota p_{1+}}{\alpha'(-\iota\xi)(\zeta+\iota\xi)}, & \mathrm{Im}\zeta < 0, \end{cases}$$

$$\frac{\gamma(\zeta)}{\alpha(\zeta)} = \begin{cases} 0, & \zeta \in \mathbf{R}, \\ \frac{\iota p_{2+}}{\alpha'(-\iota\xi)(\zeta+\iota\xi)}, & \mathrm{Im}\zeta < 0. \end{cases}$$

Observe this means the reflection coefficients are real when ζ is pure imaginary, and in addition the residues of $\left(\frac{\beta}{\alpha}\right)(\pm\zeta)$ and $\left(\frac{\gamma}{\alpha}\right)(\pm\zeta)$ at $\zeta = \mp\iota\xi$ are given by

$$\begin{aligned}
\mathrm{Res}\left\{\frac{\beta(\zeta)}{\alpha(\zeta)}; -\iota\xi\right\} &= \frac{\iota p_{1+}}{\alpha'(-\iota\xi)}, \\
\mathrm{Res}\left\{\frac{\beta(-\zeta)}{\alpha(-\zeta)}; \iota\xi\right\} &= -\frac{\iota p_{1+}}{\alpha'(-\iota\xi)}, \\
\mathrm{Res}\left\{\frac{\gamma(\zeta)}{\alpha(\zeta)}; -\iota\xi\right\} &= \frac{\iota p_{2+}}{\alpha'(-\iota\xi)}, \\
\mathrm{Res}\left\{\frac{\gamma(-\zeta)}{\alpha(-\zeta)}; \iota\xi\right\} &= -\frac{\iota p_{2+}}{\alpha'(-\iota\xi)}.
\end{aligned}$$

Matching residues once again in equations (110), (114, 115) we now find

$$\begin{aligned}
M^{11}(\tau,\zeta) &= 1 - \frac{\iota e^{\Omega_0\xi\tau}}{\alpha'(-\iota\xi)(\zeta-\iota\xi)}\left[p_{1+}M^{12}(\tau,\iota\xi) + p_{2+}M^{13}(\tau,\iota\xi)\right], \\
M^{12}(\tau,\zeta) &= -\frac{\iota e^{\Omega_0\xi\tau}}{\alpha'(-\iota\xi)(\zeta+\iota\xi)}p_{1+}M^{11}(\tau,-\iota\xi), \\
M^{13}(\tau,\zeta) &= -\frac{\iota e^{\Omega_0\xi\tau}}{\alpha'(-\iota\xi)(\zeta+\iota\xi)}p_{2+}M^{11}(\tau,-\iota\xi),
\end{aligned}$$

and so

$$M^{11}(\tau,-\iota\xi)=\left[1+\frac{\left(p_{1+}^2+p_{2+}^2\right)}{4\xi^2\alpha'^2(-\iota\xi)}e^{2\Omega_0\xi\tau}\right]^{-1}.$$

Hence

$$\begin{aligned}A_x(\tau,z) &= \frac{p_{1+}(z)}{C\alpha'(-\iota\xi)}\left\{\frac{e^{\Omega_0\xi\tau}}{1+\left[\frac{p_{1+}^2(z)+p_{2+}^2(z)}{4\xi^2\alpha'^2(-\iota\xi)}\right]e^{2\Omega_0\xi\tau}}\right\}\\ &= \frac{p_{1+}(0)}{C\alpha'(-\iota\xi)}\left\{\frac{\exp\left[\Omega_0\xi\left(\tau-\tau_0(z)\right)\right]}{1+\left[\frac{p_{1+}^2(0)+p_{2+}^2(0)}{4\xi^2\alpha'^2(-\iota\xi)}\right]\exp\left[2\Omega_0\xi\left(\tau-\tau_0(z)\right)\right]}\right\}\\ &= \frac{\xi}{C}\frac{p_{1+}(0)}{\sqrt{p_{1+}^2(0)+p_{2+}^2(0)}}\operatorname{sech}\left[\Omega_0\xi\left(\tau-\tau_1-\tau_0(z)\right)\right], \qquad (128)\end{aligned}$$

where we have used the fact that $p_{1+}(z)=p_{1+}(0)\exp\left(-\frac{2K\xi z}{\xi^2+1}\right)$, and

$$\tau_0(z):=\frac{2Kz}{\Omega_0(\xi^2+1)},\ \tau_1:=\frac{1}{\Omega_0\xi}\ln\left|\frac{2\xi\alpha'(-\iota\xi)}{\sqrt{p_{1+}^2(0)+p_{2+}^2(0)}}\right|.$$

Similarly,

$$A_y(\tau,z)=\frac{\xi}{C}\frac{p_{2+}(0)}{\sqrt{p_{1+}^2(0)+p_{2+}^2(0)}}\operatorname{sech}\left[\Omega_0\xi\left(\tau-\tau_1-\tau_0(z)\right)\right]. \qquad (129)$$

Solutions (128) and (129) are precisely the same linearly polarised 1-solitons found in Chapter 4 (cf. equation (4), p. 40).

5.2.4 Existence and Uniqueness of Solutions to the Inverse Problem.

General questions regarding existence and uniqueness of solutions to the Riemann-Hilbert problem (105) are difficult to answer categorically. However, by carrying out a simple calculation based on theory established by Gohberg and Krein (1958) [37], we can at least demonstrate existence and uniqueness in the generic sense.

First recall equation (74) for the scattering matrix $S(\zeta)$

$$S_+^{-1}(\tau,\zeta)S_-(\tau,\zeta) = S(\zeta).$$

If we define

$$\begin{aligned} G_+(\tau,\zeta) &= \alpha(-\zeta)E(\tau,\zeta)S_+^{-1}(\tau,\zeta), \quad (130) \\ G_-(\tau,\zeta) &= S_-(\tau,\zeta)E^{-1}(\tau,\zeta), \quad (131) \end{aligned}$$

then $G_\pm$ extend analytically into $\Pi^\pm$, with $G_\pm(\tau,\zeta) = I + o(1)$, as $|\zeta| \to \infty$ (cf. equations (68, 69)). Furthermore

$$\begin{aligned} G(\tau,\zeta) &:= G_+(\tau,\zeta)G_-(\tau,\zeta) \\ &= E(\tau,\zeta)\alpha(-\zeta)S_+^{-1}(\tau,\zeta)S_-(\tau,\zeta)E^{-1}(\tau,\zeta) \\ &= E(\tau,\zeta)\alpha(-\zeta)S(\zeta)E^{-1}(\tau,\zeta) \\ &= \begin{pmatrix} 1 & e^{\iota\Omega_0\zeta\tau}\eta(-\zeta), & e^{\iota\Omega_0\zeta\tau}\nu(-\zeta) \\ -e^{-\iota\Omega_0\zeta\tau}\beta(-\zeta), & \chi(\zeta) & -\mu(\zeta) \\ -e^{-\iota\Omega_0\zeta\tau}\gamma(-\zeta), & -\rho(\zeta) & \theta(\zeta) \end{pmatrix}, \quad (132) \end{aligned}$$

and $G(\tau,\zeta) = I + o(1)$ as $|\zeta| \to \infty$.

Considering that in Section 5.2.1 we also derived (105) from (74), we therefore see the equivalence of our Riemann-Hilbert problem to the matrix factorisation problem on $\text{Im}\zeta = 0$ associated with (132). Namely, given $G(\tau, z, \zeta)$, find $G_\pm(\tau, z, \zeta)$ such that

$$G_+(\tau,z,\zeta)G_-(\tau,z,\zeta) = G(\tau,z,\zeta),$$

for $\zeta \in \mathbf{R}$, where $G_\pm$ may be analytically extended into $\Pi^\pm$.

Now let $R_{3\times3}$ be the ring whose elements are 3×3 matrix functions of the form

$$F(\zeta) = cI + \int_{-\infty}^{\infty} f(s)e^{\iota\zeta s}ds,$$

where $f(s) \in L^1_{3\times3}(\mathbf{R})$, and let $R^{\pm}_{3\times3}$ be the subring of $R_{3\times3}$ having representative elements

$$F_{\pm}(\zeta) = cI + \int_0^{\infty} f_{\pm}(s)e^{\pm\iota\zeta s}ds,$$

where $f_{\pm}(s) \in L^1_{3\times3}(0,\infty)$. Then Gohberg and Krein proved the theorem below (cf. A&C., p. 449):

Theorem.

1. Every non-singular matrix $G(\zeta) \in R_{3\times3}$ possesses the factorisation

$$G(\zeta) = G_+(\zeta)\Delta(\zeta)G_-(\zeta),$$

where $G_{\pm} \in R^{\pm}_{3\times3}$, $\det G_{\pm}(\zeta) \neq 0$ for $\zeta \in \Pi^{\pm}$, and

$$\Delta(\zeta) = \operatorname{diag}\left\{\left(\frac{\zeta-\iota}{\zeta+\iota}\right)^{\kappa_1}, \left(\frac{\zeta-\iota}{\zeta+\iota}\right)^{\kappa_2}, \left(\frac{\zeta-\iota}{\zeta+\iota}\right)^{\kappa_3}\right\}.$$

Here $\kappa_1 \geq \kappa_2 \geq \kappa_3$ are integers referred to as *indices*.

2. If G possesses another factorisation

$$G(\zeta) = \tilde{G}_+(\zeta)\tilde{\Delta}(\zeta)\tilde{G}_-(\zeta),$$

then $\tilde{\Delta}(\zeta) = \Delta(\zeta)$, $\tilde{G}_+(\zeta) = G_+(\zeta)A(\zeta)$, $\tilde{G}_-(\zeta) = A^{-1}(\zeta)G_-(\zeta)$, for some non-singular matrix A.

3. $\kappa = \kappa_1 + \kappa_2 + \kappa_3 = \operatorname{ind}[\det G(\zeta)]$, where given a function $g(\zeta)$ defined on $\mathbf{R}^1$, the index of g with respect to $\mathbf{R}^1$ is

$$\operatorname{ind}[g(\zeta)] := \frac{1}{2\pi\iota}\int_{-\infty}^{\infty} \frac{d}{d\zeta}\{\ln g(\zeta)\}\,d\zeta.$$

Corollary. If it is further supposed that $G(\zeta) = I + o(1)$ as $|\zeta| \to \infty$, and

$$G_R(\zeta) = \frac{1}{2}\left[G(\zeta) + \overline{G}^t(\zeta)\right]$$

is positive definite, then $\kappa_1 = \kappa_2 = \kappa_3 = 0$ and there exists a unique factorisation

$$G(\zeta) = G_+(\zeta)G_-(\zeta),$$

where $G_\pm \in R^{\pm}_{3\times3}$, $\det G_\pm(\zeta) \neq 0$ for $\zeta \in \Pi^\pm$, and $G_\pm(\zeta) = I + o(1)$ as $|\zeta| \to \infty$.

In view of Gohberg and Krein's theory, which provides the standard method for checking existence and uniqueness of Riemann-Hilbert problems, we look again at G and $G_\pm$ specified by $(130 - 132)$:

- From Section 5.1.5 we conclude G has an integral representation

$$\begin{aligned} G(\tau,\zeta) &= I + \int_{-\infty}^{\infty} \begin{pmatrix} 0 & \widehat{\eta}(-s+\tau), & \widehat{\nu}(-s+\tau) \\ -\widehat{\beta}(-s-\tau) & 0 & 0 \\ -\widehat{\gamma}(-s-\tau) & 0 & 0 \end{pmatrix} e^{\iota\zeta s} ds \\ &+ \int_0^{\infty} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \widehat{\chi}(s) & -\widehat{\mu}(s) \\ 0 & -\widehat{\rho}(s) & \widehat{\theta}(s) \end{pmatrix} e^{\iota\zeta s} ds, \end{aligned}$$

  where $\widehat{\eta}$, $\widehat{\nu}$, $\widehat{\beta}$, $\widehat{\gamma} \in L^1(\mathrm{R})$, and $\widehat{\chi}$, $\widehat{\mu}$, $\widehat{\rho}$, $\widehat{\theta} \in L^1(0,\infty)$. Hence $G(\tau,\zeta) \in R_{3\times3}$.

- Using the integral representations (33) and (37) for the Jost solution matrices it is reasonably straightforward to prove $G_\pm(\tau,\zeta) \in R^{\pm}_{3\times3}$. For instance, equation (72) gives

$$\alpha(-\zeta)S_+^{-1}(\tau,\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ -\beta(-\zeta) & \alpha(-\zeta) & 0 \\ -\gamma(-\zeta) & 0 & \alpha(-\zeta) \end{pmatrix} T_-^t(\tau,-\zeta),$$

leading to

$$G_+^{1j}(\tau,\zeta) = \int_0^\infty \Gamma^{j1}(\tau,\tau-s)e^{\iota\Omega_0\zeta s}ds,$$

$j = 1,2,3$, where $\Gamma^{j1}(\tau,\tau-s) \in L^1(0,\infty)$ as a function of s. (Appropriate integral representations for rows two and three of $G_+(\tau,\zeta)$ are found by means of the Convolution Theorem for Laplace Transforms, and the equality $\alpha(-\zeta)S_+^{-1}(\tau,\zeta) = \text{adj}\,[S_+(\tau,\zeta)]$).

- $\text{det}G(\tau,\zeta) = \det[\alpha(-\zeta)S(\zeta)] = \alpha^2(-\zeta)\alpha(\zeta) \neq 0$ when $\zeta \in \mathbf{R}$. Whilst $\text{det}G_+(\tau,\zeta) = \alpha^2(-\zeta)$ and $\text{det}G_-(\tau,\zeta) = \alpha(\zeta)$, so that $G_\pm$ may be non-singular in their domains of analyticity.

These properties of G and $G_\pm$ are all much as we might expect. Unfortunately, due to the complicated form of $G(\tau,\zeta)$, the "hermitian positive definite" condition apparently does not furnish a clear-cut or useful criterion for existence and uniqueness. However the situation is redeemed somewhat by the following result:

$$\begin{aligned} \text{ind}\,[\text{det}G(\zeta)] &= \frac{1}{2\pi\iota}\int_{-\infty}^{\infty}\frac{d}{d\zeta}\{\ln[\text{det}G(\zeta)]\}\,d\zeta \\ &= \frac{1}{2\pi\iota}\left[\ln\left(\overline{\alpha}(\zeta)\,|\alpha(\zeta)|^2\right)\right]_{-\infty}^{\infty} = 0, \end{aligned}$$

which is a consequence of the asymptotic behaviour $\alpha(\zeta) = 1 + o(1)$ as $|\zeta| \to \infty$. According to another Gohberg and Krein theorem on stability of indices, Ablowitz & Clarkson (cf. p. 450) point out that if the overall index κ is zero, then generically speaking, the indices κ_1, κ_2, κ_3 vanish aswell. Existence and uniqueness of solutions to our Riemann-Hilbert problem is therefore guaranteed in a generic sense.

Chapter 6

Calculation of the Conserved Densities for the RMB-System.

A striking general property of integrable partial differential equations is that they possess an infinite hierarchy of local conservation laws. It will be seen in the case of the RMB-equations that the conserved quantities occur as the coefficients in an asymptotic expansion for $\log \alpha(\zeta)$, and we shall represent them as functionals of the scattering data through use of the trace formula (94) p. 79.

Let $N(\tau, z, u(\tau, z)) = 0$ be an integrable equation. Then since we have taken z to be the evolution variable and τ the transverse variable, a conservation law may be defined as an equation of the form

$$\partial_z I + \partial_\tau F = 0,$$

holding for all solutions $u(\tau, z)$ of $N = 0$, where the density I and the flux F are allowed to depend upon τ, z, u, u_τ, $u_{\tau\tau}, \ldots$, but not u_z. Providing I and F_τ are integrable as functions of τ over $\mathbb{R}$, and if $F \to$ constant as $|\tau| \to \infty$, we then find

$$\frac{d}{dz}\left(\int_{-\infty}^{\infty} I d\tau\right) = 0.$$

I is called a *conserved density* and the functional $\int_{-\infty}^{\infty} I d\tau$ is a *constant* or *invariant of the motion.*

Now in order to obtain the conserved densities for our RMB-system, we seek a gauge transformation which asymptotically reduces the matrix

$$U(\tau,\zeta) = -\frac{\iota}{\hbar}\left(\zeta H^- + H^+(\tau)\right)$$

to the diagonal form

$$U' \sim \sum_{j=-1}^{\infty} \frac{U_j'}{\zeta^j},$$

where each U_j' is a diagonal matrix.

It is simple enough to show that each U_j' is conserved. In fact, because the zero curvature compatibility condition is invariant under gauge transformations

$$U' = G_\tau G^{-1} + GUG^{-1} \quad (1), \quad V' = G_z G^{-1} + GVG^{-1},$$

we have

$$\partial_z U' - \partial_\tau V' + [U', V'] = 0.$$

The diagonal part reads

$$\partial_z U' - \partial_\tau (V')^D = 0,$$

and so integrating with respect to τ and using the boundary conditions yields

$$\frac{d}{dz}\left(\int_{-\infty}^{\infty} U' d\tau\right) = 0, \ \forall\zeta.$$

Therefore

$$\frac{d}{dz}\left(\int_{-\infty}^{\infty} U_j' d\tau\right) = 0,$$

as claimed.

Since $U(\tau,\zeta)$ is a 3×3 matrix, there are two stages involved in the diagonalisation procedure. First let us reduce U to an R-type matrix $U'(\tau,\zeta) = Z(\tau,\zeta)$ (say), by applying a gauge transformation $G_1(\tau,\zeta) = (I + W(\tau,\zeta))^{-1}$, where W is chosen to be an OR-type matrix with the asymptotic form

$$W(\tau,\zeta) = \sum_{n=1}^{\infty} \frac{W_n(\tau)}{\zeta^n} + O\left(|\zeta|^{-\infty}\right), \qquad (2)$$

as $|\zeta| \to \infty$. (The R and OR-parts of a 3×3 matrix were defined on pp. 60-61). Substituting into equation (1) it follows that

$$(I + W(\tau,\zeta))\, Z(\tau,\zeta) = -W_\tau(\tau,\zeta) - \frac{\iota}{\hbar}\left(\zeta H^- + H^+(\tau)\right)(I + W(\tau,\zeta)),$$

which is equivalent to

$$\begin{aligned} \iota\hbar Z &= \zeta H^- + H^+ W, && (3)\\ \iota\hbar W Z &= -\iota\hbar W_\tau + H^+ + \zeta H^- W, && (4) \end{aligned}$$

after splitting into R and OR-parts respectively. Equations (2) and (3) imply

$$Z(\tau,\zeta) = -\frac{\iota}{\hbar}\zeta H^- - \frac{\iota}{\hbar}\sum_{n=1}^{\infty}\frac{(H^+ W_n)(\tau)}{\zeta^n} + O\left(|\zeta|^{-\infty}\right), \qquad (5)$$

as $|\zeta| \to \infty$, whilst eliminating Z from $(3,4)$ gives

$$\iota\hbar W_\tau - H^+ + \zeta\left[W, H^-\right] + W H^+ W = 0. \qquad (6)$$

Next suppose we define $\sigma = \operatorname{diag}\{1,-1,-1\}$, and $H^s(\tau) = \begin{pmatrix} 0 & A_x, & A_y \\ A_x & 0 & 0 \\ A_y & 0 & 0 \end{pmatrix}(\tau)$.

Then equating coefficients of powers of ζ in equation (6) and employing the formulae

$$-\sigma H^s \;=\; H^s \sigma,$$

$$\left[W_n, H^-\right] = \hbar\Omega_0 \begin{pmatrix} 0 & W_n^{12}, & W_n^{13} \\ -W_n^{21} & 0 & 0 \\ -W_n^{31} & 0 & 0 \end{pmatrix} = \hbar\Omega_0\sigma W_n,$$

$$H^+ = -\iota\hbar\Omega_0 C \begin{pmatrix} 0 & A_x, & A_y \\ -A_x & 0 & 0 \\ -A_y & 0 & 0 \end{pmatrix} = -\iota\hbar\Omega_0 C\sigma H^s,$$

(where $C := eq_0/\hbar$), leads to the results listed below:

$$W_1 = -\iota C \begin{pmatrix} 0 & A_x, & A_y \\ A_x & 0 & 0 \\ A_y & 0 & 0 \end{pmatrix} = (-\iota)^1 C H^s,$$

$$W_{n+1} = \frac{\sigma}{\hbar\Omega_0}\left(-\iota\hbar W_{n\tau} - \sum_{k=1}^{n-1} W_k H^+ W_{n-k}\right), \ n \geq 1,$$

$$W_2 = -\frac{C}{\Omega_0} \begin{pmatrix} 0 & A_{x\tau}, & A_{y\tau} \\ -A_{x\tau} & 0 & 0 \\ -A_{y\tau} & 0 & 0 \end{pmatrix} = (-\iota)^2 C \left(\frac{\sigma}{\Omega_0}\right)^1 H^s_\tau,$$

$$W_3 = \iota C \left[\frac{1}{\Omega_0^2} \begin{pmatrix} 0 & A_{x\tau\tau}, & A_{y\tau\tau} \\ A_{x\tau\tau} & 0 & 0 \\ A_{y\tau\tau} & 0 & 0 \end{pmatrix} + C^2 |A|^2 \begin{pmatrix} 0 & A_x, & A_y \\ A_x & 0 & 0 \\ A_y & 0 & 0 \end{pmatrix}\right]$$

$$= (-\iota)^3 C \left(\frac{\sigma}{\Omega_0}\right)^2 \left(H^s_{\tau\tau} + \Omega_0^2 C^2 H^{s^3}\right),$$

$$W_4 = (-\iota)^4 C \left(\frac{\sigma}{\Omega_0}\right)^3 \left\{H^s_{\tau\tau\tau} + \Omega_0^2 C^2 \left[\left(H^{s^3}\right)_\tau + H^{s^2} H^s_\tau + H^s_\tau H^{s^2}\right]\right\}, \text{ etc.}$$

Clearly W_n may be written

$$W_n(\tau) = (-\iota)^n C \left(\frac{\sigma}{\Omega_0}\right)^{n-1} \omega_n(\tau), \ n \geq 1,$$

where ω_n is a real symmetric OR-matrix whose elements are polynomial functions of A_x, A_y and their τ-derivatives. The power of σ determines whether W_n

is symmetric or anti-symmetric. Hence, substituting for H^+ and W_n in (5), our expression for Z becomes

$$
\begin{aligned}
Z &= \frac{1}{3}\iota\Omega_0 \operatorname{diag}\{2,-1,-1\}\zeta \\
&+ \Omega_0^2 C^2 \sum_{n=1}^{\infty}\left(\frac{\iota\sigma}{\Omega_0}\right)^n \begin{pmatrix} A_x\omega_n^{12}+A_y\omega_n^{13} & 0 & 0 \\ 0 & A_x\omega_n^{12} & A_x\omega_n^{13} \\ 0 & A_y\omega_n^{12} & A_y\omega_n^{13} \end{pmatrix} \zeta^{-n}. \quad (7)
\end{aligned}
$$

We remark incidentally that the diagonal parts of H^+W_{2n} are total derivatives with respect to τ.

The second step is to asymptotically reduce $Z(\tau,\zeta)$ to $\Delta(\tau,\zeta)$, a diagonal matrix, using the gauge transformation $G_2(\tau,\zeta) = (I + M(\tau,\zeta))^{-1}$, where for large $|\zeta|$

$$M(\tau,\zeta) = \sum_{n=1}^{\infty} \frac{M_n(\tau)}{\zeta^n} + O\left(|\zeta|^{-\infty}\right),$$

and

$$M_n(\tau) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & M_n^{23} \\ 0 & M_n^{32} & 0 \end{pmatrix}(\tau).$$

We have

$$(I+M)\Delta = -M_\tau + Z(I+M),$$

or, in terms of diagonal and off-diagonal parts,

$$
\begin{aligned}
\Delta &= Z^D + Z^{OD}M, \quad (8) \\
M\Delta &= -M_\tau + Z^D M + Z^{OD}.
\end{aligned}
$$

Eliminating Δ therefore gives

$$M_\tau - Z^{OD} + \left[M, Z^D\right] + MZ^{OD}M = 0.$$

It is convenient for the moment to use equation (5) rather than (7) for Z, then matching coefficients of powers of ζ we obtain

$$M_{1_T} = -\frac{\iota}{\hbar}\left(H^+W_1\right)^{OD} = -\iota\Omega_0 C^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_x A_y \\ 0 & A_x A_y & 0 \end{pmatrix}, \quad (9)$$

$$\begin{aligned} M_{2_T} &= -\frac{\iota}{\hbar}\left\{\left(H^+W_2\right)^{OD} + \left[\left(H^+W_1\right)^{D}, M_1\right]\right\} \\ &= -C^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_x A_{y_T} \\ 0 & A_{x_T} A_y & 0 \end{pmatrix} + \\ &\quad \Omega_0^2 C^4 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \left(A_y^2 - A_x^2\right)\int_{-\infty}^{T} A_x A_y ds \\ 0 & \left(A_x^2 - A_y^2\right)\int_{-\infty}^{T} A_x A_y ds & 0 \end{pmatrix}, \end{aligned}$$

etc., and by equation (8)

$$\begin{aligned} \Delta &= \sum_{j=-1}^{\infty} \frac{\Delta_j}{\zeta^j} \\ &= -\frac{\iota}{\hbar}\left\{\zeta H^- + \frac{\left(H^+W_1\right)^D}{\zeta} + \frac{\left(H^+W_2\right)^D + \left(H^+W_1\right)^{OD} M_1}{\zeta^2} + \right. \\ &\quad \left. \frac{\left(H^+W_3\right)^D + \left(H^+W_1\right)^{OD} M_2 + \left(H^+W_2\right)^{OD} M_1}{\zeta^3} + O\left(|\zeta|^{-4}\right)\right\}. \end{aligned}$$

As was demonstrated at the beginning of the chapter, the diagonal matrix coefficients of this expansion for Δ are conserved densities of the RMB-system. However, after some rather laborious calculation, it becomes apparent that the coefficients of even powers of ζ are total derivatives and can be ignored.

Consider for example the coefficient $\Delta_2 = (\iota\hbar)^{-1}\left[\left(H^+W_2\right)^D + \left(H^+W_1\right)^{OD} M_1\right]$.

Plainly we have

$$\int_{-\infty}^{\infty} -\frac{\iota}{\hbar}\left(H^{+}W_{2}\right)^{D} d\tau = \int_{-\infty}^{\infty} \partial_{\tau}\left\{-\frac{C^2}{2}\left(\begin{array}{ccc} |A|^2 & 0 & 0 \\ 0 & A_x^2 & 0 \\ 0 & 0 & A_y^2 \end{array}\right)\right\} d\tau = 0.$$

Dealing with the second term

$$-\frac{\iota}{\hbar}\left(H^{+}W_{1}\right)^{OD} M_1 = -\iota\Omega_0 C^2 \mathrm{diag}\left\{0, A_x A_y M_1^{23}, A_x A_y M_1^{23}\right\}$$

is a little more complicated. From equations (19), (26) and (27) on p. 35, we find

$$\frac{d}{dz}\int_{-\infty}^{\infty} A_x A_y d\tau \propto \int_{-\infty}^{\infty} \rho_{2\tau} d\tau = 0.$$

$$\text{i.e.,} \int_{-\infty}^{\infty} A_x A_y d\tau := \kappa, \text{ (say)},$$

is also an invariant of the motion. Moreover, since equation (9) only defines $M_1^{23}(\tau, z)$ up to an arbitrary constant of integration, we may choose

$$M_1^{23}(\tau, z) = -\iota\Omega_0 C^2\left[\int_{-\infty}^{\tau}\left(A_x A_y\right)(s, z) ds - \frac{1}{2}\kappa\right].$$

Then

$$\int_{-\infty}^{\infty} -\iota\Omega_0 C^2 A_x A_y M_1^{23} d\tau = 0 = \int_{-\infty}^{\infty} -\frac{\iota}{\hbar}\left(H^{+}W_{1}\right)^{OD} M_1 d\tau.$$

It is also instructive to examine the coefficient Δ_3 of ζ^{-3}. Observe that

$$\mathrm{Tr}\left\{\int_{-\infty}^{\infty} \Delta_3 d\tau\right\} = 0,$$

which as we shall see presently , is a condition required for consistency. Indeed one can readily check

$$-\frac{\iota}{\hbar}\int_{-\infty}^{\infty}\left[\left(H^{+}W_{3}\right)^{D}+\left(H^{+}W_{1}\right)^{OD}M_{2}+\left(H^{+}W_{2}\right)^{OD}M_{1}\right]d\tau =$$

$$\iota\Omega_0 C^2\int_{-\infty}^{\infty}\left(\operatorname{diag}\{a+b,-a,-b\}+\operatorname{diag}\{0,c+d,-c-d+f_\tau\}\right)d\tau,$$

where

$$a := -\frac{1}{\Omega_0^2}A_xA_{x\tau\tau}-C^2A_x^2\left|A\right|^2,\;\; b := -\frac{1}{\Omega_0^2}A_yA_{y\tau\tau}-C^2A_y^2\left|A\right|^2,$$

$$\begin{aligned}
c &:= \Omega_0^2C^4A_xA_y\int_{-\infty}^{\tau}\left[\left(A_y^2-A_x^2\right)\left(\int_{-\infty}^{s}A_xA_ydu\right)\right]ds,\\
d &:= C^2\left[A_xA_y\left(\int_{-\infty}^{\tau}A_{x\tau}A_yds\right)+A_xA_{y\tau}\left(\int_{-\infty}^{\tau}A_xA_yds\right)\right],\\
e &:= C^2\left[A_xA_y\left(\int_{-\infty}^{\tau}A_xA_{y\tau}ds\right)+A_yA_{x\tau}\left(\int_{-\infty}^{\tau}A_xA_yds\right)\right],\\
\text{and } f &:= A_xA_y\left(\int_{-\infty}^{\tau}A_xA_yds\right),\;\; f_\tau = d+e.
\end{aligned}$$

Therefore, because $\int_{-\infty}^{\infty}f_\tau d\tau = 0$, we may set

$$\int_{-\infty}^{\infty}\Delta_3 d\tau = \iota\Omega_0C^2\int_{-\infty}^{\infty}\operatorname{diag}\{A_3+B_3,-A_3,-B_3\}\,d\tau,$$

where $A_3 := a-c-d$, $B_3 := b+c+d$.

We now introduce some notation and theory regarding the monodromy matrix $T_L(\zeta) = T(L,-L,\zeta)$. This theory helps to finalise the form of our expansion for Δ, and will enable us to represent the conserved densities in terms of the scattering data established in Chapter 5.

Let the monodromy matrix be written

$$T_L(\zeta) = \begin{pmatrix} \alpha_L & \beta_L & \gamma_L \\ \eta_L & \theta_L & \mu_L \\ \nu_L & \rho_L & \chi_L \end{pmatrix} (\zeta),$$

where

$$\alpha_L(\zeta) = (\theta_L \chi_L - \rho_L \mu_L)(-\zeta), \; \beta_L(\zeta) = (\nu_L \mu_L - \eta_L \chi_L)(-\zeta),$$

$$\gamma_L(\zeta) = (\eta_L \rho_L - \nu_L \theta_L)(-\zeta).$$

The elements of T_L satisfy the usual normalisation and orthogonality conditions, and we know $\det T_L(\zeta) = 1$. Then the reduced monodromy matrix is given by the limit

$$\begin{aligned} T(\zeta) &= \lim_{L\to\infty} \{E(-L,\zeta) T_L(\zeta) E(-L,\zeta)\} \\ &= \lim_{L\to\infty} \left\{ \begin{pmatrix} e^{-\frac{4}{3}\iota L\Omega_0\zeta}\alpha_L(\zeta), & e^{-\frac{1}{3}\iota L\Omega_0\zeta}\beta_L(\zeta), & e^{-\frac{1}{3}\iota L\Omega_0\zeta}\gamma_L(\zeta) \\ e^{-\frac{1}{3}\iota L\Omega_0\zeta}\eta_L(\zeta), & e^{\frac{2}{3}\iota L\Omega_0\zeta}\theta_L(\zeta), & e^{\frac{2}{3}\iota L\Omega_0\zeta}\mu_L(\zeta) \\ e^{-\frac{1}{3}\iota L\Omega_0\zeta}\nu_L(\zeta), & e^{\frac{2}{3}\iota L\Omega_0\zeta}\rho_L(\zeta), & e^{\frac{2}{3}\iota L\Omega_0\zeta}\chi_L(\zeta) \end{pmatrix} \right\}. \quad (10) \end{aligned}$$

Furthermore, according to equation (2.19) p. 23 of F&T., the sequence of gauge transformations G_1, G_2 transforms $T_L(\zeta)$ in such a way that

$$\begin{aligned} T_L(\zeta) &= (I + M(L,\zeta))\,(I + W(L,\zeta)) \times \\ &\quad \exp\left[\int_{-L}^{L} \Delta(\tau,\zeta)d\tau\right] (I + W(L,\zeta))^{-1}\,(I + M(L,\zeta))^{-1}. \quad (11) \end{aligned}$$

Since $\det T_L(\zeta) = 1$, this relation implicitly requires

$$\mathrm{Tr}\left\{\int_{-L}^{L} \Delta(\tau,\zeta)d\tau\right\} = 0, \; \forall\zeta, \text{ and so } \mathrm{Tr}\left\{\int_{-L}^{L} \Delta_j d\tau\right\} = 0, \; \forall j.$$

We are finally in position to draw together the results of our calculation. Equations (7) and (8) imply

$$\begin{aligned}\Delta(\tau,\zeta) &= \frac{1}{3}\iota\Omega_0 \mathrm{diag}\{2,-1,-1\}\zeta + \\ &\Omega_0^2C^2\sum_{n=1}^{\infty}\left(\frac{\iota\sigma}{\Omega_0}\right)^n \zeta^{-n}\mathrm{diag}\left\{A_x\omega_n^{12}+A_y\omega_n^{13}, A_x\omega_n^{12}, A_y\omega_n^{13}\right\} + \\ &\Omega_0^2C^2\sum_{r=2}^{\infty}\sum_{k=1}^{r-1}\left(\frac{\iota\sigma}{\Omega_0}\right)^k \zeta^{-r}\mathrm{diag}\left\{0, A_x\omega_k^{13}M_{r-k}^{32}, A_y\omega_k^{12}M_{r-k}^{23}\right\}.\end{aligned}$$

Hence for sufficiently large L we can write

$$\begin{aligned}\int_{-L}^{L}\Delta(\tau,\zeta)d\tau &= \frac{2}{3}\iota L\Omega_0 \mathrm{diag}\{2,-1,-1\}\zeta + \\ &\Omega_0C^2\int_{-L}^{L}\left[\sum_{n=1}^{\infty}\left(\frac{\iota}{\Omega_0}\right)^{2n-1}\zeta^{-2n+1}\mathrm{diag}\left\{A_x\omega_{2n-1}^{12}+A_y\omega_{2n-1}^{13},\right.\right. \\ &\left.-A_x\omega_{2n-1}^{12}, -A_y\omega_{2n-1}^{13}\right\} + \sum_{n=1}^{\infty}\sum_{k=1}^{2n}\left(\frac{\iota\sigma}{\Omega_0}\right)^k\zeta^{-2n-1}\times \\ &\left.\mathrm{diag}\left\{0, A_x\omega_k^{13}M_{2n+1-k}^{32}, -A_x\omega_k^{13}M_{2n+1-k}^{32}\right\}\right]d\tau.\end{aligned}$$

(It is of course equivalent to let

$$\mathrm{diag}\left\{0, -A_y\omega_k^{12}M_{2n+1-k}^{23}, A_y\omega_k^{12}M_{2n+1-k}^{23}\right\}$$

be the second matrix of the integrand). Then by equation (11)

$$\begin{aligned}\mathrm{Tr}\{T_L(\zeta)\} &= \mathrm{Tr}\left\{\exp\int_{-L}^{L}\Delta d\tau\right\} \\ &= \mathrm{Tr}\left\{\exp\left[\frac{2}{3}\iota L\Omega_0\zeta \mathrm{diag}\{2,-1,-1\}\right]\times\right. \\ &\exp\left[\iota\Omega_0C^2\int_{-L}^{L}\sum_{n=1}^{\infty}\zeta^{-2n+1}\mathrm{diag}\{A_{2n-1}+B_{2n-1},\right. \\ &-A_{2n-1}, -B_{2n-1}\}\,d\tau]\}\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{Tr}\left\{\exp\left[\frac{2}{3}\iota L\Omega_0\zeta \operatorname{diag}\{2,-1,-1\}\right]\times\right.\\
&\quad \left.\exp\left[\iota\Omega_0 C^2\int_{-L}^{L}\operatorname{diag}\{(A+B)(\zeta),-A(\zeta),-B(\zeta)\}\,d\tau\right]\right\}\\
&= \exp\left(\frac{4}{3}\iota L\Omega_0\zeta\right)\exp\left[\iota\Omega_0 C^2\int_{-L}^{L}(A+B)(\zeta)d\tau\right]+\\
&\quad \exp\left(-\frac{2}{3}\iota L\Omega_0\zeta\right)\exp\left[-\iota\Omega_0 C^2\int_{-L}^{L}A(\zeta)d\tau\right]+\\
&\quad \exp\left(-\frac{2}{3}\iota L\Omega_0\zeta\right)\exp\left[-\iota\Omega_0 C^2\int_{-L}^{L}B(\zeta)d\tau\right],
\end{aligned}
$$

as $L\to\infty$, where

$$
A(\tau,\zeta) := \sum_{n=1}^{\infty} A_{2n-1}(\tau)\zeta^{-2n+1},\quad B(\tau,\zeta) := \sum_{n=1}^{\infty} B_{2n-1}(\tau)\zeta^{-2n+1}.
$$

On the other hand we have

$$
\begin{aligned}
\operatorname{Tr}\{T_L(\zeta)\} &= \alpha_L(\zeta)+\theta_L(\zeta)+\chi_L(\zeta)\\
&= \alpha(\zeta)e^{\frac{4}{3}\iota L\Omega_0\zeta}+\theta(\zeta)e^{-\frac{2}{3}\iota L\Omega_0\zeta}+\chi(\zeta)e^{-\frac{2}{3}\iota L\Omega_0\zeta}+o(1)
\end{aligned}
$$

as $L\to\infty$, which is a consequence of equation (10). Comparing expressions for $\operatorname{Tr}\{T_L(\zeta)\}$ and letting $L\to\infty$ therefore yields

$$
\log\alpha(\zeta) = \log\left[(\theta\chi-\rho\mu)(-\zeta)\right] = \iota\Omega_0 C^2\int_{-\infty}^{\infty}(A+B)(\zeta)d\tau + O\left(|\zeta|^{-\infty}\right), \quad (12)
$$

$$
\log\theta(\zeta) = -\iota\Omega_0 C^2\int_{-\infty}^{\infty}A(\zeta)d\tau + O\left(|\zeta|^{-\infty}\right),
$$

$$
\log\chi(\zeta) = -\iota\Omega_0 C^2\int_{-\infty}^{\infty}B(\zeta)d\tau + O\left(|\zeta|^{-\infty}\right).
$$

Moreover, comparison of the two equations (12) and (94) p. 79 for $\log\alpha(\zeta)$ immediately gives

$$\int_{-\infty}^{\infty}(A_{2r-1}+B_{2r-1})\,d\tau = \frac{1}{\Omega_0 C^2}\left\{\frac{-1}{2\pi}\int_{-\infty}^{\infty}\log\left[1-\left(|\eta(s)|^2+|\nu(s)|^2\right)\right]s^{2r-2}ds - \frac{2\iota}{2r-1}\left[\sum_{j=1}^{n_1}\left(\iota\zeta_{\alpha_j}\right)^{2r-1}+\sum_{k=n_1+1}^{n_1+n_2}\left(\zeta_{\alpha_k}^{2r-1}-\overline{\zeta_{\alpha_k}}^{2r-1}\right)\right]\right\},$$

for $\mathbf{N}\ni r\geq 1$, where

$$A_1+B_1=|A|^2=A_x^2+A_y^2,$$

$$A_3+B_3 = -\frac{1}{\Omega_0^2}(A_xA_{x\tau\tau}+A_yA_{y\tau\tau})-C^2|A|^4, \qquad (13)$$

$$(\text{or} = \frac{1}{\Omega_0^2}|\nabla A|^2-C^2|A|^4\ ), \qquad \text{etc.}$$

Evidently this equation describes those conserved densities which are polynomial functions of A_x, A_y and their τ -derivatives, in terms of the set of scattering variables

$$\left\{|\eta(\zeta)|^2, |\nu(\zeta)|^2, \zeta_{\alpha_j}, \zeta_{\alpha_k}, \overline{\zeta_{\alpha_k}}\,\middle|\, j=1,...,n_1,\ k=n_1+1,...,n_1+n_2\right\}.$$

Chapter 7

Perturbative Effects in Real Optical Media.

So far we have been concerned with the solution of an idealised mathematical model which ignores the various loss mechanisms and inhomogeneities actually occuring in a typical optical medium. Under these simplifying assumptions we have explicitly proved the solvability of the RMB-system by the Inverse Scattering Transform method, and studied the associated hierarchy of conserved densities. We shall now exploit the exact mathematical results of the preceding chapters to derive the following evolution equation

$$\frac{d}{dz}\int_{-\infty}^{\infty}(A_{2r-1}+B_{2r-1})\,d\tau=\int_{-\infty}^{\infty}\mathbf{A}_z^t\mathbf{L}^{2r-2}\mathbf{A}\,d\tau, \qquad (1)$$

where $\mathbf{N} \ni r \geq 1$, $\mathbf{A} := (-A_x, A_x, -A_y, A_y)^t$, and $\mathbf{L}$ is a 4×4 matrix integro-differential operator to be defined later. Given a system of RMB-equations suitably altered to model an arbitrary perturbing effect, equation (1) may be used to calculate the corresponding variation with distance z of the invariant functionals

$$\int_{-\infty}^{\infty}(A_{2r-1}+B_{2r-1})\,d\tau$$

specified in Chapter 6. The main reference for this chapter is a paper by J.N. Elgin (1993) [13] entitled "Perturbations of optical solitons". Elgin proves that a similar equation to (1) holds for the simpler case of the Nonlinear Schrödinger equation, which has only a 2×2 auxiliary linear problem (cf. [13], equation (4), p. 4332).

7.1 Proof of the Evolution Equation for the Invariant Functionals.

Let us recollect some basic theory out of Chapter 5:

- The AKNS eigenvalue problem for our RMB-system can be written

$$\mathbf{f}_\tau = U\mathbf{f} = \begin{pmatrix} \frac{2}{3}\iota\Omega_0\zeta, & -\Omega_0 CA_x, & -\Omega_0 CA_y \\ \Omega_0 CA_x, & -\frac{1}{3}\iota\Omega_0\zeta & 0 \\ \Omega_0 CA_y, & 0 & -\frac{1}{3}\iota\Omega_0\zeta \end{pmatrix} \mathbf{f}, \qquad (2)$$

  where $U = U(\tau, z, \zeta)$ and $\mathbf{f} = (f_1, f_2, f_3)^t \in \mathbf{C}^3$.

- The Jost solution matrices $T_\pm$ satisfy the differential equation (2) for each z and have the asymptotic behaviour $T_\pm(\tau, z, \zeta) = E(\tau, \zeta) + o(1)$, as $\tau \to \pm\infty$. That is

$$\left(T_\pm^{(1)}, T_\pm^{(2)}, T_\pm^{(3)}\right)(\tau, z, \zeta) =$$

$$\left(\begin{pmatrix} e^{\frac{2}{3}\iota\Omega_0\zeta\tau} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e^{-\frac{1}{3}\iota\Omega_0\zeta\tau} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ e^{-\frac{1}{3}\iota\Omega_0\zeta\tau} \end{pmatrix} \right) + o(1), \qquad (3)$$

  as $\tau \to \pm\infty$.

- $T_\pm(\tau, z, \zeta)$ are connected via the completeness relation

$$T_-(\tau, z, \zeta) = T_+(\tau, z, \zeta) T(z, \zeta).$$

Columnwise, suppressing τ and z, this reads

$$T_-^{(1)}(\zeta) = \alpha(\zeta)T_+^{(1)}(\zeta) + \eta(\zeta)T_+^{(2)}(\zeta) + \nu(\zeta)T_+^{(3)}(\zeta), \quad (4)$$
$$T_-^{(2)}(\zeta) = \beta(\zeta)T_+^{(1)}(\zeta) + \theta(\zeta)T_+^{(2)}(\zeta) + \rho(\zeta)T_+^{(3)}(\zeta), \quad (5)$$
$$T_-^{(3)}(\zeta) = \gamma(\zeta)T_+^{(1)}(\zeta) + \mu(\zeta)T_+^{(2)}(\zeta) + \chi(\zeta)T_+^{(3)}(\zeta), \quad (6)$$

where $\alpha(\zeta) = (\theta\chi - \rho\mu)(-\zeta)$, $\beta(\zeta) = (\nu\mu - \eta\chi)(-\zeta)$, $\gamma(\zeta) = (\eta\rho - \nu\theta)(-\zeta)$. Since $T^{-1}(\zeta) = T^t(-\zeta)$, the inverse relations are

$$T_+^{(1)}(\zeta) = \alpha(-\zeta)T_-^{(1)}(\zeta) + \beta(-\zeta)T_-^{(2)}(\zeta) + \gamma(-\zeta)T_-^{(3)}(\zeta), \quad (7)$$
$$T_+^{(2)}(\zeta) = \eta(-\zeta)T_-^{(1)}(\zeta) + \theta(-\zeta)T_-^{(2)}(\zeta) + \mu(-\zeta)T_-^{(3)}(\zeta), \quad (8)$$
$$T_+^{(3)}(\zeta) = \nu(-\zeta)T_-^{(1)}(\zeta) + \rho(-\zeta)T_-^{(2)}(\zeta) + \chi(-\zeta)T_-^{(3)}(\zeta). \quad (9)$$

In view of the facts above, we proceed to find z-evolution equations for the transition coefficients α, β, γ, η, θ, μ, ν, ρ, χ by employing the method of variation of parameters.

Suppose we define the column vector

$$\mathbf{y} = \mathbf{f}_z = (f_{1z}, f_{2z}, f_{3z})^t \in \mathbf{C}^3.$$

Then equation (2) implies

$$\mathbf{y}_\tau - U\mathbf{y} - U_z\mathbf{f} = \vec{0}, \qquad (10)$$

where $U_z\mathbf{f} = \Omega_0 C\,(-A_{xz}f_2 - A_{yz}f_3,\ A_{xz}f_1,\ A_{yz}f_1)^t$. We already know that a suitable solution of the homogeneous equation (2), with the property $\mathbf{y} \to \vec{0}$ as $\tau \to -\infty$, is given by

$$\mathbf{f} = \kappa_1 T_-^{(1)}(\zeta) + \kappa_2 T_-^{(2)}(\zeta) + \kappa_3 T_-^{(3)}(\zeta) = \left(T_-^{(1)}, T_-^{(2)}, T_-^{(3)}\right)(\zeta)\vec{\kappa},$$

for an arbitrary constant vector $\vec{\kappa} = (\kappa_1, \kappa_2, \kappa_3)^t$. "Variation of parameters" now tells us to look for a particular solution of the inhomogeneous equation (10) having the form

$$\mathbf{y}(\mathbf{f}) = \left(T_-^{(1)}, T_-^{(2)}, T_-^{(3)}\right)(\zeta)\vec{\kappa}(\tau). \qquad (11)$$

Substituting into (10) gives

$$\begin{aligned}\vec{0} &= \left(T_-^{(1)}, T_-^{(2)}, T_-^{(3)}\right)_\tau(\zeta)\vec{\kappa}(\tau) + \left(T_-^{(1)}, T_-^{(2)}, T_-^{(3)}\right)(\zeta)\vec{\kappa}_\tau(\tau) - \\ &\quad U\left(T_-^{(1)}, T_-^{(2)}, T_-^{(3)}\right)(\zeta)\vec{\kappa}(\tau) - U_z\mathbf{f}.\end{aligned}$$

Obviously the first and third terms cancel, leaving the equation

$$\vec{\kappa}_\tau = \Omega_0 C \begin{pmatrix} T_-^{11} & T_-^{21} & T_-^{31} \\ T_-^{12} & T_-^{22} & T_-^{32} \\ T_-^{13} & T_-^{23} & T_-^{33} \end{pmatrix}(-\zeta) \begin{pmatrix} -A_{xz}f_2 - A_{yz}f_3 \\ A_{xz}f_1 \\ A_{yz}f_1 \end{pmatrix}.$$

Hence

$$\begin{aligned}\kappa_1(\tau) &= \Omega_0 C \int_{-\infty}^{\tau} \left[(-A_{xz}f_2 - A_{yz}f_3)\,T_-^{11}(-\zeta) + \right. \\ &\qquad \left. A_{xz}f_1T_-^{21}(-\zeta) + A_{yz}f_1T_-^{31}(-\zeta)\right] ds, \\ \kappa_2(\tau) &= \Omega_0 C \int_{-\infty}^{\tau} \left[(-A_{xz}f_2 - A_{yz}f_3)\,T_-^{12}(-\zeta) + \right. \\ &\qquad \left. A_{xz}f_1T_-^{22}(-\zeta) + A_{yz}f_1T_-^{32}(-\zeta)\right] ds, \\ \kappa_3(\tau) &= \Omega_0 C \int_{-\infty}^{\tau} \left[(-A_{xz}f_2 - A_{yz}f_3)\,T_-^{13}(-\zeta) + \right. \\ &\qquad \left. A_{xz}f_1T_-^{23}(-\zeta) + A_{yz}f_1T_-^{33}(-\zeta)\right] ds.\end{aligned}$$

If we choose $\mathbf{f} = T_-^{(1)}(\zeta)$, then by equation (11) we obtain

$$\begin{aligned}\frac{1}{\Omega_0 C}\frac{\partial}{\partial z}T_-^{(1)}(\zeta) &= T_-^{(1)}(\zeta)\int_{-\infty}^{\tau}\Big[\left(-A_{xz}T_-^{21}(\zeta)-A_{yz}T_-^{31}(\zeta)\right)T_-^{11}(-\zeta)+\\ &\quad A_{xz}T_-^{11}(\zeta)T_-^{21}(-\zeta)+A_{yz}T_-^{11}(\zeta)T_-^{31}(-\zeta)\Big]\,d\tau+\\ &\quad T_-^{(2)}(\zeta)\int_{-\infty}^{\tau}\Big[\left(-A_{xz}T_-^{21}(\zeta)-A_{yz}T_-^{31}(\zeta)\right)T_-^{12}(-\zeta)+\\ &\quad A_{xz}T_-^{11}(\zeta)T_-^{22}(-\zeta)+A_{yz}T_-^{11}(\zeta)T_-^{32}(-\zeta)\Big]\,d\tau+\\ &\quad T_-^{(3)}(\zeta)\int_{-\infty}^{\tau}\Big[\left(-A_{xz}T_-^{21}(\zeta)-A_{yz}T_-^{31}(\zeta)\right)T_-^{13}(-\zeta)+\\ &\quad A_{xz}T_-^{11}(\zeta)T_-^{23}(-\zeta)+A_{yz}T_-^{11}(\zeta)T_-^{33}(-\zeta)\Big]\,d\tau. \quad (12)\end{aligned}$$

Alternatively, from equations $(4-9)$ we have as $\tau \to \infty$

$$\begin{aligned}\frac{\partial}{\partial z}T_-^{(1)}(\zeta) &= \left[\alpha_z(\zeta)\alpha(-\zeta)+\eta_z(\zeta)\eta(-\zeta)+\nu_z(\zeta)\nu(-\zeta)\right]T_-^{(1)}(\zeta)+\\ &\quad \left[\alpha_z(\zeta)\beta(-\zeta)+\eta_z(\zeta)\theta(-\zeta)+\nu_z(\zeta)\rho(-\zeta)\right]T_-^{(2)}(\zeta)+\\ &\quad \left[\alpha_z(\zeta)\gamma(-\zeta)+\eta_z(\zeta)\mu(-\zeta)+\nu_z(\zeta)\chi(-\zeta)\right]T_-^{(3)}(\zeta), \quad (13)\end{aligned}$$

which follows because the matrix T_+ is independent of z for large τ (cf. equation (3)). Letting $\tau \to \infty$ and equating our two expressions for $\frac{\partial}{\partial z}T_-^{(1)}(\zeta)$ yields the required evolution equations. For instance, comparing coefficients of $T_-^{(1)}(\zeta)$ in (12) and (13), namely

$$\begin{aligned}&\Omega_0 C\int_{-\infty}^{\infty}\Big\{\left(-A_{xz}T_-^{21}(\zeta)-A_{yz}T_-^{31}(\zeta)\right)\times\\ &\left[\alpha(-\zeta)T_+^{11}(-\zeta)+\eta(-\zeta)T_+^{12}(-\zeta)+\nu(-\zeta)T_+^{13}(-\zeta)\right]+\\ &A_{xz}T_-^{11}(\zeta)\left[\alpha(-\zeta)T_+^{21}(-\zeta)+\eta(-\zeta)T_+^{22}(-\zeta)+\nu(-\zeta)T_+^{23}(-\zeta)\right]+\\ &A_{yz}T_-^{11}(\zeta)\left[\alpha(-\zeta)T_+^{31}(-\zeta)+\eta(-\zeta)T_+^{32}(-\zeta)+\nu(-\zeta)T_+^{33}(-\zeta)\right]\Big\}\,d\tau,\end{aligned}$$

and $\alpha_z(\zeta)\alpha(-\zeta)+\eta_z(\zeta)\eta(-\zeta)+\nu_z(\zeta)\nu(-\zeta)$, we see that

$$\alpha_z(\zeta) = \Omega_0 C\int_{-\infty}^{\infty}\Big\{A_{xz}\left[T_-^{11}(\zeta)T_+^{21}(-\zeta)-T_-^{21}(\zeta)T_+^{11}(-\zeta)\right]+$$

$$A_{yz}\left[T_-^{11}(\zeta)T_+^{31}(-\zeta) - T_-^{31}(\zeta)T_+^{11}(-\zeta)\right]\Big\} d\tau,$$

$$\begin{aligned}\eta_z(\zeta) &= \Omega_0 C\int_{-\infty}^{\infty}\Big\{A_{xz}\left[T_-^{11}(\zeta)T_+^{22}(-\zeta) - T_-^{21}(\zeta)T_+^{12}(-\zeta)\right] + \\ &\quad A_{yz}\left[T_-^{11}(\zeta)T_+^{32}(-\zeta) - T_-^{31}(\zeta)T_+^{12}(-\zeta)\right]\Big\} d\tau,\end{aligned}$$

$$\begin{aligned}\nu_z(\zeta) &= \Omega_0 C\int_{-\infty}^{\infty}\Big\{A_{xz}\left[T_-^{11}(\zeta)T_+^{23}(-\zeta) - T_-^{21}(\zeta)T_+^{13}(-\zeta)\right] + \\ &\quad A_{yz}\left[T_-^{11}(\zeta)T_+^{33}(-\zeta) - T_-^{31}(\zeta)T_+^{13}(-\zeta)\right]\Big\} d\tau.\end{aligned}$$

Corresponding equations for the remaining six transition coefficients are calculated in the same way after choosing either $\mathbf{f} = T_-^{(2)}(\zeta)$ or $\mathbf{f} = T_-^{(3)}(\zeta)$. In fact, if $T^{ij}(\zeta)$, $i, j \in \{1, 2, 3\}$ denotes the (i, j)-th element of the reduced monodromy matrix, then we find

$$\begin{aligned}\frac{\partial}{\partial z}T^{ij}(\zeta) &= \Omega_0 C\int_{-\infty}^{\infty}\Big\{A_{xz}\left[T_+^{2i}(-\zeta)T_-^{1j}(\zeta) - T_+^{1i}(-\zeta)T_-^{2j}(\zeta)\right] + \\ &\quad A_{yz}\left[T_+^{3i}(-\zeta)T_-^{1j}(\zeta) - T_+^{1i}(-\zeta)T_-^{3j}(\zeta)\right]\Big\} d\tau. \qquad (14)\end{aligned}$$

Suppose we now introduce the eight functions

$$\begin{aligned}a(i,j) &:= T_+^{1i}(-\zeta)T_-^{2j}(\zeta),\\ b(i,j) &:= T_+^{2i}(-\zeta)T_-^{1j}(\zeta),\\ c(i,j) &:= T_+^{1i}(-\zeta)T_-^{3j}(\zeta),\\ d(i,j) &:= T_+^{3i}(-\zeta)T_-^{1j}(\zeta),\\ k(i,j) &:= T_+^{1i}(-\zeta)T_-^{1j}(\zeta) - T_+^{2i}(-\zeta)T_-^{2j}(\zeta),\\ l(i,j) &:= T_+^{1i}(-\zeta)T_-^{1j}(\zeta) - T_+^{3i}(-\zeta)T_-^{3j}(\zeta),\\ m(i,j) &:= T_+^{2i}(-\zeta)T_-^{3j}(\zeta),\\ n(i,j) &:= T_+^{3i}(-\zeta)T_-^{2j}(\zeta).\end{aligned}$$

Then by virtue of equation (2) there is a closed system of eight first order differential equations for $a(i,j), ..., n(i,j)$ of the form

$$\begin{aligned}
a_\tau &= -\iota\Omega_0\zeta a + \Omega_0 C A_x k - \Omega_0 C A_y n, \\
b_\tau &= \iota\Omega_0\zeta b + \Omega_0 C A_x k - \Omega_0 C A_y m, \\
c_\tau &= -\iota\Omega_0\zeta c + \Omega_0 C A_y l - \Omega_0 C A_x m, \\
d_\tau &= \iota\Omega_0\zeta d + \Omega_0 C A_y l - \Omega_0 C A_x n, \\
k_\tau &= -2\Omega_0 C A_x(a+b) - \Omega_0 C A_y(c+d), \\
l_\tau &= -\Omega_0 C A_x(a+b) - 2\Omega_0 C A_y(c+d), \\
m_\tau &= \Omega_0 C\left(A_x c + A_y b\right), \\
n_\tau &= \Omega_0 C\left(A_x d + A_y a\right). \qquad (15-22)
\end{aligned}$$

Note also that equation (14) becomes

$$\begin{aligned}
\frac{\partial}{\partial z}T^{ij}(\zeta) &= \Omega_0 C\int_{-\infty}^{\infty}(-A_{xz}, A_{xz}, -A_{yz}, A_{yz})\begin{pmatrix} a(i,j) \\ b(i,j) \\ c(i,j) \\ d(i,j) \end{pmatrix} d\tau \\
&= \Omega_0 C\int_{-\infty}^{\infty}\mathbf{A}_z^t\vec{\Phi}(i,j)d\tau, \qquad (23)
\end{aligned}$$

where $\mathbf{A} = (-A_x, A_x, -A_y, A_y)^t$ and $\vec{\Phi}(i,j) := (a,b,c,d)^t(i,j)$.

Furthermore, suppose we set $i = 1 = j$. Then in particular we have

$$\begin{aligned}
k(1,1) &= T_+^{11}(-\zeta)T_-^{11}(\zeta) - T_+^{21}(-\zeta)T_-^{21}(\zeta) \\
&= \left[\alpha(\zeta)T_-^{11}(-\zeta) + \beta(\zeta)T_-^{12}(-\zeta) + \gamma(\zeta)T_-^{13}(-\zeta)\right]T_-^{11}(\zeta) - \\
&\quad \left[\alpha(\zeta)T_-^{21}(-\zeta) + \beta(\zeta)T_-^{22}(-\zeta) + \gamma(\zeta)T_-^{23}(-\zeta)\right]T_-^{21}(\zeta) \\
&= \alpha(\zeta) + o(1), \ \text{as}\ \tau \to -\infty.
\end{aligned}$$

Hence

$$\lim_{\tau\to-\infty}\{k(1,1)\} := k_-(1,1) = \alpha(\zeta).$$

Similarly $l_-(1,1) = \alpha(\zeta)$, and $m_-(1,1) = 0 = n_-(1,1)$. Since we shall be assuming $i = 1 = j$ throughout the rest of the chapter, we hereafter neglect the extra (i,j)-argument of the eight functions $a, ..., n$ to ease notation. Thus, using equations (15), (19) and (22) gives

$$\begin{aligned} \iota\Omega_0\zeta a &= -a_\tau + \Omega_0 C A_x\left(\int_{-\infty}^{\tau} k_\tau(s)ds + k_-\right) - \Omega_0 C A_y\left(\int_{-\infty}^{\tau} n_\tau(s)ds + n_-\right) \\ &= -a_\tau + \Omega_0 C A_x\left\{-\Omega_0 C\int_{-\infty}^{\tau}\left[2A_x(a+b) + A_y(c+d)\right]ds + \alpha(\zeta)\right\} - \\ &\quad \Omega_0^2 C^2 A_y\int_{-\infty}^{\tau}(A_x d + A_y a)\,ds, \end{aligned}$$

and so

$$\iota\Omega_0\zeta a - \Omega_0 C\alpha(\zeta)A_x =$$
$$\begin{pmatrix} -\partial_\tau - \Omega_0^2C^2\left[2A_x\mathrm{I}(A_x,\cdot) + A_y\mathrm{I}(A_y,\cdot)\right] \\ -2\Omega_0^2C^2A_x\mathrm{I}(A_x,\cdot) \\ -\Omega_0^2C^2A_x\mathrm{I}(A_y,\cdot) \\ -\Omega_0^2C^2\left[A_x\mathrm{I}(A_y,\cdot) + A_y\mathrm{I}(A_x,\cdot)\right] \end{pmatrix}^t \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

where I is an integral operator defined by

$$\mathrm{I}(u,\cdot)v = \mathrm{I}(u,v) = \int_{-\infty}^{\tau} u(s)v(s)ds,$$

for some representative functions u, v. Applying the same procedure to equations (16 – 18) we subsequently find

$$\zeta\vec{\Phi} - \iota C\alpha(\zeta)\mathrm{A} = \mathrm{L}\vec{\Phi} := -\frac{\iota}{\Omega_0}\times$$

$$
\begin{pmatrix}
-\partial_\tau - \Omega_0^2 C^2 \left[2A_x \mathrm{I}(A_x,\cdot) + A_y \mathrm{I}(A_y,\cdot)\right] & -2\Omega_0^2 C^2 A_x \mathrm{I}(A_x,\cdot) \\
2\Omega_0^2 C^2 A_x \mathrm{I}(A_x,\cdot) & \partial_\tau + \Omega_0^2 C^2 \left[2A_x \mathrm{I}(A_x,\cdot) + A_y \mathrm{I}(A_y,\cdot)\right] \\
-\Omega_0^2 C^2 A_y \mathrm{I}(A_x,\cdot) & -\Omega_0^2 C^2 \left[A_x \mathrm{I}(A_y,\cdot) + A_y \mathrm{I}(A_x,\cdot)\right] \\
\Omega_0^2 C^2 \left[A_x \mathrm{I}(A_y,\cdot) + A_y \mathrm{I}(A_x,\cdot)\right] & \Omega_0^2 C^2 A_y \mathrm{I}(A_x,\cdot)
\end{pmatrix.
$$

$$
\begin{matrix}
-\Omega_0^2 C^2 A_x \mathrm{I}(A_y,\cdot) & -\Omega_0^2 C^2 \left[A_x \mathrm{I}(A_y,\cdot) + A_y \mathrm{I}(A_x,\cdot)\right] \\
\Omega_0^2 C^2 \left[A_x \mathrm{I}(A_y,\cdot) + A_y \mathrm{I}(A_x,\cdot)\right] & \Omega_0^2 C^2 A_x \mathrm{I}(A_y,\cdot) \\
-\partial_\tau - \Omega_0^2 C^2 \left[A_x \mathrm{I}(A_x,\cdot) + 2A_y \mathrm{I}(A_y,\cdot)\right] & -2\Omega_0^2 C^2 A_y \mathrm{I}(A_y,\cdot) \\
2\Omega_0^2 C^2 A_y \mathrm{I}(A_y,\cdot) & \partial_\tau + \Omega_0^2 C^2 \left[A_x \mathrm{I}(A_x,\cdot) + 2A_y \mathrm{I}(A_y,\cdot)\right]
\end{pmatrix} \vec{\Phi}.
$$

It is trivial to solve this equation for $\vec{\Phi}$ asymptotically. Indeed if we look for a solution

$$\vec{\Phi} = \iota C\alpha(\zeta) \sum_{r=1}^{\infty} \vec{\Phi}_r \zeta^{-r},$$

then we quickly obtain

$$\vec{\Phi} = \sum_{r=1}^{\infty} \frac{\iota C\alpha(\zeta)}{\zeta^r} \mathrm{L}^{r-1}\mathrm{A}. \qquad (24)$$

Now according to equation (23) evaluated at $i = 1 = j$ we have

$$\frac{\partial}{\partial z}\alpha(\zeta) = \Omega_0 C \int_{-\infty}^{\infty} \mathrm{A}_z^t \vec{\Phi}\, d\tau.$$

Therefore

$$\frac{\partial}{\partial z}\alpha(\zeta) = \sum_{r=1}^{\infty} \frac{\iota\Omega_0 C^2 \alpha(\zeta)}{\zeta^{2r-1}} \int_{-\infty}^{\infty} \mathrm{A}_z^t \mathrm{L}^{2r-2}\mathrm{A}\, d\tau,$$

which follows from (24) and the symmetries contained within the matrix operator L. On the other hand, equation (12) p. 108 states that

$$
\begin{aligned}
\log\alpha(\zeta) &= \iota\Omega_0 C^2 \int_{-\infty}^{\infty} (A+B)(\zeta)\, d\tau + O\left(|\zeta|^{-\infty}\right) \\
&= \iota\Omega_0 C^2 \sum_{r=1}^{\infty} \frac{1}{\zeta^{2r-1}} \int_{-\infty}^{\infty} \left(A_{2r-1} + B_{2r-1}\right) d\tau + O\left(|\zeta|^{-\infty}\right),
\end{aligned}
$$

which implies

$$\frac{\partial}{\partial z}\alpha(\zeta) = \iota\Omega_0 C^2\alpha(\zeta)\sum_{r=1}^{\infty}\frac{1}{\zeta^{2r-1}}\int_{-\infty}^{\infty}\partial_z\left(A_{2r-1}+B_{2r-1}\right)d\tau + O\left(|\zeta|^{-\infty}\right),$$

upon differentiation with respect to z. Comparing the two expressions for $\frac{\partial}{\partial z}\alpha(\zeta)$ we at last recover equation (1):

$$\frac{d}{dz}\int_{-\infty}^{\infty}\left(A_{2r-1}+B_{2r-1}\right)d\tau = \int_{-\infty}^{\infty}\mathbf{A}_z^t\mathrm{L}^{2r-2}\mathbf{A}d\tau, \quad \mathbf{N}\ni r\geq 1.$$

Obviously the right-hand side here should only differ from zero if our original RMB-system is perturbed in some way.

In the sequel we will need the operator adjoint to L. Let us define the inner product

$$\langle\mathbf{x}|\,\mathbf{y}\rangle = \int_{-\infty}^{\infty}\mathbf{x}^t\mathbf{y}d\tau$$

for any 3×1 column vectors $\mathbf{x}, \mathbf{y}$. Then the associated adjoint operator $\hat{\mathrm{L}}$ is given by

$$\hat{\mathrm{L}} = -\frac{\iota}{\Omega_0}\times$$

$$\begin{pmatrix} \partial_\tau - \Omega_0^2C^2\left[2A_x\mathrm{I}_+(A_x,\cdot)+A_y\mathrm{I}_+(A_y,\cdot)\right] & 2\Omega_0^2C^2A_x\mathrm{I}_+(A_x,\cdot) \\ -2\Omega_0^2C^2A_x\mathrm{I}_+(A_x,\cdot) & -\partial_\tau + \Omega_0^2C^2\left[2A_x\mathrm{I}_+(A_x,\cdot)+A_y\mathrm{I}_+(A_y,\cdot)\right] \\ -\Omega_0^2C^2A_y\mathrm{I}_+(A_x,\cdot) & \Omega_0^2C^2\left[A_y\mathrm{I}_+(A_x,\cdot)+A_x\mathrm{I}_+(A_y,\cdot)\right] \\ -\Omega_0^2C^2\left[A_y\mathrm{I}_+(A_x,\cdot)+A_x\mathrm{I}_+(A_y,\cdot)\right] & \Omega_0^2C^2A_y\mathrm{I}_+(A_x,\cdot) \end{pmatrix.$$

$$\left.\begin{matrix} -\Omega_0^2C^2A_x\mathrm{I}_+(A_y,\cdot) & \Omega_0^2C^2\left[A_y\mathrm{I}_+(A_x,\cdot)+A_x\mathrm{I}_+(A_y,\cdot)\right] \\ -\Omega_0^2C^2\left[A_y\mathrm{I}_+(A_x,\cdot)+A_x\mathrm{I}_+(A_y,\cdot)\right] & \Omega_0^2C^2A_x\mathrm{I}_+(A_y,\cdot) \\ \partial_\tau - \Omega_0^2C^2\left[A_x\mathrm{I}_+(A_x,\cdot)+2A_y\mathrm{I}_+(A_y,\cdot)\right] & 2\Omega_0^2C^2A_y\mathrm{I}_+(A_y,\cdot) \\ -2\Omega_0^2C^2A_y\mathrm{I}_+(A_y,\cdot) & -\partial_\tau + \Omega_0^2C^2\left[A_x\mathrm{I}_+(A_x,\cdot)+2A_y\mathrm{I}_+(A_y,\cdot)\right] \end{matrix}\right),$$

where

$$\mathrm{I}_+(u,\cdot)v = \mathrm{I}_+(u,v) := \int_\tau^\infty u(s)v(s)ds.$$

Making use of this new notation equation (1) becomes

$$\frac{d}{dz}\int_{-\infty}^{\infty}(A_{2r-1}+B_{2r-1})\,d\tau = \left\langle \mathbf{A}_z \middle| \mathrm{L}^{2r-2}\mathbf{A}\right\rangle = \begin{cases} \left\langle \hat{\mathrm{L}}^{2r-2}\mathbf{A}_z \middle| \mathbf{A}\right\rangle, & \text{for } \mathbb{N}\ni r\geq 1, \\ \left\langle \hat{\mathrm{L}}^{2}\mathbf{A}_z \middle| \mathrm{L}^{2r-4}\mathbf{A}\right\rangle, & \text{when } r\geq 2. \end{cases} \tag{25}$$

Finally, as a consistency check for our results, we compute the right-hand side of (25) in the cases $r = 2$ and $r = 3$ using equations (18 − 27) p. 35, anticipating both answers to be zero. Applying the definitions of L and $\hat{\mathrm{L}}$ one finds

$$\mathrm{L}\mathbf{A} = \mathrm{L}\begin{pmatrix} -A_x \\ A_x \\ -A_y \\ A_y \end{pmatrix} = -\frac{\iota}{\Omega_0}\begin{pmatrix} A_{x\tau} \\ A_{x\tau} \\ A_{y\tau} \\ A_{y\tau} \end{pmatrix}, \tag{26}$$

$$\hat{\mathrm{L}}\begin{pmatrix} A_x \\ A_x \\ A_y \\ A_y \end{pmatrix} = \frac{\iota}{\Omega_0}\begin{pmatrix} -A_{x\tau} \\ A_{x\tau} \\ -A_{y\tau} \\ A_{y\tau} \end{pmatrix} = \frac{\iota}{\Omega_0}\mathbf{A}_\tau, \tag{27}$$

$$\mathrm{L}^2\mathbf{A} = -\frac{1}{\Omega_0^2}\begin{pmatrix} -A_{x\tau\tau} - 2\Omega_0^2C^2A_x\,|A|^2 \\ A_{x\tau\tau} + 2\Omega_0^2C^2A_x\,|A|^2 \\ -A_{y\tau\tau} - 2\Omega_0^2C^2A_y\,|A|^2 \\ A_{y\tau\tau} + 2\Omega_0^2C^2A_y\,|A|^2 \end{pmatrix},$$

$$\left(\text{where } |A|^2 := A_x^2 + A_y^2\right),$$

$$\hat{\mathrm{L}}\mathbf{A}_z = -\iota\left[\frac{C^2N\hbar}{2c\varepsilon_0}\begin{pmatrix} A_x \\ A_x \\ A_y \\ A_y \end{pmatrix} + \frac{CN\hbar}{2c\varepsilon_0}\begin{pmatrix} \rho_4 \\ \rho_4 \\ \rho_6 \\ \rho_6 \end{pmatrix}\right],$$

$$\hat{\mathrm{L}}^2\mathbf{A}_z = \frac{C^2N\hbar}{2c\varepsilon_0\Omega_0}\mathbf{A}_\tau + \mathbf{A}_z.$$

Observe that equations (26, 27) hold independently of whether the system is perturbed or not. Hence

$$\left\langle \mathbf{A}_z \middle| \mathbf{L}^2\mathbf{A} \right\rangle = \left\langle \hat{\mathbf{L}}^2\mathbf{A}_z \middle| \mathbf{A} \right\rangle = \left\langle \frac{C^2 N\hbar}{2c\varepsilon_0\Omega_0}\mathbf{A}_\tau \middle| \mathbf{A} \right\rangle + \left\langle \mathbf{A}_z \middle| \mathbf{A} \right\rangle = 0,$$

since

$$\left\langle \mathbf{A}_\tau \middle| \mathbf{A} \right\rangle \propto \int_{-\infty}^{\infty} \partial_\tau \left\{ |A|^2 \right\} d\tau$$

and

$$\left\langle \mathbf{A}_z \middle| \mathbf{A} \right\rangle \propto \int_{-\infty}^{\infty} \partial_z \left\{ |A|^2 \right\} d\tau \propto \int_{-\infty}^{\infty} \partial_\tau \left\{ \sqrt{3}\rho_7 + \rho_8 \right\} d\tau$$

vanish due to the boundary conditions. Moreover

$$\left\langle \hat{\mathbf{L}}^2\mathbf{A}_z \middle| \mathbf{L}^2\mathbf{A} \right\rangle = \left\langle \frac{C^2 N\hbar}{2c\varepsilon_0\Omega_0}\mathbf{A}_\tau \middle| \mathbf{L}^2\mathbf{A} \right\rangle + \left\langle \mathbf{A}_z \middle| \mathbf{L}^2\mathbf{A} \right\rangle = 0,$$

because

$$\left\langle \mathbf{A}_\tau \middle| \mathbf{L}^2\mathbf{A} \right\rangle \propto \int_{-\infty}^{\infty} \partial_\tau \left\{ |A_\tau|^2 + |A|^4 \right\} d\tau.$$

7.2 An Application of the Evolution Equation for the Invariant Functionals.

7.2.1 The Perturbed Model for an Absorbing Medium.

Our original RMB-system, given by equations (18 − 27) on p. 35, can be modified quite simply in order to take account of basic dissipative processes.

Recall from Chapter 3 that the dipole displacement and momentum operators are

$$\mathbf{q} = q_0 \left(\lambda_1, \lambda_3, 0 \right), \qquad (28)$$
$$\mathbf{p} = m q_0 \Omega_0 \left(\lambda_4, \lambda_6, 0 \right), \qquad (29)$$

respectively, and the unperturbed Hamiltonian is

$$H_0 = -\frac{1}{2}\hbar\Omega_0\left(\lambda_7 + \frac{1}{\sqrt{3}}\lambda_8\right), \qquad (30)$$

We have that $\rho_1, ..., \rho_6$ correspond to components of the dipole moment, with ρ_1 and ρ_3 further corresponding to the displacement via equation (28), and ρ_4 and ρ_6 corresponding to the momentum via (29). Clearly equation (30) says $-\frac{1}{2}\hbar\Omega_0\left(\rho_7 + \frac{1}{\sqrt{3}}\rho_8\right)$ is the expected value of each atom's internal energy.

If we now assign separate phenomenological decay lifetimes, T_1 for the atomic inversion $\rho_7 + \frac{1}{\sqrt{3}}\rho_8$, and T_2 for the dipole moment, then the perturbed RMB-system may be written

$$\begin{aligned}
\rho_{1\tau} &= -\frac{\rho_1}{T_2} + \Omega_0\left[C\left(2A_x\rho_7 - A_y\rho_2\right) + \rho_4\right], \\
\rho_{2\tau} &= -\frac{\rho_2}{T_2} + \Omega_0 C\left(A_x\rho_3 + A_y\rho_1\right), \\
\rho_{3\tau} &= -\frac{\rho_3}{T_2} + \Omega_0\left\{C\left[-A_x\rho_2 + A_y\left(\rho_7 + \sqrt{3}\rho_8\right)\right] + \rho_6\right\}, \\
\rho_{4\tau} &= -\frac{\rho_4}{T_2} + \Omega_0\left(CA_y\rho_5 - \rho_1\right), \\
\rho_{5\tau} &= -\frac{\rho_5}{T_2} + \Omega_0 C\left(A_x\rho_6 - A_y\rho_4\right), \\
\rho_{6\tau} &= -\frac{\rho_6}{T_2} - \Omega_0\left(CA_x\rho_5 + \rho_3\right), \\
\rho_{7\tau} + \frac{1}{\sqrt{3}}\rho_{8\tau} &= -\frac{1}{T_1}\left[\rho_7 - \rho_7^0 + \frac{1}{\sqrt{3}}\left(\rho_8 - \rho_8^0\right)\right] - \\
&\quad 2\Omega_0 C\left(A_x\rho_1 + A_y\rho_3\right), \\
\rho_{7\tau} - \sqrt{3}\rho_{8\tau} &= -\frac{1}{T_2}\left[\rho_7 - \rho_7^0 - \sqrt{3}\left(\rho_8 - \rho_8^0\right)\right] - \\
&\quad 2\Omega_0 C\left(A_x\rho_1 - A_y\rho_3\right). \qquad (31-38)
\end{aligned}$$

$$A_{xz} = -\frac{N\hbar C}{2c\varepsilon_0}\rho_1, \; A_{yz} = -\frac{N\hbar C}{2c\varepsilon_0}\rho_3, \qquad (39, 40)$$

where $\rho_7^0 = 1$ and $\rho_8^0 = 1/\sqrt{3}$. Equivalently, equations (37, 38) can be set down as

$$
\begin{aligned}
4\rho_{7_T} &= -\left(\frac{3}{T_1}+\frac{1}{T_2}\right)\left(\rho_7-\rho_7^0\right)-\sqrt{3}\left(\frac{1}{T_1}-\frac{1}{T_2}\right)\left(\rho_8-\rho_8^0\right)- \\
&\quad 4\Omega_0 C\left(2A_x\rho_1+A_y\rho_3\right), \\
\frac{4}{\sqrt{3}}\rho_{8_T} &= -\left(\frac{1}{T_1}-\frac{1}{T_2}\right)\left(\rho_7-\rho_7^0\right)-\frac{1}{\sqrt{3}}\left(\frac{1}{T_1}+\frac{3}{T_2}\right)\left(\rho_8-\rho_8^0\right)- \\
&\quad 4\Omega_0 C A_y\rho_3.
\end{aligned}
$$

Observe that rotational symmetry requirements constrain the nature of the decay terms in $(31-38)$. In fact, applying the rotation $R=e^{\iota\phi\lambda_6}$ to the density matrix

$$
\rho=\sum_{j=1}^{8}\rho_j\lambda_j=
$$

$$
\begin{pmatrix}
\rho_7+\rho_8/\sqrt{3} & \rho_1-\iota\rho_4 & \rho_3-\iota\rho_6 \\
\rho_1+\iota\rho_4 & \frac{-\left(\rho_7+\rho_8/\sqrt{3}\right)-\left(\rho_7-\sqrt{3}\rho_8\right)}{2} & \rho_2-\iota\rho_5 \\
\rho_3+\iota\rho_6 & \rho_2+\iota\rho_5 & \frac{-\left(\rho_7+\rho_8/\sqrt{3}\right)+\left(\rho_7-\sqrt{3}\rho_8\right)}{2}
\end{pmatrix}
$$

according to the transformation $\rho\mapsto R\rho R^{-1}$, we find that $\rho_7+\rho_8/\sqrt{3}$ and ρ_5 are rotationally invariant, whereas the vectors (ρ_1,ρ_3) and (ρ_1,ρ_4) are rotated through the angle ϕ ,

$$
\begin{aligned}
(\rho_1,\rho_3) &\mapsto (\rho_1\cos\phi+\rho_3\sin\phi,-\rho_1\sin\phi+\rho_3\cos\phi), \\
(\rho_4,\rho_6) &\mapsto (\rho_4\cos\phi+\rho_6\sin\phi,-\rho_4\sin\phi+\rho_6\cos\phi),
\end{aligned}
$$

and the vector $\left(\rho_2,\frac{1}{2}\left(\rho_7-\sqrt{3}\rho_8\right)\right)$ is rotated through 2ϕ,

$$
\begin{aligned}
\left(\rho_2,\frac{1}{2}\left(\rho_7-\sqrt{3}\rho_8\right)\right) \mapsto & \left(\rho_2\cos 2\phi+\frac{1}{2}\left(\rho_7-\sqrt{3}\rho_8\right)\sin 2\phi,\right. \\
& \left.-\rho_2\sin 2\phi+\frac{1}{2}\left(\rho_7-\sqrt{3}\rho_8\right)\cos 2\phi\right).
\end{aligned}
$$

Therefore, if we follow standard atomic physics theory by taking each of the off-diagonal components $\rho_1, ..., \rho_6$ of ρ to have a rapid dipole dephasing lifetime T_2, then for consistency we must also have $\rho_7 - \sqrt{3}\rho_8$ decaying at the same rate T_2^{-1}.

7.2.2 First Order Dissipative Effects on 1-Solitons of the RMB-System.

Consider equation (25) and set $r = 2$:

$$\frac{d}{dz}\int_{-\infty}^{\infty}(A_3 + B_3)\,d\tau = \left\langle \hat{\mathrm{L}}^2\mathbf{A}_z \middle| \mathbf{A}\right\rangle.$$

This time we shall evaluate the inner product on the right-hand side by means of the perturbed RMB-equations $(31 - 40)$. The calculations are generally routine albeit lengthy, so we have opted to summarise the individual steps rather than provide full details. Notice that we will be freely using the assumed boundary conditions and ignoring any terms which are $O\left((\Omega_0 T_2)^{-n}\right)$ or $O\left((\Omega_0 T_1)^{-n}\right)$, where $n \geq 2$.

To begin with we obtain

$$\begin{aligned}
\hat{\mathrm{L}}\mathbf{A}_z &= \hat{\mathrm{L}}\begin{pmatrix} -A_{xz} \\ A_{xz} \\ -A_{yz} \\ A_{yz} \end{pmatrix} \\
&= -\frac{\iota C^2 N\hbar}{2c\varepsilon_0}\left\{\frac{1}{C}\begin{pmatrix} \rho_4 \\ \rho_4 \\ \rho_6 \\ \rho_6 \end{pmatrix} + 2\begin{pmatrix} A_x \\ A_x \\ A_y \\ A_y \end{pmatrix} - \frac{1}{C\Omega_0 T_2}\begin{pmatrix} \rho_1 \\ \rho_1 \\ \rho_3 \\ \rho_3 \end{pmatrix} + \right. \\
&\quad \frac{1}{2T_2}\begin{pmatrix} -2A_y\int_\tau^\infty \rho_2 ds + A_x\int_\tau^\infty \left[\rho_7 - \rho_7^0 - \sqrt{3}\left(\rho_8 - \rho_8^0\right)\right] ds \\ -2A_y\int_\tau^\infty \rho_2 ds + A_x\int_\tau^\infty \left[\rho_7 - \rho_7^0 - \sqrt{3}\left(\rho_8 - \rho_8^0\right)\right] ds \\ -2A_x\int_\tau^\infty \rho_2 ds - A_y\int_\tau^\infty \left[\rho_7 - \rho_7^0 - \sqrt{3}\left(\rho_8 - \rho_8^0\right)\right] ds \\ -2A_x\int_\tau^\infty \rho_2 ds - A_y\int_\tau^\infty \left[\rho_7 - \rho_7^0 - \sqrt{3}\left(\rho_8 - \rho_8^0\right)\right] ds \end{pmatrix} +
\end{aligned}$$

$$\frac{3}{2T_1}\begin{pmatrix} A_x\int_\tau^\infty \left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]ds \\ A_x\int_\tau^\infty \left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]ds \\ A_y\int_\tau^\infty \left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]ds \\ A_y\int_\tau^\infty \left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]ds \end{pmatrix}\Bigg\}.$$

In addition, the following formulae are easy to verify for arbitrary functions f and g:

$$\begin{aligned} \hat{\mathrm{L}}\begin{pmatrix} f \\ f \\ g \\ g \end{pmatrix} &= -\frac{\iota}{\Omega_0}\begin{pmatrix} f_\tau+\Omega_0^2C^2A_y\int_\tau^\infty\left(A_xg-A_yf\right)ds \\ -f_\tau-\Omega_0^2C^2A_y\int_\tau^\infty\left(A_xg-A_yf\right)ds \\ g_\tau-\Omega_0^2C^2A_x\int_\tau^\infty\left(A_xg-A_yf\right)ds \\ -g_\tau-\Omega_0^2C^2A_x\int_\tau^\infty\left(A_xg-A_yf\right)ds \end{pmatrix}, \\ \mathrm{A}^t\hat{\mathrm{L}}\begin{pmatrix} f \\ f \\ g \\ g \end{pmatrix} &= -\frac{\iota}{\Omega_0}\mathrm{A}^t\begin{pmatrix} f_\tau \\ -f_\tau \\ g_\tau \\ -g_\tau \end{pmatrix}. \qquad (41) \end{aligned}$$

Applying relation (41) therefore gives

$$\begin{aligned} \mathrm{A}^t\hat{\mathrm{L}}^2\mathrm{A}_z &= -\frac{C^2N\hbar}{2c\varepsilon_0\Omega_0}\left\{\frac{\mathrm{A}^t}{C}\begin{pmatrix} \rho_{4\tau} \\ -\rho_{4\tau} \\ \rho_{6\tau} \\ -\rho_{6\tau} \end{pmatrix}+2\mathrm{A}^t\begin{pmatrix} A_{x\tau} \\ -A_{x\tau} \\ A_{y\tau} \\ -A_{y\tau} \end{pmatrix}-\frac{\mathrm{A}^t}{C\Omega_0T_2}\begin{pmatrix} \rho_{1\tau} \\ -\rho_{1\tau} \\ \rho_{3\tau} \\ -\rho_{3\tau} \end{pmatrix}+\right. \\ & \frac{\mathrm{A}^t}{2T_2}\partial_\tau\begin{pmatrix} -2A_y\int_\tau^\infty\rho_2ds+A_x\int_\tau^\infty\left[\rho_7-\rho_7^0-\sqrt{3}\left(\rho_8-\rho_8^0\right)\right]ds \\ 2A_y\int_\tau^\infty\rho_2ds-A_x\int_\tau^\infty\left[\rho_7-\rho_7^0-\sqrt{3}\left(\rho_8-\rho_8^0\right)\right]ds \\ -2A_x\int_\tau^\infty\rho_2ds-A_y\int_\tau^\infty\left[\rho_7-\rho_7^0-\sqrt{3}\left(\rho_8-\rho_8^0\right)\right]ds \\ 2A_x\int_\tau^\infty\rho_2ds+A_y\int_\tau^\infty\left[\rho_7-\rho_7^0-\sqrt{3}\left(\rho_8-\rho_8^0\right)\right]ds \end{pmatrix}+ \end{aligned}$$

$$\frac{3\mathbf{A}^t}{2T_1}\partial_\tau\left(\begin{array}{c} A_x\int_\tau^\infty\left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]ds \\ -A_x\int_\tau^\infty\left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]ds \\ A_y\int_\tau^\infty\left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]ds \\ -A_y\int_\tau^\infty\left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]ds \end{array}\right)\Bigg\}. \tag{42}$$

We now integrate with respect to τ over the interval $(-\infty,\infty)$ to gain an expression for $\left\langle \widehat{\mathbf{L}}^2\mathbf{A}_z \middle| \mathbf{A}\right\rangle$. Let us deal in turn with each of the five terms on the right-hand side of (42):

1.

$$\begin{aligned}\int_{-\infty}^{\infty}\mathbf{A}^t\left(\begin{array}{c}\rho_{4\tau}\\ -\rho_{4\tau}\\ \rho_{6\tau}\\ -\rho_{6\tau}\end{array}\right)d\tau &= -2\int_{-\infty}^{\infty}\left(A_x\rho_{4\tau}+A_y\rho_{6\tau}\right)d\tau \\ &= -\frac{1}{CT_1}\int_{-\infty}^{\infty}\left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]d\tau + \\ &\quad \frac{2}{T_2}\int_{-\infty}^{\infty}\left(A_x\rho_4+A_y\rho_6\right)d\tau. \end{aligned} \tag{43}$$

2.

$$\langle \mathbf{A} | \mathbf{A}_\tau\rangle = 0.$$

3.

$$\begin{aligned}\frac{1}{\Omega_0^2T_2}\int_{-\infty}^{\infty}\mathbf{A}^t\left(\begin{array}{c}\rho_{1\tau}\\ -\rho_{1\tau}\\ \rho_{3\tau}\\ -\rho_{3\tau}\end{array}\right)d\tau &= -\frac{2}{\Omega_0T_2}\int_{-\infty}^{\infty}\left(A_x\rho_4+A_y\rho_6\right)d\tau - \\ &\quad \frac{2C}{\Omega_0T_2}\int_{-\infty}^{\infty}\Big[2A_x^2\rho_7-2A_xA_y\rho_2+\end{aligned}$$

$$A_y^2\left(\rho_7+\sqrt{3}\rho_8\right)\Big]\,d\tau+O\left(\frac{1}{\Omega_0^2T_2^2}\right). \quad (44)$$

4.

$$\int_{-\infty}^{\infty}\left\{\mathbf{A}^t\partial_\tau\begin{pmatrix}-2A_y\int_\tau^\infty\rho_2ds+A_x\int_\tau^\infty\left[\rho_7-\rho_7^0-\sqrt{3}\left(\rho_8-\rho_8^0\right)\right]ds\\ 2A_y\int_\tau^\infty\rho_2ds-A_x\int_\tau^\infty\left[\rho_7-\rho_7^0-\sqrt{3}\left(\rho_8-\rho_8^0\right)\right]ds\\ -2A_x\int_\tau^\infty\rho_2ds-A_y\int_\tau^\infty\left[\rho_7-\rho_7^0-\sqrt{3}\left(\rho_8-\rho_8^0\right)\right]ds\\ 2A_x\int_\tau^\infty\rho_2ds+A_y\int_\tau^\infty\left[\rho_7-\rho_7^0-\sqrt{3}\left(\rho_8-\rho_8^0\right)\right]ds\end{pmatrix}\right\}d\tau=$$

$$\int_{-\infty}^{\infty}\left\{-4A_xA_y\rho_2+\left(A_x^2-A_y^2\right)\left[\rho_7-\rho_7^0-\sqrt{3}\left(\rho_8-\rho_8^0\right)\right]\right\}d\tau, \quad (45)$$

where we have used the identities

$$\left(A_xA_y\right)_\tau\int_\tau^\infty\rho_2ds=A_xA_y\rho_2+\partial_\tau\left\{A_xA_y\int_\tau^\infty\rho_2ds\right\},$$

and

$$\begin{aligned}\left(A_x^2\right)_\tau\int_\tau^\infty\left[\rho_7-\rho_7^0-\sqrt{3}\left(\rho_8-\rho_8^0\right)\right]ds &= A_x^2\left[\rho_7-\rho_7^0-\sqrt{3}\left(\rho_8-\rho_8^0\right)\right]+\\ &\quad\partial_\tau\left\{A_x^2\int_\tau^\infty\left[\rho_7-\rho_7^0-\right.\right.\\ &\quad\left.\left.\sqrt{3}\left(\rho_8-\rho_8^0\right)\right]ds\right\}.\end{aligned}$$

5.

$$\int_{-\infty}^{\infty}\left\{\mathbf{A}^t\partial_\tau\begin{pmatrix}A_x\int_\tau^\infty\left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]ds\\ -A_x\int_\tau^\infty\left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]ds\\ A_y\int_\tau^\infty\left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]ds\\ -A_y\int_\tau^\infty\left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]ds\end{pmatrix}\right\}d\tau=$$

$$\int_{-\infty}^{\infty}\left\{\left(A_x^2+A_y^2\right)\left[\rho_7-\rho_7^0+\frac{1}{\sqrt{3}}\left(\rho_8-\rho_8^0\right)\right]\right\}d\tau. \quad (46)$$

By virtue of equations (13) p. 109 and (43 – 46) we find

$$\begin{aligned}
\left\langle \hat{L}^2 A_z \middle| A \right\rangle &= -\frac{d}{dz}\int_{-\infty}^{\infty}\left[\frac{1}{\Omega_0^2}\left(A_x A_{x\tau\tau} + A_y A_{y\tau\tau}\right) + C^2 |A|^4\right] d\tau \\
&= -\frac{C^2 N\hbar}{2c\varepsilon_0\Omega_0}\left\{\frac{4}{CT_2}\int_{-\infty}^{\infty}\left(A_x\rho_4 + A_y\rho_6\right) d\tau - \right. \\
&\quad \frac{1}{C^2 T_1}\int_{-\infty}^{\infty}\left[\rho_7 - \rho_7^0 + \frac{1}{\sqrt{3}}\left(\rho_8 - \rho_8^0\right)\right] d\tau + \\
&\quad \frac{2}{T_2}\int_{-\infty}^{\infty}\left[2A_x^2\rho_7 - 2A_x A_y \rho_2 + A_y^2\left(\rho_7 + \sqrt{3}\rho_8\right)\right] d\tau + \\
&\quad \frac{1}{2T_2}\int_{-\infty}^{\infty}\left(A_x^2 - A_y^2\right)\left[\rho_7 - \rho_7^0 - \sqrt{3}\left(\rho_8 - \rho_8^0\right)\right] d\tau + \\
&\quad \frac{3}{2T_1}\int_{-\infty}^{\infty}\left(A_x^2 + A_y^2\right)\left[\rho_7 - \rho_7^0 + \frac{1}{\sqrt{3}}\left(\rho_8 - \rho_8^0\right)\right] d\tau - \\
&\quad \left.\frac{2}{T_2}\int_{-\infty}^{\infty} A_x A_y \rho_2 d\tau\right\} + O\left(\frac{1}{\Omega_0^2 T_2^2}\right). \qquad (47)
\end{aligned}$$

Suppose next that we replace the integrands A_x, A_y, ρ_2, ρ_4, ρ_6, ρ_7, ρ_8 appearing in the above equation, by their basic 1-solitonic forms which were derived in Chapter 4 (cf. (4) p. 40, and (8 – 15) p. 42). We choose the Greek letter ξ instead of η as the soliton parameter, thus avoiding confusion with the transition coefficient $\eta(z, \zeta)$. In other words, let

$$A_x(\tau, z) = \frac{\xi a}{C}\operatorname{sech}\Omega_0\xi\left(\tau - \tau_0(z)\right), \; A_y(\tau, z) = \frac{\xi b}{C}\operatorname{sech}\Omega_0\xi\left(\tau - \tau_0(z)\right),$$

and so forth, where $a^2 + b^2 = 1$,

$$\tau_0(z) = \int_0^z \frac{2K}{\Omega_0\left[\xi^2(u) + 1\right]} du,$$

and

$$K = \frac{Ne^2 q_0^2}{2\hbar c\varepsilon_0} = \frac{N\hbar C^2}{2c\varepsilon_0}.$$

Then integration reduces equation (47) eventually to

$$\xi_z = -\frac{2K}{\xi\left(\xi^2+1\right)}\left(\frac{\xi^2-1}{\Omega_0 T_1}+\frac{\xi^2+2}{\Omega_0 T_2}\right)+O\left(\frac{1}{\Omega_0^2 T_2^2}\right), \qquad (48)$$

since $\xi = \xi(z)$ in the presence of our dissipative perturbation. (The integrals

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{sech}^4\left[\Omega_0\xi\left(\tau-\tau_0\right)\right] d\tau &= \frac{4}{3\Omega_0\xi},\\ \int_{-\infty}^{\infty} \operatorname{sech}^2\left[\Omega_0\xi\left(\tau-\tau_0\right)\right] d\tau &= \frac{2}{\Omega_0\xi},\end{aligned}$$

are helpful when checking (48)). "Separation of variables" can hence be employed to solve this differential equation for ξ to first order. If we define

$$\varepsilon_1 = \frac{1}{\Omega_0 T_1},\ \varepsilon_2 = \frac{1}{\Omega_0 T_2},\ \text{and } \varepsilon_2 - \varepsilon_1 = \Gamma,$$

then from (48) we have the approximate equation

$$\xi_z = \frac{R\xi^2+S}{\xi\left(\xi^2+1\right)},$$

where

$$R := -2K\left(2\varepsilon_2-\Gamma\right) \text{ and } S := -2K\left(\varepsilon_2+\Gamma\right).$$

Equivalently, separating variables,

$$\left[\xi + \frac{(R-S)\,\xi}{R\xi^2+S}\right] d\xi = R dz,$$

which integrates to give

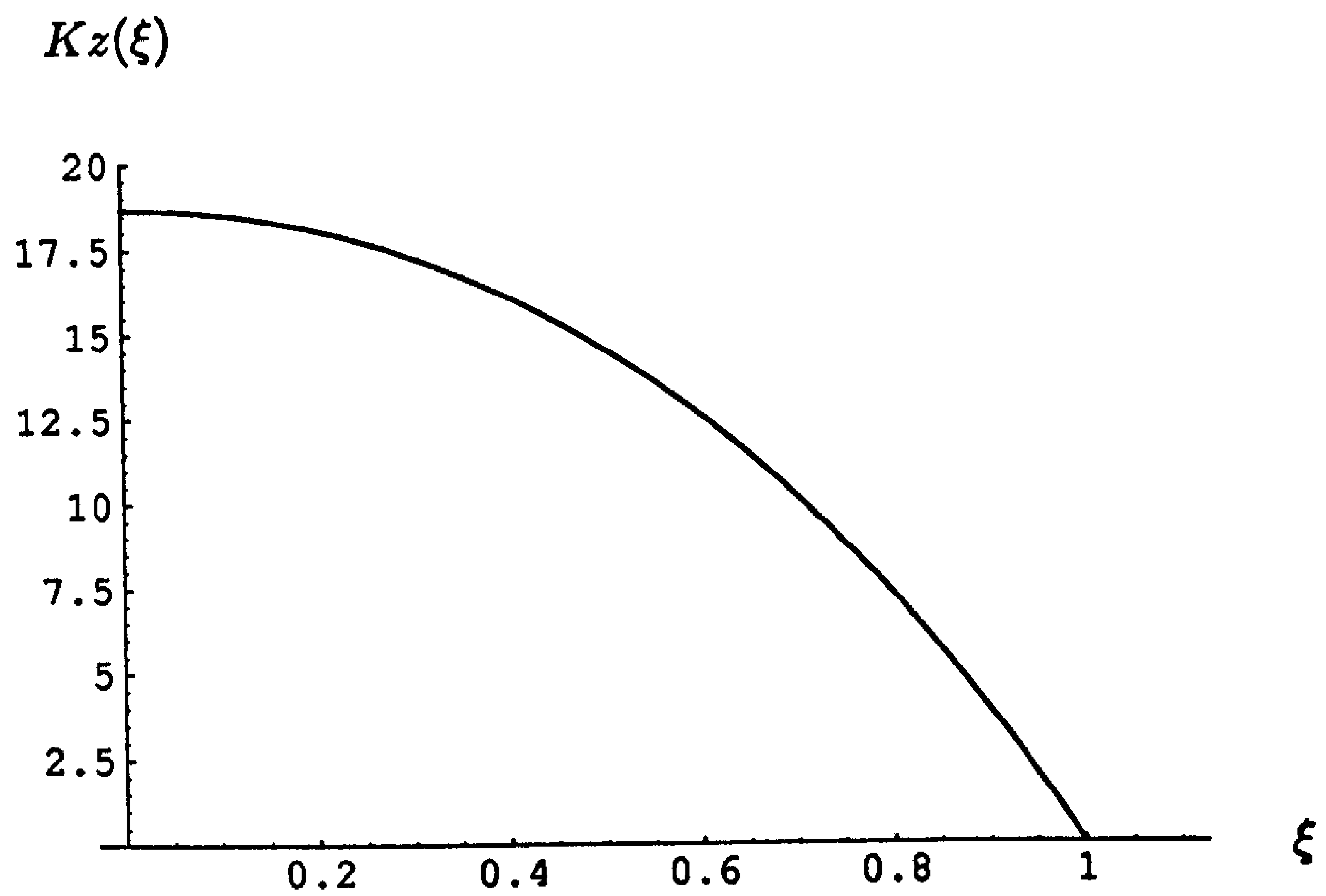
$$R\left(\xi^2-\xi_0^2\right)+(R-S)\ln\left|\frac{R\xi^2+S}{R\xi_0^2+S}\right| = 2R^2 z, \qquad (49)$$

where $\xi_0 = \xi(0)$. Written out in full and taking $\xi(0) = 1$, (49) reads as follows

$$
\begin{aligned}
4K(2\varepsilon_2 - \Gamma)^2 z &= (2\varepsilon_2 - \Gamma)\left(1 - \xi^2\right) - \\
&\quad (\varepsilon_2 - 2\Gamma)\ln\left|\frac{(2\varepsilon_2 - \Gamma)\,\xi^2 + \varepsilon_2 + \Gamma}{3\varepsilon_2}\right|. \qquad (50)
\end{aligned}
$$

The function $Kz(\xi)$ is shown below for the case when the parameters Ω_0, T_1, T_2 take the representative numerical values

$$
\begin{aligned}
\Omega_0 &= 4\pi \times 10^{14}\mathrm{s}^{-1},\\
T_1 &= 100 \times 10^{-12}\mathrm{s},\\
T_2 &= 100 \times 10^{-15}\mathrm{s}.
\end{aligned}
$$



$K \simeq N[\mathrm{m}^{-3}] \times (7.27 \times 10^{-23})\,[\mathrm{m}^2] = (7.27N \times 10^{-23})\,[\mathrm{m}^{-1}]$, where N is the number density of atoms, and the dimensions of the constants are indicated in square brackets. Clearly the validity of our approximate solution runs out before the non-physical point at $\xi = 0$ is reached.

Incidentally, it will probably have been noticed that equation (1) evaluated at $r = 1$ is tautological. However, an expression for $\partial_z \left(\int_{-\infty}^{\infty} |A|^2 \, d\tau\right)$ is easily obtained directly from equations (37), (39) and (40):

$$\partial_z \left(\int_{-\infty}^{\infty} |A|^2 \, d\tau\right) = \frac{N\hbar}{2c\varepsilon_{00}\Omega_0 T_1} \int_{-\infty}^{\infty} \left[\rho_7 - \rho_7^0 + \frac{1}{\sqrt{3}} \left(\rho_8 - \rho_8^0\right)\right] d\tau.$$

Replacing A_x, A_y, ρ_7, ρ_8 as we did before by their 1-soliton forms, we find

$$\xi_z = -\frac{2K\varepsilon_1 \xi}{\xi^2 + 1},$$

which implies

$$\xi^2 - 1 + \ln\left(\xi^2\right) = -4K\varepsilon_1 z,$$

if $\xi(0) = 1$.

Chapter 8

Conclusions.

Throughout this thesis we have focused on the resolution of two demanding mathematical problems connected with a new model for the propagation of electromagnetic plane waves in a 3-level optical medium. The Reduced Maxwell-Bloch equations governing the model are notable for being formulated in terms of the vector potential field, as opposed to the electric field of the plane wave, and because they possess *analytic* fully polarised optical soliton solutions, whereas previous comparable treatments (i.e., those integrable at the carrier timescale) only admit linearly polarised solitons. Yet prior to our own study, the most fundamental property of the RMB-equations, implied by their existence as the compatibility condition of an overdetermined "AKNS-pair" of linear differential equations [8], namely their complete integrability using the Inverse Scattering Transform method, had never been rigorously investigated. Thus we were presented with the first major task: to establish an Inverse Scattering scheme providing a method for analytic solution of the general initial-value problem for the RMB-system with rapidly decreasing boundary conditions. Such a scheme was put forward in Chapter 5, and furthermore its validity was successfully tested when the 1-soliton solutions it predicted exactly matched those found by the technique of Darboux-Bäcklund transforms in Chapter 4. As has already been mentioned, this is the first example of an Inverse Scattering scheme for a system whose scattering problem is characterised by a 3×3 matrix

$$U(\tau, z, \zeta) = \iota\zeta J + Q(\tau, z),$$

where

$$J = \mathrm{diag}\,\{J_1, J_2, J_2\} \in \mathbf{R}$$

and Q is off-diagonal (cf. equation (5) p. 12), and which possesses "sine-Gordon-type" symmetries (the zeros of the transition coefficient $\alpha(\zeta)$ occur in symmetric pairs about the imaginary axis of the complex ζ-plane).

Widening the perspective, we remark that the scheme provides a strong mathematical foundation for further research work. For instance one might use it to analyse collisions between vector solitons having different polarisation states, or as a basis for perturbation theory with which to construct solutions of near-integrable systems. A case in point, and a good potential application for the RMB-equations, is the modelling of pulse formation in laser cavities. Here we have an environment which cannot possibly be represented by an exactly integrable system, because of the amplification and dissipation processes taking place.

Bearing in mind these considerations, we turned to the problem of determining the influence of non-Hamiltonian perturbations upon the soliton parameters. By extending the ideas of Elgin [13], we managed to derive an exact evolution equation for the hierarchy of conserved functionals associated with the RMB-system in the presence of an arbitrary perturbation. More precisely, if the inner-product (involving a 4×4 matrix integro-differential operator) on the right-hand side of this evolution equation is calculated according to a suitably perturbed set of RMB-equations, and if the functions A_x, A_y, $\rho_1, ..., \rho_8$ as they subsequently appear in the equation are replaced by their 1-soliton forms, then one obtains a first order differential equation describing the variation of the single soliton parameter with propagation distance z. Perturbative effects on two-parameter solitons would be handled by carrying out the same procedure at two separate values of r (cf. (1)

p. 110), in order to get a determined system of two first-order differential equations. Whilst we accept that the computations required are quite laborious, this method at least provides a general and systematic way of evaluating the impact of non-conservative perturbations.

Regarding future work, there remain various open problems and avenues of research which could be addressed; a couple of possibilities have already been mentioned. We point out two more:

1. How to extend the inverse scattering scheme to deal with the case of transition coefficients possessing higher-order zeros.

2. How to describe the evolution of the background radiation field generated in the presence of perturbing effects.

Appendix.

Lie Algebras.

Let $\mathbf{F}$ be the field of either the real or complex numbers. Then a Lie algebra $\mathbf{L}$ over the field $\mathbf{F}$ is an n-dimensional vector space over $\mathbf{F}$ with an additional multiplicative operation $[\,,\,]$, called the Lie product or commutator, having the following properties for all $\alpha, \beta, \gamma \in \mathbf{L}$, c_1, $c_2 \in \mathbf{F}$:

1.
$$[\alpha, \beta] \in \mathbf{L},$$

2.
$$[c_1\alpha + c_2\beta, \gamma] = c_1\,[\alpha, \gamma] + c_2\,[\beta, \gamma]\,,$$

3.
$$[\alpha, \beta] = -\,[\beta, \alpha]\,,$$

4. the Jacobi Identity holds i.e.,

$$[\alpha, [\beta, \gamma]] + [\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]] = 0.$$

Now according to Ado's Theorem (cf. Jacobson [38], Chapter VI), every abstract Lie algebra is isomorphic to a Lie algebra of matrices. Therefore it is enough for our purposes to exclusively think of the elements of $\mathbf{L}$ as being matrices of some kind; in which case the Lie product becomes the ordinary matrix commutator

$$[\alpha, \beta] = \alpha\beta - \beta\alpha.$$

If we suppose that $\{\epsilon_1, ..., \epsilon_n\}$ is a basis for L, then the *structure constants* relative to this basis are defined through the relations

$$\mathrm{L} \ni [\epsilon_j, \epsilon_k] = \sum_{l=1}^{n} f_{jkl}\epsilon_l, \quad j, k = 1, ..., n. \quad (1)$$

Given the set of basis elements, the n^3 scalars $f_{jkl} \in \mathbf{F}$ specify the Lie algebra completely.

An *automorphism* of L is an invertible linear transformation $\Theta : \mathrm{L} \to \mathrm{L}$ which transforms the basis elements $\epsilon_1, ..., \epsilon_n$ in a non-trivial way whilst leaving invariant the form of the commutator relations (1).

Lastly, we define $\mathrm{u}(N)$ to be the real Lie algebra consisting of anti-hermitian $N \times N$ matrices. It has an $(N^2 - 1)$-dimensional subalgebra $\mathrm{su}(N)$ of $N \times N$ traceless matrices. Every member of $\mathrm{su}(N)$ can be expressed as a linear combination with real coefficients of the matrices

$$\epsilon_l = -\frac{1}{2}\iota\lambda_l, \quad l = 1, ...N^2 - 1,$$

where $\{\lambda_1, ..., \lambda_{N^2-1}\}$ is a basis for the space of hermitian traceless matrices. Hence, using (1) we see that the relations

$$[\lambda_j, \lambda_k] = 2\iota \sum_{l=1}^{N^2-1} f_{jkl}\lambda_l, \quad j, k = 1, ..., N^2 - 1,$$

determine the structure constants $f_{jkl} \in \mathbf{R}$ of $\mathrm{su}(N)$. We remark (but do not prove) that any odd permutation of the indices of f_{jkl} changes its sign.

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