Approximation Algorithms for Distance Independent Set and Induced Matching Problems

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Contents

Abstract

This thesis deals with the following two problems, the Maximum Distance-d Independent Set problem (MaxDdIS for short) and the Maximum Induced Matching problem (MaxIM for short), where $d \geq 3$. We design some approximation algorithms to solve MaxDdIS and MaxIM.

(1) We first study MaxDdIS. Our main results for MaxDdIS are as follows: (i) It is NP-hard to approximate MaxD3IS on 3-regular graphs within 1.00105 unless P=NP. (ii) For every fixed integers $d \geq 3$ and $r \geq 3$, MaxDdIS on r-regular graphs is APX-hard, and show the inapproximability of MaxDdIS on r-regular graphs. (iii) We design polynomial-time $O(r^{d-1})$-approximation and $O(r^{d-2}/d)$-approximation algorithms for MaxDdIS on r-regular graphs. (iv) We sharpen the above $O(r^{d-2}/d)$-approximation algorithms when restricted to $d = r = 3$, and give a polynomial-time 2-approximation algorithm for MaxD3IS on cubic graphs. (v) Furthermore, we design a polynomial-time 1.875-approximation algorithm for MaxD3IS on cubic graphs. (vi) Finally, we consider planar graphs and obtain that MaxDdIS admits a polynomial-time approximation scheme (PTAS) for planar graphs.

(2) We then investigate MaxIM on r-regular graphs. For subclasses of r-regular graphs, several better approximation algorithms are known. The previously known best approximation ratios for MaxIM on C_5-free r-regular graphs and $\{C_3, C_5\}$-free r-regular graphs are $\left(\frac{3r}{4} - \frac{1}{8} + \frac{3}{16r-8}\right)$ and $(0.7084r + 0.425)$, respectively. We design a $\left(\frac{2r}{3} + \frac{1}{3}\right)$-approximation algorithm, whose approximation ratio is strictly smaller/better than the previous one for C_5-free r-regular graphs when $r \geq 6$, and for $\{C_3, C_5\}$-free r-regular graphs when $r \geq 3$.

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Chapter 1

Introduction

In theoretical computer science and combinatorial optimization, one of the most important and most investigated computational problems is the Maximum Independent Set problem (MaxIS for short). There is a huge number of its applications in diverse fields, such as scheduling, computer vision, pattern recognition, coding theory, map labeling, and computational biology; many different problems have been modeled using independent sets. Let G be an unweighted graph; we denote by $V(G)$ and $E(G)$ the sets of vertices and edges, respectively, and let $n = |V(G)|$. An *independent set* (or *stable set*) of G is a subset $S \subseteq V(G)$ of vertices such that $\{u, v\} \notin E$ holds for all $u, v \in S$. Then, given a graph G, the goal of MaxIS is to find an independent set S of maximum cardinality in G. MaxIS is one of the most popular NP-hard problems. Therefore, there is a large literature on the approximability/inapproximability of MaxIS. Here, we define the distance between two vertices, that is, for any pair of vertices $u, v \in S$, the distance (i.e., the number of edges) of any path between u and v is at least d in G. Then, MaxIS is also named the Maximum Distance-2 Independent Set problem.

The Maximum Matching problem (MaxM for short) is also one of the most important graph optimization problems. For a simple unweighted graph $G = (V, E)$, two edges are called *adjacent* if they have a common vertex. A *matching* in the graph G is a subset of edges, no two of which are adjacent. Given a graph G, the goal of MaxM is to find a matching S of maximum cardinality in G. It is well known that the Maximum Matching problem is in P, i.e., the problem can be solved by a polynomial time algorithm.

In this thesis, we study two generalized variants of the maximum independent set and maximum matching problems, which are named maximum distance-d independent set problem and maximum induced Matching problem, respectively.

1.1 Maximum Distance-d Independent Set

In the chapter 3, we firstly consider MaxDdIS when $d \geq 3$. For an integer $d \geq 2$, a *distance-d independent set* of an unweighted graph G is a subset $DdIS \subseteq V(G)$ of vertices such that for an integer $d \geq 2$, the distance of any pair of vertices $u, v \in DdIS$ is at least d in G. Then, MaxDdIS is formulated as the following class of problems [1, 9]:

> **MAXIMUM DISTANCE-d INDEPENDENT SET (MAXDdIS)**
>
> **Input:** An unweighted graph G
>
> **Output:** A distance-d independent set of G with the maximum cardinality

When $d = 2$, MaxDdIS (i.e., MaxD2IS) is equivalent to the original MaxIS. Zuckerman [23] proved that MaxD2IS cannot be approximated in polynomial time, unless P = NP, within a factor of $n^{1-\varepsilon}$ for any $\varepsilon > 0$. Moreover, MaxD2IS remains NP-hard even if the input graph is a cubic planar graph, a triangle-free graph, or a graph with large girth. Chlebík and Chlebíková [7] proved the 1.0107, 1.0216, 1.0225, and 1.0236-inapproximability for MaxD2IS on 3-regular, 4-regular, 5-regular, and r-regular ($r \geq 6$) graphs, respectively. Fortunately, however, it is well known that MaxD2IS can be solved in polynomial time when restricted to, for example, bipartite graphs [15], chordal graphs [12], circular-arc graphs [13], comparability graphs [14], and many other classes [20, 19, 6]. On the other hand, we can obtain polynomial-time 1.2, 1.4, and 1.6-approximation algorithms for MaxD2IS on 3-regular, 4-regular, and 5-regular graphs, respectively, by applying the $\frac{\Delta+3}{5}$-approximation algorithm proposed by Berman and Fujito [5] for the problem on general graphs of maximum degree $\Delta \leq 613$. We note that, for a larger maximum degree Δ (and hence general r), Halldórsson and Radhakrishnan developed polynomial-time approximation algorithms within factors of $\frac{\Delta+2}{3}$ [16] and $O(\frac{\Delta}{\log\log\Delta})$ [17]. For planar graphs, it is well known that the Baker's shifting technique [3] for NP-hard optimization problems can be applied to MaxD2IS on planar graphs; it yields a polynomial-time approximation scheme (PTAS). Thus, MaxD2IS can be approximated within an arbitrarily small factor for planar graphs.

Table 1.1: Previous and new approximation ratios for MaxDdIS

Maximum Distance-d Independent set(MaxDdIS)		
MaxD2IS	r-regular($r \leq 613$)	$(r+3)/5$ [Berman and Fujito., 1999]
	Planar graphs	(1+ ε) [B.S.Baker., 1994]
MaxD3IS	3 **-regular**	**2.4 [This Thesis]**
		2+ ε [This Thesis]
		2 [This Thesis]
		1.875 [This Thesis]
MaxDdIS($d \geq 3$)	r **-regular**	**O**(r^{d-1}) **[This Thesis]**
		O(r^{d-2}/d) **[This Thesis]**
	Planar graphs	**1+ ε [This Thesis]**

When $d \geq 3$, Eto, Guo, and Miyano [9] proved that MaxDdIS is NP-hard even for planar bipartite graphs of maximum degree three. Furthermore, they showed that it is NP-hard to approximate MaxDdIS on bipartite graphs and chordal graphs within a factor of $n^{1/2-\varepsilon}$ ($\varepsilon > 0$) for every fixed integer $d \geq 3$ and every fixed *odd* integer $d \geq 3$, respectively. On the other hand, interestingly, they showed that MaxDdIS on chordal graphs is solvable in polynomial time for every fixed *even* integer $d \geq 3$. As the other positive results, Agnarsson, Damaschke, and Halldórsson [1] showed the tractability of MaxDdIS on interval graphs, trapezoid graphs, and circular-arc graphs.

Our main results are obtained in the chapter 3: (i) It is NP-hard to approximate MaxD3IS on 3-regular graphs within 1.00105 unless P=NP. (ii) For every fixed integers $d \geq 3$ and $r \geq 3$, we show the inapproximability of MaxDdIS on r-regular graphs, where $d \geq 3$ and $r \geq 3$. (iii) We design polynomial-time $O(r^{d-1})$-approximation and $O(r^{d-2}/d)$-approximation algorithms for MaxDdIS on r-regular graphs. (iv) We sharpen the above $O(r^{d-2}/d)$-approximation algorithms when restricted to $d = r = 3$, and give a polynomial-time 2-approximation algorithm for MaxD3IS on cubic graphs. (v) Furthermore, we design a polynomial-time 1.875-approximation algorithm for MaxD3IS on cubic graphs. (vi) Finally, we consider planar graphs and obtain that MaxDdIS admits a polynomial-time approximation scheme (PTAS) for planar graphs.

Here is a list of previous and new results on approximation ratios in Table 1.1(

ε is denoted to be any positive number).

1.2 Maximum Induced Matching

In the chapter 4, we then consider MaxIM. MaxIM is a generalized problem of Maximum Matching problem. A matching M is *induced* if no two vertices belonging to different edges of M are adjacent. In other words, an induced matching M in G is formed by the edges of a 1-regular induced subgraph of G. An induced matching is often called the *strong matching* [28, 30]. Then, the Maximum Induced Matching problem (MaxIM) is that of finding an induced matching of maximum cardinality in an input graph. Then, our problem is formulated as follows:

Maximum Induced Matching (MaxIM)

Input: An unweighted graph G

Output: An induced matching of G with the maximum cardinality

The MaxIM problem was originally introduced by Stockmeyer and Vazirani [37] as a variant of the Maximum Matching problem and motivated as the Risk-Free Marriage problem. Induced matchings have applications in the areas of concurrent transmission of messages in wireless ad hoc networks [24], secure communication channels in broadcast networks [29], communication network testing [37], and many other fields. Thus, MaxIM has received much attention in recent years.

The MaxIM problem is generally intractable. Stockmeyer and Vazirani [37], and Cameron [25] independently proved that MaxIM is NP-hard. Also, it remains NP-hard for several graph classes such as planar graphs of vertex degree at most four [32], bipartite graphs of vertex degree at most three [34, 36], line graphs, chair-free graphs, Hamiltonian graphs [33], and r-regular graphs for $r \geq 3$ [26].

In this thesis, we focus only on C_5-free r-regular graphs as input and consider the approximability of MaxIM on C_5-free r-regular graphs. On r-regular graphs, Zito [38] proved that a natural greedy strategy yields an approximation algorithm for MaxIM on r-regular graphs with approximation ratio $r - \frac{1}{2} + \frac{1}{4r-2}$. Then, Duckworth, Manlove, and Zito [26] improved the approximation ratio slightly into $\frac{n(r-1)}{n-2}$, *i.e.*, asymptotically $r - 1$ for r-regular graphs of n vertices. Subsequently, Gotthilf and Lewenstein [31] provided a $\left(\frac{3r}{4} + 0.15\right)$-approximation algorithm for MaxIM on r-regular graphs by combining a greedy approach with a local search. For

Table 1.2: Previous and new approximation ratios for MaxIM

Maximum Induced Matching	
General r -regular	$0.75r$+0.15 [Z. Gotthilf et al., 2005]
{C3,C5}-free r-regular	$0.7084r$+0.425 [D.Rautenbach ,2015]
{C3,C4}-free r-regular	$(\frac{r}{2} + \frac{r}{4r-2})$ [M. Furst et al., 2018]
{C4}-free r-regular	$(\frac{9r}{16} + \frac{33}{80})$ [M. Furst et al., 2018]
{C5}-free r-regular	$(\frac{3r}{4} + \frac{1}{8} + \frac{3}{16r-8})$ [M. Furst et al., 2018]
{C3,C5} or {C5}-free r-regular	$(\frac{2r}{3} + \frac{1}{3})$ **[This Thesis]**

subclasses of r-regular graphs, several better approximation algorithms are known. Rautenbach [35] designed a $(0.7084r + 0.425)$-approximation algorithm for MaxIM on $\{C_3, C_5\}$-free r-regular graphs. F*ü*rst, Leichter, and Rautenbach [27] provided approximation algorithms for the following three subclasses of r-regular graphs: a $\left(\frac{9r}{16} + \frac{33}{80}\right)$-approximation algorithm for C_4-free r-regular graphs, a $\left(\frac{r}{2} + \frac{1}{4} + \frac{1}{8r-4}\right)$-approximation algorithm for $\{C_3, C_4\}$-free r-regular graphs, and a $\left(\frac{3r}{4} - \frac{1}{8} + \frac{3}{16r-8}\right)$-approximation algorithm for C_5-free r-regular graphs.

The inapproximability results on MaxIM for graph subclasses are also known. Duckworth, Manlove, and Zito [26] proved that for any $\varepsilon > 0$, it is NP-hard to approximate MaxIM on graphs of maximum degree three within $\frac{475}{474} - \varepsilon$, 3-regular graphs within $\frac{2375}{2374} - \varepsilon$, and bipartite graphs of maximum degree three within $\frac{6600}{6659} - \varepsilon$. On the other hand, polynomial-time algorithms for MaxIM have been developed, for example, for chordal graphs, interval graphs [25], trees [28], circular-arc graphs [30], trapezoid graphs, k-interval-dimension graphs, and cocomparability graphs [29].

The goal of this thesis is to improve the previously best known $\left(\frac{3r}{4} - \frac{1}{8} + \frac{3}{16r-8}\right)$-approximation algorithm for C_5-free r-regular graphs [27], and we design a $\left(\frac{2r}{3} + \frac{1}{3}\right)$-approximation algorithm, whose approximation ratio is strictly smaller/better than the previously best one when $r \geq 6$. It is important to note that our approximation algorithm works also for $\{C_3, C_5\}$-free r-regular graphs, *i.e.*, MaxIM on $\{C_3, C_5\}$-free r-regular graphs can be better (than [35]) approximated within an approximation ratio of $\left(\frac{2r}{3} + \frac{1}{3}\right)$ for $r \geq 3$.

Here, we give a list of previous and new results on approximation ratios in Table 1.2.

Chapter 2

Preliminaries

In this chapter, we introduce some theoretic terminologies on approximation algorithms and graph theoretic definitions, which will be utilized throughout the following chapters.

First, some theoretic terminologies on approximation algorithms are shown in the following.

1. **α-approximation algorithm [22]:** For maximum problems on graphs, an algorithm ALG is defined a α- approximation algorithm when the approximation ratio of ALG is α, that is, $OPT(G)/ALG(G) \leq \alpha$ holds for each graph G, where $OPT(G)$ and $ALG(G)$ are a solution by the ALG and a optimal solution, respectively.

2. **Gap-preserving reduction [22]:** Two maximum problems are **MaxA** and **MaxB**. More specifically, we are given an instance P_1 of the problem **MaxA** and another instance P_2 of the problem **MaxB**. A gap-preserving reduction from **MaxA** to **MaxA** is a set of functions $(\alpha_1(n_1), \alpha_2(n_2), c_1(n_1), c_2(n_2))$ such that if $OPT(P_1) \geq g_1(P_1)$, then $OPT(P_2) \geq g_2(P_2)$, and if $OPT(P_1) < g_1(P_1)/\alpha(|P_1|)$, then $OPT(P_2) < g_2(P_2)/\beta(|P_2|)$, where g_1, g_2, α, and β are four functions, and $OPT(P_1)$ and $OPT(P_2)$ are the cost of an optimal solution of instances P_1 and P_2, respectively. Then, we can say that no polynomial time $\beta(|P_2|)$- approximation algorithm unless P=NP.

3. **Polynomial-time approximation scheme(PTAS for short) [22]:** A PTAS is an algorithm which takes an instance of an optimization problem and a

parameter $\alpha > 0$ and, in polynomial time, produces a solution that is within a factor $1 + \alpha$ of being optimal (or $1 - \alpha$ for maximization problems).

Then, we introduce graph theoretic definitions, which are used throughout this thesis:

1. **Degree [15]:** The degree of a vertex of a graph is the number of edges incident to the vertex.

2. **Regular Graph [15]:** A graph is *r-regular* graph if the degree $deg(v)$ of every vertex v is exactly $r \geq 0$.

3. **Cubic Graph [15]:** A 3-regular graph is often called *cubic* graph.

4. **Planar graph [15]:** A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

Chapter 3

Maximum Distance-d Independent Set problem

In this chapter, we focus on the problem of MaxDdIS on regular graphs and planar graphs. First, study inapproximability of MaxDdIS on regular graphs for a fixed integer $d \geq 3$. Then, we design approximation algorithms to solve MaxDdIS on regular graphs and planar graphs for a fixed integer $d \geq 3$.

3.1 Preliminaries

In this section, we introduce some definitions, which will be utilized in this chapter. For a graph $G = (V, E)$, we denote an edge with endpoints u and v by $\{u, v\}$. For a pair of vertices u and v, the length of a shortest path from u to v, i.e., the distance between u and v is denoted by $dist_G(u, v)$, and the diameter G is defined as $diam(G) = \max_{u,v \in V} dist_G(u, v)$.

For a graph G and its vertex v, we denote the (open) neighborhood of v in G by $D_1(v) = \{u \in V(G) \mid \{v, u\} \in E(G)\}$, i.e., for any $u \in D_1(v)$, $dist_G(v, u) = 1$ holds. More generally, for $d \geq 1$, let $D_d(v) = \{w \in V(G) \mid dist_G(v, w) = d\}$ be the subset of vertices that are distance-d away from v. Similarly, let $D_1(S)$ be the open neighborhood of a subset S of vertices, $D_2(S)$ be the open neighborhood of $D_1(S) \cup S$, and so on. That is, $D_k(S) = D_1\left(\bigcup_{i=1}^{k-1} D_i(S) \cup S\right)$. The degree of v is denoted by $deg(v) = |D_1(v)|$.

A graph G_S is a subgraph of a graph G if $V(G_S) \subseteq V(G)$ and $E(G_S) \subseteq E(G)$. For a subset of vertices $U \subseteq V$, let $G[U]$ be the subgraph induced by U. For a positive integer $d \geq 1$ and a graph G, the dth power of G, denoted by $G^d = (V(G), E^d)$, is the graph formed from $V(G)$, where all pairs of vertices $u, v \in G$ such that $dist_G(u, v) \leq d$ are connected by edges $\{u, v\}$'s. Note that $E(G) \subseteq E^d$, i.e., the original edges in $E(G)$ are retained.

3.2 Inapproximability of MaxDdIS for reguar graphs

In this section, we discuss inapproximability of MaxDdIS for regular graphs, which these results can give some advice for designing approximation algorithm. Our main results are summarized as follows:

(i) For every fixed integers $d \geq 3$ and $r \geq 3$, we analyze that it is NP-hard to approximate MaxDdIS on r-regular graphs.

(ii) In particular, when restricted to $d = r = 3$, we show that it is NP-hard to approximate MaxD3IS on 3-regular graphs within 1.00105.

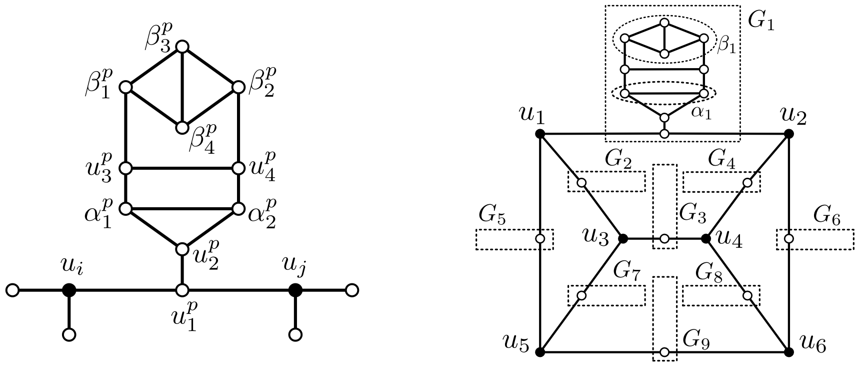
3.2.1 MaxD3IS for cubic graphs

First, we prove the following lower bound of the approximability of MaxD3IS on cubic (i.e., 3-regular) graphs.

Theorem 1. There exists no σ-approximation algorithm for MaxD3IS on cubic graphs for constant $\sigma < 1.00105 < \frac{950}{949}$ unless P = NP.

Proof. The hardness of approximation of MaxD3IS on cubic graphs is shown by a *gap-preserving reduction* from MaxD2IS on cubic graphs. It is known [7] that there exists no σ'-approximation algorithm for the latter problem for constant $\sigma' < \frac{95}{94}$ unless P = NP. Consider an input cubic graph $G_0 = (V_0, E_0)$ with n-vertices and m edges of MaxD2IS. Then, we construct another cubic graph $G = (V, E)$ as an instance of MaxD3IS on cubic graphs from G_0.

Let $\#OPT_2(G_0)$ (and $\#OPT_3(G)$, resp.) denote the number of vertices of an optimal distance-2 independent set in the cubic graph G_0 (and one of an optimal distance-3 independent set in G, resp.). Let $V_0 = \{v_1, v_2, \cdots, v_n\}$ and $E_0 = \{e_1, e_2, \cdots, e_m\}$ be vertex and edge sets of G_0, respectively. Also, let $g(n)$ be a parameter function of the instance G_0, meaning a solution size. Then, we provide the gap preserving reduction such that (C1) if $\#OPT_2(G_0) \geq g(n)$, then



(a) (b)

Figure 3.1: (a) two vertices u_i, u_j and edge-gadget $G_p^{5,3}$ and (b) reduced graph G

$\#OPT_3(G) \geq g(n) + 2m$, and (C2) if $\#OPT_2(G_0) < \frac{g(n)}{\gamma'}$ for a constant $\gamma' > 1$, then $\#OPT_3(G) < \frac{g(n)}{\gamma'} + 2m$.

From G_0, we construct the cubic graph G which consists of (i) n vertices, u_1 through u_n, which are associated with n vertices in V_0, v_1 through v_n, respectively, and (ii) m subgraphs, G_1 through G_m, which are associated with m edges in E_0, e_1 through e_m, respectively. We often call those subgraphs *edge-gadgets* in the following. See Figure 3.1(a). For every p, $1 \leq p \leq m$, the pth diamond-shape gadget G_p contains ten vertices $V(G_p) = \{u_1^p, u_2^p, u_3^p, u_4^p\} \cup \{\alpha_1^p, \alpha_2^p\} \cup \{\beta_1^p, \beta_2^p, \beta_3^p, \beta_4^p\}$, and the pth edge set $E(G_p)$ has 14 edges as illustrated in Figure 3.1(a). (iii) If $e_i = \{v_i, v_j\} \in E_0$, then we introduce two edges $\{u_1^p, u_i\}$ and $\{u_1^p, u_j\}$. As shown in Figure 3.1(b), all the edges are replaced with edge-gadgets. This completes the reduction. One can see that the constructed graph G is cubic. Also, the above construction can be accomplished in polynomial time.

For the above construction of G, we show that G has a distance-3 independent set S such that $|S| \geq g(n) + 2m$ if and only if G_0 has a distance-2 independent set S_0 such that $|S_0| \geq g(n)$. Suppose that the graph G_0 of MaxD2IS has the distance-2 independent set $S_0 = \{v_{1^*}, v_{2^*}, \cdots, v_{g(n)^*}\}$ in G_0, where $\{1^*, 2^*, \cdots, g(n)^*\} \subseteq \{1, 2, \cdots, n\}$. Then, we select a subset of vertices $S' = \{u_{1^*}, u_{2^*}, \cdots, u_{g(n)^*}\}$ and two vertices in each edge-gadget, arbitrary one of the four pairs $\{\alpha_1^p, \beta_3^p\}$, $\{\alpha_1^p, \beta_4^p\}$, $\{\alpha_2^p, \beta_3^p\}$, and $\{\alpha_2^p, \beta_4^p\}$. Let S'' be the set of vertices in edge-gadgets. Hence $|S'| = g(n)$ and $|S''| = 2m$. One can see that $S = S' \cup S''$ is a distance-3 independent set in G since the pairwise distance in S' is at least four, the pairwise distance in S'' is at least six, and the distance between α_1^p (or α_2^p) in S'' and every vertex in S' is at least three for each p.

Conversely, suppose that the graph G has the distance-3 independent set S such that $|S| \geq g(n) + 2m$. Take a look at Figure 3.1(a) again. One can verify that we can select at most two vertices as the distance-3 independent set from the subgraph G_p, at most one of $\{\beta_1^p, \beta_2^p, \beta_3^p, \beta_4^p\}$ and at most one of $\{\alpha_1^p, \alpha_2^p, u_1^p, u_2^p\}$. Thus, the maximum size of the distance-3 independent set in $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_m)$ is at most $2m$, which means that $|S \cap \{u_1, u_2, \ldots, u_n\}| \geq g(n)$. Let $\{u_{1^*}, u_{2^*}, \cdots, u_{g(n)^*}\}$ be a subset of $g(n)$ vertices in $S \cap \{u_1, u_2, \cdots, u_n\}$. Then, the pairwise distance in the corresponding subset of vertices $\{v_{1^*}, v_{2^*}, \cdots, v_{g(n)^*}\}$ of G_0 is surely at least 2, i.e., G_0 has a distance-2 independent set S_0 such that $|S_0| \geq g(n)$. Hence, the reduction satisfies the conditions (C1) and (C2). This implies that MaxD3IS on cubic graphs cannot be approximated within

$$\gamma = \frac{g(n) + 2m}{g(n)/\gamma' + 2m}.$$

In the remaining we obtain the value of γ: Note that a cubic graph has $m = \frac{3n}{2}$ edges. Thus,

$$\frac{g(n) + 2m}{g(n)/\gamma' + 2m} = \frac{g(n) + 3n}{g(n)/\gamma' + 3n}$$

It is important to note that any optimal solution of MaxD2IS on a cubic graph with $n \geq 5$ is at least $\frac{n}{3}$ since Brooks' theorem says [2] that such a graph has a (proper) coloring using three colors, and hence has an independent set of cardinality at least $\frac{n}{3}$. Thus, $g(n) \geq \frac{n}{3}$, and

$$\gamma = \frac{g(n) + 3n}{g(n)/\gamma' + 3n} \geq \frac{10\gamma'}{9\gamma' + 1}$$

since $\gamma' > 1$. By setting $\gamma' = \sigma' = \frac{95}{94}$, we obtain $\gamma \geq \frac{950}{949} > 1.00105$, i.e., the approximation gap remains at least 1.00105. This completes the proof of this theorem. □

3.2.2 MaxDdIS for r-regular graphs

Next, we give the inapproximability for MaxDdIS on r-regular graphs:

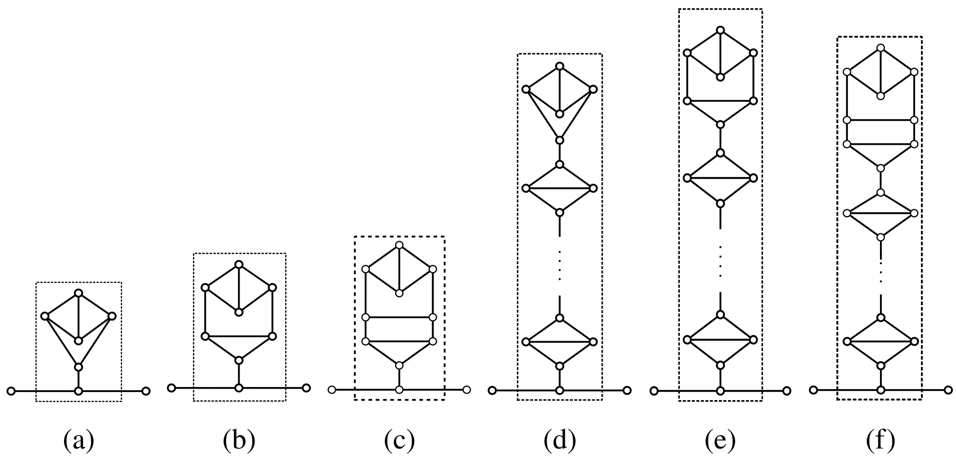
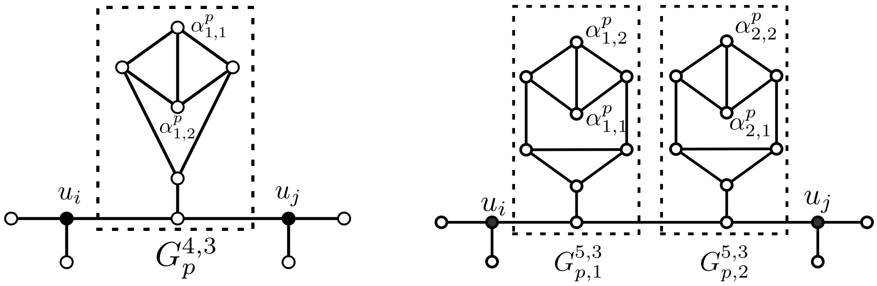


Figure 3.2: Edge-gadgets (a) $G^{4,3}$, (b) $G^{5,3}$, (c) $G^{6,3}$, (d) $G^{d,3}$ for d mod 3 = 1, (e) $G^{d,3}$ for d mod 3 = 2, and (f) $G^{d,3}$ for d mod 3 = 0

Theorem 2. There exists no σ-approximation algorithm for MaxDdIS on r-regular graphs (i) for $d = 3$, $r \geq 3$ and $\sigma < \frac{95r^2(r-1)+190}{95r^2(r-1)+188}$, (ii) for $d = 4$, $r \geq 3$ and $\sigma < \frac{95r^2(r-2)+190}{95r^2(r-2)+188}$, and (iii) for $d \geq 5$, $r \geq 3$ and $\sigma < \frac{95r^2(\lceil d/2 \rceil-1)+190}{95r^2(\lceil d/2 \rceil-1)+188}$, unless P = NP.

Proof. Similarly to the proof of Theorem 1, the hardness of approximation of MaxDdIS on r-regular graphs is shown by a gap-preserving reduction from MaxD2IS on r-regular graphs. Let $G_0 = (V_0, E_0)$ be an input cubic graph with n-vertices and m edges of MaxD2IS on r-regular graphs. Then, we construct another r-regular graph $G = (V, E)$ as an instance of MaxDdIS on r-regular graphs from G_0. In the following, we first give basic ideas of the gap-preserving reductions to prove lower bounds of the approximation ratio for MaxDdIS on r-regular graphs. All we have to do is replace the subgraph illustrated in Figure 3.1-(a) with several gadgets illustrated in Figures 3.2 through 3.7. In the figure, each subgraph is referred to as $G^{d,r}$, which is used for the proof for MaxDdIS on r-regular graphs.

(1) Firstly, we focus only on 3-regular graphs. For MaxD4IS (MaxD5IS and MaxD6IS, resp.), we use a graph in Figure 3.2-(a) ((b) and (c), resp.) as an edge-gadget. For d mod 3 = 1 (2 and 0, resp.), the edge-gadget is illustrated in Figure 3.2-(d) ((e) and (f), resp.). Now take a look at Figure 3.3. In the case of MaxD4IS on 3-regular graphs, we replace one edge, say, $e_p = \{u_i, u_j\}$, of an instance of MaxD2IS



(a) (b)

Figure 3.3: Edge-gadgets for (a) MaxD4IS on 3-regular graphs, and (b) MaxD5IS on 3-regular graphs

on 3-regular graphs with one edge gadget $G^{4,3}_p$, which consists of six vertices. Note that $dist_G(u_i, v) \le 4$ and $dist_G(u_j, v) \le 4$ for any $v \in V(G^{4,3}_p)$, and $diam(G_p) = 3$. Therefore, we can select at most one vertex as the distance-4 independent set from the subgraph $G^{4,3}_p$. In the case of MaxD5IS on 3-regular graphs, two $G^{5,3}$'s, say, $G^{5,3}_{p,1}$ and $G^{5,3}_{p,2}$, are replaced with one original edge e_p as shown in Figure 3.3(b). From each $G^{5,3}$, we can find at most one solution vertex for MaxD5IS. For larger $d \ge 6$, one edge $e_p = \{u_i, u_j\}$ is replaced with the subgraph, say, $G^{d,3}_p$, which consists of many edge-gadgets like Figure 3.4. For example, when $d = 6$, one original edge e_p is replaced with two $G^{6,3}_p$'s in Figure 3.2(c). When $d = 7$, the edge e_p is replaced with two $G^{7,3}_p$'s and one $G^{6,3}_p$. The important points are: $dist_G(u_i, u_j) = \lceil d/2 \rceil$, $dist_G(u_i, \alpha^p_{1,1}) = dist_G(u_i, \alpha^p_{2,1}) = \cdots = dist_G(u_i, \alpha^p_{1/2(\lceil d/2 \rceil - 1),1}) = d$ and so on. From each subgraph $G^{d,3}_p$ shown in Figure 3.4, we can select at most one vertex in each "tower," i.e., at most $\lceil \frac{d}{2} - 1 \rceil$ vertices in total as the distance d-independent set. It is important to note that both u_i and u_j cannot be selected into the distance-d independent set as before.

(2) Secondly, we consider 4-regular graphs. For MaxD3IS on 4-regular graphs, we prepare a graph, say, $G^{3,4}_p$, illustrated in Figure 3.5-(a) as an edge-gadget, which has 17 vertices. One can verify that we can select at most three vertices as the distance-3 independent set from $G^{3,4}_p$.

(3) Thirdly, consider r-regular graphs. For MaxD3IS on r-regular graphs, a graph, say, $G^{3,r}_p$, in Figure 3.5-(b) is used in our reduction, where K_{r-1} and K_{r-2} denote complete graphs of $r-1$ and $r-2$ vertices, i.e., $(r-2)$-regular and $(r-3)$-regular graphs, respectively. The edge-gadget $G^{3,r}_p$ includes $(r-2)$ K_{r-1}'s, C_1

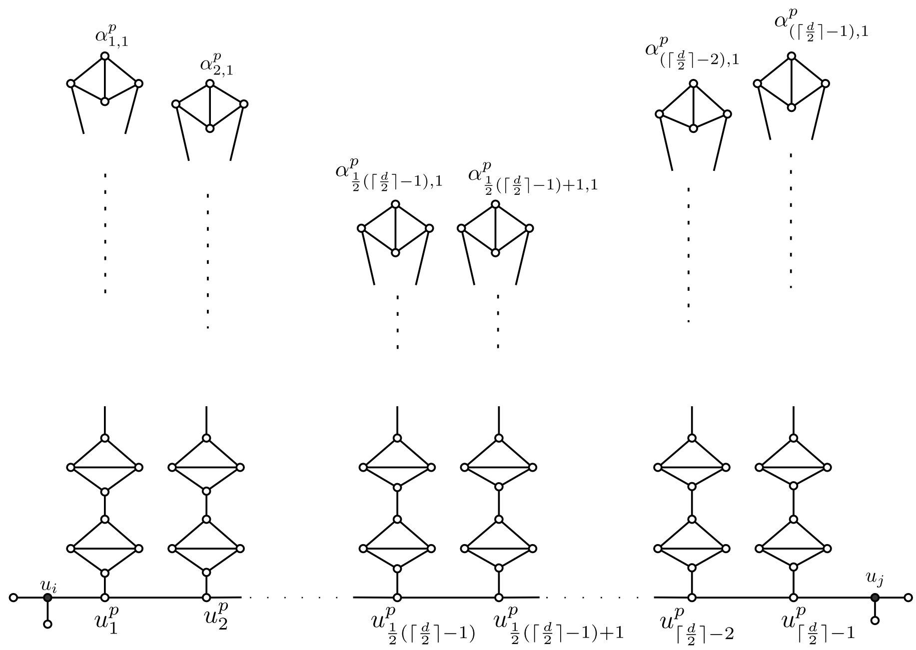


Figure 3.4: Edge-gadget $G_p^{d,3}$ for MaxDdIS on 3-regular graphs

through C_{r-2}, at the top in Figure 3.5-(b). For example, the top and rightmost vertex has $(r-1)$ edges, each of which is incident to each vertex in C_1, and the bottom vertex has $(r-1)$ edges, each of which is incident to each vertex in K_{r-2}. The number of vertices in $G_p^{3,r}$ is $(r-2)(r-1+2)+4+(r-2)+1 = r^2+1$. Note that we can select at most $(r-2)+1 = r-1$ vertices as the distance-3 independent set from $G_p^{3,r}$, one from C_i $(1 \leq i \leq r-2)$ and one from the lower part in $G_p^{3,r}$. Edge-gadgets $G_p^{4,r}$ and $G_p^{5,r}$ for MaxD4IS and MaxD4IS on r-regular graphs are shown in Figure 3.6(a) and (b), respectively. The edge-gadget $G_p^{4,r}$ has $(r-2)$ complete graphs K_{r-1} of $(r-1)$ vertices, C_1 through C_{r-2}, and every vertex in C_i is connected to two vertices, say, $u_{i,1}$ and $u_{i,2}$ outside of C_i The ith vertex, say, u_i, in K_{r-2} is connected to the bottom center vertex and two vertices $u_{i,1}$ and $u_{i,2}$ at the top. Note that at most $(r-2)$ vertices can be selected as the distance-4 independent set from $G_p^{4,r}$, one from C_i for $1 \leq i \leq r-2$. In $G_p^{5,r}$, every vertex in K_{r-2} is connected to three vertices, the bottom center vertex and two upper vertices. Note that at most one vertex can be selected as the distance-5 independent set from $G_p^{5,r}$, i.e., one from the top complete graph K_{r-1}.

(4) Finally, for more general $d \geq 5$ and $r \geq 3$, the edge-gadgets in Figure 3.7 are used in our reduction. When d mod $3 = 0$ (d mod $3 = 1$ and d mod $3 = 2$,

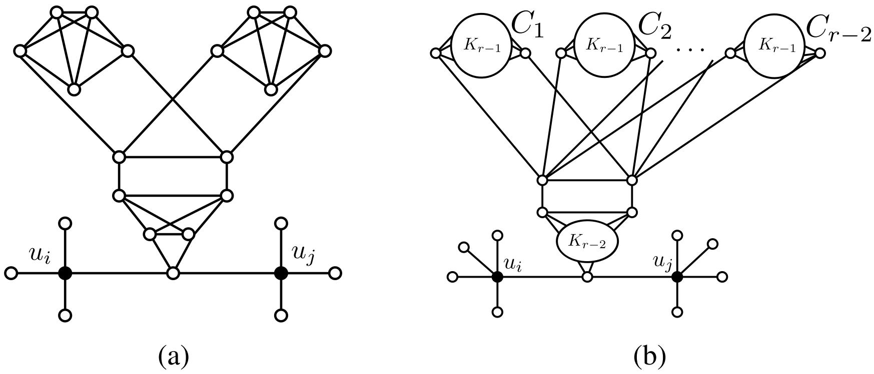


Figure 3.5: Edge-gadgets (a) $G_p^{3,4}$ for MaxD3IS on 4-regular graphs and (b) $G_p^{3,r}$ for MaxD3IS on r-regular graphs

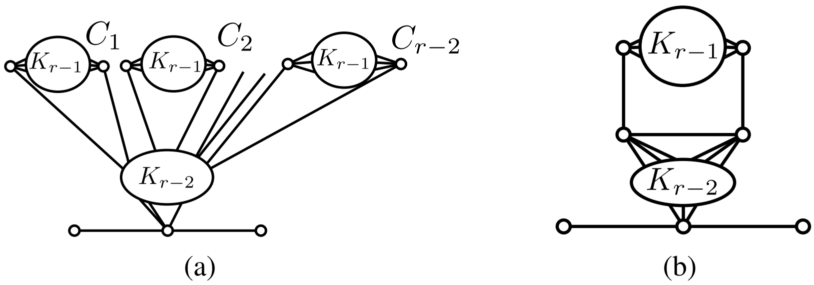


Figure 3.6: Edge-gadgets (a) $G_p^{4,r}$ for MaxD4IS on r-regular graphs and (b) $G_p^{5,r}$ for MaxD5IS on r-regular graphs

resp.) and $d \geq 5$, the edge-gadget $G^{d,r}$ shown in Figure 3.7(a) ((b) and (c), resp.) is prepared. Note that the diameter $diam(G^{d,r}) \leq d-1$ holds, and thus we can select at most one vertex from $G^{d,r}$ as the distance-d independent set. By using the similar construction to one of the subgraph $G_p^{d,3}$ shown in Figure 3.4, every edge in G_0 is replaced with $\lceil \frac{d}{2} - 1 \rceil$ edge-gadgets.

All the above reduction can be done in polynomial time. In the following, we show that our reduction still preserves the approximation gap of $\frac{95}{94}$ for MaxD2IS on r-regular graphs ($r \geq 3$) shown in [7]. Let $\#OPT_2(G_0)$ (and $\#OPT_d(G)$, resp.) denote the number of vertices of an optimal distance-2 independent set in the r-regular graph G_0 (and one of an optimal distance-d independent set in G, resp.). Let $g(n)$ be a parameter function of the instance G_0, meaning a solution size. From Brooks' theorem, we can assume that $g(n) \geq n/r$ holds [2].

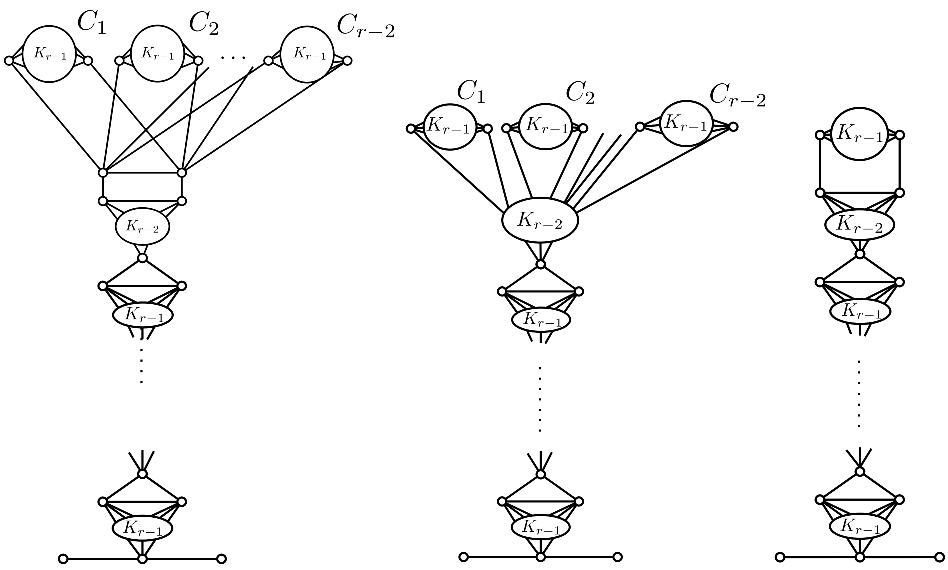


Figure 3.7: Edge-gadgets (a) $G^{d,r}$ for d mod 3 = 0, (b) $G^{d,r}$ for d mod 3 = 1, and (c) $G^{d,r}$ for d mod 3 = 2

(i) Assume that $d = 3$. See again $G_p^{3,r}$ in Figure 3.5-(b), and recall that we can select at most $(r-1)$ vertices as the distance-3 independent set from $G_p^{3,r}$ for each $1 \leq p \leq m$. By the similar arguments to ones of the proof of Theorem 1, we can show that the above reduction satisfies the following condition: (C1) If $\#OPT_2(G_0) \geq g(n)$, then $\#OPT_d(G) \geq g(n) + m(r-1)$, and (C2) if $\#OPT_2(G_0) < \frac{g(n)}{\gamma'}$ for a constant $\gamma' > 1$, then $\#OPT_d(G) < \frac{g(n)}{\gamma'} + m(r-1)$. Therefore, MaxD$d$IS on r-regular graphs cannot be approximated within

$$\frac{g(n) + m(r-1)}{g(n)/\gamma' + m(r-1)} \leq \frac{95r^2(r-1) + 190}{95r^2(r-1) + 188}$$

by setting $m = \frac{n}{2r}$, $\gamma' = \frac{95}{94}$ and $g(n) \geq \frac{n}{r}$.

(ii) Next, assume that $d = 4$. Since at most $(r-2)$ vertices can be selected as the distance-4 independent set from $G_p^{4,r}$ in Figure 3.6(a), the approximation gap is obtained as follows:

$$\frac{g(n) + m(r-2)}{g(n)/\gamma' + m(r-2)} \leq \frac{95r^2(r-2) + 190}{95r^2(r-2) + 188}.$$

(iii) Now assume that $d \geq 5$. Recall that each edge in G_0 is replaced with $\lceil \frac{d}{2} - 1 \rceil$ edge-gadgets shown in Figures 3.7(a), (b), and (c), and also recall that at most one vertex can be selected from $G^{d,r}$ as the distance-d independent set. Hence, the approximation gap is obtained as follows:

$$\frac{g(n) + m(\lceil d/2 \rceil - 1)}{g(n)/\gamma' + m(\lceil d/2 \rceil - 1)} \leq \frac{95r^2(\lceil d/2 \rceil - 1) + 190}{95r^2(\lceil d/2 \rceil - 1) + 188}.$$

This completes the proof of this theorem. □

3.3 Approximability of MaxDdIS for reguar graphs

In this section, we design some approximation algorithms to solve MaxDdIS on r-regular graphs, and furthermore, concentrate on a special regular graph of cubic graph. Moreover, we study MaxDdIS on planar graphs.

Our main results are summarized as follows:

(i) For MaxDdIS on r-regular graphs, we design polynomial-time $O(r^{d-1})$-approximation and $O(r^{d-2}/d)$-approximation algorithms. (The approximation ratio of each algorithm will be analyzed precisely.) Note that the running time of each algorithm is independent from r and d.

(ii) Fixing $d = r = 3$, we give a polynomial-time 2-approximation algorithm for MaxD3IS on 3-regular graphs. We note that the simple applications of the above $O(r^{d-2}/d)$-approximation algorithm yields an approximation ratio strictly greater than two. To improve the ratio to two, we sharpen and precisely analyze the approximation algorithm. Finally, we design an improved 1.875-approximation algorithm.

(iii) By employing the Baker's shifting technique [3], we show that MaxDdIS on planar graphs admits a PTAS for every fixed constant $d \geq 3$.

3.3.1 MaxDdIS for r-reguar graphs

We design two approximation algorithms for MaxDdIS on r-regular graphs. The first one finds a (distance-2) independent set from the $(d - 1)$th power of an input graph by using the previously known approximation algorithm for MaxIS. The second one iteratively executes the following: (i) Picks one vertex v into a solution and (ii) removes all vertices whose distance from the "center" vertex v is less than d. Then, we show that, from the point of view of the approximation ratio, the latter is better than the former for sufficiently large d and/or r.

Power-graph-based algorithms

In this section we design an $(\frac{r(r-1)^{d-1}+2r-6}{5(r-2)} + \varepsilon)$-approximation algorithm for MaxDdIS on r-regular graphs, which uses the following approximation algorithm for MaxIS, i.e., MaxD2IS as a subroutine:

Proposition 1 ([4])**.** There exists a polynomial-time $\frac{\Delta+3}{5} + \varepsilon$-approximation algorithm for MaxD2IS on graphs with the maximum degree Δ, where ε is a constant.

Let $\mathtt{ALG}_2$ be such a rough $\frac{\Delta+3}{5} + \varepsilon$-approximation algorithm for MaxD2IS on graphs with the maximum degree Δ. The above proposition immediately suggests the following simple algorithm: First, construct the $(d-1)$th power G^{d-1} of an input graph G, and then obtain a distance-2 independent set of G^{d-1}. The following is a description of the algorithm $\mathtt{POWER}_d$.

Algorithm $\mathtt{POWER}_d$

Input: r-regular graph $G = (V(G), E(G))$

Output: Distance-d independent set $DdIS(G)$ in G

Step 1. Obtain the $(d-1)$th power G^{d-1} of G by the following:

- **(1-1)** Compute $dist_G(u, v)$ for any pair $u, v \in V$.
- **(1-2)** Add an edge $\{u, v\}$ if $dist_G(u, v) \leq d - 1$.

Step 2. Apply $\mathtt{ALG}_2$ to G^{d-1}, and then obtain a distance-2 independent set $ALG_2(G^{d-1})$ in G^{d-1}.

Step 3. Output $DdIS(G) = ALG_2(G^{d-1})$ as a solution.

Theorem 3. The algorithm $\mathtt{POWER}_d$ runs in polynomial time, and achieves a $(\frac{r(r-1)^{d-1}+2r-6}{5(r-2)} + \varepsilon)$ -approximation ratio for MaxDdIS on r-regular graphs, where ε is a constant.

Proof. First, we must verify that the output $DdIS(G) = ALG_2(G^{d-1})$ of $\mathtt{POWER}_d$ is a feasible solution for MaxDdIS, i.e., the distance-2 independent set in G^{d-1} is a distance-d independent set in G. Suppose for contradiction that there is a pair of vertices $u, v \in ALG_2(G^{d-1})$ (i.e., $dist_{G^{d-1}}(u, v) \geq 2$) such that $dist_G(u, v) \leq d - 1$. Since $dist_G(u, v) \leq d - 1$, in **Step 1** of $\mathtt{POWER}_d$, an edge $\{u, v\}$ must be added between u and v. That is, $dist_{G^{d-1}}(u, v) = 1$ holds, which is a contradiction. Therefore, the output of $\mathtt{POWER}_d$ is always feasible.

Next, we show the approximation ratio of $\mathtt{POWER}_d$ by estimating the maximum degree of the $(d - 1)$th power graph G^{d-1}. Now consider a vertex $v \in V(G)$. Since G is an r-regular graph, v has r neighbor vertices, i.e., $|D_1(v)| = r$. Also, $|D_2(v)| \leq r(r - 1)$ holds since each neighbor vertex $u \in D_1(v)$ has at most $r - 1$ neighbors, each of which is not v. That is, $|D_i(v)| \leq r(r - 1)^{i-1}$ holds for each $1 \leq i \leq d - 1$. Therefore, the maximum degree Δ of G^{d-1} is at most:

$$\begin{aligned}\Delta &\leq r + r(r-1) + r(r-1)^2 + \cdots + r(r-1)^{d-2} \\ &= \frac{r}{r-2}\{(r-1)^{d-1} - 1\}.\end{aligned}$$

Since $\mathtt{POWER}_d$ applies the $(\frac{\Delta+3}{5} + \varepsilon)$ -approximation algorithm $\mathtt{ALG}_2$ for G^{d-1}, the approximation ratio of $\mathtt{POWER}_d$ is as follows:

$$\frac{r(r-1)^{d-1} + 2r - 6}{5(r-2)} + \varepsilon.$$

The algorithm clearly runs in polynomial time and hence this completes the proof of this theorem. □

Roughly, the approximation ratio of $\mathtt{POWER}_d$ is $O(r^{d-1})$.

Iterative-pick-one algorithms

Next, we consider a naive algorithm for MaxDdIS on r-regular graphs, which iteratively picks a vertex v into the distance-d independent set and eliminates all the vertices in $D_1(v) \cup D_2(v) \cup \cdots \cup D_{d-1}(v)$ from candidates of the solution. Then we show its approximation ratio. Here is a description of the "pick-one" algorithm, where $DdIS(G)$ stores vertices in the distance-d independent set, B contains vertices which are determined to be *not* candidates of the solution, and W does vertices which can be picked in the next iteration:

Algorithm $\texttt{PICK\_ONE}_d$

Input: r-regular graph $G = (V(G), E(G))$

Output Distance-d independent set $DdIS(G)$

Step 1. Set $DdIS(G) = \emptyset$, $B = \emptyset$, and $W = V(G)$.

Step 2. If $W \neq \emptyset$, then repeat the following; else goto **Step 3**:
Select one arbitrary vertex v from W. Then, let $B_i = \{v\} \cup \bigcup_{1 \leq i \leq d-1} D_i(v)$ for the ith iteration of this step, update $DdIS(G) = DdIS(G) \cup \{v\}$, $B = B \cup B_i$, and set $W = D_1(B) \setminus B$.

Step 3. Terminate and output $DdIS(G)$ as a solution.

In order to prove the approximation ratio of the algorithm $\texttt{PICK\_ONE}_d$, we now provide an upper bound of the maximum number of vertices in the distance-d independent set in an input graph G with n vertices:

Lemma 1. Consider an r-regular graph $G = (V, E)$ with $|V| = n$ vertices. Then, if $r \geq 3$ and $d \geq 4$, then the size $\#OPT_d(G)$ of optimal solutions of MaxDdIS satisfies the following inequality:

$$\#OPT_d(G) \leq \begin{cases} \dfrac{3n}{r(d-2)} & d\ is\ even, \\ \dfrac{3n}{r(d-1)} & otherwise. \end{cases}$$

Proof. Given an r-regular graph G, let $OPT_d(G) = \{v_1^*, v_2^*, \cdots, v_L^*\}$ be an optimal solution of MaxDdIS and let $\#OPT_d(G) = L$. Then, if d is even, then, for every $1 \leq i \leq L$, consider a ball $Ball(v_i^*) = D_1(v_i^*) \cup D_2(v_i^*) \cup \cdots \cup D_{(d-2)/2}(v_i^*)$, where the center of the ball is v_i^* and its radius is $(d-2)/2$ (or, equivalently, its diameter is $(d-2)$). If d is odd, then we consider a ball $Ball(v_i^*) = D_1(v_i^*) \cup D_2(v_i^*) \cup \cdots \cup D_{(d-1)/2}(v_i^*)$ of diameter $(d-1)$. Since, for every pair of i and j ($i \neq j$), $dist_G(v_i^*, v_j^*) \geq d$ holds from the feasibility of the solution, $Ball(v_i^*) \cap Ball(v_j^*) = \emptyset$ is surely satisfied for every pair i and j. It follows that $\sum_{i=1}^{L} |Ball(v_i^*)| \leq n$.

Now, we estimate the value of $\sum_{i=1}^{L} |Ball(v_i^*)|$ by considering the "smallest" r-regular graph of diameter $diam$, that is, a lower bound of the size of $|Ball(v_i^*)|$. Recently, Knor has proven [21] that the minimum number of vertices in an r-regular graph of diameter $diam$ is at least $\frac{r \cdot diam}{3}$ if $r \geq 3$ and $diam \geq 4$. As a result, the following inequality holds:

$$\sum_{i=1}^{L} |Ball(v_i^*)| \geq \frac{r \cdot diam}{3} \times L.$$

Then, we have

$$\#OPT_d(G) = L \leq \frac{3n}{r \cdot diam},$$

where $diam = d - 2$ if d is even and $diam = d - 1$ if d is odd as mentioned above. This completes the proof of this lemma. □

Now we calculate the number $\#ALG_d(G)$ of vertices in $DdIS(G)$ output by `PICK_ONE`$_d$, and obtain the following lemma:

Lemma 2. Assume that `PICK_ONE`$_d$ finds a solution of size $\#ALG_d(G)$, give an r-regular graph with n vertices. Then, the following is satisfied:

$$\#ALG(G) \geq \begin{cases} \dfrac{n(r-2) - r(r-1)^{\frac{d}{2}-1} + 2}{r(r-1)^{d-1} - r(r-1)^{\frac{d}{2}-1}} & d \text{ } is \text{ } even, \\ \dfrac{n(r-2) - 2(r-1)^{\frac{d-1}{2}} + 2r - 2}{r(r-1)^{d-1} - 2(r-1)^{\frac{d-1}{2}} + 2r - 4} & otherwise. \end{cases}$$

Proof. Let $DdIS(G) = \{s_1, s_2, \cdots, s_\ell\}$ be an output of `PICK_ONE`$_d$, and assume that `PICK_ONE`$_d$ picks those ℓ vertices into $DdIS(G)$ in this order, i.e., first s_1, next s_2, and so on. In the ith iteration of **Step 2** in `PICK_ONE`$_d$, we select s_i into a solution, remove B_i from the candidate vertices V of the distance-d independent set since $dist_G(s_i, v) \leq d - 1$ for $v \in B_i$, and merge B_i to B. Note that the current $B = \bigcup_{1 \leq j \leq i-1} B_j$ and B_i have the common vertices, i.e., $B \cap B_i \neq \emptyset$ is already removed from V before the ith iteration. Then, we call vertices in $B_i \setminus B$ the *ith newly conflict vertices* of s_i. Since all the vertices in the graph G are eventually merged into B, we can easily get the following:

$$\left| \bigcup_{1 \leq i \leq \ell} B_i \right| = n.$$

In the following, we estimate an upper bound of the number, say, Γ_i, of the ith newly conflict vertices in $B_i \setminus \bigcup_{1 \leq j \leq i-1} B_j$:

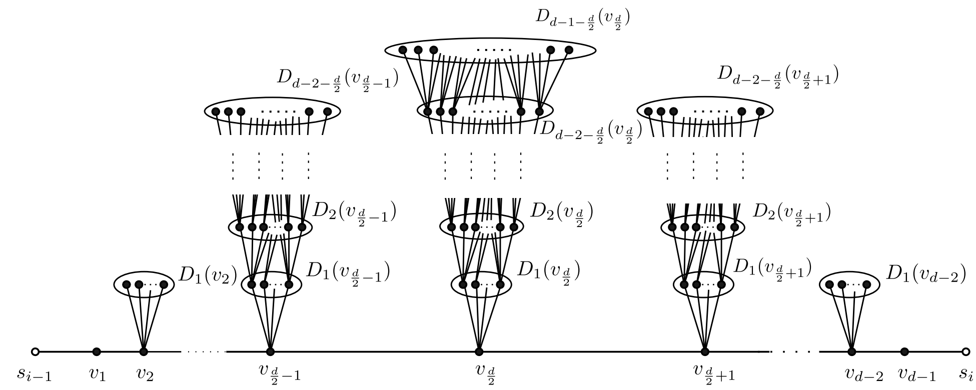


Figure 3.8: B_{i-1} and B_i share all the black vertices

(1) An upper bound of the number Γ_1 of the first newly conflict vertices in B_1 is bounded as follows:

$$\Gamma_1 = |B_1| \leq 1 + r + r(r-1) + \cdots + r(r-1)^{d-2} = \frac{r(r-1)^{d-1} - 2}{r-2}.$$

(2) We then consider an upper bound of $\Gamma_i = |B_i \setminus \bigcup_{1 \leq j \leq i-1} B_j|$ for s_i. In the ith iteration, s_i is selected into a solution, and then set $B_i = \{s_i\} \cup \bigcup_{1 \leq i \leq d-1} D_i(s_i)$. The upper bound of the size of B_i is the same as above:

$$|B_i| \leq 1 + r + r(r-1) + \cdots + r(r-1)^{d-2} = \frac{r(r-1)^{d-1} - 2}{r-2}. \tag{3.1}$$

But, in the $(i-1)$th iteration, s_{i-1} was selected and all the "black" vertices $B_1 \cup \cdots \cup B_{i-1}$ have been already removed from V as illustrated in Figure 3.8. Namely, those black vertices are doubly counted in the above inequality 3.1; we make an estimate of the number of black vertices in the following.

Now take a look at two vertices s_{i-1} and s_i. Suppose that the path of length d from s_{i-1} to s_i is denoted by $P_{s_{i-1},s_i} = \langle s_{i-1}, v_1, v_2, \cdots, v_{d-2}, v_{d-1}, s_i \rangle$. Then, for $1 \leq j \leq d-1$, every vertex v_j on the path P_{s_{i-1},s_i} is included in B_{i-1} since $dist_G(s_{i-1}, v_j) \leq d-1$ for every j. Also, every v_j is included into B_i since $dist_G(s_i, v_j) \leq d-1$ for every j. Moreover, for example, the vertices in $D_1(v_3) \cup D_2(v_3)$ are also "shared" by B_{i-1} and B_i. We consider two cases in the following: (Case 1) d is even and (Case 2) d is odd:

(Case 1) Let $d = 2h$ ($h \geq 1$). Then, the center vertex of the path P_{s_{i-1},s_i} is denoted by $v_{\frac{d}{2}}$. One can see that the neighbor vertices $D_1(v_2)$ of v_2 and $D_1(v_{d-1})$

of v_{d-1}, vertices in $D_1(v_3) \cup D_2(v_3)$ and ones in $D_1(v_{d-3}) \cup D_2(v_{d-3})$ and so on are shared by B_{i-1} and B_i. Then, $|D_1(v_3) \cup D_2(v_3)| = |D_1(v_{d-3}) \cup D_2(v_{d-3})|$, $|D_1(v_4) \cup D_2(v_4) \cup D_3(v_4)| = |D_1(v_{d-4}) \cup D_2(v_{d-4}) \cup D_3(v_{d-4})|$, and so on. Therefore, the number Λ of those black vertices shared by B_{i-1} and B_i is calculated as follows:

$$
\begin{aligned}
\Lambda &= 2 \times (1 + (1 + |D_1(v_2)|) + (1 + |D_1(v_3) \cup D_2(v_3)|) \\
&\quad +(1 + |D_1(v_4) \cup D_2(v_4) \cup D_3(v_4)|) \\
&\quad + \cdots + (1 + |D_1(v_{d/2-1}) \cup \cdots \cup D_{d-2-\frac{d}{2}}(v_{d/2-1})|)) \\
&\quad +(1 + |D_1(v_{d/2}) \cup \cdots \cup D_{d-1-\frac{d}{2}}(v_{d/2})|) \\
&= 2\frac{(r-1)^{\frac{d}{2}-1} - 1}{r-2} + (r-1)^{\frac{d}{2}-1} \\
&= \frac{r(r-1)^{\frac{d}{2}-1} - 2}{r-2}
\end{aligned}
$$

Therefore, we obtain the number of the ith newly conflict vertices:

$$
\begin{aligned}
\Gamma_i &\leq |B_i(s_i)| - \Lambda \\
&\leq \frac{r(r-1)^{d-1} - 2}{r-2} - \frac{r(r-1)^{\frac{d}{2}-1} - 2}{r-2} \\
&= \frac{r(r-1)^{d-1} - r(r-1)^{\frac{d}{2}-1}}{r-2}.
\end{aligned}
$$

The above arguments on Γ_i are applied to every i, $2 \leq i \leq \ell$. Now we know that $\Gamma_1 + (\ell - 1)\Gamma_i \geq n$, and thus,

$$
\ell \geq \frac{n(r-2) - r(r-1)^{\frac{d}{2}-1} + 2}{r(r-1)^{d-1} - r(r-1)^{\frac{d}{2}-1}}.
$$

(Case 2) Let $d = 2h+1$ ($h \geq 1$). Similarly to Case 1, we can show the following inequality on the number of the ith newly conflict vertices:

$$
\begin{aligned}
\Gamma_i &\leq |B_i(s_i)| - \Lambda \\
&\leq \frac{r(r-1)^{d-1} - 2}{r-2} - 2\frac{(r-1)((r-1)^{d-h-2} - 1)}{r-2} \\
&= \frac{r(r-1)^{d-1} - 2(r-1)^{d-h-1} + 2r - 4}{r-2}.
\end{aligned}
$$

Since $\Gamma_1 + (\ell - 1)\Gamma_i \geq n$, we can get

$$\begin{aligned}\ell &\geq \frac{n(r-2) - 2(r-1)^{d-h-1} + 2r - 2}{r(r-1)^{d-1} - 2(r-1)^{d-h-1} + 2r - 4} \\ &= \frac{n(r-2) - 2(r-1)^{\frac{d-1}{2}} + 2r - 2}{r(r-1)^{d-1} - 2(r-1)^{\frac{d-1}{2}} + 2r - 4}\end{aligned}$$

This completes the proof of this lemma. □

Theorem 4. The approximation ratio σ of $\mathtt{PICK\_ONE}_d$ is as follows:

$$\sigma = \begin{cases} \dfrac{3(r-1)^{d-1} - 3(r-1)^{\frac{d}{2}-1}}{(r-2)(d-2)} + O(\dfrac{1}{n}) & d \text{ is even}, \\ \dfrac{3r(r-1)^{d-1} - 6(r-1)^{\frac{d-1}{2}} + 6r - 12}{r(r-2)(d-1)} + O(\dfrac{1}{n}) & \text{otherwise}. \end{cases}$$

Proof. The approximation ratio σ is bounded by $\#OPT_d(G)/\#ALG_d(G)$. From the upper bound of $\#OPT_d(G)$ and the lower bound of $\#ALG_d(G)$ shown in Lemmas 1 and 2, respectively, we can obtain this theorem. □

That is, the approximation ratio of $\mathtt{PICK\_ONE}_d$ is $O(r^{d-2}/d)$, while the approximation ratio of $\mathtt{POWER}_d$ is $O(r^{d-1})$.

3.3.2 MaxD3IS for cubic graphs

In this section, as a special case, we study the approximability of MaxD3IS on cubic graphs, i.e., $d = 3$ and $r = 3$ and show the approximation ratios of $\mathtt{POWER}_3$ and $\mathtt{PICK\_ONE}_3$. Furthermore, by a slight modification, we obtain a 2-approximation algorithm for MaxD3IS on cubic graphs.

Power-graph-based algorithm

First, as an immediate consequence of Theorem 3, we have the following corollary:

Corollary 1. The algorithm $\mathtt{POWER}_3$ achieves a 2.4-approximation ratio for MaxD3IS on cubic graphs.

Proof. There exists a polynomial-time $(\frac{\Delta+3}{5}$-approximation algorithm for MaxD2IS on graphs with the maximum degree $\Delta \leq 613$ [5]. Since the maximum degree of the second power G^2 of an input 3-regular graph G is nine, the approximation ratio is $12/5 = 2.4$. □

Iterative-pick-one algorithm

In this section, we prove that $\texttt{PICK\_ONE}_3$ achieves $2 + O(1/n)$-approximation ratio, and furthermore, the ratio can be improved into exactly 2 by a slight modification of $\texttt{PICK\_ONE}_3$ and careful observations.

Recall that the upper bound of optimal solutions of $\mathsf{MaxD}d\mathsf{IS}$ on r-regular graphs provided in Lemma 1 holds only for the case where $d \geq 4$. Then, we give an estimation of the upper bound of the maximum number of vertices in an optimal solution for the case where $r = 3$ and $d = 3$:

Lemma 3. Consider a cubic graph $G = (V, E)$ with $|V| = n$ vertices. Then, the size $\#OPT_3(G)$ of every optimal solution of $\mathsf{MaxD3IS}$ satisfies the following inequality:

$$\#OPT_3(G) \leq \frac{n}{4}.$$

Proof. Given a 3-regular graph G of n vertices, let $OPT_3(G) = \{v_1^*, v_2^*, \cdots, v_L^*\}$ be an optimal solution of $\mathsf{MaxD3IS}$ and let $\#OPT_3(G) = L$. Also, let $\overline{OPT_3(G)}$ be the set of vertices not in $OPT_3(G)$, i.e., $\overline{OPT_3(G)} = V(G) \setminus OPT_3(G)$. Then, three edges, say, $\{\{v_i^*, u_{i,1}\}, \{v_i^*, u_{i,2}\}, \{v_i^*, u_{i,3}\}\}$, are incident to every vertex $v_i^* \in OPT_3(G)$ for $1 \leq i \leq L$, and $u_{i,1}, u_{i,2}, u_{i,3} \in \overline{OPT_3(G)}$. Therefore, $|\overline{OPT_3(G)}| \geq 3L$. From the definition, $|\overline{OPT_3(G)}| = n - L$ holds. As a result, the following inequality is obtained:

$$\#OPT_3(G) = L \leq \frac{n}{4}.$$

This completes the proof of this lemma. □

Consider a graph $D_2 = (V(D_2), E(D_2))$ of eight vertices such that

$$\begin{aligned} V(D_2) &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \\ E(D_2) &= \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \\ &\quad \{v_5, v_6\}, \{v_5, v_7\}, \{v_6, v_7\}, \{v_6, v_8\}, \{v_7, v_8\}, \\ &\quad \{v_4, v_5\}, \{v_8, v_1\}\}. \end{aligned}$$

That is, D_2 consists of two diamond graphs and two edges. One can verify that D_2 is cubic and $|OPT_3(D_2)| = 2 = 8/4$. Similarly, by circularly joining diamond graphs, we can obtain an infinite family of tight examples for Lemma 3; for a graph D_ℓ having ℓ diamond graphs (4ℓ vertices), $|OPT_3(D_\ell)| = \ell$.

Theorem 5. The algorithm $\mathtt{PICK\_ONE}_3$ achieves a $\left(2 + \frac{4}{n-2}\right)$-approximation ratio for MaxD3IS on cubic graphs.

Proof. Let $D3IS(G) = \{s_1, s_2, \cdots, s_\ell\}$ be an output of $\mathtt{PICK\_ONE}_3$, and without loss of generality, assume that $\mathtt{PICK\_ONE}_3$ picks those ℓ vertices into $D3IS(G)$ in this order, i.e., first s_1, next s_2, and so on.

(i) In the first iteration of **Step 2** of $\mathtt{PICK\_ONE}_3$, the first vertex s_1 is selected into $D3IS(G)$, then $B_1 = \{s_1\} \cup D_1(s_1) \cup D_2(s_1)$ are removed from $V(G)$, and set $V = V(G) \setminus B_1$. One can see that the number of vertices in B_1 is at most 10 since s_1 has at most three neighbors, i.e., $|D_1(s_1)| \leq 3$, and each vertex in $D_1(s_1)$ has at most two other vertices, i.e., $|D_2(s_1)| \leq 6$.

(ii) In the second iteration, the second vertex s_2 is selected from neighbor vertices of B_1 into $D3IS(G)$, and then $B_2 = \{s_2\} \cup D_1(s_2) \cup D_2(s_2)$ are removed from V updated in **Step 2**. The number of vertices in B_2 is again at most 10, but $|B_1 \cap B_2| \geq 2$ because there must exist at least two vertices between s_1 and s_2 from the fact $dist_G(s_1, s_2) \geq 3$. That is, $|B_2 \setminus B_1| \leq 8$ and thus at most eight vertices currently in V are removed from V in the second iteration. Similarly, when s_i for $3 \leq i \leq \ell$ are selected into $D3IS(G)$, at most eight vertices in V are removed from V. Therefore,

$$|B_1| + |B_2 \setminus B_1| + \cdots + |B_\ell \setminus (\bigcup_{1 \leq i \leq \ell-1} B_i)| \leq 10 + 8(\ell - 1).$$

At the time when $\mathtt{PICK\_ONE}_3$ terminates, $V = \emptyset$ and thus the following inequality holds since the value of the left-hand side of the above inequality is equal to n:

$$10 + 8(\ell - 1) \geq n.$$

Namely,

$$\ell \geq \frac{n-2}{8}.$$

Since $\#OPT_3(G) \leq \frac{n}{4}$, the approximation ratio of `PICK_ONE`$_3$ is as follows:

$$\frac{\#OPT_3(G)}{\ell} \leq 2 + \frac{4}{n-2}.$$

$\square$

REV_PICK_ONE$_3$

To improve the above ratio of $2 + \varepsilon$ ($\varepsilon > 0$) to 2, we slightly modify **Step 2** of `PICK_ONE`$_3$, and get the following algorithm, called `REV_PICK_ONE`$_3$:

> **Algorithm** `REV_PICK_ONE`$_3$:
>
> **Input:** 3-regular graph $G = (V(G), E(G))$
>
> **Output:** Distance-3 independent set $D3IS(G)$
>
> **Step 1.** Set $D3IS(G) = \emptyset$, $B = \emptyset$, and $W = V(G)$.
>
> **Step 2.** If $W \neq \emptyset$, then repeat the following; else goto **Step 3**:
> Select one vertex v from W such that $|(D_1(v) \cup D_2(v)) \setminus B|$ is minimum among all vertices in W. Then, let $B_i = \{v\} \cup D_1(v) \cup D_2(v)$ in the ith iteration of this step, update $D3IS(G) = D3IS(G) \cup \{v\}$, $B = B \cup B_i$, and set $W = D_1(B) \setminus B$.
>
> **Step 3.** Terminate and output $D3IS$ as a solution.

Recall that `PICK_ONE`$_3$ selects an arbitrary vertex v in each iteration in **Step 2**. On the other hand, `REV_PICK_ONE`$_3$ selects a vertex v such that $|(D_1(v) \cup D_2(v)) \setminus B|$ is minimum among all vertices in W in each iteration, only which is the difference between `PICK_ONE`$_3$ and `REV_PICK_ONE`$_3$.

Theorem 6. The algorithm `REV_PICK_ONE`$_3$ runs in polynomial time, and achieves a 2-approximation ratio for MaxD3IS on cubic graphs.

Proof. Again, let $D3IS(G) = \{s_1, s_2, \cdots, s_\ell\}$ be an output of `REV_PICK_ONE`$_3$, and assume that `REV_PICK_ONE`$_3$ picks those ℓ vertices into $D3IS(G)$ in this order. That is, in the first iteration, `REV_PICK_ONE`$_3$ picks s_1 such that $|(D_1(s_1) \cup D_2(s_1))|$ is minimum among all vertices in $V(G)$ since $B = \emptyset$. Then, we update $B = B_1 =$

$\{s_1\} \cup D_1(s_1) \cup D_2(s_1)$. We have the following three cases according to the size of $|B_1|$: (i) $|B_1| \leq 8$, (ii) $|B_1| = 9$, and (iii) $|B_1| = 10$.

(i) First consider the case where $|B_1| \leq 8$. Similarly to the proof of Theorem 5, in the second iteration of **Step 2**, the second vertex s_2 is selected from neighbor vertices of B_1 into $D3IS(G)$, and then $B_2 = \{s_2\} \cup D_1(s_2) \cup D_2(s_2)$ are removed from V updated in **Step 2**. Recall that $|B_2 \setminus B_1| \leq 8$. Similarly, when s_i for $3 \leq i \leq \ell$ are selected into $D3IS(G)$, $|B_i \setminus (\bigcup_{1 \leq j \leq i-1} B_j)| \leq 8$ holds. Therefore,

$$|B_1| + |B_2 \setminus B_1| + \cdots + |B_\ell \setminus (\bigcup_{1 \leq i \leq \ell-1} B_i)| \quad \leq \quad 8\ell. \tag{3.2}$$

Namely,

$$\ell \geq \frac{n}{8}.$$

Since $\#OPT_3(G) \leq \frac{n}{4}$, the approximation ratio of `REV_PICK_ONE`$_3$ is as follows:

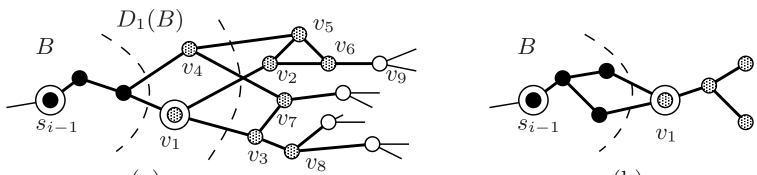
$$\frac{\#OPT_3(G)}{\ell} \leq 2.$$

(ii) Next suppose that $|B_1| = 9$. Similarly, again $|B_i \setminus (\bigcup_{1 \leq j \leq i-1} B_j)| \leq 8$ holds for the ith iteration, $2 \leq i \leq \ell$. It is now important to note that the number n of vertices in the cubic graph G must be even since the degree r is odd. Thus, actually, at least one of $|B_i \setminus (\bigcup_{1 \leq j \leq i-1} B_j)|$ for $2 \leq i \leq \ell$ must be at most seven. Therefore, the left-hand side of the inequality (3.2) is at most $9 + 7 + 8(\ell - 2) = 8\ell$. As a result, the inequality (3.2) holds again, which means that the approximation ratio is two.

(iii) Finally, suppose that $|B_1| = 10$, which implies that $|\{s_i\} \cup D_1(s_i) \cup D_2(s_i)| = 10$ for every vertex s_i since $|\{s_1\} \cup D_1(s_1) \cup D_2(s_1)|$ is minimum. Indeed, for example, $|\{v\} \cup D_1(v) \cup D_2(v)| = 10$ holds for any vertex v in a *C_4-free cubic graph* (i.e., the graph including no induced cycles of length 3 and 4). Fortunately, if at least one, say, $|B_i \setminus (\bigcup_{1 \leq j \leq i-1} B_j)|$ is seven, then there must exist at least one iteration, say, i' ($\neq i$) such that $|B_{i'} \setminus (\bigcup_{1 \leq j \leq i'-1} B_j)| \leq 7$ holds since n is even. That is, the inequality (3.2) is true as well. Unfortunately, however, if $|B_i \setminus (\bigcup_{1 \leq j \leq i-1} B_j)| = 8$ holds for every $2 \leq i \leq \ell$, then the ratio of `REV_PICK_ONE`$_3$ is $2 + 4/(n-2)$ similarly to `PICK_ONE`$_3$. Now, as the worst case, we suppose that in the second through the $(\ell - 1)$th iterations, s_2 through $s_{\ell-1}$ are selected and $|B_2 \setminus B_1|$ through $|B_{\ell-1} \setminus (\bigcup_{1 \leq j \leq \ell-2} B_j)|$ are all eight. Then, we take a look at the last iteration in detail. (iii-1) If the current V has at least nine vertices, then we can get further two vertices in the distance-3 independent set since $|B_\ell \setminus (\bigcup_{1\le j\le \ell-1} B_j)| \le 8$, which is a contradiction from the assumption of $|D3IS(G)| = \ell$. Thus, (iii-2) we can assume that the number of the remaining vertices in V is at most eight after the $(\ell - 1)$th iteration. Then, one can see that if those eight vertices are connected, then we again get two vertices in the distance-3 independent set, which is another contradiction. (iii-3) Now suppose that the remaining graph $G[V]$ has at least two connected components. Then, there must exist a vertex s_ℓ such that $|B_\ell \setminus (\bigcup_{1\le j\le \ell-1} B_j)| \le 5$. As a result, again we can obtain the inequality (3.2), which follows that the approximation ratio is two. This completes the proof of this theorem. □

Improved 1.875-Approximation Algorithm

Then, we design an improved approximation algorithm, which achieves the approximation ratio of 1.875 for MaxD3IS on cubic graphs. Now we make a simple observation; see figure 3.9(a). In the previous algorithm in [10], if s_{i-1} is selected in the $(i-1)$st iteration and black vertices are removed from the solution candidates, then we select, for example, v_1 into a solution $D3IS(G)$ in the ith iteration since $dist_G(s_{i-1}, v_1) = 3$, and remove eight "gray" vertices, v_1 through v_8, from the solution candidates. In other words, we can select one vertex v_1 into the solution among (at most) eight candidates in $\{v_1\} \cup D_1(v_1) \cup D_2(v_1) \setminus B$, where B is a set of "non-candidate vertices." For the case in figure 3.9, however, if we select a *neighbor* v_2 of v_1 into $D3IS(G)$, then at most seven vertices in $\{v_2\} \cup D_1(v_2) \cup D_2(v_2) \setminus B$ (= $\{v_1, v_2, v_3, v_4, v_5, v_6, v_9\}$) are removed; now we could select one among seven candidates. As a desirable example, if we can averagely select one vertex into $D3IS(G)$ among seven vertices in an iteration, then we can find a solution of size $n/7$, i.e., we achieve the 7/4-approximation ratio. Hence, it is our goal to find a vertex s such that $|\{s\} \cup D_1(s) \cup D_2(s) \setminus B|$ is as small as possible in each iteration. As another desirable example, if v_1 has two neighbors in B as shown in figure 3.9(b), then $|\{v_1\} \cup D_1(v_1) \cup D_2(v_1) \setminus B| \le 4$. In the following, we show that we can averagely select one vertex among "15/2" vertices, which implies the approximation ratio of $(n/4)/(2n/15) = 15/8 = 1.875$. Our new algorithm ALG basically selects (i) the *first candidate* vertex v_f from $D_1(B)$ if $|\{v_f\} \cup D_1(v_f) \cup D_2(v_f) \setminus B| \le 7$, but (ii) a neighbor u of v_f if $|\{v_f\} \cup D_1(v_f) \cup D_2(v_f) \setminus B| \ge 8$. Unfortunately, however, there are *special* subgraphs such that for any neighbor $u \in D_1(v_f)$ of the first candidate v_f, $|\{u\} \cup D_1(u) \cup D_2(u) \setminus B| \ge 8$ must hold. Therefore, ALG initially finds such special

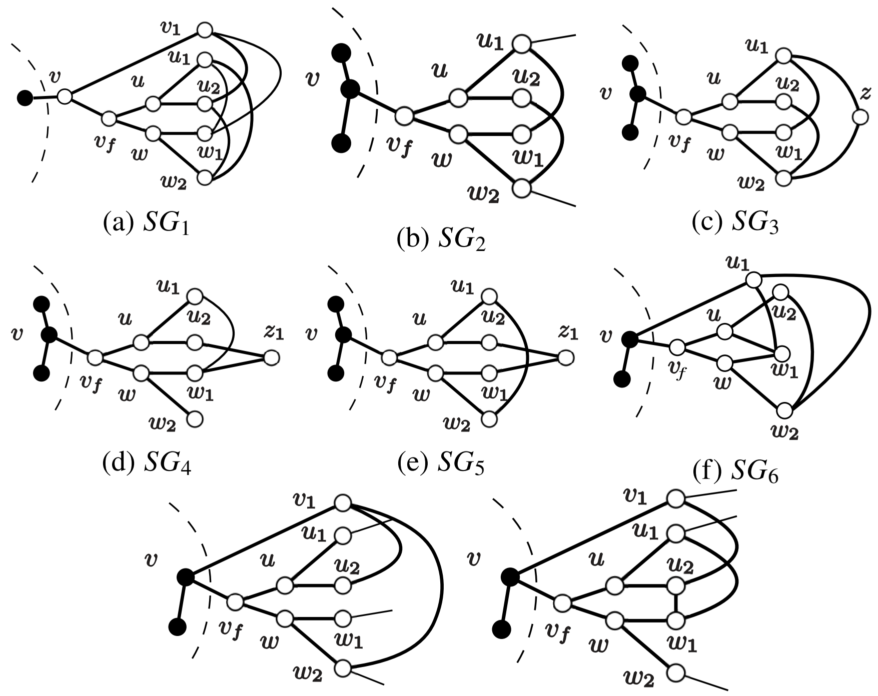


(a) (b)

Figure 3.9: Observations (a) and (b)

subgraphs and gives some special treatments to them as preprocessing, which these procedure can be clearly implemented in polynomial time.

There are eight special subgraphs, SG_1, SG_2, SG_3, SG_4, SG_5, SG_6, SG_7 and SG_8, which are illustrated in figures 3.10(a), (b), (c), (d), (e), (f), (g) and (h), respectively. The first special subgraph SG_1 consists of nine "white" vertices, the second one SG_2 consists of seven white vertices, and so on.



(a) SG_1 (b) SG_2 (c) SG_3 (d) SG_4 (e) SG_5 (f) SG_6 (g) SG_7 (h) SG_8

Figure 3.10: Special subgraphs (a) SG_1, (b) SG_2, (c) SG_3, (d) SG_4, (e) SG_5, (f) SG_6, (g) SG_7 and (h) SG_8

(a) See figure 3.10(a). The first special subgraph SG_1 has nine white vertices, the first candidate v_f, its three neighbor vertices v, u and w, two neighbors u_1 and u_2 of u, two neighbors w_1 and w_2 of w, and the top vertex v_1, where $dist_G(v_f, v_1) = 2$, and vertices of $\{v_f, u, w, v_1, u_1, u_2, w_1, w_2\}$ are not in set B and maybe v is in the set B. The vertex v_1 is connected to either of u_1 and u_2 and either of w_1 and w_2. As shown in figure 3.10(a), assume that the graph has two edges $\{v_1, u_2\}$ and $\{v_1, w_1\}$. Furthermore, there are three edges, $\{u_1, w_1\}$, $\{u_1, w_2\}$, and $\{u_2, w_2\}$. For SG_1, if v is not removed, then our algorithm ALG selects u_1, which is not connected to v_1, and v into $D3IS(G)$, and eliminates nine vertices in $V(SG_1)$ and three vertices in $(D_1(v) \cup D_2(v)) \setminus V(SG_1)$, i.e., (at most) 12 vertices in $\{v, u_1\} \cup D_1(\{v, u_1\}) \cup D_2(\{v, u_1\})$ from the solution candidates; else our algorithm ALG only selects u_1, which is not connected to v_1 into $D3IS$.

(b) See figure 3.10(b). The second special subgraph SG_2 has seven white vertices, the first candidate v_f, its two neighbors u and w, two neighbors u_1 and u_2 of u, two neighbors w_1 and w_2 of w, and moreover, these vertices are not in the set B. **(b1)** Neither of u_1 and u_2 (w_1 and w_2, resp.) is connected to w (u, resp.), and **(b2)** u_1 is connected to either w_1 or w_2, and u_2 is connected to the other. Without loss of generality, assume that u_1 (u_2, resp.) is connected to w_1 (w_2, resp.) as shown in figure 3.10(b). **(b3)** Either of $dist_G(u_1, w_2) = 1$ and $dist_G(u_2, w_1) \geq 3$, $dist_G(u_1, w_2) = dist_G(u_2, w_1) = 1$, and $dist_G(u_1, w_2) \geq 3$ and $dist_G(u_2, w_1) \geq 3$ holds. Note that the case of $dist_G(u_1, w_2) \geq 3$ and $dist_G(u_2, w_1)) = 1$ is essentially the same as the case of $dist_G(u_1, w_2) = 1$ and $dist_G(u_2, w_1)) \geq 3$. Then, (i) If $dist_G(u_1, w_2) \geq 3$ and $dist_G(u_2, w_1) \geq 3$, then ALG selects u_2 and w_1 into $D3IS(G)$. (ii) If $dist_G(u_1, w_2) = 1$ and $dist_G(u_2, w_1) \geq 3$, then ALG selects u_2 and w_1 into $D3IS(G)$. (iii) If $dist_G(u_1, w_2) = dist_G(u_2, w_1) = 1$, then ALG selects one arbitrary vertex in $\{u_1, u_2, w_1, w_2\}$ into $D3IS(G)$. One can see that the case where $dist_G(u_1, w_2) = dist_G(u_2, v_1) = dist_G(w_1, v_1) = 1$ is essentially equivalent to SG_1, where $dist_G(v_1, v_f) = 2$ and $v \notin \{v_f, u, w, u_1, u_2, w_1, w_2\}$.

(c) See figure 3.10(c). The third special subgraph SG_3 has eight white vertices, the first candidate v_f, its two neighbors u and w, two neighbors u_1 and u_2 of u, two neighbors w_1 and w_2 of w, and z, where $dist_G(z, v_f) \geq 3$, and moreover, vertices of $\{v_f, u, w, u_1, u_2, w_1, w_2\}$ are not in set B and maybe z is in set B. The conditions **(c1)** and **(c2)** are the same as **(b1)** and **(b2)**, respectively. **(c3)** The conditions on $dist_G(u_1, w_2)$ and $dist_G(u_2, w_1)$ are different from the above: $dist_G(u_1, w_2) = 2$ or $dist_G(u_2, w_1) = 2$ holds. That is, there is one vertex z between u_1 and w_2(or one vertex z between u_2 and w_1). For SG_3 with $dist_G(u_1, w_2) = 2$ in figure 3.10(c), if

z is not removed, then ALG selects v_f and z into $D3IS(G)$; else ALG selects u_1.

(d) See figure 3.10(d). The fourth special subgraph SG_4 consists of nine white vertices, the first candidate v_f, its two neighbors u and w, two neighbors u_1 and u_2 of u, two neighbors w_1 and w_2 of w, and z_1, and moreover, vertices v_f, u, w, u_1, u_2, w_1 and w_2 are not removed into set B. **(d1)** is the same as **(b1)**. **(d2)** $dist_G(u_1, w_1) = 1$ and $dist_G(u_2, w_1) = 2$ hold. **(d3)** u_2 and w_1 are intersected at the vertex z_1. Since this subgraph is not contained in SG_3, it holds $dist_G(u_2, w_2) \geq 2$, and then ALG selects w and u_2 into $D3IS(G)$.

(e) See figure 3.10(e). The fifth special subgraph SG_5 consists of eight white vertices, v_f, its two neighbors u and w, two neighbors u_1 and u_2 of u, two neighbors w_1 and w_2 of w, and z_1, and moreover, vertices v_f, u, w, u_1, u_2, w_1, w_2 are not eliminated into set B. **(e1)** is the same as **(b1)**. **(e2)** $dist_G(z_1, v_f) \geq 3$ holds. **(e3)** $dist_G(u_2, z_1) = dist_G(w_1, z_1) = 1$ and $dist_G(u_1, w_2) = 1$ hold. Then, SG_5 is not contained in subgraphs SG_2 and SG_3, and $dist_G(w_1, u_2) \geq 2$. Then, ALG selects u and w_1 into $D3IS(G)$.

(f) See figure 3.10(f). The sixth special subgraph SG_6 has seven white vertices, v_f, its two neighbors u and w, two neighbors u_2 and w_1 of u, two neighbors w_1 and w_2 of w, and u_1 whose $dist_G(u_1, v_f) = 2$ holds, and these vertices are not removed into set B. **(f1)** u and w are intersected at a same vertex w_1. **(f2)** There are three edges, $\{u_1, w_1\}$, $\{u_2, w_2\}$, and $\{u_1, w_2\}$. ALG selects w_1 into $D3IS(G)$.

(g) See figure 3.10(g). The seventh special subgraph SG_7 consists of eight white vertices, v_f, its two neighbors u and w, two neighbors u_1 and u_2 of u, two neighbors w_1 and w_2 of w, and v_1, where $dist_G(v_f, v_1) = 2$. Vertices v_f, u, w, v_1, u_2, w_2 are not eliminated into B, and maybe vertex u_1 or w_1 is eliminated. **(g1)** is the same as **(b1)**. **(g2)** The vertex v_1 is connected to one of u_1 and u_2, and one of w_1 and w_2. Now, without loss of generality, we assume that there are two edges $\{v_1, u_2\}$ and $\{v_1, w_2\}$ as shown in figure 3.10(g). Then, **(g3)** There is no edge $\{u_2, w_2\}$. **(g4)** Possibly, there is one edge,$\{u_1, w_1\}$, $\{u_1, w_2\}$ or $\{u_2, w_1\}$. Note that maybe u_1 or w_1 is eliminated, and thus, any vertex of $\{u_1, w_1\}$ is not selected into $D3IS(G)$ in the algorithm. (i) If $dist_G(u_1, w_1) = 1$ and $dist_G(u_1, w_2) \geq 2$ (i.e., SG_2 or SG_3 implies no the edge $\{u_1, w_2\}$), then ALG selects two vertices w_2 and u into $D3IS(G)$. (ii) If $dist_G(u_1, w_2) = 1$ and $dist_G(u_2, w_1) \geq 2$(i.e., SG_2 or SG_3 implies no the edge $\{u_1, w_2\}$), then ALG selects two vertices w and u_2 into $D3IS(G)$. (iii) If $dist_G(u_2, w_1) = 1$ and then $dist_G(u_1, w_2) \geq 2$(i.e., SG_2 or SG_3 implies no the edge $\{u_1, w_2\}$), then ALG selects two vertices w_2 and u into $D3IS(G)$. (iv) there are no three edges, $\{u_1, w_1\}$, $\{u_1, w_2\}$, and $\{u_2, w_1\}$, then ALG selects two vertices w_2 and u into $D3IS(G)$.

(h) See figure 3.10(h). The eighth special subgraph SG_8 consists of eight white vertices, the first candidate vertex v_f, two neighbors u and w, two neighbors u_1 and u_2 of u, two neighbors w_1 and w_2 of w, and v_1, and these vertices are not removed into set B. **(h1)** is the same as **(b1)**. **(h2)** There are three edges, $\{v_1, u_2\}$, $\{u_2, w_1\}$, and $\{u_1, w_1\}$. One can verify that if the graph has an edge $\{v_1, w_2\}$, then it can be regarded as SG_7, and if there is an edge $\{u_1, w_2\}$, then it can be regarded as SG_1 or SG_2. Therefore, all the three vertices v_1, u_1 and w_2 have neighbors which are not in SG_8. Note that $v = D_1(v_f) \setminus \{u, w\}$ holds. Then, (i) If the black vertex v is not removed, then ALG selects v and w_1 into $D3IS(G)$, and $|\{v, w_1\} \cup D_1(\{v, w_1\}) \cup D_2(\{v, w_1\})| \leq 13$. (ii) If v is removed, then ALG selects w and v_1 into $D3IS(G)$.

Recall that our algorithm ALG first finds every special subgraph and determines a (part of) solution in the special subgraphs as the preprocessing phase. After that, ALG iteratively executes the general phase, that is, it selects (i) the first candidate vertex v_f from $D_1(B)$ if $|\{v_f\} \cup D_1(v_f) \cup D_2(v_f) \setminus B| \leq 7$, but (ii) a neighbor u of v_f if $|\{v_f\} \cup D_1(v_f) \cup D_2(v_f) \setminus B| \geq 8$ into the distance-3 independent set. The following is the detailed description of ALG. In the preprocessing phase (**Phase 1**), the first candidate vertex v_f is selected and removed from a set F; the subgraph induced by $\{v_f\} \cup D_1(v_f) \cup D_2(v_f)$ is repeatedly checked whether it is identical to SG_1; after all SG_1's have been processed, the subgraph induced by $\{v_f\} \cup D_1(v_f) \cup D_2(v_f)$ is checked whether it is one of the seven special subgraphs SG_2, SG_3, SG_4, SG_5, SG_6, SG_7 and SG_8; and v_f is stored into a set C of "already checked" vertices. The vertex s_i in the distance-3 independent set is stored in $D3IS(G)$; its (closed) neighbors in $\{s_i\} \cup D_1(s_i) \cup D_2(s_i)$ are eliminated from V and stored into B.

Algorithm ALG

Input: Cubic graph $G = (V, E)$.

Output: Distance-3 independent set $D3IS(G)$ of G.

Initialization: Set $C = \emptyset$, $B = \emptyset$, $D3IS(G) = \emptyset$, and $F = \emptyset$.

Phase 1. Find all special subgraphs and determine a partial solution in them.

/* The vertices in all the special subgraphs SG_1, SG_2, SG_3, SG_4, SG_5, SG_6,

SG_7 and SG_8 are labeled as shown in figures 3.10(a), (b), (c), (d),(e),(f), (g) and (h), respectively. */

Step 0. Select arbitrarily one vertex v from V and set $F = F \cup \{v\}$.

Step 1 (SG_1). (i) If $B \cup C \neq V$ and thus $F \neq \emptyset$, then select arbitrarily one vertex $v_f \in F$, and set $F = F \setminus \{v_f\}$ and $C = C \cup \{v_f\}$. If the induced subgraph $G[\{v_f\} \cup D_1(v_f) \cup D_2(v_f)]$ includes SG_1 as its subgraph, then if $v \notin B$, then set $D3IS(G) = D3IS(G) \cup \{v, u_1\}$, $B = B \cup \{v, u_2\} \cup D_1(\{v, u_2\}) \cup D_2(\{v, u_2\})$, elseif $v \in B$, then set $D3IS(G) = D3IS(G) \cup \{u_1\}$, $B = B \cup \{u_1\} \cup D_1(\{u_1\}) \cup D_2(\{u_1\})$. $F = D_1(B \cup C) \setminus B$. Repeat **Step 1**. (ii) If $B \cup C = V$, then set $C = \emptyset$ and $F = D_1(B) \setminus B$, and goto **Step 2**.

Step 2. (i) If $B \cup C \neq V$, then select $v_f \in F$ and set $F = F \setminus \{v_f\}$ and $C = C \cup \{v_f\}$. If the induced subgraph $G[\{v_f\} \cup D_1(v_f) \cup D_2(v_f)]$ does not include any of the special subgraphs SG_2, SG_3, SG_4, SG_5, SG_6, SG_7 and SG_8, then set $F = D_1(B \cup C)$ and repeat **Step 2** (i.e., select a vertex $v'_f \neq v_f$ from F in the next iteration of **Step2**). If $G[\{v_f\} \cup D_1(v_f) \cup D_2(v_f)]$ includes SG_2, SG_3, SG_4, SG_5, SG_6, SG_7 and SG_8, then execute **Case 2-1**, **Case 2-2**, **Case 2-3**, **Case 2-4**, **Case 2-5**,**Case 2-6**, **Case 2-7** and **Case 2-8**, respectively. (ii) If $B \cup C = V$, then goto **Phase 2**.

Case 2-1 (SG_2): (i) If $dist_G(u_1, w_2) = dist_G(u_2, w_1) = 3$, then set $D3IS(G) = D3IS(G) \cup \{u_2, w_1\}$ and $B = B \cup \{u_2, w_1\} \cup D_1(\{u_2, w_1\}) \cup D_2(\{u_2, w_1\})$. (ii) If $dist_G(u_1, w_2) = 1$ and $dist_G(u_2, w_1) = 3$, then set $D3IS(G) = D3IS(G) \cup \{u_2, w_1\}$ and $B = B \cup \{u_2, w_1\} \cup D_1(\{u_2, w_1\}) \cup D_2(\{u_2, w_1\})$. (iii) If $dist_G(u_1, w_2) = dist_G(u_2, w_1) = 1$, then set $D3IS(G) = D3IS(G) \cup \{u_1\}$ and $B = B \cup \{u_1\} \cup D_1(\{u_1\}) \cup D_2(\{u_1\})$. Set $F = D_1(B \cup C)$ and goto **Step 2**. Set $F = D_1(B \cup C) \setminus B$ and goto **Step 2**.

Case 2-2 (SG_3): (i) if there is a z, which is not removed, then Set $D3IS(G) = D3IS(G) \cup \{v_f, z\}$ and $B = B \cup \{v_f, z\} \cup D_1(\{v_f, z\}) \cup D_2(\{v_f, z\}))$; else $D3IS(G) = D3IS(G) \cup \{u_1\}$ and $B = B \cup \{u_1\} \cup D_1(\{u_1\}) \cup D_2(\{u_1\}))$. Set $F = D_1(B \cup C) \setminus B$ and goto **Step 2**.

Case 2-3 (SG_4): $D3IS(G) = D3IS(G) \cup \{w, u_2\}$ and $B = B \cup \{w, u_2\} \cup D_1(w, u_2) \cup D_2(w, u_2)$. Set $F = D_1(B \cup C) \setminus B$ and goto **Step 2**.

Case 2-4 (SG_5): Set $D3IS(G) = D3IS(G) \cup \{u, w_1\}$ and $B = B \cup \{u, w_1\} \cup D_1(\{u, w_1\}) \cup D_2(\{u, w_1\})$. Set $F = D_1(B \cup C) \setminus B$ and goto **Step 2**.

Case 2-5 (SG_6): Set $D3IS(G) = \{w_1\} \cup D3IS$ and $B = B \cup \{w_1\} \cup D_1(\{w_1\}) \cup D_2(\{w_1\})$. Set $F = D_1(B \cup C) \setminus B$ and goto **Step 2**.

Case 2-6 (SG_7): (i) If $dist_G(u_1, w_1) = 1$ and $dist_G(u_1, w_2) \geq 2$, then $D3IS(G) \cup \{w_2, u\}$ and $B = B \cup \{u, w_2\} \cup D_1(\{u, w_2\}) \cup D_2(\{u, w_2\})$. (ii) If $dist_G(u_1, w_2) = 1$ and $dist_G(u_2, w_1) \geq 2$, then $D3IS(G) \cup \{w, u_2\}$ and $B = B \cup \{w, u_2\} \cup D_1(\{w, u_2\}) \cup D_2(\{w, u_2\})$. (iii) If $dist_G(u_2, w_1) = 1$ and then $dist_G(u_1, w_2) \geq 2$, then $D3IS(G) \cup \{w_2, u\}$ and $B = B \cup \{w_2, u\} \cup D_1(\{w_2, u\}) \cup D_2(\{w_2, u\})$. (iv) If there are no three edges $\{u_1, w_1\}$, $\{u_1, w_2\}$, and $\{u_2, w_1\}$, then $D3IS(G) \cup \{w_2, u\}$ and $B = B \cup \{w_2, u\} \cup D_1(\{w_2, u\}) \cup D_2(\{w_2, u\})$. Set $F = D_1(B \cup C) \setminus B$ and goto **Step 2**.

Case 2-7 (SG_8): (i) If the black vertex v is not in B, then Set $D3IS(G) = D3IS(G) \cup \{v, w_1\}$ and $B = B \cup \{v, w_1\} \cup D_1(\{v, w_1\}) \cup D_2(\{v, w_1\})$. (ii) If the black vertex v is in B, then $D3IS(G) = D3IS(G) \cup \{w, v_1\}$ and $B = B \cup \{w, v_1\} \cup D_1(\{w, v_1\}) \cup D_2(\{w, v_1\})$. Set $F = D_1(B \cup C) \setminus B$ and goto **Step 2**.

Phase 2. If $B \neq V$, then $F = D_1(B) \setminus B$ repeat the following **Step 3**. Otherwise, goto **Termination**.

Step 3. Select one candidate vertex v_f from F such that $|\{v_f\} \cup D_1(v_f) \cup D_2(v_f) \setminus B|$ is minimum among all vertices in F.

Case 3-1: If $|(\{v_f\} \cup D_1(v_f) \cup D_2(v_f) \setminus B| \leq 7$, then set $D3IS(G) = D3IS(G) \cup \{v_f\}$ and $B = B \cup \{v_f\} \cup D_1(\{v_f\}) \cup D_2(\{v_f\})$. Goto **Phase 2**.

Case 3-2: /* Reselect a new candidate vertex from unremoved neighbors of set B at some time */
(i) If $|(\{v_f\} \cup D_1(v_f) \cup D_2(v_f) \setminus B| = 8$ and $|D_1(D_2(v_f) \setminus B) \cap B| = 1$, then $T_B^1 = D_1(B) \cap (D_2(v_f) \setminus B)$ and $T_B = D_1(D_2(v_f) \setminus B) \cap B$, and furthermore, if $D_1(D_1(T_B) \cap (D_2(v_f) \setminus B)) \cap (D_1(T_B^1) \cap (D_2(v_f) \setminus B)) \neq \phi$, then select one new candidate vertex v_f from T_B^1, and then go to Case 3-3. (ii) If $|(\{v_f\} \cup D_1(v_f) \cup D_2(v_f) \setminus B| = 8$, $|D_1(D_2(v_f) \setminus B) \cap B| \geq 2$ and there are two vertices in $D_1(D_2(v_f) \setminus B) \cap B$ such that each vertex of these two vertices in $D_1(D_2(v_f) \setminus B) \cap B$ is connected to two vertices

in $D_2(v_f) \setminus B$, then $D_2^+ = D_2(v_f) \setminus B$ and select one new candidate vertex from $D_1(B) \cap D_2^+$, and then go to Case 3-3.

Case 3-3: If $|(\{v_f\} \cup D_1(v_f) \cup D_2(v_f) \setminus B| \geq 8$ and at most one vertex in $D_2(v_f) \setminus B$ is adjacent to vertices in $B \cup D_2(v_f)$, then $D3IS(G) = D3IS(G) \cup \{v_f\}$ and $B = B \cup \{v_f\} \cup D_1(\{v_f\}) \cup D_2(\{v_f\})$. Goto **Phase 2**.

Case 3-4: If $|\{v_f\} \cup D_1(v_f) \cup D_2(v_f) \setminus B| \geq 8$ and at least two vertices in $D_2(v_f) \setminus B$ are adjacent to vertices in $B \cup D_2(v_f)$, then select one, say, u, of two vertices in $D_1(v_f)$ such that $|\{u\} \cup D_1(u) \cup D_2(u) \setminus B|$ is minimum. Goto **Phase 2**.

Case 3-5: If $|\{v_f\} \cup D_1(v_f) \cup D_2(v_f) \setminus B| \geq 8$ and $|\{u\} \cup D_1(u) \cup D_2(u) \setminus B| = |\{w\} \cup D_1(w) \cup D_2(w) \setminus B| = 7$ for $u, w \in D_1(v_f)$ and u is in a cycle $\langle u, u_1, u_2 \rangle$, then set $D3IS(G) = D3IS(G) \cup \{u\}$ and $B = B \cup \{u\} \cup D_1(\{u\}) \cup D_2(\{u\})$. Goto **Phase 2**.

Termination. Terminate and output $D3IS(G)$ as a solution.

[End of ALG]

Approximation ratio. The algorithm ALG always outputs a feasible solution since ALG eliminates all vertices in $\{s\} \cup D_1(s) \cup D_2(s)$ from the solution candidates if s is in the solution. In this section, we will investigate the approximation ratio of ALG. We first give notation used in the following. Suppose that given a graph G, ALG outputs $ALG(G) = D3IS(G) = \{s_1, s_2, \cdots, s_\ell\}$. Also, without loss of generality, suppose that ALG selects those ℓ vertices into $D3IS(G)$, one by one in the order, i.e., first s_1, next s_2, and so on. We say a vertex as a first candidate vertex, which is picked up from neighbors of the set B before a solution vertex is selected. Let v_i denote the first candidate vertex when the ith vertex s_i is selected into $D3IS(G)$, and it is called the *ith first candidate*. Also, we call s_i the ith *solution vertex*. Note that if **Case 3-2** of ALG was executed, then the previous first candidate vertex of the set $D_1(B) \setminus B$, say v_f, is changed to anther vertex, say $v_f^{'}$, which is also in the set $D_1(B) \setminus B$, and then we say that the *ith first candidate* is changed to the vertex $v_f^{'}$, and in other words, $v_i = v_f$ is modified to $v_i = v_f^{'}$, and the vertex v_f is not a first candidate vertex, otherwise v_f is a first candidate vertex v_i.

For a vertex v, let $B(v) = \{v\} \cup D_1(v) \cup D_2(v)$ be a set of vertices such that $dist_G(u, v) \leq 2$ for any $u \in B(v)$. We say that a block is a set of vertices. Especially, for the ith solution vertex s_i in $ALG(G)$ ($i = 1, \cdots, \ell$), we call $B(s_i)$ the *ith solution*

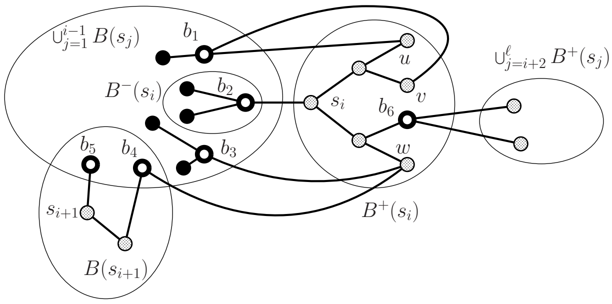


Figure 3.11: Blocks, and near/far boundary vertices

block. Let $B^-(s_i) = B(s_i) \cap (\bigcup_{j=1}^{i-1} B(s_j))$ and $B^+(s_i) = B(s_i) \setminus (\bigcup_{j=1}^{i-1} B(s_j))$, and we call $B^-(s_i)$ and $B^+(s_i)$ the *ith old solution block* and the *ith new solution block*, respectively. Let $D_1^+(s_i) = D_1(s_i) \cap B^+(s_i)$ and $D_2^+(s_i) = D_2(s_i) \cap B^+(s_i)$. Consider the time when the ith solution s_i is selected and $\bigcup_{j=1}^{i} B(s_j)$ are removed from V. Then, we define the set of *boundary vertices* in the block $B(s_i)$ $(= B^-(s_i) \cup B^+(s_i))$ by $BV(s_i) = D_1(V \setminus (\bigcup_{j=1}^{i} B(s_j))) \cap B^+(s_i)$ for each i $(1 \leq i \leq \ell - 1)$. Let $BV(ALG) = \bigcup_{i=1}^{\ell-1} BV(s_i)$ be the set of all the boundary vertices, and a vertex in $BV(ALG)$ is a boundary vertex. Also, we define the *near boundary vertices* from s_i by $BV_{near}(s_i) = (D_1(s_i) \cup D_2(s_i)) \cap (\bigcup_{j=1}^{i-1} BV(s_j))$. Note that $BV_{near}(s_i)$ is *not in* $B^+(s_i)$. Let $B^*(s_i) = B^+(s_i) \cup BV_{near}(s_i)$. Moreover, let $BV_{near}(ALG) = \bigcup_{i=1}^{\ell-1} BV_{near}(s_i)$ and $BV_{far} = BV(ALG) \setminus BV_{near}(ALG)$ be the sets of all the *near boundary* and all the *far boundary* vertices, respectively.

For example, take a look at figure 3.11, which illustrates the first $i-1$ blocks $\bigcup_{j=1}^{i-1} B(s_j)$, the ith block $B(s_i) = B^-(s_i) \cup B^+(s_i)$, the $(i+1)$st block $B(s_{i+1})$, and the remaining new blocks $\bigcup_{j=i+2}^{\ell} B^+(s_j)$. The five vertices b_1 through b_5 are the boundary vertices in $\bigcup_{j=1}^{i-1} B(s_j)$, i.e., $\bigcup_{j=1}^{i-1} BV(s_j) = \{b_1, b_2, b_3, b_4, b_5\}$ since those five vertices are connected to vertices in $V \setminus (\bigcup_{j=1}^{i-1} B(s_j))$. Also, the vertex b_6 is the boundary vertex in $B(s_i)$ since there is at least one edge between b_6 and a vertex in $\bigcup_{j=i+2}^{\ell} B^+(s_j)$. The three vertices b_2, b_4, and b_5 are the near boundary vertices since $dist_G(s_i, b_2) \leq 2$, $dist_G(s_{i+1}, b_4) \leq 2$, and $dist_G(s_{i+1}, b_5) \leq 2$ hold. Furthermore, three vertices b_2, b_4, and b_5 are in set $BV_{near}(s_i) \cup BV_{near}(s_{i+1})$. The vertex b_1 is a far boundary vertex since $dist_G(s_i, b_1) \geq 3$ holds (in other words, "b_1 is far from all the new blocks").

Next, consider ℓ integers, δ_1 through δ_ℓ, which are associated with ℓ new solution blocks, $B^+(s_1)$ through $B^+(s_\ell)$, and initially set $\delta_1 = \cdots = \delta_\ell = 0$. Recall that each far boundary vertex bv in $BV_{far} \cap B(s_i)$ must be connected to one or two vertices not in $B(s_i)$. Suppose that the far boundary vertex bv is connected to two vertices in $B^+(s_j)$. Then, we set $\delta_j = 1$. Suppose that the far boundary vertex bv is connected to two vertices, one in $B^+(s_j)$ and one in $B^+(s_k)$ for $j \neq k$. Then, if $j > k$, then we set $\delta_j = 1$; otherwise, $\delta_k = 1$. Therefore, $\sum_{i=1}^{\ell} \delta_i = |BV_{far}|$ holds. Now see figure 3.11 again. Suppose that b_1 and b_3 are far boundary vertices. Since the ith new block $B^+(s_i)$ is connected to two far boundary vertices b_1 and b_3, we set $\delta_i = 2$.

Lemma 4. For a first candidate v_i, where $2 \leq i \leq \ell$, we can observe that $|B^+(v_i)| \leq 8$ holds. Then, Suppose that the ith solution vertex s_i is selected in **Phase 2** of ALG, and s_i is not the first candidate v_i. Also, suppose that $|B^+(v_i)| = 8$. Then, $|B^+(s_i)| \leq 7$ holds, and furthermore, if $|B^+(s_i)| = 7$ occurs, then s_i must be in a cycle of length at most three.

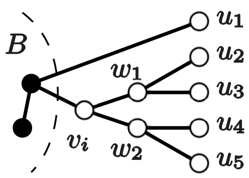


Figure 3.12: $B(v_i) \setminus \bigcup_{j=1}^{i-1} B(s_j)$

Proof. See figure 3.12. For ease of exposition, take a look at a graph consisting of vertices in $B^+(v_i) = \{v_i, w_1, w_2, u_1, u_2, u_3, u_4, u_5\}$. The algorithm implies that s_i is a selected vertex of set $\{w_1, w_2\}$ into the solution. Now suppose that all special subgraphs have been already processed in **Phase 1** of ALG. Then, from the assumption that s_i is not v_i and **Phase 2** of ALG is executed, at least two vertices, say, u_{i_1} and u_{i_2}, in $\{u_1, u_2, u_3, u_4, u_5\}$ are only adjacent to vertices in $\bigcup_{j=1}^{i-1} B(s_j) \cup \{u_1, u_2, u_3, u_4, u_5\}$. Without loss of generality, we only need to consider the following three cases on the edge $\{u_{i_1}, u_{i_2}\}$: Case (1) $\{u_{i_1}, u_{i_2}\} = \{u_1, u_2\}$, Case (2) $\{u_{i_1}, u_{i_2}\} = \{u_2, u_3\}$, and Case (3) $\{u_{i_1}, u_{i_2}\} = \{u_3, u_4\}$. For example, $\{u_{i_1}, u_{i_2}\} = \{u_2, u_5\}$ is essentially the same as (3).

See case (1). Now suppose that u_1 and u_2 are only adjacent to vertices in $\bigcup_{j=1}^{i-1} B(s_j) \cup \{u_1, u_2, u_3, u_4, u_5\}$. Note that the following arguments can be applied for the cases where $\{u_1, u_3\}$, $\{u_1, u_4\}$, and $\{u_1, u_5\}$. If u_1 is connected to a vertex in

$\bigcup_{j=1}^{i-1} B(s_j)$, then u_1 has two neighbors in $\bigcup_{j=1}^{i-1} B(s_j)$ and then $|B^+(v_i)| \leq |B^+(u_1) \leq 6$, which is a contradiction. Therefore, we can assume that u_1 is connected to two vertices in the set $\{u_2, u_3, u_4, u_5\}$. (1-1) Suppose that u_1 is connected to u_2 and u_3 (a pair of u_4 and u_5 is essentially equivalent). Then, it holds $|B^+(u_1)| \leq 7$, which is a contradiction again. Moreover, the remaining cases (except for the essentially equivalent ones) contain that u_1 is connected to either (1-2) u_3 and u_4, or (1-3) u_2 and u_4. (1-2) Suppose that u_1 is connected to u_3 and u_4. If u_2 is connected to two vertices in the set $\bigcup_{j=1}^{i-1} B(s_j)$, then we can verify $|B^+(u_2) \leq 6$, which is also contradictory. Thus, u_2 is connected to at least one vertex of the set $\{u_3, u_4, u_5\}$. Then, if u_2 is connected to u_3, then ALG should select w_1 into $D3IS(G)$ and w_1 is in a cycle $\langle w_1, u_2, u_3 \rangle$ of length three. If u_2 is connected to u_4 or u_5, then the graph is equivalent to SG_7 or SG_1, contradiction. (1-3) Suppose that u_1 is connected to u_2 and u_4. If u_2 is connected to a vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, then $|B^+(v_i)| \leq |B^+(u_2)| \leq 7$ holds, which is a contradiction. Then, u_2 is connected to one vertex of the set $\{u_3, u_4, u_5\}$. If u_2 is connected to u_3 and ALG selects w_1 as a solution vertex, then w_1 is in a cycle $\langle w_1, u_2, u_3 \rangle$ of length three. If $|B^+(w_2)| < |B^+(w_1)| \leq 7$, then ALG might select w_2. One can verify that w_2 must be again in a cycle $\langle w_2, u_4, u_5 \rangle$ of length three when $s_i = w_2$ and $|B^+(w_2)| = 7$. The case, where u_2 is connected to u_4, is also a contradiction since $|B^+(u_1)| \leq 6$. Finally, if u_2 is connected to u_5, then the graph is again equivalent to SG_7 or SG_1, contradiction.

Consider case (2). Next suppose that u_2 and u_3 are only adjacent to vertices in $\bigcup_{j=1}^{i-1} B(s_j) \cup \{u_1, u_2, u_3, u_4, u_5\}$. First, if u_1 is connected to two vertices in $\{u_2, u_3, u_4, u_5\}$, then u_1 and u_2 are adjacent to vertices in $\bigcup_{j=1}^{i-1} B(s_j) \cup D_2(v_i)$, and the case has been discussed in the previous Case (1). Thus, we can only consider cases, where u_1 is connected to at most one vertex of the set $\{u_2, u_3, u_4, u_5\}$. Then, except equivalent cases, all cases contain: (2-1) u_1 is connected to u_2, and u_1 is connected to neither u_3, u_4, nor u_5(which the following analyses can be applied for the case that u_1 is connected to u_3, and u_1 is connected to neither u_2, u_4 nor u_5.), and (2-2) u_1 is connected to neither u_2 nor u_3.

(2-1) Suppose that $dist_G(u_1, u_2) = 1$, $dist_G(u_1, u_3) \geq 2$, $dist_G(u_1, u_4) \geq 2$, and $dist_G(u_1, u_5) \geq 2$. Since u_2 is only connected to vertices in $\bigcup_{j=1}^{i-1} B(s_j) \cup D_2(v_i)$, this case is further divided to three cases: (i) $dist_G(u_2, u_3) = 1$ is satisfied. Then, ALG should select w_1 since $|\{w_1\} \cup D_1(w_1) \cup D_2(w_1) \setminus (\bigcup_{j=1}^{i-1} B(s_j))| \leq 7$, or select w_2 if $|B^+(w_2)| \leq 6$, and w_1 is in a cycle of length three. (ii) $dist_G(u_2, u_4) = 1$ occurs. If u_3 is connected to u_4 and u_5, then the graph is equivalent to SG_2, contradiction.

If u_3 is connected to a vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ and u_4 (u_5, resp.), then u_3 must be the first candidate (the graph is equivalent to SG_2 or SG_3, resp.), contradiction. (iii) $dist_G(u_2, u_5) = 1$ holds. If u_3 is connected to both u_4, then the graph contain SG_2 or SG_3. Then, u_3 is only connected to vertices in $\{u_5\} \cup \bigcup_{j=1}^{i-1} B(s_j)$, and furthermore, if u_3 is only connected to two vertices in $\bigcup_{j=1}^{i-1} B(s_j)$, then $|B^+(v_i)| \leq |B^+(u_3)| \leq 6$ occurs, which is contradictory. Then, u_3 must be connected to u_5 and one vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and we can count $|B^+(v_i)| \leq |B^+(u_3)| = 7$, contradiction. (iv) u_2 is connected to one vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and then $|B^+(u_2)| = 7$ holds, which implies $|B^+(v_i)| \leq |B^+(u_2)| = 7$, contradiction.

(2-2) Suppose that u_1 is not connected to any vertex in $\{u_2, u_3\}$. Then, it occurs $dist_G(u_2, u_1) \geq 2$ and $dist_G(u_3, u_1) \geq 2$, and since u_2 or u_3 is not connected to two vertices in $\bigcup_{j=1}^{i-1} B(s_j)$, and thus, u_2 must be connected to one vertex of the set $\{u_3, u_4, u_5\}$, and u_3 must be connected to one vertex of the set $\{u_2, u_4, u_5\}$. Concentration on the vertex u_2, the cases of (i) $dist_G(u_2, u_3) = 1$ and (ii) $dist_G(u_2, u_3) \neq 1$ and $dist_G(u_2, u_4) = 1$(equivalently, $dist_G(u_2, u_5) = 1$) are need to be considered. (i) Suppose $dist_G(u_2, u_3) = 1$. First, we can verify $|B^+(s_i)| \leq |B^+(w_1)| \leq 7$ and w_1 is in a cycle of length three. Then, if $s_i = w_1$ occurs, then s_i is in a cycle of length three. If $|B^+(s_i)| = 7$ and $s_i = w_2$ hold, then one can verify that w_2 is also in a cycle of length three. (ii) Suppose $dist_G(u_2, u_3) \neq 1$ and $dist_G(u_2, u_4) = 1$. Recall that u_3 must be connected to one vertex of the set $\{u_2, u_4, u_5\}$, u_3 must be connected to one in $\{u_4, u_5\}$. If $dist_G(u_3, u_5) = 1$ holds, then the block $B^+(v_i)$ contains a subgraph of SG_2 or SG_3, contradiction. Thus, u_3 must be connected to u_4. Recall that u_3 is not connected to u_1, u_2 or u_5, and thus, u_3 must be connected to one vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and can find $|B^+(u_3)| \leq 7$, and algorithm should select a vertex v_i, and $|B^+(v_i)| \leq |B^+(u_3)| \leq 7$ holds, which is contradictory.

See case (3). Finally, suppose that u_3 and u_4 are adjacent to vertices in $\bigcup_{j=1}^{i-1} B(s_j) \cup \{u_1, u_2, u_3, u_4, u_5\}$. Since all cases, where u_1 is connected to two vertices in $\{u_2, u_3, u_4, u_5\}$, has been discussed in case (1). Thus, can further suppose that u_1 can be connected to at most one vertex in $\{u_3, u_4\}$, we consider the following two cases: (3-1) u_1 is connected to one vertex in $\{u_3, u_4\}$, and (3-2) u_1 is not connected to any in $\{u_3, u_4\}$.

(3-1) Suppose that u_1 is connected to u_3, and it is equivalent with another assumption, which u_1 is connected to u_4. If u_1 is connected to u_3, then u_1 is not connected to u_4, and u_3 can be connected to one vertex in $\{u_2, u_4, u_5\}$ or one vertex in $\bigcup_{j=1}^{i-1} B(s_j)$. (i) If $dist_G(u_3, u_2) = 1$, then we can verify $|B^+(s_i)| \leq |B^+(w_1)| \leq 7$, and w_1 is in a cycle of length three, and thus, if $|B^+(s_i)| = 7$ holds, then the operation of this algorithm implies that s_i must be in a cycle of length three. (ii) Consider the case $dist_G(u_3, u_4) = 1$. Moreover, u_4 is connected to u_2, u_5 or one vertex in $\bigcup_{j=1}^{i-1} B(s_j)$. If u_4 is connected to u_2, then $B^+(v_i)$ contain the subgraph of SG_8. If u_4 is connected to u_5, then $|B^+(s_i)| \leq |B^+(w_2)| \leq 7$ holds and w_2 is in a cycle of length three. When $|B^+(s_i)| = 7$ occurs, this algorithm should select s_i, which is in a cycle of length three. If u_4 is connected to one vertex b in $\bigcup_{j=1}^{i-1} B(s_j)$, then for $dist_G(v_1, b) \leq 2$, we should verify $|B^+(v_i)| \leq |B^+(v_1)| \leq 7$, which is contradictory. Thus, $dist_G(v_1, b) = 3$, and then **Case 3-2(i)** of this algorithm should be executed, and then, v_1 is picked up as a new first candidate vertex substituting the previous first candidate vertex v_i, i.e., the vertex v_i in the figure 3.12 is not a first candidate vertex, and then, v_1 is the first candidate vertex v_i, and since $|B^+(v_i)| = |B^+(v_1)| = 8$, we can obviously find that at least four vertices in $D_2^+(v_1)$ is connected to $\bigcup_{j=i+1}^{\ell} B^+(s_j)$ and $s_i = v_i$ occurs, i.e., this algorithm selects the first candidate vertex as a solution vertex s_i. (iii) Suppose that $dist_G(u_3, u_5) = 1$. Then, we only need to consider two cases, that is, (iii-1) u_4 is connected to both u_2 and u_5, or (iii-2) u_4 is connected to one in $\{u_2, u_5\}$ and another in $\bigcup_{j=1}^{i-1} B(s_j)$. (iii-1) If u_4 is connected to both u_2 and u_5, then the graph is SG_2 or SG_3, contradiction. (iii-2) If u_4 is connected to u_2 and one vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, then the graph is equivalent to SG_2 or SG_3, again contradiction. If u_4 is connected to u_5 and one vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, then we can verify $|B^+(u_4)| \leq 7$, and `ALG` should choose the vertex v_i as a first candidate vertex such that $|B^+(v_i)| \leq |B^+(u_4)| \leq 7$, contradiction.

(3-2) Suppose that u_1 is not connected to any in $\{u_3, u_4\}$. Obviously, each vertex of the set $D_2^+(v_i)$ is connected to at most one vertex in $\bigcup_{j=1}^{i-1} B(s_j)$. Thus, u_3 and u_4 must be connected to one vertex in $\{u_2, u_3, u_4, u_5\}$. (i) If $dist_G(u_2, u_3) = 1$ ($dist_G(u_4, u_5) = 1$, resp.), then w_1 (w_4, resp.) is selected and it is in a cycle of length three. Therefore,$|B^+(w_1)| \leq 7$ ($|B^+(w_2)| \leq 7$, resp.) holds. Then, it implies that if $|B^+(s_i)| = 7$ is satisfied, then s_i is in a cycle of length three. Then, (ii) suppose that $dist_G(u_2, u_3) \geq 2$ and $dist_G(u_4, u_5) \geq 2$. Then, there are two cases: (ii-1) u_3 is connected to u_4 and u_5, and (ii-2) u_3 is connected to one vertex in $\{u_4, u_5\}$ and another vertex in $\bigcup_{j=1}^{i-1} B(s_j)$. (ii-1) If u_3 is connected to both u_4 and u_5, then u_4 can be connected to u_2 or one vertex in $\bigcup_{j=1}^{i-1} B(s_j)$. If $dist_G(u_4, u_2) = 1$, then the graph is equivalent to SG_2 or SG_3, or if u_4 is connected to one vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and then, $|B^+(v_i)| \leq |B^+(u_4)| \leq 7$ holds, contradiction. (ii-2) Suppose that u_3 is connected to one vertex in $\{u_4, u_5\}$ and another vertex in $\bigcup_{j=1}^{i-1} B(s_j)$. Then, if

$dist_G(u_3, u_4) = 1$ and another vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, then ALG selects u_3 as the first candidate vertex since $|B^+(u_3)| \leq 7$. If $dist_G(u_3, u_5) = 1$ and another vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, then recall that $dist_G(u_4, u_5) \geq 2$, $dist_G(u_4, u_1) \geq 2$ and u_4 must be connected to one vertex in $\{u_2, u_3, u_4, u_5\}$, and then, u_4 must be connected to u_2 and thus the block $B^+(v_i)$ contain a subgraph SG_2 or SG_3, which is a contradiction. This completes the proof of this lemma. □

Lemma 5. Suppose that s_i ($2 \leq i \leq \ell$) is selected into $D3IS(G)$ in **Phase 2** of ALG. Then, $|B^*(s_i)| \leq 9$ holds.

Proof. (1) First, suppose that s_i is identical to v_i, which is the first candidate. Then, $dist_G(s_i, s_j) \geq 3$ holds for $1 \leq j < i$, i.e., there must exist the path, say, $\langle s_j, u, v, s_i \rangle$ of length three. One can see that v is a boundary vertex in $B(s_j)$, but u is not. Since $|B(s_i)| \leq 10$, we obtain $|B^*(s_i)| \leq 10 - 1 = 9$. (2) Then, suppose that s_i is not identical to v_i. (2-1) If $|B^+(s_i)| = 7$, then from Lemma 4, we can know that s_i is in a cycle of length three, and $|\{s_i\} \cup D_1(s_i) \cup D_2(s_i)| \leq 8$. Therefore, it holds $|B^*(s_i)| \leq |B(s_i)| \leq 8$. (2-2) Next assume that $|B^+(s_i)| \leq 6$. Since s_i is not identical to v_i, $|B^+(v_i)| = 8$ holds, and s_i is in $D_1^+(v_i)$, and we can verify that no vertex in $D_1(s_i)$ are in $\bigcup_{j=1}^{i-1} B(s_j)$, and $|D_1(s_i)| = |D_1^+(s_i)| = 3$. If a vertex u in $D_1(s_i)$ is connected to at least two vertices in $\bigcup_{j=1}^{i-1} B(s_j)$, then one can verify $|B^+(v_i)| \leq |B^+(u)| \leq 6$, and then, s_i is identical to v_i, contradiction. Thus, each vertex of the set $D_1(s_i)$ is connected to at most one vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and $|B(s_i) \cap \bigcup_{j=1}^{i-1} B(s_j)| \leq 3$. Then, we further obtain $|B(s_i)| = |B^+(s_i)| + |B(s_i) \cap \bigcup_{j=1}^{i-1} B(s_j)| \leq 6 + 3$. Thus, obtain $|B^*(s_i)| \leq |B(s_i)| \leq 9$.This completes the proof of this lemma. □

Lemma 6. Suppose that given a graph $G = (V(G), E(G))$, only **Phase 1** is executed in ALG. Then, $|V(G)|/|ALG(G)| \leq 7.5$ is satisfied.

Proof. (1) Suppose that ALG finds SG_1 in figure 3.10(a). Note that in this step, only SG_1 is verified and processed. If ALG selects one vertex u_1 and the vertex v into $D3IS(G)$, and eliminates at most 12 vertices in $\{\{v, u_1\} \cup D_1(\{v, u_1\}) \cup D_2(\{v, u_1\})\}$. Then, if algorithm ALG only selects u_1, which is not connected to v_1, into $D3IS$, and then v is in the set B. We find that if three subgraphs SG_1 are connected to one same neighbor vertex of the vertex v of each subgraph SG_1, then ALG verifies these three subgraph SG_1 successively, and algorithm ALG must select an optimal solution, and thus, without loss of generality, consider that at most two subgraphs SG_1 are connected to one same neighbor vertex of the vertex v

of each subgraph SG_1, which one subgraph generates two solution vertices v' and u_1' into $D3IS$, and another subgraph generate a solution vertex u_1 into $D3IS$. Furthermore, if we can regard such two subgraphs as an unit, and then for the unit, since $|\{u_1, v', u_1'\} \cup D_1(\{u_1, v', u_1'\}) \cup D_2(\{u_1, v', u_1'\})| \leq 8 + 12 < 21$, we can know that after selecting three solution vertices u_1, v' and u_1', 21 vertices are removed into the set B. That is, we can averagely select one vertex among seven ones. On the average, we can select one vertex among seven vertices for all subgraphs SG_1.

(2) Suppose that ALG finds SG_2 as labeled in figure 3.10(b). (i) If $dist_G(u_1, w_2) = dist_G(u_2, w_1) = 3$, then ALG selects u_2 and w_1 into $D3IS(G)$ and eliminates vertices in $\{u_2, w_1\} \cup D_1(\{u_2, w_1\}) \cup D_2(\{u_2, w_1\})$. Note that u_2 (and w_1) has one neighbor not in $V(SG_2)$, which has at most two neighbors. Furthermore, v may be in $D_2(\{u_2, w_1\})$. Therefore, $|\{u_2, w_1\} \cup D_1\{u_2, w_1\} \cup D_2\{u_2, w_1\}| \leq |V(SG_2)| + 7 = 15$ holds. That is, we can select two vertices among 15 ones; on the average, one among 7.5. (ii) If $(dist_G(u_1, w_2), dist_G(u_2, w_1))$ (or $(dist_G(u_2, w_1), dist_G(u_1, w_2))$) $= (1, 3)$, then ALG selects u_2 and w_1 (or u_1 and w_2) into $D3IS(G)$. Similarly, $|\{u_2, w_1\} \cup D_1\{u_2, w_1\} \cup D_2\{u_2, w_1\}| \leq |V(SG_2)| + 7 = 15$ holds. (iii) If $dist_G(u_1, w_2) = dist_G(u_2, w_1) = 1$, then ALG selects one arbitrary vertex in $\{u_1, u_2, w_1, w_2\}$ into $D3IS(G)$. Let u_1 be selected. Then, $|\{u_1\} \cap D_1(u_1) \cup D_2(u_1)| = 7$.

(3) For SG_3 in figure 3.10(c), ALG selects v_f and v_f', and $|B^+(v_f) \cup B^+(v_f')| \leq 14$ since v_f' has further one neighbor, which has two neighbors. That is, ALG finds one solution vertex among seven vertices on the average.

(4) For SG_4 in figure 3.10(d), ALG selects w, u_2 into $D3IS(G)$ and $|\{w, u_2\} \cup D_1(w, u_2) \cup D_2(w, u_2)| \leq 14$. That is, ALG finds one solution vertex among at most 7 vertices on the average.

(5) For SG_5 in figure 3.10(f), ALG selects u, w_1 into $D3IS(G)$, and $|\{u, w_1\} \cup D_1(u, w_1) \cup D_1(u, w_1)| \leq 7 + 8 \leq 15$. ALG finds one solution vertex among at most 7.5 vertices on the average.

(6) For SG_6 in figure 3.10(e),ALG selects w_1 into $D3IS(G)$, and $|\{w_1\} \cup D_1(\{w_1\}) \cup D_2(\{w_1\}) \setminus B| \leq ||\{w_1\} \cup D_1(\{w_1\}) \cup D_2(\{w_1\})| \leq 7$. As above shown, ALG finds one solution vertex among at most 7 vertices on the average.

(7) See again SG_7 in figure 3.10(g). (i) If$dist_G(u_1, w_1) = 1$ and $dist_G(u_1, w_2) \geq 2$, then $D3IS(G) \cup \{w_2, u\}$ and $|\{u, w_2\} \cup D_1(\{u, w_2\}) \cup D_2(\{u, w_2\})| < 15$. (ii) If $dist_G(u_1, w_2) = 1$ and $dist_G(u_2, w_1) \geq 2$, then $D3IS(G) \cup \{w, u_2\}$ and $|\{w, u_2\} \cup D_1(\{w, u_2\}) \cup D_2(\{w, u_2\})| < 15$. (iii) If $dist_G(u_2, w_1) = 1$ and then $dist_G(u_1, w_2) \geq$

2, then $D3IS(G) \cup \{w_2, u\}$ and $|\{w_2, u\} \cup D_1(\{w_2, u\}) \cup D_2(\{w_2, u\})| < 15$. (iv) There are no three edges, $\{u_1, w_1\}$, $\{u_1, w_2\}$, and $\{u_2, w_1\}$, then $D3IS(G) \cup \{w_2, u\}$ and $|\{w_2, u\} \cup D_1(\{w_2, u\}) \cup D_2(\{w_2, u\})| \leq 15$. As above shown, `ALG` finds one solution vertex among at most 7.5 vertices on the average.

(8) Consider SG_8 in figure 3.10(h). If the black vertex v is not in B, then `ALG` selects v and w_1 into $D3IS(G)$, and $|\{v, w_1\} \cup D_1(\{v, w_1\}) \cup D_2(\{v, w_1\})| \leq 13$. If v is in B, then `ALG` selects w and v_1 into $D3IS(G)$, and $|\{w, v_1\} \cup D_1(\{w, v_1\}) \cup D_2(\{w, v_1\}) \setminus B| \leq 13$. That is, `ALG` finds one solution vertex among at most 7.5 vertices on the average.

As a result, `ALG` selects one solution vertex among at most 7.5 *vertices on the average.* □

Now observe a block $B^+(s_i)$, and then, we can find that any far boundary vertex can be connected to at least one vertex in $D_2^+(s_i)$ or at most two vertices of $D_2^+(s_i)$. Here, we give two kinds for boundary vertices. For a boundary vertex, if it is connected to two vertices in $D_2^+(s_i)$, then we define this boundary vertex be a far-2 boundary vertex, else say this boundary vertex be a far-1 boundary vertex. For a block $B^+(s_i)$, any far boundary vertex, where is connected to the block $B^+(s_i)$, is in set $\bigcup_{j=1}^{i-1} B(s_j)$. Then, we can obtain observed results for boundary vertices as follows:

Observation 1. All neighbors except vertices of $D_2^+(s_i)$ of any far boundary vertex containing itself are in $\bigcup_{j=1}^{i-1} B(s_j)$.

Observation 2. Each boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ is connected to at least one vertex in $\bigcup_{j=1}^{i-1} B(s_j)$.

Observation 3. If a vertex u in the set $D_2^+(s_i)$ is connected two boundary vertices of $\bigcup_{j=1}^{i-1} B(s_j)$, then $|B^+(s_i)| \leq |B^+(v)| \leq 6$ holds. If a vertex u in the set $D_2^+(s_i)$ is connected three boundary vertices in $\bigcup_{j=1}^{i-1} B(s_j)$, then $|B^+(s_i)| \leq |B^+(v)| \leq 4$. We can observe this result since Observation 2.

Observation 4. If two far-1 boundary vertices bv' and bv_1' are intersected to one same vertex u in $D_2^+(s_i)$, then $|B^+(s_i)| \leq |B^+(u)| \leq 4$ holds since Observation 1.

Observation 5. If a far-1 boundary vertex bv' and a far-2 boundary vertex bv'' are intersected at one same vertex u in $D_2^+(s_i)$, then $|B^+(s_i)| \leq |B^+(u)| \leq 5$ holds since Observation 1.

Observation 6. If one far-2 boundary vertex is intersected with one far-2 boundary vertex at only one vertex u in $D_2^+(s_i)$, then we can find $|B^+(s_i)| \leq |B^+(u)| \leq 6$. We can observe this result since Observation 3.

Observation 7. If one far-2 boundary vertex is intersected with another far-2 boundary vertex at both vertices u_1, u_2 in the set $D_2^+(s_i)$, then we can verify $|B^+(s_i)| \leq |B^+(u)| \leq 5$, where $u \in \{u_1, u_2\}$. We can observe this result since Observation 1.

Observation 8. If one far-1 boundary vertex bv' in $D_2^+(s_i)$ is connected to one vertex u in $D_2^+(s_i)$ and moreover, this vertex u is connected to one other boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ except the vertex bv', then we can find $|B^+(s_i)| \leq |B^+(u)| \leq 5$. We can observe this result since 1 and 2.

Above observations are some important keys to discuss the lower bound value of $|BV(s_i)| - \delta_i$. Obviously, $|B^+(s_i)| \leq 8$ for $2 \leq i \leq \ell$ holds, and then we analyse each lower bound value of $|BV(s_i)| - \delta_i$ when $|B^+(s_i)| = 8, 7, 6, 5$ or $|B^+(s_i)| \leq 4$ holds.

Lemma 7. If $\delta_i = 0$, i.e., no far boundary vertex is connected to this block $B^+(s_i)$, then we can obtain $|BV(s_i)| - \delta_i \geq 4$ for $|B^+(s_i)| = 8$, and $|BV(s_i)| - \delta_i \geq 0$ for $|B^+(s_i)| = 7, 6, 5$ or $|B^+(s_i)| \leq 4$.

Proof. If $|B^+(s_i)| = 8$ holds, then this algorithm implies that four vertices in $D_2^+(s_i)$ are connected to vertices in $\bigcup_{j=i+1}^{\ell} B^+(s_j)$, and $|BV(s_i)| \geq 4$. $\delta_i = 0$ holds and thus, we can know $|BV(s_i)| - \delta_i \geq 4$ for $|B^+(s_i)| = 8$. If $|B^+(s_i)| \leq 7$ holds, then $\delta_i = 0$ holds and $|BV(s_i)| - \delta_i = |BV(s_i)| \geq 0$ is known, obviously. This lemma is proved. □

Then, for the convenience of discussion, without loss of generality, we first consider a block, and at least one far-1 boundary vertex must be connected to this block from Lemma 4 to Lemma 11, and then discuss other blocks of the remaining case, where no far-1 boundary vertex and only some far-2 boundary vertices are connected to the block in Lemma 13.

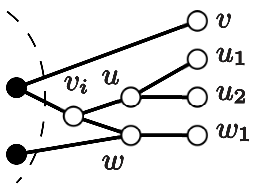
From previous Lemma 4, we know that if $|B^+(s_i)| = 8$ holds for $2 \leq i \leq \ell$, and thus, suppose that s_i is always identical to v_i for the case, where $|B^+(s_i)| = 8$ occurs. Note that v_i is a first candidate vertex in this block $B^+(s_i)$. Then, we can obtain the following lemma:

Lemma 8. Suppose that $|B^+(v_i)| = 8$ for $2 \le i \le \ell$ and v_i is selected into $D3IS(G)$ in **Phase 2** of ALG, i.e., $s_i = v_i$. Then, this block $B^+(s_i)$ is not connected to any far-1 vertex. That is, if a far-1 boundary vertex is connected to a vertex of $D_2^+(s_i)$, then $|B^+(v_i)| \le 7$ holds.

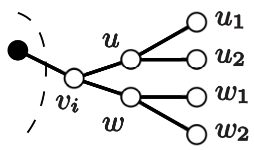
Proof. Suppose a far-1 boundary vertex bv' is connected one vertex, say u, in $D_2^+(s_i)$, and then since Observation 1, can know $|B^+(u)| \le 7$ holds, which implies that for the first candidate v_i in $B^+(s_i)$, $|B^+(v_i)| \le |B^+(u)| \le 7$, which implies that the algorithm should select the first candidate vertex v_i into this solution. Thus, if a far-1 boundary vertex is connected to this block $B^+(s_i)$, then $|B^+(s_i)| \le 7$ holds. Hence, this completes the proof of this lemma. □

Lemma 9. Suppose that $s_i \in D3IS(G)$ is selected in **Phase 2** of ALG. Then, if $|B^+(s_i)| = 7$, then it always holds $\delta_i \le |BV(s_i)|$, i.e., $|BV(s_i)| - \delta_i \ge 0$.

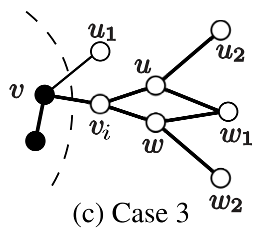
Proof. Obviously, any case contains either $s_i = v_i$ or $s_i \ne v_i$. Since this block is connected to at least far-1 boundary vertex and by Observation 1, implies that there is one vertex u in $D_2^+(s_i)$ such that $|B^+(u)| \le 7$ holds, where u is connected to a far-1 boundary vertex. Thus, $|B^+(s_i)| \le |B^+(v_i)| \le |B^+(u)| \le 7$ holds, and algorithm should select the first candidate vertex v_i. Thus, $s_i = v_i$ always occurs. By observation, we can find that at least two vertices in $BV(v_i)$ are in $\bigcup_{j=1}^{i-1} B(s_j)$, and then, get $|B^+(v_i)| = 7 \le |BV(v_i)| - 2$, and furthermore, $|B(v_i)| \ge 9$ holds, which it implies that v_i is not in any cycle of length three and is in at most one cycle of length four. Thus, there are only three cases, which can be illustrated in figure 3.13, where $s_i = v_i$ must hold. When v_i is not in a cycle of length four, case (1) and case (2) are illustrated in figure 3.13(a) and figure 3.13(b), respectively. When v_i is in a cycle of length four, the case (3) is shown in figure 3.13(c).



(a) Case 1



(b) Case 2



(c) Case 3

Figure 3.13: Three cases in the proof of Lemma 9.

For case (1), see figure 3.13(a), which a block $B^+(v_i)$ contains a set $\{v_i, u, w, v, u_1, u_2, w_1\}$ of vertices. From Observation 3, we take note that v is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and then, any far-1 and any far-2 boundary vertex are not connected to v since Observation 3. Moreover, if a far-1 is connected to w_1, then from Observation 1, we can verify $|B^+(s_i)| \leq |B^+(w_1)| \leq 6$, contradiction. Thus, w_1 is not connected to any far-1 boundary vertex. Furthermore, any far-1 boundary vertex is connected to u_1 or u_2. Without loss of generality, we can suppose that a far-1 boundary vertex, say bv', is connected to u_1, which is equivalent to $dist_G(bv', u_2)$. Then, the block can be connected to some far-2 boundary vertices except far-1 boundary vertices, and thus, only two cases are further generated, that is, (i) some far-2 boundary vertices are connected to the block, or (ii) no far-2 boundary vertex is connected to the block. See (i). Recall that any far-1 and any far-2 boundary vertex are not connected to the vertex v, and from Observation 3, we know that u_1 is not connected to any other boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ except bv'. Then, any far-2 boundary vertex is connected to both vertices u_2 and w_1, and furthermore, any vertex of the set $\{u_2, w_1\}$ is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ since Observation 3, and thus, except the far-1 boundary vertex bv', at most one far-2 boundary vertex is connected to both vertices u_2 and w_1, and say this far-2 boundary vertex be bv''. Then, $\delta_i \leq 2$ holds. We find that if $|BV(s_i)| - \delta_i < 0$ holds, then implies that at least one vertex p of the set $\{u_1, u_2, w_1\}$ is not in $BV(s_i)$, and recall that each vertex of the set $\{u_1, u_2, w_1\}$ is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and then, the vertex p is connected to one vertex except itself of the set $\{v, u_1, u_2, w_1\}$. If $dist_G(u_1, v) = 1, dist_G(u_1, u_2) = 1$ or $dist_G(u_1, w_1) = 1$ holds, then we can verify $|B^+(s_i)| \leq |B^+(u_1)| \leq 6$ since Observation 1, contradiction. Thus, it must be $dist_G(u_1, v) \geq 2, dist_G(u_1, u_2) \geq 2$ and $dist_G(u_1, w_1) \geq 2$, and then the vertex p is not denoted to the vertex u_1, and it must be $p \in \{u_2, w_1\}$. Then, if $p = w_1$ holds, then w_1 is connected to one vertex of the set $\{v, u_1, u_2\}$, and then since Observation 1, we can always verify $|B^+(s_i)| \leq |B^+(w_1)| \leq 6$, contradiction. Thus, it must be $p = u_2$, and then u_2 is connected to one vertex of the set $\{v, u_1, w_1\}$, and we find that if it holds $dist_G(u_2, u_1) = 1$ or $dist_G(u_2, w_1) = 1$, then since Observation 1, it holds $|B^+(s_i)| \leq |B^+(u_2)| \leq 6$, contradiction. Thus, $dist_G(u_2, u_1) \geq 2$ and $dist_G(u_2, w_1) \geq 2$ hold. Then, it must be $dist_G(u_2, v) = 1$, and then, since SG_7 does not appear and recall $dist_G(v, u_1) \geq 2$ holds, $dist_G(v, w_1) \geq 2$ holds. Moreover, recall $dist_G(w_1, u_1) \geq 2$, and furthermore, Observation 3 shows that v or w_1 is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and then, two vertices v and w_1 are connected to vertices in $\bigcup_{j=i+1}^{\ell} B^+(s_j)$, i.e., $v \in BV(s_i)$ and $w_1 \in BV(s_i)$. Then, it holds $|BV(s_i)| \geq 2$. Recall $\delta_i \leq 2$ holds, and then the previous assumption of $|BV(s_i)| - \delta_i < 0$ does not hold and this lemma is hold in the case (i). Then, see (ii) and no far-2 boundary vertex is connected to the block. By previous analyses, we can know that any far-1 boundary vertex is connected to u_1 or u_2. Here, suppose a contradiction of $|BV(s_i)| - \delta_i < 0$, which implies that a vertex p of $D_2^+(s_i)$ is connected to a far-1 boundary vertex and meanwhile, this vertex p is not in $BV(s_i)$. Without loss of generality, we can suppose $p = u_1$, that is, a far-1 boundary vertex bv' is connected to u_1(equivalently, u_2). Then, Observation 3 shows that u_1 must be connected to one vertex of the set $\{v, u_2, w_1\}$. If $dist_G(u_1, v) = 1$ or $dist_G(u_1, u_2) = 1$ holds, then it holds $|B^+(s_i)| \le |B^+(u_1)| \le 6$ since Observation 1, contradiction. Thus, it must be $dist_G(u_1, w_1) = 1$. If $dist_G(w_1, u_2) = 1$ holds or w_1 is connected to one boundary vertex in $\cap \bigcup_{j=1}^{i-1} B(s_j)$, then one can verify $|B^+(s_i)| \le |B^+(u_1)| \le 6$ since Observation 1, contradiction. Thus, $dist_G(w_1, u_2) \ge 2$ holds and w_1 is connected to one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$. Then, only two cases are further generated, that is, either (ii-1) $dist_G(w_1, v) = 1$ holds or (ii-2) $w_1 \in BV(s_i)$ holds. Then, for (ii-1) $dist_G(w_1, v) = 1$ holds, and then since SG_7 does not appear, $dist_G(v, u_2) \ge 2$ holds, and then Observation 3 shows that v or u_2 is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and thus, u_2 can be connected to at most one far-1 boundary vertex and $u_2 \in BV(s_i)$, v is not connected to any far-1 boundary vertex and it also holds $v \in BV(s_i)$, and moreover, at most two far-1 boundary vertex are connected to u_1 and u_2, respectively, and thus, $\delta_i \le 2$ and $|BV(s_i)| \ge 2$ hold, and thus, the previous assumption of $|BV(s_i)| - \delta_i < 0$ does not hold. For (ii-2), $w_1 \in BV(s_i)$ holds, which w_1 is not connected to any vertex of the set $\{v, u_1, u_2, w_1\}$. Recall v or w_2 is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and thus, v is connected to vertices of the set $\{u_2\} \cup \bigcup_{j=i+1}^{\ell} B^+(s_j)$ and at most one addition far-1 boundary vertex is connected to w_2, and thus $\delta_i \le 2$, and v must be in $BV(s_i)$, and furthermore, recall $w_1 \in BV(s_i)$ also holds, and thus $|BV(s_i)| = 2$, and then obtain $|BV(s_i)| - \delta_i \ge 0$ since $\delta_i \le 2$. Thus, the previous assumption of $|BV(s_i)| - \delta_i < 0$ does not hold and $|BV(s_i)| - \delta_i \ge 0$ is satisfied in this case (ii-2). In the final, we can know that all cases, which can be illustrated in figure 3.13(a), hold this lemma.

See case (2) in figure 3.13(b), which vertices in $B^+(s_i)$ are v_i, u, w, u_1, u_2, w_1 and w_2. Similarly, all cases contain that (i) at least one far-1 vertex and far-2 boundary vertices are connected to this block, or (ii) no far-2 boundary vertex is connected to this block. See (i), and then we can default that a far-1 boundary vertex, say bv', is connected to u_1, which is equivalent to $dist_G(bv', u_2) = 1, dist_G(bv', u_3) = 1$ or $dist_G(bv', u_4) = 1$. Then, u_1 is not connected to any other boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ except bv', and moreover, u_2, w_1 or w_2 is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ since Observation 3, and thus, at most one far-2 boundary vertex is connected to two vertices of the set $\{u_2, w_1, w_2\}$. Thus, suppose the far-2 boundary vertex is bv'', and then $dist_G(bv', u_1) = 1$ holds, and moreover, either (i-1) $dist_G(bv'', u_2) = dist_G(bv'', w_1) = 1$ holds, which is equivalent to $dist_G(bv'', u_2) = dist_G(bv'', w_2) = 1$, or (i-2) $dist_G(bv'', w_2) = dist_G(bv'', w_1) = 1$ holds. For (i-1), from Observation 3, we can know that for the block, except the far-1 boundary vertex bv' and the far-2 boundary vertex bv'', at most one additional far-1 boundary vertex except the far-1 vertex bv' is connected to w_2, and then $2 \leq \delta_i \leq 3$, and then, if $|BV(s_i)| - \delta_i < 0$ holds, then implies that at least two vertices p and q of the set $\{u_1, u_2, w_1, w_2\}$ such that p and q are not in $BV(s_i)$, and then, it holds that for $2 \leq \delta_i \leq 3$, at least one vertex, say p of the set $\{u_1, u_2, w_1\}$, is not in $BV(s_i)$, and since Observation 3, the vertex p is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and thus, the vertex p is connected to one vertex except itself of the set $\{u_1, u_2, w_1, w_2\}$. We find that if $dist_G(w_2, w_1) = 1, dist_G(u_2, w_1) = 1$ or $dist_G(u_2, u_1)$ holds, then one can verify $|B^+(s_i)| \leq 6$ since Observation 1, contradiction. Thus, it must be $dist_G(w_2, w_1) \geq 2, dist_G(u_2, w_1) \geq 2$ and $dist_G(u_2, u_1) \geq 2$. Then, if $dist_G(u_1, w_1) = 1, dist_G(u_1, w_2) = 1, dist_G(w_2, u_1) = 1$ or $dist_G(w_2, u_2) = 1$ holds, then this block contains SG_4 or SG_5, contradiction. Then, it must be $dist_G(u_1, w_1) \geq 2, dist_G(u_1, w_2) \geq 2, dist_G(w_2, u_1) \geq 2$ and $dist_G(w_2, u_2) \geq 2$. Thus, u_1, w_1 or u_2 is not connected to any vertex the set $\{u_1, u_2, w_1, w_2\}$. Then, such a vertex p of the set $\{u_1, u_2, w_1\}$ dose not exist, contradiction, and thus the assumption of $|BV(s_i)| - \delta_i < 0$ does not occur. Hence, $|BV(s_i)| - \delta_i \geq 0$ for this case (i-1). For (i-2), $dist_G(bv', u_1) = 1$ and $dist_G(bv'', w_2) = dist_G(bv'', w_1) = 1$ holds. Here, suppose a contradiction that $|BV(s_i)| - \delta_i < 0$ holds again. Recall three vertices are connected to at most two far boundary vertices, i.e., the far-1 boundary vertex bv' and the far-2 boundary vertex bv''. If u_2 is in $BV(s_i)$, then $|BV(s_i)| - \delta_i < 0$ holds, which implies that at most one vertex of the set $\{u_1, w_1, w_2\}$ is in $BV(s_i)$, and then at least two vertices of the set $\{u_1, w_1, w_2\}$ are not in $BV(s_i)$. Then, we can only consider that either u_1 or both vertices w_1, w_2 is not in $BV(s_i)$. Then, when u_1 is not in $BV(s_i)$, and recall Observation 3, we can know that u_1 is connected to one vertex of the set $\{u_2, w_1, w_2\}$, and we always verify $|B^+(s_i)| \leq |B^+(u_1)| \leq 6$ since

Observation 1, contradiction, and thus, both w_1, w_2 are not in $BV(s_i)$. Similarly, Observation 3 shows that each vertex of the set $\{w_1, w_2\}$ is connected to one vertex except itself of the set $\{u_1, u_2, w_1, w_2\}$. If $dist_G(w_1, w_2) = 1$, $dist_G(u_1, w_1) = 1$ or $dist_G(u_1, w_2) = 1$ holds, then we can alway find one vertex p of the set $\{u_1, w_1, w_2\}$ such that $|B^+(s_i)| \leq |B^+(p)| \leq 6$ holds since Observation 1, contradiction. Thus, it must be $dist_G(w_1, w_2) \geq 2$, $dist_G(u_1, w_1) \geq 2$ and $dist_G(u_1, w_2) \geq 2$. Then, w_1 is connected to one vertex u_2 and w_2 is also connected to the vertex u_2, and then, one can verify $|B^+(s_i)| \leq |B^+(w_2)| \leq 6$ since Observation 1, contradiction. Thus, the previous assumption of $|BV(s_i)| - \delta_i < 0$ does not hold, and thus, $|BV(s_i)| - \delta_i \geq 0$ is hold in this case (i-2). Finally, see (ii) and no far-2 boundary vertex is connected to this block. Without loss of generality, still suppose that $|BV(s_i)| - \delta_i < 0$ holds. If $|BV(s_i)| - \delta_i < 0$ holds, then there is one vertex p of the set $\{u_1, u_2, w_1, w_2\}$, which the vertex p is connected to a far-1 boundary vertex and meanwhile, the vertex p is not in $BV(s_i)$. Without loss of generality, suppose that the vertex p is u_1, i.e., a far-1 boundary vertex, say bv', is connected to u_1. Since Observation 3, u_1 is not connected to other boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ except bv', and then, u_1 is connected to one vertex of the set $\{u_2, w_1, w_2\}$. We find that if $dist_G(u_1, u_2) = 1$ holds, then since Observation 1, $|B^+(s_i)| \leq |B^+(u_1)| \leq 6$ holds, contradiction. Thus, u_1 is connected to one vertex of the set $\{w_1, w_2\}$. Then, without loss of generality, can suppose $dist_G(u_1, w_1) = 1$(equivalently, $dist_G(u_1, w_2) = 1$). Then, if w_1 is connected to one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, then since Observation 1, $|B^+(s_i)| \leq |B^+(u_1)| \leq 6$ holds, contradiction. Thus, generates only two cases, that is, (ii-1) w_1 is connected to one vertex of $\bigcup_{j=i+1}^{\ell} B^+(s_j)$ or (ii-2) vertex u_2. Then, (ii-1) if w_1 is connected to one vertex of $\bigcup_{j=i+1}^{\ell} B^+(s_j)$, i.e., $w_1 \in BV(s_i)$, then since SG_2 and SG_3 do not exist, $dist_G(u_2, w_2) \geq 2$ holds. From Observation 3, u_2 or w_2 is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and then we find that u_2 and w_2 are in $BV(s_i)$, and thus, at most three far-1 boundary vertices are connected to this block, i.e., $\delta_i \leq 3$, and three vertices w_1, u_2 and w_2 are in $BV(s_i)$, and thus $|BV(s_i)| - \delta_i \geq 0$ holds and the previous assumption of $|BV(s_i)| - \delta_i < 0$ does not hold. (ii-2) In the final, if w_1 is connected to u_2, then since Observation 1, we can verify $|B^+(s_i)| \leq |B^+(u_1)| \leq 6$ holds, contradiction. Thus, the previous assumption of $|BV(s_i)| - \delta_i < 0$ does not hold and $|BV(s_i)| - \delta_i \geq 0$ holds for this case (ii). In conclusion, all cases, which can be described in figure 3.13(b), hold this lemma.

Consider case (3) in figure 3.13(c), which vertices in $B^+(s_i)$ are v_i, u, w, u_1, u_2, w_1

and w_2. By Observation 3, any vertex of the set $\{u_1, u_2, w_1, w_2\}$ is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ and u_1 is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and thus, u_1 is not connected to any far boundary vertex. From Observation 1, if w_1 is connected to a far-1 boundary vertex, then we can find $|B^+(s_i)| \leq |B^+(w_1)| \leq 6$, contradiction. Thus, any far-1 boundary vertex must be connected to u_2 or w_2. Here, we can suppose that a far-1 boundary vertex, say bv', is connected to u_2, and we take note that it is equivalent to cases of $dist_G(bv', w_2) = 1$. If there is a far-2 boundary vertex bv'', then the Observation 3 shows that u_1 or u_2 is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and thus can know that any far-2 boundary vertex, say bv'', must be connected to both vertices w_1 and w_2, and by Observation 1, we can verify $|B^+(s_i)| \leq |B^+(w_1)| \leq 6$, contradiction. Thus, no far-2 boundary vertex is connected to this block. Without loss of generality, suppose a contradiction that $|BV(s_i)| - \delta_i < 0$ holds. Then, implies that there is a vertex p of the set $\{u_2, w_2\}$, which is connected to a far-1 boundary vertex and meanwhile, is not in $BV(s_i)$. We can default that p is denoted to the vertex u_2. Then, Observation 3 shows that u_2 is connected to one vertex of the set $\{u_1, w_1, w_2\}$. If it holds $dist_G(u_2, u_1) = 1$ or $dist_G(u_2, w_1) = 1$, then since Observation 1, we alway find $|B^+(s_i)| \leq |B^+(u_2)| \leq 6$, contradiction. Thus, it must be $dist_G(u_2, w_2) = 1$. Here, only two cases are further generated, that is, either (i) $w_2 \in BV(s_i)$ holds or (ii) $w_2 \notin BV(s_i)$ holds. For (i), we can know $|BV(s_i)| \geq 1$. Recall that any vertex of the set $\{u_1, u_2, w_1, w_2\}$ is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ and w_1 is not connected to any far-1 boundary vertex, and thus, we can know that at most one far-1 boundary vertex is connected to this block, and $\delta_i \leq 1$. Then, the previous assumption is not satisfied, and $|BV(s_i)| - \delta_i \geq 0$ holds for this case (i). In the final, for (ii), $w_2 \notin BV(s_i)$ holds. Then, w_2 is connected to one vertex of the set $\{u_1, w_1\} \cup \bigcup_{j=1}^{i-1} B(s_j)$. If w_2 is connected to one vertex of $\bigcup_{j=1}^{i-1} B(s_j)$, then we can find $|B^+(s_i)| \leq |B^+(u_2)| \leq 6$ by Observation 1, contradiction. Thus, w_2 is connected to one vertex of the set $\{u_1, w_1\}$. Then, if $dist_G(w_2, w_1) = 1$ holds, then since Observation 1, also find $|B^+(s_i)| \leq |B^+(u_2)| \leq 6$, contradiction. Thus, it must be $dist_G(w_2, u_1) = 1$. Since u_1, w_1 and w_2 are not connected to any far-1 boundary vertex and Observation 3 shows that u_2 is connected to at most one far-1 boundary vertex, we can know that $\delta_i \leq 1$ holds. Since SG_6 does not occur, it must be $dist_G(u_1, w_1) \geq 2$ and u_1 is connected to one vertex of the set $\bigcup_{j=1}^{i-1} B(s_j) \cup \bigcup_{j=i+1}^{\ell} B^+(s_j)$, and furthermore u_1 is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and thus u_1 must be connected to one vertex of set $\bigcup_{j=i+1}^{\ell} B^+(s_j)$, i.e., $u_1 \in BV(s_i)$. Thus, $|BV(s_i)| \geq 1$ holds, and recall $\delta_i \leq 1$, and thus, the previous assumption of $|BV(s_i)| - \delta_i < 0$ does not hold. Thus, $|BV(s_i)| - \delta_i \geq 0$ is also satisfied for this case (ii). Therefore, this lemma is proved. $\square$

Lemma 10. Suppose that $s_i \in D3IS(G)$ is selected in **Phase 2** of `ALG`. If $|B^+(s_i)| \geq 6$, then $\delta_i \leq |D_2^+(s_i)|$.

Proof. First, if a far-1 boundary vertex is intersected with a far-2 boundary vertex at a vertex w of $D_2(s_i) \cap B^+(s_i)$, then we can verify $|B^+(s_i)| \leq B^+(w) \leq 5$. This lemma holds. Thus, can suppose that the set of vertices, which are connected to far-2 boundary vertices, is $D_{2-sub}(s_i)$. Then, vertices of the set $D_{2-sub}(s_i)$ are only connected to far-2 boundary vertices. Since each far-2 boundary vertex is connected to two vertices in the set $D_{2-sub}(s_i)$, at most $|D_{2-sub}(s_i)|$ far-2 boundary vertices are connected to vertices of the set $D_{2-sub}(s_i)$. For each vertex of the set $D_2(s_i) \setminus D_{2-sub}(s_i)$, if there is a vertex, which is connected to at least two far-1 boundary vertices, then by Observation 4, one can verify $|B^+(s_i)| \leq 4$. Thus, each vertex of $D_2(s_i) \setminus D_{2-sub}(s_i)$ is connected to at most one far-1 boundary vertex, and the number of far-1 boundary vertices is at most $|D_2(s_i) \setminus D_{2-sub}(s_i)|$. Then, the number of far boundary vertices is at most $|D_{2-sub}(s_i)|+|D_2(s_i) \setminus D_{2-sub}(s_i)|$. Thus, we can obtain $\beta_i \leq |D_{2-sub}(s_i)|+|D_2(s_i) \setminus D_{2-sub}(s_i)| = |D_2^+(s_i)|$. Therefore, the lemma is proved. $\square$

Lemma 11. Suppose that $s_i \in D3IS(G)$ is selected in **Phase 2** of `ALG`. Then, (1) if $|B^+(s_i)| = 6$, then $|BV(s_i)| - \delta_i \geq -2$. (2) $|B^+(s_i)| = 5$, then $|BV(s_i)| - \delta_i \geq -3$. (3) If $|B^+(s_i)| = 4$, then $|BV(s_i)| - \delta_i \geq -4$.

Proof. (1) First consider $|B^+(s_i)| = 6$. From Observation 3, we know that s_i is connected to at most two boundary vertices in $\bigcup_{j=1}^{i-1} B(s_j)$. Then, only three cases are shown: (i) $|D_1(s_i) \cap \bigcup_{j=1}^{i-1} B^+(s_j)| = 0$, i.e., $|D_1^+(s_i)| = 3$; (ii) $|D_1(s_i) \cap \bigcup_{j=1}^{i-1} B^+(s_j)| = 1$, i.e., $|D_1^+(s_i)| = 1$; (iii) $|D_1(s_i) \cap \bigcup_{j=1}^{i-1} B^+(s_j)| = 2$, i.e., $|D_1^+(s_i)| = 2$. (i) if $|D_1^+(s_i)| = 3$ holds, then obtain $|D_2^+(s_i) = B^+(s_i) \setminus (\{s_i\} \cup D_1^+(s_i))| = 2$. Since Lemma 10, $\delta_i \leq |D_2^+(s_i)| \leq 2$ holds, and can know $|BV(s_i)| - \delta_i \geq -2$ since $|BV(s_i)| \geq 0$. Thus, this lemma holds. (ii) If $|D_1(s_i) \cap \bigcup_{j=1}^{i-1} B^+(s_j)| = 1$ occurs, then obtain $|D_2^+(s_i) = B^+(s_i) \setminus (\{s_i\} \cup D_1^+(s_i))| = 3$, which implies that there are three vertices in $D_2^+(s_i)$, say u_1, u_2 and u_3. Since Lemma 10, we can get $\delta_i \leq |D_2^+(s_i)| \leq 3$. For $\delta_i \leq 2$, obviously, $|BV(s_i)| - \delta_i \geq -2$ is obviously satisfied since $|BV(s_i)| \geq 0$ holds. Thus, we can only consider $\delta_i = 3$. We know that at least one far-1 boundary vertex is connected to one vertex of the set $\{u_1, u_2, u_3\}$, and then, without loss of generality, suppose that a far-1 boundary vertex bv' is connected to u_1. Then, from Observations 4 and 5, we can know that any far-2 boundary vertex, or other far-1 boundary vertex except the vertex bv' is not connected to u_1. Then, if there is a far-2 boundary vertex, which is connected to two vertices of the set $\{u_1, u_2, u_3\}$, then from Observation 7, we can know that at most one far-2 boundary vertex is connected to both vertices u_2 and u_3, and recall that only one far boundary, i.e., bv' is connected to u_1. Thus, at most two far boundary vertices are connected to vertices in the set $D_2^+(s_i)$, i.e., $\delta_i \leq 2$ is always satisfied, which is contradictory for the previous assumption of $\delta_i = 3$. Thus, we can then suppose that $\delta_i = 3$ holds and no far-2 boundary vertex is connected to this block. Then, the Observation 4 implies that each vertex of the set $\{u_1, u_2, u_3\}$ is connected to one far-1 boundary vertex. Thus, except the far-1 boundary vertex bv', which is connected to u_1, suppose that two additional far-1 boundary vertices bv_2' and bv_3' are connected to u_2 and u_3, respectively. Then, by Observation 8, we can know that except far boundary vertices, u_1, u_2 and u_3 are not connected to other boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$. Then, suppose $|BV(s_i)| = 0$ and thus, u_1, u_2 and u_3 are connected with each other, and we find that u_1, u_2 and u_3 are not connected with each other since degree of vertices is three, and implies that it does not occur $|BV(s_i)| = 0$. Thus, if $\delta_i = 3$ holds, it must be $|BV(s_i)| \geq 1$, and $|BV(s_i)| - \delta_i \geq -2$ holds. As above shown, if $|D_1(s_i) \cap \bigcup_{j=1}^{i-1} B^+(s_j)| = 1$ occurs, then this lemma is proved. (iii) If $|D_1(s_i) \cap \bigcup_{j=1}^{i-1} B^+(s_j)| = 2$ holds, then we can know $|D_2^+(s_i) = B^+(s_i) \setminus (\{s_i\} \cup D_1^+(s_i))| = 4$. all blocks, whose $|D_2^+(s_i) = 4$ holds, can be illustrated in figure 3.14, where s_i is denoted to the vertex v_i and four vertices in $D_2^+(s_i)$ are w_1, w_2, u_2 and u_1. Note that w_1 and w_2 are connected to one boundary vertex b_1 and b_2 in $\bigcup_{j=1}^{i-1} B(s_j)$, respectively. Then, from Observation 8,

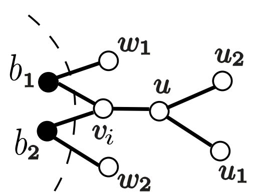


Figure 3.14: A block with six un-removed vertices

we can know that any far-1 boundary vertex is not connected to w_1 or w_2, and then, since the Observation 4, each vertex of the set $\{u_1, u_2\}$ is connected to at most one far-1 boundary vertex. Thus, at most two far-1 boundary vertices are connected to vertices in $B^+(s_i)$. Furthermore, we only need to consider cases, that is, (iii-1) there is only one far-1 boundary vertex, or (iii-2) there are two far-1 boundary vertices, and then, these two far-1 boundary vertices are connected u_2 and u_1, respectively. (iii-1) Without loss of generality, suppose a far-1 boundary vertex, say bv', is connected to u_2, which is equivalent to $dist_G(bv', u_1) = 1$. Since Observation 8, u_2 is not connected to other boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ except the vertex bv'. Thus, any far-2 boundary vertex is connected to two vertices of the set $\{w_1, w_2, u_1\}$. We find that if a far-2 boundary vertex bv'' is connected to both vertices w_1 and w_2, then w_1 or w_2 is not connected to other boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ except the bv'', and furthermore, u_2 is not connected to other boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ except the vertex bv', and thus, at most two far boundary vertices, i.e., the far-2 boundary vertex bv'' and the far-1 boundary vertex bv' are connected to vertices in $B^+(s_i)$. Thus, can get $\delta_i \leq 2$. Then, if there is a far-2 boundary vertex bv'', then the bv'' is connected a vertex u_1 and another vertex of the set $\{w_1, w_2\}$. Furthermore, we can find that there are at most two far-2 boundary vertices, and $\delta_i \leq 3$. Suppose that there are two far-2 boundary vertices, and then besides only one far-1 boundary vertex, three far-boundary vertices are connected to vertices in $D_2^+(s_i)$, and obtain that $\delta_i = 3$ must be hold. Then, only one possibility is that the two far-2 boundary vertices are connected to two vertices u_1 and w_1 and both vertices u_1, w_2, respectively. Here, suppose $|BV(s_i)| = 0$, which implies that all vertices of the set $\{w_1, w_2, u_1, u_2\}$ are not in $BV(s_i)$. The Observation 3 implies that w_1 must be connected to u_2 or w_1 when w_1 is not in $BV(s_i)$. Then, we always find $|B^+(s_i)| \leq |B^+(w_1)| \leq 5$ since Observation 1, contradiction. Thus, if $\delta_i = 3$ occurs, then it must be $|BV(s_i)| \geq 1$, and $|BV(s_i)| - \delta_i \geq -2$ holds. Next, if $\delta_i \leq 2$ holds, then $|BV(s_i)| - \delta_i \geq -2$ obviously holds. Thus, the case (iii-1) holds this lemma. (iii-2) Suppose that two far-1 boundary vertices bv_1' and bv_2' are connected to u_1 and u_2, respectively. From Observation 5, we can know that any far-2 boundary vertex must be connected to both vertices w_1 and w_2, and moreover, u_1, u_2, w_1 or w_2 is connected to at most one far boundary vertex from Observation 3, and then, one far-2 boundary vertex must be connected to w_1 and w_2, and moreover, two far-1 boundary vertices bv_1' and bv_2' are connected to u_1 and u_2, respectively. Thus, we know $\delta_i \leq 3$. Then, suppose that a far-2 boundary vertex is connected to w_1 and w_2. Recall two far-1 boundary vertices are connected to u_1 and u_2, respectively. Thus, $\delta_i = 3$ holds.

If $|BV(s_i)| = 0$ holds, then w_1 is not $BV(s_i)$, and furthermore, the Observation 3 shows that w_1 must be connected to one vertex of the set $\{w_2, u_1, u_2\}$, and then, we always find $|B^+(s_i)| \leq |B^+(w_1)| \leq 5$ since Observation 1, contradiction. Thus, if $\delta_i = 3$ is satisfied, then it must be $|BV(s_i)| \geq 1$, and $|BV(s_i)| - \delta_i \geq -2$ holds, and then, for $\delta_i \leq 2$, it must be $|BV(s_i)| - \delta_i \geq -2$. Thus, for this case (iii-2), it holds $|BV(s_i)| - \delta_i \geq -2$. Therefore, we obtain this lemma for $|B^+(s_i)| = 6$.

(2) Then, consider $|B^+(s_i)| = 5$. $|B^+(s_i)| = 5$ holds, and since Observation 3, $|D_1(s_i) \cap \bigcup_{j=1}^{i-1} B^+(s_j)| \leq 2$ holds, and then, $|D_1^+(s_i)| = 3, 2$, or 1. Then, by $|D_2^+(s_i)| = |B^+(s_i) \setminus (s_i \cup D_1^+(s_i))|$, we further need to consider only three cases, i.e., (i) $|D_2^+(s_i)| = 1$, (ii) $|D_2^+(s_i)| = 2$ or (iii) $|D_2^+(s_i)| = 3$. (i) If $|D_2^+(s_i)| = 1$ holds, then the Observation 5 shows that the vertex in the set $D_2^+(s_i)$ is connected to at most one far-1 boundary vertex, and obviously, no far-2 boundary vertex is connected to this block. Thus, $\delta_i \leq 1$ holds, and $|BV(s_i)| - \delta_i \geq -1$ is satisfied. This lemma holds. (ii) $|D_2^+(s_i)| = 2$ holds, and we denote two vertices of the set $D_2^+(s_i)$ to u_1 and u_2. We know that at least one far-1 boundary vertex is connected to one vertex of the set $\{u_1, u_2\}$ and from Observation 5, u_1 or u_2 is connected to at most one far-1 boundary vertex, and moreover, since degree of vertices is three, we can verify that at most three far boundary vertices are connected to vertices in $B^+(s_i)$, i.e., one far-2 boundary vertices are connected to both vertices u_1 and u_2, only one far-1 boundary vertex is connected to u_1 and only one other far-1 boundary vertex is connected to u_2, and $\delta_i \leq 3$. Thus, $|BV(s_i)| - \delta_i \geq -3$ and holds this lemma. (iii) $|D_2^+(s_i)| = 3$ holds and by simply observing, we can find that s_i is connected to two boundary vertices in $\bigcup_{j=1}^{i-1} B(s_j)$ and say that s_i is connected to two boundary vertices b_1, b_2. Furthermore, we can find that at least one vertex in $D_2^+(s_i)$ is connected to one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$. We can suppose that vertices in $D_2^+(s_i)$ are w_1, w_2 and w_3, and the vertex w_1 is connected to b_1. Then, from Observation 8, we know that w_1 is not connected to any far-1 boundary vertex, and furthermore, the Observation 3 shows that w_1 is connected to at most one far-2 boundary vertex by one edge. Moreover, w_2 or w_3 can be connected to at most two far boundary vertices by two edges. Thus, w_1, w_2 and w_3 are connected to far boundary vertices by at most five edges. Here, give two variables x_1, x_2. Then, x_1 is the number of far-1 boundary vertices, and x_2 is the number of far-2 boundary vertices. Then, it holds $\delta_i = x_1 + x_2$. Since one far-1 boundary vertex is connected to one vertex of the set $D_2^+(s_i)$ by an edge, and one far-2 boundary vertex is two vertices of the set $D_2^+(s_i)$ by two edges. Recall that w_1, w_2 and w_3 are connected to

far boundary vertices by at most five edges, and thus, $2x_2 + x_1 \leq 5$ holds. Since w_1 is not connected to any far-1 boundary vertex and each of $\{w_2, w_3\}$ is connected to at most one far-1 boundary vertex by Observation 4, and then, we can obtain $x_1 \leq 2$. Note that this block is connected to at least one far-1 boundary vertex, and we can know $1 \leq x_1 \leq 2$. Then, for $x_1 = 1$ or $x_1 = 2$, can get $x_2 \leq 2$ or $x_2 \leq 1.5$, and $x_1 + x_2 \leq 3$. Thus, $\delta_i \leq 3$, and $|BV(s_i)| - \delta_1 \geq -3$ holds for the case(iii). Therefore, if $|B^+(s_i)| = 5$ holds, then the lemma is hold.

(3) Finally, consider $|B^+(s_i)| \leq 4$. When $|B^+(s_i)| \leq 3$ holds, $|D_2^+(s_i)| \leq 2$ is satisfied. Then, know that there are two vertices in the set $D_2^+(s_i)$. Obviously, vertices in $D_2^+(s_i)$ are connected to at most 4 far boundary vertices, that is, each vertex in $D_2^+(s_i)$ is connected to at most two far-1 boundary vertices and $\delta_i \leq 4$. Thus, can obtain $|BV(s_i)| - \delta_i \geq -4$. We now consider $|B^+(s_i)| = 4$. By observation, we can easily obtain $|D_2^+(s_i)| \leq 3$. When $|D_2^+(s_i)| \leq 2$ holds, we can verify that vertices in $D_2^+(s_i)$ are connected to at most 4 far boundary vertices, and $\delta_i \leq 4$. In the following, only consider $|D_2^+(s_i)| = 3$. Without loss of generality, suppose that three vertices in the $D_2^+(s_i)$ are w_1, w_2 and w_3. In this case, s_i is connected to three boundary vertices in $\bigcup_{j=1}^{i-1} B(s_j)$, say b_1, b_2 and b_3, and furthermore, suppose that b_1, b_2 and b_3 are connected to w_1, w_2 and w_3, respectively. There is at least one far-1 boundary vertex bv', which is connected to one vertex of set $\{w_1, w_2, w_3\}$, and then, we can firstly suppose this bv' is connected to w_1(equivalently, w_2 or w_3). Then, if w_1 is connected to one far-2 boundary vertex bv'' or other far-1 boundary vertex except bv', then can verify $|B^+(s_i)| \leq |B^+(w_1)| \leq 3$ by observation, and thus, w_1 is not connected to other far boundary vertex except bv'. Similarly, if w_2 or w_3 is connected to one far-1 boundary vertex bv', then w_2 or w_3 is not connected to other far boundary vertex except bv'. Moreover, we find that at most far-2 boundary vertices are connected to w_2 and w_3 and meanwhile, w_2 or w_3 is not connected to any far-1 boundary vertex since degree of vertices is three. Thus, it implies that either at most two far-1 boundary vertices are connected to w_2 and w_3, or at most one far-2 boundary vertex is connected to w_2 and w_3. Thus, besides the far-1 boundary bv', which is connected to w_1, at most four far boundary vertices are connected to vertices in $B^+(s_i)$, and $\delta_i \leq 4$. Thus, it holds $|BV(s_i)| - \delta_i \geq -4$. Therefore, this lemma holds. □

So far, we have discussed all cases that at least one far-1 boundary vertex is connected to one vertex in $B^+(s_i)$. From now on, we investigate the lower bound value of $|BV(s_i)| - \delta_i$ of remaining cases, where no far-1 boundary vertex is connected to vertices in block $B^+(s_i)$ and at least one far-2 boundary vertex is connected to two vertices in block $B^+(s_i)$, when $|B^+(s_i)| = 8, 7, 6, 5$ or $|B^+(s_i)| \leq 4$ holds.

Lemma 12. If $|B^+(s_i)| = 8$ holds and any far-1 boundary vertex is not connected to vertices in $D_2^+(s_i)$, then there is at most one far-2 boundary vertex, which is connected to vertices of the set $D_2^+(s_i)$.

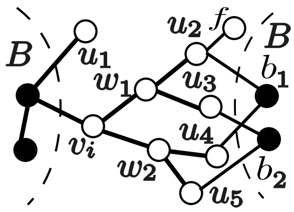


Figure 3.15: Each of two far boundary vertices is connected to two vertices of the set $D_2^+(v_i)$.

Proof. For $|B^+(s_i)| = 8$, if there is at least three boundary vertices in $\bigcup_{j=1}^{i-1} B(s_j)$, which is connected to two vertices of set $D_2^+(s_i)$, then there are two boundary vertices in $\bigcup_{j=1}^{i-1} B(s_j)$ such that these two boundary vertices are intersected at one same vertex in set $D_2^+(s_i)$, and then, find that at least one vertex in set $D_2^+(s_i)$ is connected to two boundary vertices in $\bigcup_{j=1}^{i-1} B(s_j)$ by observation, and then, it implies $|B^+(v_i)| \leq |B^+(v_i)| \leq 6$ since Observation 3. Then, without loss of generality, suppose that there are two far-2 boundary vertices, and then, **Case 3-2(ii)** of this algorithm is executed. Here, suppose that before changing the first candidate vertex, the first candidate vertex is v_f', and after implementing **Case 3-2(ii)** of this algorithm, the vertex v_f substitutes v_f' as a first candidate vertex. By Lemma 4, the algorithm selects a vertex $s_i = v_f$ into the solution. The block $B^+(v_f')$ can be illustrated in figure 3.12, where v_f' is denoted to v_i. As figure 3.12 is shown, the block $B^+(v_f')$ contains $v_i, v_1, u, w, u_1, u_2, w_1$ and w_2. Then, **Case 3-2(ii)** of this algorithm is executed and implies that two boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, say b_1 and b_2, which are in the $D_3(v_i)$, are connected to two vertices of the set $\{u_2, u_3, u_4, u_5\}$, respectively. If a boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ is connected to u_1, two vertices u_2 and u_3, or two vertices u_4 and u_5, then we can verify $|B^+(s_i)| \leq 7$, contradiction. Furthermore, the Observation 3 shows that two boundary vertices in

$\bigcup_{j=1}^{i-1} B(s_j)$ are not intersected at one same vertex of the set $\{u_2, u_3, u_4, u_5\}$. Thus, without loss of generality, we can suppose $dist_G(b_1, u_2) = dist_G(b_1, u_4) = 1$, and then $dist_G(b_2, u_3) = dist_G(b_2, u_5) = 1$ holds, which the case can be illustrated by figure 3.15. The **Case 3-1(ii)** of this algorithm is executed, and v_f is one vertex of the set $\{u_2, u_3, u_4, u_5\}$. Considering equivalent cases, without loss of generality, v_f can be denoted to u_2. Here, we denote another vertex except w_1 in $D_1^+(v_f)$ to be the vertex f. When f is denoted to u_1, we can find $|B^+(u_1)| \leq 7$, and thus, $|B^+(s_i)| \leq |B^+(u_1)| \leq 7 holds$, contradiction. If f is connected to u_3 or u_4, then we can find $|B^+(u_2)| \leq 7$, and thus, implies $|B^+(v_i)| \leq 7$, contradiction. Thus, vertices v_i, u_3 and u_4 in figure 3.15 are contained in $D_2^+(v_f)$, i.e., in $D_2^+(s_i)$. Observation 3 shows that u_3 and u_4 are not connected to vertices $\bigcup_{j=1}^{i-1} B(s_j)$ except vertices of the set $\{b_1, b_2\}$, and furthermore, u_3 is not connected to any far-2 boundary vertex. Each far-2 boundary vertex must be two vertices in $D_2^+(s_i)$, and b_1 is not far-2 boundary vertex. Here, b_2 can be one far-2 boundary vertex. If b_2 is not one far-2 boundary vertex, then u_4 is not connected to any far-2 boundary vertex, and recall that u_3 is not connected to any far-2 boundary vertex, any far-2 boundary vertex must be connected to v_i and other two vertices, which are not u_3 and u_4, and furthermore, if there are at least two far-2 boundary vertices, then there are two far-2 boundary vertices such that they are intersected to one same vertex x, which is in $D_2^+(s_i)$ and not in set $\{u_3, u_4\}$, and then it implies $|B^+(s_i)| \leq |B^+(x)| \leq 6$ since Observation 3. Thus, at most one far-2 boundary vertex is connected to vertices in $D_2^+(s_i)$. Then, if b_2 is one far-2 boundary vertex, then besides u_3, u_5 must be also in $D_2^+(s_i)$, and thus, f must be connected to u_5. By Observation 3, we further know that u_3 or u_5 is not connected to other far-2 boundary vertex except b_2. Recall that u_4 is not connected to any far-2 boundary vertex, and then u_3, u_5 and u_4 are connected to at most one far-2 boundary vertex, i.e., b_2. Now, suppose that there is one far-2 boundary vertex bv'' except b_2. Then, bv'' must be connected to vertices in $D_2^+(s_i) \setminus \{u_3, u_4, u_5\}$, where $D_2^+(s_i)$ contains u_3, u_4, u_5, v_i and one vertex f_1 of $D_1(f) \setminus \{w_1, u_5\}$, and furthermore, the far-2 boundary vertex bv'' must be connected to one vertex of $D_1(f) \setminus \{w_1, u_5\}$ and v_i. Then, we can verify that $|B^+(f_1)| \leq 7$, and implies $|B^+(s_i)| \leq |B^+(v_i)| \leq |B^+(f_1)| \leq 7$, contradiction, and thus, the far-2 boundary vertex bv'' does not appear, and at most one far-2 boundary vertex is connected to vertices in $D_2^+(s_i)$. Therefore, if $|B^+(s_i)| = 8$ holds, then at most one far-2 boundary vertex is connected to vertices in $D_2^+(s_i)$, and this lemma holds. □

Lemma 13. If there is one far boundary vertices bv which connects with two vertices of $D_2^+(s_i)$, then some equalities are shown: (1) $|BV(s_i)| - \delta_i \geq 4$ when $|B^+(s_i)| = 8$, (2) $|BV(s_i)| - \delta_i \geq 0$ when $|B^+(s_i)| = 7$, (3) $|BV(s_i)| - \delta_i \geq -2$ when $|B^+(s_i)| = 6$, (4) $|BV(s_i)| - \delta_i \geq -3$ when $|B^+(s_i)| = 5$, and (5) $|BV(s_i)| - \delta_i \geq -4$ when $|B^+(s_i)| \leq 4$.

Proof. (1) Consider $|B^+(s_i)| = 8$. From Lemma 4, we can know that it always holds $s_i = v_i$, where v_i is the first candidate vertex, and can use figure 3.12 again, which vertices in $B^+(s_i)$ are $v_i, w_1, w_2, u_1, u_2, u_3, u_4$ and u_5. From Lemma 12, we can know that at most one far-2 boundary vertex is connected to this block $B^+(s_i)$. Note that no far-1 boundary vertex is connected to this block $B^+(s_i)$ and then $\delta_i \leq 1$. Without loss of generality, suppose that the far-2 boundary vertex is bv''. If the bv'' is connected to the vertex u_1, two vertices u_2 and u_3, or two vertices u_4 and u_5, then by Observations 1 and 3, we can verify $|B^+(s_i)| \leq 7$, contradiction. Thus, bv'' is connected to one vertex of the set $\{u_2, u_3\}$ and one vertex of the set $\{u_4, u_5\}$. Except equivalent cases, without loss of generality, can suppose that the vertex bv'' is connected to two vertices u_2 and u_4. If u_2 or u_4 is not in $BV(s_i)$, then we can find that u_2 or u_4 is connected to one other boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ except the vertex bv'' or one vertex of the set $\{u_1, u_2, u_3, u_4, u_5\}$, and then, observing each possibility, we can verify that either there is a vertex u' of the set $\{u_1, u_2, u_3, u_4, u_5\}$ such that $|B^+(s_i)| \leq |B^+(u')| \leq 6$, or the block $B^+(v_i)$ contains the subgraph SG_3, SG_4 or SG_5, which is preprocessed in **Phase 1** of `ALG`. Thus, u_2 and u_4 are in $BV(s_i)$. Then, observe two vertices u_3 and u_5. The vertex u_3 is equivalent to the vertex u_5, and thus, we can only discuss the vertex u_3. From Observation 3, u_3 is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$. Then, if u_3 is not in $BV(s_i)$, then u_3 must be connected to one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ or two vertices of the set $\{u_1, u_2, u_4, u_5\}$. Since the subgraph SG_3, SG_4 or SG_5 does not occur in **Phase 2** of `ALG`, u_3 is not connected to u_4 or u_5, and furthermore, if u_3 is connected to u_2, then one can verify $|B^+(s_i)| \leq |B^+(u_2)| \leq 6$ since Observation 1, contradiction. Thus, u_3 is not connected to u_2, u_4 or u_5, and u_3 is connected to one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ or one vertex u_1. If u_3 is connected to u_1, then u_3 must be connected to one vertex in set $\bigcup_{j=1}^{i-1} B^+(s_j)$, and one can verify $|B^+(s_i)| \leq |B^+(u_2)| \leq 6$, contradiction. Thus, u_3 is not connected to any vertex of the set $\{u_1, u_2, u_3, u_4, u_5\}$. Recall that u_3 is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and thus, u_3 must be connected to one vertex in the set $\bigcup_{j=i+1}^{\ell} B^+(s_j)$, and u_3 is in $BV(s_i)$. Then, u_3 is equivalent to the vertex u_5, and similarly, u_5 is also in $BV(s_i)$. Thus, any vertex of the set $\{u_1, u_2, u_3, u_4, u_5\}$ is in the $BV(s_i)$, and $|BV(s_i)| = 5$ is satisfied. Recall $\delta_i \leq 1$, and thus, $|BV(s_i)| - \delta_i \geq 4$ holds. Therefore, if $|B^+(s_i)| = 8$ holds, then this lemma is proved.

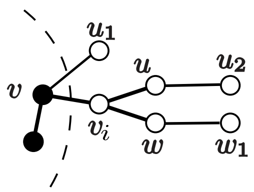
(2) Consider $|B^+(s_i)| = 7$. For $|B^+(s_i)| = 7$, all cases are further generated, i.e., (i) $s_i \neq v_i$, or (ii) $s_i = v_i$. Consider (i). Since $s_i \neq v_i$, this algorithm implies $|B^+(v_i)| = 8$. Then, this block $B^+(s_i)$ can be illustrated by figure 3.12 and note $v_i \neq s_i$. By Lemma 4, it can show that $s_i = w_1$ or $s_i = w_2$, and s_i is in a cycle of length three. Without loss of generality, can suppose $s_i = w_1$, where w_i is in a cycle of length three. We can observe that w_2 is in $D_2^+(s_i)$ and is not connected any far boundary vertex. Since SG_2, SG_3 does not exist, w_2 must be in $BV(s_i)$, and $|BV(s_i)| \geq 1$. Since $|D_1^+(s_i) \cup \{s_i\}| = 4$, we can know $|D_2^+(s_i)| = 3$ and w_2 in $D_2^+(s_i)$ is not connected to any far boundary vertex by observation, and then, we can find that some far-2 boundary vertices must be connected to two vertices of set $D_2^+(s_i) \setminus w_2$. Since Observation 3, each vertex in $D_2^+(s_i) \setminus w_2$ is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and thus, we can know that there is at most one far-2 boundary vertex, which must be connected to vertices in $D_2^+(s_i) \setminus w_2$. Recall $|BV(s_i)| \geq 1$, and can get $\delta_i \leq 1$. Thus, $|BV(s_i)| - \delta_i \geq 0$ holds. Consider (ii), which it occurs $s_i = v_i$. Similarly, all three cases can be illustrated in figure 3.13. Firstly, discuss cases, which can be illustrated by figure 3.13(a), and no far-1 boundary vertex is connected to this block. We can find that if a far-2 boundary vertex is connected to v(or w_1), then one can verify $|B^+(s_i)| \leq |B^+(v)| \leq 6$(or $|B^+(s_i)| \leq |B^+(w)| \leq 6$, resp.) from Observation 3 (or Observations 1 and 2, resp.), contradiction. Thus, any far-2 boundary vertex must be connected to both vertices u_1 and u_2. Observation 7 shows that at most one far-2 boundary vertex is connected to both vertices u_1 and u_2 and furthermore, only one far-2 boundary vertex is connected to vertices in $B^+(s_i)$. Thus, we can know $\delta_i \leq 1$. Here, we can suppose $|BV(s_i)| = 0$ and a far-2 boundary vertex bv'' is connected to two vertices u_1 and u_2. Then, Observation 3 shows that u_1 or u_2 is connected to one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ or one vertex in $D_2^+(s_i)$, and furthermore, since Observation 3, any vertex of $\{u_1, u_2\}$ is connected to one vertex in $D_2^+(s_i)$. If u_1 or u_2 is connected to v, then one can verify $|B^+(s_i)| \leq |B^+(u_1)| \leq 6$ or $|B^+(s_i)| \leq |B^+(u_2)| \leq 6$ since Observations 1 and 2. Since Observation 1, if u_1 is connected to u_2, then we can verify $|B^+(s_i)| \leq |B^+(u_1)| \leq 6$, and thus, u_1 is not connected to u_2. Thus, u_1 and u_2 must be connected to w_1. Then, one can verify $|B^+(s_i)| \leq |B^+(u_1)| \leq 6$ since Observations 1 and 2, contradiction. Thus, the

assumption of $|BV(s_i)| = 0$ does not occur, and recall $\delta_i \leq 1$, and thus, this lemma holds. Secondly, consider cases, which can be illustrated by figure 3.13(b), and the block $B^+(s_i)$ contains v_i, u, w, u_1, u_2, w_1 and w_2, which no far-1 boundary vertex is connected to this block. Recall Observation 3, we can know that any vertex of the set $\{u_1, u_2, w_1, w_2\}$ is connected to at most one boundary vertex, and implies that two far-2 boundary vertices are not intersected at one same vertex of the set $\{u_1, u_2, w_1, w_2\}$, and at most two far-2 boundary vertices are connected to vertices of the set $\{u_1, u_2, w_1, w_2\}$, and $\delta_i \leq 2$. If there are two far-2 boundary vertices, then the two far-2 boundary vertices are connected to two different vertices of the set the set $\{u_1, u_2, w_1, w_2\}$, respectively. Then, Observation 3 shows that each of the set $\{u_1, u_2, w_1, w_2\}$ is not connected to other boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ except far-2 boundary vertices. If $dist_G(u_1, u_2) = 1$ or $dist_G(w_1, w_2)$ holds, then one can verify $|B^+(s_i)| \leq |B^+(u_1)| \leq 6$ or $|B^+(s_i)| \leq |B^+(w_1)| \leq 6$ since Observation 1, contradiction. Thus, it must be $dist_G(u_1, u_2) \geq 2$ and $dist_G(w_1, w_2) \geq 2$. If no vertex of the set $\{u_1, u_2, w_1, w_2\}$ is connected to one vertex of the set $\{u_1, u_2, w_1, w_2\}$, then recall that each of the set $\{u_1, u_2, u_3, u_4\}$ is not connected to other boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ except the far-2 boundary vertex, and thus, each vertex of the set $\{u_1, u_2, w_1, w_2\}$ is connected to one vertex in the $\bigcup_{i+1}^{\ell} B^+(s_j)$, i.e., is in $BV(s_i)$, and $|BV(s_i)| = 4$. Obviously, it holds $|BV(s_i)| - \delta_i \geq 0$ since $\delta_i \leq 2$. Then, if at least one vertex of the set $\{u_1, u_2, w_1, w_2\}$ is connected to one vertex except itself of the set $\{u_1, u_2, w_1, w_2\}$, and without loss of generality, suppose that u_1 is connected to one vertex of the set $\{u_2, w_1, w_2\}$, and then recall $dist_G(u_1, u_2) \geq 2$ holds, and then u_1 is connected to one vertex of the set $\{w_1, w_2\}$, and then, when $dist_G(u_1, w_1) = 1$(or $dist_G(u_1, w_2) = 1$) holds, SG_2 or S_3 appears in **Phase 2** of this algorithm, and then it holds $dist_G(u_2, w_2) \geq 2$(or equivalently, $dist_G(u_2, w_1) \geq 2$). Then, we can only discuss $dist_G(u_2, w_2) \geq 2$, and then recall $dist_G(u_1, u_2) \geq 2$ and $dist_G(w_1, w_2) \geq 2$ hold and each of the set $\{u_1, u_2, u_3, u_4\}$ is not connected to other boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$ except the far-2 boundary vertices, and thus, we can verify that u_2 and w_2 are connected to a vertex in the $\bigcup_{i+1}^{\ell} B^+(s_j)$ and are in $BV(s_i)$, and $|BV(s_i)| \geq 2$, and thus, $|BV(s_i)| - \delta_i \geq 0$ holds. Thus, if there are two far-2 boundary vertices are connected to vertices of the set $\{u_1, u_2, u_3, u_4\}$, then $|BV(s_i)| - \delta_i \geq 0$ is satisfied. In the final, there is one far-2 boundary vertex, which is connected to two vertices of of the set $\{u_1, u_2, u_3, u_4\}$. Then, $\delta_i = 1$ holds. Suppose that the far-2 boundary vertex is bv'' and then, find that either bv'' is connected to two vertices u_1 and u_2(or w_1 and w_2) or bv'' is connected to one of the set $\{u_1, u_2\}$

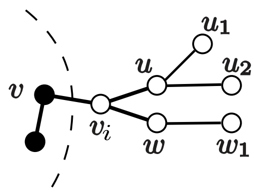
and one of the set $\{w_1, w_2\}$. Then, except equivalent cases, without loss of generality, we can only consider two cases, that is, (i) bv'' is connected to u_1 and u_2, or (ii) bv'' is connected to u_1 and w_1. (i) Suppose a contradiction that $|BV(s_i)| - \delta_i < 0$ holds. Then, it shows $|BV(s_i)| < \delta_i \leq 1$, i.e., u_1 and u_2 are not in $BV(s_i)$. Furthermore, the Observation 3 shows that any vertex of the set $\{u_1, u_2\}$ is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and thus, u_1 and u_2 are connected to one vertex of the set $\{u_1, u_2, w_1, w_2\}$. If $dist_G(u_1, u_2) = 1$ holds, then one can verify $|B^+(s_i)| \leq |B^+(u_1)| \leq 6$ since Observation 1, contradiction. Thus, u_1 and u_2 are connected to a vertex of the set $\{w_1, w_2\}$. Since SG_2 and SG_3 do not appear in **Phase 2** of this algorithm, u_1 and u_2 are connected to one same vertex of the set $\{w_1, w_2\}$, and then we can find $|B^+(s_i)| \leq |B^+(u_1)| \leq 6$ since Observation 1, contradiction. Thus, $|BV(s_i)| - \delta_i < 0$ does not hold, and $|BV(s_i)| - \delta_i \geq 0$ is satisfied in this case(i). (ii) If $dist_G(u_1, w_1) = 1$, $dist_G(u_1, u_2) = 1$ or $dist_G(w_1, w_2) = 1$ holds, then $|B^+(s_i)| \leq 6$ is verified since Observation 1, contradiction. Thus, it shows $dist_G(u_1, w_1) \geq 2$, $dist_G(u_1, u_2) \geq 2$ and $dist_G(w_1, w_2) \geq 2$. Since bv'' is connected to u_1 and w_1 and furthermoreSG_2 and SG_3 do not appear in **Phase 2** of this algorithm, u_1 is not connected to w_1 or w_2, and u_2 is not connected to w_1 or w_2, and then, recall $dist_G(u_1, w_1) \geq 2$, $dist_G(u_1, w_2) \geq 2$ and $dist_G(w_1, w_2) \geq 2$, and then we can find that any vertex of the set $\{u_1, w_1\}$ is not connected to any vertex of the set $\{u_1, u_2, w_1, w_2\}$. Observation 3 shows that any vertex of the set $\{u_1, w_1\}$ is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and thus, any vertex of the set $\{u_1, w_1\}$ is connected to a vertex of the set $\bigcup_{j=i+1}^{\ell} B^+(s_j)$, and thus, u_1 and w_1 are in $BV(s_i)$, and $|BV(s_i)| \geq 2$. Recall $\delta_i = 1$, and thus $|BV(s_i)| - \delta_i \geq 0$ holds for this case(ii). As the conclusion, all cases, which can be illustrated in figure 3.13(b), hold the lemma. Finally, consider cases, which can be showed in figure 3.13(c). u_1 is connected to one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and Observation 3 shows that u_1 is not connected to any far boundary vertex. If a far-2 boundary vertex is connected to w_1 and one vertex of $\{u_2, w_2\}$, then one can verify $|B^+(s_i)| \leq |B^+(w_1)| \leq 6$ holds, contradiction. Then, any far-2 boundary vertex must be connected to both vertices u_2 and w_2. Here, suppose that a far-2 boundary vertex bv'' is connected to both vertices u_2, w_2, and without loss of generality, let $|BV(s_i)| - \delta_i < 0$ hold. Since Observation 3, u_2 or w_2 is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, and then, can know that except bv'', no other far-2 boundary vertex is connected to w_2 or u_2, and furthermore, recall that u_1 or w_1 is not connected to any far-2 boundary vertex and this block is not connected to far-1 boundary vertex, and then, at most only one far boundary vertex, i.e., the bv'' is connected to both vertices u_2 and w_2, and get $\delta_i \leq 1$. Since $|BV(s_i)| - \delta_i < 0$ holds, we can know $|BV(s_i)| = 0$, and then u_2 or w_2 is connected to one vertex of $\{u_1, u_2, w_1, w_2\}$ except itself, where u_2 or w_2 is connected to at most one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$. If u_2 is connected to w_1 or w_2, then since Observation 1, $|B^+(s_i)| \leq |B^+(u_2)| \leq 6$ holds, contradiction. Similarly, if w_2 is connected to w_1 or u_2, then since Observation 1, $|B^+(s_i)| \leq |B^+(w_2)| \leq 6$ holds. Thus, for the block, if $|BV(s_i)| - \delta_i < 0$ holds, then u_2 must be connected to u_1, and w_2 is connected to u_1, and then, since Observation 1, $|B^+(s_i)| \leq |B^+(u_1)| \leq 6$ holds, contradiction. Thus, the assumption of $|BV(s_i)| - \delta_i < 0$ does not occur, and $|BV(s_i)| - \delta_i \geq 0$ holds for any block, which can be showed by figure 3.13(c). Therefore, if $|B^+(s_i)| = 7$ occurs, then this lemma is hold.

(3) Consider $|B^+(s_i)| = 6$. Since Observation 3, we can know that only three cases are shown as $D_1^+(s_i)| =$ (i) 3, (ii) 2 or (iii) 1. First, consider (i). If $D_1^+(s_i)| = 3$, then $|D_2^+(s_i)| = |B^+(s_i) \setminus (\{s_i\} \cup D_1^+(s_i)| = 2$. Lemma 10 shows $\delta_i \leq |D_2^+(s_i)| \leq 2$. Thus, $|BV(s_i)| - \delta_i \geq -2$ is hold. Next, consider (ii). If $D_1^+(s_i)| = 2$, then $|D_2^+(s_i)| = |B^+(s_i) \setminus (\{s_i\} \cup D_1^+(s_i)| = 3$. Without loss of generality, we say three vertices in $D_2^+(s_i)$ to be u_1, u_2 and w_1. Since Observation 2, at most one vertex in $D_2^+(s_i)$ is connected to one boundary vertex in $\bigcup_{j=1}^{i-1} B(s_j)$, say b, and meanwhile, this b is connected to the vertex s_i. Then, two cases are generated, that is, (ii-1) one vertex, which is denoted to u_1, is connected to the boundary vertex b in $\bigcup_{j=1}^{i-1} B(s_j)$, or (ii-2) no vertex of $D_2^+(s_i)$ is connected to the boundary vertex b in $\bigcup_{j=1}^{i-1} B(s_j)$. Here,no far-1 boundary vertex is connected to the block, and then, set one value x_2, which x_2 is the number of far-2 boundary vertices. Then, it holds $\delta_i = x_2$. See (ii-1), which can be depicted in figure 3.16(a). Since Observation 3, at most one edge containing u_1 is connected to one far-1 boundary vertex. For u_2 and w_1, at most two edges containing u_2 or w_1 are connected to far-2 boundary vertices. At most five edges are incident with far-2 boundary vertices. Furthermore, one far-2 boundary vertex is incident with two edges, and thus, $2x_2 \leq 5$. Then, we can know $\delta_i = x_2 \leq 2$, and for the case(ii-1), it is satisfied for $|BV(s_i)| - \delta_i \geq -2$. See (ii-2). By observing, it must occur that two vertices, say u_1, u_2 of $D_2^+(s_i)$ are intersected to one same vertex of $D_1^+(s_i)$, which the same vertex is u. Then, it can be illustrated in figure3.16(b). Since Lemma 10, we can know $\delta_i \leq |D_2^+(s_i)| \leq 3$. If $\delta_i = 3$ holds, it implies that each vertex of $D_2^+(s_i)$ is connected to some far-2 boundary vertices, and we find that if no far-2 boundary vertex is connected to u_1 and u_2, then $\delta_i = 3$

does not appear and $\delta_i \leq 2$ holds, and then $|BV(s_i)| - delta_i \geq -2$ is satisfied since $|BV(s_i)| = 0$. Thus, if $\delta_i = 3$ holds, then we finds that it must occur that u_1 and u_2 is connected to one far-2 boundary vertex, and then the second far-2 boundary vertex is connected to u_1 and w_1, and the third far-2 boundary vertex is connected to u_2 and w_1, and then one can verify $|B^+(s_i)| \leq |B^+(u_1)| \leq 5$ since Observation 1, which is contradiction. Thus,$|BV(s_i)| - \delta_i \geq -2$ is satisfied in the case(ii-2). In the final, $|BV(s_i)| - \delta_i \geq -2$ is hold for the case(ii). Consider (iii). If $D_1^+(s_i)| = 1$,



(a) Case 1



(b) Case 2

Figure 3.16: Two cases are shown.

then $|D_2^+(s_i)| = |B^+(s_i) \setminus (\{s_i\} \cup D_1^+(s_i)| = 4$. By observation, it can be illustrated in figure 3.14 again, where $s_i = v_i$, and four vertices in $D_2^+(s_i)$ are w_1, u_2, u_1 and w_2. Since Observation 3, w_1 or w_2 is connected to at most one far-2 boundary vertex, and thus, at most one edge containing w_1 or w_2 is connected to far-2 boundary vertices. For u_2 or u_1, we can clearly know that at most two edges containing u_2 or u_1 are connected to far-2 boundary vertices. No far-1 boundary vertex is connected to this block. Thus, at most five edges are incident with far-2 boundary vertices. Suppose that the number of far-2 boundary vertices is x_2, and then $\delta_i = x_2$. Recall that at most five edges are incident with far-2 boundary vertices and each far-2 boundary vertex is incident with two edges, and thus, it holds $2x_2 \leq 5$. Since x_2 is integer, we can know $\delta_i = x_2 \leq 2$. Thus,$|BV(s_i)| - delta_i \geq -2$ is also hold for this case(iii). Therefore, if $|B^+(s_i)| = 6$, then it holds $|BV(s_i)| - \delta_i \geq -2$

(4) Consider $|B^+(s_i)| = 5$. Since Observation 3, we know $|D_1^+(s_i)| =$ 0, 1 or 2. Then, three cases are generated, i.e., $|D_2^+(s_i)| =$ (i) 1, (ii) 2, or (iii) 3. Consider (i). Since a far-2 vertex must be connected to two vertices of $D_2^+(s_i)$, obviously it does not occur. Consider (ii). Since two vertices in $D_2^+(s_i)$ can be connected to at most two far-2 boundary vertices and then $\delta_i \leq 2$, this lemma obviously holds. Consider (iii).Since three vertices in $D_2^+(s_i)$ can be connected to at most three far-2 boundary vertices, and then $\delta_i \leq 3$, and this lemma obviously holds. Therefore, if $|B^+(s_i)| = 5$, then it holds this lemma, i.e., $|BV(s_i)| - \delta_i \geq -3$ holds.

(5) Consider $|B^+(s_i)| \leq 4$. For $|B^+(s_i)| \leq 4$, we can get $0 \leq |D_2 +^(s_i)| \leq 3$.

Since each far-2 boundary vertex is connected to two vertices in $D_2^+(s_i)$, one can verify that three vertices in $D_2^+(s_i)$ can be connected to at most three far-2 boundary vertices, and $\delta_i \leq 3$. Therefore, $|BV(s_i)| - \delta_i \geq -4$ is satisfied, and this lemma is proved. □

From above analyses from the Lemma 7 to Lemma 13, we can obtain the following remark:

Remark 1. In conclusion (i) if $|B^+(s_i)| = 8$ occurs, then $|BV(s_i)| - \delta_i \geq 4$ holds, (ii) if $|B^+(s_i)| = 7$ holds, then $|BV(s_i)| - \delta_i \geq 0$ is satisfied, (iii) if it holds $|B^+(s_i)| = 6$, $|BV(s_i)| - \delta_i \geq -2$ is hold, (iv) if $|B^+(s_i)| = 5$ is satisfied, then it holds $|BV(s_i)| - \delta_i \geq -3$. (v) when $|B^+(s_i)| \leq 4$, $|BV(s_i)| - \delta_i \geq -4$ holds.

In the following, we assume that ALG selects ℓ_1 vertices, s_1 through s_{ℓ_1}, and ℓ_2 vertices, s_{ℓ_1+1} through $s_{\ell_1+\ell_2}$, into $D3IS(G)$ in **Phase 1** and **Phase 2**, respectively. That is, $\ell = \ell_1 + \ell_2$. Let i_k denote the number of the solution vertices s_i such that $|B^+(s_i)| = k$ for $5 \leq k \leq 8$. Also, let $i_{\leq 4}$ denote the number of the solution vertices s_i such that $|B^+(s_i)| \leq 4$. Let $BV'(ALG) = \bigcup_{i=\ell_1+1}^{\ell} BV(s_i)$ and $BV'_{near}(ALG) = \bigcup_{i=\ell_1+1}^{\ell} BV_{near}(s_i)$. Then, if **Phase 1** is executed (i.e., at least one special subgraph is included in the input graph G), then let p be the number of vertices which are put into B in **Phase 1** and connected to vertices in $\bigcup_{i=\ell_1+1}^{\ell} B^+(s_i)$; otherwise, i.e., if no special subgraphs are not included in G and thus **Phase 1** is not executed, then let p be equal to $|BV(s_1)|$.

Lemma 14. (1) If **Phase 1** of ALG is not executed, then $|BV_{near}(ALG)| \geq p + 4i_8 - 2i_6 - 3i_5 - 4i_{\leq 4}$ is satisfied. (2) Suppose that **Phase 1** is executed and $s_i \in D3IS(G)$ is selected in **Phase 2** for $\ell_1 + 1 \leq i \leq \ell$. Then $|BV'_{near}(ALG)| \geq p + 4i_8 - 2i_6 - 3i_5 - 4i_{\leq 4}$ is satisfied.

Proof. (1) We first assume that **Phase 1** is *not* executed. Since $|BV_{near}(ALG)| = |BV(ALG)| - |BV_{far}|$, it satisfies $|BV_{near}(ALG)| = |BV(ALG)| - |BV_{far}| \geq \sum_{i=1}^{\ell} |BV(s_i)| - \sum_{i=1}^{\ell} \delta_i = \sum_{i=1}^{\ell}(|BV(s_i)| - \delta_i)$. By Remark 1, we can know $|BV_{near}(ALG)| \geq \sum_{i=1}^{\ell}(|BV(s_i)| - \delta_i) \geq (|BV(s_1)| - 0) + \sum_{i=2}^{\ell}(|BV(s_i)| - \delta_i) \geq p + 4i_8 - 2i_6 - 3i_5 - 4i_{\leq 4}$. (2) Suppose that **Phase 1** is executed and $s_i \in D3IS(G)$ is selected in **Phase 2** for $\ell_1 + 1 \leq i \leq \ell$. Then, $|BV'_{near}(ALG)| = (p + |BV'(ALG)|) - |BV_{far}| \geq p + \sum_{i=\ell_1+1}^{\ell}(|BV(s_i)| - \delta_i)$. By Remark 1, $|BV'_{near}(ALG)| \geq p + 4i_8 - 2i_6 - 3i_5 - 4i_{\leq 4}$. This completes the proof of this lemma. □

Corollary 2. (1) If **Phase 1** of `ALG` is not executed, then it satisfies $4i_8 \le 9\ell + 1 + 2i_6 + 3i_5 + 4i_{\le 4} - n - p$. (2) Suppose that **Phase 1** is executed and $s_i \in D3IS(G)$ is selected in **Phase 2** for $\ell_1 + 1 \le i \le \ell$. Let $n_2 = |\bigcup_{i=\ell_1+1}^{\ell} B^+(s_i)|$. Then, $4i_8 \le 9\ell_2 + 2i_6 + 3i_5 + 4i_{\le 4} - n_2 - p$ is satisfied.

Proof. (1) Suppose that **Phase 1** is *not* executed. From Lemma 14, $\sum_{i=1}^{\ell}(|B^*(s_i)| - |B^+(s_i)|) \ge |BV_{near}(ALG)| \ge p + 4i_8 - 2i_6 - 3i_5 - 4i_{\le 4}$. Since $|B^*(s_i)| \le 9$ holds for $i \ge 2$ from Lemma 5, $10 + 9(\ell - 1) \ge |B^+(s_1)| + 9(\ell - 1) - n \ge p + 4i_8 - 2i_6 - 3i_5 - 4i_{\le 4}$ and we can obtain the inequality $4i_8 \le 9\ell + 1 + 2i_6 + 3i_5 + 4i_{\le 4} - n - p$. (2) Suppose that **Phase 1** is executed. From Lemma 14, we know $\sum_{i=\ell_1+1}^{\ell}(|B^*(s_i)| - |B^+(s_i)|) \ge |BV'_{near}(ALG)| \ge p + 4i_8 - 2i_6 - 3i_5 - 4i_{\le 4}$. Furthermore, since $|B^*(s_i)| \le 9$ holds for $i \ge 2$ from Lemma 5, the following inequality holds: $9\ell_2 - n_2 \ge \sum_{i=\ell_1+1}^{\ell}(|B^*(s_i)| - |B^+(s_i)|) \ge p + 4i_8 - 2i_6 - 3i_5 - 4i_{\le 4}$. Hence, we get $4i_8 \le 9\ell_2 + 2i_6 + 3i_5 + 4i_{\le 4} - n_2 - p$. □

Theorem 7. `ALG` achieves an approximation ratio of $1.875 + O(\frac{1}{n})$.

Proof. We need to investigate the following three situations: (1) $1 \le \ell_1 < \ell$, i.e., both **Phase 1** and **Phase 2** are executed, (2) $\ell_1 = 0$, i.e., **Phase 1** is not executed, and (3) $\ell_1 = \ell$, i.e., **Phase 2** is not executed.

(1) One can see that $7.5\ell_1 + 8i_8 + 7i_7 + 6i_6 + 5i_5 + 4i_{\le 4} \ge n$ holds. From $\ell = \ell_1 + i_8 + i_7 + i_6 + i_5 + i_{\le 4}$, we obtain $4\ell + i_5 + 2i_6 + 3i_7 + 4i_8 + 3.5\ell' \ge n$. Furthermore, since $i_7 = \ell - \ell_1 - i_8 - i_6 - i_5 - i_{\le 4}$ holds, we get $4\ell + i_5 + 2i_6 + 3(\ell - \ell_1 - i_8 - i_6 - i_5 - i_{\le 4}) + 4i_8 + 3.5\ell_1 \ge n$. That is, $7\ell - 2i_5 - i_6 - 3i_{\le 4} + i_8 + 0.5\ell_1 \ge n$ holds. Recall that $4i_8 \le 9\ell_2 + 2i_6 + 3i_5 + 4i_{\le 4} - n_2 - p$ as shown in Corollary 2. Since $\ell_2 = \ell - \ell_1$ and $n_2 \ge n - 7.5\ell_1$, we get $4i_8 \le 9\ell + 2i_6 + 3i_5 + 4i_{\le 4} - n - 1.5\ell_1 - p$. Since $\ell_1 \le \ell - 1$, we obtain $\ell \ge (5n + 1.5)/37.5 > n/7.5$. (2) $\ell_2 = \ell$ and $n_2 = n$. Obviously, $p \ge 1$. From $|B^+(s_1)| \le 10$ and the definitions on i_k, $10 + 8i_8 + 7i_7 + 6i_6 + 5i_5 + 4i_{\le 4} \ge |B^+(s_1)| + 8i_8 + 7i_7 + 6i_6 + 5i_5 + 4i_{\le 4} \ge n$ holds. Note that $1 + i_8 + i_7 + i_6 + i_5 + i_{\le 4} = \ell$. Hence, we obtain $7\ell + i_8 - 2i_5 - i_6 - 3i_{\le 4} + 3 \ge n$. From Corollary 2, $7\ell + (9\ell + 2i_6 + 3i_5 + 4i_{\le 4} - n)/4 - 2i_5 - i_6 - 3i_{\le 4} + 3 \ge n$ holds. Therefore, we obtain $\ell \ge (5n - 12)/37 > (5n - 12)/37 \ge n/7.5 - 12/37$. (3) From Lemma 6, $\ell \ge n/7.5$.

Since $|OPT(G)| \le \frac{n}{4}$ holds from Lamma 3, `ALG` achieves the approximation ratio of $1.875 + O(1/n)$. □

3.4 PTAS algorithm of MaxDdIS for planar graphs

For planar graphs, we find that there is a PTAS algorithm for MaxDdIS on planar graphs. An *outerplanar* graph (often called a 1-*outerplanar* graph) is a graph that can be drawn in the plane without any edge-crossing such that all vertices lie on the unbounded face. A planar graph G is said to be k-*outerplanar* for $k \geq 2$ if it has a plane-embedding such that by removing the vertices on the unbounded face, we obtain a $(k-1)$-outerplanar graph; the deleted vertices form the *kth layer* of G. Note that every planar graph G can be regarded as a k-outerplanar graph for some integer k, although k can be $\Omega(\sqrt{|V(G)|})$. Also note that the treewidth of a k-outerplanar graph is at most $3k+1$. The outerplanar factor k plays an important role in many polynomial-time approximation schemes based on the *Baker's shifting technique* for NP-hard optimization problems on planar graphs [3]. The Baker's shifting technique can be applied to MaxDdIS on planar graphs, as follows:

Algorithm $\texttt{SHIFTING}_d$

Input: D-outerplanar graph G

Output: Distance-d independent set $DdIS(G)$ of G

Step 1. For each $i \in \{1, 2, \ldots, k\}$, repeat the following:

- **(1-1)** Delete all vertices in layers i through $i + (d-2)$, $k + i + (d-2)$ through $k + i + 2(d-2)$, $2k + i + 2(d-2)$ through $2k + i + 3(d-2)$, and so on. Let G_i be the resulting graph.

  /* Note that each connected component of G_i is a $(k-1)$-outerplanar graph, and hence its treewidth is at most $3k-2$. */

- **(1-2)** Solve MaxDdIS for each connected component of G_i, and obtain an optimal distance-d independent set S_i^* of G_i.

Step 2. Output the best S^* among the k obtained distance-d independent sets S_1^* through S_k^* as the solution $DdIS(G)$.

Theorem 8. For a fixed constant $d \geq 2$, MaxDdIS admits a polynomial-time approximation scheme for planar graphs.

Proof. As a seminal result of Courcelle [8], it is known that every problem definable in monadic second-order logic can be solved for graphs with bounded treewidth in time linear in the number of vertices of the graph. By a simple extension of the independent set problem (i.e., MaxD2IS), MaxDdIS can be also defined in monadic second order logic. Therefore, MaxDdIS can be solved in linear time (although its running time depends exponentially on the treewidth and the distance d). Thus, the algorithm $\texttt{SHIFTING}_d$ runs in time polynomial in n, which is the number of vertices. Let S be any optimal distance-d independent set in a given planar graph. Let S_i be the distance-d independent set obtained from S by deleting all vertices in layers i through $i+(d-2)$, $k+i+(d-2)$ through $k+i+2(d-2)$, $2k+i+2(d-2)$ through $2k+i+3(d-2)$, and so on. Let S^* be the output of the algorithm $\texttt{SHIFTING}_d$, and S_i^* be the distance-d independent set of G_i (and hence of G) obtained by Step 1-2. From the definitions of these sets, both $|S_i| \leq |S_i^*|$ and $|S_i^*| \leq |S^*|$ hold for every $i \in \{1, 2, \ldots, k\}$. Then, since $|S_i| \leq |S_i^*|$ for every $i \in \{1, 2, \ldots, k\}$, we have

$$|S_1| + |S_2| + \cdots + |S_k| \quad \leq \quad |S_1^*| + |S_2^*| + \cdots + |S_k^*|.$$

Next, since G_i (or S_i) does not include any vertices in layers i through $i + (d-2)$, $k+i+(d-2)$ through $k+i+2(d-2)$, $2k+i+2(d-2)$ through $2k+i+3(d-2)$, and so on, the following inequality holds:

$$|S_1| + |S_2| + \cdots + |S_k| \quad \geq \quad (k - (d-1))|S|.$$

Since $|S^*| = \max\{|S_i^*| : 1 \leq i \leq k\}$, we have

$$|S_1^*| + |S_2^*| + \cdots + |S_k^*| \quad \leq \quad k|S^*|.$$

Therefore, the following holds:

$$(k - (d-1))|S| \leq k|S^*|,$$

that is,

$$\frac{|S|}{|S^*|} \leq 1 + \frac{d-1}{k-(d-1)}.$$

Thus, by setting $k = \lceil \frac{d-1}{\varepsilon} \rceil + d - 1$, we can conclude that $\texttt{SHIFTING}_d$ is a $(1+\varepsilon)$-approximation algorithm, that is, it is a polynomial-time approximation scheme for MaxDdIS on planar graphs. This completes the proof.

Chapter 4

Maximum Induced Matching Problem

In this chapter, we design an algorithm for the maximum induced matching on C_5-free r-regular graphs, which is better than the previous algorithm.

4.1 Preliminaries

In this section, we introduce some definitions, which will be utilized in this chapter. Still, let $G = (V, E)$ be a simple, unweighted, and undirected graph, where V and E denote the set of vertices and the set of edges, respectively. $V(G)$ and $E(G)$ also denote the vertex set and the edge set of G, respectively. Throughout the paper, let $n = |V|$ and $m = |E|$ for any given graph. Let $G[V']$ denote a vertex-induced subgraph of $G = (V, E)$, consisting of a subset $V' \subseteq V$ and all the edges connecting pairs of vertices in V'. Also, let $G[E']$ denote an edge-induced subgraph of $G = (V, E)$, consisting of a subset $E' \subseteq E$ and the vertices that are endpoints of edges in E'. Let H be a set of graphs. A graph is *H-free* if it does not contain any graph in H as a vertex-induced subgraph.

For a vertex v in a graph G, the *open neighborhood* of v in G is $N_G(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$ and the *closed neighborhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v in G is denoted by $deg_G(v) = |N_G(v)|$. A graph G is *r-regular* if all the vertices in G have degree r. Throughout the paper, we assume that $r \geq 3$ since MaxIM on 1-regular and 2-regular graphs can be solved in polynomial time.

A (simple) *path* P_k with k vertices $v_1, v_2, \cdots, v_k$ is represented as a sequence $\langle v_1, v_1, \cdots, v_k \rangle$ of those k vertices where $\{v_i, v_{i+1}\}$ is an edge in P_k for each $i = 1, 2, \cdots, k-1$. The *length* of the path P is the number of edges in P, *i.e.*, the length of P_k with k vertices is $k-1$. A *cycle* C_k with k vertices is similarly written as $C_k = \langle v_1, v_2, \cdots, v_k, v_1 \rangle$.

For a pair of vertices v and v' in G, the *distance* between v and v' is the length of a shortest path from v to v', which is denoted by $dist_G(v, v')$. For the path $P = \{v_1, v_2, v_3, v_4, v_5, \cdots, v_k\}$ of length $k-1$, for example, $dist_P(v_1, v_1) = 0$, $dist_P(v_1, v_2) = 1$, $dist_P(v_1, v_3) = 2$ and so on. If $dist_G(v, v') = \ell$ for two vertices v and v', then v' is called a distance-ℓ vertex of v. Let $DV_\ell(v)$ be a set of distance-ℓ vertices of v. Similarly, for a pair of edges e and e' in $E(G)$, we define the distance $dist_G(e, e')$ between two edges e and e': The *line graph* $L(G)$ of G is the graph whose vertices are the edges of G, and in which two vertices are adjacent only if they share an incident vertex as edges of G. Then, the distance $dist_G(e, e')$ between two edges e and e' in G is defined as $dist_{L(G)}(e, e')$ between two vertices e and e' in $L(G)$, *i.e.*, the length of a shortest path from e to e' in the line graph $L(G)$ of G. For example, for P, $dist_P(\{v_1, v_2\}, \{v_1, v_2\}) = 0$, $dist_P(\{v_1, v_2\}, \{v_2, v_3\}) = 1$, $dist_P(\{v_1, v_2\}, \{v_3, v_4\}) = 2$, and so on. If $dist_G(e, e') = \ell$ for two edges e and e', then e' is called a distance-ℓ edge of e. Let $DE_\ell(e)$ be a set of distance-ℓ edges of e. Furthermore, we define the distance between an edge e and a vertex v as the length of a shortest path from one endpoint of e to v, *i.e.*, $dist_G(e, v) = \min\{dist_G(v_e, v), dist_G(v'_e, v)\}$ for $e = \{v_e, v'_e\}$. For example, $dist_P(\{v_2, v_3\}, v_1) = 1$, $dist_P(\{v_2, v_3\}, v_4) = 1$, $dist_P(\{v_2, v_3\}, v_5) = 2$, and so on.

We say that an edge $e \in E(G)$ is *in conflict with* another edge $e' \in E(G)$ if $dist_G(e, e') \leq 2$ and the edge $e \in E(G)$ is called a *conflict edge* of $e' \in E(G)$ Then, for an edge e of a graph G, let

$$\begin{aligned} C_G(e) &= \{e' \in E(G) \mid dist_G(e, e') \leq 2\} \\ &= \{e\} \cup DE_1(e) \cup DE_2(e). \end{aligned}$$

be the set of all the conflict edges of e. Also, the set of all the conflict edges of a set $E' \subseteq E(G)$ is defined as follows:

$$C_G(E') = \bigcup_{e \in E'} C_G(e).$$

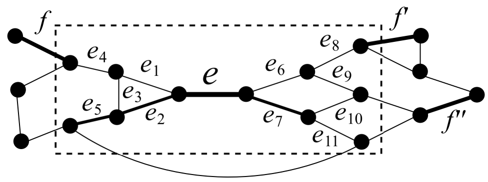


Figure 4.1: Edges $e_1, e_2, \cdots, e_{11}$ and e in the dotted-line rectangle are conflict edges of e. If $M = \{e, f, f', f''\}$, then the private conflict edges of e to M are e_2, e_5, e_7 and e.

For a subset $E' \subseteq E(G)$ of edges and an edge e in G, let

$$PC_G(E', e) = C_G(e) \setminus \bigcup_{e' \in E' \setminus \{e\}} C_G(e')$$

be the set of edges that are in conflict with e but *not* in conflict with every $e' \in E' \setminus \{e\}$. The edge in $PC_G(E', e)$ is called a *private conflict edge* of e to the set E'. For example, for the graph G shown in Figure 4.1, the conflict edges of e are e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, and e. Also, the private conflict edges of e to the set $M = \{e, f, f', f''\}$ are e_2, e_5, e_7, and e.

4.2 Induced Matching on C_5-free r-regular graphs

In this section we design a $\left(\frac{2r}{3} + \frac{1}{3}\right)$-approximation algorithm for MaxIM on C_5-free r-regular graphs. Here is an outline of our approximation algorithm for an input C_5-free r-regular graph G, which mainly consists of two steps. (i) In the first step, the algorithm initially finds a *maximal* induced matching M by iteratively picking an edge e into the induced matching M, and eliminating all the edges in $C_G(e)$ from the candidates of the solution. (ii) In the second step, the algorithm tries to find a larger induced matching from the temporally obtained induced matching M by a "small modification" as follows: Let M be the set of induced matching edges currently obtained. The algorithm picks one edge e from M. Then, if there exist (at least) two edges e' and e'' in $PC_G(M, e) \setminus \{e\}$ such that $dist_G(e', e'') > 2$, then the algorithm updates the "old" induced matching M to the "new" $M = (M \setminus \{e\}) \cup \{e', e''\}$. If there does not exist such an edge e in M, then the algorithm tries to find an edge e_{min} from $PC_G(M, e)$ such that $|C_G(e_{min})|$ is the minimum among $|C_G(e')|$ for every $e' \in PC_G(M, e)$. If the algorithm finds e_{min}, then it swaps e and e_{min}, *i.e.*, updates $M = (M \setminus \{e\}) \cup \{e_{min}\}$.

4.2.1 Approximation Algorithm

The following is a description of our algorithm ALG, where let M be the induced matching obtained by ALG:

Algorithm ALG

Input: A C_5-free r-regular graph $G = (V, E)$.

Output: An induced matching M of G.

Initialization: Set $M = \emptyset$, and obtain $C_G(e)$ and $|C_G(e)|$ for every edge $e \in E$.

Step 1. /* Find an initial maximal set M of induced matching edges. */
If $C_G(M) = E$, then go to **Step 2**; otherwise, arbitrarily select an edge e from $E \setminus C_G(M)$, set $M = M \cup \{e\}$ and repeat **Step 1**.

Step 2. /* Find a larger set M of induced matching edges */
Obtain $PC_G(M, e)$ for every $e \in M$.

(i) If there exists an edge e such that the size of a *maximal* induced matching $MAX(e)$ in $PC_G(M, e) \setminus \{e\}$ is at least two, then set $M = (M \setminus \{e\}) \cup MAX(e)$ and repeat **Step 2**.

(ii) If there exists a pair of edges $e \in M$ and $e' \in PC_G(M, e)$ such that $|C_G(e)| > |C_G(e')|$ and $|C_G(e')|$ is the minimum among $|C_G(e'')|$ for every $e'' \in PC_G(M, e)$, then set $M = (M \setminus \{e\}) \cup \{e'\}$ and repeat **Step 2**.

(iii) Otherwise, go to **Termination**.

Termination. Output the solution M and halt.

[End of ALG]

Here is a detailed implementation of **Step 2**(i): Suppose that $PC_G(M, e)$ has k edges and let $PC_G(M, e) = \{e, e_1, e_2, \cdots, e_{k-1}\}$. Also, for each $1 \leq i \leq k - 1$, let $MAX(e, e_i)$ be a maximal induced matching which is obtained by first selecting e_i from $PC_G(M, e) \setminus \{e\}$ and then selecting induced matching edges from $(PC_G(M, e) \setminus \{e\}) \setminus C_G(e_i)$ if such induced matching edges exist. In **Step 2**(i), ALG first obtains $k - 1$ maximal induced matchings $MAX(e, e_1)$ through $MAX(e, e_{k-1})$, and then finds the set of maximum cardinality among those $k-1$ sets as $MAX(e)$. One can see that if there exists at least one maximal matching which has at least two induced matching edges, then ALG surely finds it in polynomial time.

Now we show the feasibility of the induced matching M output by ALG. One can see that if an edge e is selected into M, then all the edges in $C_G(e)$ are eliminated from candidates of the solution. Moreover, we can verify that each edge in $PC_G(M, e)$ is not in conflict with any edge in M except the edge e. Thus, the distance of any two edges in M is at least three and thus all the edges in the output M are induced matching edges. That is, ALG can always output a feasible induced matching M.

Next, we bound the running time of ALG: Clearly, **Initialization** and **Step 1** can be executed in $O(m^2)$ time. In each execution of **Step 2**(i), the number of induced matching edges in M is incremented at least by one. Hence the total number of executions of **Step 2**(i) is at most $O(m)$. Each iteration of **Step 2**(i) can be done in $O(m^2)$. Therefore, the total computational complexity of **Step 2**(i) is $O(m^3)$. As for **Step 2**(ii), if $|M| = i$ at some time point, then ALG has to check i private conflict edge sets, $PC_G(M, e_1)$ through $PC_G(M, e_i)$, in **Step 2**(ii). That is, the total number of executions of **Step 2**(ii) is at most $O(m^2)$. **Step 2**(ii) can be implemented in $O(m)$ time. Hence the total comutational complexity of **Step 2**(ii) is again $O(m^3)$. In the beginning of each iteration of **Step 2** we need $O(m^2)$ time to obtain $PC_G(M, e)$ for every $e \in M$. Since the iteration of **Step 2** is bounded in $O(m^2)$, the time complexity of **Step 2** is $O(m^4)$. Therefore, ALG runs in $O(m^4)$.

We make a detailed observation on **Step 2**: From the maximality of M, $\bigcup_{e \in M} C_G(e) = E(G)$ holds after **Step 1**. Now suppose that in some iteration of **Step 2**(i), ALG finds an edge e_1 such that a maximal induced matching $MAX(e_1)$ in $PC_G(M, e_1)$ has at least two induced matching edges. At this moment, $\bigcup_{e \in M \setminus \{e_1\}} C_G(e) = E(G) \setminus PC_G(M, e_1)$ holds since all the edges in $PC_G(M, e_1)$ are in conflict only with e_1. Moreover, from the maximality of $MAX(e_1)$, $PC_G(M, e_1) \subseteq \bigcup_{e' \in MAX(e_1)} C_G(e')$ must hold. Since ALG obtains a new temporal solution M' by setting $M' = (M \setminus \{e_1\}) \cup MAX(e_1)$ in **Step 2**(i), $\bigcup_{e \in M'} C_G(e) = E(G)$ is satisfied again for M'. Note that **Step 2**(ii) guarantees that when M is eventually output by ALG, $|C_G(e)| \leq |C_G(e')|$ must hold for every edge $e' \in PC_G(M, e)$. Therefore, from the termination condition of ALG, the following should be remarked:

Remark 2. When ALG terminates and outputs an induced matching M for an input graph G, the following three properties must be satisfied:

1. As for every private conflict edge set $PC_G(M, e)$ of e to M, any two edges in $PC_G(M, e)$ must be in conflict with each other;
2. For every edge $e' \in PC_G(M, e)$, $|C_G(e)| \leq |C_G(e')|$ holds; and
3. $\bigcup_{e \in M} C_G(e) = E(G)$ holds, *i.e.*, M must be a maximal set of induced matching edges.

4.2.2 Approximation ratio

In this section, we investigate the approximation ratio of the algorithm ALG. Now suppose that given a graph $G = (V, E)$, ALG finally outputs a set M of induced matching edges, and $|ALG(G)| = |M|$. Note that the output M by ALG cannot be enlarged by picking other two or more edges from $PC_G(M, e)$ if edge e is in M. We can obtain the following relationship between $|C_G(e)|$ and $|PC_G(M, e)|$:

Lemma 15. For any maximal set M of induced matching edges in a graph $G = (V, E)$, the following inequality is satisfied:

$$\sum_{e \in M} (|C_G(e)| - |PC_G(M, e)|) \geq 2(|E| - \sum_{e \in M} |PC_G(M, e)|).$$

Proof. Consider an edge e in a subset M of edges, the conflict edge set $C_G(e)$ of e, and the private conflict edge set $PC_G(M, e)$ of e to M. From the definitions, we know

$$\bigcup_{e \in M} (C_G(e) \setminus PC_G(M, e)) = E \setminus \left(\bigcup_{e \in M} PC_G(M, e) \right).$$

Since the private conflict edge sets are independent, the following equality holds:

$$\left| E \setminus \left(\bigcup_{e \in M} PC_G(M, e) \right) \right| = |E| - \sum_{e \in M} |PC_G(M, e)| \,.$$

Recall that every edge in $C_G(e) \setminus PC_G(M, e)$ must be included in at least one different conflict edge set, say, $C_G(e')$ of $e' \in M$ for $e' \neq e$. Therefore, the inequality holds. $\square$

Now we can estimate the maximum number Γ_d of conflict edges of an edge e in r-regular graphs, which was shown in [26]:

Proposition 2 (Theorem 3.1 in [26])**.** For any edge e in a r-regular graph G, the number $|C_G(e)|$ of conflict edges is at most $2r^2 - 2r + 1$.

Let Γ_d be the upper bound of $|C_G(e)|$ of conflict edges over all of the edges $e \in E(G)$. One can see that the number $|C_G(e)|$ of conflict edges of the edge e gets much smaller than $2r^2 - 2r + 1$ if an edge e' in $C_G(e)$ is in a short cycle, for example, C_3 or C_4. Indeed, the following results are known [31]:

Proposition 3 (Lemmas 4 and 6 in [31])**.** If a cycle C_3 of length three contains an edge e in $C_G(e)$ of a r-regular graph G, then the cycle C_3 decreases the upper bound Γ_d of $|C_G(e)|$ by at least r. Moreover, if a cycle C_4 of length four contains an edge e in $C_G(e)$, then the cycle C_4 decreases the upper bound Γ_d by at least one.

Take a look at an edge $e = \{t, u\}$ illustrated in Figure 4.2. If two neighbor vertices, w_1 and w_2, of the edge e are connected by an edge $e' = \{w_1, w_2\}$, then e' is called the *triangle edge* of e, and we say that e owns the triangle edge e' or e' is the triangle edge of e. Then, we can obtain Lemma 16:

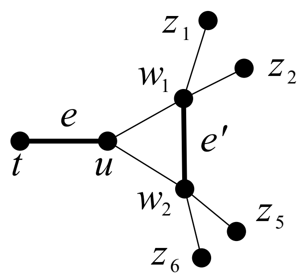


Figure 4.2: An edge $e = \{t, u\}$ owns a triangle edge $e' = \{w_1, w_2\}$.

Lemma 16. If an edge e in a graph G owns a triangle edge e', then e' decreases the upper bound Γ_d of $|C_G(e)|$ by at least one.

Proof. This lemma can be obtained by a simple observation on two graphs illustrated in Figure 4.3. The right graph does not have any triangle edge but the left one has one triangle edge $e' = \{w_1, w_2\}$. That is, we can think that two edges $\{w_1, z_3\}$ and $\{w_2, z_4\}$ in the right graph are replaced with one triangle edge $\{w_1, w_2\}$, or two edges are combined into one edge. Therefore, the value of Γ_d must decrease by at least one, because of the triangle edge e'. □

Now consider an edge $e = \{t, u\}$ in the solution M and the private conflict edges of e to M, $PC_G(M, e)$. Then, let $U_G(e) = (\{e' \mid dist_G(e', u) \leq 1\} \cap$

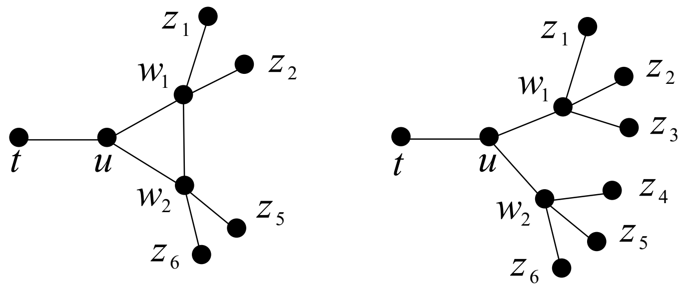


Figure 4.3: Since an edge $e = \{t, u\}$ owns a triangle edge $e' = \{w_1, w_2\}$, $e' = \{w_1, w_2\}$ decreases the upper bound Γ_d of $|C_G(e)|$ by at least one.

$PC_G(M, e)) \setminus \{e\}$ and $T_G(e) = (\{e' \mid dist_G(e', t) \leq 1\} \cap PC_G(M, e)) \setminus \{e\}$. Roughly speaking, $U_G(e)$ and $T_G(e)$ are the "u-side" subset and the "t-side" subset of edges in $PC_G(M, e)$, respectively. Note that $PC_G(M, e) = U_G(e) \cup T_G(e) \cup \{e\}$ and $U_G(e) \cap T_G(e)$ may be non-empty. Moreover, let $U^0_G(e) = \{e' \in U_G(e) \mid dist_G(e', u) = 0\}$, $U^1_G(e) = U_G(e) \setminus U^0_G(e)$, $T^0_G(e) = \{e' \in T_G(e) \mid dist_G(e', t) = 0\}$, and $T^1_G(e) = T_G(e) \setminus T^0_G(e)$.

From now on, let $|PC_G(M, e)| = \beta$. Without loss of generality, we assume that $|U_G(e)| \geq |T_G(e)|$ holds in the following. Then, we obtain the following lemma, which is quite trivial but plays a key role to estimate the approximation ratio of ALG:

Lemma 17. For each $e \in M$, $|U^1_G(e)| \geq \frac{\beta-1}{2} - (r - 1)$ holds.

Proof. Clearly $|U^0_G(e)| \leq r - 1$ holds. Since $|U_G(e) \cup T_G(e)| = \beta - 1$ and $|U_G(e)| \geq |T_G(e)|$ by the assumptions, $|U_G(e)| \geq \frac{\beta-1}{2}$ is satisfied. Hence, we can obtain $|U^1_G(e)| = |U_G(e) \setminus U^0_G(e)| \geq \frac{\beta-1}{2} - (r - 1)$. □

See Figure 4.4. Let $W_G(e) = V(G[U_G(e)]) \cap DV_1(u) = \{w_1, w_2, \cdots, w_\delta\}$ be a set of δ neighbor vertices of u, where $\delta \leq |DV_1(u)| - 1$ holds (where "-1" comes from the edge $\{t, u\}$). Then, we define $U^1_G(e, w_i) = \{(w_i, v) \mid v \in DV_1(w_i)\} \cap U^1_G(e)$ for each $w_i \in W_G(e)$. Without loss of generality, we assume that $|U^1_G(e, w_1)| \geq |U^1_G(e, w_i)|$ for each $i = 2, \cdots, \delta$. Now, we consider the case where $|U^1_G(e, w_1)| \leq 1$ holds. Then, we obtain the following lemma:

Lemma 18. Suppose that $|U^1_G(e, w_1)| \leq 1$ and the algorithm ALG outputs a solution M. Then $|PC_G(M, e)| \leq 4r - 3$ and $|C_G(e)| + |PC_G(M, e)| \leq 2r^2 + 2r - 2$ hold for every induced matching edge $e \in M$.

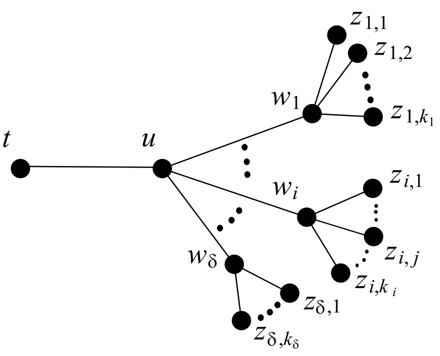


Figure 4.4: $W_G(e) = V(G[U_G(e)]) \cap DV_1(u) = \{w_1, w_2, \cdots, w_\delta\}$ where w_i has k_i neighbors, $z_{i,1}$ through z_{i,k_i}.

Proof. From the definition, $PC_G(M, e) = \{e\} \cup U_G(e) \cup T_G(e)$. Then, by the assumption $|U_G(e)| \geq |T_G(e)|$, the following inequality holds:

$$\begin{aligned} |PC_G(M, e)| &\leq 1 + |U_G(e)| + |T_G(e)| \\ &\leq 1 + 2|U_G(e)|. \end{aligned}$$

For a r-regular graph G, $|U^0_G(e)| \leq r-1$ holds. The assumption $|U^1_G(e, w_1)| \leq 1$ means that $|U^1_G(e, w_i)| \leq 1$ holds for each i, $2 \leq i \leq \delta$. It follows that $|U^1_G(e)| \leq r-1$ and $|U_G(e)| = |U^0_G(e)| + |U^1_G(e)| \leq 2(r-1)$. Therefore, $|PC_G(M, e)| \leq 1 + 4(r-1) = 4r - 3$ holds.

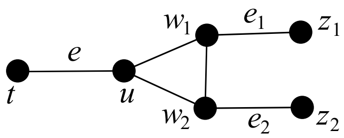
Since $|C_G(e)| \leq 2r^2 - 2r + 1$ as shown in Proposition 2, the inequality

$$\begin{aligned} |C_G(e)| + |PC(M, e)| &\leq (2r^2 - 2r + 1) + (4r - 3) \\ &= 2r^2 + 2r - 2 \end{aligned}$$

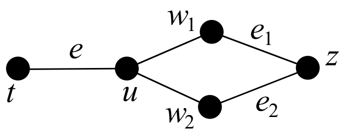
is obtained. □

Next, suppose that $|U^1_G(e, w_1)| \geq 2$ holds. We first depict all possible conflict ways of an edge of $U^1_G(e, w_1)$ and another edge of $U^1_G(e, w_i)$, where $i \neq 1$.

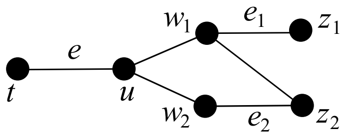
Recall that any two edges in $PC_G(M, e)$ (and thus any two edges in $U^1_G(e)$) are in conflict with each other to the solution M of `ALG`. There are five types of conflicts of two edges, say, e_1 and e_2, in $U^1_G(e)$ as follows: (a) triangle-conflict, (b) $\diamond$-quadrangle-conflict, (c) σ-quadrangle-conflict, (d) ρ-quadrangle-conflict, and (e) pentagon-conflict. See Figure 4.5 and consider two edges $e_1 = \{w_1, z_1\}$ and



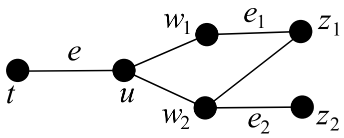
(a) triangle-conflict



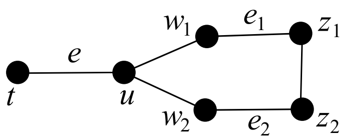
(b) $\diamond$-quadrangle-conflict



(c) σ-quadrangle-conflict



(d) ρ-quadrangle-conflict



(e) pentagon-conflict

Figure 4.5: Five types of conflicts of two edges e_1 and e_2 in $U_G^1(e)$

$e_2 = \{w_2, z_2\}$ in $U_G^1(e)$. (a) If e_1 is in conflict with e_2 since there exists the edge $\{w_1, w_2\}$ as shown in Figure 4.5(a), then we say that e_1 and e_2 are in *triangle-conflict* with each other by the edge $\{w_1, w_2\}$. (b) See Figure 4.5(b). If e_1 and e_2 are incident to a common vertex z and $U_G^1(e)$ does not have the edge $\{w_1, w_2\}$, then we say that e_1 and e_2 are in $\diamond$*-quadrangle-conflict* with each other. Note that if the graph shown in Figure 4.5(b) has the edge $\{w_1, w_2\}$, then we regard the conflict of e_1 and e_2 as the triangle conflict caused by $\{w_1, w_2\}$. (c) If there exists the edge $\{w_1, z_2\}$ but does not exist the edge $\{w_1, w_2\}$ as shown in Figure 4.5(c), then we say that e_1 and e_2 are in σ*-quadrangle-conflict* with each other by $\{w_1, z_2\}$. (d) If there exists the edge $\{w_2, z_1\}$ but does not exist the edge $\{w_1, w_2\}$ as shown in Figure 4.5(d), then we say that e_1 and e_2 are in ρ*-quadrangle-conflict* with each other by $\{w_2, z_1\}$.

(e) See Figure 4.5(e). If there exists the edge $\{z_1, z_2\}$ but does not exist the edge $\{w_1, w_2\}$, then we say that e_1 and e_2 are in *pentagon-conflict* with each other by $\{z_1, z_2\}$. Recall, however, that all the input graphs are now C_5-free. It follows that the induced cycle $\langle u, w_1, z_1, z_2, w_2, u\rangle$ of length 5 must have at least one edge inside of it. For example, the graph has the edge $\{w_1, z_2\}$, then we regard the conflict of e_1 and e_2 as the σ-quadrangle-conflict caused by $\{w_1, z_2\}$. Therefore, we do not need to take the pentagon-conflict into account.

In the following, we slightly change the previous definition of *triangle edges*. (We call the previously defined triangle edge the *original triangle edge* in the following.) An edge in $U_G^1(e)$ is called a *triangle edge* of the edge e if its one endpoints is w_i and the other is w_j in $W_G(e) \setminus \{w_i\}$, where $w_i \neq w_1$, $w_j \neq w_1$, and $w_i \neq w_j$. That is, for example, an edge $\{w_1, w_3\}$ is not regarded as a triangle edge since its one endpoint is w_1. Let $TE_G(e)$ be the set of triangle edges. Then, we define as follows:

$$A_G(e) = U_G^1(e) \setminus (U_G^1(e, w_1) \cup TE_G(e)).$$

Every edge e_2 in $A_G(e)$ is in conflict with every edge e_1 in $U_G^1(e, w_1)$, and $|U_G^1(e, w_1)| \geq |U_G^1(e, w_i)|$ from the definition. Then, all the edges in $A_G(e)$ are divided into the following two sets, the sets of *triangle-conflict edges* and *quadrangle-conflict edges*.

Triangle-Conflict edge: If an edge e' in $A_G(e)$ is in triangle-conflict with an edge in $U_G^1(e, w_1)$, then we say that e' is a *triangle-conflict* edge. Let $TC_G(e)$ be the set of triangle-conflict edges.

Quadrangle-Conflict edge: If an edge e' in $A_G(e)$ is in $\diamond$-quadrangle, σ-quadrangle, or ρ-quadrangle-conflict with an edge in $U_G^1(e, w_1)$, then we simply say that the edge e' is a *quadrangle-conflict* edge. Let $QC_G(e)$ be the set of quadrangle-conflict edges.

From the definitions, $U_G^1(e) = TC_G(e) \cup QC_G(e) \cup U_G^1(e, w_1) \cup TE_G(e)$ and $TC_G(e) \cap QC_G(e) = \emptyset$ hold.

Recall that we are now assuming that $|U_G^1(e, w_1)| \geq 2$. We take a look at the edge $e' = \{u, w_1\}$ and calculate the cardinality of the set $C_G(e')$ of conflict edges of e'. Note that each edge in $TC_G(e)$ creates one cycle C_3 of length three, which contains e', and each edge in $QC_G(e)$ creates one cycle C_4 of length four, which contains e'. Also, each edge in $TE_G(e)$ must be an original triangle edge of e'. It follows that each edge in $TC_G(e) \cup QC_G(e) \cup TE_G(e)$ causes decrease of the upper bound Γ_d of $|C_G(e')|$ by at least one from Proposition 3 and Lemma 16.

Lemma 19. Suppose that $|U^1_G(e, w_1)| \geq 2$. Also, suppose that the algorithm `ALG` outputs a solution M. Then, $|C_G(e')| \leq 2r^2 - \frac{\beta}{2} - \frac{1}{2}$ holds, where $e' = \{u, w_1\}$.

Proof. See Figure 4.4 again and take a look at triangle-conflict, quadrangle-conflict, and (original) triangle edges in the following:

(i) Suppose that p vertices in $\{w_2, w_3, \cdots, w_\delta\}$ of $\delta - 1$ vertices are endpoints of *triangle-conflict* edges. Then, we can verify that there are p cycles of length three which contain the edge $e' = \{u, w_1\}$. Therefore, by Proposition 3, the value of the upper bound Γ_d of e' is reduced by at least pd. Since each of those p vertices is connected to at most $r - 1$ edges in $TC_G(e)$, $|TC_G(e)| \leq p(r-1) \leq pr$ holds. Namely, we can estimate that each edge in $TC_G(e)$ reduces the value of Γ_d of e' by at least one on average.

(ii) Each edge in $QC_G(e)$ obviously generates one cycle of length four which contains the edge $e' = \{u, w_1\}$. Thus, by Proposition 3, we can also estimate that each edge in $QC_G(e)$ decreases the value of Γ_d of e' by at least one.

(iii) Clearly, each edge in $TE_G(e)$ is a triangle edge of e. Also, it is an *original* triangle edge of $e' = \{u, w_1\}$. Then, by Lemma 16, we can estimate that each edge in $TE_G(e)$ decreases the value of Γ_d of e' by at least one.

Consequently, we can estimate that each edge in $TC_G(e) \cup QC_G(e) \cup TE_G(e)$ decreases the value of Γ_d of e' by at least one. Thus, all the edges in $TC_G(e) \cup QC_G(e) \cup TE_G(e)$ decrease the value of Γ_d of e' by at least $|TC_G(e) \cup QC_G(e) \cup TE_G(e)|$ in total.

Now, recall that $U^1_G(e) = TC_G(e) \cup QC_G(e) \cup U^1_G(e, w_1) \cup TE_G(e)$. Then,

$$\begin{aligned} &|TC_G(e) \cup QC_G(e) \cup TE_G(e)| \\ = &|U^1_G(e) \setminus U^1_G(e, w_1)| \\ \geq &|U^1_G(e)| - (r-1) \end{aligned}$$

holds since $|U^1_G(e, w_1)| \leq r - 1$. Furthermore, since $|U^1_G(e)| \geq \frac{\beta - 1}{2} - (r-1)$ as shown in Lemma 17, we obtain the following:

$$
\begin{aligned}
&|TC_G(e) \cup QC_G(e) \cup TE_G(e)| \\
&\geq |U_G^1(e)| - (r-1) \\
&\geq \left(\frac{\beta - 1}{2} - (r-1)\right) - (r-1) \\
&= \frac{\beta - 1}{2} - 2(r-1).
\end{aligned}
$$

Therefore, the upper bound Γ_d of e' decreases by at least $\frac{\beta-1}{2} - 2r + 2$.

From Proposition2, we obtain the following inequalities:

$$
\begin{aligned}
|C_G(e')| &\leq 2r^2 - 2r + 1 - \left(\frac{\beta - 1}{2} - 2r + 2\right) \\
&= 2r^2 - \frac{1}{2} - \frac{\beta}{2}.
\end{aligned}
$$

This completes the proof of this lemma. □

From Lemma 19, we can get the following corollary:

Corollary 3. Suppose that $|U_G^1(e, w_1)| \geq 2$ and the algorithm ALG outputs a solution M. Then, $|C_G(e)| \leq 2r^2 - \frac{1}{2} - \frac{\beta}{2}$ for every induced matching edge $e \in M$.

Proof. From Lemma 19, we know that there is an edge e' in $U_G(e)$ of $PC_G(M, e)$ such that $|C_G(e')| \leq 2r^2 - \frac{\beta}{2} - \frac{1}{2}$ for any induced matching edge e. Furthermore, Remark 2 shows that $|C_G(e)| \leq |C_G(e')|$ must be satisfied for e and e'. Therefore, $|C_G(e)| \leq 2r^2 - \frac{1}{2} - \frac{\beta}{2}$ holds. □

The above corollary gives us the following lemma:

Lemma 20. Suppose that $|U_G^1(e, w_1)| \geq 2$ and the algorithm ALG outputs a solution M. Then, $|PC_G(M, e)| \leq \frac{4r^2-1}{3}$, and $|C_G(e)| + |PC_G(M, e)| \leq \frac{8r^2-2}{3}$ hold for every induced matching edge $e \in M$.

Proof. From Corollary 3, we know that for each $e \in M$, $|C_G(e)| \leq 2r^2 - \frac{1}{2} - \frac{\beta}{2}$ holds. From the definitions, $PC_G(M, e) \subseteq C_G(e)$ holds. Therefore, we obtain

$$
|PC_G(M, e)| = \beta \leq |C_G(e)| \leq 2r^2 - \frac{\beta}{2} - \frac{1}{2}.
$$

That is, $\beta \leq 2r^2 - \frac{\beta}{2} - \frac{1}{2}$ holds and hence β is bounded from above as follows:

$$\beta \leq \frac{4r^2 - 1}{3}. \tag{4.1}$$

By the definition $|PC_G(M, e)| = \beta$,

$$\begin{aligned}|C_G(e)| + |PC_G(M, e)| &\leq 2r^2 - \frac{\beta}{2} - \frac{1}{2} + \beta \\ &= 2r^2 + \frac{\beta}{2} - \frac{1}{2} \\ &\leq \frac{8r^2 - 2}{3},\end{aligned}$$

where the last inequality comes from the above (4.1). This completes the proof of this lemma. □

From Lemmas 18 and 20, we have the following corollary:

Corollary 4. Suppose that a solution M is obtained by the algorithm `ALG`. Then, $|C_G(e)| + |PC_G(M, e)| \leq \frac{8r^2-2}{3}$ holds for every induced matching edge $e \in M$.

Proof. By Lemma 20, we know that for $|U_G^1(e, w_1)| \geq 2$,

$$|C_G(e)| + |PC_G(M, e)| \leq \frac{8r^2 - 2}{3}.$$

From the assumption $r \geq 3$ and Lemma 18, we obtain the following inequality also for $|U_G^1(e, w_1)| \leq 1$:

$$\begin{aligned}|C_G(e)| + |PC_G(M, e)| &\leq 2r^2 + 2r - 2 \\ &\leq \frac{8r^2 - 2}{3}.\end{aligned}$$

This completes the proof of this corollary. □

The following is our main theorem:

Theorem 9. The algorithm `ALG` is a $\left(\frac{2r}{3} + \frac{1}{3}\right)$-approximation algorithm for MaxIM on C_5-free r-regular graphs, whose running time is $O(m^4)$.

Proof. From Remark 2, the solution for an input C_5-free r-regular graph $G = (V, E)$

satisfies the inequality in Lemma 15, that is, we have obtained

$$\sum_{e \in M} (|C_G(e)| - |PC_G(M, e)|)$$
$$\geq 2(|E| - \sum_{e \in M} |PC_G(M, e)|),$$

or equivalently,

$$\sum_{e \in M} (|C_G(e)| + |PC_G(M, e)|) \geq 2|E|. \tag{4.2}$$

From Corollary 4 and $|ALG(G)| = |M|$, we obtain:

$$\sum_{e \in M} (|C_G(e)| + |PC_G(M, e)|)$$
$$\leq \frac{|ALG(G)|(8r^2 - 2)}{3} \tag{4.3}$$

Suppose that $|V| = n$, and hence $|E| = \frac{nr}{2}$. Then, the above (4.2) and (4.3) give the following inequality:

$$\frac{|ALG(G)|(8r^2 - 2)}{3} \geq nr.$$

Thus,

$$|ALG(G)| \geq \frac{3nr}{8r^2 - 2}.$$

It is known [38] that the size $|OPT(G)|$ of an optimal solution is at most $\frac{nr}{4r-2}$. Therefore, the approximation ratio is as follows:

$$\frac{|OPT(G)|}{|ALG(G)|} \leq \frac{2r}{3} + \frac{1}{3}.$$

□

4.3 Remark

On the approximability of MaxIM on C_5-free r-regular graphs. The previously best known approximation ratio was $(\frac{3r}{4} - \frac{1}{8} + \frac{3}{16r-8})$. In this thesis, we have provided a $\left(\frac{2r}{3} + \frac{1}{3}\right)$-approximation algorithm ALG. One can verify that the new approximation ratio of `ALG` is strictly better than the old one when $r \geq 6$. Recall that `ALG` initially finds a maximal induced matching M in **Step 1**. However, it is important to note that **Step 1** can be replaced with the $\left(\frac{3r}{4} - \frac{1}{8} + \frac{3}{16r-8}\right)$-approximation algorithm as a subroutine. **Step 2** surely finds an induced matching of the same or larger size than the initial induced matching. This implies that the "hybrid" approximation algorithm achieves the approximation ratio of $\min\left\{\frac{3r}{4} - \frac{1}{8} + \frac{3}{16r-8}, \frac{2r}{3} + \frac{1}{3}\right\}$ for MaxIM on C_5-free r-regular graphs for every $r \geq 3$.

Chapter 5

Conclusion

In the chapter 3, we have studied the problem of MaxDdIS and have obtained (in)approximability of MaxDdIS on r-regular graphs, where $d \geq 3$ and $r \geq 3$. On inapproximability of MaxDdIS on r-regular graphs, we have proved that it is NP-hard to approximate MaxD3IS on 3-regular graphs within 1.00105 unless P=NP. Furthermore, restricting $d \geq 3$ and $r \geq 3$, we get results that there exists no σ-approximation algorithm for MaxDdIS on r-regular graphs unless P=NP: (i) for $d = 3$, $r \geq 3$ and $\sigma < \frac{95r^2(r-1)+190}{95r^2(r-1)+188}$, (ii) for $d = 4$, $r \geq 3$ and $\sigma < \frac{95r^2(r-2)+190}{95r^2(r-2)+188}$, and (iii) for $d \geq 5$, $r \geq 3$ and $\sigma < \frac{95r^2(\lceil d/2 \rceil-1)+190}{95r^2(\lceil d/2 \rceil-1)+188}$. On approximability of MaxDdIS on regular graphs, we first concentrate on MaxDdIS on r-regular graphs, and design $O(r^{d-1})$-approximation and an improved $O(r^{d-2}/d)$-approximation algorithms. Then, restricting $r = d = 3$, we focus on MaxD3IS on cubic graphs, and we have designed four approximation algorithms with the approximation ratios 2.4 , $2 + \frac{4}{n-2}$, 2 and 1.875, respectively. Moreover, we have produced a PTAS algorithm for planar graphs.

In the chapter 4, we have studied MaxIM. On C_5-free r-regular graphs, we have designed an improved approximation algorithm with the perform factor of $\frac{2r+2}{3}$. On general r-regular graphs, our algorithm can be utilized, and unfortunately, we can not ensure that whether this algorithm is strictly better than the previous approximation algorithm. Thus, restricted general regular graphs, it is still open for designing a better algorithm than the previous best approximation algorithm. Moreover, some variants of maximum matching problem is also open.

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