# Free products and continuous bundles of $C^*$-algebras

by

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# Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow.

The results of Chapter 1 are not the original work of the author. The alternative proof in section 3.3.4 was suggested by Simon Wassermann. The methods used in section 5.5 were suggested by Étienne Blanchard.

Every other result, unless otherwise indicated, is the original work of the author, under the direction of Simon Wassermann.

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# Summary


The purpose of this thesis is to investigate the properties of free products of $C^*$-algebras and continuous bundles of $C^*$-algebras. We also consider how these two areas are connected.

In the first chapter we present background material relevant to the thesis. We discuss nuclearity, exactness and Hilbert $C^*$-modules. Then we review the definitions and properties of bundles and free products of $C^*$-algebras.

The second chapter considers reduced amalgamated free products of $C^*$-algebras. We show that, if the initial conditional expectations involved are all faithful, then the resulting free product conditional expectation is also faithful.

In the third chapter we are interested in the properties of reduced free product $C^*$-algebras. We introduce the orthounitary basis concept for unital $C^*$-algebras with faithful traces and show that reduced free products of $C^*$-algebras with orthounitary bases are, except in a few special cases, not nuclear. Building on this, we then determine the ideals in a certain tensor product $C \otimes_\nu C^{op}$ of the reduced free product with its opposite $C^*$-algebra. In the second half of the chapter, we use Cuntz-Pimsner $C^*$-algebras to study reduced free products of nuclear $C^*$-algebras with respect to pure states. We show that, if the G.N.S. representations of the $C^*$-algebras involved contain the compact operators, then the reduced free product $C^*$-algebra is also nuclear.

Chapter four looks at the minimal tensor product operation on continuous bundles of $C^*$-algebras. We construct, for any non-exact $C^*$-algebra $C$, a continuous bundle $\mathcal{A}$ on the unit interval $[0, 1]$ such that $\mathcal{A} \otimes C$ is not continuous. This leads to a new characterisation of exactness for $C^*$-algebras. These results are then extended to allow for *any* compact infinite metric space as the base space.

Finally, we introduce free product operations on bundles of $C^*$-algebras in chapter five. Both full and reduced free product bundles are constructed. We show that taking the free

product (full or reduced) of two continuous bundles gives another continuous bundle, at least when the bundle $C^*$-algebras are exact.

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# Introduction

Here we discuss the main points of this thesis. We start with free products, then move on to bundles of $C^*$-algebras.

Free products are becoming increasingly important in the theory of operator algebras, $C^*$-algebras in particular. There are essentially two kinds of free product, the full free product and the reduced free product. The full free product appears to be the most natural, being defined by a universal property in the same way as free products of groups are defined. It has been a part of $C^*$-algebra theory for a considerable time. Unfortunately, the full free product is not often very well behaved. In particular, it is rarely nuclear or exact, and hence cannot be approximated well by finite dimensional $C^*$-algebras.

More recently, the reduced free product of $C^*$-algebras (and von Neumann algebras) has been defined. It was defined in certain special cases by Ching [11] and Avitzour [4]. However, the theory really took off with Voiculescu [61], who also introduced the reduced *amalgamated* free product for the first time.

In order to define the reduced amalgamated free product of a family $(A_\iota)_{\iota\in I}$ of unital $C^*$-algebras containing a common unital $C^*$-subalgebra $B$, it is necessary to have conditional expectations $\phi_\iota : A_\iota \to B$ for every $\iota \in I$. So we are really dealing with pairs $(A, \phi)$ where $A$ is a unital $C^*$-algebra containing a copy of $B$ and $\phi : A \to B$ is a conditional expectation. Such a pair is known as a $B$-probability space, one reason for this terminology being the following. Suppose that $(\Omega, \Sigma, \mu)$ is a probability space. Let $A = L^\infty(\Omega, \Sigma, \mu)$, $B = \mathbb{C}$ and $\phi(f) = \int f d\mu$. With these definitions, $(A, \phi)$ is a $\mathbb{C}$-probability space. This construction shows that, for a general $C^*$-algebra $A$, any pair $(A, \phi)$ may be thought of as a *non-commutative* probability space. This leads to connections between $C^*$-algebra theory and probability theory, though we do not go into this here.

Given a family $((A_\iota, \phi_\iota))_{\iota\in I}$ of $B$-probability spaces, the reduced amalgamated free product

$$(A, \phi) = *_{\iota \in I}(A_\iota, \phi_\iota) \tag{1}$$

is another $B$-probability space (see chapter 1 for the actual definition). One of the most obvious questions related to this construction is whether or not the free product conditional expectation $\phi$ is faithful. Generalising techniques of Dykema [18], we show that $\phi$ is faithful precisely when all the $\phi_\iota$ are faithful. This is done in chapter 2.

Next, we consider various properties of the reduced free product, restricting to the case where $B = \mathbb{C}$. Then, all the conditional expectations become states. One question that might be asked is, when is the reduced free product an exact $C^*$-algebra? Dykema [20] has shown that, in equation (1) above, $A$ is exact if and only if all the $A_\iota$ are exact. This result is true for arbitrary $B$.

However, nuclearity is certainly not preserved in the same way. We always have a conditional expectation $A \to A_\iota$ for every $\iota \in I$. So if $A$ is nuclear then every $A_\iota$ is nuclear. The converse is false. For example, $C_r^*(\mathbb{Z})$ is nuclear (indeed commutative), but the reduced free product

$$C_r^*(\mathbb{Z}) *_r C_r^*(\mathbb{Z}) = C_r^*(\mathbb{F}_2)$$

(with respect to the canonical traces) is certainly not nuclear.

In an attempt to generalise the above example of an exact but non-nuclear reduced free product, we introduce the orthounitary basis concept for a unital $C^*$-algebra $A$ with a faithful trace $\tau$. An orthounitary basis is really a generalisation of a group. If $\mathcal{O}$ is an orthounitary basis for $A$ then $\mathcal{O}$ is a subset of $A$ such that, if $a_1, a_2 \in \mathcal{O}$ then the product $a_1 a_2 = \lambda a_3$ for some $a_3 \in \mathcal{O}$ and some $\lambda \in \mathbb{T}$, the unit circle. See chapter 3 for the details of this. Any orthounitary basis $\mathcal{O}$ has an underlying group $G$. The orthounitary basis corresponds to a unitary projective representation

$$\pi : G \to P\mathcal{U}(H).$$

This is just a group homomorphism from $G$ into the quotient $\mathcal{U}(H)/\mathbb{T}$ where $\mathcal{U}(H)$ is the group of unitary operators on $H = L^2(A, \tau)$ and $\mathbb{T}$ is the normal subgroup of complex numbers of modulus one. See chapter 9 of de la Harpe and Jones [36] for more on unitary projective representations of groups.

We show that most reduced free products of $C^*$-algebras with orthounitary bases are non-nuclear. This is done by showing that the reduced free product $C$ also has an orthounitary basis. Moreover, the orthounitary basis for $C$ is of a special form. This enables us to show that a certain tensor product $C \otimes_\nu C^{op}$ (see chapter 3 for details) contains the compact operators, using methods based on those of Wassermann [65]. As $C \otimes_\nu C^{op}$ is therefore non-simple, and yet $C$ itself *is* simple, it then follows that $C$ cannot be nuclear.

The natural question to ask following this is, which $C^*$-algebras have orthounitary bases? Fortunately, there are plenty of examples. The reduced group $C^*$-algebra of a discrete group is the most obvious example, but matrix algebras, U.H.F. algebras and irrational rotation algebras also have orthounitary bases.

Following on from this work, we then investigate the tensor product $C \otimes_\nu C^{op}$ more deeply. We know that this tensor product contains the ideal of compact operators. Akemann and Ostrand [1] were able to show that this is actually the only non-trivial ideal, in the case when $C = C_r^*(\mathbb{F}_2)$. Using their ideas, we show that the compact operators constitute the unique non-trivial ideal of $C \otimes_\nu C^{op}$ when we start from $C^*$-algebras whose orthounitary bases are either finite or free, i.e. the underlying groups are either finite or free. The methods fail to work in other cases, one problem being that it is difficult to define a suitable *length function* on a group that is neither finite nor free.

In the second half of chapter 3, we consider reduced free products of nuclear $C^*$-algebras with respect to pure states. It is suspected that all such reduced free products are nuclear. Certainly, Kirchberg [41] has shown that a reduced free product of matrix algebras with respect to pure states is nuclear. Here we show that the reduced free product is nuclear if in addition the G.N.S. representations of the $C^*$-algebras involved contain the compact operators.

This is done using Cuntz-Pimsner $C^*$-algebras. Dykema and Shlyakhtenko [24] showed that the reduced free product $A$ embeds into a certain Cuntz-Pimsner $C^*$-algebra $E(H)$. If we are taking a reduced free product of nuclear $C^*$-algebras, then this Cuntz-Pimsner $C^*$-algebra turns out also to be nuclear. Assuming further that the G.N.S. representations contain the compact operators, it is possible to construct a conditional expectation $\Psi : E(H) \to A$. This shows that $A$ is nuclear. An alternative proof of this result is then given, using Kirchberg's work on reduced free products of matrix algebras [41] and methods from the proof of the equivalence of nuclear embeddability and exactness.

Now we move on to the second main topic of this thesis, namely continuous bundles of $C^*$-algebras. These have been a part of $C^*$-algebra theory for a long time. They are often known as continuous fields of $C^*$-algebras. More recently, the framework of $C(X)$-algebras has provided another viewpoint on the theory of continuous bundles of $C^*$-algebras. For this thesis, we are primarily interested in operations on bundles. The main question is whether or not continuity of the bundle is preserved by the operation in question.

Chapter 4 looks at the minimal tensor product operation on continuous bundles. If we have a continuous bundle $\mathcal{A}$ on the space $X$, and $B$ is a fixed $C^*$-algebra, then there is an obvious minimal tensor product bundle $\mathcal{A} \otimes B$ on the same space $X$. It is certainly not obvious that $\mathcal{A} \otimes B$ is also continuous. This *is* true when $B$ is exact. However, Kirchberg and Wassermann [44] showed that, if $B$ is not exact, then there exists a continuous bundle $\mathcal{A}$ on the one point compactification $\widehat{\mathbb{N}}$ such that $\mathcal{A} \otimes B$ is not continuous. The space $\widehat{\mathbb{N}}$ is the most simple non-discrete space.

Here, given a non-exact $C^*$-algebra $B$, we construct a continuous bundle $\mathcal{A}$ on the unit interval $[0,1]$ such that $\mathcal{A} \otimes B$ is not continuous. This is done by embedding $\widehat{\mathbb{N}}$ into $[0,1]$, then modifying the methods of Kirchberg and Wassermann to produce a bundle on the space $[0,1]$ instead of $\widehat{\mathbb{N}}$. This construction results in a new characterisation of exactness in terms of continuous bundles of $C^*$-algebras with base space $[0,1]$. After this, we extend this bundle construction to bundles over any infinite compact metric space $(X,d)$. Such bundles are constructed via the induced bundle construction, which is a well-known construction in the context of topological fibre bundles. We fix a non-isolated point $x \in X$ and define $\eta : X \to \mathbb{R}_+$ by

$$\eta(y) = d(x,y).$$

As $X$ is compact we can assume that $\eta(X) \subset [0,1]$. This enables us to use the bundle constructed on $[0,1]$ to create a bundle on $X$.

In the final chapter of this thesis we attempt to combine free products and continuous bundles, by considering free product operations on continuous bundles of $C^*$-algebras. This is done in an attempt to perhaps obtain new characterisations of exactness or nuclearity, as in chapter 4. We first consider the full free product. Given a bundle $\mathcal{A} = (X, \pi_x : A \to A_x, A)$ and a fixed $C^*$-algebra $B$, a full free product bundle $\mathcal{A} * B$ is constructed. This has fibres $A_x * B$, as you might expect. Using results of Blanchard [9] (and Kirchberg) we show that, for any continuous bundle $\mathcal{A}$ (on a compact metric space with unital separable exact bundle $C^*$-algebra), the bundle $\mathcal{A} * B$ is also continuous, regardless of $B$. The reason for this is that the assumed conditions imply that $\mathcal{A}$ is subtrivial. This in turn implies that $\mathcal{A} * B$ is also subtrivial, hence certainly continuous.

The situation for the reduced free product is more complicated. To construct a reduced free product bundle $\mathcal{A} *_r B$, we first need to attach a state to $B$ and a continuous field of states to the fibres of $\mathcal{A}$. We are then left with two possible definitions for the reduced free product bundle. There is the upper semicontinuous bundle $C^u$ and the lower semicontinuous bundle $C^l$. $C^u$ is the most natural from the $C(X)$-algebra point of view. However, it is not clear that the fibres are always $A_x *_r B$. The bundle $C^l$ has the advantage that its fibres are always $A_x *_r B$, which is what you would expect from a reduced free product bundle.

We discuss possible methods for proving that $C^l$ is continuous. It turns out that $C^l$ is continuous precisely when the bundles $C^u$ and $C^l$ coincide. Using methods inspired by a result of Effros and Haagerup [25] we show that $C^l$ is indeed continuous, at least in certain special cases. The main idea is to construct a unital completely positive lifting $\rho : A_x \to A$. This is complicated by the requirement that $\rho$ must respect the state on $A_x$ and the conditional expectation on $A$ (in a sense made precise in section 5.4).

Finally, we consider the continuity of $C^u$. We show that, if $\mathcal{A}$ is continuous and has exact bundle $C^*$-algebra then $C^u$ is continuous, regardless of $B$. This is done using the work of Dykema and Shlyakhtenko [24]. Interestingly, this is the same work that was used in showing the nuclearity of certain reduced free products in section 3.3. We embed $C^u$ into a Cuntz-Pimsner $C^*$-algebra $E(H)$, where $H$ is a Hilbert $D$-bimodule, $D$ being the bundle $C^*$-algebra of the minimal tensor product bundle $\mathcal{A} \otimes B$. The Cuntz-Pimsner $C^*$-algebra is also a bundle, and we use the structure of the Cuntz-Pimsner $C^*$-algebra $E(H)$ to show that it is actually a continuous bundle. We start from the fact that $E(H)$ contains the minimal tensor product bundle $D$, which is continuous because $\mathcal{A}$ has exact bundle $C^*$-algebra. From the continuity of $E(H)$ , it follows that $C^u$ itself is continuous. Unfortunately, it is not clear if continuity of $C^u$ implies anything about the continuity of $C^l$.

# Chapter 1

# Preliminaries

This Chapter brings together background material which is relevant to this thesis. We first detail the notation and conventions used. Then Section 1.1 provides the definitions and some properties of the important concepts of nuclearity and exactness. These ideas are used heavily in the rest of the thesis, with Chapter 3 focussing on nuclearity and Chapter 4 providing new characterisations of exactness. Exactness appears again in Chapter 5.

Section 1.2 looks at Hilbert $C^*$-modules, especially the interior tensor product, which appears throughout the rest of this thesis. In Section 1.3 we define continuous bundles of $C^*$-algebras. We note the alternative viewpoints provided by continuous fields of $C^*$-algebras and $C(X)$-algebras. Finally, in Section 1.4 we define free products of $C^*$-algebras. We consider the full and reduced cases, as well as amalgamated free products.

## Notation and conventions

If $A$ and $B$ are $C^*$-algebras then $A \otimes B$ denotes the minimal or spatial tensor product, while $A \otimes_{max} B$ denotes the maximal tensor product. Generally, $A \odot B$ denotes the algebraic tensor product. All ideals are closed and two-sided, unless stated otherwise. The $C^*$-algebra of bounded linear operators on a Hilbert space $H$ is denoted by $B(H)$. The $C^*$-algebra of $n \times n$ matrices over the complex numbers is denoted $M_n$. Other notation will be introduced in later sections of this chapter.

## 1.1 Nuclearity and exactness

Nuclearity and exactness are two of the most important ways of approximating an infinite dimensional $C^*$-algebra by finite dimensional ones. Much information about these concepts can be found in Wassermann's monograph [66]. Their definitions are as follows.

**Definition 1.1.1.** *A $C^*$-algebra $A$ is **nuclear** if, for every $C^*$-algebra $B$, there is a unique $C^*$-norm on the algebraic tensor product $A \odot B$.*

**Definition 1.1.2.** *A $C^*$-algebra $A$ is **exact** if*

$$0 \to A \otimes J \to A \otimes B \to A \otimes (B/J) \to 0$$

*(with the canonical maps) is an exact sequence for every $C^*$-algebra $B$ and for every ideal $J$ of $B$.*

The above definition of nuclearity is not always easy to work with, so other characterisations in terms of approximation properties have been considered. The following definition makes use of completely positive maps. For more information on these, and the related completely bounded maps, see Paulsen [51].

**Definition 1.1.3.** *A unital $C^*$-algebra $A$ has the **completely positive approximation property** if there are $n_\lambda \in \mathbb{N}$ and nets of unital completely positive maps $\psi_\lambda : A \to M_{n_\lambda}$, $\phi_\lambda : M_{n_\lambda} \to A$ such that*

$$\lim_\lambda \phi_\lambda \psi_\lambda(a) = a \ \ \forall\, a \in A.$$

If $A$ is not unital, the approximating maps are required to be complete contractions.

It is not too difficult to show that, if $A$ has the completely positive approximation property, then $A$ is nuclear. The converse was proved by Kirchberg [39] and by Choi and Effros [13], thus giving the following.

**Theorem 1.1.4.** *A $C^*$-algebra is nuclear if and only if it has the completely positive approximation property.*

It is now more clear from the above theorem that nuclearity is a form of approximation by finite dimensional $C^*$-algebras. All finite dimensional $C^*$-algebras are nuclear, as are all commutative $C^*$-algebras.

Furthermore, the class of nuclear $C^*$-algebras is closed under the taking of inductive limits and quotients. Another important property, which follows from Theorem 1.1.4, is the following. Suppose $B$ is a $C^*$-subalgebra of the nuclear $C^*$-algebra $A$ and that there exists a conditional expectation from $A$ onto $B$. Then $B$ is also nuclear. This property is essential in Section 3.3. Perhaps surprisingly, there are examples of $C^*$-subalgebras of nuclear $C^*$-algebras which are not nuclear: see section 3.1.1.

Moving on, it is natural to ask if exactness can also be reformulated in a similar way to the reformulation of nuclearity given in Theorem 1.1.4. This is indeed the case. A unital $C^*$-algebra $A$ is said to be *nuclearly embeddable* if for some Hilbert space $H$, there is a nuclear embedding $\iota : A \hookrightarrow B(H)$. That is, there are $n_\lambda \in \mathbb{N}$ and nets $\psi_\lambda : A \to M_{n_\lambda}$, $\phi_\lambda : M_{n_\lambda} \to B(H)$ of unital completely positive maps such that

$$\lim_\lambda \phi_\lambda \psi_\lambda(a) = \iota(a) \;\; \forall\, a \in A.$$

In the non-unital case, the unital completely positive maps are replaced by completely positive contractions. We have the following result.

**Theorem 1.1.5.** *A $C^*$-algebra is exact if and only if it is nuclearly embeddable.*

*Proof.* See Theorem 4.1 of [42] and chapter 7 of [66]. □

This result makes it clear that exactness is also a form of approximation by finite dimensional $C^*$-algebras. Comparing with Theorem 1.1.4, it is obvious that any nuclear $C^*$-algebra is exact. We also have that any $C^*$-subalgebra of an exact $C^*$-algebra is exact. This means that a $C^*$-subalgebra of a nuclear $C^*$-algebra is always exact.

Not all $C^*$-algebras are exact. An example is the full group $C^*$-algebra of the free group on two generators, $C^*(\mathbb{F}_2)$ (see example 1.4.1). In fact, $B(H)$ is not exact when $H$ is infinite dimensional.

Another property of some importance is the following. A $C^*$-algebra is said to be *residually finite-dimensional* if it has a separating family of finite dimensional representations. As with nuclearity and exactness, all commutative and finite-dimensional $C^*$-algebras are residually finite-dimensional. However, the class of residually finite-dimensional $C^*$-algebras is also closed under taking full free products, unlike the classes of nuclear $C^*$-algebras and exact $C^*$-algebras (see section 1.4). In this thesis, though, we will mostly be concerned with the concepts of nuclearity and exactness.

## 1.2 Hilbert $C^*$-modules

In this section we provide an overview of the most basic features of Hilbert $C^*$-modules. Further information can be found in [47] or [67].

Suppose that $A$ is a $C^*$-algebra. An *inner-product $A$-module* is a complex vector space $E$ which is also a right $A$-module and has an inner-product map $\langle\, ,\rangle : E \times E \to A$ satisfying

- $\langle x, \lambda y + \mu z\rangle = \lambda\langle x, y\rangle + \mu\langle x, z\rangle\ \forall\, x, y, z \in E\ \forall\, \lambda, \mu \in \mathbb{C}$
- $\langle x, ya\rangle = \langle x, y\rangle a\ \forall\, x, y \in E\ \forall\, a \in A$
- $\langle y, x\rangle = \langle x, y\rangle^*\ \forall\, x, y \in E$
- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ implies $x = 0$.

This definition implies that the inner-product map is conjugate-linear in the first variable and linear in the second variable. Occasionally, in this thesis, we may make use of $\mathbb{C}$-valued inner products which are linear in the first variable, but this should not cause any confusion.

**Definition 1.2.1.** *Let $A$ be a $C^*$-algebra. An inner-product $A$-module $E$ is said to be a* ***Hilbert $A$-module*** *if it is complete with respect to the norm defined by setting $\|x\| = \|\langle x, x\rangle\|^{1/2}$ for $x \in E$.*

The theory of such modules really took off with the work of Paschke [50]. We now provide some simple examples of Hilbert $C^*$-modules.

*Example 1.2.1.* Taking $A = \mathbb{C}$, we obtain the usual Hilbert spaces. Also, any $C^*$-algebra $A$ can be made into a Hilbert $A$-module over itself in an obvious way. If $H$ is a Hilbert space and $A$ is a $C^*$-algebra, then we can form the Hilbert $A$-module $H \otimes A$, which is the closure of the algebraic tensor product of $H$ and $A$, with inner-product given by

$$\langle \xi \otimes a, \eta \otimes b\rangle = \langle \xi, \eta\rangle a^* b\ \ \forall\, \xi, \eta \in H,\ \forall\, a, b \in A.$$

A closed submodule $F$ of a Hilbert $A$-module $E$ is said to be *complemented* if $E = F \oplus F^\perp$ where $F^\perp$ is defined as for Hilbert spaces. In contrast to the theory of Hilbert spaces, closed submodules may not in general be complemented.

Given a Hilbert $A$-module $E$, we shall be interested in the set $L(E)$ of adjointable maps on $E$. This is the set of maps $t : E \to E$ such that there exists a map $t^* : E \to E$

with

$$\langle tx, y \rangle = \langle x, t^* y \rangle \; \forall \, x, y \in E.$$

Such maps $t$ are automatically $A$-linear and bounded. If $A$ happens to be $\mathbb{C}$ we get the usual set of bounded linear operators on a Hilbert space. In general, $L(E)$ shares many of the properties of $B(H)$, but not everything. For example (as noted in chapter 2), we might expect that for $t \in L(E)$ we would have $(\mathrm{im} t)^{\perp} = \ker t^*$. However, whilst $(\mathrm{im} t)^{\perp} \subset \ker t^*$ always holds, the equality fails in general.

Contained inside $L(E)$ is the ideal of compact operators $K(E)$. This is the closed span of $\{\theta_{x,y} : x, y \in E\}$ where $\theta_{x,y}(z) = x\langle y, z\rangle$ for $z \in E$. When $E = A$, $K(E)$ is isomorphic to $A$ and $L(E)$ is isomorphic to the multiplier algebra of $A$.

The usual G.N.S. construction for a state on a $C^*$-algebra gives rise to a representation of the $C^*$-algebra as operators on a Hilbert space (see section 3.4 of Murphy [49]). We will often be interested in the situation where $A$ is a unital $C^*$-algebra, $B$ is a unital $C^*$-subalgebra of $A$, and $\phi : A \to B$ is a conditional expectation. The conditional expectation gives rise to a kind of generalised G.N.S. construction, where $A$ is represented as operators on a Hilbert $B$-module. This generalised G.N.S. construction is described in chapter 5 of Lance [47] and section 1 of Dykema [20]. Here we give a brief description of the details of the construction, sufficient for our purposes.

We let $E = L^2(A, \phi)$ be the Hilbert $B$-module obtained from $A$ by separation and completion with respect to the seminorm

$$||a|| = ||\phi(a^* a)||^{1/2}.$$

There is a canonical map $A \to E$, which we denote by $a \longmapsto \widehat{a}$. By construction, $\widehat{A}$ is dense in $E$, and the inner-product map is defined on $E$ by

$$\langle \widehat{a}_1, \widehat{a}_2 \rangle = \phi(a_1^* a_2)$$

for $a_1, a_2 \in A$. We represent $A$ on $E$ via $\pi : A \to L(E)$, where $\pi(a)(\widehat{a}_1) = \widehat{a a_1}$ for $a, a_1 \in A$. There is a vacuum vector $\xi = \widehat{1} \in E$. This is the generalisation of the G.N.S. construction that we shall require. When taking reduced amalgamated free products of $C^*$-algebras, we will require, for the pair $(A, \phi)$, that the corresponding G.N.S. representation is faithful. By this, we mean that we require $\pi$ to be faithful. This is equivalent to asking that, for all $a \in A$ with $a \neq 0$, there is some $a_1 \in A$ such that $\phi(a_1^* a^* a a_1) \neq 0$.

We shall be particularly interested in tensor products of Hilbert $C^*$-modules. There are basically two kinds of tensor product. Suppose $E$ is a Hilbert $A$-module and $F$ is a Hilbert $B$-module, where $A$ and $B$ are $C^*$-algebras. Then the *exterior tensor product* $E \otimes F$ is a Hilbert $A \otimes B$-module. It is defined to be the completion of the algebraic tensor product of $E$ and $F$ with respect to the norm induced from

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \otimes \langle y_1, y_2 \rangle$$

where $x_1, x_2 \in E$ and $y_1, y_2 \in F$. Taking $B = \mathbb{C}$ and $E = A$, so that $F$ is a Hilbert space, we obtain the tensor product of a Hilbert space and a $C^*$-algebra, as considered previously.

Although the exterior tensor product would appear to be the most natural construction, we will rarely consider it. Instead we shall use the *interior tensor product* of $E$ and $F$ most of the time. For this we require a $*$-homomorphism $\phi : A \to L(F)$. This makes $F$ into a left $A$-module, so we can form the algebraic tensor product of $E$ and $F$ over $A$. We complete this with respect to the norm induced by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1, \phi(\langle x_1, x_2 \rangle) y_2 \rangle$$

where $x_1, x_2 \in E$ and $y_1, y_2 \in F$. The resulting interior tensor product is denoted by $E \otimes_\phi F$. It is a Hilbert $B$-module in the obvious way. Usually the $*$-homomorphism $\phi$ will be obvious in any given context, so we will avoid explicitly mentioning it.

*Example 1.2.2.* Perhaps the simplest example is when $F = B$ and $A = \mathbb{C}$. The $*$-homomorphism $\phi : \mathbb{C} \to M(B)$ is the usual unital embedding, and the interior tensor product then becomes the tensor product of a Hilbert space and a $C^*$-algebra, as considered previously.

*Example 1.2.3.* A more interesting example is the following. Suppose that $X$ is a compact Hausdorff space. Then we can consider a Hilbert $C(X)$-module $E$. Suppose that $x \in X$. Then we have the evaluation map $\mathrm{ev}_x : C(X) \to \mathbb{C}$. So we can form the interior tensor product $E_x = E \otimes_{\mathrm{ev}_x} \mathbb{C}$, which is actually a Hilbert space. We think of $E$ as a continuous field of Hilbert spaces over the base space $X$, with $E_x$ being the Hilbert space attached to the point $x \in X$. In the case where $E = H \otimes C(X)$, $H$ being a Hilbert space, we do of course get $E_x = H$ for every $x \in X$.

The above example can also be viewed in terms of the concept of *localisation.* This is discussed at the end of chapter 5 of Lance [47]. Suppose that $E$ is a Hilbert $A$-module with inner product $\langle\, , \rangle_A$, where $A$ is a unital $C^*$-algebra. Suppose that $A$ contains a unital $C^*$-subalgebra $B$ and that there is a conditional expectation $\psi : A \to B$. Then $E$ can be made into a semi-inner-product $B$-module by defining

$$\langle x, y\rangle_B = \psi(\langle x, y\rangle_A).$$

The usual completion process results in a Hilbert $B$-module $E_\psi$, known as the localisation of $E$ with respect to $\psi$.

This is useful because it is possible to define a $*$-homomorphism $\pi_\psi : L(E) \to L(E_\psi)$. This is defined in the obvious way. It is injective if $\psi$ is faithful. To obtain the above example, take $A = C(X)$, $B = \mathbb{C}$ and $\psi = \mathrm{ev}_x$ for some $x \in X$.

Finally, we consider what is generally known as Kasparov's stabilisation theorem. Let $H = \ell^2(\mathbb{N})$ and denote by $H_A$ the Hilbert $A$-module $H \otimes A$ discussed earlier. It turns out that $H_A$ has the following remarkable properties. For proofs, see chapter 6 of [47] or the original paper of Kasparov [38]. A Hilbert $A$-module $E$ is said to be *countably generated* if there is a countable set $S \subset E$ such that the smallest closed submodule of $E$ containing $S$ is the whole of $E$.

**Theorem 1.2.2.** *Suppose that $A$ is a $C^*$-algebra and that $E$ is a countably generated Hilbert $A$-module. Then $E \oplus H_A \cong H_A$.*

**Corollary 1.2.3.** *Suppose that $E$ is a countably generated Hilbert $A$-module. Then $E$ is (unitarily equivalent to) a fully complemented submodule of $H_A$.*

## 1.3 Bundles of $C^*$-algebras

In this section we consider bundles of $C^*$-algebras. Bundles are very important in chapters 4 and 5 of this thesis. They have been prominent in $C^*$-algebra theory for some time. As algebras of operator fields, they were studied by Fell [29]. As continuous fields of $C^*$-algebras, they were studied extensively by Dixmier [16].

**Definition 1.3.1.** *Let $X$ be a locally compact Hausdorff space. Then a* ***bundle*** *of $C^*$-algebras over $X$ is a triple $\mathcal{A} = (X, \pi_x : A \to A_x, A)$ where $A$ is the bundle $C^*$-algebra, and $A_x$ is the fibre $C^*$-algebra at $x \in X$. The maps $\pi_x$ are surjective $*$-homomorphisms such that*

- *the family $\{\pi_x : x \in X\}$ is faithful.*

- *$A$ is a left $C_0(X)$-module with $\pi_x(fa) = f(x)\pi_x(a)$ for $x \in X$, $f \in C_0(X)$, $a \in A$.*

We often write $a_x$ for $\pi_x(a)$. The bundle is said to be **continuous** if, for every $a \in A$, the function $x \longmapsto \|a_x\|$ is in $C_0(X)$.

*Example 1.3.1.* There are two motivating examples for the above definitions. First there is the *trivial bundle* with fibre $A$ (a $C^*$-algebra) on the space $X$. This has bundle $C^*$-algebra $C_0(X, A)$, with fibre $A$ at every $x \in X$. The maps $\pi_x$ are just the evaluation maps. Such a bundle is clearly continuous.

Now suppose that $X$ is a discrete space and that for every $x \in X$ we have a $C^*$-algebra $A_x$. Then $\oplus_{x \in X} A_x$ can be made into a continuous bundle of $C^*$-algebras over the space $X$, with fibre $A_x$ at $x \in X$ and the obvious maps.

In general, the bundle $\mathcal{A} = (X, \pi_x : A \to A_x, A)$ is said to be **trivial** if there are a $C^*$-algebra $B$ and $*$-isomorphisms $\theta_x : A_x \to B$, $\theta : A \to C_0(X, B)$ such that, for every $x \in X$, we have $\theta_x \circ \pi_x = \mathrm{ev}_x \circ \theta$. The bundle $\mathcal{A}$ is **subtrivial** if the maps $\theta$, $\theta_x$ are not necessarily surjective.

*Example 1.3.2.* An interesting example of a continuous bundle is given by Elliott, Natsume and Nest in [27]. This paper proves the Bott periodicity theorem via a description of the $C^*$-algebra of the Heisenberg group as a continuous bundle of $C^*$-algebras. The base space is $\mathbb{R}$. The fibre at 0 is $C_0(\mathbb{R}^2)$, whilst every other fibre is $K(L^2(\mathbb{R}))$.

Recently, Blanchard [9] has obtained some remarkable results on the subtriviality of continuous bundles of $C^*$-algebras. These are best described within the framework of $C(X)$-algebras: see section 1.3.2.

In this thesis, we shall be particularly interested in *operations* on continuous bundles of $C^*$-algebras. Let $\mathcal{A} = (X, \pi_x : A \to A_x, A)$ be a continuous bundle of $C^*$-algebras and fix a $C^*$-algebra $B$. Then the minimal tensor product bundle

$$\mathcal{A} \otimes B = (X, \pi_x \otimes \mathrm{id} : A \otimes B \to A_x \otimes B, A \otimes B)$$

can be constructed. There is also the maximal tensor product bundle

$$\mathcal{A} \otimes_{max} B = (X, \pi_x \otimes_{max} \mathrm{id} : A \otimes_{max} B \to A_x \otimes_{max} B, A \otimes_{max} B).$$

The constructions of both these bundles are explained in [44].

In [44] Kirchberg and Wassermann obtained a new characterisation of exactness of $C^*$-algebras in terms of minimal tensor product bundles over the one-point compactification of the natural numbers $\widehat{\mathbb{N}}$.

**Theorem 1.3.2.** *Suppose that $B$ is a $C^*$-algebra. Then $B$ is exact if and only if for any continuous bundle $\mathcal{A}$ of $C^*$-algebras over $\widehat{\mathbb{N}}$ (with separable bundle $C^*$-algebra), $\mathcal{A} \otimes B$ is continuous.*

They also found a similar characterisation of nuclearity in terms of maximal tensor product bundles over $\widehat{\mathbb{N}}$.

**Theorem 1.3.3.** *Suppose that $B$ is a $C^*$-algebra. Then $B$ is nuclear if and only if for any continuous bundle $\mathcal{A}$ of $C^*$-algebras over $\widehat{\mathbb{N}}$ (with separable bundle $C^*$-algebra), $\mathcal{A} \otimes_{max} B$ is continuous.*

In the above results, $B$ is regarded as fixed and the initial bundle is allowed to vary. Fixing the bundle and allowing $B$ to vary, we get a characterisation of exactness of the bundle $C^*$-algebra. The following appears as part of Theorem 4.6 in [44].

**Theorem 1.3.4.** *Let $\mathcal{A} = (X, \pi_x : A \to A_x, A)$ be a continuous bundle of $C^*$-algebras with exact fibres. Then the bundle $C^*$-algebra $A$ is exact if and only if for any $C^*$-algebra $B$, $\mathcal{A} \otimes B$ is continuous.*

### 1.3.1 Continuous fields of $C^*$-algebras

Here we mention an alternative viewpoint for the theory of continuous bundles. We make the following definition.

**Definition 1.3.5.** *Let $X$ be a locally compact Hausdorff space. A continuous field of $C^*$-algebras $\mathcal{A} = ((A(x))_{x \in X}, \Gamma)$ consists of a family of $C^*$-algebras $A(x)$, indexed by the set $X$, together with a subset $\Gamma \subset \prod_{x \in X} A(x)$. The subset $\Gamma$ satisfies the following:*

- *$\Gamma$ is a $*$-subalgebra of $\prod_{x \in X} A(x)$.*
- *For every $x \in X$, $\{s(x) : s \in \Gamma\}$ is dense in $A(x)$.*
- *For every $s \in \Gamma$, $x \longmapsto \|s(x)\|$ is continuous.*
- *Suppose $t \in \prod_{x \in X} A(x)$. If, for every $x \in X$ and $\epsilon > 0$ there exists $s \in \Gamma$ such that $\|s(z) - t(z)\| \leq \epsilon$ for $z$ in some neighbourhood of $x$, then $t \in \Gamma$.*

The subset $\Gamma$ is the set of continuous sections for $\mathcal{A}$. If $X$ is compact then $\Gamma$ is a $C^*$-algebra in the obvious way. If $X$ is not compact then $\Gamma_0 \subset \Gamma$, the subset of continuous sections which vanish at infinity, is a $C^*$-algebra.

As explained in the introduction of [44], the concepts of continuous field and continuous bundle are essentially equivalent and it is easy to pass between the two concepts. Sometimes continuous bundles are more useful in the context of a particular problem, other times continuous fields seem more natural.

For further information on continuous fields of $C^*$-algebras, see chapter 10 of Dixmier [16]. Continuous fields are also used extensively by, for example, Kirchberg and Phillips [43].

### 1.3.2 $C(X)$-algebras

This section considers the point of view provided by $C(X)$-algebras. First, recall the definition of a bundle of $C^*$-algebras. We required the bundle $C^*$-algebra $A$ to be a left $C_0(X)$-module with $\pi_x(fa) = f(x)\pi_x(a)$. As explained in [3], this can be replaced by assuming the existence of a $*$-homomorphism $\theta : C_0(X) \to \mathcal{Z}M(A)$ (where $\mathcal{Z}M(A)$ denotes the centre of the multiplier algebra of $A$) such that

$$\pi_x(\theta(f)a) = f(x)\pi_x(a).$$

This motivates the following definition, taken from section 2.2 of Blanchard [8].

**Definition 1.3.6.** *Let $X$ be a locally compact Hausdorff space. A $C_0(X)$-**algebra** consists of a $C^*$-algebra $A$ together with a non-degenerate $*$-homomorphism $C_0(X) \to \mathcal{Z}M(A)$.*

We are usually interested in the case where $X$ is compact and the non-degenerate $*$-homomorphism is actually an embedding. In this case, the non-degeneracy is equivalent to asking that the $*$-homomorphism is unital.

Now consider

$$C_x(X)A = \{fa : f \in C_x(X),\, a \in A\}.$$

By [8] Corollary 1.9 this is a closed vector subspace of $A$ and hence it is an ideal. Define $A_x$ to be the quotient $A/C_x(X)A$. It is now clear how a $C_0(X)$-algebra can be made into a bundle over $X$. By [8] Proposition 2.8 the family of quotient maps $\{A \to A/C_x(X)A\}_{x\in X}$ is faithful. So we do indeed obtain a bundle of $C^*$-algebras over $X$. Moreover, the definition implies that this bundle is automatically upper semicontinuous (this is explained in, for example, section 1.1 of [9]).

Going in the other direction, suppose that we have a bundle of $C^*$-algebras. Then, as remarked at the beginning of this section, we get a $*$-homomorphism $C_0(X) \to \mathcal{Z}M(A)$ in a natural way. Using this $*$-homomorphism, we obtain a new bundle over $X$ with the same bundle $C^*$-algebra as the original bundle but with possibly different fibres. By [44] Lemma 2.3, the new and original bundles coincide precisely when the original bundle is upper semicontinuous.

It will be useful to consider representations of $C_0(X)$-algebras on Hilbert $C_0(X)$-modules. We use the following definition from section 2.3 of [8].

**Definition 1.3.7.** *Suppose that $A$ is a $C_0(X)$-algebra. A* ***$C_0(X)$-representation*** *$\pi : A \to L(\mathcal{E})$ is a $C_0(X)$-linear $*$-homomorphism from $A$ into the adjointable operators on some Hilbert $C_0(X)$-module $\mathcal{E}$.*

This means that, for every $x \in X$, $\pi$ induces a Hilbert space representation $\pi_x : A_x \to L(\mathcal{E}_x)$ where $\mathcal{E}_x = \mathcal{E} \otimes_{\mathrm{ev}_x} \mathbb{C}$. If every $\pi_x$ is faithful, $\pi$ is said to be a *field of faithful representations.* As remarked in [8] section 2.3, the function $x \longmapsto \|\pi_x(a_x)\|$ is always lower semicontinuous.

A $C_0(X)$-algebra is said to be a *continuous field of $C^*$-algebras* if the bundle which it defines is continuous. In this situation, we will often simply say that the $C_0(X)$-algebra is *continuous.*

Given a $C_0(X)$-algebra $A$, we let $S(A)$ be the state space of $A$ and we denote by $S_X(A)$ the set of states $\phi$ such that $\phi|C_0(X)$ is pure (and hence a character on $C_0(X)$). So there is an obvious map $p : S_X(A) \to X$. We have the following theorem: see section 3.1 of [8].

**Theorem 1.3.8.** *Let $X$ be a locally compact Hausdorff space. Suppose $A$ is a separable $C_0(X)$-algebra with every $A_x$ non-zero. Then the following are equivalent:*

**(1)** *$A$ is a continuous field of $C^*$-algebras.*

**(2)** *$p : S_X(A) \to X$ is open.*

**(3)** *There is a family $\{\phi_\lambda\}$ of continuous fields of states on $A$ such that for every $x \in X$, $\{ev_x \circ \phi_\lambda\}$ is a faithful family of states on $A_x$.*

**(4)** *$A$ admits a field of faithful representations.*

Here a *continuous field of states* on $A$ is simply a $C_0(X)$-linear positive map $\phi : A \to C_0(X)$ such that for every $x \in X$, $\phi_x = \mathrm{ev}_x \circ \phi$ is a state on $A_x$.

By Lemma 3.8 of [8], if $X$ is compact and $A$ is unital (a case we will often be interested in) then item (3) above may be replaced by

**(3')** There is a continuous field of faithful states on $A$.

The existence of such a continuous field of faithful states can also be seen from [6], in the case where $X$ is a metric space as well as being compact.

This theorem will be of use in chapter 5, where we consider the reduced free product operation on continuous bundles of $C^*$-algebras.

Next, we mention tensor products in the context of $C(X)$-algebras. Just as there are (maximal and minimal) tensor product operations on bundles of $C^*$-algebras, so it also possible to take tensor products of $C(X)$-algebras. We will rarely consider such tensor products. For more details, see the work of Blanchard [7], [8].

Finally, we mention the following result of Blanchard [9].

**Theorem 1.3.9.** *Suppose that $X$ is a compact metric space and that $A$ is a unital separable $C(X)$-algebra with unital embedding $C(X) \hookrightarrow \mathcal{Z}(A)$. Then the following are equivalent:*

**(1)** *$A$ is a continuous field of nuclear $C^*$-algebras over $X$.*

**(2)** *There is a unital $C(X)$-linear monomorphism $\alpha : A \hookrightarrow C(X, \mathcal{O}_2)$ and a unital $C(X)$-linear completely positive map $E : C(X, \mathcal{O}_2) \to A$ such that $E \circ \alpha = id_A$.*

Here, of course, $\mathcal{O}_2$ is the Cuntz algebra on two generators. Basically, this theorem says that any continuous bundle of nuclear $C^*$-algebras is subtrivial.

Kirchberg (in an appendix to [9]) showed that a continuous field of $C^*$-algebras with *exact* bundle $C^*$-algebra is subtrivial in the same way as in the above theorem (but of course the map $E$ need not exist).

Kirchberg and Phillips obtained similar results in [43]. They obtained more information about the embedding $\alpha$ (see the statement of the theorem above). However, they had to restrict the base space somewhat more than in Blanchard's result.

*Example 1.3.3.* An interesting example of a bundle is given by the rotation $C^*$-algebras. Let $\theta$ be a real number between 0 and 1. Then, by definition, the rotation $C^*$-algebra $A_\theta$ is the universal $C^*$-algebra generated by a pair of unitaries $u$, $v$ satisfying $vu = e^{2\pi i\theta}uv$. See example 3.1.2 later in this thesis.

Elliott [26] showed that the family of rotation $C^*$-algebras is actually a continuous field. That is, there is a $C^*$-algebra $A$ and surjective $*$-homomorphisms $\pi_\theta : A \to A_\theta$ such that $\theta \longmapsto \|\pi_\theta(a)\|$ is continuous for every $a \in A$.

Haagerup and Rørdam [35] were able to show that the rotation $C^*$-algebras form a subtrivial continuous bundle. This was done as follows. Suppose $H$ is a separable infinite-dimensional Hilbert space. Then they obtained continuous paths $u, v : [0,1] \to U(H)$ with $u(0) = u(1)$, $v(0) = v(1)$ and $u(\theta)v(\theta) = e^{2\pi i\theta}v(\theta)u(\theta)$ for every $\theta \in [0,1]$. This subtriviality is not surprising, given Theorem 1.3.9 above and the fact that all the rotation $C^*$-algebras are nuclear.

## 1.4 Full and reduced free products of $C^*$-algebras

There are basically two types of free products of $C^*$-algebras, full free products and reduced free products. Both types will be of some importance in this thesis.

### 1.4.1 Full free products

Full free products of $C^*$-algebras are defined in chapter 1 of Voiculescu, Dykema and Nica [62]. See also Blackadar [5]. For unital $C^*$-algebras (these are our main concern) we have the following definition.

**Definition 1.4.1.** *Let $(A_\iota)_{\iota\in I}$ be a family of unital $C^*$-algebras. The **free product** $C^*$-algebra $*_{\iota\in I}A_\iota$ is the unique unital $C^*$-algebra $A$ with unital embeddings $\psi_\iota : A_\iota \hookrightarrow A$ such that, given any unital $C^*$-algebra $B$ and unital $*$-homomorphisms $\phi_\iota : A_\iota \to B$ there exists a unique unital $*$-homomorphism $\Phi : A \to B$ such that $\phi_\iota = \Phi \circ \psi_\iota$ for every $\iota \in I$. That is, $\Phi$ extends all the $\phi_\iota$.*

Non-unital free products are defined in an analogous manner.

It is necessary to show that the free product $A$ actually exists. To do this, we consider the algebraic free product $A_0$, which is a $*$-algebra. Define the following norm for $x \in A_0$:

$$\|x\| = \sup\{\|\pi(x)\| : \pi \text{ a unital } *\text{-representation of } A_0 \text{ on a Hilbert space}\}$$

Completing $A_0$ with respect to this norm, we obtain the free product $A$.

Full free products relate very nicely to free products of groups.

**Proposition 1.4.2.** *Let $(G_\iota)_{\iota\in I}$ be a family of groups. Then $C^*(*_{\iota\in I}G_\iota) = *_{\iota\in I}C^*(G_\iota)$.*

*Proof.* See Proposition 1.4.3 of [62]. □

Another important property of the full free product is the following. Suppose $A$ and $B$ are unital $C^*$-algebras with $A \subset B$ unitally. Then, if $C$ is another unital $C^*$-algebra, we have a canonical inclusion $A * C \subset B * C$. The minimal tensor product of $C^*$-algebras satisfies a similar property. However, the maximal tensor product of $C^*$-algebras does not have this kind of property in general.

Full free products also satisfy the following. Recall the residually finite-dimensional property from section 1.1.

**Theorem 1.4.3.** *Let $A$ and $B$ be unital $C^*$-algebras. Then the unital full free product $A*B$ is residually finite-dimensional if and only if $A$ and $B$ are residually finite-dimensional.*

*Proof.* See Theorem 3.2 of Exel and Loring [28]. □

Unfortunately, taking full free products does not preserve exactness or nuclearity.

*Example 1.4.1.* Consider $C(\mathbb{T})$ where $\mathbb{T}$ is the unit circle. Now $C(\mathbb{T})$ is commutative so is certainly nuclear and exact. But $C(\mathbb{T}) \cong C^*(\mathbb{Z})$ so

$$C(\mathbb{T}) * C(\mathbb{T}) \cong C^*(\mathbb{Z}) * C^*(\mathbb{Z}) \cong C^*(\mathbb{F}_2).$$

In [63] and [64] Wassermann showed that $C^*(\mathbb{F}_2)$ fails even to be exact. Indeed, letting $J$ denote the kernel of the canonical $*$-homomorphism $C^*(\mathbb{F}_2) \to C^*_r(\mathbb{F}_2)$, the sequence

$$0 \to C^*(\mathbb{F}_2) \otimes J \to C^*(\mathbb{F}_2) \otimes C^*(\mathbb{F}_2) \to C^*(\mathbb{F}_2) \otimes C^*_r(\mathbb{F}_2) \to 0$$

is not exact.

Finally, note that there is also the concept of a full amalgamated free product of $C^*$-algebras. This involves amalgamating over a common $C^*$-subalgebra. The definition is very similar to Definition 1.4.1. Essentially the difference is that we only take $*$-homomorphisms $\phi_\iota : A_\iota \to B$ which agree on the common $C^*$-subalgebra.

In fact, the unital full free product of $C^*$-algebras that we have been considering is, strictly speaking, an amalgamated free product. We are amalgamating over the common $C^*$-subalgebra $\mathbb{C}1$. Full amalgamated free products will also be considered in chapter 5 of this thesis.

### 1.4.2 Reduced free products

Reduced free products of $C^*$-algebras have been important recently, both in $C^*$-algebra theory and in free probability theory (see the book [62] for more on this). Reduced free products were introduced by Voiculescu [61]. Earlier they had been constructed in a less general manner by Avitzour [4] and Ching [11]. As we are particularly interested in reduced *amalgamated* free products, we shall define and construct the reduced amalgamated free product. See chapter 1 of [62] for the unamalgamated case, chapter 3 of [62] for the amalgamated case. Here we follow the notation and conventions of Dykema [20].

The concept of freeness (a non-commutative analogue of independence) is important for the definition of reduced free products.

**Definition 1.4.4.** *Let $A$ be a unital $C^*$-algebra with unital $C^*$-subalgebra $B$ and a conditional expectation $\phi : A \to B$. Consider a family of intermediate $C^*$-subalgebras $B \subset A_\iota \subset A$. The family $(A_\iota)_{\iota \in I}$ is said to be* **free** *with respect to $\phi$ if $\phi(a_1 a_2 \cdots a_n) = 0$ whenever $a_j \in A_{\iota_j}$, $\phi(a_j) = 0$ and $\iota_1 \neq \iota_2 \neq \cdots \neq \iota_n$.*

We now define the reduced amalgamated free product.

**Definition 1.4.5.** *Suppose $B$ is a unital $C^*$-algebra and that for every $\iota \in I$, $A_\iota$ is a unital $C^*$-algebra containing $B$ and with a conditional expectation $\phi_\iota : A_\iota \to B$. Assuming that the corresponding G.N.S. representations are faithful, the reduced amalgamated free product $C^*$-algebra is the unique unital $C^*$-algebra $A$ containing $B$, with conditional expectation $\phi : A \to B$ and embeddings $A_\iota \hookrightarrow A$ restricting to the identity on $B$, such that*

**(1)** *$\phi$ extends all the $\phi_\iota$;*

**(2)** *the family $(A_\iota)_{\iota \in I}$ is free with respect to $\phi$;*

**(3)** *$A$ is generated by the union of the $A_\iota$;*

**(4)** *the G.N.S. representation of $\phi$ is faithful on $A$.*

The reduced amalgamated free product is written

$$(A, \phi) = *_{\iota \in I}(A_\iota, \phi_\iota).$$

Sometimes the reduced free product (using states) of two $C^*$-algebras $A$ and $B$ is written $A *_r B$ if it is obvious which states are involved.

A $C^*$-algebra $A$ with the above properties is unique. The question is, does there exist such an $A$? The construction goes as follows. Recall section 1.2 on Hilbert $C^*$-modules.

Let $E_\iota = L^2(A_\iota, \phi_\iota)$ be the Hilbert $B$-module obtained (via the generalised G.N.S. construction described in section 1.2) from the pair $(A_\iota, \phi_\iota)$. As usual, we denote the canonical map $A_\iota \to E_\iota$ by $a \longmapsto \widehat{a}$. Let $\pi_\iota : A_\iota \to L(E_\iota)$ be the corresponding (generalised) G.N.S. representation. We are assuming that every $\pi_\iota$ is faithful.

Now consider the vacuum vector $\xi_\iota = \widehat{1} \in E_\iota$. The subspace $\xi_\iota B$ is a complemented submodule of $E_\iota$. Indeed, $\theta_{\xi_\iota,\xi_\iota}$ is the projection onto $\xi_\iota B$. Let $E_\iota^o$ be the complementing submodule. That is, $E_\iota^o = \{x \in E_\iota : \langle x, \xi_\iota \rangle = 0\}$. Note that this is just the closure of $\widehat{A_\iota^o}$ in $E_\iota$, where $A_\iota^o = \ker\phi_\iota$. Define

$$E = \xi B \oplus \bigoplus_{\substack{n\in\mathbb{N} \\ \iota_1,\dots,\iota_n\in I \\ \iota_1\neq\iota_2\neq\cdots\neq\iota_n}} E_{\iota_1}^o \otimes_B E_{\iota_2}^o \otimes_B \cdots \otimes_B E_{\iota_n}^o$$

where $\xi B$ is just a copy of $B$, considered as a Hilbert $B$-module, with $\xi = 1$. The tensor products are interior tensor products arising from restricting the maps $\pi_\iota$ to $B$. The Hilbert $B$-module $E$ is known as the *free product* Hilbert $B$-module, and we write

$$(E, \xi) = *_{\iota\in I}(E_\iota, \xi_\iota).$$

Next, take $\iota \in I$ and define

$$E(\iota) = \eta_\iota B \oplus \bigoplus_{\substack{n\in\mathbb{N} \\ \iota_1,\dots,\iota_n\in I \\ \iota_1\neq\iota_2\neq\cdots\neq\iota_n \\ \iota_1\neq\iota}} E_{\iota_1}^o \otimes_B E_{\iota_2}^o \otimes_B \cdots \otimes_B E_{\iota_n}^o$$

where $\eta_\iota B$ is another copy of $B$ with $\eta_\iota = 1$. Let $V_\iota : E_\iota \otimes_B E(\iota) \to E$ be the natural unitary operator defined by

$$\xi_\iota \otimes \eta_\iota \longmapsto \xi$$

$$\zeta \otimes \eta_\iota \longmapsto \zeta$$

$$\xi_\iota \otimes (\zeta_1 \otimes \cdots \otimes \zeta_n) \longmapsto \zeta_1 \otimes \cdots \otimes \zeta_n$$

$$\zeta \otimes (\zeta_1 \otimes \cdots \otimes \zeta_n) \longmapsto \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_n$$

where $\zeta \in E_\iota^o$, $\zeta_j \in E_{\iota_j}^o$ with $\iota \neq \iota_1 \neq \iota_2 \neq \cdots \neq \iota_n$.

The unitary $V_\iota$ enables us to define a representation of $A_\iota$ on $E$ rather than $E_\iota$. That is, we get a $*$-homomorphism $\lambda_\iota : A_\iota \to L(E)$ given by

$$\lambda_\iota(a) = V_\iota(\pi_\iota(a) \otimes 1)V_\iota^*.$$

We then define the reduced amalgamated free product $A$ to be the $C^*$-algebra generated by $\bigcup_{\iota \in I} \lambda_\iota(A_\iota)$. We let $\phi : A \to B$ be the conditional expectation $\phi(a) = \langle a\xi, \xi \rangle$. Note that in the case where $B = \mathbb{C}$, all the Hilbert $C^*$-modules become Hilbert spaces and the conditional expectations become states. (We also tend to write $H_\iota$ for the Hilbert space $E_\iota$ in this special case.) But, for a general reduced amalgamated free product, it is necessary to use the theory of Hilbert $C^*$-modules.

It is not difficult to check that $(A, \phi)$ satisfies the desired properties. Consider, for example, the property of freeness. Suppose that $a_j \in A_{\iota_j}$ and $\phi_{\iota_j}(a_j) = 0$ for $j = 1, 2, \ldots, n$. This means that $\langle a_j \xi_{\iota_j}, \xi_{\iota_j} \rangle = 0$. Direct calculation reveals that, assuming $\iota_1 \neq \iota_2 \neq \cdots \neq \iota_n$,

$$a_1 a_2 \cdots a_n \xi = \widehat{a}_1 \otimes \widehat{a}_2 \otimes \cdots \otimes \widehat{a}_n$$

where we have omitted the representations $\lambda_\iota$. Clearly the right-hand side of the above equation is orthogonal to $\xi$, hence $\phi(a_1 a_2 \cdots a_n) = 0$. This proves that the freeness condition is satisfied.

This completes our review of the construction of the reduced amalgamated free product. Note that $A$ is the closed span of $B$ together with all *reduced words*. These are words of the form $a_1 a_2 \cdots a_n$ where $a_j \in A_{\iota_j}$, $\phi_{\iota_j}(a_j) = 0$, $\iota_1 \neq \cdots \neq \iota_n$ and $n \in \mathbb{N}$. We will often write $A_\iota^o$ for $A_\iota \cap \ker\phi_\iota$.

The canonical example of a reduced amalgamated free product is the one associated to an amalgamated free product of groups. See Example 2.3.1 for details of the construction.

We also have the following interesting result due to Dykema [20].

**Theorem 1.4.6.** *Suppose $B$ is a unital $C^*$-algebra, $I$ is a set, and for $\iota \in I$ we have a unital $C^*$-algebra $A_\iota$ containing $B$ and having a conditional expectation $\phi_\iota : A_\iota \to B$ whose G.N.S. representation is faithful. Let*

$$(A, \phi) = *_{\iota \in I}(A_\iota, \phi_\iota)$$

*be the reduced amalgamated free product. Then $A$ is exact if and only if all the $A_\iota$ are exact.*

So reduced amalgamated free products preserve exactness. The extent to which they preserve nuclearity is one of the questions asked in this thesis.

Finally we mention the $K$-theory of free products of $C^*$-algebras. Germain has investigated the $K$-theory of both full [32] and reduced [30] free products of unital $C^*$-algebras (amalgamating only over the units). See also Dykema and Rørdam [23] for more on the $K$-theory of the free product.

It turns out that, at least if the $C^*$-algebras involved are nuclear, then the $K$-theory is the same for both the reduced and full free products. Moreover, the $K$-groups can be calculated from a simple six-term exact sequence. For details of this, see Germain [32], [30].

*Example 1.4.2.* Consider the reduced free product $A$ of $M_n$ and $M_m$ (with respect to chosen states) for $n, m \in \mathbb{N}$. The results of Germain imply that $K_1(A)$ is always zero. On the other hand, $K_0(A) = \mathbb{Z} \oplus \mathbb{Z}/\mathrm{im}\alpha$ where the group homomorphism $\alpha : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is given by $1 \longmapsto n \oplus (-m)$. It follows that $K_0(A) \cong \mathbb{Z} \oplus \mathbb{Z}_k$ where $k$ is the highest common factor of $m$ and $n$. The $K$-theory of the full free product of $M_n$ and $M_m$ will be exactly the same.

So, $K$-theory is of some use in understanding free products, though we do not consider such methods in the remainder of this thesis.

# Chapter 2

# Free product conditional expectations

## 2.1 Introduction

In this chapter we consider the faithfulness of the free product conditional expectation obtained in the construction of the reduced amalgamated free product of a collection of $C^*$-algebras. First we review the case where amalgamation takes place only over the complex numbers. We then go on to show that the free product conditional expectation is faithful precisely when the initial conditional expectations are faithful. Following this, we consider some examples of reduced amalgamated free products and their associated conditional expectations.

As in Section 1.4, let $I$ be a set and, for $\iota \in I$, let $A_\iota$ be a unital $C^*$-algebra. We suppose that each $A_\iota$ contains the unital $C^*$-subalgebra $B$ and that we have conditional expectations $\phi_\iota : A_\iota \to B$ with faithful G.N.S. representations. Let

$$(A, \phi) = *_{\iota \in I}(A_\iota, \phi_\iota)$$

be the reduced amalgamated free product $C^*$-algebra.

Consider first the case where $B$ is the complex numbers. Then the conditional expectations are states. It is clear that the state $\phi$ inherits certain properties of the initial states $\phi_\iota$. From the reduced free product construction, it follows that $\phi$ has faithful G.N.S. representation if all the $\phi_\iota$ have faithful G.N.S. representations. It is easy to show that $\phi$ is a trace if every $\phi_\iota$ is a trace. It is also true that $\phi$ is pure precisely when every $\phi_\iota$ is pure (see Section 3.3).

Voiculescu proved that the free product state in the reduced free product of von Neumann algebras is faithful, if the initial states are faithful. It follows that, in the $C^*$-algebra case we are dealing with, $\phi$ is faithful if for all $\iota \in I$ the vacuum vector is cyclic for the commutant of $A_\iota$ (in the G.N.S. representation for $(A_\iota, \phi_\iota)$). This is always true when $\phi_\iota$ is a faithful trace. However, in [18] Dykema has constructed an example where $\phi_\iota$ is faithful, not a trace, and with the vacuum vector not cyclic for the commutant of $A_\iota$.

Despite this, it is shown in [18] that $\phi$ is faithful if all the initial states are faithful. In view of this, it is natural to ask whether the same result is true for the conditional expectations when amalgamating over an arbitrary common $C^*$-subalgebra. This is indeed the case. Sakamoto [56] states this without proof, in the case when the common $C^*$-subalgebra involved has a faithful state. Here we make no assumptions on the common $C^*$-subalgebra over which we are amalgamating.

The methods used in what follows are based on but at the same time generalise those found in [18]. The generalisation provides some clarification of the calculations and definitions of [18]. We use standard reduced amalgamated free product notation, as contained in Section 1.4.

## 2.2 Faithfulness of the free product conditional expectation

In the following, we assume that $A_\iota \neq B$ for every $\iota \in I$, in order to avoid the consideration of trivial cases. As remarked in [18], this is not really a restriction.

Let $n \in \mathbb{N}$ with $n \geq 2$ and take indices $\iota_1 \neq \iota_2 \neq \cdots \neq \iota_n$ in $I$. For $1 \leq j \leq n-1$ let $\zeta_j = \widehat{a}_{\iota_j} \in E^o_{\iota_j}$. Define $V : E_{\iota_n} \to E$ by

$$V = a_{\iota_1} a_{\iota_2} \cdots a_{\iota_{n-1}} | E_{\iota_n}.$$

For $n = 1$ we define $V$ to be simply the canonical embedding of $E_{\iota_n}$ into $E$. This embedding sends $x \oplus \xi_{\iota_n} b \in E^o_{\iota_n} \oplus \xi_{\iota_n} B = E_{\iota_n}$ to $x \oplus \xi b \in E$.

This is equivalent to the definition of $V$ given in [18], when $B = \mathbb{C}$. $V$ is adjointable, with adjoint

$$V^* = P \circ (a_{\iota_1} a_{\iota_2} \cdots a_{\iota_{n-1}})^* : E \to E_{\iota_n}$$

where $P$ is the orthogonal projection onto $E_{\iota_n}$, which is considered as a submodule of $E$ via the canonical embedding defined above.

These maps $V$ will be very useful in what follows. Note that $V$ is not necessarily an isometry (unlike in [18] ), but this does not matter for our purposes. In fact, we have

$$V^*V = \phi(a^*_{\iota_{n-1}} \cdots a^*_{\iota_1} a_{\iota_1} \cdots a_{\iota_{n-1}})1.$$

With $V$ defined as above, we now perform some calculations involving $V$. The results are as follows.

**Lemma 2.2.1.** *Suppose $m \in \mathbb{N}$ and $k_1 \neq k_2 \neq \cdots \neq k_m$, with $d_j \in A^\circ_{k_j}$ for $1 \leq j \leq m$. Then:*

- *If $m = 2p-1$ where $1 \leq p < n$, and $k_m = \iota_1 = k_1$, $k_{m-1} = \iota_2 = k_2$, $\ldots$, $k_{p+1} = \iota_{p-1} = k_{p-1}$, $k_p = \iota_p$ then*

$$V^*d_1 \cdots d_m V = \phi(a^*_{\iota_{n-1}} \cdots a^*_{\iota_{p+1}} a^*_{\iota_p} b d_p b' a_{\iota_p} a_{\iota_{p+1}} \cdots a_{\iota_{n-1}})1$$

  *where $b, b' \in B$ are given by*

$$b = \phi(a^*_{\iota_{p-1}} \phi(a^*_{\iota_{p-2}} (\cdots \phi(a^*_{\iota_1} d_1) \cdots) d_{p-2}) d_{p-1})$$

$$b' = \phi(d_{p+1} \phi(\cdots \phi(d_{m-1} \phi(d_m a_{\iota_1}) a_{\iota_2}) \cdots) a_{\iota_{p-1}})$$

- *If $m = 2n-1$ and $k_m = \iota_1 = k_1$, $k_{m-1} = \iota_2 = k_2$, $\ldots$, $k_{n+1} = \iota_{n-1} = k_{n-1}$, $k_n = \iota_n$ then*

$$V^*d_1 \cdots d_m V = b d_n b'$$

  *where $b, b' \in B$ are given by*

$$b = \phi(a^*_{\iota_{n-1}} \phi(a^*_{\iota_{n-2}} (\cdots \phi(a^*_{\iota_1} d_1) \cdots) d_{n-2}) d_{n-1})$$

$$b' = \phi(d_{n+1} \phi(\cdots \phi(d_{m-1} \phi(d_m a_{\iota_1}) a_{\iota_2}) \cdots) a_{\iota_{n-1}})$$

- *Otherwise*

$$V^*d_1 \cdots d_m V = 0.$$

*Proof.* This is similar to the corresponding proof in [18]. We use induction on $n$.

First consider the case $n = 1$. Here $V$ is just the inclusion of Hilbert $B$-modules $E_{\iota_1} \hookrightarrow E$. If $m = 1$ and $k_m = \iota_1$ then $V^*d_1V = d_1$. If $k_m \neq \iota_1$ and $m \geq 1$ then

$$d_1 \cdots d_m \xi = \widehat{d_1} \otimes \cdots \otimes \widehat{d_m} \quad \perp \xi B \oplus E^\circ_{\iota_1} \tag{1}$$

and

$$d_1 \cdots d_m E^o_{\iota_1} = \hat{d}_1 \otimes \cdots \otimes \hat{d}_m \otimes E^o_{\iota_1} \quad \perp \xi B \oplus E^o_{\iota_1}.$$

So $V^* d_1 \cdots d_m V = 0$.

If $k_m = \iota_1$ and $m \geq 2$ then (1) holds and we find that

$$d_1 \cdots d_m E^o_{\iota_1} \subset (\hat{d}_1 \otimes \cdots \otimes \hat{d}_{m-1}) B + \hat{d}_1 \otimes \cdots \otimes \hat{d}_{m-1} \otimes E^o_{\iota_1} \quad \perp \xi B \oplus E^o_{\iota_1}.$$

So again $V^* d_1 \cdots d_m V = 0$. Hence the statements of the lemma are valid in the case $n = 1$.

Now suppose that $n > 1$. As $V$ is an operator on Hilbert $B$-modules, we can no longer be certain that $(\mathrm{im} V)^\perp = \ker V^*$. However, $(\mathrm{im} V)^\perp \subset \ker V^*$ is still valid, and this is all that we shall need.

If $k_m \neq \iota_1$ then we obtain

$$d_1 \cdots d_m (\zeta_1 \otimes \cdots \otimes \zeta_{n-1}) \quad \perp \mathrm{im} V$$

$$d_1 \cdots d_m (\zeta_1 \otimes \cdots \otimes \zeta_{n-1} \otimes E^o_{\iota_n}) \quad \perp \mathrm{im} V$$

since there cannot be any reduction of words. Hence $V^* d_1 \cdots d_m V = 0$. Taking adjoints implies that the same is true if $k_1 \neq \iota_1$. So let us assume that $k_m = \iota_1 = k_1$. This implies that either $m = 1$ or $m \geq 3$.

If $n = 2$ then we obtain

$$d_1 \cdots d_m (\zeta_1) = \hat{d}_1 \otimes \cdots \otimes \hat{d}_{m-1} \otimes (d_m \zeta_1 - \xi_{\iota_1} \phi(d_m a_{\iota_1}))$$
$$+ d_1 \cdots d_{m-1} \phi(d_m a_{\iota_1}) \xi \tag{2}$$

and $\forall \zeta = \hat{a} \in E^o_{\iota_2}$,

$$d_1 \cdots d_m (\zeta_1 \otimes \zeta) = \hat{d}_1 \otimes \cdots \otimes \hat{d}_{m-1} \otimes (d_m \zeta_1 - \xi_{\iota_1} \phi(d_m a_{\iota_1})) \otimes \zeta$$
$$+ d_1 \cdots d_{m-1} \phi(d_m a_{\iota_1}) a \xi \tag{3}$$

If $n > 2$ we obtain

$$d_1 \cdots d_m (\zeta_1 \otimes \cdots \otimes \zeta_{n-1})$$
$$= \hat{d}_1 \otimes \cdots \otimes \hat{d}_{m-1} \otimes (d_m \zeta_1 - \xi_{\iota_1} \phi(d_m a_{\iota_1})) \otimes \zeta_2 \otimes \cdots \otimes \zeta_{n-1}$$
$$+ d_1 \cdots d_{m-1} \phi(d_m a_{\iota_1}) a_{\iota_2} \cdots a_{\iota_{n-1}} \xi \tag{4}$$

and $\forall \zeta = \hat{a} \in E^o_{\iota_n}$,

$$d_1 \cdots d_m(\zeta_1 \otimes \cdots \otimes \zeta_{n-1} \otimes \zeta)$$

$$= \widehat{d}_1 \otimes \cdots \otimes \widehat{d}_{m-1} \otimes (d_m\zeta_1 - \xi_{\iota_1}\phi(d_m a_{\iota_1})) \otimes \zeta_2 \otimes \cdots \otimes \zeta_{n-1} \otimes \zeta$$

$$+ d_1 \cdots d_{m-1}\phi(d_m a_{\iota_1}) a_{\iota_2} \cdots a_{\iota_{n-1}} a\xi \tag{5}$$

If $m = 1$ then the second term of the right hand sides of equations (2), (3), (4) and (5) is orthogonal to $\mathrm{im}V$. We can now apply $V^*$. For example, applying $V^*$ to equation (5) gives

$$P(a^*_{\iota_{n-1}} \cdots a^*_{\iota_2} a^*_{\iota_1}(d_1 a_{\iota_1} - \phi(d_1 a_{\iota_1})1) a_{\iota_2} \cdots a_{\iota_{n-1}} a\xi)$$

$$= P(a^*_{\iota_{n-1}} \cdots a^*_{\iota_2} a^*_{\iota_1} d_1 a_{\iota_1} a_{\iota_2} \cdots a_{\iota_{n-1}} a\xi) \tag{6}$$

Now, if $x = a^*_{\iota_{n-1}} \cdots a^*_{\iota_1} d_1 a_{\iota_1} \cdots a_{\iota_{n-1}}$ then

$$x - \phi(x)1 \quad \in \ker\phi$$

so

$$(x - \phi(x)1)\xi \quad \perp \xi$$

and hence, $x - \phi(x)1$ is a sum of reduced words, all of which must end in some element of $A^o_{\iota_{n-1}}$. Since $a \in A^o_{\iota_n}$ and $\iota_{n-1} \neq \iota_n$ it follows that

$$P((x - \phi(x)1)a\xi) = 0.$$

Hence (6) becomes

$$P(\phi(a^*_{\iota_{n-1}} \cdots a^*_{\iota_1} d_1 a_{\iota_1} \cdots a_{\iota_{n-1}})a\xi)$$

$$= \phi(a^*_{\iota_{n-1}} \cdots a^*_{\iota_1} d_1 a_{\iota_1} \cdots a_{\iota_{n-1}})a\xi.$$

Very similar things happen on applying $V^*$ to equations (2), (3) and (4), the result being that

$$V^* d_1 V = \phi(a^*_{\iota_{n-1}} \cdots a^*_{\iota_1} d_1 a_{\iota_1} \cdots a_{\iota_{n-1}})1.$$

If $m \geq 3$, the first term in the right hand sides of equations (2), (3), (4) and (5) is orthogonal to $\mathrm{im}V$. Hence we see that

$$V^* d_1 \cdots d_m V = V^* d_1 \cdots d_{m-2} d_{m-1} \phi(d_m a_{\iota_1}) U$$

where $U$ is defined in the same way as $V$ but using only the vectors $\zeta_2, \ldots, \zeta_{n-1}$, not $\zeta_1$.

For a moment, consider the calculation of $U^* d_1 \cdots d_m V$ where $m \geq 2$ and $k_m = \iota_1$. We get the same equations (2), (3), (4) and (5): although there is a possibility that the condition $k_1 = \iota_1$ is not satisfied, this is not relevant to these calculations. We now want to apply $U^*$. Since $m \geq 2$ the first term on the right hand side of each equation is orthogonal to $\operatorname{im} U$. So we obtain

$$U^* d_1 \cdots d_m V = U^* d_1 \cdots d_{m-1} \phi(d_m a_{\iota_1}) U.$$

The above paragraph shows that, letting $\tilde{d}_{m-1} = d_{m-1}\phi(d_m a_{\iota_1})$, we have

$$\begin{aligned} V^* d_1 \cdots d_m V &= V^* d_1 \cdots d_{m-2} \tilde{d}_{m-1} U \\ &= (U^* \tilde{d}_{m-1}^* d_{m-2}^* \cdots d_1^* V)^* \\ &= (U^* \tilde{d}_{m-1}^* d_{m-2}^* \cdots d_2^* \phi(d_1^* a_{\iota_1}) U)^* \\ &= U^* \phi(a_{\iota_1}^* d_1) d_2 \cdots d_{m-2} d_{m-1} \phi(d_m a_{\iota_1}) U. \end{aligned}$$

Now apply the inductive hypothesis to finish the proof. To give an idea of how the elements $b, b' \in B$ from the statement of the Lemma build up, consider the situation where $m \geq 5$ and $n \geq 3$. Then

$$\begin{aligned} V^* d_1 \cdots d_m V &= U^* \phi(a_{\iota_1}^* d_1) d_2 \cdots d_{m-2} d_{m-1} \phi(d_m a_{\iota_1}) U \\ &= T^* \phi(a_{\iota_2}^* \phi(a_{\iota_1}^* d_1) d_2) d_3 \cdots d_{m-2} \phi(d_{m-1} \phi(d_m a_{\iota_1}) a_{\iota_2}) T \end{aligned}$$

where $T$ is defined as for $V$ but using the vectors $\zeta_3, \ldots, \zeta_{n-1}$. □

It is not clear that Lemma 1.3 of [18], which states that $V^*AV = A_{\iota_n}$, holds in this generality. However, it is clear from the proof that we still have $V^*AV \subset A_{\iota_n}$, given the calculations performed in Lemma 2.2.1 above. This allows us to prove the following.

**Theorem 2.2.2.** *Let*

$$(A, \phi) = *_{\iota \in I}(A_\iota, \phi_\iota)$$

*be the reduced amalgamated free product of $C^*$-algebras. Assume that for every $\iota \in I$, $A_\iota \neq B$ and that the conditional expectation $\phi_\iota$ is faithful. Then the free product conditional expectation $\phi$ is faithful on $A$.*

*Proof.* Suppose that $a \in A$, $a \geq 0$, $a \neq 0$ and $\phi(a) = 0$. Then $\langle \xi, a\xi \rangle = 0$. Let $p_{\iota_1, \ldots, \iota_n}$ denote the projection from $E$ onto its direct summand $E_{\iota_1}^o \otimes \cdots \otimes E_{\iota_n}^o$, where of course $\iota_1 \neq \iota_2 \neq \cdots \neq \iota_n$.

Since $a \geq 0$ but $a \neq 0$ it follows that for some $\iota_1, \iota_2, \ldots, \iota_n$ we have

$$p_{\iota_1,\ldots,\iota_n} a p_{\iota_1,\ldots,\iota_n} \neq 0.$$

Let $n$ be the smallest such that this holds. Then we can find

$$z = \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n \in E^o_{\iota_1} \otimes \cdots \otimes E^o_{\iota_n}$$

such that

$$\langle z, p_{\iota_1,\ldots,\iota_n} a p_{\iota_1,\ldots,\iota_n} z \rangle \neq 0.$$

In fact, since $\widehat{A}^o_\iota$ is dense in $E^o_\iota$ for all $\iota$, we can assume $\zeta_j = \widehat{a}_{\iota_j}$ for some $a_{\iota_j} \in A^o_{\iota_j}$, for all $1 \leq j \leq n$. Now define $V$ using the vectors $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$. Then $\langle \zeta_n, V^* a V \zeta_n \rangle \neq 0$, so $V^* a V \neq 0$ but is $\geq 0$. Now we know $V^* a V \in A_{\iota_n}$ so, because $\phi_{\iota_n}$ is faithful, we have

$$\langle \xi_{\iota_n}, V^* a V \xi_{\iota_n} \rangle = \phi_{\iota_n}(V^* a V) \geq 0 \text{ but } \neq 0.$$

If $n = 1$ then

$$\langle \xi_{\iota_n}, V^* a V \xi_{\iota_n} \rangle = \langle \xi, a\xi \rangle = \phi(a) = 0$$

which is a contradiction. If $n > 1$ then

$$\langle \zeta_1 \otimes \cdots \otimes \zeta_{n-1}, a(\zeta_1 \otimes \cdots \otimes \zeta_{n-1}) \rangle = \langle \xi_{\iota_n}, V^* a V \xi_{\iota_n} \rangle \geq 0 \text{ but } \neq 0.$$

It follows that

$$p_{\iota_1,\ldots,\iota_{n-1}} a p_{\iota_1,\ldots,\iota_{n-1}} \neq 0.$$

This contradicts the minimality of $n$. Hence we conclude that $\phi$ must be faithful. □

## 2.3 Some examples

In this section we consider some examples of reduced amalgamated free products and the related $C^*$-algebras.

*Example 2.3.1.* This is the main motivating example for reduced amalgamated free product $C^*$-algebras. We suppose that for every $\iota$ in some index set $I$, we have a discrete group $G_\iota$. Suppose further that these groups all possess a common subgroup $H$. Then we can define the amalgamated free product of groups $G = (*_H)_{\iota \in I} G_\iota$ in the usual way. This is a standard construction in group theory: see, for example, Chapter 4 of Magnus, Karrass and Solitar [48] for details.

Define $A = C_r^*(G) \subset B(\ell^2(G))$ and let $B = \overline{span}\,\lambda(H) \subset A$, where $\lambda$ is the left regular representation of $G$. If we let $A_\iota = \overline{span}\,\lambda(G_\iota) \subset A$, then $B \cong C_r^*(H)$ and $A_\iota \cong C_r^*(G_\iota)$. For $\iota \in I$ we define the conditional expectation $\tau_H^{G_\iota} : A_\iota \to B$ via

$$\tau_H^{G_\iota}(\lambda_g) = \begin{cases} \lambda_g & \text{for } g \in H \\ 0 & \text{for } g \notin H \end{cases} \quad \text{where } g \in G_\iota.$$

We similarly define the conditional expectation $\tau_H^G : A \to B$.

It can easily be shown that

$$(C_r^*(G), \tau_H^G) = *_{\iota \in I}(C_r^*(G_\iota), \tau_H^{G_\iota}).$$

Letting $\tau^G : A \to \mathbb{C}$ and $\tau^H : B \to \mathbb{C}$ denote the canonical faithful traces, we have $\tau^G = \tau^H \circ \tau_H^G$. So in this case the free product conditional expectation is faithful, as are all the initial conditional expectations. This is compatible with the results of Section 2.2.

As a concrete example, suppose we have the two groups $\mathbb{Z}_4 = \langle a;\, a^4 = 1\rangle$ and $\mathbb{Z}_6 = \langle b;\, b^6 = 1\rangle$. Then $\mathbb{Z}_4$ contains $\{1, a^2\} \cong \mathbb{Z}_2$ and $\mathbb{Z}_6$ contains $\{1, b^3\} \cong \mathbb{Z}_2$. Hence we can define the free product $G$ of $\mathbb{Z}_4$ and $\mathbb{Z}_6$, amalgamating over the common subgroup $\mathbb{Z}_2$. Clearly $G = \langle a, b;\, a^4 = 1,\, a^2 = b^3\rangle$. It is well known that in fact $G \cong SL_2(\mathbb{Z})$ where $a$ can be identified with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $b$ can be identified with $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

In terms of $C^*$-algebras we have

$$C_r^*(SL_2(\mathbb{Z})) = C_r^*(\mathbb{Z}_4) *_{C_r^*(\mathbb{Z}_2)} C_r^*(\mathbb{Z}_6).$$

This is significant because, even though $C_r^*(\mathbb{Z}_4)$ and $C_r^*(\mathbb{Z}_6)$ are finite dimensional (commutative even), the reduced amalgamated free product is not even nuclear. We will have more to say about this in Chapter 3.

*Example 2.3.2.* Here we consider an important class of $C^*$-algebras, namely the Cuntz-Pimsner $C^*$-algebras. These will be of use in later chapters, so we review their construction. They were first considered in [52]. We use the same notation as in [24].

First take a unital $C^*$-algebra $B$ and let $H$ be a Hilbert $B$-bimodule. That is, $H$ is a right Hilbert $B$-module with an (injective) $*$-homomorphism $B \to L(H)$. We assume

$H$ is full, so that the image of the $B$-valued inner product map on $H$ generates $B$ as a $C^*$-algebra. Let $\mathcal{F}(H) = B \oplus \bigoplus_{n \geq 1} H^{(\otimes_B)n}$ be the Fock space over $H$, where $H^{(\otimes_B)n}$ refers to the canonical $n$-fold interior tensor product of $H$ with itself. Clearly $\mathcal{F}(H)$ is also a Hilbert $B$-bimodule in a natural way.

For $h \in H$ define the corresponding creation operator $\ell(h) \in L(\mathcal{F}(H))$ by

$$\ell(h) h_1 \otimes \cdots \otimes h_n = h \otimes h_1 \otimes \cdots \otimes h_n, \quad h_1, \ldots, h_n \in H$$

$$\ell(h) b = hb, \quad b \in B$$

These operators are clearly adjointable. Indeed the adjoint of $\ell(h)$ is given by

$$\ell(h)^* h_1 \otimes \cdots \otimes h_n = \langle h, h_1 \rangle \, h_2 \otimes \cdots \otimes h_n, \quad h_1, \ldots, h_n \in H$$

$$\ell(h)^* b = 0, \quad b \in B.$$

Moreover,

$$\ell(h)^* \ell(g) = \langle h, g \rangle, \quad h, g \in H$$

$$b_1 \ell(h) b_2 = \ell(b_1 h b_2) \ \ h \in H, \quad b_1, b_2 \in B.$$

We shall be mostly interested in the extended Cuntz-Pimsner algebra $E(H)$, which is defined to be the $C^*$-algebra generated by $\{\ell(h) : h \in H\}$ in $L(\mathcal{F}(H))$. The fullness of $H$ implies that $E(H)$ contains a canonical copy of $B$. We also have a canonical conditional expectation $\mathcal{E} : E(H) \to B$ given by

$$\mathcal{E}(x) = \langle 1_B, x 1_B \rangle \quad x \in E(H).$$

The Cuntz-Pimsner $C^*$algebra $\mathcal{O}(H)$ is defined to be the image of $E(H)$ in the quotient $L(\mathcal{F}(H))/J$ where the ideal $J$ is the $C^*$-algebra generated by $\{L(\oplus_{n=0}^{k} H^{(\otimes_B)n}) : k \in \mathbb{N}\}$. It turns out that $J$ is the ideal $K(\mathcal{F}(H))$ of compact operators if $H$ is finitely generated as a right $B$-module. Taking $B = \mathbb{C}$ and $H$ to be a finite-dimensional Hilbert space, $\mathcal{O}(H)$ becomes one of the Cuntz algebras introduced in [14]. Taking $B$ to be finite-dimensional and commutative, with $H$ finitely generated, $\mathcal{O}(H)$ becomes one of the Cuntz-Krieger algebras considered in [15].

In [58] Speicher has shown that, if we take two Hilbert $B$-bimodules $H_1, H_2$, then

$$(E(H_1 \oplus H_2), \mathcal{E}_{H_1 \oplus H_2}) \cong (E(H_1), \mathcal{E}_{H_1}) * (E(H_2), \mathcal{E}_{H_2})$$

where the conditional expectations are the canonical ones mentioned above. In contrast to our first example of an amalgamated free product, none of the conditional expectations involved here are faithful. Indeed, for $h \in H$ we have $\mathcal{E}(\ell(h)\ell(h)^*) = 0$. Looking at the case where $B = \mathbb{C}$ we find that these conditional expectations become pure states. Note that pureness and faithfulness are mutually disjoint properties for states defined on a $C^*$-algebra of dimension $\geq 2$: this follows easily from Theorem 5.3.4 of Murphy [49], which states that a state $\tau$ on a $C^*$-algebra $A$ is pure if and only if

$$\ker\tau = N_\tau + N_\tau^*$$

where $N_\tau = \{a \in A : \tau(a^*a) = 0\}$.

We shall have more to say about the Cuntz-Pimsner algebras in Chapter 3.

# Chapter 3

# Nuclearity and other properties of reduced free product $C^*$-algebras

In this chapter we consider certain classes of reduced free product $C^*$-algebras. We are particularly interested in the nuclearity of these reduced free products, but we also consider other properties.

Section 3.1 looks at reduced free products of $C^*$-algebras with orthounitary bases, using faithful traces. Most such free products turn out to be non-nuclear (although simple). Section 3.2 builds on these results. We consider the ideals in a certain tensor product $C \otimes_\nu C^{op}$ of the reduced free product $C$ with its opposite $C^*$-algebra $C^{op}$. The methods used are based on ideas in a paper of Akemann and Ostrand [1].

Finally, in Section 3.3 we look at reduced free products of nuclear $C^*$-algebras, using pure states. Many of these reduced free products are found to be nuclear (and not simple). There is a close connection here with the results on Cuntz-Pimsner $C^*$-algebras contained in [24]. The results of this section contrast sharply with those of Section 3.1.

## 3.1 $C^*$-algebras with orthounitary bases

### 3.1.1 Introduction

This section considers the reduced free product of $C^*$-algebras with orthounitary bases, using faithful traces. Recall that Takesaki [60] gave the first example of a non-nuclear $C^*$-algebra, namely $C_r^*(\mathbb{F}_2)$. Subsequently, Choi [12] gave an example of a non-nuclear $C^*$-algebra embedded into a nuclear $C^*$-algebra. Indeed, he was able to show that $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$

is not nuclear and yet it can be embedded into the Cuntz algebra $\mathcal{O}_2$.

Later, Wassermann [65] gave an explicit proof of the non-nuclearity of $C_r^*(\mathbb{F}_2) = C_r^*(\mathbb{Z}) *_r C_r^*(\mathbb{Z})$. Here we show that many other reduced free product $C^*$-algebras are in fact non-nuclear. The proofs are based on the methods in [65]. We construct a $C^*$-norm $\nu$ on $C \odot C^{op}$, where $C$ is the reduced free product $C^*$-algebra, such that $\nu$ differs from the spatial norm. Clearly, this implies non-nuclearity of the reduced free product in question.

Our results always assume the existence of orthounitary bases, a concept first introduced by Ching in [11]. Note that Dykema's results on various reduced free products [17] can also be used to show that certain $C^*$-algebra free products (including many that we consider here) are not nuclear. However, our approach here is quite different and perhaps more elementary.

### 3.1.2 The free product orthounitary basis

Consider unital $C^*$-algebras $A_1, A_2$ with faithful traces $\tau_1, \tau_2$ respectively. We take the reduced free product $(C, \tau) = (A_1, \tau_1) * (A_2, \tau_2)$ as defined in Section 1.4. Our main purpose at the moment is to show that, if $A_1$ and $A_2$ have orthounitary bases, then the reduced free product $C$ also has an orthounitary basis.

For $i = 1, 2$ we let $A_i$ act on the G.N.S. Hilbert space $H_i$, via the representation $\pi_i$ and with vacuum vector $\xi_i$. If $a \in A_i$, we write $\widehat{a}$ for the corresponding element of $H_i$. Let the free product Hilbert space be $(H, \xi) = (H_1, \xi_1) * (H_2, \xi_2)$. The trace is defined by

$$\tau(c) = \langle c\xi, \xi \rangle \quad c \in C.$$

$\tau_1$ and $\tau_2$ are faithful traces, so $\tau$ is also a faithful trace.

We make the following definition, which is adapted slightly from that found in [11]. Let $\mathcal{U}(A)$ denote the unitary group of the unital $C^*$-algebra $A$.

**Definition 3.1.1.** *Let $A$ be a unital $C^*$-algebra with faithful trace. We suppose $A$ to be acting on the associated G.N.S. Hilbert space with vacuum vector $\xi$. We say that $\mathcal{O} = (u_\alpha)_{\alpha \in I}$ is an* ***orthounitary basis*** *for $A$ if $\mathcal{O} \subset \mathcal{U}(A)$ and*

1. *$\langle u_{\alpha_1}\xi, u_{\alpha_2}\xi \rangle = 0$ if $\alpha_1, \alpha_2 \in I$ and $\alpha_1 \neq \alpha_2$,*

2. *$u_\alpha^* = c(\alpha) u_{\nu(\alpha)}$ for some $u_{\nu(\alpha)} \in \mathcal{O}$ and $c(\alpha) \in \mathbb{T}$, where $\mathbb{T}$ is the unit circle,*

3. *$u_{\alpha_1} u_{\alpha_2} = c(\alpha_1, \alpha_2) u_{\mu(\alpha_1, \alpha_2)}$ for some $u_{\mu(\alpha_1, \alpha_2)} \in \mathcal{O}$, some $c(\alpha_1, \alpha_2) \in \mathbb{T}$,*

4. $\overline{span}\, \mathcal{O} = A$.

Note that some related objects, *standard orthonormal bases*, are used extensively in [21]. These are similar to orthounitary bases, except that we only insist on the first and fourth conditions to be satisfied (and the condition $\mathcal{O} \subset \mathcal{U}(A)$ is also removed).

The question of existence of an orthounitary basis seems to be quite difficult. Zorn's Lemma always gives a maximal (with respect to inclusion) subset of $\mathcal{U}(A)$ satisfying the first three conditions of the definition, but such a subset need not satisfy the fourth condition. In fact, $\mathbb{C} \oplus \mathbb{C}$ with a non-canonical trace does not even have any zero-trace unitaries (this example is mentioned in [4]). However, $C_r^*(G)$, with the canonical faithful trace, clearly has an orthounitary basis when $G$ is discrete. This is given by the set of unitaries obtained from the left regular representation of $G$. We also have the following example.

*Example 3.1.1.* Consider $M_n(\mathbb{C})$ with the canonical trace. Let $w \in M_n(\mathbb{C})$ be the matrix with

$$w_{n,1} = w_{1,2} = w_{2,3} = \cdots = w_{n-1,n} = 1$$

and all other entries zero. Let $v \in M_n(\mathbb{C})$ be the matrix with

$$v_{j,j} = e^{2\pi i(j-1)/n} \quad (1 \leq j \leq n)$$

and all other entries zero. Clearly $v$ and $w$ are both unitaries. Let $\mathcal{O} = \{w^k v^j \ : 1 \leq j, k \leq n\}$.

It can easily be checked that these unitaries are pairwise orthogonal (with respect to the trace). Using the relation $vw = e^{2\pi(n-1)i/n} wv$, it can be checked that $\mathcal{O}$ is actually an orthounitary basis for $M_n(\mathbb{C})$ (see [11]).

Now, suppose that $A_1$ has orthounitary basis $\mathcal{O}_1 = (u_\alpha)_{\alpha \in I}$ and $A_2$ has orthounitary basis $\mathcal{O}_2 = (v_\beta)_{\beta \in J}$. We may as well suppose that each orthounitary basis contains 1. This is because the fourth condition for an orthounitary basis implies that it is non-empty. The second and third conditions then imply that there is an element in the orthounitary basis of the form $\lambda 1$ for some $\lambda \in \mathbb{T}$. The first condition implies that there is a unique such element. Multiplying every element of the orthounitary basis by $\overline{\lambda}$ gives an orthounitary basis containing 1. So, fix $\alpha_0 \in I$ such that $u_{\alpha_0} = 1$ and fix $\beta_0 \in J$ such that $v_{\beta_0} = 1$. We wish to construct an orthounitary basis for the free product $C$, following the same approach as in Ching's paper [11].

Define

$$\mathcal{O} = \bigcup_{n \geq 1} \{u_{\alpha_1} v_{\beta_1} \cdots u_{\alpha_n} v_{\beta_n} : \alpha_i \neq \alpha_0 \text{ for } i > 1, \beta_i \neq \beta_0 \text{ for } i < n\}.$$

**Lemma 3.1.2.** *$\mathcal{O}$ is an orthounitary basis for the reduced free product $C$.*

*Proof.* The second and third conditions are easily seen to be satisfied since, for example,

$$(u_{\alpha_1} v_{\beta_1} \cdots u_{\alpha_n} v_{\beta_n})^* = 1 v_{\beta_n}^* u_{\alpha_n}^* v_{\beta_{n-1}}^* \cdots u_{\alpha_1}^* 1,$$

which is again of the required form once any unnecessary 1's have been eliminated.

For the first condition, consider two distinct elements of $\mathcal{O}$,

$$u_1 = u_{\alpha_1} v_{\beta_1} \cdots u_{\alpha_n} v_{\beta_n},$$

$$u_2 = u_{\alpha'_1} v_{\beta'_1} \cdots u_{\alpha'_m} v_{\beta'_m}.$$

We suppose that $m \geq n$. Clearly

$$\langle u_1 \xi, u_2 \xi \rangle = \langle v_{\beta'_m}^* u_{\alpha'_m}^* \cdots v_{\beta'_1}^* (u_{\alpha'_1}^* u_{\alpha_1}) v_{\beta_1} \cdots u_{\alpha_n} v_{\beta_n} \xi, \xi \rangle.$$

Consider first the pair $\alpha_1, \alpha'_1$. If $\alpha_1 = \alpha'_1$ then $u_{\alpha'_1}^* u_{\alpha_1} = 1$ so we move on to the next pair of indices. If $\alpha_1 \neq \alpha'_1$ then, since $\mathcal{O}_1$ is an orthounitary basis,

$$\tau(u_{\alpha'_1}^* u_{\alpha_1}) = \langle u_{\alpha'_1} \xi, u_{\alpha_1} \xi \rangle = 0.$$

Assuming also $v_{\beta_n} \neq 1, v_{\beta'_m} \neq 1$, we know that

$$\tau(v_{\beta'_m}^*) = \tau(u_{\alpha'_m}^*) = \cdots = \tau(v_{\beta'_1}^*) = \tau(v_{\beta_1}) = \cdots = \tau(u_{\alpha_n}) = \tau(v_{\beta_n}) = 0.$$

So, by freeness,

$$\tau(v_{\beta'_m}^* u_{\alpha'_m}^* \cdots v_{\beta'_1}^* (u_{\alpha'_1}^* u_{\alpha_1}) v_{\beta_1} \cdots u_{\alpha_n} v_{\beta_n}) = 0.$$

Hence $\langle u_1 \xi, u_2 \xi \rangle = 0$. Very similar things happen if $v_{\beta_n} = 1$ or $v_{\beta'_m} = 1$. Now, either we meet a pair of indices which are not the same (and then the above shows that $\langle u_1 \xi, u_2 \xi \rangle = 0$) or (if all of $u_1$ is 'used up') we obtain $\langle u_1 \xi, u_2 \xi \rangle = \langle u_3^* \xi, \xi \rangle$ where $u_3$ is an end-portion of $u_2$. A similar calculation to the one above then gives $\langle u_3^* \xi, \xi \rangle = 0$. So the first condition is satisfied.

Finally, note that the span of $\mathcal{O}$ contains the span of all products from $\mathcal{O}_1$ and $\mathcal{O}_2$. It follows that the fourth condition is satisfied. Hence $\mathcal{O}$ is an orthounitary basis for the free product $C$. $\square$

### 3.1.3 Non-nuclearity of the free product

Here we show that the free products we have been considering are not nuclear. In order to do this, we must first consider the G.N.S. representations of the free product $C$ and its opposite $C^*$-algebra $C^{op}$. There is an injection

$$C \hookrightarrow H \quad ; \quad c \longmapsto c\xi$$

with dense image, so we may consider $H$ to be the G.N.S. Hilbert space associated with $(C, \tau)$ and $\xi$ to be the associated vacuum vector.

We have the G.N.S. representation of $C$, given by

$$\lambda : C \longrightarrow B(H) \quad ; \quad \lambda(c)(a\xi) = ca\xi \quad \text{for} \quad a \in C.$$

In other words, $\lambda = \mathrm{id}$. There is also the G.N.S. representation of $C^{op}$,

$$\rho : C^{op} \longrightarrow B(H) \quad ; \quad \rho(c)(a\xi) = ac\xi \quad \text{for} \quad a \in C$$

where $C^{op}$ is the same $C^*$-algebra as $C$ but with the multiplication reversed. As usual, $[\lambda(C), \rho(C^{op})] = 0$.

Write the orthounitary basis $\mathcal{O} = (e_\alpha)_{\alpha \in S}$ for some indexing set $S$. Since $\overline{C\xi} = H$, $(e_\alpha\xi)_{\alpha \in S}$ is an orthonormal basis for $H$. So, if $c \in C$ and we define $c_\alpha = \langle c\xi, e_\alpha\xi \rangle$ then $c\xi = \sum_{\alpha \in S} c_\alpha e_\alpha \xi$. Applying $\rho(a)$ to both sides of this equation for $a \in C$ shows that $\sum_{\alpha \in S} c_\alpha e_\alpha$ converges strongly to $c$ on the dense subspace $C\xi$ of $H$. In fact, this is valid for any $c$ in the double commutant of $C$. This enables us to use the proof of Theorem 1 from [11], something we do in Proposition 3.1.4.

**Lemma 3.1.3.** *The mapping $\sum x_i \otimes y_i \longmapsto \sum \lambda(x_i)\rho(y_i)$ is an embedding of $C \odot C^{op}$ into $B(H)$ if $dim A_1 \geq 2$ and $dim A_2 \geq 3$.*

*Proof.* To show that the mapping is injective, it suffices to show that $C \odot C^{op}$ is simple. A standard algebraic result (see, for example, Theorem 4.1.1 of [37]) states that this is the case so long as $C$ itself is simple. The dimension assumptions and the existence of orthounitary bases imply that the Avitzour conditions are satisfied, so by [4] $C$ is simple. Hence the mapping is indeed injective. □

From now on we assume that the dimension conditions in the above Lemma are in force. The embedding allows us to define a $C^*$-norm $\nu$ on $C \odot C^{op}$. Let the resulting tensor product $C^*$-algebra be denoted $C \otimes_\nu C^{op}$. We wish to show that $\nu$ is not the spatial norm, which is done as in [65] by making use of the '14$\epsilon$ lemma'. First, recall the following from [11].

The indexing set $S$ for $\mathcal{O}$ can be made into a group, multiplication being given by $\alpha_1\alpha_2 = \mu(\alpha_1, \alpha_2)$ where $\mu(\alpha_1, \alpha_2)$ is such that $e_{\alpha_1}e_{\alpha_2} = c(\alpha_1, \alpha_2)e_{\mu(\alpha_1,\alpha_2)}$ for some $c(\alpha_1, \alpha_2) \in \mathbb{T}$. The identity is $\iota$, where $e_\iota = 1$.

Define $F \subset S$ to be the set of $\alpha$ such that $e_\alpha$ ends in a non-trivial $v_\beta$ (recall the definition of $\mathcal{O}$). Choose non-trivial $v_\beta \in \mathcal{O}_2$ and let $e_{r_0} = 1v_\beta$. Then $F \cup r_0Fr_0^{-1} = S\backslash\{\iota\}$. Next, choose distinct non-trivial $u_{\alpha_1}, u_{\alpha_2} \in \mathcal{O}_1$ and let $e_{r_1} = u_{\alpha_1}1, \quad e_{r_2} = u_{\alpha_2}1$. Then $F, r_1Fr_1^{-1}, r_2Fr_2^{-1}$ are pairwise disjoint subsets of $S\backslash\{\iota\}$. We are now ready to prove the following.

**Proposition 3.1.4.** *$C\otimes_\nu C^{op} \subset B(H)$ contains the compact operators $K(H)$ if $dimA_1 \geq 2$ and $dimA_2 \geq 3$.*

*Proof.* We first show that $C \otimes_\nu C^{op}$ contains a rank 1 projection, obtained by applying the continuous functional calculus to $x \in C \otimes_\nu C^{op}$ given by

$$x = \frac{1}{8}(\lambda(e_{r_0})\rho(e^*_{r_0}) + \lambda(e^*_{r_0})\rho(e_{r_0}) + 2\lambda(e_{r_1})\rho(e^*_{r_1})$$

$$+2\lambda(e^*_{r_1})\rho(e_{r_1}) + \lambda(e_{r_2})\rho(e^*_{r_2}) + \lambda(e^*_{r_2})\rho(e_{r_2})).$$

In the same way as in Lemma 2 of [65], we take $\zeta \in (\mathbb{C}\xi)^\perp$ of norm 1 and define $K = ||\zeta - x\zeta||$. We hope to get a numerical lower bound on $K$. If

$$x_1 = \frac{1}{2}(\lambda(e_{r_0})\rho(e^*_{r_0}) + \lambda(e^*_{r_0})\rho(e_{r_0}))$$

$$x_2 = \frac{1}{2}(\lambda(e_{r_1})\rho(e^*_{r_1}) + \lambda(e^*_{r_1})\rho(e_{r_1}))$$

$$x_3 = \frac{1}{2}(\lambda(e_{r_2})\rho(e^*_{r_2}) + \lambda(e^*_{r_2})\rho(e_{r_2}))$$

then $x = \frac{1}{4}(x_1 + 2x_2 + x_3)$ so

$$||\zeta - \frac{1}{4}(x_1 + 2x_2 + x_3)\zeta|| = K.$$

Applying Lemma 1 of [65] twice gives, for $i = 1, 2, 3$

$$||x_i\zeta - \zeta|| \leq 3K^{1/4}.$$

Applying Lemma 1 again gives

$$||\lambda(e_{r_i})\rho(e^*_{r_i})\zeta - \zeta|| \leq 4K^{1/8}$$

for $i = 0, 1, 2$.

Now we wish to apply the '$14\epsilon$ lemma', stated as Lemma 4 of [11], in order to obtain an upper bound on $\|\zeta\|$. Decompose $\zeta$ with respect to the orthonormal basis $(e_\alpha\xi)_{\alpha\in S}$:

$$\zeta = \sum_{\alpha\in S} f(\alpha)e_\alpha\xi.$$

Then $f \in \ell^2(S)$, and since $\zeta \in (\mathbb{C}\xi)^\perp$, it follows that $f(\alpha_0) = 0$, where $\alpha_0$ is such that $e_{\alpha_0} = 1$.

We have

$$\begin{aligned}
4K^{1/8} &\geq \|\zeta - \lambda(e_{r_0})\rho(e_{r_0}^*)\zeta\| \\
&= \left\|\sum_{\alpha\in S} f(\alpha)(e_\alpha\xi - e_{r_0}e_\alpha e_{r_0}^*\xi)\right\| \\
&= \left\|\sum_{\alpha\in S} f(\alpha)e_\alpha\xi - \sum_{\alpha\in S} f(\alpha)c_\alpha e_{r_0\alpha r_0^{-1}}\xi\right\| \\
&= \left\|\sum_{\beta\in S} f(r_0\beta r_0^{-1})e_{r_0\beta r_0^{-1}}\xi - \sum_{\alpha\in S} f(\alpha)c_\alpha e_{r_0\alpha r_0^{-1}}\xi\right\| \\
&= \left(\sum_{\alpha\in S} |f(r_0\alpha r_0^{-1}) - c_\alpha f(\alpha)|^2\right)^{1/2}.
\end{aligned}$$

Here $c_\alpha \in \mathbb{T}$ is defined by $e_{r_0}e_\alpha e_{r_0}^* = c_\alpha e_{r_0\alpha r_0^{-1}}$.

Similar calculations work for $e_{r_1}$ and $e_{r_2}$, so Lemma 4 of [11] now gives $\|\zeta\| \leq 56K^{1/8}$. As $\zeta$ is of norm 1, it follows that $K \geq 56^{-8}$.

As in Lemma 2 of [65], this shows that $x = x^*$ is a contraction with $x|\mathbb{C}\xi = 1$ and $\sigma(x|(\mathbb{C}\xi)^\perp) \subset [-1, 1 - 56^{-8}]$. If $f : [-1,1] \longrightarrow [0,1]$ is a continuous function with $f(1) = 1$ and $f(t) = 0$ for $t \leq 1 - 56^{-8}$, then $f(x)$ is the rank 1 projection onto $\mathbb{C}\xi$. So $C \otimes_\nu C^{op}$ contains a rank 1 projection.

Finally we must show that $C \odot C^{op}$ acts irreducibly on $H$. Suppose $E$ is a closed invariant subspace of $H$ and let $p$ be the projection onto it. Then $p \in (C \odot C^{op})'$ so

$$p \in \lambda(C)' \cap \rho(C^{op})' = \lambda(C)' \cap \lambda(C)'' = \mathcal{Z}(\lambda(C)'').$$

But the proof of Theorem 1 in [11] shows that $\mathcal{Z}(\lambda(C)'')$ is trivial, so $p = 0$ or $p = 1$. Hence $C \odot C^{op}$ acts irreducibly on $H$ and $C \otimes_\nu C^{op}$ contains the compact operators. □

**Corollary 3.1.5.** *If $dim A_1 \geq 2$ and $dim A_2 \geq 3$ then the reduced free product $C$ is not nuclear.*

*Proof.* $C$ is simple so $C \otimes C^{op}$ is also simple. But $C \otimes_\nu C^{op}$ contains the compact operators and hence cannot be simple. So $\nu$ cannot be the spatial norm. $\square$

Note that Lemma 1.1 of [10] shows that there is a conditional expectation from the free product $C$ onto each of $A_1$ and $A_2$. So nuclearity of $C$ implies nuclearity of both $A_1$ and $A_2$. Hence the above Corollary is only really interesting when $A_1$ and $A_2$ are nuclear. It also means that there is no interest in extending the statement of the Corollary to reduced free products of more than two $C^*$-algebras.

Recall from [20] that reduced free products of exact $C^*$-algebras are exact. So, in the case where $A_1$ and $A_2$ are nuclear, it is always the case that $C$ is exact.

Finally, note that Theorem 3.8 of [21] implies, in particular, that all the reduced free products we consider here are of stable rank 1.

Which $C^*$-algebras can we take for $A_1$ and $A_2$ ? We have assumed that $A_1$ and $A_2$ have faithful traces, orthounitary bases, and have $\dim A_1 \geq 2$, $\dim A_2 \geq 3$. So either $A_1$ or $A_2$ can be $M_n(\mathbb{C})$ $(n \geq 2)$ or $C_r^*(G)$ for some discrete amenable group $G$ with at least 3 elements. Another example is the following.

*Example 3.1.2.* Consider the irrational rotation algebra $A_\theta$, where $\theta$ is an irrational number between 0 and 1. $A_\theta$ is the universal $C^*$-algebra generated by a pair of unitaries $u, v$ satisfying $vu = e^{2\pi i\theta}uv$. $A_\theta$ turns out to be simple and nuclear. These algebras were extensively studied by Rieffel [55].

Given $(\lambda, \mu) \in \mathbb{T}^2$ there is an automorphism $\alpha_{\lambda,\mu}$ of $A_\theta$ sending $u$ to $\lambda u$ and $v$ to $\mu v$. This enables us to define a (unique) faithful trace $\tau$ on $A_\theta$ via

$$\tau(x)1 = \iint_{\mathbb{T}^2} \alpha_{\lambda,\mu}(x)\, d\lambda d\mu \quad x \in A_\theta.$$

With respect to this trace, the elements of $\mathcal{O} = \{u^i v^j : i, j \in \mathbb{Z}\}$ are pairwise orthogonal. It then easily follows that $\mathcal{O}$ is an orthounitary basis for $A_\theta$. Hence, either $A_1$ or $A_2$ can be an irrational rotation algebra in Corollary 3.1.5 above.

The following example is also of some interest.

*Example 3.1.3.* Let $M$ be a UHF algebra. Then $M = \otimes_{i=1}^{\infty} M_{s(i)}$ for some positive integers $s(i)$. Let $\mathcal{O}_i$ be the usual orthounitary basis for $M_{s!(i)}$ (see Example 3.1.1), where

$$s!(i) = s(1)s(2)\cdots s(i).$$

We have $\mathcal{O}_i \subset \mathcal{O}_{i+1}$ for every $i$, so let $\mathcal{O} = \cup_{i=1}^{\infty} \mathcal{O}_i$. It is easy to see that $\mathcal{O}$ is an orthounitary basis for $M$, with respect to the usual faithful trace. Hence, either $A_1$ or $A_2$ can be a UHF algebra in Corollary 3.1.5. Reduced free products of hyperfinite von Neumann algebras were considered by Dykema: see Theorem 4.6 of [17].

## 3.2 The ideals of $C \otimes_\nu C^{op}$

### 3.2.1 Introduction

This section is closely related to Section 3.1. Here we continue to look at the $C^*$-algebra $C \otimes_\nu C^{op}$ defined in the previous section. Proposition 3.1.4 showed that this $C^*$-algebra contains the compact operators as an ideal, when $\dim A_1 \geq 2$ and $\dim A_2 \geq 3$. We show that $K(H)$ is actually the unique ideal of $C \otimes_\nu C^{op}$, generalising a result of Akemann and Ostrand in [1]. However, it seems necessary to assume that the underlying groups of the orthounitary bases for $A_1$ and $A_2$ are either finite or free.

First let us review the case where $C = C_r^*(\mathbb{F}_2) = C_r^*(\mathbb{Z}) *_r C_r^*(\mathbb{Z})$. Theorem 1 of [1] shows that in this case the compact operators are contained in $C \otimes_\nu C^{op}$. The proof of this uses the results of the extensive calculations done in [2]. However, Wassermann [65] showed how to obtain this result without referring to the calculations of [2], and we have used similar methods to prove the generalisations of Section 3.1.

Theorem 3 of [1] states that the compact operators are the only non-trivial ideal of $C \otimes_\nu C^{op}$ when $C = C_r^*(\mathbb{F}_2)$. The proof makes no use of the calculations of [2] or the proof of Theorem 1, and as mentioned above, the result can be generalised to certain other situations where $C$ is not $C_r^*(\mathbb{F}_2)$.

### 3.2.2 The results

Let $A_1$ and $A_2$ satisfy the conditions required in Section 3.1, so that the compact operators is an ideal of $C \otimes_\nu C^{op}$. We consider the case where $A_1$ and $A_2$ are finite dimensional, which is equivalent to asking that the orthounitary bases of $A_1$ and $A_2$ are finite. We now use similar methods to those in [1] to show that the compact operators form the unique ideal of $C \otimes_\nu C^{op}$.

Let $\theta$ be the embedding of Lemma 3.1.3. Consider the inverse mapping $\phi$. This is a $*$-isomorphism from a dense $*$-subalgebra of $C \otimes_\nu C^{op}$ onto a dense $*$-subalgebra of $C \otimes C^{op}$, with norm 1. Hence $\phi$ extends to a surjective $*$-homomorphism $\phi : C \otimes_\nu C^{op} \to C \otimes C^{op}$, where $C \otimes_\nu C^{op}$ is considered as a $C^*$-subalgebra of $B(H)$. The following lemma is required.

**Lemma 3.2.1.** *The kernel of $\phi$ is precisely the compact operators $K(H)$.*

*Proof.* First show that $K(H) \subset \ker\phi$. We know that $\ker\phi \cap K(H)$ must be 0 or $K(H)$ since $K(H)$ is simple. If the intersection is $K(H)$ then we are done. If it is zero, then $\phi(K(H))$ is an ideal of $C \otimes C^{op}$, which is simple as remarked in Section 3.1. So either $\phi(K(H)) = 0$, which is what we want, or $\phi(K(H)) = C \otimes C^{op}$. This last equality leads to a contradiction, since $\phi(K(H))$ is non-unital, but $C \otimes C^{op}$ is unital. So the conclusion must be that $K(H) \subset \ker\phi$.

Next we show that $\ker\phi \subset K(H)$. Suppose $a \in \ker\phi$ and take $\epsilon > 0$. Choose $b = \sum_{i=1}^{n} b_i \lambda(x_i)\rho(y_i)$ such that $\|a - b\| < \epsilon$. Here $x_i, y_i \in \mathcal{O}$, the orthounitary basis for $C$, whilst $b_i \in \mathbb{C}$. Since $\phi(a) = 0$ and $\|\phi\| = 1$ we have $\|\phi(b)\| < \epsilon$. We will find a compact operator $c$ with $\|b - c\| < \sqrt{2}\|\phi(b)\| < \sqrt{2}\epsilon$, in order to show that $a$ is compact.

In order to define $c$, we need to consider a length function $\ell$ defined on the orthounitary basis for $C$. This function is defined in the obvious way, with $\ell(u_{\alpha_1} v_{\beta_1} \cdots u_{\alpha_n} v_{\beta_n}) = 2n$, $\ell(u_{\alpha_1}) = 1$, and so on. This is just the 'block length' for reduced words in the free product $C$. Define $S_i = \{x \in \mathcal{O} : \ell(x) < i\}$ and $T_i = \mathcal{O} - S_i$.

Let $p$ be the maximum length of all elements of $\mathcal{O}$ appearing in the expression for $b$. Let $q$ denote projection from $H$ onto $(\mathrm{span} S_{6p})\xi$. Since $A_1$ and $A_2$ are finite dimensional, $q$ is finite rank. So $c = bq$ is compact. Also,

$$\|b - c\| = \sup\{\|bx\|_2 : x \in \mathbb{C}T_{6p}\xi \text{ with } \|x\|_2 = 1\}.$$

Fix $x = \sum_{j=1}^{t} \sigma_j w_j \xi$ in $\mathbb{C}T_{6p}\xi$, with the $w_j$ distinct.

For $z \in \mathcal{O}$, let

$$I(z) = \{(i,j) : 1 \le i \le n, 1 \le j \le t \text{ and } \exists c_{ij} \in \mathbb{T} \text{ such that } x_i w_j y_i = c_{ij} z\}.$$

Then let $H = \{z \in \mathcal{O} : I(z) \neq \emptyset\}$. Clearly $H$ is a finite set. For $z \in H$ we let $\mu_z = \sum_{(i,j) \in I(z)} b_i \sigma_j c_{ij}$. Then

$$bx = \sum_{i,j} b_i \sigma_j x_i w_j y_i \xi = \sum_{z \in H} \mu_z z \xi.$$

Hence $\|bx\|_2 = (\sum_{z\in H} |\mu_z|^2)^{1/2}$.

We claim that there is $\zeta$ in the algebraic tensor product $\mathbb{C}\mathcal{O}\xi \otimes \mathbb{C}\mathcal{O}\xi$ with $\ell^2$-norm 1 and such that $\|bx\|_2 \leq \sqrt{2}\|\phi(b)\zeta\|_2$. Clearly, if this is the case, then the Lemma will be proved.

To define $\zeta$, we need to define certain truncation functions. Let $h = h_1 h_2 \cdots h_m \in \mathcal{O}$, so that each $h_i$, for $1 \leq i \leq m$, is either in $\mathcal{O}_1$ or $\mathcal{O}_2$ (the orthounitary bases for $A_1$ and $A_2$). We define, for $1 \leq i \leq m$, $f_i(h) = h_1 h_2 \cdots h_i$. We let $f_0(h) = 1$. Define also $g_i(h) = f_i(h)^{-1}h$.

Decompose $\mathbb{C}\mathcal{O}\xi \otimes \mathbb{C}\mathcal{O}\xi$ into orthogonal subspaces $K_z$ for $z \in \mathcal{O}$, where $K_z$ has orthonormal basis

$$\{u\xi \otimes v\xi : uvz^{-1} \in \mathbb{T}1\}.$$

For $j \leq t$ define $\zeta_j \in K_{w_j}$ by

$$\zeta_j = \frac{1}{\sqrt{4p}} \sum_{k=p}^{5p-1} f_k(w_j)\xi \otimes g_k(w_j)\xi.$$

Then let $\zeta = \sum_{j=1}^t \sigma_j \zeta_j$. Since $\|\zeta_j\|_2 = 1$ for all $j$, and the subspaces $K_{w_j}$ are orthogonal, we have $\|\zeta\|_2 = (\sum_{j=1}^t |\sigma_j|^2)^{1/2} = 1$.

Let $z \in H$ and $(i,j) \in I(z)$. Then

$$(x_i \otimes y_i)\zeta_j = \frac{1}{\sqrt{4p}} \sum_{k=p}^{5p-1} x_i f_k(w_j)\xi \otimes g_k(w_j) y_i \xi.$$

Now $x_i w_j y_i = c_{ij} z$ and $\ell(w_j) \geq 6p$. So for $p \leq k \leq 5p-1$, $x_i f_k(w_j)$ is, up to an element of $\mathbb{T}$, an 'initial portion' of $z$. Moreover, it is clear that the element of $\mathbb{T}$ is independent of $k$, as is the amount of cancellation between $x_i$ and $f_k(w_j)$. So $x_i f_k(w_j) = s_{ij} f_{k+r(i,j)}(z)$ for $p \leq k \leq 5p-1$, where $|r(i,j)| \leq p$ and $s_{ij} \in \mathbb{T}$.

We also obtain

$$\begin{aligned} g_k(w_j) y_i &= f_k(w_j)^{-1} w_j y_i \\ &= f_k(w_j)^{-1} x_i^{-1} c_{ij} z \\ &= \overline{s_{ij}} f_{k+r(i,j)}(z)^{-1} c_{ij} z \\ &= c_{ij} \overline{s_{ij}} g_{k+r(i,j)}(z) \end{aligned}$$

Therefore

$$(x_i \otimes y_i)\zeta_j = \frac{c_{ij}}{\sqrt{4p}} \sum_{k=p+r(i,j)}^{5p-1+r(i,j)} f_k(z)\xi \otimes g_k(z)\xi$$

and this is an element of $K_z$ for every $(i,j) \in I(z)$.

Now let $Q_z$ denote the orthogonal projection from $K_z$ onto the subspace spanned by $\{f_k(z)\xi \otimes g_k(z)\xi : 2p \leq k \leq 4p-1\}$. Define

$$\Delta_z = \frac{1}{\sqrt{4p}} \sum_{k=2p}^{4p-1} f_k(z)\xi \otimes g_k(z)\xi.$$

We have $\|\Delta_z\|_2^2 = \frac{1}{2}$ and $Q_z((x_i \otimes y_i)\zeta_j) = c_{ij}\Delta_z$ for all $(i,j) \in I(z)$.

Finally, we use the projections $Q_z$ to estimate $\|\phi(b)\zeta\|_2$. We get

$$\begin{aligned}
\phi(b)\zeta &= \sum_{i=1}^{n} b_i \sum_{j=1}^{t} \sigma_j (x_i \otimes y_i)\zeta_j \\
&= \sum_{z\in H} \sum_{(i,j)\in I(z)} b_i\sigma_j (x_i \otimes y_i)\zeta_j
\end{aligned}$$

Now $(x_i \otimes y_i)\zeta_j \in K_z$ so

$$\begin{aligned}
\|\phi(b)\zeta\|_2^2 &= \sum_{z\in H} \| \sum_{(i,j)\in I(z)} b_i\sigma_j (x_i \otimes y_i)\zeta_j\|_2^2 \\
&\geq \sum_{z\in H} \|Q_z( \sum_{(i,j)\in I(z)} b_i\sigma_j (x_i \otimes y_i)\zeta_j)\|_2^2 \\
&= \sum_{z\in H} \| \sum_{(i,j)\in I(z)} b_i\sigma_j c_{ij}\Delta_z\|_2^2 \\
&= \sum_{z\in H} \|\mu_z\Delta_z\|_2^2 \\
&= \frac{1}{2}\sum_{z\in H} |\mu_z|^2 \\
&= \frac{1}{2}\|bx\|_2^2.
\end{aligned}$$

This proves the Lemma.

□

**Proposition 3.2.2.** *Let $A_1$ and $A_2$ be finite-dimensional $C^*$-algebras with faithful traces and orthounitary bases, and $\dim A_1 \geq 2$, $\dim A_2 \geq 3$. Then, if $C$ is the reduced free product, $K(H)$ constitutes the unique ideal of $C \otimes_\nu C^{op}$.*

*Proof.* Let $I$ be an ideal of $C \otimes_\nu C^{op}$. Then $\phi(I)$ is an ideal of $C \otimes C^{op}$, which is simple, so either $\phi(I) = 0$ or $\phi(I) = C \otimes C^{op}$. If $\phi(I) = 0$ then $I$ is an ideal of $K(H)$ (by the previous Lemma), so $I$ is either zero or $K(H)$. If $\phi(I) = C \otimes C^{op}$ then $I \neq 0$. So

$$0 \neq K(H)I \subset I \cap K(H).$$

As $I \cap K(H) \neq 0$, we must have $I \cap K(H) = K(H)$ and this implies that $I = C \otimes_\nu C^{op}$. Thus $K(H)$ is the only non-trivial ideal of $C \otimes_\nu C^{op}$. $\square$

This result can also be obtained in the case when the groups underlying the orthounitary bases of $A_1$ and $A_2$ are both free. Indeed, Akemann and Ostrand look at the case where the underlying groups are both $\mathbb{Z}$. The freeness of the groups involved ensures that a suitable definition of length can be obtained (more refined than the 'block length' considered in the Lemma above). For more general groups (ones that are neither finite nor free), it is not clear how to obtain a similar result.

## 3.3 Reduced free products using pure states

### 3.3.1 Introduction

Here we consider reduced free products of $C^*$-algebras with respect to pure states. These were considered by Kirchberg in [41]. In the course of showing that reduced amalgamated free products of finite dimensional $C^*$-algebras are exact, he showed that the reduced free product of some matrix algebra with itself ( a finite number of times), using the same pure state on each copy of the matrix algebra, is in fact an extension of a Cuntz-Krieger algebra by the compact operators. It follows that such reduced free products are nuclear.

In the following we exploit the connection between reduced free products and Cuntz-Pimsner $C^*$-algebras which is described by Dykema and Shlyakhtenko in [24]. We show that many reduced free products of nuclear $C^*$-algebras with respect to pure states are in fact nuclear. This includes all reduced free products of matrix algebras (with pure states attached).

### 3.3.2 Nuclearity of the reduced free product

Let $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$ be the reduced free product of two unital $C^*$-algebras $A_1$ and $A_2$. We assume that the states $\phi_1$ and $\phi_2$ are pure (and of course have faithful G.N.S. representations). We assume also that $A_1$ and $A_2$ are nuclear, this being a necessary condition for the reduced free product to be nuclear.

Now let $\pi_i : A_i \to B(H_i)$ be the G.N.S. representation corresponding to $\phi_i$, for $i = 1, 2$. Let the corresponding vacuum vectors be denoted $\xi_i$. We can show the following.

**Theorem 3.3.1.** *Suppose that $K(H_i) \subset \pi_i(A_i)$ for $i = 1, 2$. Then the reduced free product $A$ is nuclear (and not simple).*

The extra condition required in the above Theorem is automatic when $A_1$ and $A_2$ are matrix algebras, it being equivalent to pureness of the corresponding state. The condition is in fact clearly satisfied for any type I $C^*$-algebras $A_1$ and $A_2$. So, this Theorem covers all reduced free products of matrix algebras, as well as many infinite dimensional examples (see later in this Section).

We now show how to prove this theorem, by relating the reduced free product $A$ to the Cuntz-Pimsner $C^*$-algebras. These were introduced in Example 2.3.2.

First note that, by Voiculescu's characterisation of the commutant of the reduced free product (see Theorem 1.6.5 in [62]), it follows that the free product state $\phi$ is pure.

Now let $B = A_1 \otimes A_2$ and let $\rho$ be the tensor product state on $B$ given by $\phi_1 \otimes \phi_2$. Let $H$ be the Hilbert $B$-bimodule $L^2(B, \rho) \otimes_{\mathbb{C}} B$, and let $E(H)$ be the associated Cuntz-Pimsner $C^*$-algebra. Finally, we let $\mathcal{E} : E(H) \to B$ be the canonical vacuum expectation. Proposition 4.2 in [24] shows that there is an embedding

$$\pi : A \to E(H) \tag{1}$$

which is state-preserving in the sense that $\rho \circ \mathcal{E} \circ \pi = \phi$. The following result is essential.

**Lemma 3.3.2.** *The Cuntz-Pimsner $C^*$-algebra $E(H)$ is nuclear.*

*Proof.* The tensor product $B$ is certainly nuclear. We can then use the proof of Theorem 3.1 in [24] to give a proof of the nuclearity of $E(H)$, using the usual closure properties of the class of nuclear $C^*$-algebras.

Alternatively, use Theorem 2.4 of [33]. □

Hence, to show nuclearity of the reduced free product $A$, it suffices to provide a conditional expectation $E(H) \to A$.

The proof of Proposition 4.2 in [24] shows that

$$(E(H), \rho \circ \mathcal{E}) = (C, \psi) * (B, \rho) \tag{2}$$

where $C$ is the $C^*$-subalgebra of $E(H)$ generated by the non-unitary isometry $\ell(\widehat{1} \otimes 1)$, $\widehat{1} \otimes 1$ being an element of $H$. The state $\psi$ is the (scalar valued) restriction of $\mathcal{E}$ to $C$.

Clearly, $C$ is isomorphic to the Toeplitz algebra. By the Toeplitz algebra, we mean the universal $C^*$-algebra generated by a non-unitary isometry (see, for example, section 3.5 of Murphy [49] for more details on this).

Let $\mathcal{H}$ be the free product Hilbert space corresponding to the free product (2), with vacuum vector $\xi$. Now $\pi$ is defined by

$$\pi(a_1) = ua_1u^{-1} \quad a_1 \in A_1$$

$$\pi(a_2) = u^2a_2u^{-2} \quad a_2 \in A_2$$

where $u \in C$ is a Haar unitary, in other words $\psi(u^k) = 0$ for all $k > 0$.

So, $\mathcal{K} = \overline{\pi(A)\xi}$ is a closed subspace of $\mathcal{H}$. Let $P$ denote the orthogonal projection from $\mathcal{H}$ onto this subspace, and let

$$\Psi : E(H) \to B(\mathcal{K}) \tag{3}$$

be compression with respect to this projection $P$.

The embedding (1) is state-preserving, so we may identify $\mathcal{K}$ with the G.N.S. Hilbert space $L^2(A, \phi)$. In fact, it is easy to see that $\Psi|\pi(A)$ is an isomorphism between $\pi(A)$ and $A$. So, to show that $\Psi$ is the required conditional expectation, it suffices to prove the following.

**Lemma 3.3.3.** *The image of $\Psi$ is contained in $\Psi(\pi(A))$ (which we identify with $A$).*

*Proof.* This is divided into parts as follows.

**A. Decomposition of elements of $B$**

Since $\Psi(1) = 1$ it suffices to consider $\Psi(x)$ where $x$ is a reduced word in the free product (2). By linearity and continuity of $\Psi$, we can assume

$$x = c_1(a_1 \otimes b_1)c_2(a_2 \otimes b_2)\cdots c_n(a_n \otimes b_n)c_{n+1} \tag{4}$$

where $c_j \in C$, $a_j \in A_1$, $b_j \in A_2$, $\rho(a_j \otimes b_j) = 0\ \forall j$, and $\psi(c_j) = 0\ \forall j$. (We could possibly have $c_1 = 1$ or $c_{n+1} = 1$, depending on what type of reduced word $x$ happens to be.)

Write $a_j \otimes b_j = (\phi_1(a_j)1 + \overline{a_j}) \otimes (\phi_2(b_j)1 + \overline{b_j})$. Here, if $a \in (A, \phi)$ then by $\overline{a}$ we mean $a - \phi(a)1$. Doing this allows us to assume that each tensor $a_j \otimes b_j$ is one of the following three types:

(a) $a_j \otimes b_j$ with $\phi_1(a_j) = \phi_2(b_j) = 0$.

(b) $a_j \otimes 1 = a_j$ with $\phi_1(a_j) = 0$.

(c) $1 \otimes b_j = b_j$ with $\phi_2(b_j) = 0$.

### B. Projections in the free product $A$

By assumption, $A_1$ contains the finite rank projection $P_{\mathbb{C}\xi_1}$. Hence $A_1$ also contains $1 - P_{\mathbb{C}\xi_1} = P_{H_1^o}$. Let $P_2$ be the image of this projection under the canonical embedding $A_1 \hookrightarrow A$. Then $P_2 \in A$ is the projection from $(H,\xi) = (H_1,\xi_1) * (H_2,\xi_2)$ onto

$$\bigoplus_{\substack{n\geq 1\\ \iota_1\neq\iota_2\neq\cdots\neq\iota_n\\ \iota_1=1}} H_{\iota_1}^o \otimes \cdots \otimes H_{\iota_n}^o.$$

Doing a similar thing for $A_2$ gives a projection $P_3 \in A$, from $H$ onto

$$\bigoplus_{\substack{n\geq 1\\ \iota_1\neq\iota_2\neq\cdots\neq\iota_n\\ \iota_1=2}} H_{\iota_1}^o \otimes \cdots \otimes H_{\iota_n}^o.$$

We also define $P_1 = P_{\mathbb{C}\xi} = 1 - P_2 - P_3 \in A$. These three projections, contained in $A$, will be useful in what follows.

### C. Evaluation of $P$

$\Psi$ is defined as compression by $P$, so to understand $\Psi$ it is useful to know how $P$ can be evaluated. Suppose we have a vector $x\xi \in \mathcal{H}$, where $x$ is of the form (4) and each tensor is either of type (a), (b) or (c).

First note that, since $P$ projects onto $\overline{\pi(A)\xi}$, it follows that

$$P(\pi(a)x\xi) = \pi(a)P(x\xi) \tag{5}$$

for all $a \in A$.

Next, note that $P(x\xi) = 0$ if $x$ contains any tensors of type (a). This involves showing that $\langle\pi(a)x\xi,\xi\rangle = 0$ for all reduced words $a$ in the algebraic free product of $A_1$ and $A_2$ (as well as $a = 1$). To get a non-zero result for $\langle\pi(a)x\xi,\xi\rangle$, we must have complete reduction of the word $\pi(a)x$. But this involves either $a_1 \in A_1^o$ or $b_1 \in A_2^o$ meeting a tensor $a_j \otimes b_j$ of type (a), the result of which is $a_1a_j \otimes b_j$ or $a_j \otimes b_1b_j$. These tensors, however, are still reduced. That is, $\rho(a_1a_j \otimes b_j) = 0$ and $\rho(a_j \otimes b_1b_j) = 0$. This means that reduction between the words $\pi(a)$ and $x$ 'stops' at $a_j \otimes b_j$, hence complete reduction can never occur and we must have $\langle\pi(a)x\xi,\xi\rangle = 0$.

So we can assume that all tensors in $x$ are actually elements of $A_1^o$ or $A_2^o$, in other words of type (b) or (c). That is,

$$x = c_1a_1c_2a_2\cdots c_na_nc_{n+1} \tag{6}$$

where $a_j \in A^o_{n(j)}$ and $n(j) \in \{1,2\}$. Of course, consecutive integers $n(j)$ could well be the same.

If $c_1 = 1$ then $P(x\xi) = 0$, regardless of $c_{n+1}$. This is because the concatenation of $\pi(a)$ and $x$ is already a reduced word ($\pi(a)$ always ends with a non-zero power of $u$).

If $\psi(c_1) = 0$ and $n(1) = 1$ then we claim that

$$P(c_1 a_1 \cdots c_{n+1}\xi) = \psi(u^{-1}c_1)P(ua_1 \cdots c_{n+1}\xi). \tag{7}$$

For this, we need

$$\langle \pi(a)c_1a_1 \cdots c_{n+1}\xi, \xi\rangle = \psi(u^{-1}c_1)\langle \pi(a)ua_1 \cdots c_{n+1}\xi, \xi\rangle$$

where, as usual, $a$ is a reduced word from the algebraic free product of $A_1$ and $A_2$. If $a$ ends with an element of $A^o_2$, both sides are zero. On the other hand, if $a$ ends with an element of $A^o_1$ then both sides are seen to agree.

If $n(1) = 2$ then we get the corresponding formula

$$P(c_1 a_1 \cdots c_{n+1}\xi) = \psi(u^{-2}c_1)P(u^2a_1 \cdots c_{n+1}\xi). \tag{7'}$$

in the same way.

Once (7) or (7') has been used, we can then write (in the case n(1)=1)

$$\begin{aligned} P(ua_1c_2a_2 \cdots c_{n+1}\xi) &= P(\pi(a_1)uc_2a_2 \cdots c_{n+1}\xi) \\ &= \pi(a_1)P(uc_2a_2 \cdots c_{n+1}\xi) \end{aligned}$$

by (5). Continuing this process allows us to evaluate $P(x\xi)$. For example, if $a_1 \in A^o_1$, $b_1 \in A^o_2$ and $c_1, c_2, c_3$ are reduced then

$$P(c_1a_1c_2b_1c_3\xi) = \psi(u^{-1}c_1)\psi(u^{-1}c_2)\psi(u^2c_3)\pi(a_1b_1)\xi.$$

It is apparent that, if two consecutive integers $n(j)$ are equal, then the result will be zero. Also, we made no assumptions on $c_{n+1}$. Clearly we again get zero if $c_{n+1}$ happens to be 1. So the use of (5), (7) and (7') allow us to evaluate $P(x\xi)$ fully for any type of word $x$.

### D. Words containing tensors of type (a)

Now consider $\Psi(x)$ where $x$ is of the form (4). We are assuming here that all tensors in $x$ are of types (a), (b) or (c). Suppose $x$ contains a tensor of type (a). Then we claim that $\Psi(x) = 0$.

Indeed, $P(x\xi) = 0$ by the comments in part C above, since $x$ contains a tensor of type (a). Now consider $P(x\pi(a)\xi)$ where $a$ is a reduced word in the algebraic free product of $A_1$ and $A_2$. Consider the reduction of $x\pi(a)$ into a sum of reduced words. Any words containing a tensor of type (a) will be sent to zero on application of $P$. The only chance of obtaining something non-zero is when a tensor from $x$ meets an element, say $a_1 \in A_1^o$, from $a$.

In this case we obtain the following sequence

$$\cdots c(a_j \otimes b_j)a_1 u^k \cdots$$

where $c \in C$, $a_j \otimes b_j$ is a tensor of type (a) from $x$, and $k = -1$ if there is nothing after $a_1$ in $a$, while $k = 1$ if there *is* something afterwards.

Applying $P$ gives

$$P(\cdots c(a_j a_1 \otimes b_j)u^k \cdots) = \phi_1(a_j a_1)P(\cdots c b_j u^k \cdots)$$

(see part C). Apply the process mentioned in part C to evaluate $P(\cdots c b_j u^k \cdots)$ : after removing $\pi(b_j)$ using (5), we evaluate $\psi(u^2 u^k)$ (when $k = -1$) or $\psi(u^{-2}(u^2 u^k))$ (when $k = 1$). In either case we obtain zero.

An entirely analogous calculation works for the case where $a_j \otimes b_j$ meets $b_1 \in A_2^o$. So $\Psi(x) = 0$.

## E. Unsymmetrical words

Having dealt with any words containing tensors of type (a), we can now restrict to $x$ of the form (6), where everything is reduced, except possibly $c_1$ and $c_{n+1}$ (these are permitted to be 1).

It is clear from part C that $\Psi(x) = 0$ when both $c_1$ and $c_{n+1}$ are equal to 1. Now we consider the unsymmetrical words where one (but not both) of $c_1$ and $c_{n+1}$ is equal to 1.

Since $\Psi(x) = (\Psi(x^*))^*$, it suffices to consider the case where $c_{n+1} = 1$. Suppose $a_n \in A_1^o$ (a very similar argument works for $a_n \in A_2^o$). Suppose $a$, a reduced word in the algebraic free product of $A_1$ and $A_2$, begins with $a_1' \in A_1^o$. Then

$$P(x\pi(a)\xi) = P(c_1 \cdots a_n u a_1' u^k \cdots).$$

Here $k$ depends on whether there is anything after $a_1'$ in $a$ but is always non-zero.

Using the usual evaluation process, after removing $\pi(a_n)$ using (5) we have to evaluate $\psi(u^{-1}u^2) = 0$. Similarly, in the case when $a$ begins with an element of $A_2^o$, we are required to evaluate $\psi(u^{-2}u^3) = 0$, so again $P(x\pi(a)\xi) = 0$. Therefore, since $P(x\xi) = 0$ also, we get $\Psi(x) = 0$.

**F. Evaluation of $\Psi(C)$**

We now wish to determine $\Psi(c)$ for $c \in C$. In fact,

$$\Psi(c) = \psi(c)P_1 + \psi(u^{-1}cu)P_2 + \psi(u^{-2}cu^2)P_3 \tag{8}$$

where $P_1$, $P_2$ and $P_3$ are the projections from part B. As these projections are actually in $A$ (which we identify with $\pi(A)$), it follows that $\Psi(C) \subset \Psi\pi(A)$. Note that, from now on in the proof of this Lemma, $u^k$ denotes a non-zero power of $u$ in some reduced word. The actual value of $k$ will be unimportant, and depends on the type of reduced word involved, but is always non-zero.

To prove (8), note that $\Psi(c)\xi = P(c\xi) = \psi(c)\xi$ (this is easy to check). Considering a word $a$ beginning with $a_1' \in A_1^\circ$ as in part E, we obtain

$$\begin{aligned} P(c\pi(a)\xi) &= P(cua_1'u^k\cdots\xi) \\ &= \psi(u^{-1}cu)P(ua_1'u^k\cdots\xi) \\ &= \psi(u^{-1}cu)\pi(a)\xi \\ &= \psi(u^{-1}cu)P_2\pi(a)\xi. \end{aligned}$$

Similarly for a word $a$ beginning with an element of $A_2^\circ$. Hence (8) is proved.

**G. Induction formula for evaluation of $\Psi$**

We are considering words of the form

$$x = c_1a_1c_2a_2\cdots c_na_nc_{n+1} \tag{9}$$

where every letter is reduced and $a_j \in A_{n(j)}^\circ$, $n(j) \in \{1, 2\}$. Let $w = c_1a_1c_2a_2\cdots c_n$, so that $x = wa_nc_{n+1}$. We show that $\Psi(x)$ can be written in terms of $\Psi(x')$ for various words $x'$ of length less than that of $x$. Since $\Psi(C) \subset \Psi\pi(A)$, it will then follow by induction that the entire image of $\Psi$ is contained in $\Psi\pi(A)$.

We claim that, if $a_n \in A_1^\circ$, then

$$\Psi(x) = \Psi(wu^{-1})\pi(a_n)(\psi(uc_{n+1})P_1 + \psi(c_{n+1}u)P_2 + \psi(u^{-1}c_{n+1}u^2)P_3). \tag{10}$$

Note $wu^{-1}$ does indeed have length less than that of $x$. To prove (10) we need to evaluate both sides at various points, and check that equality holds.

At $\xi$, the L.H.S. of (10) is

$$P(wa_nc_{n+1}\xi) = \psi(uc_{n+1})P(wa_nu^{-1}\xi)$$

using the usual evaluation of $P$ procedure from part C. But this is precisely what you obtain from the R.H.S. of (10).

At $\pi(a_1')\xi,\ a_1' \in A_1^\circ$,

$$\begin{aligned}\text{L.H.S.} &= P(wa_nc_{n+1}ua_1'u^{-1}\xi) \\ &= \psi(c_{n+1}u)P(wa_na_1'u^{-1}\xi).\end{aligned}$$

It is easy to check that this is precisely the R.H.S.

At $\pi(b_1')\xi,\ b_1' \in A_2^\circ$,

$$\begin{aligned}\text{L.H.S.} &= P(wa_nc_{n+1}u^2b_1'u^{-2}\xi) \\ &= P(wa_n\overline{c_{n+1}u^2}b_1'u^{-2}\xi).\end{aligned}$$

Using the usual evaluation procedure, after removing $\pi(a_n)$ we obtain

$$\begin{aligned}P(u\overline{c_{n+1}u^2}b_1'u^{-2}\xi) &= \psi(u^{-2}u\overline{c_{n+1}u^2})\pi(b_1')\xi \\ &= \psi(u^{-1}c_{n+1}u^2)\pi(b_1')\xi.\end{aligned}$$

On the other hand, the R.H.S. gives

$$\psi(u^{-1}c_{n+1}u^2)\Psi(wu^{-1})\pi(a_nb_1')\xi = \psi(u^{-1}c_{n+1}u^2)P(wa_nub_1'u^{-2}\xi),$$

which by the previous comments can easily be seen to be equal to the L.H.S.

At $\pi(a_1'b_1'\cdots)\xi$ ($a_1' \in A_1^\circ$, $b_1' \in A_2^\circ$), a word beginning with an element of $A_1^\circ$ and of length $\geq 2$, we have

$$\begin{aligned}\text{L.H.S.} &= P(wa_nc_{n+1}ua_1'ab_1'u^k\cdots\xi) \\ &= \psi(c_{n+1}u)P(wa_na_1'ub_1'u^k\cdots\xi)\end{aligned}$$

while

$$\begin{aligned}\text{R.H.S.} &= \psi(c_{n+1}u)\Psi(wu^{-1})ua_na_1'ub_1'u^k\cdots\xi \\ &= \psi(c_{n+1}u)P(wa_na_1'ub_1'u^k\cdots\xi) \\ &= \text{L.H.S.}\end{aligned}$$

At $\pi(b_1'a_1'\cdots)\xi$ ($b_1' \in A_2^\circ$, $a_1' \in A_1^\circ$), a word beginning with an element of $A_2^\circ$ and of length $\geq 2$, we have

$$\begin{aligned}\text{L.H.S.} &= P(wa_nc_{n+1}u^2b_1'u^{-1}a_1'u^k\cdots\xi) \\ &= P(wa_n\overline{c_{n+1}u^2}b_1'u^{-1}a_1'u^k\cdots\xi).\end{aligned}$$

Evaluating this, after removing $\pi(a_n)$ we obtain

$$P(u\overline{c_{n+1}u^2}b_1'u^{-1}a_1'u^k\cdots\xi) = \psi(u^{-1}c_{n+1}u^2)\pi(b_1'a_1'\cdots)\xi.$$

On the other hand,

$$\begin{aligned}\text{R.H.S.} &= \psi(u^{-1}c_{n+1}u^2)\Psi(wu^{-1})\pi(a_nb_1'a_1'\cdots)\xi \\ &= \psi(u^{-1}c_{n+1}u^2)P(wa_nub_1'u^{-1}a_1'u^k\cdots\xi)\end{aligned}$$

which is now seen to be precisely the same as the L.H.S. This proves (10).

If we have $a_n \in A_2^\circ$ in (9), a similar formula is obtained by very similar methods:

$$\Psi(x) = \Psi(wu^{-2})\pi(a_n)(\psi(u^2c_{n+1})P_1 + \psi(uc_{n+1}u)P_2 + \psi(c_{n+1}u^2)P_3) \tag{11}$$

So, (10) and (11), along with part F, can be used to give a proof (by induction on the length of words) that all words $x$ of the form (9) satisfy $\Psi(x) \in \Psi\pi(A)$. We have now considered all possible words in the free product. Hence the Lemma is proved. □

The above Lemma is of interest in itself, providing some insight into the structure of the reduced free product (2), as well as showing how the reduced free product $A$ and the tensor product $B$ (both contained in $E(H)$) interact. The Lemma also allows us to prove Theorem 3.3.1.

*Proof of Theorem 3.3.1.* By Lemma 3.3.3, we have a conditional expectation $\Psi : E(H) \to A$. By Lemma 3.3.2, $E(H)$ is nuclear. Hence $A$ is nuclear. In part B of the proof of Lemma 3.3.3, we showed that the finite rank projection $P_{\mathbb{C}\xi}$ was contained in $A$. As remarked earlier, the free product state $\phi$ is pure so the corresponding G.N.S. representation $\pi_\phi : A \to B(H)$ is irreducible. As $\pi_\phi(A) \cap K(H) \neq 0$, it follows that $K(H) \subset \pi_\phi(A)$. So $A$ is certainly not simple. □

### 3.3.3 Examples

Which $C^*$-algebras are covered by Theorem 3.3.1 ? As already remarked, all matrix algebras are covered, in particular the reduced free products considered by Kirchberg in [41]. We can also consider any unital nuclear $C^*$-algebra $A$. Let $\pi_\phi : A \to B(H_\phi)$ be the G.N.S. representation corresponding to some state $\phi$ on $A$. Then $\pi_\phi(A) + K(H_\phi)$ is a $C^*$-algebra to which the Theorem can be applied.

Let $\widetilde{O}_d$ denote the Cuntz-Pimsner $C^*$-algebra for the Hilbert $\mathbb{C}$-bimodule $\mathbb{C}^d$ (where $d$ is finite), with corresponding vacuum expecation (actually a state) $\phi_\xi$ (see example 2.3.2). Now $\widetilde{O}_d$ contains the compact operators $K(\mathcal{F}(\mathbb{C}^d))$, and it is readily seen that $\widetilde{O}_d/K(\mathcal{F}(\mathbb{C}^d)) \cong O_d$, the Cuntz algebra on $d$ generators. Thus $\widetilde{O}_d$ is an extension of $O_d$ by the compact operators, and we have

$$(\widetilde{O}_{d_1}, \phi_\xi) * (\widetilde{O}_{d_2}, \phi_\xi) = (\widetilde{O}_{d_1+d_2}, \phi_\xi)$$

where $d_1, d_2 \in \mathbb{N}$. This is a special case of Speicher's result (see Example 2.3.2). The states $\phi_\xi$ are pure, so Theorem 3.3.1 is saying that $\widetilde{O}_{d_1+d_2}$ is nuclear and not simple, which is certainly true.

Note that if $A_1$ and $A_2$ satisfy the required conditions for the Theorem, then $A_1 \otimes A_2$ with the tensor product state does as well. Also, the reduced free product $A$ of $A_1$ and $A_2$ satisfies these same conditions.

Reduced free products with respect to non-faithful states were also considered in [22]. Theorem 3.1 in this paper states that certain free products

$$(\mathfrak{A}, \phi) = (A, \phi_A) * (M_N(\mathbb{C}) \otimes B, \phi_N \otimes \phi_B)$$

are simple (and purely infinite), where $\phi_N(e_{11}) = 1$. However, this conclusion is only valid for pairs $A, B$ which satisfy the so-called property $Q$, which, roughly speaking, excludes the compact operators from the G.N.S. representations of $A$ and $B$. So the conditions of Theorem 3.3.1 do not apply.

### 3.3.4 An alternative proof

Here we give an alternative proof of Theorem 3.3.1. The methods used are perhaps less elementary, since they use a fair amount of the existing theory. They are related to the proof of the equivalence of nuclear embeddability and exactness: see Theorem 4.1 of [42], chapter 7 of [66], as well as section 1 of [53].

The method is, roughly speaking, as follows. We know that reduced free products of matrix algebras (with pure states attached) are nuclear [41]. Now $A_1$ and $A_2$ are nuclear, so they can be approximated by matrix algebras. Hence we can show that $A$ is nuclear if we can take the reduced free product of the approximating maps for $A_1$ and $A_2$. For this, we need these approximating maps to be state-preserving (see [10]). Unfortunately, this is not necessarily the case, so we need to modify the approximating maps somewhat, in order to ensure that they are state-preserving.

*Remark 3.3.1.* We obtain some inspiration for this alternative proof by looking at reduced free products of UHF algebras with pure states (of a certain special form) attached.

Suppose that $M$ is a UHF algebra, so that $M = \otimes_{i=1}^{\infty} M_{s(i)}$ for some positive integers $s(i)$. Let $\phi_i$ be a pure state on $M_{s(i)}$. Then $\phi = \otimes_{i=1}^{\infty} \phi_i$ is a pure state on $M$.

For $n \in \mathbb{N}$, define $\Phi_n : M \to M_{s!(n)}$ where $s!(n) = s(1)s(2) \cdots s(n)$. We let $\Phi_n$ act as the identity on the first $n$ factors. That is $\Phi_n | M_{s(i)} = \mathrm{id}$ for $i \leq n$. Then, for $i > n$, we let $\Phi_n | M_{s(i)} = \phi_i$. Thus $\Phi_n$ is a tensor product of unital completely positive maps, and is therefore a unital completely positive map. It is also clearly state-preserving, in the sense that $\phi = (\otimes_{i=1}^{n} \phi_i) \circ \Phi_n$ ( $\phi$ being the state on $M$, $\otimes_{i=1}^{n} \phi_i$ being the state on $M_{s!(n)}$).

Now let $\Psi_n : M_{s!(n)} \to M$ be the inclusion. Obviously this is also a state-preserving unital completely positive map, in the sense that $\otimes_{i=1}^{n} \phi_i = \phi \circ \Psi_n$. With the above definitions, we now obtain

$$\lim_{n \to \infty} \|\Psi_n \Phi_n(x) - x\| = 0 \; \forall x \in M.$$

Indeed, this is clearly true when $x$ is in $\cup_{n=1}^{\infty} M_{s!(n)}$, and such $x$ are dense in $M$.

So we have the equivalent of Proposition 3.3.6 for UHF algebras (with states of the above form attached). Now the alternative proof of Theorem 3.3.1 (see after Proposition 3.3.6) shows that reduced free products of UHF algebras with pure states (of the above form) attached are nuclear.

This kind of procedure has also been considered by Haagerup in [34]. Here a semidiscrete $\mathrm{II}_1$-factor is approximated by matrix algebras, and it is shown that the approximating maps can be taken to preserve the canonical traces involved. As the canonical trace on a matrix algebra is somewhat different in nature from the pure states on matrix algebras that we are considering, it turns out that the methods used in [34] are quite different.

We start with the following simple Lemma.

**Lemma 3.3.4.** *Suppose $B_1$ and $B_2$ are unital $C^*$-algebras, with states $\psi_i \in S(B_i)$ for $i = 1, 2$, whose G.N.S. representations are faithful. Let $(B, \psi)$ be the reduced free product of $B_1$ and $B_2$. Then $B$ may be embedded into*

$$(B_1 \otimes B_2, \psi_1 \otimes \psi_2) * (B_1 \otimes B_2, \psi_1 \otimes \psi_2). \tag{1}$$

*Moreover, there is a conditional expectation from the reduced free product (1) onto $B$.*

*Proof.* Clearly $B_1$ embeds into $B_1 \otimes B_2$ via $b_1 \longmapsto b_1 \otimes 1$, and $(\psi_1 \otimes \psi_2)(b_1 \otimes 1) = \psi_1(b_1)$. Similarly, $B_2$ embeds into $B_1 \otimes B_2$ in a state-preserving manner. Hence by [10] Theorem 1.3 there is an embedding of $B$ into the reduced free product (1).

On the other hand, the map $\mathrm{id}_{B_1} \otimes \psi_2 : B_1 \otimes B_2 \to B_1$ is unital completely positive and state-preserving (that is, $\psi_1 \otimes \psi_2 = \psi_1 \circ (\mathrm{id}_{B_1} \otimes \psi_2)$), similarly for $\psi_1 \otimes \mathrm{id}_{B_2}$. So by [10] Theorem 2.2 there exists a unital completely positive map from (1) to $B$, and it can readily be seen that this map is in fact a conditional expectation onto $B$. □

In order to carry out the previously mentioned process of modifying the approximating maps, we also need the following simple result.

**Lemma 3.3.5.** *Suppose $e$ and $p$ are projections and that $\epsilon > 0$. If $\|(epe)^2 - epe\| < \epsilon$ then $\|\,[e,p]\,\| < 2\sqrt{\epsilon}$.*

*Proof.* We have

$$\|(epe - pe)^*(epe - pe)\| = \|epe - (epe)^2\| < \epsilon.$$

So $\|epe - pe\| < \sqrt{\epsilon}$ and $\|epe - ep\| < \sqrt{\epsilon}$. Hence $\|ep - pe\| < 2\sqrt{\epsilon}$. □

Next we consider the approximation of a nuclear $C^*$-algebra with matrix algebras, using maps which preserve the pure states involved. We let $\phi_0 \in S(M_n)$ be the pure state on $M_n$ given by $\phi_0(e_{11}) = 1$.

**Proposition 3.3.6.** *Let $A$ be a separable nuclear $C^*$-algebra and suppose $\phi \in S(A)$ is pure, but with faithful G.N.S. representation $\pi$. Suppose $\pi(A) \supset K(H)$, where $H$ is the G.N.S Hilbert space. Then for every finite dimensional operator system $X \subset A$ and $\epsilon > 0$, there exist $n \in \mathbb{N}$ and unital completely positive maps $\Phi : A \to M_n$, $\Psi : M_n \to A$ such that $\Phi$, $\Psi$ are state-preserving (in the sense that $\phi_0 \circ \Phi = \phi$, $\phi \circ \Psi = \phi_0$) and $\|(\Psi\Phi - id)|_X\| < \epsilon$.*

*Proof.* This is certainly true if $A$ is finite dimensional. Indeed, since $\phi$ is pure, $\pi(A)' = \mathbb{C}1$ so $A$ has trivial centre. Hence $A \cong M_p$ for some $p \in \mathbb{N}$. Now $\pi(M_p) = \pi(M_p)'' = B(H)$

so $H \cong \mathbb{C}^p$. So we have a $*$-isomorphism $\pi : M_p \to M_p$ such that $\phi = \phi_0 \circ \pi$ (choosing an orthonormal basis for $\mathbb{C}^p$ whose first element is the vacuum vector). This means that we can take $n = p$, $\Phi = \pi$ and $\Psi = \pi^{-1}$.

If $A$ is not finite-dimensional, we can suppose that $H = \ell^2(\mathbb{N})$, with orthonormal basis $e_1, e_2, \ldots$. Here we can assume that $e_1$ is the vacuum vector $\xi$.

As in Lemma 2 of [53], let $\Phi_n : A \to M_n$ be the unital completely positive map given by compression with respect to the projection onto $\operatorname{span}\{e_i : i \leq n\}$. As $e_1 = \xi$, these maps are also state-preserving.

Now consider $\Phi_n|X$. For large enough $n$ this is going to be injective. So if $n \geq n_0$ say, then we have an inverse mapping $W_n : \Phi_n(X) \to X$. The proof of Theorem 1 in [53] tells us that the $W_n$ are completely bounded and in fact

$$\overline{\lim}_{n\to\infty} \|W_n\|_{cb} = 1.$$

So for some large $n$ we can take $\|W_n\|_{cb} < 1 + \epsilon$.

We need to use the nuclearity of $A$. Nuclearity implies that there are $m \in \mathbb{N}$ and unital completely positive maps $U : A \to M_m$, $V : M_m \to A$ with

$$\|(VU - \mathrm{id})|_X\| < \epsilon.$$

Consider $UW_n : \Phi_n(X) \to M_m$. As $U$ is completely contractive, we have $\|UW_n\|_{cb} < 1+\epsilon$. The Wittstock extension theorem (see Theorem 1.13 in [66] for example) implies that there is a completely bounded self-adjoint extension $W : M_n \to M_m$ with $\|W\|_{cb} < 1 + \epsilon$. Now Proposition 1.19 of [66] gives a unital completely positive map $T : M_n \to M_m$ with $\|T - W\|_{cb} < \epsilon$. Let $\Psi_1 = VT : M_n \to A$. This is a unital completely positive map.

Since $\pi(A) \supset K(H)$, we can assume that $X$ contains the rank one projection $e_{11} \in K(H)$. We want $\Psi_1$ to be state-preserving. Now $\Phi_n(e_{11}) = e_{11} \in M_n$, so we would like $\Psi_1(e_{11})$ to be $e_{11}$. What we can say is that, letting $\Psi_1(e_{11}) = z$, we have $\|z - e_{11}\| < 2\epsilon$.

Define $\Psi_0 : M_n \to A$ by

$$\Psi_0(x) = e_{11}\Psi_1(e_{11}xe_{11})e_{11} + e_{11}^{\perp}\Psi_1(e_{11}^{\perp}xe_{11})e_{11} + e_{11}\Psi_1(e_{11}xe_{11}^{\perp})e_{11}^{\perp} + e_{11}^{\perp}\Psi_1(e_{11}^{\perp}xe_{11}^{\perp})e_{11}^{\perp}.$$

Here $e_{11}^{\perp} = 1 - e_{11}$. Certainly, $\Psi_0$ is completely positive. Also, since $e_{11}$ is rank one, we have

$$\Psi_0(e_{11}) = e_{11}ze_{11} = \lambda e_{11}$$

where $\lambda \geq 0$ satisfies $|\lambda - 1| < 2\epsilon$. Although $\Psi_0$ is not unital, we do have

$$\Psi_0(1) = e_{11}ze_{11} + e_{11}^{\perp}(1 - z)e_{11}^{\perp} = t, \text{ say.}$$

Note that $||t-1|| < 2\epsilon$ so $t$ is positive and invertible.

The result of this is that we can define a map $\Psi : M_n \to A$ by $\Psi(x) = t^{-1/2}\Psi_0(x)t^{-1/2}$. With this definition, $\Psi$ is unital and completely positive. It can be seen that $[t, e_{11}] = 0$, so $[t^{1/2}, e_{11}] = 0$ and hence

$$\Psi(e_{11}) = t^{-1/2}e_{11}ze_{11}t^{-1/2} = e_{11}.$$

It follows that $e_{11}$ is in the multiplicative domain of $\Psi$. Hence if $x \in M_n$ then

$$\Psi(e_{11}xe_{11}) = e_{11}\Psi(x)e_{11} = \phi(\Psi(x))e_{11}.$$

On the other hand,

$$\Psi(e_{11}xe_{11}) = \Psi(\phi_0(x)e_{11}) = \phi_0(x)e_{11}.$$

Thus $\Psi$ is state-preserving.

So, define $\Phi$ to be $\Phi_n$. Then $\Phi$ and $\Psi$ are unital completely positive state-preserving maps. What is $||(\Psi\Phi - \mathrm{id})|_X||$ ?

Well, $\Psi_1 : M_n \to A \subset B(H)$ is unital completely positive. Hence Stinespring's theorem gives a representation $\sigma : M_n \to B(K)$ (where $K$ is a Hilbert space containing $H$) such that, letting $E$ denote projection from $K$ onto $H$, we have $\Psi_1(x) = E\sigma(x)E$. Now $||E\sigma(e_{11})E - e_{11}|| < 2\epsilon$ so it follows that

$$||(E\sigma(e_{11})E)^2 - E\sigma(e_{11})E|| < 6\epsilon.$$

Lemma 3.3.5 now implies that $||\,[E, \sigma(e_{11})]\,|| < 2\sqrt{6\epsilon}$. Also,

$$||\,[E, \sigma(e_{11}^{\perp})]\,|| = ||\,[E, \sigma(e_{11})]\,|| < 2\sqrt{6\epsilon}.$$

Using these estimates and the expression defining $\Psi_0(x)$, we get

$$||\Psi_0(x) - \Psi_1(x)|| \leq (16\epsilon + 16\sqrt{6\epsilon})||x||.$$

Hence, $||\Psi_0 - \Psi_1|| \leq f(\epsilon)$ say, where $f(\epsilon) \to 0$ as $\epsilon \to 0$.

Since $||t-1|| < 2\epsilon$, it follows (by functional calculus arguments) that $||t^{-1/2}-1|| < g(\epsilon)$, where $g(\epsilon) \to 0$ as $\epsilon \to 0$. So

$$\begin{aligned} ||\Psi(x) - \Psi_0(x)|| &\leq ||t^{-1/2}\Psi_0(x)t^{-1/2} - \Psi_0(x)t^{-1/2}|| + ||\Psi_0(x)t^{-1/2} - \Psi_0(x)|| \\ &\leq (||t^{-1/2}||g(\epsilon) + g(\epsilon))||\Psi_0(x)|| \\ &\leq ((1+g(\epsilon))g(\epsilon) + g(\epsilon))(1+2\epsilon)||x|| \\ &= h(\epsilon)||x|| \text{ say,} \end{aligned}$$

where $h(\epsilon) \to 0$ as $\epsilon \to 0$.

Now put everything together. We know that $W|\Phi(X) = UW_n$ and that $\|VT - VW\| < \epsilon$. So $\|VT|\Phi(X) - VUW_n|\Phi(X)\| < \epsilon$. Hence

$$\|(\Psi_1\Phi - VU)|_X\| = \|(VT\Phi - VUW_n\Phi)|_X\| < \epsilon.$$

But $\|(VU - \mathrm{id})|_X\| < \epsilon$ so $\|(\Psi_1\Phi - \mathrm{id})|_X\| < 2\epsilon$. As $\|\Psi_1 - \Psi\| \leq f(\epsilon) + h(\epsilon)$, we can conclude that

$$\|(\Psi\Phi - \mathrm{id})|_X\| < f(\epsilon) + h(\epsilon) + 2\epsilon.$$

As $\epsilon$ is arbitrary, the result follows. □

The above Proposition allows us to give an alternative proof of Theorem 3.3.1.

*Alternative proof of Theorem 3.3.1* Take $x_1, \dots, x_n$ in the algebraic free product of $A_1$ and $A_2$, and take $\delta > 0$. We wish to show that there are a nuclear $C^*$-algebra $N$ and unital completely positive maps $\alpha : A \to N$, $\beta : N \to A$ such that $\|\beta\alpha(x_i) - x_i\| < \delta$ for every $i \leq n$. From this it will follow, by standard techniques, that $A$ is nuclear. Indeed, as $N$ is nuclear, Theorem 1.1.4 implies that $N$ has the completely positive approximation property. So the identity map id : $N \to N$ may be approximately factorised as $\sigma \circ \nu$ for some unital completely positive $\sigma : N \to M_p$, $\nu : M_p \to N$ (and some $p \in \mathbb{N}$). Hence the identity map id : $A \to A$ may be approximately factorised using $\sigma \circ \alpha : A \to M_p$ and $\beta \circ \nu : M_p \to A$. It follows that $A$ has the completely positive approximation property, and so by Theorem 1.1.4 $A$ is nuclear.

Each $x_i$ has an expression as a sum of reduced words plus a multiple of 1. We know that the maximum length $L$ of all the words involved is finite. Let $L'$ be the maximum number of reduced words appearing in an $x_i$. Let $F_j$ be the set of elements of $A_j$ appearing in the expressions for the $x_i$ ($j = 1, 2$). Apply the above Proposition to $A_j$ with $\epsilon = \delta/2LL'$ and with $X$ being the operator system generated by $F_j$ ($j = 1, 2$).

For $j = 1, 2$ we get unital completely positive state-preserving maps $\alpha_j : A_j \to M_{n_j}$, $\beta_j : M_{n_j} \to A_j$ with the properties stated in the Proposition. By [10] Theorem 2.2, we can take the reduced free product of these maps to get unital completely positive $\alpha : A \to M_{n_1} *_r M_{n_2}$, $\beta : M_{n_1} *_r M_{n_2} \to A$. We can certainly assume that the norms of all elements in $F_1$ and $F_2$ are $\leq 1$. Now $\alpha$ is defined via

$$\alpha(a_1 \cdots a_k) = \alpha_{n(1)}(a_1) \cdots \alpha_{n(k)}(a_k)$$

where (for $1 \leq j \leq k$) $a_j \in A^o_{n(j)}$ and $n(j) \in \{1, 2\}$ with $n(1) \neq n(2) \neq \cdots \neq n(k)$. As $\beta$ is defined in a similar fashion, it can easily be seen that

$$\|\beta\alpha(x_i) - x_i\| \leq LL'\epsilon < \delta$$

for all $i \leq n$.

Finally, Lemma 7.6 of [41] implies that all reduced free products $M_k *_r M_k$ (with pure states attached and $k$ any positive integer) are nuclear. Lemma 3.3.4 then shows that the reduced free product $M_{n_1} *_r M_{n_2}$ mentioned above is nuclear. This means that we can take the nuclear $C^*$-algebra $N$ mentioned in the first paragraph to be the reduced free product $M_{n_1} *_r M_{n_2}$. Now standard techniques show that $A$ is nuclear. □

### 3.3.5 The general result

It appears likely that any reduced free product of nuclear $C^*$-algebras with pure states attached is also nuclear, but it is not obvious how to go about proving this.

The original proof of Theorem 3.3.1 does not work when the G.N.S. representations involved fail to contain the compact operators. It may at first seem possible to prove the general result via a generalisation of Proposition 3.3.6, where we do not assume that the G.N.S. representation contains the compact operators. The problem is that, without the compact operators around, it is not clear how to ensure that the unital completely positive maps involved are state-preserving.

Finally, another approach could be to modify the proof that nuclearity implies the completely positive approximation property (as contained in Kirchberg [39] or Choi and Effros [13]). This modification would attempt to ensure that the approximating maps $A \to M_n \to A$, where $A$ is the nuclear $C^*$-algebra concerned, are state-preserving with respect to the pure state on $A$ and some canonical pure state on $M_n$.

One of the problems with doing this is that the proofs in [39] and [13] depend on the fact that the set of compositions $A \to M_n \to A$ (where $n$ can vary and the maps are unital completely positive) is a convex subset of the set of all unital completely positive maps from $A$ to $A$. If we insist that we only allow state-preserving maps in these compositions, then convexity is lost.

# Chapter 4

# The tensor product operation on continuous bundles of $C^*$-algebras

## 4.1 Introduction

This chapter looks at continuous bundles of $C^*$-algebras. See Section 1.3 for the necessary background material. We are particularly interested in the minimal tensor product operation on bundles. The most important question is whether or not continuity of the bundle is preserved by this operation. As remarked by Kirchberg and Wassermann [44] it was at one time thought that continuity was always preserved.

In [44] bundles with base space $\widehat{\mathbb{N}}$ were constructed such that continuity was not always preserved. In Section 4.2, we construct a bundle on the unit interval $[0, 1]$ such that continuity is not always preserved. In Section 4.3, we extend this, giving a construction of a bundle on any fixed compact infinite metric space $X$ such that continuity is not always preserved. These constructions give rise to new characterisations of exactness in terms of the continuity of certain minimal tensor product bundles.

## 4.2 Continuous bundles on the unit interval

Here we construct a continuous bundle on $[0, 1]$ such that continuity is not always preserved by the minimal tensor product operation. First of all we review the properties of the minimal tensor product bundle and the situation for the base space $\widehat{\mathbb{N}}$. Then we provide a fairly general procedure for constructing a continuous bundle of $C^*$-algebras on $[0, 1]$, starting from a sequence of $C^*$-algebras. Finally we use this procedure to obtain a bundle on $[0,1]$ with the required discontinuity properties, thus giving a new characterisation of exactness for $C^*$-algebras.

Let $\mathcal{A} = (X, \pi_x : A \to A_x, A)$ be a continuous bundle of $C^*$-algebras, and fix a $C^*$-algebra $B$. The minimal tensor product bundle $\mathcal{A} \otimes B$ is given by

$$(X, \pi_x \otimes \mathrm{id} : A \otimes B \to A_x \otimes B, A \otimes B).$$

So if $\mathcal{A}$ has fibres $A_x$ then $\mathcal{A} \otimes B$ has fibres $A_x \otimes B$. If $\mathcal{A}$ is continuous then $\mathcal{A} \otimes B$ is always lower semicontinuous (see Lemma 2.5 of [44]). It turns out that, so long as $A$ is exact, $\mathcal{A} \otimes B$ is actually continuous (Theorem 4.6 of [44]). Also, if $B$ is exact then $\mathcal{A} \otimes B$ is always continuous (see Theorem 4.5 in [44]).

However, if $B$ is not exact then there exists a continuous bundle $\mathcal{A}$ on $\widehat{\mathbb{N}}$ such that $\mathcal{A} \otimes B$ fails to be continuous. This is described in Proposition 4.3 of [44]. The base space $\widehat{\mathbb{N}}$ is clearly the simplest metric space on which there exist discontinuous functions. The question is, can we replace $\widehat{\mathbb{N}}$ with the unit interval $[0,1]$ ? This is certainly possible in the corresponding situation for maximal tensor product bundles (see Remarks 3.3 of [44]).

Now we provide a fairly general procedure for constructing continuous bundles of $C^*$-algebras on $[0,1]$, starting from a sequence of $C^*$-algebras. In fact, we always consider bundles on $\widehat{\mathbb{R}}_+$, the one point compactification of the non-negative reals $\mathbb{R}_+$. As $[0,1]$ is homeomorphic to $\widehat{\mathbb{R}}_+$, this can certainly be done. The main reason for doing it is to make the construction a bit easier.

Start with a sequence $A_1, A_2, A_3, \ldots$ of separable $C^*$-algebras. We let $\oplus_{i=1}^{\infty} A_i$ denote the direct product of this sequence. That is,

$$\oplus_{i=1}^{\infty} A_i = \left\{ (x_i)_{i=1}^{\infty} : x_i \in A_i \ \forall i, \ \sup_{i \geq 1} \|x_i\| < \infty \right\}.$$

We also assume we have embeddings $A_n \hookrightarrow A_{n+1}$, and we suppose we have a separable $C^*$-algebra $A$ such that $I_0 \lhd A \subset \oplus_{i=1}^{\infty} A_i$, where $I_0$ is the ideal of sequences in $\oplus_{i=1}^{\infty} A_i$ tending to zero.

Define $B$ to be the set of $f = (f_n) \in \oplus_{n=1}^{\infty} C([n-1, n], A_n)$ such that

- $\sup_{n \in \mathbb{N}} \|f_n\| < \infty$ (actually part of the definition of $\oplus_{n=1}^{\infty}$)
- $f_n(n) = f_{n+1}(n) \ \forall n \in \mathbb{N}$
- $(f_n(n - \alpha))_{n=1}^{\infty} \in A \ \forall \alpha \in [0,1]$

- $\alpha \longmapsto (f_n(n-\alpha))_{n=1}^{\infty}$ is a continuous function from $[0,1]$ to $A$.

We can (and will) think of elements of $B$ as bounded functions on $\mathbb{R}_+$. To be explicit, to $f = (f_n) \in B$, we associate the function $g$ on $\mathbb{R}_+$ given by

$$g(x) = f_{[x]+1}(x).$$

The second clause in the definition of $B$ ensures that the definition of $g$ is unambiguous at integer values. The first clause of the definition of $B$ ensures that $g$ is norm-bounded.

The final clause of the definition of $B$ implies that there is an embedding $\iota : B \hookrightarrow C([0,1], A)$. As $C([0,1], A)$ is separable, it follows that $B$ must also be separable.Clearly $B$ is a closed $*$-subalgebra of $\oplus_{n=1}^{\infty} C([n-1,n], A_n)$, in other words $B$ is a $C^*$-algebra.

For $x \in \mathbb{R}_+$, let

$$B_x = B/\{f \in B : f|[x,\infty) = 0\}.$$

We think of elements of $B_x$ as functions on $\mathbb{R}_+$ that are zero except possibly on the interval $[x, \infty)$. Also, define

$$B_\infty = B/\{f \in B : \lim_{z\to\infty} \|f(z)\| \text{ exists and is zero}\}.$$

There is an embedding $j : B \hookrightarrow \oplus_{x\in\widehat{\mathbb{R}}_+} B_x$, via the quotient maps $\pi_x : B \to B_x$, $\pi_\infty : B \to B_\infty$.

Note that, if $f \in B$ then

$$\|\pi_\infty(f)\| = \overline{\lim}_{z\to\infty} \|f(z)\|.$$

The proof of this is similar to the proof that $\|\pi(s)\| = \overline{\lim}_{n\to\infty} |s_n|$, where $s = (s_n) \in \ell^\infty$ is a bounded sequence of complex numbers and $\pi$ is the quotient map corresponding to the ideal $c_0$ of sequences $s$ with $s_n \to 0$ as $n \to \infty$. Now, this formula for $\|\pi(s)\|$ is obtained using truncations of a sequence $s$. That is, we consider sequences $s'$ of the form $s'_n = s_n$ for $n \leq N$ (and $s'_n = 0$ for $n > N$) for some $N$. What we are really using is the fact that $\ell^\infty$ is closed under multiplication by elements of $c_0$.

In a similar way, in order to prove the formula for $\|\pi_\infty(f)\|$ we need to show that $B$ is closed under multiplication by elements of $C_0(\mathbb{R})$, where this multiplication is defined in the obvious way. So, if $h \in C_0(\mathbb{R})$, we need to show that $hf \in B$.

The first clause in the definition of $B$ is satisfied since $h$ is bounded. The second clause is satisfied since $h$ is continuous. For the third clause we need to show that $((hf)(n-\alpha))_{n\geq 1} = (h(n-\alpha)f(n-\alpha))_{n\geq 1}$ is an element of $A$. It is certainly a member of $I_0$, so it must belong to $A$. For the fourth clause we need to show that $\alpha \longmapsto (h(n-\alpha)f(n-\alpha))_{n\geq 1}$ is continuous. This is true because this is basically a product of two continuous functions. It follows that $hf \in B$, and hence the required formula for $\|\pi_\infty(f)\|$ can indeed be proved.

Denote by $B'$ the image $j(B)$. Now let $\sigma_x : \oplus_{y\in\widehat{\mathbb{R}}_+} B_y \to B_x$ denote the $x$'th coordinate map. Then

$$B' = \{\gamma \in \oplus_x B_x : \exists f \in B \text{ with } \pi_x(f) = \sigma_x(\gamma) \;\; \forall x \in \widehat{\mathbb{R}}_+\}.$$

We usually write $\gamma_x$ for $\sigma_x(\gamma)$.

Any $f \in C(\widehat{\mathbb{R}}_+)$ acts on $\oplus_x B_x$ by sending $\gamma \in \oplus_x B_x$ to $f\gamma$ where $(f\gamma)_x = f(x)\gamma_x$. Now $B'$ is not necessarily invariant under this action, so we enlarge $B'$ to $B''$, the smallest $C^*$-subalgebra of $\oplus_x B_x$ such that $B' \subset B''$ and $C(\widehat{\mathbb{R}}_+)B'' \subset B''$.

We can now define our continuous bundle on $\widehat{\mathbb{R}}_+$. The bundle algebra is $B'' \subset \oplus_{x\in\widehat{\mathbb{R}}_+} B_x$. The fibre at $x \in \widehat{\mathbb{R}}_+$ is $B_x$, with the fibre map $B'' \to B_x$ being $\sigma_x|B''$. This definition means that the required module properties are trivially satisfied. Faithfulness is also clearly satisfied. The fibre maps are surjective since

$$\sigma_x(B'') \supset \sigma_x(B') = \pi_x(B) = B_x.$$

Finally we must check continuity.

Now $B''$ is the closure of

$$\left\{\sum g_i\gamma_i : g_i \in C(\widehat{\mathbb{R}}_+), \gamma_i \in B', \text{ sum finite}\right\}.$$

So first consider a typical $\sum g_i\gamma_i$ where $\gamma_i \in B'$ and so $(\gamma_i)_x = \pi_x(f_i)$ for some $f_i \in B$. We wish to consider continuity of the function

$$\begin{aligned} x \quad \longmapsto \quad & \left\|(\sum g_i\gamma_i)_x\right\| \\ = \quad & \left\|\sum g_i(x)\pi_x(f_i)\right\| \\ = \quad & \left\|\pi_x(\sum g_i(x)f_i)\right\| \\ = \quad & \sup_{y\geq x}\left\|\sum g_i(x)f_i(y)\right\| \text{ if } x < \infty \end{aligned}$$

All the functions involved are bounded continuous functions on $\mathbb{R}_+$, so this function is clearly continuous at all finite $x$.

To show continuity at $\infty$, we need to show that (as $x \to \infty$),

$$\sup_{y\geq x}\left\|\sum g_i(x)f_i(y)\right\| \longrightarrow \overline{\lim}_{z\to\infty}\left\|\sum g_i(\infty)f_i(z)\right\|.$$

Here, we're using the fact, discussed previously, that $\|\pi_\infty(f)\| = \overline{\lim}_{z\to\infty}\|f(z)\|$ for $f \in B$.

Now, $\sum g_i(\infty)f_i \in B$ so

$$\sup_{y\geq x}\left\|\sum g_i(\infty)f_i(y)\right\| \longrightarrow \overline{\lim}_{z\to\infty}\left\|\sum g_i(\infty)f_i(z)\right\|.$$

As $x \to \infty$, $|g_i(x) - g_i(\infty)| \to 0$, so

$$\left|\left\|\sum g_i(x)f_i(y)\right\| - \left\|\sum g_i(\infty)f_i(y)\right\|\right| \longrightarrow 0$$

uniformly in $y$, which implies what we need to show. So we have continuity for the element $\sum g_i\gamma_i$. Now an $\epsilon/3$ argument shows that we have continuity for all elements of $B''$. Hence the bundle is continuous. Thus we have now constructed a continuous bundle on $\widehat{\mathbb{R}}_+$ from an initial sequence of $C^*$-algebras.

The above construction does not contain any analogue of Lemma 4.1 in [44]. It appears that such an analogue is not necessary for the above construction. It can be shown that there exists a continuous function $d : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sup_{y\in[x,x+d(x)]}\|f(y)\| \to \overline{\lim}_{z\to\infty}\|f(z)\| \quad \text{as } x \to \infty$$

for all $f \in B$. This is in some sense a continuous analogue of Lemma 4.1 in [44]. However, this is not required in the above construction, where we have essentially taken $d(x) = \infty$ for all $x$.

We are now in a position to prove the following.

**Proposition 4.2.1.** *Suppose that $C$ is a non-exact $C^*$-algebra. Then there exists a continuous bundle of $C^*$-algebras $\mathcal{A}$ on the unit interval such that $\mathcal{A} \otimes C$ is not continuous.*

*Proof.* Non-exactness of $C$ implies that the canonical sequence

$$0 \to I_0 \otimes C \hookrightarrow M \otimes C \to M/I_0 \otimes C \to 0$$

is not exact (see Kirchberg [40]). Here $M = \oplus_{i=1}^{\infty} M_i$ and $I_0$ is the set of sequences in this direct sum which tend to zero. Hence, denoting the quotient map from $M$ to $M/I_0$ by $\pi$, there exists $x \in M \otimes C$ with $(\pi \otimes \mathrm{id}_C)(x) = 0$ but $x \notin I_0 \otimes C$.

Now we let $A_n = M_n$ in the construction described above, with the usual embeddings $M_n \hookrightarrow M_{n+1}$. Define $A = C^*(I_0, \{x_{nj}\})$. Here we have chosen $x_n \in M \odot C$ with $x_n \to x$, and we have then written $x_n$ as the finite sum $\sum_j x_{nj} \otimes c_{nj}$. Note that $A$ is separable, and that its definition ensures that $x \in A \otimes C$.

The above construction gives a continuous bundle

$$\mathcal{A} = (\widehat{\mathbb{R}}_+, \sigma_x|B'' : B'' \to B_x, B'').$$

We wish to show that the bundle $\mathcal{A} \otimes C$ is discontinuous. This entails finding a 'bad' element in $B'' \otimes C$ which is somehow related to $x$.

First note that $B'' \otimes C \supset j(B) \otimes C \cong B \otimes C$. Also, we have an embedding $\iota : B \hookrightarrow C[0,1] \otimes A$ and hence there is an embedding

$$\iota \otimes \mathrm{id}_C : B \otimes C \hookrightarrow C[0,1] \otimes A \otimes C.$$

Also, given $a = (a_n) \in A$ we can construct a corresponding element of $B$: if $\Lambda \in C[0,1]$ is given by

$$\Lambda(t) = \begin{cases} 2t & \text{for } t \leq 1/2 \\ 2 - 2t & \text{for } t \geq 1/2 \end{cases}$$

then we can consider $\Lambda \otimes a \in C[0,1] \otimes A$. Each $x_{nj} \in A$, so we obtain corresponding elements $\Lambda \otimes x_{nj} \in B$. Then $\Lambda \otimes x_{nj} \otimes c_{nj} \in B \otimes C$, so summing over $j$ gives $\Lambda \otimes x_n \in B \otimes C$. Taking the limit as $n \to \infty$ we get $\Lambda \otimes x \in B \otimes C$.

We claim that $\Lambda \otimes x$, considered as an element of $B'' \otimes C$, is the required 'bad' element. That is, we claim $(\sigma_\infty \otimes \mathrm{id})(\Lambda \otimes x) = 0$ while

$$\overline{\lim}_{y \in \mathbb{R}_+} \|(\sigma_y \otimes \mathrm{id})(\Lambda \otimes x)\| > 0.$$

We know that $(\pi \otimes \mathrm{id})(x) = 0$ so $(\pi \otimes \mathrm{id})(x_n) \to 0$ as $n \to \infty$, hence $\lim_n \|\sum_j \pi(x_{nj}) \otimes c_{nj}\| = 0$. Consider the map $\sigma : A/I_0 \to B_\infty$ given by $\pi(a) \longmapsto \sigma_\infty(\Lambda \otimes a)$. This function is well-defined, linear and positive. In fact, it is easy to see that $\sigma$ is completely positive, which implies that $\sigma \otimes \mathrm{id}_C$ is also completely positive, and hence bounded. Therefore

$$\begin{aligned} \left\| \sum_j \sigma_\infty(\Lambda \otimes x_{nj}) \otimes c_{nj} \right\| &= \left\| (\sigma \otimes \mathrm{id}_C) \sum_j \pi(x_{nj}) \otimes c_{nj} \right\| \\ &\leq \|\sigma \otimes \mathrm{id}_C\| \left\| \sum_j \pi(x_{nj}) \otimes c_{nj} \right\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

So $(\sigma_\infty \otimes \mathrm{id})(\Lambda \otimes x_n) \to 0$ as $n \to \infty$. Hence, $(\sigma_\infty \otimes \mathrm{id})(\Lambda \otimes x) = 0$, which is the first part of what we wanted to show.

Denote by $\pi_n : A \to M_n$ the coordinate map sending $(a_n) \in A$ to $a_n \in M_n$. As $x \notin I_0 \otimes C$ we have

$$0 < \mathrm{dist}(x, I_0 \otimes C) = \overline{\lim}_n \|(\pi_n \otimes \mathrm{id})x\|.$$

Consider the map $\zeta_n : B_{n-1/2} \to M_n$ given by $\sigma_{n-1/2}(f) \longmapsto f(n-1/2)$ where $f \in B$. This is a well-defined $*$-homomorphism. Now,

$$\begin{aligned}
0 &< \overline{\lim}_n \|(\pi_n \otimes \mathrm{id})x\| \\
&= \overline{\lim}_n \lim_N \|\sum_j \pi_n(x_{Nj}) \otimes c_{Nj}\| \\
&= \overline{\lim}_n \lim_N \|\sum_j \zeta_n(\sigma_{n-1/2}(\Lambda \otimes x_{Nj})) \otimes c_{Nj}\| \\
&= \overline{\lim}_n \lim_N \|(\zeta_n \otimes \mathrm{id}_C)\sum_j \sigma_{n-1/2}(\Lambda \otimes x_{Nj}) \otimes c_{Nj}\| \\
&\le \overline{\lim}_n \lim_N \|\sum_j \sigma_{n-1/2}(\Lambda \otimes x_{Nj}) \otimes c_{Nj}\| \\
&= \overline{\lim}_n \lim_N \|(\sigma_{n-1/2} \otimes \mathrm{id}_C)(\Lambda \otimes x_N)\| \\
&= \overline{\lim}_n \|(\sigma_{n-1/2} \otimes \mathrm{id}_C)(\Lambda \otimes x)\| \\
&\le \overline{\lim}_{y\in\mathbb{R}_+} \|(\sigma_y \otimes \mathrm{id}_C)(\Lambda \otimes x)\|.
\end{aligned}$$

Thus $y \longmapsto \|(\sigma_y \otimes \mathrm{id}_C)(\Lambda \otimes x)\|$ is discontinuous at $y = \infty$. Hence the bundle $\mathcal{A} \otimes C$ is not continuous. □

**Corollary 4.2.2.** *Fix a $C^*$-algebra $B$. Then $B$ is exact if and only if for any continuous bundle $\mathcal{A}$ of $C^*$-algebras on $[0,1]$ (with separable bundle $C^*$-algebra), $\mathcal{A} \otimes B$ is continuous.*

*Proof.* This follows from Theorem 4.5 of [44], together with the above Proposition. □

So, we now have a new characterisation of exactness of $C^*$-algebras, in terms of the continuity of bundles with base space $[0,1]$. The bundle obtained in the above Proposition can also be modified to give examples of other bundles on $[0,1]$ with interesting properties.

Let $\mathcal{A} = ([0,1], \pi_t : A \to A_t, A)$ be the continuous bundle defined in the proof of the above Proposition, except that we assume the base space to be $[0,1]$. Now define a new bundle $\mathcal{B}$ on $[0,2]$. We suppose that $\mathcal{B}$ has bundle $C^*$-algebra

$$B = \{a \oplus a' \in A \oplus A : \pi_1(a) = \pi_1(a')\}.$$

For $y \in [0,2]$ we define the fibre map $\sigma_y : B \to B_y$ as follows:

$$\sigma_y(a \oplus a') = \begin{cases} \pi_y(a) & \text{for } y \le 1 \\ \pi_{2-y}(a') & \text{for } y \ge 1 \end{cases}$$

For $f \in C[0,2]$ and $a \oplus a' \in B$, the module action is defined via

$$f(a \oplus a') = f_1 a \oplus f_2 a'$$

where $f_1 \in C[0,1]$ is given by $f_1(t) = f(t)$ and $f_2 \in C[0,1]$ is given by $f_2(t) = f(2-t)$. With these definitions, it is easy to see that $\mathcal{B}$ is a continuous bundle on $[0,2]$.

If $C$ is a non-exact $C^*$-algebra, then $\mathcal{A} \otimes C$ is discontinuous at $\infty \in \widehat{\mathbb{R}}_+$, which corresponds to $1 \in [0,1]$. So $\mathcal{B} \otimes C$ is discontinuous at $1 \in [0,2]$. Since any two bounded closed intervals are homeomorphic, this means that for any $t \in [0,1]$ we can construct a continuous bundle of $C^*$-algebras $\mathcal{A}^{(t)}$ on $[0,1]$ such that $\mathcal{A}^{(t)} \otimes C$ fails to be continuous at $t$.

Taking finite direct sums of these bundles $\mathcal{A}^{(t)}$, we can construct a continuous bundle on $[0,1]$ such that, on tensoring with $C$, the bundle is discontinuous at any chosen finite set of points in $[0,1]$.

In fact, we can also take countable direct sums. Suppose that we have a sequence $(t_n)_{n=1}^{\infty}$ in $[0,1]$. Letting $A_n$ be the bundle $C^*$-algebra corresponding to the bundle $\mathcal{A}^{(t_n)}$, we can then form a bundle from the restricted direct sum of these bundle $C^*$-algebras. This consists of sequences $(a_n)_{n=1}^{\infty}$ where $a_n \in A_n$ for every $n$, and $\|a_n\| \to 0$ as $n \to \infty$.

Using the restricted direct sum gives a bundle that is still continuous and yet, on tensoring with $C$, gives a bundle that is discontinuous at every $t_n \in [0,1]$. The reason that the restricted direct sum bundle is continuous is as follows. If $f_n \in C[0,1]$ is positive for every $n \in \mathbb{N}$, then $f = \sup_{n \in \mathbb{N}} f_n$ is not necessarily continuous. However, if we also insist that $\|f_n\|_\infty \to 0$ as $n \to \infty$ (which is the condition imposed by the restricted direct sum), then $f$ is guaranteed to be continuous.

It would be interesting to know if there was a continuous bundle $\mathcal{B}$ on $[0,1]$ such that $\mathcal{B} \otimes C$ is discontinuous at every point of $[0,1]$.

## 4.3 Continuous bundles on infinite compact metric spaces

In this Section we construct a continuous bundle of $C^*$-algebras on any fixed infinite compact metric space with properties analogous to those of the bundles on $[0,1]$ constructed above. This is done via the induced or pullback bundle construction. We first review this induced bundle construction. Then we use it to construct continuous bundles on any fixed infinite compact metric space. Finally we show how this leads to new characterisations of exactness for $C^*$-algebras.

First we look at the induced bundle construction. This is a well-known construction in the context of topological fibre bundles (see, for example, page 47 of [59]). Kirchberg and Phillips consider this construction in the context of continuous fields of $C^*$-algebras (see Lemma 1.3 of [43]). Here we consider these ideas from the point of view of bundles of $C^*$-algebras.

Suppose $\mathcal{A} = (X, \pi_x : A \to A_x, A)$ is a continuous bundle of $C^*$-algebras and that $\eta : Y \to X$ is a continuous map. Here we assume for simplicity that $X$ and $Y$ are compact metric spaces. We wish to define a bundle $P_\eta(\mathcal{A})$ on $Y$, the pullback of $\mathcal{A}$ via $\eta$. For $y \in Y$ define $B_y = A_{\eta(y)}$. Define $\sigma_y : A \to B_y$ by $a \longmapsto \pi_{\eta(y)}(a)$. If we assume that $\eta$ is surjective then

$$\sigma_y(a) = 0 \;\; \forall y \in Y \Rightarrow a = 0.$$

So there is an embedding $j : A \hookrightarrow \oplus_{y \in Y} B_y$ which sends $a$ to $(\sigma_y(a))_{y \in Y}$.

Now enlarge $j(A)$ to $B$, the smallest $C^*$-subalgebra of $\oplus_{y \in Y} B_y$ such that $A \subset B$ and $B$ is closed under the obvious action of $C(Y)$ on $\oplus_{y \in Y} B_y$. We define the pullback of $\mathcal{A}$ to be $(Y, \tau_y : B \to B_y, B)$ where $\tau_y : \oplus_{z \in Y} B_z \to B_y$ is the usual coordinate map. Faithfulness and the $C(Y)$-module properties are clear for this bundle. Is the bundle continuous?

If $a \in A$ then $y \longmapsto \|\tau_y(j(a))\|$ is continuous, from the continuity of the bundle $\mathcal{A}$ and the continuity of the map $\eta$. Now $B$ is the closure of the set of finite sums $\sum g_i j(a_i)$ where $a_i \in A$ and $g_i \in C(Y)$. So to show that the bundle is continuous, it suffices to show that $y \longmapsto \|\tau_y(\sum g_i j(a_i))\| = \|\sigma_y(\sum g_i(y) a_i)\|$ is continuous.

Fixing $y \in Y$ and $\epsilon > 0$, consider

$$\left| \left\| \sigma_y(\sum g_i(y) a_i) \right\| - \left\| \sigma_{y'}(\sum g_i(y') a_i) \right\| \right|$$

which is

$$\leq \left| \left\| \sigma_y(\sum g_i(y) a_i) \right\| - \left\| \sigma_{y'}(\sum g_i(y) a_i) \right\| \right| + \left| \left\| \sigma_{y'}(\sum g_i(y) a_i) \right\| - \left\| \sigma_{y'}(\sum g_i(y') a_i) \right\| \right|.$$

Continuity of $z \longmapsto \|\sigma_z(\sum g_i(y) a_i)\|$ $(z \in Y)$ at $z = y$ shows that the first term is $< \epsilon/2$ for $y'$ suitably close to $y$. For even closer $y'$ we can assume that, for every $i$, we have

$$|g_i(y) - g_i(y')| < \frac{\epsilon}{2(1 + \sum \|a_i\|)}.$$

Then the second term is

$$\begin{aligned} &\leq \left\| \sigma_{y'}(\sum g_i(y) a_i) - \sigma_{y'}(\sum g_i(y') a_i) \right\| \\ &\leq \left\| \sum (g_i(y) - g_i(y')) a_i \right\| \\ &\leq \epsilon/2. \end{aligned}$$

This shows that the pullback bundle is indeed continuous.

We also require the notion of restricting a bundle. Suppose that $\mathcal{A} = (X, \pi_x : A \to A_x, A)$ is continuous and that $Z$ is a closed or open subset of the compact metric space $X$. We can then define the continuous bundle $\mathcal{A}|Z = (Z, \sigma_z : A' \to A_z, A')$. Here $A' = A/\{a \in A : \pi_z(a) = 0 \ \forall z \in Z\}$ while $\sigma_z(\overline{a}) = \pi_z(a)$. The module action is given by $f\overline{a} = \overline{f_1 a}$ where $f_1$ is any continuous extension of $f \in C_0(Z)$ to $X$ (which exists by Tietze's extension theorem).

We can now prove the following.

**Proposition 4.3.1.** *Suppose that $(X, d)$ is an infinite compact metric space and that $C$ is a non-exact $C^*$-algebra. Then there exists a continuous bundle $\mathcal{B}$ on $X$ such that $\mathcal{B} \otimes C$ is not continuous.*

*Proof.* As $X$ is not discrete, there exists a sequence $x_n$ ($n \in \mathbb{N}$) and $x \in X$ such that, if $d_n = d(x_n, x)$, then the $d_n$ are distinct non-zero and satisfy $d_n \downarrow 0$. In particular $x_n \to x$ as $n \to \infty$.

We know that there exists a continuous bundle $\mathcal{B} = ([0,1], \sigma_x : B \to B_x, B)$ such that $\mathcal{B} \otimes C$ is not continuous. As remarked in Section 4.2, we can allow the point of discontinuity to be any point of $[0,1]$. It is simplest to suppose that the discontinuity occurs at 0. The proof of Proposition 4.2.1 then gives $z \in B \otimes C$ for which $(\sigma_0 \otimes \mathrm{id}_C)z = 0$ but $\overline{\lim}_{x \to 0} \|(\sigma_x \otimes \mathrm{id}_C)z\| > 0$.

There is also a map $\eta : X \to \mathbb{R}_+$ defined by $y \longmapsto d(x, y)$. By scaling we can assume that $\eta(X) \subset [0,1]$ since $X$ is compact. So we have $\eta : X \to [0,1]$. In order to pullback we need $\eta$ to be surjective so we consider the map $\eta : X \to \eta(X)$. Now $\eta(X)$ is a compact subset of $[0,1]$ so we may define the restriction bundle $\mathcal{B}|\eta(X)$. This is a continuous bundle on $\eta(X)$. Let the bundle algebra be denoted by $B'$ and let the fibre maps be denoted by $\sigma'_y$ for $y \in \eta(X)$.

Clearly $0 \in \eta(X)$ and we claim that $(\mathcal{B}|\eta(X)) \otimes C$ is not continuous at 0. The 'bad' element is $(q \otimes \mathrm{id}_C)z = z'$ say, where $q$ is the quotient map $B \to B'$. Indeed,

$$(\sigma'_0 \otimes \mathrm{id}_C)z' = (\sigma'_0 q \otimes \mathrm{id}_C)z = (\sigma_0 \otimes \mathrm{id}_C)z = 0.$$

On the other hand, we have the sequence $x_n \in X$ with $x_n \to x$. Let $y_n = \eta(x_n)$ so that $y_n \to 0$. Then

$$\|(\sigma'_{y_n} \otimes \mathrm{id}_C)z'\| = \|(\sigma_{y_n} \otimes \mathrm{id}_C)z\|$$

and $\overline{\lim}_{n\to\infty}\|(\sigma_{y_n}\otimes \mathrm{id}_C)z\| > 0$, looking back at the proof of Proposition 4.2.1. (Here we have implicitly used a homeomorphism $\mathbb{R}_+ \cong (0,1]$. We must make sure that the sequence $(n-1/2)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ is mapped onto the sequence $(y_n)_{n\in\mathbb{N}}$ in $(0,1]$ by this homeomorphism.) So the bundle $(\mathcal{B}|\eta(X))\otimes C$ is not continuous at 0.

Now consider the pullback bundle $P_\eta(\mathcal{B}|\eta(X))$, which is continuous. Let the bundle algebra be $B''$ and let the fibre maps be $\tau_z : B'' \to B''_z$ for $z \in X$. Recall the embedding $j : B' \hookrightarrow \oplus_{z\in X} B''_z$ from the pullback construction. We now have the 'bad' element $z'' = (j\otimes \mathrm{id}_C)z' \in B''\otimes C$. Now

$$(\tau_x \otimes \mathrm{id}_C)z'' = (\sigma'_{\eta(x)} \otimes \mathrm{id}_C)z' = (\sigma'_0 \otimes \mathrm{id}_C)z' = 0.$$

On the other hand,

$$\overline{\lim}_{n\to\infty}\|(\tau_{x_n}\otimes \mathrm{id}_C)z''\| = \overline{\lim}_{n\to\infty}\|(\sigma'_{y_n}\otimes \mathrm{id}_C)z'\| > 0.$$

So $(P_\eta(\mathcal{B}|\eta(X)))\otimes C$ is not continuous at $x \in X$. This proves the Proposition. □

This construction provides the following characterisation of exactness of $C^*$-algebras.

**Corollary 4.3.2.** *Fix an infinite compact metric space $X$ and a $C^*$-algebra $B$. Then $B$ is exact if and only if for any continuous bundle $\mathcal{A}$ of $C^*$-algebras on $X$ (with separable bundle $C^*$-algebra), $\mathcal{A}\otimes B$ is continuous.*

*Proof.* This follows from Theorem 4.5 of [44], along with the above proposition. Note that if a bundle has separable bundle $C^*$-algebra then so does any restriction or pullback of it. Hence the statement about separability is valid in the above Corollary. □

This result can probably be extended to more general spaces $X$.

# Chapter 5

# Free product bundles

## 5.1 Introduction

In this chapter we look at free product bundles. To be more precise, we look at free product operations on continuous bundles of $C^*$-algebras, in the same way that Chapter 4 considered tensor product operations on continuous bundles of $C^*$-algebras.

Why consider free product operations on continuous bundles? Well, in Chapter 4 we looked at the minimal tensor product operation on continuous bundles, and in doing so we obtained new characterisations of exactness in terms of the continuity of certain bundles. In [44] maximal tensor product bundles were also considered, giving a new characterisation of nuclearity. So it is natural to consider other operations which can be applied to continuous bundles of $C^*$-algebras, perhaps in the hope of obtaining new characterisations of such properties as nuclearity and exactness.

The crossed product operation has been considered by Kirchberg and Wassermann in [45]. In fact there are two crossed product operations, corresponding to the full and reduced crossed products. Continuity of the full crossed product bundle is closely related to amenability of the group involved, whereas continuity of the reduced crossed product bundle is closely related to the exactness of the group. As remarked in [46], although it is not obvious that the reduced crossed product bundle is continuous, there is no known example where continuity fails. This underlines the interesting nature of the bundles constructed in Chapter 4.

This Chapter considers other operations on continuous bundles of $C^*$-algebras. In Section 5.2 we first look at a very simple operation, namely taking the multiplier algebra of a continuous bundle. This multiplier algebra is again a bundle, but continuity is not preserved in general. We also look at full free product bundles and consider when continuity is preserved under this operation.

Neither of the above constructions seems to lead to characterisations of exactness or nuclearity, so we then turn our attention to reduced free product bundles. Study of these combines reduced free products, as studied in Chapters 2 and 3, with the continuous bundles studied in Chapter 4. Recall that nuclearity and exactness are concepts defined in terms of $C^*$-algebra tensor products. So it is not surprising that these concepts should be related to continuity of the minimal and maximal tensor product operations on continuous bundles of $C^*$-algebras.

Why should these concepts be related to some reduced free product operation on continuous bundles? Well, it is certainly not obvious why this should be so. However, it is certainly known now that exactness is in some way connected with reduced free products. Most importantly, it was shown by Dykema [20] that a reduced amalgamated free product $C^*$-algebra is exact precisely when all the factors are exact $C^*$-algebras, a result which fails to hold in the case of full free products. This gives us some reason to expect a connection between exactness and a reduced free product operation on continuous bundles.

In Section 5.3 we construct a suitable definition of a reduced free product bundle. It turns out that there are two bundles which may reasonably be called a reduced free product bundle. These are denoted by $C^u$ and $C^l$. Section 5.4 considers the continuity of $C^l$. This bundle is always lower semicontinuous. We show that, at least in certain special cases, it is actually continuous.

Section 5.5 considers the continuity of $C^u$, which is always upper semicontinuous. Assuming exactness of the $C^*$-algebras involved, we show that this bundle is actually continuous. The proof makes use of the Cuntz-Pimsner $C^*$-algebras. These $C^*$-algebras were important in Chapter 3, so this work provides an interesting connection between Chapter 3 and Chapter 5.

Finally, in Section 5.6 we consider the relationship between continuity of $C^u$ and continuity of $C^l$. We look at possible applications of these results, such as the embedding of a continuous bundle into a larger continuous bundle whose fibres are simple.

Note that free products of $C(X)$-algebras, amalgamating over $C(X)$, have been considered before by Germain [31]. However, Germain considered these free products from a somewhat different viewpoint and, for example, there is no reference to the continuity of the $C(X)$-algebras involved.

## 5.2 Multiplier algebra bundles and full free product bundles

In this Section we first look at the construction and continuity of the multiplier algebra bundle, before moving on to the full free product bundle.

Let $(X, \pi_x : A \to A_x, A)$ be a continuous bundle of $C^*$-algebras on a locally compact Hausdorff space $X$. Consider the multiplier algebra $M(A)$. We show how this can be made into a bundle. We use the double centraliser interpretation of $M(A)$, as contained in Wegge-Olsen [67] for example.

Assume the bundle algebra $A$ is separable. Then the surjective $*$-homomorphism $\pi_x : A \to A_x$ can be extended to a $*$-homomorphism $\widetilde{\pi}_x : M(A) \to M(A_x)$ which is also surjective.

Suppose $(L, R) \in M(A)$ and $\widetilde{\pi}_x((L, R)) = 0$ for all $x \in X$. If $(L_x, R_x) = \widetilde{\pi}_x((L, R)) \in M(A_x)$ then

$$\forall a \in A \, \forall x \in X \quad L_x(a_x) = \pi_x(L(a)) = 0.$$

So $L(a) = 0$ for all $a \in A$. Hence $L = 0$, and similarly $R = 0$. Hence $\{\widetilde{\pi}_x : x \in X\}$ is faithful.

Now we check that the required module property is satisfied. Since $A$ is a bundle over $X$, we have $C_0(X) \subset \mathcal{Z}(M(A))$. Let $f \in C_0(X)$. Write $f = (L_f, R_f) \in M(A)$, where $L_f(a) = fa$ and $R_f(a) = af$. If $\widetilde{\pi}_x(f) = (L_x, R_x)$ then

$$L_x(a_x) = \pi_x(L_f(a)) = \pi_x(fa) = f(x)a_x$$

and similarly $R_x(a_x) = f(x)a_x$. Hence $\widetilde{\pi}_x(f) = f(x)1$. So, if $m \in M(A)$, then

$$\widetilde{\pi}_x(fm) = \widetilde{\pi}_x(f)\widetilde{\pi}_x(m) = f(x)\widetilde{\pi}_x(m).$$

So $(X, \widetilde{\pi}_x : M(A) \to M(A_x), M(A))$ is a bundle of $C^*$-algebras on the space $X$. Is it continuous if the original bundle is continuous? Of course, if $A$ is unital then the answer is trivially yes. In general, the answer is no.

**Proposition 5.2.1.** *The multiplier algebra bundle operation does not preserve continuity of bundles.*

*Proof.* Consider the trivial bundle $C(\widehat{\mathbb{N}}, K(\ell^2(\mathbb{N})))$, which is certainly continuous. It can be seen that the multiplier algebra of this is $C_b(\widehat{\mathbb{N}}, B(\ell^2)_\beta)$ where $\beta$ denotes the strict topology on $B(\ell^2)$. (Indeed this is an exercise in [67].) This is the set of functions $\widehat{\mathbb{N}} \to B(\ell^2)_\beta$ which are continuous and norm-bounded. The fibre maps are given by evaluations at each point of $\widehat{\mathbb{N}}$.

Another exercise in [67] considers the projections $Q_n \in B(\ell^2)$, where $Q_n$ projects onto $\mathbb{C}e_n$. Here $e_n$ is the $n$'th element of the usual orthonormal basis for $\ell^2$. Now $\|Q_n\| = 1$ for all $n$, but $Q_n \to 0$ in the strict topology. So define $f : \widehat{\mathbb{N}} \to B(\ell^2)_\beta$ via $n \longmapsto Q_n$ and $\infty \longmapsto 0$. With this definition, $f \in C_b(\widehat{\mathbb{N}}, B(\ell^2)_\beta)$ but $n \longmapsto \|f(n)\|$ is clearly not continuous at $\infty$. That is to say, the multiplier algebra bundle fails to be continuous, even though we started off with a trivial bundle. □

We now move on to the construction of full free product bundles. Let $\mathcal{A} = (X, \pi_x : A \to A_x, A)$ be a continuous bundle of $C^*$-algebras. For simplicity we assume that $A$ is unital and that $X$ is compact Hausdorff. Let $B$ be a fixed unital $C^*$-algebra. We wish to define a full free product bundle $\mathcal{A} * B$ over the same space $X$.

We let the bundle algebra be $C = A *_{C(X)} C(X, B)$, while the fibre at $x \in X$ is given by $C_x = A_x * B$. How do we obtain the fibre maps? The universal property of the full amalgamated free product implies that there is a surjective $*$-homomorphism

$$\sigma_x = \pi_x * \mathrm{ev}_x : C \to A_x * B$$

such that $\sigma_x|A = \pi_x$ and $\sigma_x|C(X, B) = \mathrm{ev}_x$.

Now we check the module property. As $C(X) \subset \mathcal{Z}(A)$ and $C(X) \subset \mathcal{Z}(C(X, B))$, it follows that $C(X) \subset \mathcal{Z}(C)$. So the maps $\sigma_x$ clearly satisfy the required module property for a bundle:

$$\sigma_x(fc) = \sigma_x(f)\sigma_x(c) = f(x)\sigma_x(c) \quad \forall f \in C(X),\ \forall c \in C.$$

Now consider the quotient $C/C_x(X)C$. Note $A_x$ and $B$ both embed in this quotient, and that $C^*(A_x, B) = C/C_x(X)C$. It is easily checked that $C/C_x(X)C$ satisfies the universal property required of $A_x * B$. Hence $C/C_x(X)C \cong A_x * B$ canonically. This means that $C$ has the structure of a $C(X)$-algebra, and the maps $\sigma_x$ may be thought of as quotient maps $C \to C/C_x(X)C$. Proposition 2.8 of [8] then implies that the family $\{\sigma_x\}_{x \in X}$ is faithful. Hence $C$ is indeed a bundle over $X$.

It also follows from the above paragraph that $C$ is always upper semicontinuous. When is it continuous?

First note that, if $A$ and $B$ are $C^*$-algebras, then

$$C(X,A) *_{C(X)} C(X,B) \cong C(X, A * B).$$

It follows that if $\mathcal{A}$ is a trivial bundle over $X$, with fibre $A$, then $\mathcal{A} * B$ is the trivial bundle over $X$ with fibre $A * B$. Thus the full free product operation preserves the triviality of bundles. In fact, it preserves the subtriviality of bundles too, and this enables us to prove the following.

**Proposition 5.2.2.** *Let $\mathcal{A} = (X, \pi_x : A \to A_x, A)$ be a continuous bundle of $C^*$-algebras with $A$ unital separable and $X$ a compact metric space. If $A$ is exact, then the full free product bundle $\mathcal{A} * B$ is continuous, regardless of $B$.*

*Proof.* The results of [9] imply that there is a $C(X)$-linear embedding $\alpha : A \hookrightarrow C(X, \mathcal{O}_2)$. In particular, $A$ is subtrivial.

We consider $\alpha$ as a map $\alpha : A \hookrightarrow C(X, \mathcal{O}_2 * B)$. We also have the obvious embedding $\mathrm{id} : C(X,B) \hookrightarrow C(X, \mathcal{O}_2 * B)$. The universal property of the full amalgamated free product then provides a $*$-homomorphism $\alpha * \mathrm{id} : C \to C(X, \mathcal{O}_2 * B)$. We claim that this is isometric, or equivalently injective.

Recall that if $\beta : E \to F$ is a $C(X)$-linear $*$-homomorphism between $C(X)$-algebras, then $\beta$ is injective precisely when all the induced maps $\beta_x : E_x \to F_x$ are injective. So, to prove the claim, it suffices to show that any map $(\alpha * \mathrm{id})_x : C_x \to \mathcal{O}_2 * B$ is injective.

Now, $(\alpha * \mathrm{id})_x = \alpha_x * \mathrm{id}_B$. Both $\alpha_x$ and $\mathrm{id}_B$ are injective, hence their free product is too. So $(\alpha * \mathrm{id})_x$ is injective for any $x$, and the claim is true.

Note also that $(\alpha * \mathrm{id})_x \circ \sigma_x = \mathrm{ev}_x \circ (\alpha * \mathrm{id})$ on $C$. This is because $C = C^*(A, C(X,B))$ and it is easily seen that the two $*$-homomorphisms agree on $A$ and $C(X,B)$.

Therefore, for $c \in C$, the function

$$\begin{aligned} x \quad \longmapsto \quad & \|\sigma_x(c)\| \\ = \quad & \|(\alpha * \mathrm{id})_x \circ \sigma_x(c)\| \\ = \quad & \|\mathrm{ev}_x \circ (\alpha * \mathrm{id})(c)\| \end{aligned}$$

is certainly continuous. Hence, the full free product bundle is continuous. □

Clearly the proof of the above result depends only upon the subtriviality of the bundle $\mathcal{A}$. So it can be seen that any subtrivial continuous bundle $\mathcal{A}$ will have its continuity preserved by the full free product operation. This is in contrast to the situation for maximal tensor products. Kirchberg and Wassermann [44] construct a subtrivial continuous bundle $\mathcal{D}$ such that $\mathcal{D} \otimes_{max} C$ is not continuous, where $C$ is a non-nuclear $C^*$-algebra.

One reason for this difference is that if $A_1 \subset A_2$ and $B$ is another $C^*$-algebra, then $A_1 * B \subset A_2 * B$ but $A_1 \otimes_{max} B$ is not necessarily contained in $A_2 \otimes_{max} B$. So full free product bundles are better behaved than you might at first expect. It seems unlikely that continuity is always preserved by the full free product operation, and it would be interesting to obtain an example of a continuous bundle for which continuity is not preserved. Such a bundle would necessarily be non-subtrivial.

## 5.3 Construction of reduced free product bundles

Here we construct the two reduced free product bundles $C^u$ and $C^l$. Suppose that $\mathcal{A} = (X, \pi_x : A \to A_x, A)$ is a continuous bundle of $C^*$-algebras. We assume that $A$ is unital and that $X$ is compact Hausdorff. We wish to take the reduced free product of $\mathcal{A}$ with some fixed unital $C^*$-algebra $B$. For the construction, we need to assume that $A$ and $B$ are separable.

To define a reduced free product, it is necessary to attach states (or conditional expectations) to the $C^*$-algebras involved. For this reason, we assume the existence of a continuous field of faithful states on $A$. That is, we assume the existence of faithful $f_x \in S(A_x)$ for all $x \in X$, imposing the condition that $x \longmapsto f_x(a_x)$ is continuous for every $a \in A$. The existence of such states is guaranteed by results in section 3 of Blanchard's thesis [8].

The continuous field of faithful states gives in particular a faithful conditional expectation $f : A \to C(X)$ where $f(a)(x) = f_x(a_x)$. We suppose we have a faithful state $\phi \in S(B)$. As $B$ is separable, such a state is guaranteed to exist. Now we get a faithful conditional expectation $R_\phi : C(X, B) \to C(X)$ where $R_\phi(g) = \phi \circ g$. Using the reduced amalgamated free product construction (see section 1.4) we obtain

$$(C, \psi) = (A, f) *_{C(X)} (C(X, B), R_\phi).$$

The properties of the reduced amalgamated free product imply that, as $A$ is a $C(X)$-algebra, $C$ is also a $C(X)$-algebra in the obvious way.

Theorem 2.2.2 implies that the conditional expectation $\psi$ is faithful. This can also be seen in a more direct way: see the construction of $C^l$. We now consider the two possible reduced free product bundles.

**Construction of the bundle $C^u$**

As $C$ is a $C(X)$-algebra, we can make it into a bundle in the usual way. That is, we let

$$C^u = (X, q_x : C \to C/C_x(X)C, C)$$

where $q_x$ is the quotient map. By construction, this bundle is automatically upper semi-continuous. The fibres are $C/C_x(X)C$ for $x \in X$. For a reduced free product bundle, we would like the fibres to be $A_x *_r B$ (recall from Section 1.4 that this notation is used as a shorthand for $(A_x, f_x) * (B, \phi)$) for $x \in X$. This requirement motivates a second possible construction.

**Construction of the bundle $C^l$**

Consider the reduced free products $(C_x, \psi_x) = (A_x, f_x) * (B, \phi)$. We want $C^l$ to have fibres $C_x$ for $x \in X$, so we need a $*$-epimorphism $C \to C_x$. Now $C$ acts on the free product Hilbert $C(X)$-module $E$, so we have a $C(X)$-representation $P : C \to L(E)$.

We have a representation $P_x = P \otimes \mathrm{ev}_x : C \to L(E_x)$, where $E_x$ is the interior tensor product $E \otimes_{\mathrm{ev}_x} \mathbb{C}$. Can we identify the Hilbert space $E_x$?

Well, suppose that $E_1 = L^2(A, f)$ and $E_2 = L^2(C(X, B), R_\phi)$. We then have $E_{1x} = E_1 \otimes_{\mathrm{ev}_x} \mathbb{C} = L^2(A_x, f_x)$ and $E_{2x} = E_2 \otimes_{\mathrm{ev}_x} \mathbb{C} = L^2(B, \phi)$ for $x \in X$. Then

$$E = \xi C(X) \oplus \bigoplus_{\substack{n \geq 1 \\ \iota_1 \neq \cdots \neq \iota_n}} E^o_{\iota_1} \otimes \cdots \otimes E^o_{\iota_n}.$$

Now $E^o_{\iota_1} \otimes \cdots \otimes E^o_{\iota_n} \otimes_{\mathrm{ev}_x} \mathbb{C}$ is canonically isomorphic to $E^o_{\iota_1 x} \otimes \cdots \otimes E^o_{\iota_n x}$. Indeed, $E^o_{\iota_1} \otimes \cdots \otimes E^o_{\iota_n} \otimes_{\mathrm{ev}_x} \mathbb{C}$ is spanned by elements $a_{\iota_1} \cdots a_{\iota_n} \xi \otimes 1$ where $a_{\iota_j} \in A^o$ or $a_{\iota_j} \in C(X, B)^o$ according as $\iota_j = 1$ or $\iota_j = 2$. This corresponds to

$$\widehat{a}_{\iota_1 x} \otimes \cdots \otimes \widehat{a}_{\iota_n x} \in E^o_{\iota_1 x} \otimes \cdots \otimes E^o_{\iota_n x}.$$

So

$$\begin{aligned} E_x &= \xi C(X) \otimes_{\mathrm{ev}_x} \mathbb{C} \oplus \bigoplus_{\substack{n \geq 1 \\ \iota_1 \neq \cdots \neq \iota_n}} E^o_{\iota_1} \otimes \cdots \otimes E^o_{\iota_n} \otimes_{\mathrm{ev}_x} \mathbb{C} \\ &= \xi \mathbb{C} \oplus \bigoplus_{\substack{n \geq 1 \\ \iota_1 \neq \cdots \neq \iota_n}} E^o_{\iota_1 x} \otimes \cdots \otimes E^o_{\iota_n x} \\ &= L^2(C_x, \psi_x). \end{aligned}$$

Thus $E_x$ is the G.N.S. Hilbert space for the pair $(C_x, \psi_x)$.

It can now be checked that, if $a \in A$ then $P_x(a) = a_x$ where we consider $a_x$ as an element of the reduced free product $C_x$. Similarly $P_x(g) = g(x)$ for $g \in C(X, B)$. Since $P_x$ is a $*$-homomorphism, it follows that the image of $P_x$ can be identified with the reduced free product $C_x$. Hence it makes sense to define

$$C^l = (X, P_x : C \to A_x *_r B, C).$$

It is easy to check that $C^l$ satisfies the requirements for a bundle. Suppose $c \in C$ and $P_x(c) = 0$ for all $x \in X$. This implies that $\psi(c^*c)(x) = 0$ $\forall x \in X$. Hence $\psi(c^*c) = 0$ and so $c = 0$ since $\psi$ is faithful. The module property is also satisfied, because the ideal $C_x(X)C$ is contained in $\ker P_x$. Hence $C^l$ is indeed a bundle over $X$.

Note that the above definition of $P_x$ implies that, for $c \in C$, $\psi_x(P_x(c)) = \psi(c)(x)$. That is, $\mathrm{ev}_x \circ \psi = \psi_x \circ P_x$. Being a reduced free product of two faithful states, $\psi_x$ is also a faithful state for every $x \in X$ (by Dykema [18]). It then follows that $\psi$ is a faithful conditional expectation, as remarked earlier in this Section.

Clearly the above constructions can be extended to the case where we are taking the reduced free product of two continuous bundles of $C^*$-algebras, $A_1$ and $A_2$ say. If we attach continuous fields of faithful states $f_1$, $f_2$ to $A_1$, $A_2$ respectively, we can define

$$(C, \psi) = (A_1, f_1) *_{C(X)} (A_2, f_2).$$

The bundles $C^u$ and $C^l$ are then defined in exactly the same manner as above.

## 5.4 Continuity of $C^l$

In this Section we consider the continuity of the reduced free product bundle $C^l$. We discuss one possible method for showing that this bundle is continuous. Then we provide an example which uses this method.

Since $C^l$ is constructed via a $C(X)$-representation of $C$, it follows (see remarks in section 2.3 of [8]) that $C^l$ is automatically lower semicontinuous.

As noted in the construction of $C^l$, $C_x(X)C \subset \ker P_x$. Are these two ideals equal? By Lemma 2.3 of [44], they are equal $\forall x \in X$ precisely when $C^l$ is upper semicontinuous. So, to show continuity of $C^l$, we need to show that $C_x(X)C = \ker P_x$ $\forall x \in X$. It is perhaps reasonable to suggest that $C^l$ is continuous if the original bundle $C^*$-algebra $A$ is exact, regardless of the $C^*$-algebra $B$. We now describe a possible method for proving the continuity of $C^l$, inspired in part by the methods of Effros and Haagerup [25] .

We suppose that $A$ is exact. Fix $x \in X$. Then, as explained in section 4.3 of [44], the sequence

$$0 \to C_x(X)A \otimes B \to A \otimes B \to A_x \otimes B \to 0$$

is exact for arbitrary $B$. By results of Effros and Haagerup (see Theorem 3.2 of [25]) it follows that $\pi_x : A \to A_x$ is locally liftable. That is, given any finite dimensional operator system $Z \subset A_x$ there is a unital completely positive isometry $\rho : Z \to A$ such that $\pi_x \circ \rho = \mathrm{id}_Z$.

In the following, the methods are inspired by the ideas of Effros and Haagerup: see [25] and the first part of the proof of Proposition 6.8 in [66]. As has already been noted, proving that $C^l$ is continuous is equivalent to showing that (for every $x \in X$) the sequence

$$0 \to C_x(X)C \hookrightarrow C \to C_x = A_x *_r B \to 0$$

(where $P_x$ provides the map from $C$ to $C_x$) is exact. Now, as $C_x(X)C \subset \ker P_x$, there is an induced $*$-homomorphism

$$\overline{P}_x : C/C_x(X)C \to A_x *_r B.$$

We want to show that $\overline{P}_x$ is actually isometric. Take $y \in C/C_x(X)C$ where $y$ is represented by a finite sum of reduced words, plus possibly a multiple of the identity. We want to show that $\|\overline{P}_x(y)\| \geq \|y\|$ because, as such $y$ are dense in $C/C_x(X)C$, it will then follow that $\overline{P}_x$ is isometric as required.

For simplicity of notation, assume that $y$ is represented by a sum of the form $\sum a_i g_i a_i'$ where $a_i, a_i' \in A$ and $g_i \in C(X, B)$ are reduced. It will be clear that the methods apply whatever happens to be the form of $y$.

Define $\sigma : B \to C(X, B)$ by $b \longmapsto 1 \otimes b$. Let $Z$ be the finite dimensional operator system generated by the $a_{ix}, a_{ix}'$ in $A_x$, where $a_{ix}$ means $\pi_x(a_i)$ and so on. We know that there exists a unital completely positive isometry $\rho : Z \to A$ such that $\pi_x \circ \rho = \mathrm{id}_Z$. So we have

$$\|y\| = \left\|\sum q(a_i g_i a_i')\right\|_{C/C_x(X)C} = \left\|\sum q(\rho(a_{ix})\sigma(g_i(x))\rho(a_{ix}'))\right\|_{C/C_x(X)C}$$

since $\pi_x(\rho(a_{ix}) - a_i) = 0$, $\mathrm{ev}_x(g_i - \sigma(g_i(x))) = 0$, and so on. This is

$$\leq \left\| \sum \rho(a_{ix})\sigma(g_i(x))\rho(a'_{ix}) \right\|_C$$

by definition of the quotient norm. We would like to show that this is

$$\leq \left\| \sum a_{ix} g_i(x) a'_{ix} \right\|_{A_x *_r B}.$$

To do this, we really want to consider a reduced free product map $\rho * \sigma$. Unfortunately, there are two problems with this. Firstly, the domain of $\rho$ is in general not a $C^*$-algebra, so it is not even clear what the domain of such a free product map should be. Secondly, recall from [10] that, in order to take the reduced free product of two maps, it is necessary for them to preserve the states or conditional expectations involved. In this case, this would seem to require the following condition to be satisfied by $\rho$:

$$f(\rho(z)) = f_x(z)1 \; \forall \, z \in Z. \tag{1}$$

The question is, when can (1) be satisfied?

We consider the following example, where both these problems can be solved.

*Example 5.4.1.* Take $A = C([0,1], \mathbb{C}^2)$ to be the trivial bundle on $[0,1]$ with fibre $\mathbb{C}^2$. So $A_x = \mathbb{C}^2$ for every $x \in [0,1]$. We do not restrict the fixed $C^*$-algebra $B$ in any way. Whilst $A$ may be trivial, we take a non-trivial continuous field of faithful states on $A$, defined by

$$f_x(z_1 \oplus z_2) = \lambda_x z_1 + (1 - \lambda_x) z_2$$

where $\lambda_x = \frac{2x+1}{4}$.

Fix $x \in [0,1]$. Note that as the fibres of $A$ in this example are finite dimensional, the first problem mentioned above does not exist. To solve the second problem, we construct $\rho : A_x = \mathbb{C}^2 \to A$ satisfying the requirements of equation (1).

Define $\rho$ by letting $\rho(1 \oplus 1) = 1$ and $\rho(1 \oplus 0) = g$. Here, for $y \in [0,1]$, we have $g(y) = g(y)_1 \oplus g(y)_2$ where

$$g(y)_1 = \begin{cases} \frac{\lambda_x}{\lambda_y} & \text{for } y \geq x \\ 1 & \text{for } y < x \end{cases}$$

and

$$g(y)_2 = \begin{cases} 0 & \text{for } y \geq x \\ \frac{\lambda_x - \lambda_y}{1 - \lambda_y} & \text{for } y < x \end{cases}$$

We now show that the map $\rho$ satisfies the required conditions. Note that $g \in A$, $g \geq 0$ and $1 - g \geq 0$. Also $g(x) = 1 \oplus 0$, so that $\rho$ is a genuine lifting. We also have, for $y \in [0,1]$,

$$\lambda_y g(y)_1 + (1 - \lambda_y) g(y)_2 = \lambda_x$$

so $f_y(g(y)) = f_x(1 \oplus 0)$. It follows that $\rho$ satisfies equation (1). Clearly $\rho$ is unital. Is it completely positive? As the domain $\mathbb{C}^2$ is a commutative $C^*$-algebra, it suffices to show that $\rho$ is positive. Suppose $z_1 \oplus z_2 \geq 0$ in $\mathbb{C}^2$. Then

$$\begin{aligned} \rho(z_1 \oplus z_2) &= z_2 \rho(1 \oplus 1) + (z_1 - z_2)\rho(1 \oplus 0) \\ &= z_2 + (z_1 - z_2) g \\ &= z_1 g + z_2(1 - g) \geq 0. \end{aligned}$$

Next we describe how to construct the required free product map in the case (as in the above example) where each $A_x$ is finite dimensional and a suitable map $\rho$ can be found.

First, note that there exists a $*$-homomorphism $\mu : C(X) \otimes A \to A$ given by multiplication, in other words $\mu(f \otimes a) = fa$. Define $\theta = \mu \circ (\mathrm{id}_{C(X)} \otimes \rho) : A_x \otimes C(X) \to A$. Then $\theta$ is a unital completely positive map. Since we're assuming that $\rho$ satisfies equation (1), it follows that $f_x \otimes \mathrm{id}_{C(X)} = f \circ \theta$ on $A_x \otimes C(X)$. It is also easy to check that $\theta$ is a $C(X), C(X)$-bimodule map. Let $\iota : B \otimes C(X) \to B \otimes C(X)$ be the identity mapping.

Theorem 2.2 of [10] shows that there exists a free product map

$$\Phi : (A_x \otimes C(X), f_x \otimes \mathrm{id}) *_{C(X)} (B \otimes C(X), \phi \otimes \mathrm{id}) \to C$$

which extends both $\theta$ and $\iota$. We also have that $\Phi$ is unital completely positive. The domain of $\Phi$ can clearly be identified as

$$((A_x, f_x) * (B, \phi)) \otimes C(X).$$

So,

$$\begin{aligned} \left\| \sum a_{ix} g_i(x) a'_{ix} \right\|_{A_x *_r B} &= \left\| \left(\sum a_{ix} g_i(x) a'_{ix}\right) \otimes 1 \right\|_{(A_x *_r B) \otimes C(X)} \\ &\geq \left\| \Phi\left(\left(\sum a_{ix} g_i(x) a'_{ix}\right) \otimes 1\right) \right\|_C \\ &= \left\| \Phi\left(\sum (a_{ix} \otimes 1)(g_i(x) \otimes 1)(a'_{ix} \otimes 1)\right) \right\|_C \\ &= \left\| \sum \Phi(a_{ix} \otimes 1)\Phi(g_i(x) \otimes 1)\Phi(a'_{ix} \otimes 1) \right\|_C \\ &= \left\| \sum \theta(a_{ix} \otimes 1)(g_i(x) \otimes 1)\theta(a'_{ix} \otimes 1) \right\|_C \\ &= \left\| \sum \rho(a_{ix}) \sigma(g_i(x)) \rho(a'_{ix}) \right\|_C \end{aligned}$$

where the inequality is valid because $\Phi$ must necessarily be contractive. Thus $C^l$ is certainly continuous when the fibres of $A$ are finite dimensional and there exist unital completely positive liftings $\rho : A_x \to A$ satisfying equation (1).

We now return to our example.

*Example 5.4.2.* This is a continuation of Example 5.4.1. Note that even though $A = C([0,1], \mathbb{C}^2)$ is trivial, the fibres of the reduced free product bundle $C^l$ are not necessarily isomorphic. For example, take $B = \mathbb{C}^2$ with the trace defined by $\rho(\lambda, \mu) = \frac{1}{2}(\lambda + \mu)$ for $\lambda, \mu \in \mathbb{C}$. Then, by Proposition 2.7 of [19], the fibres of $C^l$ are isomorphic to either $\mathbb{C}^2 \oplus C([0,1], M_2)$ (at every $x \neq 1/2$) or

$$\{f : [0,1] \to M_2 \mid f \text{ continuous, } f(0) \text{ and } f(1) \text{ diagonal}\}$$

at $x = 1/2$. The centres of the above two $C^*$-algebras are $\mathbb{C}^2 \oplus C[0,1]$ and $C[0,1]$ respectively, so they are certainly not isomorphic. Despite this $C^l$ is continuous.

It would be interesting to know if the methods described in the above Section are more widely applicable. It would also be of interest to find an explicit example where continuity of $C^l$ fails.

## 5.5 Continuity of $C^u$

In this section, we consider the continuity of the bundle $C^u$, which is automatically upper semicontinuous. We consider the reduced free product $(A_1, f_1) *_{C(X)} (A_2, f_2)$ of two unital continuous bundles of $C^*$-algebras over the compact Hausdorff space $X$, with continuous fields of faithful states attached. We show that, if $A_1$ is also exact, then $C^u$ is continuous.

The strategy is as follows. First we obtain a $C(X)$-version of [24] Proposition 4.2. This enables us to embed $C^u$ in a Cuntz-Pimsner $C^*$-algebra $E(H)$ (which is also a $C(X)$-algebra). We then use the analysis of Cuntz-Pimsner $C^*$-algebras provided in [24] to show that the Cuntz-Pimsner $C^*$-algebra involved is actually a continuous field. It will then follow that $C^u$ is continuous.

First we require a $C(X)$-version of [24] Lemma 4.1.

**Lemma 5.5.1.** *Let $A_1$, $A_2$ be unital continuous bundles of $C^*$-algebras over the compact Hausdorff space $X$, with continuous fields of faithful states $f_1$, $f_2$ respectively. Let $(C, \psi)$ be the reduced amalgamated free product. Let $B = A_1 \otimes_{C(X)} A_2$ (for the definition of this* *see [44]) be the amalgamated minimal tensor product. Attach to $B$ the tensor product conditional expectation $\rho = f_1 \otimes f_2$.*

*Let $D_1$ be a unital $C(X)$-algebra with $C(X)$-valued conditional expectation $g_1$ attached, such that the G.N.S. representation corresponding to $(D_1, g_1)$ is faithful. Suppose there is a Haar unitary in $D_1$, i.e. there exists a unitary $u \in D_1$ such that $g_1(u^n) = 0$ for every non-zero integer $n$. Let $(D, g) = (D_1, g_1) *_{C(X)} (B, \rho)$ be the reduced amalgamated free product. Denote by $\pi_k : A_k \to D$ the $C(X)$-linear embedding given by $\pi_k(a) = u^k a u^{-k}$ ($k = 1, 2$). Then there is a $C(X)$-linear embedding $\pi : C \to D$ extending the $\pi_k$, such that $g \circ \pi = \psi$.*

*Proof.* As $u$ is a Haar unitary, it follows that $u$ has full spectrum. That is, $C^*(u) \cong C(\mathbb{T})$. This is because the Haar unitary condition implies that $u$ acts as the bilateral shift operator on the G.N.S. Hilbert space $L^2(C^*(u), g_1|C^*(u)) \cong \ell^2(\mathbb{Z})$.

The assumptions on $D_1$ then tell us that

$$(D_1, g_1) \supset \left(C(X) \otimes C(\mathbb{T}), \mathrm{id}_{C(X)} \otimes \int \cdot d\lambda\right)$$

where $\int \cdot d\lambda$ denotes integration with respect to the Haar measure on $\mathbb{T}$. The results of [10] allow us to assume that

$$(D_1, g_1) = \left(C(X) \otimes C(\mathbb{T}), \mathrm{id}_{C(X)} \otimes \int \cdot d\lambda\right).$$

In $(D, g)$, the family $(u^k B u^{-k})_{k \in \mathbb{Z}}$ is free. Consider $\overline{B} = C^*(\cup_{k \in \mathbb{Z}} u^k B u^{-k}) \subset D$. Conjugation by $u$ gives the free shift on $\overline{B}$. As $\overline{B} \cup \{u\}$ generates $D$, it follows that $D = \overline{B} \rtimes \mathbb{Z}$. As $g$ is a faithful conditional expectation, it is certainly true that the G.N.S. representation corresponding to $(\overline{B}, g|\overline{B})$ is faithful. Therefore

$$(\overline{B}, g|\overline{B}) = *_{k \in \mathbb{Z}}(u^k B u^{-k}, g|u^k B u^{-k})$$

where the right-hand side is a reduced free product, amalgamating over $C(X)$.

Now the embedding results of [10] imply that we have an embedding $\pi : C \hookrightarrow \overline{B} \subset D$ extending every $\pi_k$. This embedding preserves the conditional expectations on the $C^*$-algebras involved, so we get $g \circ \pi = \psi$. It is also clear that $\pi$ is $C(X)$-linear (because all the $\pi_k$ are). $\square$

The above lemma enables us to prove a $C(X)$-version of [24] Proposition 4.2. Note that, although we are working with *amalgamated* reduced free products here, we avoid use of section 5 in Dykema and Shlyakhtenko [24].

**Proposition 5.5.2.** *For $i = 1, 2$ let $A_i$ be a unital $C(X)$-algebra where $X$ is a compact Hausdorff space. Let $\phi_i : A_i \to C(X)$ be a continuous field of faithful states ($i = 1, 2$). Define $(C, \psi)$ to be the reduced amalgamated free product of $A_1$ and $A_2$. We let $B = A_1 \otimes_{C(X)} A_2$, as in the above lemma, with $\rho = \phi_1 \otimes \phi_2$ the tensor product conditional expectation. Then there exists a Hilbert $B$-bimodule $H$ such that the Cuntz-Pimsner $C^*$-algebra $E(H)$ is a $C(X)$-algebra and there exists an injective $C(X)$-linear $*$-homomorphism $\pi : C \to E(H)$ such that $\rho \circ \mathcal{E} \circ \pi = \psi$ where $\mathcal{E} : E(H) \to B$ is the canonical vacuum expectation.*

*Proof.* Consider the interior tensor product $H = L^2(B, \rho) \otimes_{C(X)} B$ where the left action of $C(X)$ on $B$ is given by the canonical inclusion of $C(X)$ into $B$ (recall that $B$ is a $C(X)$-algebra). Also, since $C(X) \subset \mathcal{Z}(B)$, the left and right actions of $C(X)$ on $H$ are the same. This results in $E(H)$ being a $C(X)$-algebra.

Let $\xi$ be the element of $H$ given by $\xi = \widehat{1} \otimes 1$. Let $D$ be the $C^*$-subalgebra of $E(H)$ generated by $C(X)$ and $\ell(\xi)$. Consider the conditional expectation $\phi = \mathcal{E}|D$. It follows from the definition of $\mathcal{E}$ that $\mathcal{E}|C^*(\ell(\xi))$ is $\mathbb{C}$-valued. Thus $\phi$ is a $C(X)$-valued conditional expectation.

We claim that $D$ and $B$ are free with amalgamation over $C(X)$, with respect to $\rho \circ \mathcal{E}$. This essentially follows from Shlyakhtenko [57] Theorem 2.3, but here we provide the details for this particular situation.

Now $D \cong C(X) \otimes C^*(\ell(\xi))$ and $\ker\phi$ is the closed span of elements of the form $f \otimes L^n(L^*)^m$ where $n, m \geq 0$, $n + m > 0$, $f \in C(X)$ and $L = \ell(\xi)$. So, to show freeness, it suffices to show that

$$\rho \circ \mathcal{E}(b_0 f_1 L^{n_1}(L^*)^{m_1} b_1 f_2 L^{n_2}(L^*)^{m_2} \cdots f_k L^{n_k}(L^*)^{m_k}) = 0$$

where $k \in \mathbb{N}$ and $f_j \in C(X)$, $b_j \in B$, $\rho(b_j) = 0$, $n_j, m_j \geq 0$, $n_j + m_j > 0$ for every $j$. As $\rho \circ \mathcal{E}$ is $C(X)$-linear, we can assume that every $f_j = 1$. If our word contains a subword of the form $L^* b_j L$ then, since $L^* b_j L = \rho(b_j) = 0$, we obtain the required result. If there are no such subwords then our word is necessarily of the form

$$b'_1 L b'_2 L \cdots b'_p L b''_1 L^* b''_2 \cdots L^* b''_{q+1}$$

where $b'_j, b''_j \in B$ for all $j$ and $p, q \geq 0$, $p + q > 0$. However, it is readily seen that such words are contained in the kernel of $\mathcal{E}$ (from the definition of $\mathcal{E}$). Hence we have the desired freeness and the above claim is true.

It follows that $(E(H), \rho \circ \mathcal{E}) = (D, \phi) *_{C(X)} (B, \rho)$. Now $\phi|C^*(\ell(\xi))$ is $\mathbb{C}$-valued, and in the same way as in [24] Proposition 4.2 we get a unitary $u \in C^*(\ell(\xi)) \subset D$ such that $\phi(u^k) = 0$ for every non-zero integer $k$.

Applying Lemma 5.5.1 now gives the required $C(X)$-linear embedding of the reduced free product $C$ into $E(H)$. $\square$

So, the above Proposition gives an injective $C(X)$-linear $*$-homomorphism $\pi : C \to E(H)$. As $\pi$ is $C(X)$-linear, the induced maps $\pi_x : C_x \to E(H)_x$ are also injective. Hence there is an embedding of bundles $C^u \hookrightarrow E(H)$. So, in order to prove continuity of the reduced free product bundle $C^u$, it suffices to prove that the Cuntz-Pimsner $C^*$-algebra $E(H)$ mentioned above is continuous. In order to do this, we use the following lemma.

**Lemma 5.5.3.** *Suppose that*

$$0 \to I \to A \to B \to 0$$

*is an exact sequence of $C(X)$-algebras. We assume that $A$ is unital and that all the maps involved are $C(X)$-linear $*$-homomorphisms. We denote the quotient map from $A$ to $B$ by $q$. Suppose that $I$ and $B$ are continuous fields. Then $A$ is also a continuous field.*

*Proof.* Blanchard's characterisation of continuous fields (as described in Theorem 1.3.8) implies that we have continuous fields of faithful representations $\pi : B \to L(\mathcal{E})$ and $\sigma : I \to L(\mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ are Hilbert $C(X)$-modules.

We would like to extend $\sigma$ to $A$. For this, we need $\sigma$ to be non-degenerate. Let $\mathcal{F}'$ be $\overline{\sigma(I)\mathcal{F}}$, the closed span of $\{\sigma(z)\eta : z \in I, \eta \in \mathcal{F}\}$. Then $\sigma$ defines a representation $\widetilde{\sigma} : I \to L(\mathcal{F}')$, which is still $C(X)$-linear. Suppose $x \in X$ and suppose $(\widetilde{\sigma})_x(z_x) = 0$ for some $z \in I$. Then $(\sigma(z)\eta)_x = 0$ for every $\eta \in \sigma(I)\mathcal{F}$. Taking an approximate unit for $I$, we get $(\sigma(z)\eta)_x = 0$ for all $\eta \in \mathcal{F}$. So $\sigma_x(z_x) = 0$ and $z_x = 0$. Hence $(\widetilde{\sigma})_x$ is faithful. This means we can assume that $\sigma$ is non-degenerate.

Proposition 2.1 in Lance [47] implies that $\sigma$ has a unique extension $\sigma_1 : A \to L(\mathcal{F})$. Also, $\sigma_1$ is $C(X)$-linear since $\sigma$ is.

Now define $\nu : A \to L(\mathcal{E} \oplus \mathcal{F})$ via $a \longmapsto \pi(q(a)) \oplus \sigma_1(a)$. This is $C(X)$-linear. Consider $\nu_x : A_x \to L(\mathcal{E}_x \oplus \mathcal{F}_x)$ for some $x \in X$. Suppose that $\nu_x(a_x) = 0$ for some $a_x \in A_x$. Then $\pi_x q_x(a_x) = 0$. Now $\pi_x$ is faithful, so $q_x(a_x) = q(a)_x = 0$. Hence $q(a) \in C_x(X)B$. Therefore, we can write $q(a) = fq(a')$ where $f \in C_x(X)$ and $a' \in A$. Then $q(a - fa') = 0$ so $a - fa' \in I$. That is, $a - fa' = z$ for some $z \in I$. So

$$0 = (\sigma_1)_x(a_x) = (\sigma_1)_x(z_x) = (\sigma_1(z))_x = (\sigma(z))_x = \sigma_x(z_x).$$

As $\sigma_x$ is faithful, we find that $a_x = z_x = 0$. So $\nu$ is a continuous field of faithful representations of $A$. Hence, using Theorem 1.3.8 again, we see that $A$ is also a continuous field. $\square$

We are now in a position to prove the following.

**Proposition 5.5.4.** *Let $B$ be a unital separable $C(X)$-algebra which is actually a continuous field of $C^*$-algebras over the compact Hausdorff space $X$. Suppose that $H$ is a countably generated Hilbert $B$-bimodule, and that the left and right actions of $C(X)$ on $H$ are the same. Then the Cuntz-Pimsner $C^*$-algebra $E(H)$ is a continuous field of $C^*$-algebras over $X$.*

*Proof.* As the left and right actions of $C(X)$ on $H$ are the same, it follows that $E(H)$ is a $C(X)$-algebra. We must now show that this $C(X)$-algebra is continuous.

As in the proof of [24] Theorem 3.1, let $\widetilde{H} = H \oplus B$. Since $E(H) \subset E(\widetilde{H})$, it suffices to prove that $E(\widetilde{H})$ is continuous. In the proof of [24] Theorem 3.1, various $C^*$-subalgebras of $E(\widetilde{H})$ are considered. In particular, an increasing sequence of $C^*$-subalgebras $A_0, A_1, A_2, \ldots$ is defined. Here, $A_0 = B$ and we have (split) exact sequences

$$0 \to I_n \to A_n \to A_{n-1} \to 0$$

for every $n$. It is easily checked that, in the present context, the maps involved are $C(X)$-linear. Also $I_n \cong K(\widetilde{H}^{(\otimes_B)n})$.

Our separability assumptions imply that $\widetilde{H}^{(\otimes_B)n}$ is a countably generated Hilbert $B$-module. Hence the Kasparov stabilisation theorem (see Theorem 1.2.2) implies that $\widetilde{H}^{(\otimes_B)n}$ is a closed complemented submodule of $\ell^2(\mathbb{N}) \otimes B$. It follows that

$$I_n \subset K(\ell^2(\mathbb{N}) \otimes B) \cong K(\ell^2(\mathbb{N})) \otimes B.$$

As $K(\ell^2(\mathbb{N}))$ is exact and $B$ is continuous, it is seen (see Theorem 4.5 of [44]) that $I_n$ is actually a continuous field for every $n$. Using Lemma 5.5.3, we can now show that every $A_n$ is a continuous field.

The $A_n$ are increasing, hence $A = \overline{\cup_{n \in \mathbb{N}} A_n}$ is also a continuous field over $X$. It follows that the inductive limit $\overline{A}$ described in section 2 of [24] is continuous. As $\mathbb{Z}$ is an amenable group, it follows (for example, see [44] Remarks 2.6) that any crossed product bundle $\overline{A} \rtimes \mathbb{Z}$ is continuous. Hence the crossed product $A \rtimes_\Psi \mathbb{N}$ of [24] Theorem 3.1 is continuous, it being a $C^*$-subalgebra of a crossed product of $\overline{A}$ by $\mathbb{Z}$. But this crossed product $A \rtimes_\Psi \mathbb{N}$ is isomorphic to $E(\widetilde{H})$. Thus $E(\widetilde{H})$ is continuous, and that is what we wanted to prove. $\square$

**Corollary 5.5.5.** *Let $C^u$ denote the upper semicontinuous bundle (as constructed in section 5.3) arising from the reduced amalgamated free product $(A_1, f_1) *_{C(X)} (A_2, f_2)$ of two unital separable continuous bundles of $C^*$-algebras over the compact Hausdorff space $X$ (with continuous fields of faithful states attached). Then, if $A_1$ is exact, $C^u$ is continuous.*

*Proof.* By [44] Theorem 4.6, $B = A_1 \otimes_{C(X)} A_2$ is continuous. By Proposition 5.5.2, we have a $C(X)$-linear embedding of the reduced amalgamated free product $C$ into some Cuntz-Pimsner $C^*$-algebra $E(H)$. By Proposition 5.5.4, this Cuntz-Pimsner $C^*$-algebra is a continuous field of $C^*$-algebras over $X$. Hence $C$ must be a continuous field over $X$. That is, $C^u$ is a continuous bundle over $X$. □

It is interesting that, in the above proofs, we have used the same work on Cuntz-Pimsner $C^*$-algebras as was used to show nuclearity of certain reduced free products in Section 3.3.

## 5.6 Concluding remarks

We have now considered the continuity of both $C^u$ and $C^l$. The continuity of $C^l$ is equivalent to asking for $\ker P_x = C_x(X)C$ for every $x \in X$. So if $C^l$ is continuous, then $C^u$ and $C^l$ coincide, hence $C^u$ is certainly continuous. On the other hand, it is not at all clear if the continuity of $C^u$ implies anything about the continuity of $C^l$.

One thing we can say is the following. Assume that the reduced amalgamated free product $C$ is separable (as we have been doing throughout). Then Proposition 2.12 of Blanchard [8] implies that the set of $x \in X$ for which $\ker P_x = C_x(X)C$ is dense in $X$. It is perhaps reasonable to conjecture that, if $C$ is exact, then $C^u$ and $C^l$ coincide, and hence by Corollary 5.5.5 $C^l$ is continuous.

Finally, we can mention one possible application of the above results. This is to the problem of embedding a continuous bundle into another continuous bundle whose fibres are simple. The work of Blanchard [9] shows that this is always possible if the bundle $C^*$-algebra is exact. It has been clear for some time now that reduced free products are often simple (see for example the work of Powers [54], Avitzour [4] and Dykema [19]). So a way of embedding a continuous bundle into a continuous bundle with simple fibres would be to take the reduced free product of the continuous bundle with an appropriate fixed $C^*$-algebra. We then have to show continuity of the resulting bundle $C^l$. Unfortunately, as remarked earlier, the results of Section 5.5 do not seem to provide any help in this direction.

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