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Rafał Kapica and Janusz Morawiec

Abstract. Given measure spaces $(\Omega_1, \mathcal{A}_1, \mu_1), \ldots, (\Omega_N, \mathcal{A}_N, \mu_N)$, functions $\varphi_1 \colon \mathbb{R}^m \times \Omega_1 \to \mathbb{R}^m, \ldots, \varphi_N \colon \mathbb{R}^m \times \Omega_N \to \mathbb{R}^m$ and $g \colon \mathbb{R}^m \to \mathbb{R}$, we present results on the existence of solutions $f \colon \mathbb{R}^m \to \mathbb{R}$ of the inhomogeneous poly-scale refinement type equation of the form

$$f(x) = \sum_{n=1}^{N} \int_{\Omega_n} \big|\det(\varphi_n)'_x(x, \omega_n)\big| f\big(\varphi_n(x, \omega_n)\big) \, d\mu_n(\omega_n) + g(x)$$

in some special classes of functions. The results are obtained by Banach fixed point theorem applied to a perturbed Markov operator connected with the considered inhomogeneous poly-scale refinement type equation.

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1. Introduction

Fix $m, N \in \mathbb{N}$, measure spaces $(\Omega_1, \mathcal{A}_1, \mu_1), \ldots, (\Omega_N, \mathcal{A}_N, \mu_N)$ and functions $\varphi_1 \colon \mathbb{R}^m \times \Omega_1 \to \mathbb{R}^m, \ldots, \varphi_N \colon \mathbb{R}^m \times \Omega_N \to \mathbb{R}^m, g \colon \mathbb{R}^m \to \mathbb{R}$. We are interested in solutions $f \colon \mathbb{R}^m \to \mathbb{R}$ of the inhomogeneous poly-scale refinement type equation

$$f(x) = \sum_{n=1}^{N} \int_{\Omega_n} \big|\det(\varphi_n)'_x(x, \omega_n)\big| f\big(\varphi_n(x, \omega_n)\big) \, d\mu_n(\omega_n) + g(x) \qquad (1.1)$$

in the space of all pth power Lebesgue integrable functions as well as in the space of all continuous and bounded functions. If for every $n \in \{1, \ldots, N\}$

the function $\varphi_n \colon \mathbb{R}^m \times \Omega_n \to \mathbb{R}^m$ is of the form

$$\varphi_n(x, \omega_n) = K_n(\omega_n)x - M_n(\omega_n) \tag{1.2}$$

with given functions

$$K_n \colon \Omega_n \to \mathbb{R}^{m \times m} \quad \text{and} \quad M_n \colon \Omega_n \to \mathbb{R}^m,$$

then equation (1.1) becomes the inhomogeneous poly-scale refinement type equation

$$f(x) = \sum_{n=1}^{N} \int_{\Omega_n} \big| \det K_n(\omega_n) \big| f\big(K_n(\omega_n)x - M_n(\omega_n)\big) \, d\mu_n(\omega_n) + g(x). \tag{1.3}$$

Taking $g = 0$ in (1.3) we obtain the following homogeneous poly-scale refinement type equation

$$f(x) = \sum_{n=1}^{N} \int_{\Omega_n} \big| \det K_n(\omega_n) \big| f\big(K_n(\omega_n)x - M_n(\omega_n)\big) \, d\mu_n(\omega_n).$$

This equation extends the discrete poly-scale refinement equation which has been studied in [5, 26, 27]. If $N = 1$, then equation (1.3) reduces to the inhomogeneous refinement type equation

$$f(x) = \int_{\Omega} \big| \det K(\omega) \big| f\big(K(\omega)x - M(\omega)\big) \, d\mu(\omega) + g(x). \tag{1.4}$$

Inhomogeneous refinement equations are motivated by constructions of multiwavelets, multichannel filters and constructions of wavelets on a finite interval (see, e.g., [4, 10, 23]). Various inhomogeneous forms of equation (1.4) have been investigated in [6, 12, 24, 25].

Several problems from different areas of pure and applied mathematics lead to the problem of the existence of nontrivial Lebesgue integrable solutions of refinement type equation (1.4) with $g = 0$, i.e., the homogeneous refinement type equation

$$f(x) = \int_{\Omega} \big| \det K(\omega) \big| f\big(K(\omega)x - M(\omega)\big) \, d\mu(\omega) \tag{1.5}$$

(for more details see the survey [15] and the references therein). It turns out that in some applications, continuous and bounded or continuous and compactly supported solutions of homogeneous refinement equations are important. Such solutions have significant applications in wavelet theory, approximation theory, theory of subdivision schemes, computer graphics, physics, combinatorial number theory and many others (see, e.g., [2, 7, 8, 9, 18, 19, 22]).

From the point of view of applications, all results on the existence of "good" solutions of homogeneous as well as of inhomogeneous refinement equations are very important (see [3] where it is showed how nonexistence of "good" solutions of a refinement equation can lead to anomalous behavior of numerical methods for a construction of wavelets). Let us note that refinement equations always have plenty of "bad" solutions, even extremely strange (see, e.g., [1, 15, 20]).

To find conditions under which homogeneous refinement type equation (1.5) has a nontrivial Lebesgue integrable solution is rather difficult; some results in this direction can be found in [14, 21]. To the best of our knowledge, there is no result concerning the existence of a nontrivial Lebesgue integrable solution of the homogeneous poly-scale refinement type equation

$$f(x) = \sum_{n=1}^{N} \int_{\Omega_n} \big| \det(\varphi_n)'_x(x, \omega_n) \big| f\big(\varphi_n(x, \omega_n) \big) \, d\mu_n(\omega_n). \qquad (1.6)$$

Nevertheless, it is possible to formulate conditions under which homogeneous poly-scale refinement type equation (1.6) has no nontrivial Lebesgue integrable solution (see [13] for the case where $N = 1$). Therefore, if homogeneous poly-scale refinement type equation (1.6) has no nontrivial Lebesgue integrable solution, we can ask if its inhomogeneous counterpart (1.1), obtained by adding to the right-hand side of (1.6) a perturbation function g, has such a solution.

If $\mathcal{F}$ is a given class of functions, then the existence of a solution $f \in \mathcal{F}$ of inhomogeneous poly-scale refinement type equation (1.1) is a consequence of the existence of a fixed point of the operator $P \colon \mathcal{F} \to \mathcal{F}$ given by

$$Pf = \sum_{n=1}^{N} \int_{\Omega_n} \big| \det(\varphi_n)'_x(\cdot, \omega_n) \big| f\big(\varphi_n(\cdot, \omega_n) \big) \, d\mu_n(\omega_n) + g. \qquad (1.7)$$

It turns out that the operator P happens (under suitable assumptions) to be the Markov operator in the case where $g = 0$; for the definition of the Markov operator and more information on it see [17]. Asymptotically stability of Markov operators has been explored in [21] to study the problem of the existence of nontrivial Lebesgue integrable solutions of homogeneous refinement type equation (1.5). In this paper, we are going to examine the Banach fixed point theorem to obtain results on the existence of a pth power Lebesgue integrable solution as well as a continuous and bounded solution of equation (1.1) and of its special case (1.3). Thus we are looking for conditions, on the spaces $(\Omega_1, \mathcal{A}_1, \mu_1), \ldots, (\Omega_N, \mathcal{A}_N, \mu_N)$ and the given functions $\varphi_1, \ldots, \varphi_N$ and g, guaranteeing that the operator P is well defined and satisfies assumptions of the Banach fixed point theorem.

2. Notation

Given a real number $p \geq 1$, we write L^p to denote the Banach space of all pth power Lebesgue measurable functions $f \colon \mathbb{R}^m \to \mathbb{R}$ with the standard norm

$$\|f\|_p = \left(\int_{\mathbb{R}^m} |f(x)|^p \, dx \right)^{1/p}.$$

If $f \in L^p$, then $f = f_+ - f_-$, where f_+ and f_- are the positive and negative parts of f; i.e., $f_+ = \max\{f, 0\}$ and $f_- = \max\{-, 0\}$. We say that a point $x \in \mathbb{R}^m$ is in the support of $f \in L^p$ if for every neighborhood $U_x \subset \mathbb{R}^m$ of x

we have
$$l_m\big(\{y \in U_x : f(y) \neq 0\}\big) > 0;$$
here and throughout, l_m denotes the Lebesgue measure on $\mathbb{R}^m$. Given $f \in L^p$, it is easy to see that the set
$$\operatorname{supp} f = \{x \in \mathbb{R}^m : x \text{ is in the support of } f\}$$
is closed; it is called the support of f and if it is compact we say that f is compactly supported.

We write C_B to denote the Banach space of all continuous and bounded functions $f\colon \mathbb{R}^m \to \mathbb{R}$ with the supremum norm
$$\|f\|_{\sup} = \sup\big\{|f(x)| : x \in \mathbb{R}^m\big\}.$$
Since $\mathbb{R}^m$ is a noncompact space, we can consider the subspace $C_C \subset C_B$ of all functions $f\colon \mathbb{R}^m \to \mathbb{R}$ with compact support; in contrast to the definition of the support of pth power Lebesgue measurable functions, we say that a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ has a compact support if the set
$$\operatorname{supp} f = \operatorname{cl}\{x \in \mathbb{R}^m : f(x) \neq 0\},$$
called the support of f, is compact. The space C_C endowed in the supremum norm is not, in general, complete. The completion of C_C is the space C_0 consisting of all those continuous functions $f\colon \mathbb{R}^m \to \mathbb{R}$ that vanish at infinity; i.e., $f \in C_0$ if and only if $f \in C_B$ and the set $\operatorname{cl}\{x \in \mathbb{R}^m : |f(x)| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$. It is easy to prove that $f \in C_0$ if and only if f is continuous and $\lim_{\|x\| \to +\infty} f(x) = 0$, where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^m$.

The symbol $\mathcal{B}$ will denote the family of all Borel subsets of $\mathbb{R}^m$.

From now on we assume that $(\Omega_1, \mathcal{A}_1, \mu_1), \ldots, (\Omega_N, \mathcal{A}_N, \mu_N)$ are complete finite measure spaces, and P denotes the operator given by (1.7).

3. L^1-solutions

Throughout this section we assume that $g \in L^1$.

As it was mentioned earlier, our aim is to use the Banach fixed point theorem to the operator P defined by (1.7). For this purpose we need to know that P transforms the space L^1 into itself. Before we give conditions on φ_n's under which $P(L^1) \subset L^1$, we recall that $f \in L^1$ is called an L^1-solution of equation (1.1), if every representative of f satisfies (1.1) for almost all $x \in \mathbb{R}^m$ with respect to l_m.

Lemma 3.1. *Assume that for every $n \in \{1, \ldots, N\}$ the function*
$$\varphi_n \colon \mathbb{R}^m \times \Omega_n \to \mathbb{R}^m$$
satisfies the following conditions:

(3.1) *$\varphi_n(\cdot, \omega_n)$ is a diffeomorphism from $\mathbb{R}^m$ onto $\mathbb{R}^m$ for every $\omega_n \in \Omega_n$;*
(3.2) *$\varphi_n(x, \cdot)$ is $\mathcal{A}_n$-measurable for every $x \in \mathbb{R}^m$;*
(3.3) *$(l_m \otimes \mu_n)(\varphi_n^{-1}(B)) = 0$ for every Borel set $B \subset \mathbb{R}^m$ with $l_m(B) = 0$.*

Then

(i) *if $f \colon \mathbb{R}^m \to \mathbb{R}$ is a Lebesgue integrable function, then so is the function*

$$\mathbb{R}^m \ni x \longmapsto \sum_{n=1}^{N} \int_{\Omega_n} \big|\det(\varphi_n)'_x(x, \omega_n)\big| f\big(\varphi_n(x, \omega_n)\big) \, d\mu_n(\omega_n); \qquad (3.4)$$

(ii) *if $f, \tilde{f} \colon \mathbb{R}^m \to \mathbb{R}$ are Lebesgue integrable functions such that $f(x) = \tilde{f}(x)$ for almost all $x \in \mathbb{R}^m$ and f satisfies equation (1.1) for almost all $x \in \mathbb{R}^m$, then also $\tilde{f}$ satisfies equation (1.1) for almost all $x \in \mathbb{R}^m$;*

(iii) *we have $P(L^1) \subset L^1$.*

Proof. (i) Fix $n \in \{1, \dots, N\}$.

Conditions (3.1) and (3.2) imply that both functions φ_n and $\det(\varphi_n)'_x$ are $\mathcal{B} \otimes \mathcal{A}_n$-measurable (see [11] or [16]).

Fix a Lebesgue integrable function $f \colon \mathbb{R}^m \to [0, +\infty)$ and a set $B \in \mathcal{B}$. From condition (3.3) we obtain that the set $(f \circ \varphi_n)^{-1}(B)$ belongs to the completion $\overline{\mathcal{B} \otimes \mathcal{A}_n}$ of $\mathcal{B} \otimes \mathcal{A}_n$. Consequently, the function $|\det(\varphi_n)'_x|(f \circ \varphi_n)$ is $\overline{\mathcal{L}_m \otimes \mathcal{A}_n}$-measurable and

$$\int_{\mathbb{R}^m \times \Omega_n} \big|\det(\varphi_n)'_x(x, \omega_n)\big| f\big(\varphi_n(x, \omega_n)\big) \, d\left(\overline{l_m \otimes \mu_n}\right)(x, \omega_n)$$

$$= \mu_n(\Omega_n) \int_{\mathbb{R}^m} f(x) \, dx = \mu_n(\Omega_n) \|f\|_1 < +\infty.$$

Since $\mathcal{A}_n$ is complete, we conclude from the Fubini theorem that the function

$$\big|\det(\varphi_n)'_x(x, \cdot)\big|(f \circ \varphi_n)(x, \cdot)$$

is μ_n-integrable for almost all $x \in \mathbb{R}^m$ and that the function

$$\int_{\Omega_n} \big|\det(\varphi_n)'_x(\cdot, \omega_n)\big| f\big(\varphi_n(\cdot, \omega_n)\big) \, d\mu_n(\omega_n)$$

is Lebesgue integrable.

Fix now an arbitrary Lebesgue integrable function $f \colon \mathbb{R}^m \to \mathbb{R}$. Since both the functions f_+ and f_- are nonnegative and Lebesgue integrable, we conclude that both the functions

$$\int_{\Omega_n} \big|\det(\varphi_n)'_x(\cdot, \omega_n)\big| f_+\big(\varphi_n(\cdot, \omega_n)\big) \, d\mu_n(\omega_n)$$

and

$$\int_{\Omega_n} \big|\det(\varphi_n)'_x(\cdot, \omega_n)\big| f_-\big(\varphi_n(\cdot, \omega_n)\big) \, d\mu_n(\omega_n)$$

are Lebesgue integrable. In consequence, the function

$$\int_{\Omega_n} \big|\det(\varphi_n)'_x(\cdot, \omega_n)\big| f\big(\varphi_n(\cdot, \omega_n)\big) \, d\mu_n(\omega_n)$$

is also Lebesgue integrable.

Finally, the function given by (3.4) is Lebesgue integrable, because it is a finite sum of Lebesgue integrable functions.

(ii) Fix two Lebesgue integrable functions $f, \tilde{f} \colon \mathbb{R}^m \to \mathbb{R}$ and assume that there exists a set $B \in \mathcal{B}$ with $l_m(B) = 0$ such that $f(x) = \tilde{f}(x)$ for every $x \notin B$. Since the right-hand side of equation (1.1) is a Lebesgue integrable function of variable x by assertion (i), we can assume that (1.1) holds for every $x \notin A$ with $l_m(A) = 0$. By (3.3) for every $n \in \{1, \ldots, N\}$ we have

$$0 = (l_m \otimes \mu_n)\big(\varphi_n^{-1}(B)\big) = \int_{\mathbb{R}^m} \mu\big(\varphi_n^{-1}(B)_x\big)\, dx.$$

Then for every $n \in \{1, \ldots, N\}$ there exists a Lebesgue measurable set C_n such that $l_m(C_n) = 0$ and $\mu_n(\varphi_n^{-1}(B)_x) = 0$ for every $x \notin C_n$. Hence, if

$$x \notin A \cup B \cup \bigcup_{n=1}^{N} C_n,$$

then

$$\tilde{f}(x) = f(x)$$
$$= \sum_{n=1}^{N} \int_{\Omega_n} \big|\det(\varphi_n)_x'(x, \omega_n)\big| f\big(\varphi_n(x, \omega_n)\big)\, d\mu_n(\omega_n) + g(x)$$
$$= \sum_{n=1}^{N} \int_{\Omega_n \setminus \varphi_n^{-1}(B)_x} \big|\det(\varphi_n)_x'(x, \omega_n)\big| f\big(\varphi_n(x, \omega_n)\big)\, d\mu_n(\omega_n) + g(x)$$
$$= \int_{\Omega_n \setminus \varphi_n^{-1}(B)_x} \big|\det(\varphi_n)_x'(x, \omega_n)\big| \tilde{f}\big(\varphi_n(x, \omega_n)\big)\, d\mu_n(\omega_n) + g(x)$$
$$= \sum_{n=1}^{N} \int_{\Omega_n} \big|\det(\varphi_n)_x'(x, \omega_n)\big| \tilde{f}\big(\varphi_n(x, \omega_n)\big)\, d\mu_n(\omega_n) + g(x);$$

i.e., $\tilde{f}$ satisfies (1.1) for almost all $x \in \mathbb{R}^m$.

(iii) Fix $f \in L^1$ and choose two representatives f_1 and f_2 of f. By assertion (i) we infer that both the functions Pf_1 and Pf_2 are Lebesgue integrable. Now by the same arguments as in the proof of assertion (ii) we conclude that $Pf_1(x) = Pf_2(x)$ for almost all $x \in \mathbb{R}^m$. $\square$

Observe that assertion (ii) of the above lemma says that the definition of L^1-solutions of equation (1.1) is well posed.

In the case where the functions φ_n's are of the form (1.2) we do not need any special condition guaranteeing that $P \colon L^1 \to L^1$. More precisely, we have the following observation.

Remark 3.2. Let $K_n \colon \Omega_n \to \mathbb{R}^{m \times m}$ and $M_n \colon \Omega_n \to \mathbb{R}^m$ be Lebesgue measurable functions. If $\det K_n(\omega) \neq 0$ for every $\omega_n \in \Omega_n$, then the function $\varphi_n \colon \mathbb{R}^m \times \Omega_n \to \mathbb{R}^m$ given by (1.2) satisfies conditions (3.1)–(3.3).

Now we are in a position to formulate our first result on the existence of a unique L^1-solution of equation (1.1) in the space L^1.

Theorem 3.3. *Assume that for every $n \in \{1, \ldots, N\}$ the function*

$$\varphi_n \colon \mathbb{R}^m \times \Omega_n \to \mathbb{R}^m$$

satisfies conditions (3.1)–(3.3). If $\sum_{n=1}^{N} \mu(\Omega_n) < 1$, then equation (1.1) has exactly one L^1-solution f and for every $f_0 \in L^1$ the sequence $(P^n f_0)_{n\in\mathbb{N}}$ converges to f in L^1. Moreover,

(i) if g is of constant sign, then so is f;
(ii) if $\int_{\mathbb{R}^m} g(x)\,dx = 0$, then $\int_{\mathbb{R}^m} f(x)\,dx = 0$;
(iii) if there exists a closed set $Z \subset \mathbb{R}^m$ such that

$$\varphi_n(Z \times \Omega_n) \subset Z \quad \text{for every } n \in \{1, \dots, N\} \tag{3.5}$$

and if $\operatorname{supp} g \subset Z$, then $\operatorname{supp} f \subset Z$; in particular, if Z is compact, then f is compactly supported.

Proof. By Lemma 3.1 we have $P\colon L^1 \to L^1$. Moreover, for all $f_1, f_2 \in L^1$ we obtain

$$\|Pf_1 - Pf_2\|_1 \leq \sum_{n=1}^{N} \mu_n(\Omega_n)\|f_1 - f_2\|_1.$$

The Banach fixed point theorem completes the proof of the main part of the theorem.

(i) Clearly, both the sets

$$L_+^1 = \{h \in L^1 : h \geq 0\} \quad \text{and} \quad L_-^1 = \{h \in L^1 : h \leq 0\}$$

are closed subspaces of L^1. Now it is enough to observe that $P\colon L_+^1 \to L_+^1$ and $P\colon L_-^1 \to L_-^1$.

(ii) Observe first that the set $\{h \in L^1 : \int_{\mathbb{R}^m} h(x)dx = 0\}$ is a closed subspace of L^1.

Taking the Fourier transform of both sides of equation (1.1), we obtain

$$\widehat{f}(y) = \int_{\mathbb{R}^m} e^{ixy} f(x)\,dx$$

$$= \sum_{n=1}^{N} \int_{\Omega_n} \int_{\mathbb{R}^m} e^{ixy} \big|\det(\varphi_n)_x'(x, \omega_n)\big| f\big(\varphi_n(x, \omega_n)\big)\,dx\,d\mu_n(\omega_n) + \widehat{g}(y)$$

for every $y \in \mathbb{R}$. Hence for $y = 0$ we get

$$\left[1 - \sum_{n=1}^{N} \mu_n(\Omega_n)\right] \widehat{f}(0) = \widehat{g}(0) = \int_{\mathbb{R}^m} g(x)\,dx = 0,$$

and in consequence,

$$\int_{\mathbb{R}^m} f(x)\,dx = \widehat{f}(0) = 0.$$

(iii) Since Z is closed, it follows that

$$X = \{h \in L^1 : \operatorname{supp} h \subset Z\}$$

is a closed subspace of L^1.

Fix $h \in X$. By (3.5) we have $\operatorname{supp}(h \circ \varphi_n) \subset Z$ for every $n \in \{1, \dots, N\}$. This jointly with $\operatorname{supp} g \subset Z$ implies $\operatorname{supp} Ph \subset Z$ and, in consequence, we obtain $P\colon X \to X$. $\qquad\square$

Corollary 3.4. *Assume that for every $n \in \{1, \ldots, N\}$ the functions*

$$K_n \colon \Omega_n \to \mathbb{R}^{m \times m} \quad and \quad M_n \colon \Omega_n \to \mathbb{R}^m$$

are Lebesgue measurable. If $\sum_{n=1}^{N} \mu(K_n \neq 0) < 1$, then equation (1.3) has exactly one L^1-solution f and for every $f_0 \in L^1$ the sequence $(P^n f_0)_{n \in \mathbb{N}}$ converges to f in L^1. Moreover,

(i) *if g is of constant sign, then so is f;*
(ii) *if $\int_{\mathbb{R}^m} g(x)\,dx = 0$, then $\int_{\mathbb{R}^m} f(x)\,dx = 0$;*
(ii) *if there exists a closed set $Z \subset \mathbb{R}^m$ such that*

$$K(\omega_n)Z \subset Z + M(\omega_n) \quad for\ all\ \omega_n \in \Omega_n\ and\ n \in \{1, \ldots, N\} \tag{3.6}$$

and if $\operatorname{supp} g \subset Z$, then $\operatorname{supp} f \subset Z$; in particular, if Z is compact, then f is compactly supported.

Proof. Putting

$$\widetilde{\Omega}_n = \{\omega_n \in \Omega_n : \det K_n(\omega_n) \neq 0\}$$

for every $n \in \{1, \ldots, N\}$ we get $\sum_{n=1}^{N} \mu_n(\widetilde{\Omega}_n) < 1$. Now it is enough to apply Remark 3.2 and Theorem 3.3. □

In the case where $g = 0$, Theorem 3.3 says that the trivial function is the only L^1-solution of equation (1.6). To see it in another way suppose that, on the contrary, f is a nontrivial L^1-solution of equation (1.6). Then

$$\|f\|_1 \leq \sum_{n=1}^{N} \int_{\Omega_n} \int_{\mathbb{R}^m} \left| \det(\varphi_n)'_x(x, \omega_n) f\big(\varphi_n(x, \omega_n)\big) \right| dx\,d\mu_n(\omega_n)$$

$$= \|f\|_1 \sum_{n=1}^{N} \mu_n(\Omega_n) < \|f\|_1,$$

which is impossible.

4. L^p-solutions

Throughout this section we assume that $g \in L^p$ with $p > 1$.

Recall that $f \in L^p$ is called an L^p-solution of equation (1.1), if every representative of f satisfies (1.1) for almost all $x \in \mathbb{R}^m$ with respect to l_m.

We begin with a counterpart of Lemma 3.1 for the space L^p.

Lemma 4.1. *Assume that for every $n \in \{1, \ldots, N\}$ the function*

$$\varphi_n \colon \mathbb{R}^m \times \Omega_n \to \mathbb{R}^m$$

satisfies conditions (3.1)–(3.3) and

$$\sup_{x \in \mathbb{R}^m} \left| \det(\varphi_n)'_x(x, \cdot) \right|^{p-1} \leq (\Phi_n)^p \quad with\ \Phi_n \in L^1(\Omega_n). \tag{4.1}$$

Then

(i) *if $f \colon \mathbb{R}^m \to \mathbb{R}$ is a pth power Lebesgue integrable function, then so is the function given by formula (3.4);*

(ii) *if* $f, \tilde{f} \colon \mathbb{R}^m \to \mathbb{R}$ *are pth power Lebesgue integrable functions such that* $f(x) = \tilde{f}(x)$ *for almost all* $x \in \mathbb{R}^m$ *and* f *satisfies equation* (1.1) *for almost all* $x \in \mathbb{R}^m$, *then also* $\tilde{f}$ *satisfies equation* (1.1) *for almost all* $x \in \mathbb{R}^m$;

(iii) *we have* $P(L^p) \subset L^p$.

Proof. (i) Fix $n \in \{1, \dots, N\}$ and pth power Lebesgue integrable function $f \colon \mathbb{R}^m \to \mathbb{R}$. In the same way as in the proof of assertion (i) of Lemma 3.1, conditions (3.1)–(3.3) imply that the function

$$\left| \det(\varphi_n)'_x \right| (f \circ \varphi_n)$$

is $\overline{\mathcal{L}_m \otimes \mathcal{A}_n}$-measurable. By the Minkowski inequality for integrals and (4.1) we have

$$\left(\int_{\mathbb{R}^m} \left[\int_{\Omega_n} \left| \det(\varphi_n)'_x(x, \omega_n) \right| f_+ \big(\varphi_n(x, \omega_n) \big) \, d\mu_n(\omega_n) \right]^p dx \right)^{1/p}$$

$$\leq \int_{\Omega_n} \left(\int_{\mathbb{R}^m} \left[\left| \det(\varphi_n)'_x(x, \omega_n) \right| f_+ \big(\varphi_n(x, \omega_n) \big) \right]^p dx \right)^{1/p} d\mu_n(\omega_n)$$

$$\leq \int_{\Omega_n} \left(\int_{\mathbb{R}^m} \left[\Phi_n(\omega_n) f_+(y) \right]^p dy \right)^{1/p} d\mu_n(\omega)$$

$$= \| \Phi_n \|_1 \| f_+ \|_p < +\infty,$$

which shows that the function

$$\int_{\Omega_n} \left| \det(\varphi_n)'_x(\cdot, \omega_n) \right| f_+ \big(\varphi_n(\cdot, \omega_n) \big) \, d\mu_n(\omega_n)$$

is pth power Lebesgue integrable. In a similar way we can prove that the function

$$\int_{\Omega_n} \left| \det(\varphi_n)'_x(\cdot, \omega_n) \right| f_- \big(\varphi_n(\cdot, \omega_n) \big) \, d\mu_n(\omega_n)$$

is pth power Lebesgue integrable. Consequently, the function

$$\int_{\Omega_n} \left| \det(\varphi_n)'_x(\cdot, \omega_n) \right| f \big(\varphi_n(\cdot, \omega_n) \big) \, d\mu_n(\omega_n)$$

is pth power Lebesgue integrable. Finally, the function given by (3.4) is pth power Lebesgue integrable, because it is a finite sum of pth power Lebesgue integrable functions.

Assertions (ii) and (iii) can be proved in the same manner as assertions (ii) and (iii) of Lemma 3.1. □

In the case where the functions φ_n's are of the form (1.2) we need a condition on the functions K_n's guaranteeing that (4.1) holds.

Remark 4.2. Let $K_n \colon \Omega_n \to \mathbb{R}^{m \times m}$ and $M_n \colon \Omega_n \to \mathbb{R}^m$ be Lebesgue measurable functions. If

$$(\det K_n)^{(p-1)/p} \in L^1(\Omega_n),$$

then the function $\varphi_n \colon \mathbb{R}^m \times \Omega_n \to \mathbb{R}^m$ given by (1.2) satisfies condition (4.1).

Now we can formulate a result on the existence of the unique L^p-solution of equation (1.1) in the space L^p.

Theorem 4.3. *Assume that for every $n \in \{1, \ldots, N\}$ the function*

$$\varphi_n \colon \mathbb{R}^m \times \Omega_n \to \mathbb{R}^m$$

satisfies conditions (3.1)–(3.3) and (4.1). If $\sum_{n=1}^{N} \|\Phi_n\|_1 < 1$, then equation (1.1) has exactly one L^p-solution f and for every $f_0 \in L^p$ the sequence $(P^n f_0)_{n \in \mathbb{N}}$ converges to f in L^p. Moreover,

  (i) *if g is of constant sign, then so is f;*
  (ii) *if there exists a closed set $Z \subset \mathbb{R}^m$ such that (3.5) holds and if $\operatorname{supp} g \subset Z$, then $\operatorname{supp} f \subset Z$; in particular, if Z is compact, then f is compactly supported.*

Proof. By Lemma 4.1 we have $P \colon L^p \to L^p$. Fix $f_1, f_2 \in L^p$. Then the Minkowski inequality for integrals and condition (4.1) imply that

$$\|Pf_1 - Pf_2\|_p$$

$$\leq \left(\int_{\mathbb{R}^m} \left[\sum_{n=1}^{N} \int_{\Omega_n} \left| \det(\varphi_n)'_x(x, \omega_n) \right. \right.$$
$$\left. \left. \times \left[f_1\big(\varphi_n(x, \omega_n)\big) - f_2\big(\varphi_n(x, \omega_n)\big) \right] \right| d\mu_n(\omega_n) \right]^p dx \right)^{1/p}$$

$$\leq \sum_{n=1}^{N} \left(\int_{\mathbb{R}^m} \left[\int_{\Omega_n} \left| \det(\varphi_n)'_x(x, \omega_n) \right. \right. \right.$$
$$\left. \left. \left. \times \left[f_1\big(\varphi_n(x, \omega_n)\big) - f_2\big(\varphi_n(x, \omega_n)\big) \right] \right| d\mu_n(\omega_n) \right]^p dx \right)^{1/p}$$

$$\leq \sum_{n=1}^{N} \int_{\Omega_n} \left(\int_{\mathbb{R}^m} \left| \det(\varphi_n)'_x(x, \omega_n) \right. \right.$$
$$\left. \left. \times \left[f_1\big(\varphi_n(x, \omega_n)\big) - f_2\big(\varphi_n(x, \omega_n)\big) \right] \right|^p dx \right)^{1/p} d\mu_n(\omega_n)$$

$$\leq \sum_{n=1}^{N} \int_{\Omega_n} \left(\int_{\mathbb{R}^m} \left[\Phi_n(\omega_n) \right]^p |f_1(y) - f_2(y)|^p dy \right)^{1/p} d\mu_n(\omega_n)$$

$$= \sum_{n=1}^{N} \|\Phi_n\|_1 \|f_1 - f_2\|_p.$$

The Banach fixed point theorem completes the proof of the main part of the theorem.

Assertions (i) and (ii) hold because all the sets

$$L_+^p = \{h \in L^p : h \geq 0\}, \ L_-^p = \{h \in L^p : h \leq 0\}, \ X = \{h \in L^p : \operatorname{supp} h \subset Z\}$$

are closed subspaces of L^p and $P(L_+^p) \subset L_+^p$, $P(L_-^p) \subset L_-^p$, $P(X) \subset X$. $\square$

An immediate consequence of Theorem 4.3 and Remarks 3.2 and 4.2 is the following corollary.

Corollary 4.4. *Assume that for every $n \in \{1, \dots, N\}$ the functions*

$$K_n \colon \Omega_n \to \mathbb{R}^{m \times m} \quad and \quad M_n \colon \Omega_n \to \mathbb{R}^m$$

are Lebesgue measurable with $(\det K_n)^{(p-1)/p} \in L^1(\Omega_n)$. If

$$\sum_{n=1}^{N} \left\| (\det K_n)^{(p-1)/p} \right\|_1 < 1,$$

then equation (1.3) has exactly one L^p-solution f and for every $f_0 \in L^1$ the sequence $(P^n f_0)_{n \in \mathbb{N}}$ converges to f in L^1. Moreover,

(i) *if g is of constant sign, then so is f;*
(ii) *if there exists a closed set $Z \subset \mathbb{R}^m$ such that (3.6) holds and if supp $g \subset Z$, then supp $f \subset Z$; in particular, if Z is compact, then f is compactly supported.*

5. Continuous and bounded solutions

Throughout this section we assume that $g \in C_B$.

In contrast to the previous section we recall that $f \in C_B$ is called solution of equation (1.1), if (1.1) holds for every $x \in \mathbb{R}^m$.

We begin with conditions on φ_n's under which $P \colon C_B \to C_B$.

Lemma 5.1. *Assume that for every $n \in \{1, \dots, N\}$ the function*

$$\varphi_n \colon \mathbb{R}^m \times \Omega_n \to \mathbb{R}^m$$

satisfies conditions (3.1)–(3.3) and

$$\sup_{x \in \mathbb{R}^m} \left| \det(\varphi_n)'_x(x, \cdot) \right| \leq \Psi_n \quad with \ \Psi_n \in L^1(\Omega_n). \tag{5.1}$$

Then $P(C_B) \subset C_B$.

Proof. Fix a function $f \in C_B$. Conditions (3.1)–(3.3) imply that the function $|\det(\varphi_n)'_x|(f \circ \varphi_n)$ is $\overline{\mathcal{L}_1 \otimes \mathcal{A}}$-measurable for every $n \in \{1, \dots, N\}$. Fix a sequence $(x_k)_{k \in \mathbb{N}}$ of reals convergent to a real number x. From (3.1) we have

$$\lim_{k \to \infty} \det(\varphi_n)'_x(x_k, \omega_n) f\big(\varphi_n(x_k, \omega_n)\big) = \det(\varphi_n)'_x(x, \omega_n) f\big(\varphi_n(x, \omega_n)\big),$$

and by (5.1) we obtain

$$\left| \det(\varphi_n)'_x(x_k, \omega_n) f\big(\varphi_n(x_k, \omega_n)\big) \right| \leq \|f\|_{\sup} \Psi_n(\omega_n)$$

for all $\omega_n \in \Omega_n$ and $n \in \{1, \dots, N\}$. Hence

$$|Pf(y)| \leq \|f\|_{\sup} \sum_{n=1}^{N} \|\Psi_n\|_1 + \|g\|_{\sup}$$

for every $y \in \mathbb{R}^m$ and the Lebesgue dominated convergence theorem implies that $\lim_{k \to \infty} Pf(x_k) = Pf(x)$. In consequence, $Pf \in C_B$. $\qquad\square$

In the case where the functions φ_n's are of the form (1.2), we need a condition on the functions K_n's guaranteeing that (5.1) holds.

Remark 5.2. Let $K_n\colon \Omega_n \to \mathbb{R}^{m\times m}$ and $M_n\colon \Omega_n \to \mathbb{R}^m$ be Lebesgue measurable functions. If $\det K_n \in L^1(\Omega_n)$, then the function $\varphi_n\colon \mathbb{R}^m \times \Omega_n \to \mathbb{R}^m$ given by (1.2) satisfies condition (5.1).

Now we can formulate a result on the existence of a unique solution of equation (1.1) in the space C_B.

Theorem 5.3. *Assume that for every $n \in \{1,\ldots,N\}$ the function*

$$\varphi_n\colon \mathbb{R}^m \times \Omega_n \to \mathbb{R}^m$$

satisfies conditions (3.1)–(3.3) and (5.1). If $\sum_{n=1}^N \|\Psi_n\|_1 < 1$, then equation (1.1) has exactly one solution $f \in C_B$ and for every $f_0 \in C_B$ the sequence $(P^n f_0)_{n\in\mathbb{N}}$ converges to f in the supremum norm. Moreover,

   (i) *if g is of constant sign, then so is f;*
  (ii) *if $g \in C_0$, then $f \in C_0$;*
 (iii) *if there exists a closed set $Z \subset \mathbb{R}^m$ such that (3.5) holds and if $\operatorname{supp} g \subset Z$, then $\operatorname{supp} f \subset Z$; in particular, if Z is compact, then f is compactly supported.*

Proof. By Lemma 5.1 we have $P\colon C_B \to C_B$. Fix $f_1, f_2 \in L^p$. Then by (5.2) we get

$$\|Pf_1 - Pf_2\|_{\sup} \leq \sum_{n=1}^N \|\Psi_n\|_1 \|f_1 - f_2\|_{\sup}.$$

The Banach fixed point theorem completes the proof of the main part of the theorem.

Assertions (i)–(iii) hold because all the sets

$$C_B^+ = \{h \in C_B : h \geq 0\},\ C_B^- = \{h \in C_B : h \leq 0\},\ C_0,$$
$$X = \{h \in C_B : \operatorname{supp} h \subset Z\}$$

are closed subspaces of C_B and $P(C_B^+) \subset C_B^+$, $P(C_B^-) \subset C_B^-$, $P(C_0) \subset C_0$, $P(X) \subset X$. To prove that $P(C_0) \subset C_0$ take $f \in C_0$. By (3.1) we have

$$(f \circ \varphi_n)(\cdot, \omega_n) \in C_0$$

for every $n \in \{1,\ldots,N\}$. This jointly with (5.1) and the Lebesgue dominated convergence theorem gives

$$\lim_{\|x\|\to+\infty} \sum_{n=1}^N \int_{\Omega_n} \Psi_n(\omega_n) \big| f\big(\varphi_n(x,\omega_n)\big)\big|\, d\mu_n(\omega_n) = 0.$$

Finally, since $g \in C_0$, it follows that $\lim_{\|x\|\to+\infty} |Pf(x)| = 0$. □

Theorem 5.3 can be easily reformulated to get the existence and uniqueness of solutions of equation (1.1) in the space L^∞.

We end this paper with the following immediate consequence of Theorem 5.3 and Remarks 3.2 and 5.2.

Corollary 5.4. *Assume that for every $n \in \{1, \ldots, N\}$ the functions*

$$K_n \colon \Omega_n \to \mathbb{R}^{m \times m} \quad and \quad M_n \colon \Omega_n \to \mathbb{R}^m$$

are Lebesgue measurable with $\det K_n \in L^1(\Omega_n)$. If $\sum_{n=1}^{N} \|\det K_n\|_1 < 1$, then equation (1.3) has exactly one solution $f \in C_B$ and for every $f_0 \in C_B$ the sequence $(P^n f_0)_{n \in \mathbb{N}}$ converges to f in the supremum norm. Moreover,

(i) *if g is of constant sign, then so is f;*
(ii) *if $g \in C_0$, then $f \in C_0$;*
(iii) *if there exists a closed set $Z \subset \mathbb{R}^m$ such that (3.6) holds and if $\operatorname{supp} g \subset Z$, then $\operatorname{supp} f \subset Z$; in particular, if Z is compact, then f is compactly supported.*

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