Foad Shokrollahi

Option pricing in fractional models

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Tiivistelmä

Väitöskirja tarkastelee fraktionaalisen Black–Scholes -mallin ja sekoitetun fraktionallisen Black–Scholes -mallin käyttöä erityyppisten optioiden arvottamisessa. Tätä tutkitaan neljässä artikkelissa. Ensimmäisessä artikkelissa tarkastellaan geometrisia aasialaisia optioita ja potenssioptioita, kun osakehinta noudattaa aikamuunnettua sekoitettua fraktionaalista mallia. Tässä mallissa sekoitun fraktionaalisen Black–Scholes -mallin käänteinen subordinaattoriprosessi korvaa fysikaalisen ajan. Kolmannen artikkelin tarkoitus on hinnoitella eurooppalainen valuuttaoptio fraktionaalisen Brownin liikkeen mallissa aikamuunnetulla strategialla. Lisäksi aika-askeleen ja pitkän aikavälin riippuvuuden vaikutusta tutkitaan transaktiokulujen alaisuudessa.

Ehdollinen keskiarvosuojaaminen fraktionaalisessa Black–Sholes -mallissa on toisen artikkelin aihe. Ehdollinen keskiarvosuojaus eurooppalaiselle vaniljaoptiolle, jolla on konveksi tai konkaavi positiivinen tuottofunktio transaktiokulujen vallitessa, on artikkelin päätulos. Neljännessä artikkelissa tutkitaan eurooppalaisia optioita diskreetissä ajassa mallissa, joka on hypyllinen sekoitettu fraktionaalinen Brownin liike. Käyttäen keskiarvoista deltasuojausstrategiaa artikkelissa johdetaan hinnoittelumalli eurooppalaisille optioille transaktiokulujen vallitessa.

Asiasanat
Optioiden hinnoittelu, Stokastinen mallinnus, Matemaattinen rahoitusteoria, Fraktionaaliset mallit, Fraktionaalinen Brownin liike

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Abstract

This thesis deals with application of the fractional Black-Scholes and mixed fractional Black-Scholes models to evaluate different type of options. These assessments are considered in four individual papers. In the first articles, the problem of geometric Asian and power options pricing is investigated when the stock price follows a time changed mixed fractional model. In this model, an inverse subordinator process in the mixed fractional Black-Scholes model replaces the physical time. The aim of the third paper is to evaluate the European currency option in a fractional Brownian motion environment by the time-changed strategy. Also, the impact of time step and long range dependence are obtained under transaction costs.

Conditional mean hedging under fractional Black-Scholes model is the propose of the second article. The conditional mean hedge of the European vanilla type option with convex or concave positive payoff under transaction costs is obtained. In the fourth article, the mixed fractional Brownian motion with jump process are incorporated to analyze European options in discrete time case. By a mean delta hedging strategy, the pricing model is proposed for European option under transaction costs.

Keywords

Option pricing , Stochastic modeling, Mathematical finance, Fractional model, Fractional Brownian motion

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Vaasa, May 2019

Foad Shokrollahi

CONTENTS

LIST OF PUBLICATIONS

The dissertation is based on the following three refereed articles:

(I) Shokrollahi, F. (2018). The evaluation of geometric Asian power options under time changed mixed fractional Brownian motion. *Journal of Computational and Applied Mathematics* 344, 716–724.

(II) Shokrollahi, F., and Sottinen, T. (2017). Hedging in fractional Black-Scholes model with transaction costs. *Statistics & Probability Letters*. 130, 85–91.

(III) Shokrollahi, F. (2018). Subdiffusive fractional Black-Scholes model for pricing currency options under transaction costs. *Cogent Mathematics & Statistics* 5, 1470145.

(IV) Shokrollahi, F. (2018). Mixed fractional Merton model to evaluate European options with transaction costs. *Journal of Mathematical Finance* 8, 623–639.

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AUTHOR'S CONTRIBUTION

Publication I: "The evaluation of geometric Asian power options under time changed mixed fractional Brownian motion"

This is an independent work of the author.

Publication II: "Hedging in fractional Black—Scholes model with transaction costs"

This article is the outcome of a joint discussion and all the results are a joint work with Tommi Sottinen.

Publication III: "Subdiffusive fractional Black-–Scholes model for pricing currency options under transaction costs "

This is an independent work of the author.

Publication IV: "Mixed fractional Merton model to evaluate European options with transaction costs"

This is an independent work of the author.

1 INTRODUCTION

In recent years, options have received increasing attention and their role has grown rapidly in most trading exchanges. Most common examples of variables underlying options are the price of stocks, bonds or commodities traded on the exchange. However, options can depend on almost any variable, from the price of pork bellies to the amount of rainfall in a certain geographic area.

Standardized options on stock prices have been traded at the exchange since 1973. There are two basic types of options. A call option gives the holder the right (and not the obligation) to buy the underlying asset at a certain time in the future for a certain price. A put option gives the holder the right to sell the underlying asset at a certain time in the future for a certain price. The price for which the asset is being exchanged is referred to as the strike price or the exercise price. The future time when the exchange takes place is referred to as the maturity of the option or the expiration date. Based on the exercise conditions, options are categorized into European options and American options. European options can only be exercised at maturity. American options can be exercised anytime during the life of the option. Throughout this section options are assumed European unless otherwise specified. Depending on the strategy, options trading can provide a variety of benefits, including the security of limited risk and the advantage of leverage. Another benefit is that options can protect or enhance your portfolio in rising, falling and neutral markets.

Since it appeared in the 1970s, the Black-Scholes (BS) model (Black & Scholes (1990)) has become the most popular method to option pricing and its generalized version has provided mathematically beautiful and powerful results on option pricing. However, they are still theoretical adoptions and not necessarily consistent with empirical features of financial returns, such as nonindependence, nonlinearity, ect. For example, Hull and White (Hull & White (1987)) introduced a bivariate diffusion model for pricing options on assets with stochastic volatilities. Heston (Heston (1993)) proposed affine stochastic volatility. Furthermore, since discontinuity or jumps is one of the significant component in financial asset pricing (see Andersen, Benzoni & Lund (2002), Chernov, Gallant, Ghysels & Tauchen (2003), Pan (2002), Eraker (2004)) and also some scholars have been represented pricing models based on the jump processes (see Merton (1976), Kou (2002), Cont & Tankov (2004), Ahn, Cho & Park (2007), Ma (2006)).

Many realistic models have been described long memory behavior in financial time series (Lo, A. W. (1991), Willinger, W., Taqqu, M. S., & Teverovsky, V. (1999) (1999), Cont, R. (2005), Dai & Singleton (2000), Berg & Lyhagen (1998), Hsieh (1991), Huang & Yang (1995)). Since, fractional Brownian motion (fBm) is a self-similar and long-range dependence process, then it can be a appropriate candidate to capture these phenomena (Wang, Zhu, Tang & Yan (2010), Wang (2010), Sottinen

(2003), Sottinen & Valkeila (2003), Wang, Wu, Zhou & Jing (2012), Xiao, Zhang, Zhang & Wang (2010), Zhang, Xiao & He (2009), Cartea & del Castillo-Negrete (2007)). Kolmogorov introduced the fBm in 1940. A representation theorem for Kolmogorov's process was introduced by Mandelbrot and Van Ness (Mandelbrot & Van Ness (1968)). Nowadays, the fBm process play a significant role in stochastic finance and different extensions of the fractional Black-Scholes formulas for pricing options based on the geometric fractional Brownian motion are proposed to capture the behavior of underlying asset (Bayraktar, Poor & Sircar (2004), Meng & Wang (2010)).

On the mathematical side, fBm is neither a semi martingale nor a Markov process (except in the Brownian motion case). Hence, the classical stochastic integration theory developed for semimartingale is not handy to analyze financial markets based on fractional Brownian motion (Hu, Y., & Øksendal, B. (2003)). Further, some authors discussed arbitrage under fractional Black-Scholes model and proposed some restrictions to exclude arbitrage in fractional markets (see Bender, Sottinen & Valkeila (2007), Bender, C., Sottinen, T., & Valkeila, E. (2008) (2008), Bender & Elliott (2004), Björk & Hult (2005)).

To better describe long memory property and fluctuations in the financial assets, the mixed fractional Brownian motion $(mfBm)$ was presented (see El-Nouty (2003), Mishura (2008), Cheridito, P. (2001), Zili (2006)). A $mfBm$ is a family of Gussian process which is a linear combination of Brownian motion and independent fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$. The pioneering work to apply the $mfBm$ in finance was presented by Cheridito, P. (2001). He proved that for $H \in (\frac{3}{4}, 1)$, the $mfBm$ is equivalent to one with Brownian motion, and then its free of arbitrage. For $H \in (\frac{1}{2}, 1)$, Mishura and Valkeila (Mishura & Valkeila (2002)) proved that the mixed model is arbitrage-free.

2 STOCHASTIC PROCESSES

In this section, we provide some definitions and auxiliary facts that are needed in this thesis (for further details, see Shiryaev, A. N., do Rosário Grossinho, M., Oliveira, P. E., & Esquível, M. L. (2006) (2006), Föllmer, H., & Schied, A. (2011), Shiryaev, A. N. (1999), Kallenberg, O. (2006), Clark & Ghosh (2004), Melnikov, Pliska (1997), Mikosch (1998)). Throughout this section all random objects are defined in the probability space $(\Omega, \mathscr{F}, P)$.

2.1 General facts

Definition 2.1. Let $T \subseteq [0, +\infty)$ be an interval. A stochastic process X indexed by interval T is a collection of random variables $(X_t)_{t\geq 0}$.

Also, for every $\omega \in \Omega$, the real valued function $t \in T \mapsto X_t(\omega)$ is called a trajectory or a sample path of the process X.

Definition 2.2. A filtration is a family $(\mathscr{F}_t)_{t\geq 0}$ of σ-algebras $\mathscr{F}_t \subseteq \mathscr{F}$ such that

$$\forall 0 \leq s \leq t \Rightarrow \mathscr{F}_s \subseteq \mathscr{F}_t.$$

Definition 2.3. Let $(\mathscr{F}_t)_{t\geq 0}$ be a filtration. A stochastic process $X = (X_t)_{t\geq 0}$ is said to be adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$ if for every $t \geq 0$, the random variable X_t is $\mathscr{F}_t$-measurable.

Definition 2.4. An stochastic process $(X_t)_{t\geq 0}$ is called a martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ (and probability measure P) if the following conditions are satisfied:

(i) X_t is $\mathscr{F}_t$-measurable for all $t \geq 0$,

(ii) $E[|X_t|] < \infty$ for all $t \geq 0$, and

(iii) $E[X_t|\mathscr{F}_s] = X_s$ for all $0 \leq s \leq t$.

Definition 2.5. An $\mathscr{F}_t$ - adapted stochastic process is called a local martingale with respect to the given filtration $(\mathscr{F}_t)_{t\geq 0}$ if there exists an increasing sequence of $\mathscr{F}_t$ - stopping times τ_k such that

$$\tau_k \to \infty \quad as \quad k \to \infty,$$

and

$$(X_{t\wedge\tau_k})_{t\geq 0}, \tag{2.1}$$

is an $\mathscr{F}_t$- martingale for all k, where $t \wedge \tau_k = \min(t, \tau_k)$.

Definition 2.6. A process $X = (X_t)_{t\geq 0}$ is called an $(\mathscr{F}_t)_{t\geq 0}$- semi-martingale, if it admits the representation

$$X_t = X_0 + M_t + A_t, \tag{2.2}$$

where M is an $(\mathscr{F}_t)_{t\geq 0}$- local martingale with $M_0 = 0$, A is a process of locally bounded variation and adapted to filteration $(\mathscr{F}_t)_{t\geq 0}$, X_0 is $\mathscr{F}_0$-measurable.

Definition 2.7. (Hölder continuous)

Let $\alpha \in (0,1]$. A function $f : R \to R$ is said to be locally α-Hölder continuous at $x \in R$, if there exists $\varepsilon > 0$ and $c = c_x$ such that

$$|f(x) - f(y)| \leq c|x-y|^{\alpha}, \quad \text{for all} \quad y \in R \quad \text{with} \quad |y-x| < \varepsilon.$$

Definition 2.8. A stochastic process $X = (X_t)_{t\geq 0}$ is said to have stationary increments if for all $s \geq 0$, and every $h > 0$,

$$(X_t - X_s)_{t\geq 0} \overset{f.d.}{=} (X_{t+h} - X_{s+h})_{t\geq 0}. \tag{2.3}$$

Here $\overset{f.d.}{=}$ denotes equality in finite dimensional distribution.

Definition 2.9. A stochastic process $X = (X_t)_{t\geq 0}$ is said to have independent increments if for every $t \geq 0$ and any choice $t_i \in T$ with $t_0 < t_1 < ... < t_n$ and $n \geq 1$,

$$X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}} \tag{2.4}$$

are independent random variables.

2.2 Lévy processes

Lévy processes are stochastic processes with independent and stationary increments.

Definition 2.10. Lévy process $(X_t)_{t>0}$ is a process with the following properties

(1) Independent increments,

(2) Stationary increments, and

(3) Continuous paths in probability: That is $\lim_{h\to 0} P(|X_{t+h} - X_t| \geq \varepsilon) = 0$ for any $\varepsilon > 0$.

Definition 2.11. (Subordinator process)

A subordinator is a real-valued Lévy process with nondecreasing sample paths.

Definition 2.12. (Stable process)

A stable process is a real-valued Lévy process $(X_t)_{t\geq 0}$ with initial value $X_0 = 0$ that satisfies the self-similarity property

$$(X_{at})_{t\geq 0} \overset{f.d.}{=} (a^{1/\alpha} X_t)_{t\geq 0} \quad \forall t > 0.$$

The parameter α is called the exponent of the process.

Definition 2.13. (Poisson process)

A Poisson process $(X_t)_{t\geq 0}$ satisfies the following conditions:

(1) $X_0 = 0$,

(2) $X_t - X_s$ are integer valued for $0 \leq s < t < \infty$ and

$$P(X_t - X_s = k) = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)} \quad \text{for} \quad k = 0, 1, 2, ... \tag{2.5}$$

(3) The increments $X_{t_2} - X_{t_1}, X_{t_4} - X_{t_3} ...$ and $X_{t_n} - X_{t_{n-1}}$ are independent for every $0 \leq t_1 < t_2 < t_3 < t_4 < ... < t_n$.

Example 2.14. The fundamental Lévy processes are the Brownian motion (defined later) and the Poisson process. The Poisson process is a subordinator, but is not stable; the Brownian motion is stable, with exponent $\alpha = 2$.

2.3 Long-range dependence and self-similarity

Definition 2.15. Let $(X_t)_{t\geq 0}$ be a process with stationary trajectories and $(r_n)_{n\in N}$ the autocovariance sequence defined by

$$\forall n \in N, \quad r_n = E[X_{n+1} X_1]. \tag{2.6}$$

Then, the process $X = (X_t)_{t\geq 0}$ is called long-range dependence if

$$\sum_{n\in N} r_n = \infty.$$

Remark 2.16. Since $(X_t)_{t\geq 0}$ is a process with stationary trajectories

$$\forall s \geq 0, \forall n \in N, \quad r_n = E[X_{n+s}X_s].$$

Definition 2.17. Let $H \in (0,1]$. A stochastic process $X = (X_t)_{t\geq 0}$ is said to be self-similar with exponent H, if for any $a > 0$,

$$(X_{at})_{t\geq 0} \overset{f.d.}{=} (a^H X_t)_{t\geq 0}.$$

2.4 Gaussian processes

Definition 2.18. A stochastic process $(X_t)_{t\in T}$ is Gaussian if all finite dimensional projections $(X_t)_{t\in T_0}$, $T_0 \subset T$ finite, are multivariate Gaussian.

2.4.1 Brownian motion

Definition 2.19. Brownian motion is a process $(B_t)_{t\geq 0}$ with the following properties:

(1) $B_0 = 0$,

(2) B_t has independent increments,

(3) $B_t - B_s \sim N(0, t-s)$ for $s < t$, here N is the normal distribution function.

Definition 2.20. (Markov Process)

The process $(X_t)_{t\in T}$ is a Markov process if

$$E[f(X_t)|\mathscr{F}_s] = E[f(X_t)|X_s], \quad \forall t > s, \quad t, s \in T,$$

where $\mathscr{F}_s = \sigma\{X_u; u \leq s\}$, and f is a bounded Borel function.

2.4.2 Fractional Brownian motion

fBm has recently become a useful choice for modeling in mathematical finance and other sciences. On purely empirical data, some believe that fBm is an ideal candidate since it enjoys two important statistical features of long memory and self-similarity. Even with its popularity, our understanding of the properties and behaviour of fBm is limited.

Kolmogorov (Kolmogorov (1941)) was the first to introduce the Gaussian process which is now known as fBm in the theory of probability. This class of processes was studied by Kolmogorov in detail and it played an essential role in the series of problem s of the statistical theory of turbulence. Yaglom (Yaglom (1955)) discussed the spectral density and correlation function of fBm. A quadratic variation formula for fBm follows from a general result of Baxter (Baxter (1956)). Gladyshev (Gladyshev (1961)) extended Baxter's result and provided a theoretical result to determine the value of the Hurst effect denoted by H. However, most of the encomium to fBm has been given to Mandelbrot and Van Ness (Mandelbrot & Van

Ness (1968)) who used fBm to model natural phenomena such as the speculative market fluctuations. For get more information about fBm, you can see, Hu, Y., & Øksendal, B. (2003), Nualart (2006), Biagini, Hu, Øksendal & Zhang (2008), Mishura (2008).

Definition 2.21. The fractional brownian motion with Hurst index $H \in (0,1)$ denoted by $(B_t^H)_{t\in R}$, is the centered Gaussian process with covariance function

$$R^H(s,t) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad s,t \in R.$$

Figure (1) shows the sample path of the fBm for different parameter.

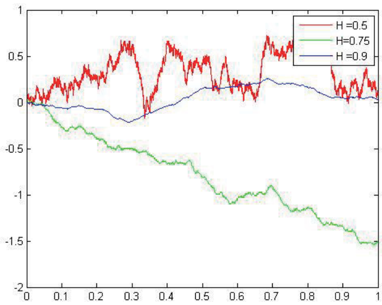

Figure 1. fBm **with different Hurst parameter** H **.**

The fBm can be represented in terms of the one-sided or two-sided standard Brownin motion see (Nualart (2006)). First, we review some special function involved in the representation results.

The Gamma function is defined by

$$\Gamma(\alpha) = \int_0^{\infty} \exp(-v) v^{\alpha-1} dv, \quad \alpha > 0.$$

The Beta function is defined by

$$\beta(\alpha,\beta) = \int_0^1 (1-v)^{\alpha-1} v^{\beta-1} dv, \quad \alpha,\beta > 0.$$

The Gauss hypergeometric function of parameters a,b,c and variable $z \in R$ is defined by the formal power series

$$F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!}.$$

where $(a)_k = a(a+1)...(a+k-1)$.

(1) The one-sided fBm can be constructed from a one-sided Brownian motion:

Theorem 2.22. *Molchan-Golosov representation (Molchan & Golosov (1969))*

For $H \in (0,1)$, it holds that

$$B_t^H = C(H) \int_0^t (t-s)^{H-\frac{1}{2}} F\left(\frac{1}{2}, H-\frac{1}{2}, H+\frac{1}{2}, \frac{s-t}{s}\right) dB_s,$$

where $C(H) = \frac{2H}{\Gamma(H+\frac{1}{2})}$.

(2) The two-sided fBm can be written in terms of one-sided Brownian motion:

Mandelbrot-Van Ness representation (Mandelbrot & Van Ness (1968))

Theorem 2.23. *For $H \in (0,1)$, it holds that*

$$B_t^H = C(H) \int_{\mathbb{R}} \left((t-s)^{H-\frac{1}{2}} \mathbf{1}_{(-\infty,t)}(s) - (-s)^{H-\frac{1}{2}} \mathbf{1}_{(-\infty,0)}(s)\right) d\tilde{B}_s,$$

here $\tilde{B}$ represents two-sided Brownian motion.

Theorem 2.24. *(Mishura (2008)) For $H \neq \frac{1}{2}$, fBm is neither a Markov process nor a semimartingale.*

Remark 2.25. Since fBm is not a semimartngale,the classical integration theory developed for semimartingale is not available, Mishura (2008). Then, many scholars introduced two different approaches for stochastic integral with respect to fBm (1) Pathwise approach (2) Malliavin calculus (Skorokhod integration) approach (Nualart (2006), Sottinen, T., & Viitasaari, L. (2016)).

Remark 2.26. Using stationarity increment of fBm, it can be shown that the autocovariance function $\gamma_n =$ of the sequence $(X_n)_{n\geq 1} := (B_{n+1}^H - B_n)_{n\geq 1}$ is given by

$$\gamma_n = \sum_{k=0}^{n} E[B_1^H(B_{k+1}^H - B_k^H)] = \frac{1}{2}\Big[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\Big], \qquad (2.7)$$

therefore

$$\gamma_n \approx H(2H-1)n^{2H-2}, \quad as \quad n \to \infty, H \neq \frac{1}{2}. \tag{2.8}$$

Notice that when

(1) $H = \frac{1}{2}, \gamma_n = 0, \forall n$, therefore fBm has independent increments.

(2) If $H > \frac{1}{2}, \gamma_n > 0$, the increments of the fBm process are positively correlated and by p-series $\sum_{n=1}^{\infty} |\gamma_n| = \infty$, therefore has long-range dependence.

(3) If $H < \frac{1}{2}, \gamma_n < 0$, the increments of the fBm process are negatively correlated and by p-series $\sum_{n=1}^{\infty} |\gamma_n| = c < \infty$, therefore has short-range dependence.

2.4.3 Mixed fractional Brownian motion

Let a and b be two real constants such that $(a,b) \neq (0,0)$.

Definition 2.27. A $mfBm$ with parameters a, b, and H is a process $M^H = (M_t^H(a,b))_{t \geq 0}$, defined by

$$M_t^H = M_t^H(a,b) = aB_t + bB_t^H, \quad \forall t \geq 0 \tag{2.9}$$

where B is a Brownian motion and B^H is an independent fBm with Hurst parameter H (Cheridito, P. (2001), van Zanten, H. (2007), Mishura (2008), Zili (2006), Marinucci & Robinson (1999)).

Proposition 2.28. *The $mfBm$ has the following properties*

(i) M^H *is a centered Gaussian process,*

(ii) for all $t \in R_+, E((M_t^H(a,b))^2) = a^2 t + b^2 t^{2H}$,

(iii) one has that

$$\begin{aligned} & Cov\Big(M_t^H(a,b), M_s^H(a,b)\Big) \\ = \;& a^2(t \wedge s) + \frac{1}{2} b^2 \Big[t^{2H} + s^{2H} - |t-s|^{2H}\Big], \forall s,t \in R_+, \end{aligned} \tag{2.10}$$

where $\quad t \wedge s = 1/2(t+s+|t-s|)$,

(iv) the increments of the $mfBm$ are stationary.

Lemma 2.29. *(Zili (2006)) For any* $h > 0, (M^H_{ht}(a,b))_{t\geq 0} \stackrel{f.d.}{=} (M^H_t(ah^{\frac{1}{2}}, bh^H)))_{t\geq 0}$. *This property will be called the mixed-self-similarity.*

Theorem 2.30. *(Zili (2006)) For all* $H \in (0,1) - \{\frac{1}{2}\}, a \in R$ *and* $b \in R - \{0\}, (M^H_t(a,b))_{t\geq 0}$ *is not a Markov process.*

Theorem 2.31. *(Cheridito, P. (2001)) For* $H \in (\frac{3}{4}, 1]$*, the* $mfBm$ *is equivalent to Brownian motion.*

Theorem 2.32. *(Zili (2006)) For all* $a \in R$ *and* $b \in R - \{0\}$*, the increments of* $(M^H_t(a,b))_{t\in R_+}$ *are positively correlated if* $\frac{1}{2} < H < 1$*, uncorrelated if* $H = \frac{1}{2}$*, and negatively correlated if* $0 < H < \frac{1}{2}$.

Lemma 2.33. *(Zili (2006)) For all* $a \in R$ *and* $b \in R - \{0\}$*, the increments of* $(M^H_t(a,b))_{t\in \mathbf{R}_+}$ *are long-range dependence if and only if* $H > \frac{1}{2}$.

Theorem 2.34. *(Hölder continuity)*

(Zili (2006)) For all $T > 0$ *and* $\gamma < \frac{1}{2} \wedge H$*, the* $mfBm$ *has a modification which sample paths have a Hölder-continuity, with order* γ*, on the interval* $[0,T]$.

3 ESSENTIALS OF STOCHASTIC ANALYSIS

3.1 Itô's lemma

Let $(X_t)_{t\geq 0}$ be a stochastic process and suppose that there exists a real number X_0 and two $(\mathscr{F}_t)_{t\geq 0}$-adapted processes $\mu = (\mu_t)_{t\geq 0}$ and $\sigma = (\sigma_t)_{t\geq 0}$ such that the following relation holds for all $t \geq 0$,

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s. \tag{3.1}$$

where the stochastic integral in (3.1) is an Itô integral, such processes are called Itô diffusions. We can write the equation as follows

$$dX_t = \mu_t dt + \sigma_t dB_t, \tag{3.2}$$

Then, we can say X satisfies the *SDE* given by (3.2) with the initial condition X_0 given. Note that the formal notation $dX_t = \mu_t dt + \sigma_t dB_t$ is only formal. It is simply a shorthand version of the expression (3.2) above.

In option pricing, we often take as given a stochastic differential equation representation for some basic quantity such as stock price. Many other quantities of interest will be functions of that basic process. To determine the dynamics of these other processes, we shall apply Itô's Lemma, which is basically the chain rule for stochastic processes (Mikosch (1998), Tong (2012), Øksendal (2003)).

Theorem 3.1. *(Itô's Lemma)*

Assume the stochastic process X_t satisfies in the following equation

$$dX_t = \mu_t dt + \sigma_t dB_t, \tag{3.3}$$

where $\mu = (\mu_t)_{t\geq 0}$ and $\sigma = (\sigma_t)_{t\geq 0}$ are adapted processes to a filtration $(\mathscr{F}_t)_{t\geq 0}$. Let Y be a new process defined by $Y_t = f(t, X_t)$ where $f(t,x)$ is a function twice differentiable in its first argument and once in its second. Then Y satisfies the stochastic differential equation:

$$dY_t = \Big(\frac{\partial f}{\partial t}(t,X_t) + \mu_t \frac{\partial f}{\partial x}(t,X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\Big)(t,X_t)dt + \sigma_t \frac{\partial f}{\partial x}(t,X_t)dB_t.$$

Theorem 3.2. *For fBm, we have Skorokhod and Föllmer types Itô's lemma (Duncan, T. E., Hu, Y., & Pasik-Duncan, B. (2000) (2000), Sottinen & Valkeila (2003))*

If $f : R \to R$ is a twice continuously differentiable function with bounded derivatives to order two, then the Skorokhod integral is

$$f(B^H_T) - f(B^H_0) = \int_0^T f'(B^H_s)\delta B^H_s + H\int_0^T s^{2H-1} f''(B^H_s)ds, \tag{3.4}$$

where δ is divergence operator connected to Brownian motion. For $H > \frac{1}{2}$, the Föllmer or pathwise integral is

$$f(B^H_T) - f(B^H_0) = \int_0^T f'(B^H_s)dB^H_s, \tag{3.5}$$

and for $H < \frac{1}{2}$, $\int_0^T f'(B^H_s)dB^H_s$ does not exist as pathwise integral in general.

3.2 Girsanov's Theorem

Definition 3.3. Two measures P and Q on a measurable space $(\Omega, \mathscr{F})$ are equivalent if

$$P(A) = 0 \Leftrightarrow Q(A) = 0, \quad \forall A \in \mathscr{F}. \tag{3.6}$$

The Radon-Nikodym derivative can be defined by using two equivalent measures as follows:

$$M_t = \frac{dQ}{dP}|_{\mathscr{F}t}, \tag{3.7}$$

which enables us to change a measure to another. It follows that for any random variable X that is $\mathscr{F}_t$-measurable

$$E_P[XM] = \int_\Omega X(w)M_t(\omega)dP(\omega) = \int_\Omega X(\omega)dQ(\omega) = E_Q[X]. \tag{3.8}$$

To change the measures for Brownian motion we can use the Girsanov's theorem.

Theorem 3.4. *(Girsanov's Theorem)*

Assume we have $(\mathscr{F}_t)_{t\geq 0}$ be a filtration on interval $[0,T]$ where $T<\infty$. Define a random process M:

$$M_t = \exp\Big[-\int_0^t \lambda_u dB_u^P - \frac{1}{2}\int_0^t \lambda_u^2 du\Big], \quad t\in[0,T]. \tag{3.9}$$

where B^P is a Brownian motion under probability measure P and λ is an $(\mathscr{F}_t)_{t\geq 0}$-adapted process that satisfies the Novikov condition

$$E\Big\{\exp\Big[\frac{1}{2}\int_0^t \lambda_u^2 du\Big]\Big\} < \infty, \quad t\in[0,T]. \tag{3.10}$$

If we define B^Q as

$$B_t^Q = B_t^P + \int_0^t \lambda_u du, \quad t\in[0,T], \tag{3.11}$$

then the following outcomes holds:

(i) M is the Radon-Nikodym martingale

$$M_t = \frac{dQ}{dP}|_{\mathscr{F}_t}.$$

(ii) B^Q is a Brownian motion under the probability measure Q.

4 FUNDAMENTAL ELEMENTS OF STOCHASTIC FINANCE

4.1 Useful financial terminologies

Definition 4.1. (Financial derivatives)

A financial derivative is a contract whose value depends on one or more securities or assets, called underlying assets. Typically the underlying asset is a stock, a bond, a currency exchange rate or the quotation of commodities such as gold, oil or wheat.

Definition 4.2. The spot price (stock price) is the current price in the marketplace at which a given asset—such as a security, commodity, or currency—can be bought or sold for immediate delivery. The strike price will be denoted by S.

Definition 4.3. The strike (exercise) price is the price at which a derivative can be exercised, and refers to the price of the derivative's underlying asset. The strike price will be denoted by K.

Definition 4.4. Expiration date (maturity time) is date on which the option can be exercised. This will be denoted by T.

Definition 4.5. A European call (Put) option grants the right to purchase (sell) a stock at a specific time called maturity T for a specific amount K called the exercise price.

The value of a European call option is denoted by $(S_T - K)_+$ where $(x)_+ = \max(x, 0)$. Similarly, the value of a European put option is $(K - S_T)_+$. This amount is called the option payoff. Here, S_T is the spot price at time T.

Definition 4.6. A risk free interest rate, denoted by r, is the rate of return on an asset that possesses no risk.

Remark 4.7. A dividend payout during the life of an option will have the affect of decreasing the value of a call and increasing the value of a put, the stock price typically falls by the amount of the dividend when it is paid. This will be denoted by D.

Definition 4.8. A currency option is a contract, which gives the owner the right but not the obligation to purchase or sell the indicated amount of foreign currency at a specified price within a specified period of time (American option) or on a fixed date (European option).

Definition 4.9. Asian options are known as path dependent options whose payoff depends on the average stock price and a fixed or floating strike price during a specific period of time before maturity. There exist two types of Asian options such as fixed strike and floating strike options. The payoff for a fixed strike price option is $(A_T - K)_+$ and $(K - A_T)_+$ for a call and put option respectively where K denotes the strike price, T is the strike time and A_T is the average price of the underlying asset over the predetermined interval. For a floating strike price option, the payoffs are $(S_T - A_T)_+$ and $(A_T - S_T)_+$, for a call and put option respectively where S_T stands for the price of stock at time T. Further, Asian options can be also categorized based on different averaging, namely that (i) geometric average that is

$$A_T = \exp\{\frac{1}{T}\int_0^T \log S_t dt\}.$$

(ii) arithmetic average

$$A_T = \frac{1}{T}\int_0^T S_t dt.$$

Consider a financial market consisting of n assets with prices $S_t^1, ..., S_t^n$, which under probability measure P are governed by the following stochastic differential equations:

$$dS_t^i = \mu_t^i dt + \sigma_t^i dB_t^i, \quad i = 1, 2, ..., n, \tag{4.1}$$

where $B = (B^1, ..., B^n)$ is an n dimentional Brownian motion and, μ^i and σ^i are adapted to the natural filtration of the Brownian motion B.

Next, we denote an n-dimensional stochastic process $\theta_t = (\delta_t^1, ..., \delta_t^n)$ as a trading strategy, where δ_t^i denotes the holding in asset i at time t. The value $V_t(\delta)$ at time t of a trading strategy δ is given by

$$V_t(\delta) = \sum_{i=1}^{n} \delta_t^i S_t^i. \tag{4.2}$$

Definition 4.10. A self-financing trading strategy is a strategy δ with the property:

$$V_t(\delta) = V_0(\delta) + \sum_{i=1}^{n} \int_0^t \delta_t^i dS_t^i, \quad t \in [0, T]. \tag{4.3}$$

Definition 4.11. An arbitrage opportunity is a self-financing trading strategy δ with

(i) $V_0(\delta) \leq 0$ a.s.

(ii) $V_T(\delta) \geq 0$ a.s. and $P[V_T(\delta) > 0] > 0$.

In words, arbitrage is a situation where it is possible to make a profit without the possibility of incurring a loss.

Definition 4.12. Time value of an option is basically the risk premium that the seller requires to provide the option buyer with the right to buy/sell the stock up to the expiration date.

Definition 4.13. Portfolio hedging describes a variety of techniques used by investment managers, individual investors and corporations to reduce risk exposure in an investment portfolio. Hedging uses one investment to minimize the negative impact of adverse price swings in another.

Hedging of options is one of the central problems in mathematical finance and it has been studied extensively in various setups. The idea of hedging is to replicate a claim $f(S_T)$ by trading only the underlying asset. In mathematical terms, we are interested in finding a predictable $\mathscr{H}$ such that

$$f(S_T) = C + \int_0^T \mathscr{H}_s dS_s \tag{4.4}$$

where the deterministic constant C is called the hedging cost. In arbitrage free models the hedging cost can be interpreted as the fair price of the option.

Definition 4.14. A risk-neutral measure is a probability measure such that each share price is exactly equal to the discounted expectation of the share price under this measure. In other words, a risk neutral measure is any probability measure, equivalent to the market measure P , which makes all discounted asset prices martingales. This is heavily used in the pricing of financial derivatives due to the fundamental theorem of asset pricing, which implies that in a complete market a derivative's price is the discounted expected value of the future payoff under the unique risk-neutral measure.

Definition 4.15. Greeks are the quantities representing the sensitivity of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. The name is used because the most common of these sensitivities are denoted by Greek letters. Traders use different Greek values, such as delta, theta, and others, to assess options risk and manage option portfolios.

Definition 4.16. Transaction costs are the costs incurred during trading – the process of selling and purchasing – on top of the price of the product that is changing hands. Transaction costs may also refer to a fee that a bank, broker, underwriter or other financial intermediary charges. The difference between what a dealer and buyer paid for a security is one of the transaction costs.

4.2 Classical Black-Scholes market model

In the classical Black-Scholes market model, two assets are traded continously over the time interval $[0,T]$. Denote by A the riskless asset, or bond, and by S the risky asset, or stock. The dynamic of asset price is governed by a geometric Brownian motion:

$$\begin{aligned} dA_t &= r_t A_t dt, \\ dS_t &= \mu_t S_t dt + \sigma S_t dB_t, \end{aligned}$$

where r is a deterministic interest rate, $\sigma > 0$ is constant, B is a Brownian motion. The function μ is the deterministic drift of the stock.

4.3 PDE approach in option pricing

Given the drift rate μ and the volatility σ, the geometric Brownian motion for the stock price process S_t is given by:

$$dS_t = \mu_t S_t dt + \sigma S_t dB_t,$$

where B is a Brownian motion.

Then, applying Itô's lemma and self financing strategy the value of an option $V(t,S_t)$ with a bounded payoff $f(S_T)$ satisfies the following *SDE*:

$$dV = \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} \right) dt + \sigma S_t \frac{\partial V}{\partial S_t} dB_t$$

Setting up a hedging portfolio with one unit in V and Δ units of the stock S_t with $\Delta = \frac{\partial V}{\partial S_t}$ gives us Black-Scholes *PDE* with the drift μ replaced by the risk-free rate r, i.e.:

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} = rV.$$

After delta hedging has removed the risk of the portfolio of one unit in V and Δ

units in stock S_t, the right stock price process to consider is the one in the risk neutral measure, as:

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

Feynman-Kac representation of *SDE* tell us that *PDE* of the Black- Scholes kind have an equivalent probabilistic representation (see, Pascucci, A. (2011)). That is, Feynman-Kac assures that one can solve for the price of the derivative $V(t,S_t)$ by either discretizing the Black-Scholes *PDE* using finite difference methods, or by exploiting the probabilistic interpretation and using Monte Carlo methods. The Black-Scholes *PDE* implies that the price of the derivative $V(t,S_t)$ at time t is equivalent to the discounted value of the expected payoff at expiration (time T). This is the famous Feynman-Kac representation:

$$V(t,S_t) = E_Q[e^{-r(T-t)}V(T,S_T)|\mathscr{F}_t].$$

4.4 (Mixed) Fractional Black-Scholes market model

4.4.1 Fractional Black-Scholes market model

The first try to involve the fractional Brownian motion in modeling of the market was simply to replace the B with B^H in the Black-Scholes model. To interpret the integrals in the pathwise way (which is possible if $H > \frac{1}{2}$), which is natural from the point of view. In this case the dynamic of asset price in the fractional Black-Scholes market model is given by

$$dS_t = \mu_t S_t dt + \sigma S_t dB_t^H.$$

4.4.2 Mixed fractional Black-Scholes market model

If one wants to introduce an economically meaningful market model with long range dependent returns, mixed models are a good option. In such models, one can have both long range dependence and no-arbitrage in the sense of (Bender et al.

(2008)). In the mixed fractional market model the price of an asset is modeled as

$$dS_t = \mu_t S_t dt + \sigma S_t dB_t + \sigma S_t dB_t^H.$$

where B is a Brownian motion and B^H is a fBm.

5 CONCLUSIONS

5.1 Summaries of the articles

I. The evaluation of geometric Asian power options under time changed mixed fractional Brownian motion

This paper deals with pricing a geometric Asian option under time changed mixed fractional Brownian motion. In this model, to get better behaving financial market model, we replace the physical time t with the inverse α- subordinator process $T_\alpha(t)$ in the mixed fractional Black-Scholes model. Then, we apply this result to propose a new model for pricing asian power options. Finally, a lower bound for asian option price is introduced.

II. Hedging in fractional Black—Scholes model with transaction costs

In this paper, we consider the discounted fractional Black–Scholes pricing model where the riskless investment, or the bond, is taken as the numeraire and risky asset $S = (S_t)_{t\in[0,T]}$ is given by the dynamics

$$dS_t = S_t \mu dt + S_t \sigma dB_t,$$

where B is the fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$. Then, we obtain the conditional mean hedge of the European vanilla type option with transaction costs.

III. Subdiffusive fractional Black—Scholes model for pricing currency options under transaction costs

A generalization of the fractional Black-Scholed model is proposed for pricing an European currency option in a discrete time market model with transaction cost. We assume that the stock price follows the subdiffusive fractional Black-Scholes model

$$S_t = S_0 \exp\left\{(r_d - r_f)T_\alpha(t) + \sigma B^H_{T_\alpha(t)}\right\}, \quad S_0 > 0,$$

where $T_\alpha(t)$ is the inverse of an α-stable subordinator, with $\alpha \in (\frac{1}{2},1), H \in [\frac{1}{2},1), \alpha + \alpha H > 1$. In this paper, an analytical pricing model for European currency option under transaction costs is introduced in discrete time case.

IV. Mixed fractional Merton model to evaluate European options with transaction costs

The problem of European option pricing in discrete time are discussed under a mixed fractional version of the Merton model with transaction costs. In this case, it has been assumed that the price dynamics S_t satisfies in the following

$$S_t = S_0 \exp\left\{\mu t + \sigma B_t + \sigma_H B_t^H + N_t \ln J\right\}, \quad S_0 > 0,$$

where $S_0, \mu, \sigma, \sigma_H$ are constant, B_t is a Brownian motion; B_t^H is a fractional Brownian motion with Hurst parameter $H \in (\frac{3}{4},1)$, N_t is a Poisson process with intensity $\lambda > 0$ and J is a positive random variable. We assume that B_t, B_t^H, N_t and J are independent. Finally, we evaluate impact of parameters on the option price.

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The evaluation of geometric Asian power options under time changed mixed fractional Brownian motion

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ABSTRACT

In this paper, the geometric Asian option pricing problem is investigated under the assumption that the underlying stock price is assumed following a mixed fractional subdiffusive Black–Scholes model, and the geometric average Asian option pricing formula is derived under this assumption. We then apply the results to value Asian power options on the stocks that pay constant dividends when the payoff is a power function. Finally, lower bound of Asian options and some special cases are provided.

1. Introduction

A standard option is a financial contract which gives the owner of the contract the right, but not the obligation, to buy or sell a specified asset at a prespecified time (maturity) for a prespecified price (strike price). The specified asset (underlying asset) can be for example stocks, indexes, currencies, bonds or commodities. The option can be either a call option, which gives the owner the right to buy the underlying asset, or it can be a put option, which gives the owner the right to sell the underlying asset. There are several types of options that are traded in a market. American option allows the owner to exercise his option at any time up to and including the strike date. European options can be exercised only on the strike date. European options are also called vanilla options. Their payoffs at maturity depend on the spot value of the stock at the time of exercise. There are other options whose values depend on the stock prices over a predetermined time interval. For an Asian option, the payoff is determined by the average value over some predetermined time interval. The average price of the underlying asset can either determine the underlying settlement price (average price Asian options) or the option strike price (average strike Asian options). Furthermore, the average prices can be calculated using either the arithmetic mean or the geometric mean. The type of Asian option that will be examined throughout this research is geometric Asian option.

Over the past three decades, academic researchers and market practitioners have developed and adopted different models and techniques for option valuation. The most popular model on option pricing was introduced by Black and Scholes (*BS*) [1] in 1973. In the *BS* model it has been assumed that the asset price dynamics are governed by a geometric Brownian motion. However, a large number of empirical studies have shown that the distributions of the logarithmic returns of financial asset usually exhibit properties of self-similarity, heavy tails, long-range dependence in both auto-correlations and cross-correlations, and volatility clustering [2–4]. Actually, the most popular stochastic process that exhibits long-range

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dependence is of course the fractional Brownian motion. Moreover, the fractional Brownian motion produces a burstiness in the sample path behavior, which is the important behavior of financial time series. Since fractional Brownian motion is neither a Markov process nor a semi-martingale, we cannot use the usual stochastic calculus to analyze it. Further, fractional Brownian motion admits arbitrage in a complete and frictionless market. To get around this problem and to take into account the long memory property, it is reasonable to use the mixed fractional Brownian motion (*mfBm*) to capture the fluctuations of the financial asset [5–7].

The *mfBm* is a linear combination of the Brownian motion and fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$, defined on the filtered probability $(\Omega, \mathscr{F}, \mathbb{P})$ for any $t \in \mathbb{R}^+$ by:

$$M_t^H(a, b) = aB(t) + bB^H(t), \tag{1.1}$$

where $B(t)$ is a Brownian motion, and $B^H(t)$ is an independent fractional Brownian motion with Hurst index H. Cheridito [7] proved that, for $H \in (\frac{3}{4}, 1)$, the mixed model is equivalent to the Brownian motion and hence it is also arbitrage free. For $H \in (\frac{1}{2}, 1)$, Mishura and Valkeila [8] demonstrated that the mixed model is arbitrage free. Rao [9] discussed geometric Asian power option under *mfBm*. To see more about the mixed model, one can refer to Refs. [6,7,10,11].

Analysis of various real-life data shows that many processes observed in economics display characteristic periods in which they stay motionless [12]. This feature is most common for emerging markets in which the number of participants, and thus the number of transactions, is rather low. Notably, similar behavior is observed in physical systems exhibiting subdiffusion. The constant periods of financial processes correspond to the trapping events in which the subdiffusive test particle gets immobilized [13]. Subdiffusion is a well known and established phenomenon in statistical physics. Its usual mathematical description is in terms of the celebrated Fractional Fokker–Planck equation (*FFPE*). This equation was first derived from the continuous-time random walk scheme with heavy-tailed waiting times [14,15,10], and since then became fundamental in modeling and analysis of complex systems exhibiting slow dynamics. Following this line, and to model the observed long range dependence and fluctuations in the financial price time series, we introduce a time-changed mixed fractional *BS* model to value Asian power option when the underlying stock is

$$\begin{aligned} S_t &= X(T_\alpha(t)) \\ &= S_0 e^{(r-q)T_\alpha(t) + M_\alpha^H(t)(\sigma,\sigma) - \frac{1}{2}\sigma^2 \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{1}{2}\sigma^2 \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H}}, \quad S_0 = X(0) > 0, \end{aligned} \tag{1.2}$$

where $M_\alpha^H(t)(\sigma, \sigma) = \sigma B(T_\alpha(t)) + \sigma B^H(T_\alpha(t))$, $\alpha \in (\frac{1}{2}, 1)$, $2\alpha - \alpha H > 1$ and $T_\alpha(t)$ is the inverse α-stable subordinator.

We then apply the result to price geometric Asian power options that pay constant dividends when the payoff is a power function. We also provide some special cases and lower bound for the Asian option price. The rest of the paper is organized as follows. In Section 2, some useful concepts and theorems of time changed mixed fractional process are introduced. In Section 3, a brief introduction of Asian options is given. Analytical valuation formula for geometric Asian options is derived in Section 4 and then applied to geometric Asian power options in Section 5. The lower bound on the price of the Asian option is proposed in Section 6.

2. Auxiliary facts

In this section, we recall some definitions and results about mixed fractional time changed process. More information about mixed fractional process can be found in [16,10].

The time-changed process $T_\alpha(t)$ is the inverse α-stable subordinator defined as below

$$T_\alpha(t) = \inf\{\tau > 0, U_\alpha(t) \geq t\}$$

here $U_\alpha(\tau)_{\tau \geq 0}$ is a strictly increasing α-stable Lévy process [17] with Laplace transform: $\mathbb{E}(e^{-uU_\alpha(\tau)}) = e^{-\tau u^\alpha}$, $\alpha \in (0, 1)$.

$U_\alpha(t)$ is $\frac{1}{\alpha}$ self-similar and $T_\alpha(t)$ is α self-similar, that is, for every $h > 0$, $U_\alpha(ht) \triangleq h^{\frac{1}{\alpha}} U_\alpha(t)$ $T_\alpha(ht) \triangleq h^\alpha T_\alpha(t)$, here $\triangleq$ indicates that the random variables on both sides have the same distribution. Specially, when $\alpha \uparrow 1$, $T_\alpha(t)$ reduces to the physical time t. You can find more details about subordinator and its inverse processes in [18,19].

Consider the subdiffusion process

$$M_\alpha^H(t)(a, b) = aW_\alpha(t) + bW_\alpha^H(t) = aB(T_\alpha(t)) + bB^H(T_\alpha(t)),$$

where $B(\tau)$ is a Brownian motion, $B^H(\tau)$ is a fractional Brownian motion with Hurst index H and $T_\alpha(t)$ is inverse α-subordinator which are supposed to be independent. When $a = 0, b = 1$, then it is the process considered in [20] and if $b = 0, a = 1$, then it is the process considered in [21]. In this research, we assume that $H \in (\frac{3}{4}, 1)$ and $(a, b) = (1, 1)$.

Remark 2.1. When $\alpha \uparrow 1$, the processes $W_\alpha(t)$ and $W_\alpha^H(t)$ degenerate to $B(t)$ and $B^H(t)$, respectively. Then, $M_\alpha^H(t)(a, b)$ reduces to the *mfBm* in Eq. (1.1).

Remark 2.2. From [20,21], we know that $\mathbb{E}(T_\alpha(t)) = \frac{t^\alpha}{\Gamma(\alpha+1)}$. Then, by applying α-self-similar and non-decreasing sample path of $T_\alpha(t)$, we have

$$\mathbb{E}[(B(T_\alpha(t)))^2] = \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{2.1}$$

$$\mathbb{E}[(B^H(T_\alpha(t)))^2] = \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H}. \tag{2.2}$$

3. Asian options

The payoff of an Asian option is based on the difference between an asset's average price over a given time period, and a fixed price called the strike price. Asian options are popular because they tend to have lower volatility than options whose payoffs are based purely on a single price point. It is also harder for big traders to manipulate an average price over an extended period than a single price, so Asian option offers further protection against risk. The Asian call and put options have a payoff that is calculated with an average value of the underlying asset over a specific period. The payoff for an Asian call and put option with strike price K and expiration time T is $(\bar{S}(T) - K)_+$ and $(K - \bar{S}(T))_+$ respectively, where $\bar{S}(T)$ is the average price of the underlying asset over the prespecified interval. Since Asian options are less expensive than their European counterparts, they are attractive to many different investors. Apart from the regular Asian option there also exists Asian strike option. An Asian strike call option guarantees the holder that the average price of an underlying asset is not higher than the final price. The option will not be exercised if the average price of the underlying asset is greater than the final price. The holder of an Asian strike put option makes sure that the average price received for the underlying asset is not less than what the final price will provide. The payoff for an Asian strike call and put option is $(\bar{S}(T)-S(T))_+$ and $(S(T)-\bar{S}(T))_+$ respectively, where $S(T)$ is the value of underlying stock at maturity date T.

Asian options are divided into two different types, when calculating the average, the geometric Asian option

$$G(T) = \exp\left\{\frac{1}{T}\int_0^T \ln S(t)dt\right\},$$

and the arithmetic Asian option.

$$A(T) = \frac{1}{T}\int_0^T S(t)dt.$$

We assume that the prespecified interval $[0, T]$ is fixed, then will price the geometric Asian option in the continuous average case under time changed mixed fractional Brownian motion environment.

4. Pricing model of geometric Asian option

In order to derive an Asian option pricing formula in a time changed mixed fractional market, we make the following assumptions:

(i) The price of underlying stock at time t is given by Eq. (1.2).
(ii) There are no transaction costs in buying or selling the stocks or option.
(iii) The risk free interest rate r and dividend rate q are known and constant through time.
(iv) The option can be exercised only at the maturity time.

From Eq. (1.2), we know that $\ln S_t \simeq N(u, v)$, where

$$u = \ln S(0) + (r-q)T_\alpha(t) - \frac{1}{2}\sigma^2\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{1}{2}\sigma^2\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H} \tag{4.1}$$

$$v = \sigma^2\frac{t^\alpha}{\Gamma(\alpha+1)} + \sigma^2\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H}. \tag{4.2}$$

Let $C(S(0), T)$ be the price of a European call option at time 0 with strike price K and that matures at time T. Then, from [16], we can get

$$C(S(0), T) = S(0)e^{-qT}\phi(d_1) - Ke^{-rT}\phi(d_2),$$

where

$$d_1 = \frac{\ln\frac{S_0}{K} + (r - q + \frac{\hat{\sigma}^2}{2})T}{\hat{\sigma}\sqrt{T}}, \quad d_2 = d_1 - \hat{\sigma}\sqrt{T},$$

$$\hat{\sigma}^2 = \sigma^2 \frac{T^{\alpha-1}}{\Gamma(\alpha)} + \sigma^2 \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H},$$

and $\phi(.)$ denotes cumulative normal density function.

Under the above assumptions (i)–(iv), we obtain the value of the geometric Asian call option by the following theorem

Theorem 4.1. *Suppose the stock price S_t satisfied Eq. (1.2). Then, under the risk-neutral probability measure, the value of geometric Asian call option $C(S(0), T)$ with strike price K and maturity time T is given by*

$$\begin{aligned} C(S(0), T) = S(0) \exp\Bigg\{ & -rT + (r-q)\frac{T^\alpha}{\Gamma(\alpha+2)} + \frac{\sigma^2(-T)^\alpha}{2\Gamma(\alpha+3)} \\ & - \frac{\sigma^2 T^{2\alpha H}}{4(2\alpha H+1)(\alpha H+1)(\Gamma(\alpha+1))^{2H}} \Bigg\} \phi(d_1) - Ke^{-qT}\phi(d_2), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} d_2 &= \frac{\mu_G - \ln K}{\sigma_G}, \quad d_1 = d_2 + \sigma_G, \\ \mu_G &= \ln S(0) + (r - q - \frac{\sigma^2}{2})\frac{T^\alpha}{\Gamma(\alpha+2)} - \frac{\sigma^2 T^{2\alpha H}}{2(2\alpha H+1)(\Gamma(\alpha+1))^{2H}}, \\ \sigma_G^2 &= \frac{\sigma^2 T^\alpha}{\Gamma(\alpha+2)} + \frac{\sigma^2(-T)^\alpha}{\Gamma(\alpha+3)} + \frac{\sigma^2 T^{2\alpha H}}{(2\alpha H+2)(\Gamma(\alpha+1))^{2H}}, \end{aligned}$$

the interest rate r and the dividend rate q are constant over time and $\phi(.)$ denotes cumulative normal density function.

Proof. Suppose

$$L(T) = \frac{1}{T}\int_0^T \ln S(t)dt.$$

Then

$$G(T) = e^{L(T)}. \tag{4.4}$$

We know that $\ln S_t \simeq N(u, v)$, then it is clear that the random variable $L(T)$ has Gaussian distribution under the risk-neutral probability measure. We will now compute its mean and variance under the risk-neutral probability measure. Let $\mathbb{E}$ denote the expectation and, μ_G and σ_G^2 denote the mean and the variance of the random variable $\mathbb{E}$ under the risk-neutral probability measure. Note that

$$\begin{aligned} \mu_G &= \mathbb{E}[L(T)] = \frac{1}{T}\int_0^T \mathbb{E}[\ln S(t)]dt \\ &= \ln S(0) + \frac{1}{T}\int_0^T (r-q)\frac{t^\alpha}{\Gamma(\alpha+1)}dt - \frac{\sigma^2}{2T}\int_0^T \left[\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha H}}{(\Gamma(\alpha+1))^{2H}}\right]dt \\ &= \ln S(0) + (r-q)\frac{T^\alpha}{\Gamma(\alpha+2)} - \frac{\sigma^2 T^\alpha}{2\Gamma(\alpha+2)} - \frac{\sigma^2 T^{2\alpha H}}{(4\alpha H+2)(\Gamma(\alpha+1))^{2H}}, \end{aligned}$$

and

$$\begin{aligned} \sigma_G^2 &= Var[L(T)] = \mathbb{E}[(L(T)-\mu_G)^2] \\ &= \frac{\sigma^2}{T^2}\int_0^T\int_0^T \big(\mathbb{E}[W_\alpha(t)W_\alpha(\tau)] + \mathbb{E}[W_\alpha^H(t)W_\alpha^H(\tau)]\big)\,dtd\tau, \end{aligned}$$

by independence of the processes $B(t)$, $B^H(t)$ and $T_\alpha(t)$, we obtain

$$\begin{aligned} &= \frac{\sigma^2}{T^2}\int_0^T\int_0^T \left(|\frac{t^\alpha}{\Gamma(\alpha+1)}| + |\frac{\tau^\alpha}{\Gamma(\alpha+1)}| - |\frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)}|\right)dtd\tau \\ &+ \frac{\sigma^2}{T^2}\int_0^T\int_0^T \left(|\frac{t^\alpha}{\Gamma(\alpha+1)}|^{2H} + |\frac{\tau^\alpha}{\Gamma(\alpha+1)}|^{2H} - |\frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)}|^{2H}\right)dtd\tau \\ &= \frac{\sigma^2 T^\alpha}{\Gamma(\alpha+2)} + \frac{\sigma^2(-T)^\alpha}{\Gamma(\alpha+3)} + \frac{\sigma^2 T^{2\alpha H}}{(2\alpha H+2)(\Gamma(\alpha+1))^{2H}}. \end{aligned}$$

From (4.4), we know that the random variable $G(T)$ is log-normally distributed, then $\ln G(T) \simeq N(\mu_G, \sigma_G^2)$. Let $I = \{x : e^x > K\}$ and $\phi(.)$ be the probability density function of a standard normal distribution, then the price of geometric Asian call option is given by the following computations

$$\begin{aligned}
C(S(0),T) &= e^{-rT}\mathbb{E}[(G(T)-K)^{+}]\\
&= e^{-rT}\int_I (e^x - K)\frac{1}{\sqrt{2\pi}\sigma_G}\exp\left\{-\frac{(x-\mu_G)^2}{2\sigma_G^2}\right\}dx\\
&= e^{-rT}\int_I (e^{\mu_G+z\sigma_G} - K)\frac{1}{\sqrt{2\pi}\sigma_G}\exp\left\{-\frac{(x-\mu_G)^2}{2\sigma_G^2}\right\}\varphi(z)dz\\
&= e^{-rT+\mu_G+\frac{1}{2}\sigma_G^2}\int_{-d_2}^{\infty} e^{-\frac{1}{2}(z-\sigma_G)^2}dz - Ke^{-rT}\int_{-d_2}^{-\infty}\varphi(z)dz\\
&= e^{-rT+\mu_G+\frac{1}{2}\sigma_G^2}\int_{-d_2-\sigma_G}^{\infty}\varphi(z)dz - Ke^{-rT}\int_{-\infty}^{d_2}\varphi(z)dz\\
&= e^{-rT+\mu_G+\frac{1}{2}\sigma_G^2}\int_{-\infty}^{d_2+\sigma_G}\varphi(z)dz - Ke^{-rT}\int_{-\infty}^{d_2}\varphi(z)dz\\
&= e^{-rT+\mu_G+\frac{1}{2}\sigma_G^2}\phi(d_1) - Ke^{-rT}\phi(d_2),\\
&= S(0)\exp\left\{-rT + (r-q)\frac{T^\alpha}{\Gamma(\alpha+2)} + \sigma^2\frac{(-T)^\alpha}{2\Gamma(\alpha+3)}\right.\\
&\quad\left. -\sigma^2\frac{T^{2\alpha H}}{4(2\alpha H+1)(\alpha H+1)(\Gamma(\alpha+1))^{2H}}\right\}\phi(d_1) - Ke^{-qT}\phi(d_2),
\end{aligned}$$

here

$$\begin{aligned}
I &= \{x : e^x > K\} = \{z : e^{\mu_G+z\sigma_G} > K\}\\
&= \{z : \mu_G + z\sigma_G > \ln K\} = \{z : z > -d_2\},
\end{aligned}$$

thus we obtain the pricing formula. □

Moreover, using the put–call parity, the valuation model for a geometric Asian put option under time changed mixed fractional *BS* model can be written

$$\begin{aligned}
P(S(0),T) &= Ke^{-qT}\phi(-d_2) - S(0)\exp\left\{-rT + (r-q)\frac{T^\alpha}{\Gamma(\alpha+2)} + \frac{\sigma^2(-T)^\alpha}{2\Gamma(\alpha+3)}\right.\\
&\quad\left. - \frac{\sigma^2 T^{2\alpha H}}{4(2\alpha H+1)(\alpha H+1)(\Gamma(\alpha+1))^{2H}}\right\}\phi(-d_1),
\end{aligned} \tag{4.5}$$

where d_1 and d_2 are defined previously.

Letting $\alpha \uparrow 1$, then the stock price follows the *mfBm* shown below

$$\begin{aligned}
S_t &= S_0\exp\left\{(r-q)T + \sigma B(t) + \sigma B^H(t)\right.\\
&\quad\left. -\frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 t^{2H}\right\}, \quad 0 < t < T,
\end{aligned} \tag{4.6}$$

and the result is presented below.

Corollary 4.1. *The value of geometric Asian call option with maturity T and strike K, whose stock price follows Eq. (4.6), is given by*

$$\begin{aligned}
&C(S(0),T) =\\
&S(0)\exp\left\{-\frac{1}{2}(r+q)T - \frac{\sigma^2 T}{12} - \frac{\sigma^2 T^{2H}}{4(2H+1)(H+1)}\right\}\phi(d_1) - Ke^{-qT}\phi(d_2),
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
d_2 &= \frac{\mu_G - \ln K}{\sigma_G}, \quad d_1 = d_2 + \sigma_G,\\
\mu_G &= \ln S(0) + \frac{1}{2}(r - q - \frac{\sigma^2}{2})T - \frac{\sigma^2 T^{2H}}{2(2H+1)},
\end{aligned}$$

$$\sigma_G^2 = \frac{\sigma^2 T}{3} + \frac{\sigma^2 T^{2H}}{(2H+2)},$$

which is consistent with result in [9].

5. Pricing model of Asian power option

In this section, we consider the pricing model of Asian power call option with strike price K and maturity time T under time changed mixed fractional *BS* model where the payoff function is $(G^n(T) - K)^+$ for some constant integer $n \geq 1$.

Theorem 5.1. *Suppose the stock price S_t satisfied Eq. (1.2). Then, under the risk-neutral probability measure the value of geometric Asian power call option $C(S(0), T)$ with strike price K, maturity time T and payoff function $(G^n(T) - K)^+$ is given by*

$$\begin{aligned} C(S(0), T) = S(0) \exp\Bigg\{ & -rT + (r-q)\frac{nT^\alpha}{\Gamma(\alpha+2)} - \frac{(n-n^2)\sigma^2 T^\alpha}{2\Gamma(\alpha+2)} + \frac{n^2\sigma^2(-T)^\alpha}{2\Gamma(\alpha+3)} \\ & - \frac{n\sigma^2 T^{2\alpha H}}{(4\alpha H+2)(\Gamma(\alpha+1))^{2H}} - \frac{n^2\sigma^2 T^{2\alpha H}}{(4\alpha H+4)(\Gamma(\alpha+1))^{2H}}\Bigg\}\phi(f_1) \\ & - Ke^{-qT}\phi(f_2), \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} f_2 &= \frac{\mu_G - \frac{1}{n}\ln K}{\sigma_G}, \quad f_1 = f_2 + n\sigma_G, \\ \mu_G &= \ln S(0) + (r - q - \frac{\sigma^2}{2})\frac{T^\alpha}{\Gamma(\alpha+2)} - \frac{\sigma^2 T^{2\alpha H}}{2(2\alpha H+1)(\Gamma(\alpha+1))^{2H}}, \\ \sigma_G^2 &= \frac{\sigma^2 T^\alpha}{\Gamma(\alpha+2)} + \frac{\sigma^2(-T)^\alpha}{\Gamma(\alpha+3)} + \frac{\sigma^2 T^{2\alpha H}}{(2\alpha H+2)(\Gamma(\alpha+1))^{2H}}, \end{aligned}$$

the interest rate r and the dividend rate q are constant over time and $\varphi(.)$ denotes cumulative normal density function.

Proof. The payoff function for Asian power option is $(G^n(T) - K)^+ = (e^{nL(T)} - K)^+$, then applying similar computation in Theorem 4.1, we obtain

$$\begin{aligned} C(S(0), T) &= e^{-rT}\mathbb{E}[(G^n(T) - K)^+] \\ &= e^{-rT}\int_I (e^{nx} - K)\frac{1}{\sqrt{2\pi}\sigma_G}\exp\left\{-\frac{(x-\mu_G)^2}{2\sigma_G^2}\right\}dx \\ &= e^{-rT}\int_I (e^{n(\mu_G + z\sigma_G)} - K)\frac{1}{\sqrt{2\pi}\sigma_G}\exp\left\{-\frac{(x-\mu_G)^2}{2\sigma_G^2}\right\}\varphi(z)dz \\ &= e^{-rT+n\mu_G+\frac{1}{2}n^2\sigma_G^2}\int_{-f_2}^{\infty} e^{-\frac{1}{2}(z-n\sigma_G)^2}dz - Ke^{-rT}\int_{-f_2}^{-\infty}\varphi(z)dz \\ &= e^{-rT+n\mu_G+\frac{1}{2}n^2\sigma_G^2}\int_{-f_2-n\sigma_G}^{\infty}\varphi(z)dz - Ke^{-rT}\int_{-\infty}^{f_2}\varphi(z)dz \\ &= e^{-rT+n\mu_G+\frac{1}{2}n^2\sigma_G^2}\int_{-\infty}^{f_2+n\sigma_G}\varphi(z)dz - Ke^{-rT}\int_{-\infty}^{f_2}\varphi(z)dz \\ &= e^{-rT+n\mu_G+\frac{1}{2}n^2\sigma_G^2}\phi(f_1) - Ke^{-rT}\phi(f_2), \\ &= S(0)\exp\Bigg\{-rT + (r-q)\frac{nT^\alpha}{\Gamma(\alpha+2)} - \frac{(n-n^2)\sigma^2 T^\alpha}{2\Gamma(\alpha+2)} + \frac{n^2\sigma^2(-T)^\alpha}{2\Gamma(\alpha+3)} \\ &\quad - \frac{n\sigma^2 T^{2\alpha H}}{(4\alpha H+2)(\Gamma(\alpha+1))^{2H}} - \frac{n^2\sigma^2 T^{2\alpha H}}{(4\alpha H+4)(\Gamma(\alpha+1))^{2H}}\Bigg\}\phi(f_1) \\ &\quad - Ke^{-qT}\phi(f_2), \end{aligned}$$

here

$$I = \{x : e^{nx} > K\} = \{z : e^{n(\mu_G + z\sigma_G)} > K\}$$
$$= \{z : \mu_G + z\sigma_G > \frac{1}{n}\ln K\} = \{z : z > -f_2\},$$

thus the proof is completed. □

The time changed mixed fractional *BS* model includes the jump behavior of price process because the subordinator process is a pure jump Levy process so it can capture the random variations in volatility. Also, it can be used for data with long range dependence and visible constant time periods characteristic for processes delayed by inverse subordinators.

6. Lower bound of the Asian option price

The aim of this section is to obtain the lower bound on the price of the Asian option. The next theorem shows that the normal distribution is stable when the random variables are jointly normal.

Theorem 6.1 ([22]). *The conditional distribution of* $\ln S_{t_i}$ *given* $\ln G(T)$ *is a normal distribution*

$$(\ln S_{t_i} | \ln G(T) = z) \simeq N(\mu_i + (z - \mu_G)\frac{\lambda_i}{\sigma_G^2}, \sigma_i^2 - \frac{\lambda_i^2}{\sigma_G^2}), \quad i = 1, \dots, n,$$

where

$$\mu_i = \ln S(0) + (r - q)T_\alpha(t_i) - \frac{1}{2}\sigma^2 \frac{t_i^\alpha}{\Gamma(\alpha+1)} - \frac{1}{2}\sigma^2\left(\frac{t_i^\alpha}{\Gamma(\alpha+1)}\right)^{2H}$$
$$\sigma_i^2 = \sigma^2 \frac{t_i^\alpha}{\Gamma(\alpha+1)} + \sigma^2\left(\frac{t_i^\alpha}{\Gamma(\alpha+1)}\right)^{2H},$$

$\lambda_i = Cov(\ln S_{t_i}, \ln G(T))$, $0 \le t_1 < t_2 < \cdots < t_n \le T$, $T_\alpha(t)$ *is inverse* α*-stable subordinator and,* μ_G *and* σ_G^2 *are defined in* Theorem 4.1.

Moreover, $(S_{t_i} | \ln G(T))$ *has a lognormal distribution and*

$$\mathbb{E}\left[S_{t_i} | \ln G(T) = z\right]$$
$$= \exp\left\{\mu_i + (z - \mu_G)\frac{\lambda_i}{\sigma_G^2} + \frac{1}{2}(\sigma_i^2 - \frac{\lambda_i^2}{\sigma_G^2})\right\} \quad i = 1, \dots, n. \tag{6.1}$$

Now, we condition on the geometric average $G(T)$ in the pricing expression of the Asian option

$$C(S(0), T) = e^{-rT}\mathbb{E}[(A(T) - K)^+] = e^{-rT}\mathbb{E}[\mathbb{E}[(A(T) - K)^+ | G(T)]]$$
$$= e^{-rT}\int_0^\infty \mathbb{E}[(A(T) - K)^+ | G(T) = z]g(z)dz,$$

where g is the lognormal density function of G. Let

$$C_1 = \int_0^K \mathbb{E}[(A(T) - K)^+ | G(T) = z]g(z)dz,$$
$$C_2 = \int_K^\infty \mathbb{E}[(A(T) - K)^+ | G(T) = z]g(z)dz,$$

then $C(S(0), T) = e^{-rT}(C_1 + C_2)$. Since the geometric average is less than arithmetic average $A(T) \ge G(T)$,

$$C_2 = \int_K^\infty \mathbb{E}[A(T) - K | G(T) = z]g(z)dz, \tag{6.2}$$

from Theorem 6.1, we can calculate C_2. Applying Jensen's inequality we obtain a lower bound on C_1

$$C_1 = \int_0^K \mathbb{E}[(A(T) - K)^+ | G(T) = z]g(z)dz$$
$$\ge \int_0^K (E[A(T) - K | G(T) = z])^+ g(z)dz$$
$$= \int_{\tilde{K}}^K \mathbb{E}[A(T) - K | G(T) = z]g(z)dz = \tilde{C}_1 \tag{6.3}$$

where $\tilde{K} = \{z | \mathbb{E}[A(T) | G(T) = z] = K\}$.

Eq. (6.1) enables us to obtain $\tilde{K}$, then we calculate the following expectation

$$\mathbb{E}[A(T)|G(T)=z] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} S_{t_i}|G(T)=z\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[S_{t_i}|G(T)=z\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n}\exp\left(\mu_i + (\log z - \mu_G)\frac{\lambda_i}{\sigma_G^2} + \frac{1}{2}(\sigma_i^2 - \frac{\lambda_i^2}{\sigma_G^2})\right).$$

Theorem 6.2. *A lower bound on the price of the Asian option with strike price K and maturity time T is given by*

$$\tilde{C}(S(0),T) = e^{-rT}(\tilde{C}_1 + C_2)$$
$$= e^{-rT}\left\{\frac{1}{n}\sum_{i=1}^{n}\exp(\mu_i + \frac{1}{2}\sigma_i^2)\phi\left(\frac{\mu_G - \ln\tilde{K} + \gamma_i}{\sigma_G}\right)\right.$$
$$\left. - K\phi\left(\frac{\mu_G - \ln\tilde{K}}{\sigma_G}\right)\right\},$$

where all parameters are defined previously.

Proof. Collecting Eqs. (6.2) and (6.3) gives

$$\tilde{C}_1 + C_2 = \int_{\tilde{K}}^{\infty}\mathbb{E}[A(T) - K|G(T)=z]g(z)dz$$
$$= \int_{\tilde{K}}^{\infty}\mathbb{E}[A(T)|G(T)=z]g(z)dz - K\int_{\tilde{K}}^{\infty}g(z)dz$$
$$= \int_{\tilde{K}}^{\infty}\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}S_{t_i}|G(T)=z\right]g(z)dz - K\int_{\tilde{K}}^{\infty}g(z)dz$$
$$= \int_{\tilde{K}}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[S_{t_i}|G(T)=z\right]g(z)dz - K\int_{\tilde{K}}^{\infty}g(z)dz$$
$$= \frac{1}{n}\sum_{i=1}^{n}\int_{\tilde{K}}^{\infty}\mathbb{E}\left[S_{t_i}|\ln G(T)=\ln z\right]g(z)dz - K\int_{\tilde{K}}^{\infty}g(z)dz.$$

From the proof of Theorem 4.1, we obtain

$$K\int_{\tilde{K}}^{\infty}g(z)dz = K\phi\left(\frac{\mu_G - \ln\tilde{K}}{\sigma_G}\right),$$

and from Eq. (6.1)

$$\int_{\tilde{K}}^{\infty}\mathbb{E}\left[S_{t_i}|\ln G(T)=\ln z\right]g(z)dz$$
$$= \int_{\tilde{K}}^{\infty}\exp\left(\mu_i + (\ln z - \mu_G)\frac{\lambda_i}{\sigma_G^2} + \frac{1}{2}(\sigma_i^2 - \frac{\lambda_i^2}{\sigma_G^2})\right)g(z)dz$$
$$= \exp\left(\mu_i + \frac{1}{2}\sigma_i^2\right)\int_{\tilde{K}}^{\infty}\exp\left((\ln z - \mu_G)\frac{\lambda_i}{\sigma_G^2} - \frac{1}{2}\frac{\lambda_i^2}{\sigma_G^2}\right)g(z)dz.$$

Using the density of the lognormal distribution, we have

$$\int_{\tilde{K}}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_G z}\exp\left((\ln z - \mu_G)\frac{\lambda_i}{\sigma_G^2} - \frac{1}{2}\frac{\lambda_i^2}{\sigma_G^2} - \frac{1}{2}(\frac{\mu_G - \ln z}{\sigma_G})^2\right)dz.$$

Making the change of variables $y = \frac{\mu_G - \ln z + \lambda_i}{\sigma_G}$ and $\frac{dy}{dz} = -\frac{1}{\sigma_G z}$, then we have

$$\int_{\frac{\mu_G - \ln z + \lambda_i}{\sigma_G}}^{-\infty} -\frac{1}{\sqrt{2\pi}}\exp\left((\frac{\lambda_i}{\sigma_G} - y)\frac{\lambda_i}{\sigma_G} - \frac{1}{2}\frac{\lambda_i^2}{\sigma_G^2} - \frac{1}{2}(y - \frac{\lambda_i}{\sigma_G})^2\right)dy$$
$$= \int_{-\infty}^{\frac{\mu_G - \ln z + \lambda_i}{\sigma_G}}\frac{1}{\sqrt{2\pi}}\exp\left(-y\frac{\lambda_i}{\sigma_G} + \frac{1}{2}\frac{\lambda_i^2}{\sigma_G^2} - \frac{1}{2}y^2 - \frac{1}{2}\frac{\lambda_i^2}{\sigma_G^2} + y\frac{\lambda_i}{\sigma_G}\right)dy$$
$$= \int_{-\infty}^{\frac{\mu_G - \ln z + \lambda_i}{\sigma_G}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y^2\right)dy = \phi\left(\frac{\mu_G - \ln\tilde{K} + \gamma_i}{\sigma_G}\right),$$

by collecting $\tilde{C}_1$ and C_2 the proof is completed. □

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Hedging in fractional Black–Scholes model with transaction costs

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ABSTRACT

We consider conditional-mean hedging in a fractional Black–Scholes pricing model in the presence of proportional transaction costs. We develop an explicit formula for the conditional-mean hedging portfolio in terms of the recently discovered explicit conditional law of the fractional Brownian motion.

1. Introduction

We consider discrete hedging in fractional Black–Scholes models where the asset price is driven by a long-range dependent fractional Brownian motion. For a convex or concave European vanilla type option, we construct the so-called conditional-mean hedge. This means that at each trading time the value of the conditional mean of the discrete hedging strategy coincides with the frictionless price. By frictionless we mean the continuous trading hedging price without transaction costs. The key ingredient in constructing the conditional mean hedging strategy is the recent representation for the regular conditional law of the fractional Brownian motion given in Sottinen and Viitasaari (2017). Let us note that there are arbitrage strategies with continuous trading without transaction costs, but not with discrete trading strategies, even in the absence of trading costs. For details of the use of fractional Brownian motion in finance and discussion on arbitrage see Bender et al. (2011).

For the classical Black–Scholes model driven by the Brownian motion, the study of hedging under transaction costs goes back to Leland (1985). See also Denis and Kabanov (2010) and Kabanov and Safarian (2009) for a mathematically rigorous treatment. For the fractional Black–Scholes model driven by the long-range dependent fractional Brownian motion, the study of hedging under transaction costs was studied in Azmoodeh (2013). In the series of articles (Shokrollahi et al., 2016; Wang, 2010a, b; Wang et al., 2010a, b) the discrete hedging in the fractional Black–Scholes model was studied by using the economically dubious Wick–Itô–Skorohod interpretation of the self-financing condition. Actually, with the economically solid forward-type pathwise interpretation of the self-financing condition, these hedging strategies are valid, not for the geometric fractional Brownian motion, but for a geometric Gaussian process where the driving noise is a Gaussian martingale

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with the same variance function as the corresponding fractional Brownian motion would have, see Gapeev et al. (2011), Shokrollahi and Kılıçman (2014), Shokrollahi and Kılıçman (2015) and Shokrollahi et al. (2015).

2. Preliminaries

We are interested in pricing of European vanilla options $f(S_T)$ of a single underlying asset $S = (S_t)_{t\in[0,T]}$, where $T > 0$ is a fixed time of maturity of the option.

We consider the discounted fractional Black–Scholes pricing model where the "riskless" investment, or the bond, is taken as the numéraire and the risky asset $S = (S_t)_{t\in[0,T]}$ is given by the dynamics

$$\frac{\mathrm{d}S_t}{S_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \tag{2.1}$$

where B is the fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$. Recall that, qualitatively, the fractional Brownian motion is the (up to a multiplicative constant) unique Gaussian process with stationary increments and self-similarity index H. Quantitatively, the fractional Brownian motion is defined by its covariance function

$$r(t, s) = \frac{1}{2}\left[t^{2H} + s^{2H} - |t - s|^{2H}\right].$$

Since the fractional Brownian motion with index $H \in (\frac{1}{2}, 1)$ has zero quadratic variation, the classical change-of-variables rule applies. Consequently, the pathwise solution to the stochastic differential equation (2.1) is

$$S_t = S_0 \mathrm{e}^{\mu t + \sigma B_t}. \tag{2.2}$$

Also, it follows from the classical change-of-variables rule that

$$f(S_T) = f(S_0) + \int_0^T f'(S_t)\,\mathrm{d}S_t, \tag{2.3}$$

where f is a convex or concave function and f' is its left-derivative. We refer to Azmoodeh et al. (2009) for details. The economic interpretation of (2.3) is that under continuous trading and no transaction costs, the replication price of a European vanilla option $f(S_T)$ is $f(S_0)$ and the replicating strategy is given by $\pi_t = f'(S_t)$, where π_t denotes the number of the shares of the risky asset S held by the investor at time t. Furthermore, we note that the value V^π of the hedging strategy $\pi = f'(S_\cdot)$ at time t is

$$\begin{aligned} V_t^\pi &= V_0^\pi + \int_0^t \pi_u\,\mathrm{d}S_u \\ &= f(S_0) + \int_0^t f'(S_u)\,\mathrm{d}S_u \\ &= f(S_t). \end{aligned}$$

Indeed, the first equality is simply the self-financing condition and the rest follows immediately from (2.3). Note that this is very different from the value in the classical Black–Scholes model, where the value is determined by the Black–Scholes partial differential equation, which in turn comes to the Itô's change-of-variables rule.

We assume that the trading only takes place at fixed preset time points $0 = t_0 < t_1 < \cdots < t_N = T$. We denote by π^N the discrete trading strategy

$$\pi_t^N = \pi_0^N \mathbf{1}_{\{0\}}(t) + \sum_{i=1}^N \pi_{t_{i-1}}^N \mathbf{1}_{(t_{i-1},t_i]}(t).$$

The value of the strategy π^N is given by

$$V_t^{\pi^N,k} = V_0^{\pi^N,k} + \int_0^t \pi_u^N\,\mathrm{d}S_u - \int_0^t kS_u|\mathrm{d}\pi_u^N|, \tag{2.4}$$

where $k \in [0, 1)$ is the proportional transaction cost.

Under transaction costs perfect hedging is not possible. In this case, it is natural to try to hedge on average in the sense of the following definition:

Definition 2.1 (*Conditional-Mean Hedge*). Let $f(S_T)$ be a European vanilla type option with convex or concave payoff function f. Let π be its Markovian replicating strategy: $\pi_t = f'(S_t)$. We call the discrete-time strategy π^N a *conditional-mean hedge*, if for all trading times t_i,

$$\mathbb{E}\left[V_{t_{i+1}}^{\pi^N,k} \mid \mathcal{F}_{t_i}\right] = \mathbb{E}\left[V_{t_{i+1}}^{\pi} \mid \mathcal{F}_{t_i}\right]. \tag{2.5}$$

Here $\mathcal{F}_{t_i}$ is the information generated by the asset price process S up to time t_i.

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Remark 2.1 (*Conditional-Mean Hedge as Tracking Condition*). Criterion (2.5) is actually a tracking requirement. We do not only require that the conditional means agree on the last trading time before the maturity, but also on all trading times. In this sense the criterion has an "American" flavor in it. From a purely "European" hedging point of view, one can simply remove all but the first and the last trading times.

Remark 2.2 (*Arbitrage and Uniqueness of Conditional-Mean Hedge*). The conditional-mean hedging strategy π^N depends on the continuous-time hedging strategy π. Since there is strong arbitrage in the fractional Black–Scholes model (zero can be perfectly replicated with negative initial wealth), the replicating strategy π is not unique. However, the strong arbitrage strategies are very complicated. Indeed, it follows directly from the change-of-variables formula that in the class of Markovian strategies $\pi_t = g(t, S_t)$, the choice $\pi_t = f'(S_t)$ is the unique replicating strategy for the claim $f(S_T)$.

We stress that the expectation in (2.5) is with respect to the true probability measure; not under any equivalent martingale measure. Indeed, equivalent martingale measures do not exist in the fractional Black–Scholes model.

To find the solution to (2.5) one must be able to calculate the conditional expectations involved. This can be done by using Sottinen and Viitasaari (2017, Theorem 3.1), a version of which we state below as Lemma 2.1 for the readers' convenience.

Lemma 2.1 (*Conditional Fractional Brownian Motion*). *The fractional Brownian motion B with index $H \in (\frac{1}{2}, 1)$ conditioned on its own past $\mathcal{F}_u^B$ is the Gaussian process $B(u) = B|\mathcal{F}_u^B$ with $\mathcal{F}_u^B$-measurable mean*

$$\hat{B}_t(u) = B_u - \int_0^u \Psi(t, s|u)\,\mathrm{d}B_s,$$

where

$$\Psi(t, s|u) = -\frac{\sin(\pi(H-\frac{1}{2}))}{\pi} s^{\frac{1}{2}-H}(u-s)^{\frac{1}{2}-H} \int_u^t \frac{z^{H-\frac{1}{2}}(z-u)^{H-\frac{1}{2}}}{z-s}\,\mathrm{d}z,$$

and deterministic covariance function

$$\hat{r}(t, s|u) = r(t, s) - \int_0^u k(t, v)k(s, v)\mathrm{d}v,$$

where

$$k(t, s) = \left(H - \frac{1}{2}\right)\sqrt{\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H-\frac{1}{2}\right)\Gamma\left(2-2H\right)}}\; s^{\frac{1}{2}-H} \int_s^t z^{H-\frac{1}{2}}(z-s)^{H-\frac{2}{3}}\,\mathrm{d}z;$$

Γ is the Euler's gamma function.

Remark 2.3 (*Conditional Asset Process*). By (2.2) we have the equality of filtrations: $\mathcal{F}_t^B = \mathcal{F}_t^S = \mathcal{F}_t$, for $t \in [0, T]$. Consequently, the conditional process $S(u) = S|\mathcal{F}_u$ is, informally, given by

$$\begin{aligned} S_t(u) &= S_0 \mathrm{e}^{\mu t + \sigma B_t(u)} \\ &= S_u \mathrm{e}^{\mu(t-u) + \sigma(B_t(u) - B_u)}. \end{aligned}$$

More formally, this means, in particular, that for $t > u$,

$$\begin{aligned} \mathbb{E}\left[f(S_t)\,\middle|\,\mathcal{F}_u^S\right] &= \mathbb{E}\left[f\left(S_0\mathrm{e}^{\mu t+\sigma B_t}\right)\,\middle|\,\mathcal{F}_u^B\right] \\ &= \int_{-\infty}^{\infty} f\left(S_0 \mathrm{e}^{\mu t + \sigma \hat{B}_t(u) + \sigma\sqrt{\hat{r}(t|u)}z}\right)\phi(z)\mathrm{d}z \\ &= \int_{-\infty}^{\infty} f\left(S_u \mathrm{e}^{\mu(t-u) + \sigma\left(\hat{B}_t(u) - B_u\right) + \sigma\sqrt{\hat{r}(t|u)}z}\right)\phi(z)\mathrm{d}z, \end{aligned}$$

where we have denoted

$$\hat{r}(t|u) = \hat{r}(t, t|u),$$

and ϕ is the standard normal density function.

3. Conditional-Mean hedging strategies

Denote

$$\Delta\hat{B}_{t_{i+1}}(t_i) = \hat{B}_{t_{i+1}}(t_i) - B_{t_i}.$$

In Theorem 3.1 we will calculate the conditional-mean hedging strategy in terms of the following conditional gains:

$$
\begin{aligned}
\Delta\hat{S}_{t_{i+1}}(t_i) &= \hat{S}_{t_{i+1}}(t_i) - S_{t_i} \\
&= \mathbb{E}\left[S_{t_{i+1}}|\mathcal{F}_{t_i}\right] - S_{t_i}, \\
\Delta\hat{V}^{\pi}_{t_{i+1}}(t_i) &= \hat{V}^{\pi}_{t_{i+1}}(t_i) - V^{\pi}_{t_i} \\
&= \mathbb{E}\left[V^{\pi}_{t_{i+1}}|\mathcal{F}_{t_i}\right] - V^{\pi}_{t_i}, \\
\Delta\hat{V}^{\pi^N,k}_{t_{i+1}}(t_i) &= \hat{V}^{\pi^N,k}_{t_{i+1}}(t_i) - V^{\pi}_{t_i} \\
&= \mathbb{E}\left[V^{\pi^N,k}_{t_{i+1}}|\mathcal{F}_{t_i}\right] - V^{\pi^N,k}_{t_i}.
\end{aligned}
$$

Lemma 3.1 states that all these conditional gains can be calculated explicitly by using the prediction law of the fractional Brownian motion.

Lemma 3.1 (*Conditional Gains*)**.**

$$
\begin{aligned}
\Delta\hat{S}_{t_{i+1}}(t_i) &= S_{t_i}\left(\int_{-\infty}^{\infty} e^{\mu\Delta t_{i+1}+\sigma\Delta\hat{B}_{t_{i+1}}(t_i)+\sigma\sqrt{\hat{r}(t_{i+1}|t_i)}z}\phi(z)\,dz - 1\right), \\
\Delta\hat{V}^{\pi}_{t_{i+1}}(t_i) &= \int_{-\infty}^{\infty} f\left(S_{t_i}e^{\mu\Delta t_{i+1}+\sigma\Delta\hat{B}_{t_{i+1}}(t_i)+\sigma\sqrt{\hat{r}(t_{i+1}|t_i)}z}\right)\phi(z)\,dz - f(S_{t_i}), \\
\Delta\hat{V}^{\pi^N,k}_{t_{i+1}}(t_i) &= \pi^N_{t_i}\Delta\hat{S}_{t_{i+1}}(t_i) - kS_{t_i}|\Delta\pi^N_{t_i}|.
\end{aligned}
$$

Proof. Let $g:\mathbb{R}\to\mathbb{R}$ be such that $\mathbb{E}\left[\left|g(B_{t_{i+1}})\right|\right] < \infty$. Then, by Lemma 2.1,

$$
\mathbb{E}\left[g(B_{t_{i+1}})\,|\,\mathcal{F}_{t_i}\right] = \int_{-\infty}^{\infty} g\left(\hat{B}_{t_{i+1}}(t_i) + \sqrt{\hat{r}(t_{i+1}|t_i)}z\right)\phi(z)\,dz.
$$

Consider $\Delta\hat{S}_{t_{i+1}}(t_i)$. By choosing

$$
g(x) = S_0e^{\mu t+\sigma x},
$$

we obtain

$$
\begin{aligned}
\hat{S}_{t_{i+1}}(t_i) &= \mathbb{E}\left[S_{t_{i+1}}\,\middle|\,\mathcal{F}_{t_i}\right] \\
&= \mathbb{E}\left[g\left(B_{t_{i+1}}\right)\,\middle|\,\mathcal{F}_{t_i}\right] \\
&= \int_{-\infty}^{\infty} S_0e^{\mu t_{i+1}+\sigma\left(\hat{B}_{t_{i+1}}(t_i)+\sqrt{\hat{r}(t_{i+1}|t_i)}z\right)}\phi(z)\,dz.
\end{aligned}
$$

The formula for $\Delta\hat{S}_{t_{i+1}}(t_i)$ follows from this.

Consider then $\Delta V^{\pi}_{t_{i+1}}(t_i)$. By choosing

$$
g(x) = f\left(S_0e^{\mu t+\sigma x}\right)
$$

we obtain

$$
\begin{aligned}
\hat{V}^{\pi}_{t_{i+1}}(t_i) &= \mathbb{E}\left[V^{\pi}_{t_{i+1}}\,|\,\mathcal{F}_{t_i}\right] \\
&= \mathbb{E}\left[f(S_{t_{i+1}})\,|\,\mathcal{F}_{t_i}\right] \\
&= \mathbb{E}\left[g(B_{t_{i+1}})\,|\,\mathcal{F}_{t_i}\right] \\
&= \int_{-\infty}^{\infty} g\left(\hat{B}_{t_{i+1}}(t_i) + \sqrt{\hat{r}(t_{i+1}|t_i)}z\right)\phi(z)\,dz \\
&= \int_{-\infty}^{\infty} f\left(S_0e^{\mu t_{i+1}+\sigma\left(\hat{B}_{t_{i+1}}(t_i)+\sqrt{\hat{r}(t_{i+1}|t_i)}z\right)}\right)\phi(z)\,dz.
\end{aligned}
$$

The formula for $\Delta\hat{V}^{\pi}_{t_{i+1}}(t_i)$ follows from this.

Finally, we calculate

$$\begin{aligned}\hat{V}^{\pi^N,k}_{t_{i+1}}(t_i) &= \mathbb{E}\left[V^{\pi^N,k}_{t_{i+1}} \,|\, \mathcal{F}_{t_i}\right] \\ &= V^{\pi^N,k}_{t_i} + \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \pi^N_u \,\mathrm{d}S_u - \int_{t_i}^{t_{i+1}} kS_u|\mathrm{d}\pi^N_u| \,\Big|\, \mathcal{F}_{t_i}\right] \\ &= V^{\pi^N,k}_{t_i} + \pi^N_{t_i}\left(\mathbb{E}\left[S_{t_{i+1}}|\mathcal{F}_{t_i}\right] - S_{t_i}\right) - kS_{t_i}|\Delta\pi^N_{t_i}| \\ &= V^{\pi^N,k}_{t_i} + \pi^N_{t_i}\Delta\hat{S}_{t_{i+1}}(t_i) - kS_{t_i}|\Delta\pi^N_{t_i}|.\end{aligned}$$

The formula for $\Delta\hat{V}^{\pi^N,k}_{t_{i+1}}(t_i)$ follows from this. □

Now we are ready to state and prove our main result. We note that, in principle, our result is general: it is true in any pricing model where the option $f(S_T)$ can be replicated. In practice, our result is specific to the fractional Black–Scholes model via Lemma 3.1.

Theorem 3.1 (*Conditional-Mean Hedging Strategy*). *The conditional mean hedge of the European vanilla type option with convex or concave positive payoff function f with proportional transaction costs k is given by the recursive equation*

$$\pi^N_{t_i} = \frac{\Delta\hat{V}^{\pi}_{t_{i+1}}(t_i) + (V^{\pi}_{t_i} - V^{\pi^N,k}_{t_i}) + kS_{t_i}|\Delta\pi^N_{t_i}|}{\Delta\hat{S}_{t_{i+1}}(t_i)}, \tag{3.1}$$

where $V^{\pi^N,k}_{t_i}$ is determined by (2.4).

Proof. Let us first consider the left hand side of (2.5). We have

$$\begin{aligned}\mathbb{E}\left[V^{\pi^N,k}_{t_{i+1}} \,|\, \mathcal{F}_{t_i}\right] &= \mathbb{E}\left[V^{\pi^N,k}_{t_i} + \int_{t_i}^{t_{i+1}} \pi^N_u \,\mathrm{d}S_u - k\int_{t_i}^{t_{i+1}} S_u|\mathrm{d}\pi^N_u| \,\Big|\, \mathcal{F}_{t_i}\right] \\ &= V^{\pi^N,k}_{t_i} + \pi^N_{t_i}\mathbb{E}\left[S_{t_{i+1}}(t_i) - S_{t_i} \,\big|\, \mathcal{F}_{t_i}\right] - kS_{t_i}|\Delta\pi^N t_i| \\ &= V^{\pi^N,k}_{t_i} + \pi^N_{t_i}\Delta\hat{S}_{t_{i+1}}(t_i) - kS_{t_i}|\Delta\pi^N_{t_i}|.\end{aligned}$$

For the right-hand-side of (2.5), we simply write

$$\mathbb{E}\left[V^{\pi}_{t_{i+1}} \,|\, \mathcal{F}_{t_i}\right] = \Delta\hat{V}^{\pi}_{t_{i+1}}(t_i) + V^{\pi}_{t_i}.$$

Equating the sides we obtain (3.1) after a little bit of simple algebra. □

Remark 3.1. Taking the expected gains $\Delta\hat{S}_{t_{i+1}}(t_i)$ to be the numéraire, one recognizes three parts in the hedging formula (3.1). First, one invests on the expected gains in the time-value of the option. This "conditional-mean Delta-hedging" is intuitively the most obvious part. Indeed, a naïve approach to conditional-mean hedging would only give this part, forgetting to correct for the tracking-errors already made, which is the second part in (3.1). The third part in (3.1) is obviously due to the transaction costs.

Remark 3.2. Eq. (3.1) for the strategy of the conditional-mean hedging is recursive: in addition to the filtration $\mathcal{F}_{t_i}$, the position $\pi^N_{t_{i-1}}$ is needed to determine the position $\pi^N_{t_i}$. Consequently, to determine the conditional-mean hedging strategy by using (3.1), the initial position π^N_0 must be fixed. The initial position is, however, not uniquely defined. Indeed, let β^N_0 be the position in the riskless asset. Then the conditional-mean criterion (2.5) only requires that

$$\beta^N_0 + \pi^N_0\mathbb{E}[S_{t_1}] - kS_0|\pi^N_0| = \mathbb{E}[f(S_{t_1})].$$

There are of course infinite number of pairs (β^N_0, π^N_0) solving this equation. A natural way to fix the initial position (β^N_0, π^N_0) for the investor interested in conditional-mean hedging would be the one with minimal cost. If short-selling is allowed, the investor is then faced with the minimization problem

$$\min_{\pi^N_0\in\mathbb{R}} v(\pi^N_0),$$

where the initial wealth v is the piecewise linear function

$$\begin{aligned}v(\pi^N_0) &= \beta^N_0 + \pi^N_0 S_0 \\ &= \begin{cases}\mathbb{E}[f(S_{t_1})] - \left(\Delta\hat{S}_{t_1}(0) - kS_0\right)\pi^N_0, & \text{if} \quad \pi^N_0 \geq 0, \\ \mathbb{E}[f(S_{t_1})] - \left(\Delta\hat{S}_{t_1}(0) + kS_0\right)\pi^N_0, & \text{if} \quad \pi^N_0 < 0.\end{cases}\end{aligned}$$

Clearly, the minimal solution π_0^N is independent of $\mathbb{E}[f(S_{t_1})]$, and, consequently, of the option to be replicated. Also, the minimization problem is bounded if and only if

$$k \geq \left| \frac{\Delta \hat{S}_{t_1}(0)}{S_0} \right|,$$

i.e. the proportional transaction costs are bigger than the expected return on $[0, t_1]$ of the stock. In this case, the minimal cost conditional mean-hedging strategy starts by putting all the wealth in the riskless asset.

We end this note by applying Theorem 3.1 to European call options.

Corollary 3.1 (*European Call Option*)**.** *Denote*

$$\hat{d}^+_{t_{i+1}}(t_i) = \frac{\ln \frac{S_{t_i}}{K} - \mu \Delta t_{i+1} - \sigma \Delta \hat{B}_{t_{i+1}}(t_i)}{\sigma \sqrt{\hat{r}(t_{i+1}|t_i)}} - \sigma \sqrt{\hat{r}(t_{i+1}|t_i)},$$

$$\hat{d}^-_{t_{i+1}}(t_i) = \frac{\ln \frac{S_{t_i}}{K} - \mu \Delta t_{i+1} - \sigma \Delta \hat{B}_{t_{i+1}}(t_i)}{\sigma \sqrt{\hat{r}(t_{i+1}|t_i)}},$$

$$\hat{X}_{t_{i+1}}(t_i) = \mu \Delta t_{i+1} + \sigma \Delta \hat{B}_{t_{i+1}}(t_i) + \frac{1}{2}\sigma^2 \hat{r}(t_{i+1}|t_i),$$

and let Φ be the cumulative distribution function of the standard normal law. Then the conditional-mean hedging strategy for the European call option with strike-price K is given by

$$\pi^N_{t_i} = \frac{S_{t_i} e^{\hat{X}_{t_{i+1}}(t_i)} \Phi(\hat{d}^+_{t_{i+1}}(t_i)) - K\Phi(\hat{d}^-_{t_{i+1}}(t_i)) - V^{\pi^N,k}_{t_i} + kS_{t_i}|\Delta \pi^N_{t_i}|}{\Delta \hat{S}_{t_{i+1}}(t_i)}. \tag{3.2}$$

Proof. First we note that

$$\hat{V}^{\text{call}}_{t_{i+1}}(t_i) = \int_{-\infty}^{\infty} \left(S_{t_i} e^{\mu \Delta t_{i+1} + \sigma \Delta \hat{B}_{t_{i+1}}(t_i) + \sigma \sqrt{\hat{r}(t_{i+1}|t_i)} z} - K \right)^+ \phi(z) \mathrm{d}z$$
$$= S_{t_i} e^{\hat{X}_{t_{i+1}}(t_i)} \Phi \left(\hat{d}^+_{t_{i+1}}(t_i) \right) - K\Phi \left(\hat{d}^-_{t_{i+1}}(t_i) \right).$$

Next we note that

$$V^{\text{call}}_{t_i} = (S_{t_i} - K)^+.$$

So,

$$\Delta \hat{V}^{\text{call}}_{t_{i+1}}(t_i) = S_{t_i} e^{\hat{X}_{t_{i+1}}(t_i)} \Phi(\hat{d}^+_{t_{i+1}}(t_i)) - K\Phi(\hat{d}^-_{t_{i+1}}(t_i)) - (S_{t_i} - K)^+,$$

and (3.2) follows from this. □

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Subdiffusive fractional Black–Scholes model for pricing currency options under transaction costs

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Abstract: A new framework for pricing European currency option is developed in the case where the spot exchange rate follows a subdiffusive fractional Black–Scholes. An analytic formula for pricing European currency call option is proposed by a mean self-financing delta-hedging argument in a discrete time setting. The minimal price of a currency option under transaction costs is obtained as time-step $\Delta t = \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{-1} \left(\frac{2}{\pi}\right)^{\frac{1}{2H}} \left(\frac{k}{\sigma}\right)^{\frac{1}{H}}$, which can be used as the actual price of an option. In addition, we also show that time-step and long-range dependence have a significant impact on option pricing.

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Keywords: subdiffusion process; currency option; transaction costs; inverse subordinator process

MR Subject classifications: 91G20; 91G80; 60G22

1. Introduction

The standard European currency option valuation model has been presented by Garman and Kohlhagen ($G-K$) (Garman & Kohlhagen, 1983). However, some papers have provided evidence of the mispricing for currency options by the $G-K$ model. The most important reason why this

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PUBLIC INTEREST STATEMENT

Subdiffusion refers to a well-known and established phenomenon in statistical physics. One description of subdiffusion is related to subordination, where the standard diffusion process is time-changed by the so-called inverse subordinator. According to the features of the subdiffusion process and the fractional Brownian motion, we propose the new model for pricing European currency options by using the fractional Brownian motion, subdiffusive strategy, and scaling time in discrete time setting, to get behavior from financial markets. Motivated by this objective, we illustrate how to price a currency options in discrete time setting for both cases: with and without transaction costs by applying subdiffusive fractional Brownian motion model. By considering the empirical data, we will demonstrate that the proposed model is further flexible in comparison with the previous models and it obtains suitable benchmark for pricing currency options. Additionally, impact of the parameters on our pricing formula is investigated.

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model may not be entirely satisfactory could be that currencies are different from stocks in important respects and the geometric Brownian motion cannot capture the behavior of currency return (Ekvall, Jennergren, & Näslund, 1997). Since then, many methodologies for currency option pricing have been proposed by using modifications of the $G-K$ model (Garman & Kohlhagen, 1983; Ho, Stapleton, & Subrahmanyam, 1995).

All this research above assumes that the logarithmic returns of the exchange rate are independent identically distributed normal random variables. However, in general, the assumptions of the Gaussianity and mutual independence of underlying asset log returns would not hold. Moreover, the empirical research has also shown that the distributions of the logarithmic returns in the financial market usually exhibit excess kurtosis with more probability mass near the origin and in the tails and less in the flanks than would occur for normally distributed data (Dai & Singleton, 2000). That is to say the features of financial return series are non-normality, non-independence, and nonlinearity. To capture these non-normal behaviors, many researchers have considered other distributions with fat tails such as the Pareto-stable distribution and the Generalized Hyperbolic Distribution. Moreover, self-similarity and long-range dependence have become important concepts in analyzing the financial time series.

There is strong evidence that the stock return has little or no autocorrelation. As fractional Brownian motion (FBM) has two important properties called self-similarity and long-range dependence, it has the ability to capture the typical tail behavior of stock prices or indexes (Borovkov, Mishura, Novikov, & Zhitlukhin, 2018; Shokrollahi & Sottinen, 2017).

The fractional Black–Scholes (FBS) model is an extension of the Black–Scholes (BS) model, which displays the long-range dependence observed in empirical data. This model is based on replacing the classic Brownian motion by the fractional Brownian motion (FBM) in the Black–Scholes model. That is

$$\hat{V}(t) = \hat{V}_0 \, exp\Big\{\mu t + \sigma \hat{B}_H(t)\Big\}, \qquad \hat{V}_0 > 0, \tag{1.1}$$

where $\mu,$ and σ are fixed, and $\hat{B}_H(t)$ is a FBM with Hurst parameter $H \in [\frac{1}{2}, 1)$. It has been shown that the FBS model admits arbitrage in a complete and frictionless market (Cheridito, 2003; Shokrollahi & Kılıçman, 2014; Sottinen & Valkeila, 2003; Wang, Zhu, Tang, & Yan, 2010; Xiao, Zhang, Zhang, & Wang, 2010). Wang (2010) resolved this contradiction by giving up the arbitrage argument and examining option replication in the presence of proportional transaction costs in discrete time setting (Mastinšek, 2006).

Magdziarz (2009a) applied the subdiffusive mechanism of trapping events to describe properly financial data exhibiting periods of constant values and introduced the subdiffusive geometric Brownian motion

$$V_\alpha(t) = V(T_\alpha(t)), \tag{1.2}$$

as the model of asset prices exhibiting subdiffusive dynamics, where $V_\alpha(t)$ is a subordinated process (for the notion of subordinated processes please refer to Refs. Janicki and Weron (1993, 1995), Kumar, Wyłomańska, Połoczański, and Sundar (2017), Piryatinska, Saichev, and Woyczynski (2005), in which the parent process $V(\tau)$ is a geometric Brownian motion and $T_\alpha(t)$ is the inverse α-stable subordinator defined as follows:

$$T_\alpha(t) = inf\{\tau > 0 : Q_\alpha(\tau) > t\}, \quad 0 < \alpha < 1. \tag{1.3}$$

Here, $Q_\alpha(t)$ is a strictly increasing α-stable subordinator with Laplace transform: $E\left(e^{-\eta Q_\alpha(\tau)}\right) = e^{-\tau\eta^\alpha}$, $0 < \alpha < 1$, where E denotes the mathematical expectation.

Magdziarz (2009a) demonstrated that the considered model is free-arbitrage but is incomplete and proposed the corresponding subdiffusive BS formula for the fair prices of European options.

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Subdiffusion is a well-known and established phenomenon in statistical physics. The usual model of subdiffusion in physics is developed in terms of *FFPE* (fractional Fokker-Planck equations). This equation was first derived from the continuous-time random walk scheme with heavy-tailed waiting times (Metzler & Klafter, 2000). It provides a useful way for the description of transport dynamics in complex systems (Magdziarz, Weron, & Weron, 2007). Another description of subdiffusion is in terms of subordination, where the standard diffusion process is time-changed by the so-called inverse subordinator (Gu, Liang, & Zhang, 2012; Guo, 2017; Janczura, Orzeł, & Wyłomańska, 2011; Magdziarz, 2009b, Magdziarz et al., 2007; Scalas, Gorenflo, & Mainardi, 2000, Shokrollahi & Kılıçman, 2014; Yang, 2017).

The objective of this paper is to study the European call currency option by a mean self financing delta hedging argument. The main contribution of this paper is to derive an analytical formula for European call currency option without using the arbitrage argument in discrete time setting when the exchange rate follows a subdiffusive *FBS*

$$S_t = \hat{V}(T_\alpha(t)) = S_0 exp\left\{\mu T_\alpha(t) + \sigma \hat{B}_H(T_\alpha(t))\right\}, \tag{1.4}$$

$$S_0 = \hat{V}(0) > 0.$$

We then apply the result to value European put currency option. We also provide representative numerical results.

Making the change of variable, $B_H(t) = \frac{\mu + r_f - r_d}{\sigma} t + \hat{B}_H(t)$, under the risk-neutral measure, we have that

$$S_t = \hat{V}(R_\beta(t)) = S_0 exp\{(r_d - r_f)(T_\alpha(t)) + \sigma B_H((T_\alpha(t))\}, \tag{1.5}$$

$$S_0 = \hat{V}(0) > 0.$$

This formula is similar to the Black–Scholes option pricing formula, but with the volatility being different.

We denote the subordinated process $W_{\alpha,H}(t) = B_H(T_\alpha(t))$, here the parent process $B_H(\tau)$ is a *FBM* and $T_\alpha(t)$ is assumed to be independent of $B_H(\tau)$. The process $W_{\alpha,H}(t)$ is called a subdiffusion process. Particularly, when $H = \frac{1}{2}$, it is a subdiffusion process presented in Karipova and Magdziarz (2017), Kumar et al. (2017), and Magdziarz (2010).

Figure 1 shows typically the differences and relationships between the sample paths of the spot exchange rate in the *FBS* model and the subdiffusive *FBS* model.

The rest of the paper proceeds as follows: In Section 2, we provide an analytic pricing formula for the European currency option in the subdiffusive *FBS* environment and some Greeks of our pricing model are also obtained. Section 3 is devoted to analyze the impact of scaling and long-range dependence on currency option pricing. Moreover, the comparison of our subdiffusive *FBS* model and traditional models is undertaken in this section. Finally, Section 4 draws the concluding remarks. The proof of Theorems are provided in Appendix.

2. Pricing model for the European call currency option

In this section, we derive a pricing formula for the European call currency option of the subdiffusive *FBS* model under the following assumptions:

(i) We consider two possible investments: (1) a stock whose price satisfies the equation:

$$S_t = S_0 exp\{(r_d - r_f)T_\alpha(t) + \sigma W_{\alpha,H}(t)\}, \quad S_0 > 0, \tag{2.1}$$

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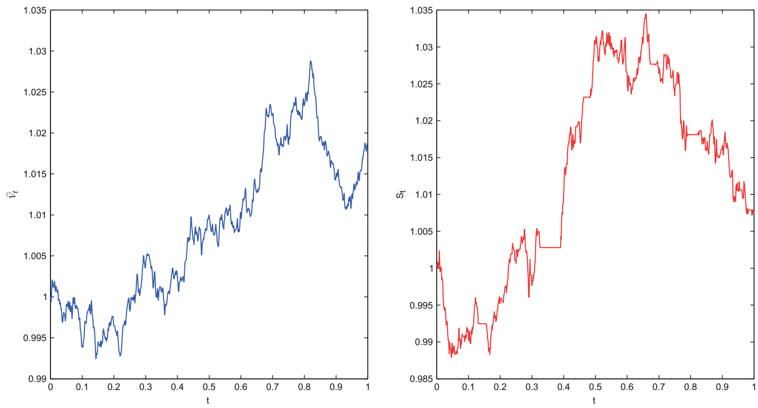

Figure 1. Comparison of the spot exchange rate' sample paths in the *FBS* model (left) and the subdiffusive *FBS* model (right) for $r_d = 0.03, r_f = 0.02, \alpha = 0.9, H = 0.8, \sigma = 0.1, S_0 = 1$.

where $\alpha \in (\frac{1}{2}, 1)$, $H \in [\frac{1}{2}, 1)$, $\alpha + \alpha H > 1$, and r_d, and r_f are the domestic and the foreign interest rates, respectively. (2) A money market account:

$$dF_t = r_d F_t dt, \tag{2.2}$$

where r_d shows the domestic interest rate.

(ii) The stock pays no dividends or other distributions, and all securities are perfectly divisible. There are no penalties to short selling. It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate. These are the same valuation policy as in the *BS* model.

(iii) There are transaction costs that are proportional to the value of the transaction in the underlying stock. Let k denote the round trip transaction cost per unit dollar of transaction. Suppose U shares of the underlying stock are bought $(U > 0)$ or sold $(U < 0)$ at the price S_t, then the transaction cost is given by $\frac{k}{2}|U|S_t$ in either buying or selling. Moreover, trading takes place only at discrete intervals.

(iv) The option value is replicated by a replicating portfolio Π with $U(t)$ units of stock and riskless bonds with value $F(t)$. The value of the option must equal the value of the replicating portfolio to reduce (but not to avoid) arbitrage opportunities and be consistent with economic equilibrium.

(v) The expected return for a hedged portfolio is equal to that from an option. The portfolio is revised every Δt and hedging takes place at equidistant time points with rebalancing intervals of (equal) length Δt, where Δt is a finite and fixed, small time-step.

Remark 2.1. From Guo and Yuan (2014), Magdziarz (2009c), we have $E(T_\alpha^m(t)) = \frac{t^{m\alpha} m!}{\Gamma(m\alpha+1)}$. Then, by using α-self-similar and non-decreasing sample paths of $T_\alpha(t)$, we can obtain that α-self-similar and non-decreasing sample paths of $T_\alpha(t)$,

$$\begin{aligned} E(\Delta T_\alpha(t)) &= E(T_\alpha(t + \Delta t) - T_\alpha(t)) \\ &= \tfrac{1}{\Gamma(1+\alpha)}[(t + \Delta t)^\alpha - t^\alpha] = \tfrac{t^{\alpha-1}}{\Gamma(\alpha)} \Delta t. \end{aligned} \tag{2.3}$$

and

$$E\left((\Delta B_H(T_\alpha(t))^2\right) = \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]^{2H} \Delta t^{2H}. \tag{2.4}$$

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Let $C = C(t, S_t)$ be the price of a European currency option at time t with a strike price K that matures at time T. Then, the pricing formula for currency call option is given by the following theorem.

Theorem 2.1. *$C = C(t, S_t)$ is the value of the European currency call option on the stock S_t satisfied (1.5) and the trading takes place discretely with rebalancing intervals of length Δt. Then, C satisfies the partial differential equation*

$$\frac{\partial C}{\partial t} + (r_d - r_f)S_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\hat{\sigma}^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} - r_d C = 0, \tag{2.5}$$

with boundary condition $C(T, S_T) = \max\{S_T - K, 0\}$. The value of the currency call option is

$$C(t, S_t) = S_t e^{-r_f(T-t)}\Phi(d_1) - Ke^{-r_d(T-t)}\Phi(d_2), \tag{2.6}$$

and the value of the put currency option is

$$P(t, S_t) = Ke^{-r_d(T-t)}\Phi(-d_2) - S_t e^{-r_f(T-t)}\Phi(-d_1), \tag{2.7}$$

where

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r_d - r_f)(T-t) + \frac{\hat{\sigma}^2}{2}(T-t)}{\hat{\sigma}\sqrt{T-t}},$$

$$d_2 = d_1 - \hat{\sigma}(t)\sqrt{T-t}, \tag{2.8}$$

$$\hat{\sigma}^2 = \sigma^2\left[\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H}\Delta t^{2H-1} + \sqrt{\frac{2}{\pi}\frac{k}{\sigma}}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{H}\Delta t^{H-1}\right], \tag{2.9}$$

where $\Phi(.)$ is the cumulative normal distribution function.

In what follows, the properties of the subdiffusive *FBS* model are discussed, such as Greeks, which summarize how option prices change with respect to underlying variables and are critically important to asset pricing and risk management. The model can be used to rebalance a portfolio to achieve the desired exposure to certain risk. More importantly, by knowing the Greeks, particular exposure can be hedged from adverse changes in the market by using appropriate amounts of other related financial instruments. In contrast to option prices that can be observed in the market, Greeks cannot be observed and must be calculated given a model assumption. The Greeks are typically computed using a partial differentiation of the price formula.

Theorem 2.2. *The Greeks can be written as follows:*

$$\Delta = \frac{\partial C}{\partial S_t} = e^{-r_f(T-t)}\Phi(d_1), \tag{2.10}$$

$$\nabla = \frac{\partial C}{\partial K} = -e^{-r_d(T-t)}\Phi(d_2), \tag{2.11}$$

$$\rho_{r_d} = \frac{\partial C}{\partial r_d} = K(T-t)e^{-r_d(T-t)}\Phi(d_2), \tag{2.12}$$

$$\rho_{r_f} = \frac{\partial C}{\partial r_f} = -S_t(T-t)e^{-r_f(T-t)}\Phi(d_1), \tag{2.13}$$

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$$\begin{aligned}\Theta = \frac{\partial C}{\partial t} &= S_t r_f e^{-r_f(T-t)}\Phi(d_1) - Kr_d e^{-r_d(T-t)}\Phi(d_2) \\ &+ S_t e^{-r_f(T-t)}\sigma^2(\alpha-1)\frac{t^{\alpha-2}}{\Gamma(\alpha)}\frac{H\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H-1}\Delta t^{2H-1}}{\widehat{\sigma}\sqrt{T-t}}(T-t)\Phi'(d_1) \\ &+ S_t e^{-r_f(T-t)}\sqrt{\frac{2}{\pi}}k\sigma(\beta-1)\frac{t^{\alpha-2}}{\Gamma(\alpha)}\frac{H\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{H-1}\Delta t^{H-1}}{2\widehat{\sigma}\sqrt{T-t}}(T-t)\Phi'(d_1) \\ &- S_t e^{-r_f(T-t)}\frac{\widehat{\sigma}}{2\sqrt{T-t}}\Phi'(d_1),\end{aligned} \tag{2.14}$$

$$\Gamma = \frac{\partial^2 C}{\partial S_t^2} = e^{-r_f(T-t)}\frac{\Phi'(d_1)}{S_t\hat{\sigma}\sqrt{T-t}}, \tag{2.15}$$

$$\vartheta_{\hat{\sigma}} = \frac{\partial C}{\partial \hat{\sigma}} = S_t e^{-r_f(T-t)}\sqrt{T-t}\Phi'(d_1). \tag{2.16}$$

Remark 2.2. The modified volatility without transaction costs $(k=0)$ is given by

$$\hat{\sigma}^2 = \sigma^2\left[\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H}\Delta t^{2H-1}\right], \tag{2.17}$$

specially if $\alpha \uparrow 1$,

$$\hat{\sigma}^2 = \sigma^2\Delta t^{2H-1}, \tag{2.18}$$

which is consistent with the result in Necula (2002).

Furthermore, from Equation (2.18), if $H \downarrow \frac{1}{2}$, then $\hat{\sigma}^2 = \sigma^2$, which is according to the results with the $G-K$ model (Garman & Kohlhagen, 1983).

Letting $\alpha \uparrow 1$, from Equation (2.9), we obtain

Remark 2.3. The modified volatility under transaction costs is given by

$$\hat{\sigma}^2 = \sigma^2\left[\Delta t^{2H-1} + \sqrt{\frac{2}{\pi}}\frac{k}{\sigma}\Delta t^{H-1}\right], \tag{2.19}$$

that is in line with the findings in Wang (2010).

3. Empirical studies

The objective of this section is to obtain the minimal price of an option with transaction costs and to show the impact of time scaling Δt, transaction costs k, and subordinator parameter α on the subdiffusive *FBS* model. Moreover, in the last part, we compute the currency option prices using our model and make comparisons with the results of the $G-K$ and *FBS* models.

As $\frac{k}{\sigma} < \sqrt{\frac{\pi}{2}}$ often holds (for example: $\sigma = 0.1, k = 0.01$), from Equation (2.9), we have

$$\begin{aligned}\frac{\hat{\sigma}^2}{\sigma^2} &= \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H}\Delta t^{2H-1} + \sqrt{\frac{2}{\pi}}\frac{k}{\sigma}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{H}\Delta t^{H-1} \\ &\geq 2\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{\frac{3}{2}H}\Delta t^{\frac{3}{2}H-1}\left(\frac{2}{\pi}\right)^{\frac{1}{4}}\left(\frac{k}{\sigma}\right)^{\frac{1}{2}},\end{aligned} \tag{3.1}$$

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where $H>\frac{1}{2}$. Then, the minimal volatility $\hat{\sigma}_{min}$ is $\sqrt{2}\sigma\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{\frac{1}{2}}\left(\frac{2}{\pi}\right)^{\frac{1}{2}-\frac{1}{4H}}\left(\frac{k}{\sigma}\right)^{1-\frac{1}{2H}}$ as $\Delta t=\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{-1}\left(\frac{2}{\pi}\right)^{\frac{1}{2H}}\left(\frac{k}{\sigma}\right)^{\frac{1}{H}}$. Thus, the minimal price of an option under transaction costs is represented as $C_{min}(t,S_t)$ with $\hat{\sigma}_{min}$ in Equation (2.8).

Moreover, the option rehedging time interval for traders to take is $\Delta t=\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{-1}\left(\frac{2}{\pi}\right)^{\frac{1}{2H}}\left(\frac{k}{\sigma}\right)^{\frac{1}{H}}$. The minimal price $C_{min}(t,S_t)$ can be used as the actual price of an option.

In particular, as $\Delta t<1, \alpha\in\left(\frac{1}{2},1\right)$ and $\frac{\partial C}{\partial\hat{\sigma}}=S_te^{-r_f(T-t)}\frac{\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d^2}{2}}>0$,

$$\begin{aligned}\frac{\partial\hat{\sigma}}{\partial H}&=\sigma\left[2\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H}\Delta t^{2H-1}+\sqrt{\frac{2}{\pi}\frac{k}{\sigma}}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{H}\Delta t^{H-1}\right]\left[\ln\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)+\ln\Delta t\right]\\&\quad\times 2\left[\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H}\Delta t^{2H-1}+\sqrt{\frac{2}{\pi}\frac{k}{\sigma}}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{H}\Delta t^{H-1}\right]^{-\frac{1}{2}}\\&=\left[2\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H}\Delta t^{2H-1}+\sqrt{\frac{2}{\pi}\frac{k}{\sigma}}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{H}\Delta t^{H-1}\right]\\&\quad\times\frac{\sigma^2\left[\ln\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)+\ln\Delta t\right]}{2\hat{\sigma}}<0,\end{aligned}\tag{3.2}$$

and $\frac{\partial C}{\partial H}=\frac{\partial C}{\partial\hat{\sigma}}\frac{\partial\hat{\sigma}}{\partial H}$, then we have

$$\frac{\partial C}{\partial H}<0 \quad as H\in[\frac{1}{2},1),\tag{3.3}$$

which displays that an increasing Hurst exponent comes along with a decrease of the option value (see Figure 2).

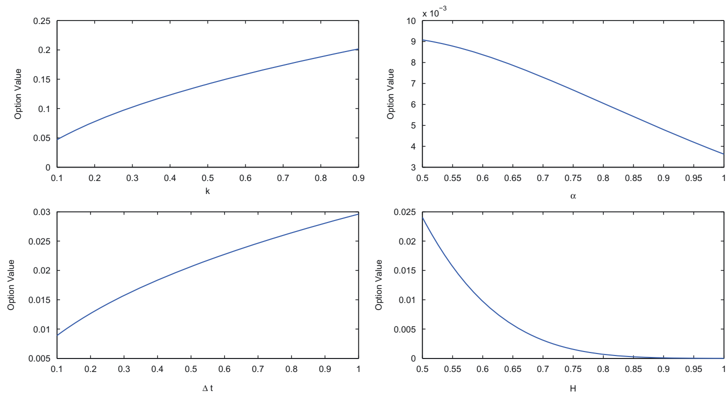
On the other hand, if $H\downarrow\frac{1}{2}$, then

$$\hat{\sigma}_{min}=\sqrt{2}\sigma\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{\frac{1}{2}}\left(\frac{2}{\pi}\right)^{\frac{1}{2}-\frac{1}{4H}}\left(\frac{k}{\sigma}\right)^{1-\frac{1}{2H}}\rightarrow\sigma\sqrt{2\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)}\tag{3.4}$$

and if $\alpha\uparrow 1$, then $\hat{\sigma}_{min}\rightarrow\sqrt{2}\sigma$ as $H\downarrow\frac{1}{2}$.

In addition, if $H\downarrow\frac{1}{2}$

Figure 2. Call currency option values.



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$$\Delta t = \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{-1}\left(\frac{2}{\pi}\right)^{\frac{1}{2H}}\left(\frac{k}{\sigma}\right)^{\frac{1}{H}} \to \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{-1}\left(\frac{2}{\pi}\right)\left(\frac{k}{\sigma}\right)^{2}, \tag{3.5}$$

and if $\alpha \uparrow 1$, then $\Delta t \to \left(\frac{2}{\pi}\right)\left(\frac{k}{\sigma}\right)^2$ as $H \downarrow \frac{1}{2}$.

Lux and Marchesi (1999) have shown that Hurst exponent $H = 0.51 \pm 0.004$ in some cases, so Equations (3.4) and (3.5) have a practical application in option pricing. For example: if $H \downarrow \frac{1}{2}, \alpha \uparrow 1, k = 2\%$ and $\sigma = 20\%$, then $\hat{\sigma}_{min} \to \frac{\sqrt{2}}{20}$, and $\Delta t \to \frac{0.02}{\pi}$; and if $H \uparrow \frac{1}{2}, \alpha \uparrow 1, k = 0.2\%$ and $\sigma = 20\%$, then $\hat{\sigma}_{min} \to \frac{\sqrt{2}}{20}$, and $\Delta t \to \frac{2}{\pi} \times 10^{-4}$.

In the following, we investigate the impact of scaling and long-range dependence on option pricing. It is well known that Mantegna and Stanley (1995) introduced the method of scaling invariance from the complex science into the economic systems for the first time. Since then, a lot of research for scaling laws in finance has begun. If $H = \frac{1}{2}$ and $k = 0$, from Equation (2.9), we know that $\hat{\sigma}^2 = \sigma^2\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)$ shows that fractal scaling Δt has not any impact on option pricing if a mean self-financing delta-hedging strategy is applied in a discrete time setting, while subordinator parameter β has remarkable impact on option pricing in this case. In particular, from Equations (3.4) and (3.5), we know that $\hat{\sigma}_{min} \to \sigma\sqrt{2\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)}$ as $H \approx \frac{1}{2}$ and $\Delta t \to \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{-1}\left(\frac{2}{\pi}\right)\left(\frac{k}{\sigma}\right)^2$, as $H \approx \frac{1}{2}$. Therefore, $C_{min}(t, S_t)$ is approximately scaling free with respect to the parameter k, if $H \approx \frac{1}{2}$, but is scaling dependent with respect to subordinator parameter α. However, $\Delta t \to \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{-1}\left(\frac{2}{\pi}\right)\left(\frac{k}{\sigma}\right)^2$, is scaling dependent with respect to parameters k and α, if $H \approx \frac{1}{2}$. On the other hand, if $H > \frac{1}{2}$ and $k = 0$, from Equation (2.17), we know that $\hat{\sigma}^2 = \sigma^2\left[\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H}\Delta t^{2H-1}\right]$, which displays that the fractal scaling Δt and sabordinator parameter α have a significant impact on option pricing. Furthermore, for $k \neq 0$, from Equation (2.8), we know that option pricing is scaling dependent in general.

Now, we present the values of currency call option using subdiffusive *FBS* model for different parameters. For the sake of simplicity, we will just consider the out-of-the-money case. Indeed, using the same method, one can also discuss the remaining cases: in-the-money and at-the-money. First, the prices of our subdiffusive *FBS* model are investigated for some Δt and prices for different exponent parameters. The prices of the call currency option versus its parameters $H, \Delta t, \alpha$ and k are revealed in Figure 2. The selected parameters are $S_t = 1.4$, $K = 1.5, \sigma = 0.1$, $r_d = 0.03$, $r_f = 0.02$, $T = 1$, $t = 0.1$, $\Delta t = 0.01$, $k = 0.01$, $H = 0.8$, $\alpha = 0.9$. Figure 2 indicates that the option price is an increasing function of k and Δt, while it is a decreasing function of H and α.

For a detailed analysis of our model, the prices calculated by the $G-K$, *FBS* and subdiffusive *FBS* models are compared for both out-of-the-money and in-the-money cases. The following parameters are chosen: $S_t = 1.2, \sigma = 0.5, r_d = 0.05, r_f = 0.01, t = 0.1, \Delta t = 0.01, k = 0.001$, and $H = 0.8$, along with time maturity $T \in [0.1, 2]$, strike price $K \in [0.8, 1.19]$ for the in-the-money case and $K \in [1.21, 1.4]$ for the out-of-the-money case. Figures 3 and 4 show the theoretical values difference by the $G-K$, *FBS*, and our subdiffusive *FBS* models for the in-the-money and out-of-the-money, respectively. As indicated in these figures, the values computed by our subdiffusive *FBS* model are better fitted to the $G-K$ values than the *FBS* model for both in-the-money and out-of-the money cases. Hence, when compared to these figures, our subdiffusive *FBS* model seems reasonable.

4. Conclusion

Without using the arbitrage argument, in this paper, we derive a European currency option pricing model with transaction costs to capture the behavior of the spot exchange rate price, where the

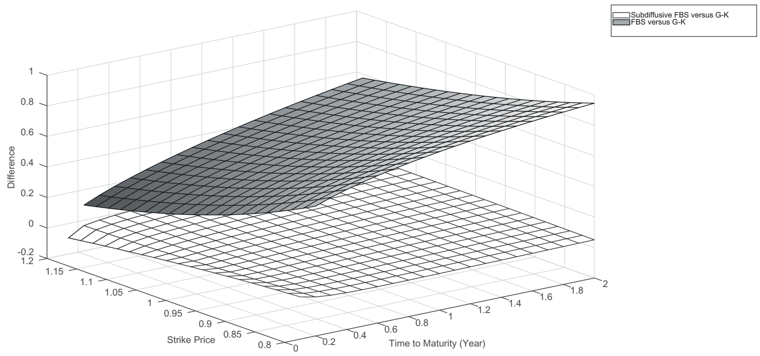

Figure 3. Relative difference between the $G-K$, FBS, and subdiffusive FBS models for the in-the-money case.

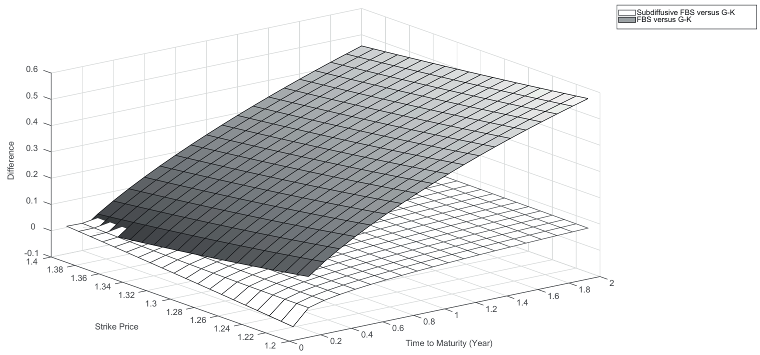

Figure 4. Relative difference between the $G-K$, FBS, and subdiffusive FBS models for the out-of-the-money case.

spot exchange rate follows a subdiffusive *FBS* with transaction costs. In discrete time case, we show that the time scaling Δt and the Hurst exponent H play an important role in option pricing with or without transaction costs and option pricing is scaling dependent. In particular, the minimal price of an option under transaction costs is obtained.

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Appendix

Proof of Theorem 2.1. The movement of S_t on time interval $[t, t+\Delta t)$ of length Δt is

$$\begin{aligned}\Delta S_t = S_{t+\Delta t} - S_t &= S_t(e^{(r_d-r_f)\Delta T_\alpha(t)+\sigma\Delta W_{\alpha,H}(t)} - 1)\\ &= S_t((r_d-r_f)\Delta T_\alpha(t) + \sigma\Delta W_{\alpha,H}(t)\\ &\quad+\frac{1}{2}((r_d-r_f)\Delta T_\alpha(t) + \sigma\Delta W_{\alpha,H}(t))^2)\\ &\quad+\frac{1}{6}S_t e^{[\theta((r_d-r_f)\Delta T_\alpha(t)+\sigma\Delta W_{\alpha,H}(t))]}\\ &\quad\times((r_d-r_f)\Delta T_\alpha(t) + \sigma\Delta W_{\alpha,H}(t))^3,\end{aligned} \tag{4.1}$$

here $\theta = \theta(t, \Delta t) \in (0, 1)$ is a random variable corresponding to process S_t.

Based on Lemmas 2.1 and 2.2 of Gu et al. (2012), we can get

$$\begin{aligned}((r_d-r_f)\Delta T_\alpha(t) + \sigma\Delta W_{\alpha,H}(t))^2 &= (o(\Delta t^{\alpha-\varepsilon}) + o(\Delta t^{\alpha H-\varepsilon}))^2\\ &= o(\Delta t^{\alpha+\alpha H-2\varepsilon}) + o(\Delta t^{2\alpha H-2\varepsilon}),\end{aligned} \tag{4.2}$$

$$\begin{aligned}&e^{[\theta((r_d-r_f)\Delta T_\alpha(t)+\sigma\Delta W_{\alpha,H}(t))]}((r_d-r_f)\Delta T_\alpha(t) + \sigma\Delta W_{\alpha,H}(t))^3\\ &\quad= o(\Delta t^{3\alpha-3\varepsilon}) + o(\Delta t^{2\alpha+\alpha H-3\varepsilon}) + o(\Delta t^{2\alpha H+\alpha-3\varepsilon}) + o(\Delta t^{3\alpha-3\varepsilon})\\ &\quad= o(\Delta t^{3\alpha H-3\varepsilon}),\end{aligned} \tag{4.3}$$

$$o(\Delta t^{\alpha+\alpha H-2\varepsilon}) + o(\Delta t^{3\alpha H-3\varepsilon}) = o(\Delta t^{\alpha+\alpha H-2\varepsilon}). \tag{4.4}$$

From the above equations, Equation (4.1) can be rewritten as follows

$$\begin{aligned}\Delta S_t &= (r_d-r_f)S_t\Delta T_\alpha(t) + \sigma S_t\Delta W_{\alpha,H}(t)\\ &\quad+\frac{1}{2}\sigma^2 S_t(\Delta W_{\alpha,H}(t))^2 + o(\Delta t^{\alpha+\alpha H-2\varepsilon}).\end{aligned} \tag{4.5}$$

By the assumption $\alpha H + \alpha > 1$, we obtain

$$\Delta S_t = (r_d-r_f)S_t\Delta T_\alpha(t) + \sigma S_t\Delta W_{\alpha,H}(t) \tag{4.6}$$

Applying the Taylor expansion to $C(t, S_t)$, we have

$$\begin{aligned}\Delta C(t,S_t) &= \frac{\partial C}{\partial t}\Delta t + \frac{\partial C}{\partial S_t}\Delta S_t + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\Delta S_t^2\\ &\quad+\frac{1}{2}\frac{\partial^2 C}{\partial t^2}\Delta t^2 + \frac{\partial^2 C}{\partial S_t\partial t}\Delta t\Delta S_t + o(\Delta t^{3\alpha H-\varepsilon})\\ &= \frac{\partial C}{\partial t}\Delta t + \frac{\partial C}{\partial S_t}\Delta S_t + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\Delta S_t^2 + o(\Delta t)\\ &= \frac{\partial C}{\partial t}\Delta t + (r_d-r_f)S_t\frac{\partial C}{\partial S_t}\Delta T_\alpha(t) + \sigma S_t\frac{\partial C}{\partial S_t}\Delta W_{\alpha,H}(t)\\ &\quad+\frac{1}{2}\sigma^2 S_t\frac{\partial C}{\partial S_t}(\Delta W_{\alpha,H}(t))^2\\ &\quad+\frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2}(\Delta W_{\alpha,H}(t))^2 + o(\Delta t).\end{aligned} \tag{4.7}$$

From Equations (4.1)–(4.5), we obtain that $\frac{\partial^2 C}{\partial S_t^2}$, $\frac{\partial^3 C}{\partial S_t^3}$, $\frac{\partial^2 C}{\partial C\partial t}$ is $o(\Delta t^{\frac{1}{2}(1-H\alpha)^{-\varepsilon}})$ and

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$$\Delta\left(\frac{\partial C}{\partial S_t}\right) = \frac{\partial^2 C}{\partial S_t \partial t}\Delta t + \frac{\partial^2 C}{\partial S_t^2}\Delta S_t + \frac{1}{2}\frac{\partial^3 C}{\partial S_t^3}\Delta S_t^2 + o(\Delta t), \tag{4.8}$$

and

$$\left|\Delta\left(\frac{\partial C}{\partial S_t}\right)\right|.S_{t+\Delta t} = \sigma S_t^2\left|\frac{\partial^2 C}{\partial S_t^2}\right|\left|\Delta W_{\alpha,H}(t)\right| + o(\Delta t). \tag{4.9}$$

Moreover, from assumptions (iii) and (iv), it is found that the change in the value of portfolio Π_t is

$$\begin{aligned}\Delta\Pi_t &= U_t(\Delta S_t + r_f S_t \Delta t) + \Delta F_t - \tfrac{k}{2}|\Delta U_t| S_{t+\Delta t} \\ &= U_t(\Delta S_t + r_f S_t \Delta t) + r_d F_t \Delta t \\ &- \tfrac{k}{2}|\Delta U_t| S_{t+\Delta t} + o(\Delta t),\end{aligned} \tag{4.10}$$

where the number of bonds U_t is constant during time-step Δt. From assumption (v), $C(t, S_t)$ is replicated by portfolio $\Pi(t)$. Thus, at time points Δt, $2\Delta t$, $3\Delta t, \ldots,$ we have $C(t, S_t) = U_t S_t + F_t$ and $U_t = \frac{\partial C}{\partial S_t}$. Therefore, according to Equations (4.5)–(4.10), we have

$$\begin{aligned}\Delta\Pi &= \frac{\partial C}{\partial S_t}\left[(r_d - r_f)S_t \Delta T_\alpha(t) + \sigma S_t \Delta W_{\alpha,H}(t) + \frac{1}{2}\sigma^2 S_t(\Delta W_{\alpha,H}(t))^2 + r_f S_t \Delta t\right] \\ &+ r_d F_t \Delta t - \frac{k}{2}\left|\Delta\left(\frac{\partial C}{\partial S_t}\right)\right|.S_{t+\Delta t} + o(\Delta t) \\ &= \frac{\partial C}{\partial S_t}\left[(r_d - r_f)S_t \Delta T_\alpha(t) + \sigma S_t \Delta W_{\alpha,H}(t) + \frac{1}{2}\sigma^2 S_t(\Delta W_{\alpha,H}(t))^2 + r_f S_t \Delta t\right] \\ &+ \left(C(t, S_t) - S_t\frac{\partial C}{\partial S_t}\right) r_d \Delta t - \frac{k}{2}\sigma S_t^2\left|\frac{\partial^2 C}{\partial S_t^2}\right| \Delta W_{\alpha,H}(t) + o(\Delta t).\end{aligned} \tag{4.11}$$

Consequently,

$$\begin{aligned}\Delta\Pi - \Delta C &= \left(r_d C - (r_d - r_f)S_t\frac{\partial C}{\partial S_t} - \frac{\partial C}{\partial t}\right)\Delta t - \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2}(\Delta W_{\alpha,H}(t))^2 \\ &- \frac{k}{2}\sigma S_t^2\left|\frac{\partial^2 C}{\partial S_t^2}\right|\Delta W_{\alpha,H}(t) + o(\Delta t).\end{aligned} \tag{4.12}$$

The time subscript, t, has been suppressed. As expected, using Equation (4.12), (iv), Remark 2.1, and (4.13), we infer

$$\begin{aligned}E(\Delta\Pi - \Delta C) &= \left(r_d C - (r_d - r_f)S_t\frac{\partial C}{\partial S_t} - \frac{\partial C}{\partial t}\right)\Delta t \\ &- \frac{1}{2}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]^{2H}\Delta t^{2H}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2} - \frac{1}{2}\sqrt{\frac{2}{\pi}}k\sigma S_t^2\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]^{H}\Delta t^{H}\left|\frac{\partial^2 C}{\partial S_t^2}\right| \\ &= \left(r_d C - (r_d - r_f)S_t\frac{\partial C}{\partial S_t} - \frac{\partial C}{\partial t} - \frac{1}{2}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]^{2H}\Delta t^{2H-1}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2}\right. \\ &\left. - \frac{1}{2}\sqrt{\frac{2}{\pi}}k\sigma S_t^2\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]^{H}\Delta t^{H-1}\left|\frac{\partial^2 C}{\partial S_t^2}\right|\right)\Delta t = 0.\end{aligned} \tag{4.13}$$

Thus, from Equation (4.13), we can derive

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$$r_dC = (r_d - r_f)S_t\frac{\partial C}{\partial S_t} + \frac{\partial C}{\partial t} + \frac{1}{2}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]^{2H}\Delta t^{2H-1}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2} + \frac{1}{2}\sqrt{\frac{2}{\pi}}k\sigma S_t^2\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]^{H}\Delta t^{H-1}\left|\frac{\partial^2 C}{\partial S_t^2}\right|. \tag{4.14}$$

We define $\hat{\sigma}^2(t)$ as follows:

$$\hat{\sigma}^2 = \sigma^2\left(\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]^{2H}\Delta t^{2H-1} + \sqrt{\frac{2}{\pi}}k\sigma^{-1}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]^{H}\Delta t^{H-1}\right). \tag{4.15}$$

where $\frac{\partial^2 C}{\partial S_t^2}$ is ever positive for the ordinary European currency call option without transaction costs, if the same conduct of $\frac{\partial^2 C}{\partial S_t^2}$ is postulated here and $\hat{\sigma}(t)$ remains fixed during the time-step $[t, \Delta t)$. Then, from Equations (4.14) and (4.15), we obtain

$$\frac{\partial C}{\partial t} + (r_d - r_f)S_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\hat{\sigma}^2 S_t^2\frac{\partial^2 C}{\partial S_t^2} - r_dC = 0. \tag{4.16}$$

Followed by

$$C = C(t, S_t) = S_te^{-r_f(T-t)}\Phi(d_1) - Ke^{-r_d(T-t)}\Phi(d_2), \tag{4.17}$$

and

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r_d - r_f)(T-t) + \frac{\hat{\sigma}^2}{2}(T-t)}{\hat{\sigma}\sqrt{T-t}}, \quad d_2 = d_1 - \hat{\sigma}\sqrt{T-t}. \tag{4.18}$$

Proof of Theorem 2.2. First, we derive a general formula. Let y be one of the influence factors. Thus

$$\frac{\partial C}{\partial y} = \frac{\partial S_te^{-(r_f)(T-t)}}{\partial y}\Phi(d_1) + S_te^{-r_f(T-t)}\frac{\partial \Phi(d_1)}{\partial y} - \frac{\partial Ke^{-r_d(T-t)}}{\partial y}\Phi(d_2) - Ke^{-r_d(T-t)}\frac{\partial \Phi(d_2)}{\partial y} \tag{4.19}$$

But

$$\begin{aligned}\frac{\partial \Phi(d_2)}{\partial y} &= \Phi'(d_2)\frac{\partial d_2}{\partial y}\\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}}\frac{\partial d_2}{\partial y}\\ &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(d_1 - \hat{\sigma}\sqrt{T-t})^2}{2}\right)\frac{\partial d_2}{\partial y}\\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\exp(d_1\hat{\sigma}\sqrt{T-t})\exp\left(-\frac{\hat{\sigma}^2(T-t)}{2}\right)\frac{\partial d_2}{\partial y}\\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\exp\left(\ln\frac{S_t}{K} + (r_d - r_f)(T-t)\right)\frac{\partial d_2}{\partial y}\\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\frac{S}{K}\exp((r_d - r_f)(T-t))\frac{\partial d_2}{\partial y}.\end{aligned} \tag{4.20}$$

Then

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$$\frac{\partial C}{\partial y} = \frac{\partial S_t e^{-(r_f)(T-t)}}{\partial y}\Phi(d_1) - \frac{\partial K e^{-r_d(T-t)}}{\partial y}\Phi(d_2) + S_t e^{-r_f(T-t)}\Phi'(d_1)\frac{\partial\hat{\sigma}\sqrt{T-t)}}{\partial y}. \tag{4.21}$$

Substituting in (4.21), we get the desired Greeks.

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Mixed Fractional Merton Model to Evaluate European Options with Transaction Costs

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Abstract

This paper deals with the problem of discrete-time option pricing by the mixed fractional version of Merton model with transaction costs. By a mean-self-financing delta hedging argument in a discrete-time setting, a European call option pricing formula is obtained. We also investigate the effect of the time-step δt and the Hurst parameter H on our pricing option model, which reveals that these parameters have high impact on option pricing. The properties of this model are also explained.

Keywords

Transaction Costs, Mixed Fractional Brownian Motion, European Option, Merton Model

1. Introduction

Over the last few years, the financial markets have been regarded as complex and nonlinear dynamic systems. A series of studies has found that many financial market time series display scaling laws and long-range dependence. Therefore, it has been proposed that the Brownian motion in the classical Black-Scholes (*BS*) model [1] should be replaced by a process with long-range dependence.

Nowadays, the *BS* model is the one most commonly used for analyzing financial data, and some scholars have presented modified forms of the *BS* model which have influential and significant outcomes on option pricing. However, they are still theoretical adaptations and not necessarily consistent with the empirical features of financial return series, such as nonnormality, long-range dependence, etc. For example, some scholars [2] [3] [4] [5] [6] have showed that returns are of long-range (or short-range) dependence, which suggests strong time-correlations between different events at different time scales [7] [8] [9]. In the search for better models for describing long-range dependence in financial return series, a mixed fractional Brownian (*MFBM*) model has been proposed as an improvement of the classical *BS* model [10]-[18]. The advantage of using the *MFBM* is that the markets are free of arbitrage. Moreover, Cheridito [10] has proved that, for $H \in \left(\frac{3}{4},1\right)$, the *MFBM* is equivalent to one with Brownian motion, and hence time-step and long-range dependence in return series have no impact on option pricing in a complete financial market without transaction costs. In addition, a number of empirical studies show that the paths of asset prices are discontinuous and that there are jumps in asset prices, both in the stock market and foreign exchange [9] [19] [20] [21] [22].

The above researches have an important implication for option pricing. Merton [23] created a revolution in option pricing when the underlying asset was governed by a diffusion process. Based on this theory, Kou [24], Cont and Tankov [25] also considered the problems of pricing options under a jump diffusion environment in a larger setting. In this paper, to capture jumps or discontinuities, fluctuations and to take into account the long memory property of financial markets, a mixed fractional version of the Merton model is introduced, which is based on a combination of Poisson jumps and *MFBM*. The mixed fractional Merton (*MFM*) model is based on the assumption that the underlying asset price is generated by a two-part stochastic process: 1) small, continuous price movements are generated by a *MFBM*, and 2) large, infrequent price jumps are generated by a Poisson process. This two-part process is intuitively appealing, as it is consistent with an impressive market in which major information arrives infrequently and randomly. This process may provide a description for empirically observed distributions of exchange rate changes that are skewed, leptokurtic, have long memory and fatter tails than comparable normal distributions and apparent nonstationary variance. Further, we will show the impact of the time-step and long-range dependence in return series exactly on option pricing, regardless of whether proportional transaction costs are considered or not in a discrete time setting.

Leland [26] is a pioneer scholar, who investigated option replication where transaction costs exist in a discrete time setting. In this view, the arbitrage-free arguments presented by Black and Scholes [1] are not applicable in a model where transaction costs occur at all moments of trading of the stock or bond. The problem is that perfect replication incurs an infinite number of transaction costs because of the infinite variation which exists in the geometric Brownian motion. In this regard, a delta hedge strategy is constructed in accordance with revision conducted a discrete number of times. Transaction costs lead to the failure of the no arbitrage principle and the continuous time trade in general: instead of no arbitrage, the principle of hedge pricing, according to which the price of an option is defined as the minimum level of initial wealth needed to hedge the option, comes into force.

According to the empirical findings obtained before and the views of behavioral finance and econophysics, we are motivated to examine the problem that exists in option pricing, while the dynamics of price S_t follows a mixed fractional jump-diffusion process under the transaction costs. We assume that S_t satisfies

$$S_t = S_0 \mathrm{e}^{\mu t + \sigma B(t) + \sigma_H B_H(t) + N_t \ln J}. \tag{1.1}$$

where S_0, μ, σ and σ_H are fixed; $B(t)$ is a Brownian motion; $B_H(t)$ is a fractional Brownian motion with Hurst parameter $H \in \left(\frac{3}{4}, 1\right)$; N_t is a Poisson process with intensity $\lambda > 0$; and J is a positive random variable. We assume that $B(t), B_H(t), N_t$ and J are independent.

This paper is organized into several sections. In Section 2, we will study the problem of option pricing with transaction costs by applying delta hedging strategy. In addition, a new framework for pricing European option is obtained when the stock price S_t is satisfied in Equation (1.1). Section 3 is devoted to empirical studies and simulations to show the performance of the *MFM* model. A conclusion is presented in Section 4.

2. Pricing Option by Mixed Fractional Version of Merton Model with Transaction Costs

Suppose $\{B(t)\}_{t \geq 0}$ be a standard Brownian motion and $\{B_H(t)\}_{t \geq 0}$ be a fractional Brownian motion with the Hurst parameter $H \in \left(\frac{3}{4}, 1\right)$, both defined on complete probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$, the absolute price jump size J is a nonnegative random variable drawn from lognormal distribution, *i.e.* $\ln(J) = N(\mu_J, \sigma_J)$, which implies

$$J \sim Lognormal\left(\mathrm{e}^{\mu_J + \frac{\sigma_J^2}{2}}, \mathrm{e}^{2\mu_J + \sigma_J^2}\left(\mathrm{e}^{\sigma_J^2} - 1\right)\right)$$

and a Poisson process $N = (N_t)_{t \geq 0}$ with rate λ. Additionally, the processes B, B_H, N and J are independent, P is the real world probability measure and $(\mathscr{F}_t)_{t \in [0,T]}$ denotes the P-augmentation of filtration generated by $(B(\tau), B_H(\tau)), \tau \leq t$.

The objective of this section is to derive a stock pricing formula under transaction costs in a discrete time setting. Consider (D, S)-market with a bond D_t and a stock S_t, where

$$D_t = D_0 \mathrm{e}^{rt}. \tag{2.1}$$

and

$$S_t = S_0 \mathrm{e}^{\mu t + \sigma B(t) + \sigma_H B_H(t) + N_t \ln J}, \quad \mu, \sigma, \sigma_H \in R, D_0, S_0, t \in R^+. \tag{2.2}$$

The groundwork of modeling the effects of transaction costs was done by Leland [26]. He adopted the hedging strategy of rehedging at every time-step

δt. That is, with every δt the portfolio is rebalanced, whether or not this is optimal in any sense. In the following proportional transaction cost option pricing model, we follow the other usual assumptions in the Black-Scholes model, but with the following exceptions:

1) The price S_t of the underlying stock at time t satisfies Equation (2.2).

2) The portfolio is revised every δt where δt is a finite and fixed, small time-step.

3) Transaction costs are proportional to the value of the transaction in the underlying. Let k denote the round trip transaction cost per unit dollar of transaction. Suppose $U > 0$ shares are bought $(U > 0)$ or sold $(U < 0)$ at the price S_t, then the transaction cost is given by $\frac{k}{2}|U|S_t$ in either buying or selling, where k is a constant. The value of k will depend on the individual investor. In the *MFM* model, where transaction costs are incurred at every time the stock or the bond is traded, the no arbitrage argument used by Black and Scholes no longer applies. The problem is that due to the infinite variation of the *MFBM*, perfect replication incurs an infinite amount of transaction costs.

4) The hedge portfolio has an expected return equal to that from an option. This is exactly the same valuation policy as earlier on discrete hedging with no transaction costs.

5) Traditional economics assumes that traders are rational and maximize their utility. However, if their behaviour is assumed to be boundedly rational, the traders' decisions can be explained both by their reaction to the past stock price, according to a standard speculative behaviour, and by imitation of other traders' past decisions, according to common evidence in social psychology. It is well known that the delta-hedging strategy plays a central role in the theory of option pricing and that it is popularly used on the trading floor. Therefore, based on the availability heuristic, suggested by Tversky and Kahneman [27], traders are assumed to follow, anchor, and imitate the Black-Scholes delta-hedging strategy to price an option. In this case, delta-hedging argument is a partial and imperfect hedging strategy, which does not eliminate all of the risk. However, as mentioned in the paper [28], in most models of stock fluctuations, except for very special cases, risk in option trading cannot be eliminated and strict arbitrage opportunities do not exist, whatever be the price of the option. The risk cannot be eliminated is furthermore the fundamental reason for the very existence of option markets.

Delta hedging is an options strategy that aims to reduce, or hedge, the risk associated with price movements in the underlying asset, by offsetting long and short positions. For example, a long call position may be delta hedged by shorting the underlying stock. This strategy is based on the change in premium, or price of option, caused by a change in the price of the underlying security. In this section we use the delta hedging strategy to obtain a pricing formula for European call option.

Let the price of European call option be denoted with expiration T and strike price K by $C(t,S_t)$ with boundary conditions:

$$C(T,S_T)=(S_T-K)^+, \quad C(t,0)=0, \quad C(t,S_t)\to S_t \quad \text{as } S_t\to\infty. \tag{2.3}$$

Then, $C(t,S_t)$ is derived by the following theorem.

Theorem 2.1. *The price at every $t\in[0,T]$ of a European call option with strike price K that matures at time T is given by*

$$C(t,S_t)=\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda'(T-t)}\left(\lambda'(T-t)\right)^n}{n!}\left[S_t\phi(d_1)-K\mathrm{e}^{-r(T-t)}\phi(d_2)\right]. \tag{2.4}$$

Moreover, $C(t,S_t)$ satisfies the following equation

$$\begin{aligned}&\frac{\partial C}{\partial t}+rS_t\frac{\partial C}{\partial S_t}+\frac{S_t^2\hat{\sigma}^2}{2}\frac{\partial^2 C}{\partial S_t^2}-rC+\lambda E\left[C(t,JS_t)-C(t,S_t)\right]\\&-\lambda E[J-1]S_t\frac{\partial C}{\partial S_t}=0,\end{aligned} \tag{2.5}$$

where

$$d_1=\frac{\ln\left(\frac{S_t}{K}\right)+r_n(T-t)+\frac{\sigma_n^2}{2}(T-t)}{\sigma_n\sqrt{T-t}}, \quad d_2=d_1-\sigma_n\sqrt{T-t}, \tag{2.6}$$

$$\lambda'=\lambda E(J)=\lambda\mathrm{e}^{\mu_J+\frac{\sigma_J^2}{2}}, \quad \sigma_n^2=\hat{\sigma}^2+\frac{n\sigma_J^2}{T-t}, \tag{2.7}$$

$$r_n=r-\lambda E(J-1)+\frac{n\ln E(J)}{T-t}=r-\lambda\left(\mathrm{e}^{\mu_J+\frac{\sigma_J^2}{2}}-1\right)+\frac{n\left(\mu_J+\frac{\sigma_J^2}{2}\right)}{T-t}, \tag{2.8}$$

$$\hat{\sigma}^2=\sigma^2+\sigma_H^2(\delta t)^{2H-1}+k\sqrt{\frac{2}{\pi}\left(\frac{\sigma^2}{\delta t}+\sigma_H^2(\delta t)^{2H-2}\right)}sign(\Gamma), \tag{2.9}$$

$sign(\Gamma)$ is the signum function of $\frac{\partial^2 C}{\partial S_t^2}$; n is the number of prices jumps; δt is a small and fixed time-step; k is the transaction costs and $\phi(.)$ is the cumulative normal distribution.

Moreover, using the put call parity, we can easily obtain the valuation model for a put currency option, which is provided by the following corollary.

Corollary 2.1. *The value of European put option with transaction costs is given by*

$$P(t,S_t)=\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda'(T-t)}\left(\lambda'(T-t)\right)^n}{n!}\left[K\mathrm{e}^{-r(T-t)}\phi(-d_2)-S_t\phi(-d_1)\right].$$

3. Properties of Pricing Formula

In this section, we present the properties of *MFM*'s log-return density. The effects of Hurst parameter and time-step on our modified volatility (σ_n^2) are also discussed in the discrete time and continuous time cases. Then we show that these parameters play a significant role in a discrete time setting, both with and without transaction costs.

3.1. Log-Return Density

In the case of *MFM* the log return jump size is assumed to be $(Y_i) = (\ln J_i) \sim N(\mu_J, \sigma_J^2)$ and the probability density of log return $x_t = \ln(S_t/S)$ is achieved as a quickly converging series of the following form:

$$P(x_t \in A) = \sum_{n=0}^{\infty} P(N_t = n) P(x_n \in A \mid N_t = n)$$

$$P(x_t) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\lambda t} (\lambda t)^n}{n!} N\left(x_t; \mu t + n\mu_J, \sigma^2 t + \sigma_H^2 t^{2H} + n\sigma_J^2\right), \tag{3.1}$$

where

$$N\left(x_t; \mu t + n\mu_J, \sigma^2 t + \sigma_H^2 t^{2H} + n\sigma_J^2\right) = \frac{1}{\sqrt{2\pi\left(\sigma^2 t + \sigma_H^2 t^{2H} + n\sigma_J^2\right)}} \exp\left[-\frac{\left(x_t - (\mu t + n\mu_J)\right)^2}{2\left(\sigma^2 t + \sigma_H^2 t^{2H} + n\sigma_J^2\right)}\right] \tag{3.2}$$

The term $P(N_t = n) = \frac{\mathrm{e}^{-\lambda t} (\lambda t)^n}{n!}$ is the probability that the asset price jumps *n* times during the time interval of length *t*. And $P(x_n \in A \mid N_t = n) = N\left(x_t; \mu t + n\mu_J, \sigma^2 t + \sigma_H^2 t^{2H} + n\sigma_J^2\right)$ is the mixed fractional normal density of log-return. It supposes that the asset price jumps *i* times in the time interval of *t*. As a result, in the *MFM* model, the log-return density can be described as the weighted average of the mixed fractional normal density by the probability that the asset price jumps *n* times.

The outstanding properties of log-return density $P(x_t)$ are observed in the *MFM*. Firstly, the μ_J sign refers to the expected log-return jump size, $E(Y) = E(\ln J) = \mu_J$, which indicates the skewness sign. If $\mu_J < 0$, the log-return density $P(x_t)$ shows negatively skewed, and if $\mu_J = 0$, it is symmetric as displayed in Figure 1 (Table 1).

Secondly, larger value of intensity λ (which means that jumps are expected to occur more frequently) makes the density fatter-tailed as illustrated in Figure 2. Note that the excess kurtosis in the case $\lambda = 20$ is much smaller than in the case $\lambda = 1$ or $\lambda = 10$. This is because excess kurtosis is a standardized measure (by standard deviation) (Table 2).

3.2. The Impact of Parameters

Mantegna and Stanley [29] as pioneer scholars proposed the scaling invariance method from the complex science of economic systems which led to numerous investigations into scaling laws in finance. The major question in economics is whether the price impact of scaling law and long-range dependence is significant in option pricing. The answer to this question is assured. For instance, one of the significant issues in finance concerning the modeling of high-frequency data is related to analyzing the volatility in different time scales.

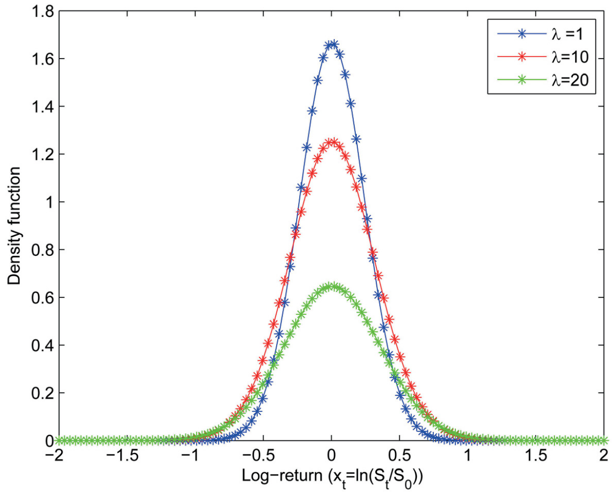

Figure 1. *MFM*'s log-return density. Fixed parameters are $\sigma = 0.25$, $\sigma_H = 0.25$, $H = 0.76$, $\sigma_J = 0.1$, $\mu_J = 0$, $\mu = 0.009$, and $t = 0.5$.

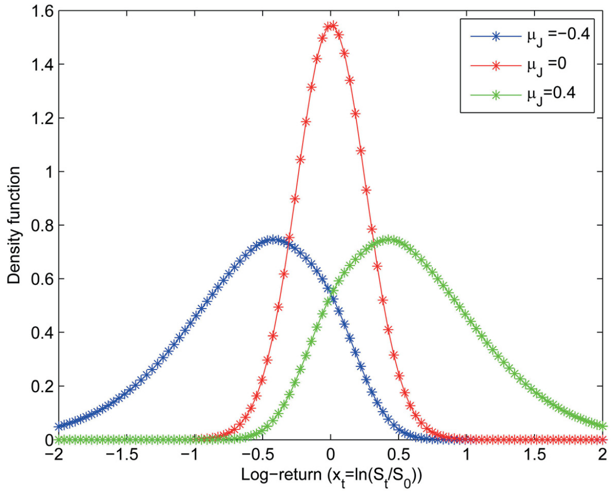

Figure 2. *MFM*'s log-return density. Fixed parameters are $\sigma = 0.25$, $\sigma_H = 0.25$, $H = 0.76$, $\sigma_J = 0.1$, $\lambda = 3$, $\mu = 0.009$, and $t = 0.5$.

Table 1. Annualized moments of Merton's log-return density in **Figure 1**.

Model	Mean	Standard Deviation	Skewness	Excess Kurtosis
$\mu_J = -0.4$	−1.1910	0.6161	−0.5082	0.2806
$\mu_J = 0$	0.0090	0.1361	0	0.706
$\mu_J = 0.4$	1.2090	0.6161	0.5082	0.2806

Table 2. Annualized moments of Merton's log-return density in Figure 2.

Model	Mean	Standard Deviation	Skewness	Excess Kurtosis
$\lambda = 1$	0.0040	0.1161	0	0.0223
$\lambda = 10$	−0.0411	0.2061	0	0.706
$\lambda = 20$	−0.0913	0.3061	0	0.0640

Remark 3.1. In a continuous time setting $(\delta t = 0, \lambda \neq 0)$ without transaction costs the implied volatility is $\hat{\sigma}_n^2 = \sigma^2 + \frac{n\sigma_J^2}{T-t}$, thus the option value is similar to the Merton jump diffusion model [19]. Moreover, if $\delta t = 0$ in the absence of transaction costs and jump case, the MFM model reduces to the BS model

$$\frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S_t} + \frac{S_t^2 \sigma^2}{2} \frac{\partial^2 C}{\partial S_t^2} - rC = 0, \tag{3.3}$$

which shows that the Hurst parameter H and time-step δt have no effect on option pricing model in a continuous time setting $(\delta t = 0)$.

Remark 3.2. In a discrete time setting without transaction costs $(k = 0, \delta t \neq 0)$, if jump occurs, the modified volatility is $\hat{\sigma}_n^2 = \sigma^2 + \sigma_H^2 (\delta t)^{2H-1} + \frac{n\sigma_J^2}{T-t}$ and when jump does not occur $(\lambda = 0)$, from Equation (2.5), we have

$$\frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S_t} + \left(\sigma^2 + \sigma_H^2 (\delta t)^{2H-1}\right) \frac{S_t^2}{2} \frac{\partial^2 C}{\partial S_t^2} - rC = 0, \tag{3.4}$$

which demonstrates that the delta hedging strategy in a discrete time case is fundamentally different in comparison with a continuous time case. It also indicates that the scaling exponent $2H-1$ and time-step δt play a significant role in option pricing theory. Figure 3 illustrates the impacts of the time-step, Hurst parameter, mean jump, and jump intensity on our option pricing model.

Remark 3.3. From [30] we infer there exists $\delta t \in \left(0, \frac{1}{M}\right)$ such that

$$\min_{\delta t \in \left(0, \frac{1}{M}\right)} \hat{\sigma}^2, \tag{3.5}$$

Holds,

where $M > 1$, k is small enough

$$\hat{\sigma}^2 = \sigma^2 + \sigma_H^2 (\delta t)^{2H-1} + k\sqrt{\frac{2}{\pi}\left(\frac{\sigma^2}{2\delta t} + \sigma_H^2 (\delta t)^{2H-2}\right)}. \tag{3.6}$$

Indeed,

$$\begin{aligned} &\sigma_H^2 (\delta t)^{2H-1} + k\sqrt{\frac{2}{\pi}\left(\frac{\sigma^2}{\delta t} + \sigma_H^2 (\delta t)^{2H-2}\right)} \\ &\geq 2\sigma_H (\delta t)^{H-\frac{1}{2}} k^{\frac{1}{2}} \left(\frac{2}{\pi}\left(\frac{\sigma^2}{\delta t} + \sigma_H^2 (\delta t)^{2H-2}\right)\right)^{\frac{1}{4}}. \end{aligned} \tag{3.7}$$

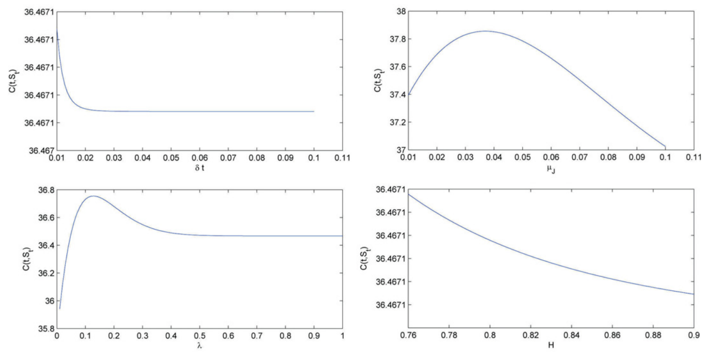

Figure 3. Modified volatility. Fixed parameters are $\sigma = 0.1$, $\sigma_H = 0.1$, $H = 0.76$, $\sigma_J = 0.03$, $T = 0.2$, $T = 0.2$, and $t = 0.1$.

Set

$$\sigma_H^2 (\delta t)^{2H-1} = \sqrt{\frac{2k^2}{\pi}\left(\frac{\sigma^2}{\delta t} + \frac{\sigma_H^2}{(\delta t)^{2-2H}}\right)}. \tag{3.8}$$

Thus

$$\sigma_H^2 (\delta t)^{2H} = \frac{\frac{2k^2}{\pi} + \sqrt{\left(\frac{2k^2}{\pi}\right)^2 + \frac{8k^2}{\pi}\sigma^2 \delta t}}{2}. \tag{3.9}$$

Suppose

$$f(x) = \sigma_H^2 x^{2H} - \frac{\frac{2k^2}{\pi} + \sqrt{\left(\frac{2k^2}{\pi}\right)^2 + \frac{8k^2}{\pi}\sigma^2 x}}{2}. \tag{3.10}$$

Since $f(0) < 0$ and

$$f\left(\frac{1}{M}\right) = \sigma_H^2 \left(\frac{1}{M}\right)^{2H} - \frac{\frac{2k^2}{\pi} + \sqrt{\left(\frac{2k^2}{\pi}\right)^2 + \frac{8k^2}{\pi}\sigma^2 \frac{1}{M}}}{2} > 0, \tag{3.11}$$

as k is small enough.

Hence, there exists a $\delta t \in \left(0, \frac{1}{M}\right)$ such that $\min_{\delta t \in \left(0, \frac{1}{M}\right)} \hat{\sigma}^2$ holds.

Suppose

$$\hat{\sigma}^2(\min) = \min_{\delta t \in \left(0, \frac{1}{M}\right)} \hat{\sigma}^2, \tag{3.12}$$

so

$$\sigma_n^2(\min) = \min_{\delta t \in \left(0, \frac{1}{M}\right)} \sigma_n^2 = \min_{\delta t \in \left(0, \frac{1}{M}\right)} \hat{\sigma}^2 + \frac{n\delta^2}{T-t}. \tag{3.13}$$

Then the minimal price of an option with respect to transaction costs is displayed as $C_{\min}(t, S_t)$ with $\sigma_n^2(\min)$ in Equation (2.4). $C_{\min}(t, S_t)$ can be applied to the real price of an option.

4. Conclusion

To capture the long memory and discontinuous property, this article focuses on the problem of pricing European option in a mixed fractional Merton environment without using the arbitrage argument. We obtain a mixed fractional version of Merton model for pricing European option with transaction costs. Some properties of mixed fractional Merton's log-return density are discussed. Moreover, we derive that the Hurst parameter H and time-step δt play a significant role in pricing option in a discrete time setting for cases both with and without transaction costs. But these parameters have no impact on option pricing in a continuous time setting.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

Proof of Theorem 2.1. We consider a replicating portfolio with $\psi(t)$ units of financial underlying asset and one unit of the option. Then, the value of the portfolio at time t is

$$P_t = \psi(t) S_t - C(t, S_t). \tag{4.1}$$

Now, the movement in S_t and P_t is considered under discrete time interval δt. In view of this, we suppose that trading takes place at the specific time points of t and $t+\delta t$. It can be said that the number of shares through the use of delta-hedging strategy and the present stock price S_t are constantly held during the rebalancing interval $[t, t+\delta t)$. Then, the movement in the value of the portfolio after time interval δt is defined as follows:

$$\delta P_t = \psi(t)\delta S_t - \delta C(t, S_t) - \frac{k}{2}\left|\delta\psi(t)\right| S_t. \tag{4.2}$$

where δS_t is the movement of the underlying stock price, $\delta\psi(t)$ is the movement of the number of units of stock held in the portfolio, and δP_t is the change in the value of the portfolio.

Since the time-step δt and the asset change are both small, according to Taylor's formulae we have if $\delta N_t = 0$ with probability $1-\lambda\delta t$, so

$$\begin{aligned}\delta S_t &= S_t \mu\delta t + S_t \delta\sigma B(t) + S_t \delta\sigma_H B_H(t) + \frac{S_t}{2}\left(\mu\delta t + \sigma\delta B(t) + \sigma_H \delta B_H(t)\right)^2 \\ &+ \frac{S_t}{6} e^{\theta\left[\mu\delta t + \sigma\delta B(t) + \sigma_H \delta B_H(t)\right]}\left(\mu\delta t + \sigma\delta B(t) + \sigma_H\delta B_H(t)\right)^3,\end{aligned} \tag{4.3}$$

where $\theta = \theta(t, w), w \in \Omega$, and $0 < \theta < 1$.

Since $B(t)$ and $B_H(t)$ are continuous, then from [28] we have

$$(\delta t)\delta B_H(t) = O\left((\delta t)^{1+H}\sqrt{\log\frac{1}{\delta t}}\right), \tag{4.4}$$

$$(\delta t)\delta B(t) = O\left((\delta t)^{\frac{3}{2}}\sqrt{\log\frac{1}{\delta t}}\right), \tag{4.5}$$

$$\frac{\delta B_H(t)}{\delta B(t)} \to 0 \quad \text{as } \delta t \to 0, \tag{4.6}$$

and

$$\begin{aligned}&e^{\theta\left[\mu\delta t + \sigma\delta B(t) + \sigma_H\delta B_H(t)\right]}\left[\mu\delta t + \sigma\delta B(t) + \sigma_H\delta B_H(t)\right]^3 \\ &= O\left((\delta t)^3\right) + O\left((\delta t)^{\frac{5}{2}}\sqrt{\log(\delta t)^{-1}}\right) + O\left((\delta t)^2 \log(\delta t)^{-1}\right) \\ &+ O\left((\delta t)^{\frac{3}{2}}\left(\log(\delta t)^{-1}\right)^{\frac{3}{2}}\right) = O\left((\delta t)^{\frac{3}{2}}\left(\log(\delta t)^{-1}\right)^{\frac{3}{2}}\right).\end{aligned}$$

Thus, we can get

$$\begin{aligned}\delta S_t &= \mu S_t \delta t + S_t\left[\sigma\delta B(t) + \sigma_H\delta B_H(t)\right] \\ &+ \frac{S_t}{2}\left[\sigma\delta B(t) + \sigma_H\delta B_H(t)\right]^2 + O\left((\delta t)^{\frac{3}{2}}\sqrt{\log(\delta t)^{-1}}\right),\end{aligned} \tag{4.7}$$

$$\left(\delta S_t\right)^2 = S_t^2\left[\sigma\delta B(t) + \sigma_H \delta B_H(t)\right]^2 + O\left(\left(\delta t\right)^{\frac{3}{2}}\sqrt{\log\left(\delta t\right)^{-1}}\right), \tag{4.8}$$

$$\delta C\left(t, S_t\right) = \frac{\partial C}{\partial t}\delta t + \frac{\partial C}{\partial S_t}\delta\left(S_t\right) + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\left(\delta S_t\right)^2 + O\left(\left(\delta t\right)^{\frac{3}{2}}\sqrt{\log\left(\delta t\right)^{-1}}\right), \tag{4.9}$$

and

$$\delta\psi(t) = \frac{\partial\psi}{\partial t}\delta t + \frac{\partial\psi}{\partial S_t}\delta S_t + \frac{1}{2}\frac{\partial^2\psi}{\partial S_t^2}\left(\delta S_t\right)^2 + O\left(\left(\delta t\right)^{\frac{3}{2}}\sqrt{\log\left(\delta t\right)^{-1}}\right). \tag{4.10}$$

If $\delta N_t = 1$ with probability $\lambda\delta t$ and the jump of N_t in $\left[t, t+\Delta t\right]$ is assumed to occur at current time *t*, then

$$S_{t^+} = S_0 \mathrm{e}^{\mu t + \sigma B(t) + \sigma_H B_H(t) + \ln J}, \tag{4.11}$$

$$S_{t+\delta t} = S_0 \mathrm{e}^{\mu(t+\delta t) + \sigma B(t+\delta t) + \sigma_H B_H(t+\delta t) + \ln J}, \tag{4.12}$$

$$\delta S_{t^+} = S_{t+\delta t} - S_{t^+} = S_{t^+}\left[\mathrm{e}^{\mu t + \sigma B(t) + \sigma_H B_H(t)} - 1\right], \tag{4.13}$$

$$\begin{aligned} \delta S_t &= S_{t+\delta t} - S_t = S_{t+\delta t} - S_{t^+} + S_{t^+} - S_t \\ &= S_{t^+}\left[\mathrm{e}^{\mu t + \sigma B(t) + \sigma_H B_H(t)} - 1\right] + \left(S_{t^+} - S_t\right) \end{aligned} \tag{4.14}$$

$$\begin{aligned} \delta C\left(t, S_t\right) &= C\left(t+\delta t, S_{t+\delta t}\right) - C\left(t, S_{t^+}\right) + C\left(t, S_{t^+}\right) - C\left(t, S_t\right) \\ &= C\left(t, S_{t^+}\right) - C\left(t, S_t\right) + \frac{\partial C}{\partial t}\delta t + \frac{\partial C}{\partial S_{t^+}}\delta\left(S_{t^+}\right) \\ &\quad + \frac{1}{2}\frac{\partial^2 C}{\partial S_{t^+}^2}\left(\delta S_{t^+}\right)^2 + O\left(\left(\delta t\right)^{\frac{3}{2}}\sqrt{\log\left(\delta t\right)^{-1}}\right), \end{aligned} \tag{4.15}$$

$$\begin{aligned} \delta\psi\left(t, S_t\right) &= \psi\left(t, S_{t^+}\right) - \psi\left(t, S_t\right) + \frac{\partial\psi\left(t, S_{t^+}\right)}{\partial t}\delta t + \frac{\partial\psi\left(t, S_{t^+}\right)}{\partial S_{t^+}}\delta\left(S_{t^+}\right) \\ &\quad + \frac{1}{2}\frac{\partial^2\psi\left(t, S_{t^+}\right)}{\partial S_{t^+}^2}\left(\delta S_{t^+}\right)^2 + O\left(\left(\delta t\right)^{\frac{3}{2}}\sqrt{\log\left(\delta t\right)^{-1}}\right), \end{aligned} \tag{4.16}$$

where $\delta S_{t^+} = S_{t+\delta t} - S_{t^+}$.

Based on the above assumptions iv and v, we have $E\left(\frac{\delta P_t}{P_t}\right) = \frac{\delta D_t}{D_t}$, *i.e.* $E\delta P_t = rP_t + O\left(\left(\delta t\right)^2\right)$. Then

$$\begin{aligned} &\left(1-\lambda\delta t\right)E\left[\psi\delta S_t - \delta C\left(t, S_t\right) - \frac{kS_t}{2}\left|\delta\psi(t)\right|\right] \\ &+\lambda\delta t E\left[S_{t^+}\left(\mathrm{e}^{\mu t + \sigma B(t) + \sigma_H B_H(t)} - 1\right)\psi(t) + \left(S_{t^+} - S_t\right)\psi(t) - \left(C\left(t, S_{t^+}\right) - C\left(t, S_t\right)\right.\right. \\ &\left.+\frac{\partial C}{\partial t}\delta t + \frac{\partial C}{\partial S_{t^+}}\delta\left(S_{t^+}\right) + \frac{1}{2}\frac{\partial^2 C}{\partial S_{t^+}^2}\left(\delta S_{t^+}\right)^2 + O\left(\left(\delta t\right)^{\frac{3}{2}}\sqrt{\log\left(\delta t\right)^{-1}}\right)\right) \\ &-\frac{kS_t}{2}\left|\psi\left(t, S_{t^+}\right) - \psi\left(t, S_t\right) + \frac{\partial\psi\left(t, S_{t^+}\right)}{\partial t}\delta t + \frac{\partial\psi\left(t, S_{t^+}\right)}{\partial S_{t^+}}\delta\left(S_{t^+}\right)\right. \\ &\left.\left.+\frac{1}{2}\frac{\partial^2\psi\left(t, S_{t^+}\right)}{\partial S_{t^+}^2}\left(\delta S_{t^+}\right)^2 + O\left(\left(\delta t\right)^{\frac{3}{2}}\sqrt{\log\left(\delta t\right)^{-1}}\right)\right|\right] = rP_t\delta t, \end{aligned}$$

i.e.

$$(1-\lambda\delta t)E\left[\psi\delta S_t-\delta C(t,S_t)\right]+\lambda\delta tE\Big[S_{t^+}\left(\mathrm{e}^{\mu t+\sigma B(t)+\sigma_H B_H(t)}-1\right)\psi(t)$$
$$+\left(S_{t^+}-S_t\right)\psi(t)-\Big(C\left(t,S_{t^+}\right)-C(t,S_t)+\frac{\partial C\left(t,S_{t^+}\right)}{\partial t}\delta t+\frac{\partial C\left(t,S_{t^+}\right)}{\partial S_{t^+}}\delta\left(S_{t^+}\right)$$
$$+\frac{1}{2}\frac{\partial^2 C\left(t,S_{t^+}\right)}{\partial S_{t^+}^2}\left(\delta S_{t^+}\right)^2+O\left((\delta t)^{\frac{3}{2}}\sqrt{\log(\delta t)^{-1}}\right)\Big)\Big]-(1-\lambda\delta t)E\left[\frac{kS_t}{2}\left|\delta\psi(t)\right|\right]$$
$$-\lambda\delta tE\Big[\frac{kS_t}{2}\Big|\psi\left(t,S_{t^+}\right)-\psi(t,S_t)+\frac{\partial\psi\left(t,S_{t^+}\right)}{\partial t}\delta t+\frac{\partial\psi\left(t,S_{t^+}\right)}{\partial S_{t^+}}\delta\left(S_{t^+}\right)$$
$$+\frac{1}{2}\frac{\partial^2\psi\left(t,S_{t^+}\right)}{\partial S_{t^+}^2}\left(\delta S_{t^+}\right)^2+O\left((\delta t)^{\frac{3}{2}}\sqrt{\log(\delta t)^{-1}}\right)\Big|\Big]=rP_t\delta t+O\left((\delta t)^2\right),$$

where the current stock price S_t is given. Since

$$E\left[\frac{kS_t}{2}\left|\delta\psi(t)\right|\right]=\frac{(1-\lambda\delta t)kS_t}{2}E\left|\delta\psi(t)\right|+\frac{(\lambda\delta t)kS_t}{2}\Big|\psi\left(t,S_{t^+}\right)$$
$$-\psi(t,S_t)+\frac{\partial\psi\left(t,S_{t^+}\right)}{\partial t}\delta t+\frac{\partial\psi\left(t,S_{t^+}\right)}{\partial S_{t^+}}\delta\left(S_{t^+}\right)$$
$$+\frac{1}{2}\frac{\partial^2\psi\left(t,S_{t^+}\right)}{\partial S_{t^+}^2}\left(\delta S_{t^+}\right)^2+O\left((\delta t)^{\frac{3}{2}}\sqrt{\log(\delta t)^{-1}}\right)\Big|$$
$$=\frac{(1-\lambda\delta t)kS_t}{2}E\Big|\frac{\partial\psi(t,S_t)}{\partial t}\delta t+\frac{\partial\psi(t,S_t)}{\partial S_t}\delta S_t+\frac{1}{2}\frac{\partial^2\psi(t,S_t)}{\partial S_t^2}(\delta S_t)^2$$
$$+O\left((\delta t)^{\frac{3}{2}}\sqrt{\log(\delta t)^{-1}}\right)\Big|+\frac{(\lambda\delta t)kS_t}{2}\Big|\psi\left(t,S_{t^+}\right)-\psi(t,S_t)+\frac{\partial\psi\left(t,S_{t^+}\right)}{\partial t}\delta t$$
$$+\frac{\partial\psi\left(t,S_{t^+}\right)}{\partial S_{t^+}}\delta\left(S_{t^+}\right)+\frac{1}{2}\frac{\partial^2\psi\left(t,S_{t^+}\right)}{\partial S_{t^+}^2}\left(\delta S_{t^+}\right)^2+O\left((\delta t)^{\frac{3}{2}}\sqrt{\log(\delta t)^{-1}}\right)\Big|$$
$$\approx\frac{kS_t^2}{2}\left|\frac{\partial\psi}{\partial S_{t^+}}\right|E\left|\sigma\delta B(t)+\sigma_H\delta B_H(t)\right|+\frac{k\lambda S_t\delta t}{2}\left|\psi\left(t,S_{t^+}\right)-\psi(t,S_t)\right|+O(\delta t)\quad(4.17)$$
$$=\frac{kS_t^2}{2}\left|\frac{\partial\psi}{\partial S_t}\right|\sqrt{\frac{2}{\pi}\left(\sigma^2\delta t+\sigma_H^2(\delta t)^{2H}\right)}+\frac{k\lambda S_t\delta t}{2}\left|\psi\left(t,S_{t^+}\right)-\psi(t,S_t)\right|+O(\delta t),$$

and $\psi=\frac{\partial C}{\partial S_t}$, from Equations (4.1) - (4.17), we can get

$$\Big[\frac{\partial C}{\partial t}+rS_t\frac{\partial C}{\partial S_t}+\frac{S_t^2}{2}\left(\sigma^2+\sigma_H^2(\delta t)^{2H-1}\right)\frac{\partial^2 C}{\partial S_t^2}$$
$$-rC+\lambda E\left[C(t,J_tS_t)-C(t,S_t)\right]-\lambda E[J-1]S_t\frac{\partial C}{\partial S_t}\quad(4.18)$$
$$+\frac{kS_t^2}{2}\sqrt{\frac{2}{\pi}\left(\frac{\sigma^2}{\delta t}+\sigma_H^2(\delta t)^{2H-2}\right)}\left|\frac{\partial^2 C}{\partial S_t^2}\right|\Big]\delta t+O(\delta t)=0.$$

Hence, we assume that

$$\begin{aligned}&\frac{\partial C}{\partial t}+rS_t\frac{\partial C}{\partial S_t}+\frac{S_t^2}{2}\left(\sigma^2+\sigma_H^2(\delta t)^{2H-1}\right)\frac{\partial^2 C}{\partial S_t^2}\\&-rC+\lambda E\left[C(t,J_tS_t)-C(t,S_t)\right]-\lambda E[J-1]S_t\frac{\partial C}{\partial S_t}\\&+\frac{kS_t^2}{2}\sqrt{\frac{2}{\pi}\left(\frac{\sigma^2}{\delta t}+\sigma_H^2(\delta t)^{2H-2}\right)}\left|\frac{\partial^2 C}{\partial S_t^2}\right|=0.\end{aligned}\tag{4.19}$$

Note that the term $\frac{kS_t^2}{2}\sqrt{\frac{2}{\pi}\left(\frac{\sigma^2}{\delta t}+\sigma_H^2(\delta t)^{2H-2}\right)}$ is nonlinear, except when $\Gamma=\frac{\partial^2 C}{\partial S_t^2}$ does not change sign for all S_t. Since it represents the degree of mishedging of the portfolio, it is not surprising to observe that Γ is involved in the transaction cost term. We may rewrite Equation (4.19) in the form which resembles the Merton equation:

$$\begin{aligned}&\frac{\partial C}{\partial t}+rS_t\frac{\partial C}{\partial S_t}+\frac{S_t^2\hat{\sigma}^2}{2}\frac{\partial^2 C}{\partial S_t^2}-rC+\lambda E\left[C(t,J_tS_t)-C(t,S_t)\right]\\&-\lambda E[J-1]S_t\frac{\partial C}{\partial S_t}=0.\end{aligned}\tag{4.20}$$

where $E[J-1]=\mathrm{e}^{\mu_J+\frac{\sigma_J^2}{2}}-1$ and the implied volatility is given by

$$\hat{\sigma}^2=\sigma^2+\sigma_H^2(\delta t)^{2H-1}+k\sqrt{\frac{2}{\pi}\left(\frac{\sigma^2}{\delta t}+\sigma_H^2(\delta t)^{2H-2}\right)}sign(\Gamma).\tag{4.21}$$

If $\hat{\sigma}^2$, Equation (4.20) becomes mathematically ill-posed. This occurs when $\Gamma<0$ and $\delta t\to 0$. However, it is known that Γ is always positive for the simple European call and put options in the absence of transaction costs. If we postulate the same sign behaviour for Γ in the presence of transaction costs, Equation (4.20) becomes linear under such an assumption so that the Merton formula becomes applicable except that the modified volatility $\hat{\sigma}$ should be used as the volatility parameter. Moreover, if $\Gamma>0$ from Equation (4.20) we obtain

$$C(t,S_t)=\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\lambda'(T-t)}\left(\lambda'(T-t)\right)^n}{n!}\left[S_t\phi(d_1)-K\mathrm{e}^{-r(T-t)}\phi(d_2)\right],$$

where

$$d_1=\frac{\ln\left(\frac{S_t}{K}\right)+r_n(T-t)+\frac{\sigma_n^2}{2}(T-t)}{\sigma_n\sqrt{T-t}},\quad d_2=d_1-\sigma_n\sqrt{T-t},$$

$$\lambda'=\lambda E(J)=\lambda\mathrm{e}^{\mu_J+\frac{\sigma_J^2}{2}},\quad \sigma_n^2=\hat{\sigma}^2+\frac{n\sigma_J^2}{T-t},$$

$$r_n=r-\lambda E(J-1)+\frac{n\ln E(J)}{T-t}=r+\lambda\left(\mathrm{e}^{\mu_J+\frac{\sigma_J^2}{2}}-1\right)+\frac{n\left(\mu_J+\frac{\sigma_J^2}{2}\right)}{T-t},$$

$$\hat{\sigma}^2 = \sigma^2 + \sigma_H^2 (\delta t)^{2H-1} + k\sqrt{\frac{2}{\pi}\left(\frac{\sigma^2}{\delta t} + \sigma_H^2 (\delta t)^{2H-2}\right)} sign(\Gamma),$$

$sign(\Gamma)$ is the signum function of $\frac{\partial^2 C}{\partial S_t^2}$, δt is a small and fixed time-step, k is the transaction costs and $\phi(.)$ is the cumulative normal distribution.