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WORDS WITHOUT NEAR-REPETITIONS

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ABSTRACT. We find an infinite word w on four symbols with the following property: Two occurrences of any block in w must be separated by more than the length of the block. That is, in any subword of w of the form xyx, the length of y is greater than the length of x. This answers a question of C. Edmunds connected to the Burnside problem for groups.

1. **Introduction.**

In their solution of the Burnside problem for groups [5], Novikov and Adjan use a result from combinatorics on words:

> There is an infinite word v on the alphabet $\{0,1\}$ such that v contains no subword of the form xxx, $x \neq \epsilon$. [2,6]

Novikov and Adjan invoke this result at the end of their notoriously long and involved proof. The bulk of their proof, filling a book of 300+ pages, involves constructions of groups. C. Edmunds [4] suggests that it may be possible to find a shorter proof by using stronger results from combinatorics on words, rather than by finding new group theoretic constructions. With this motivation, Edmunds poses the following question:

Can one find a finite alphabet S, and some infinite word w over S such that whenever xyx is a subword of w, the length of y is greater than the length of x?

We answer Edmunds' question in the affirmative. The smallest alphabet for which such a w can exist is a 4 letter alphabet.

2. **Notation.**

Our notation follows the usual notation of automata theory. Let S be a set. A *word* is a finite sequence of elements of S. We refer to S as an *alphabet*, its elements as *letters*. The set of all words over S is denoted S^*. We take a naive view of words as strings of letters; thus the concatenation of two words w and v, written wv, is simply the string of letters consisting of the letters of w followed by the letters of v.

Say that v is a *subword* of w if we can write $w = uvz$; $u, v, z \in S^*$. If $w = uv$ then we say that u is a *prefix* of w; v is a *suffix* of w. The *empty word*, denoted ϵ, is the word with no letters in it. Denote by $|w|$ the *length* of w, equal to the number of letters of w.

Let S, T be alphabets. A *substitution* $h: S^* \longrightarrow T^*$ is a function generated by its values on S. That is, suppose $w \in S^*$, $w = a_1a_2\cdots a_m$; $a_i \in S$ for $i = 1$ to m. Then $h(w) = h(a_1)h(a_2)\cdots h(a_m)$.

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Let S be an alphabet, $w \in S^*$ a word over S. If we can write $w = uxyxv$ with $|y| \leq |x|$, $u, v, x, y \in S^*$, we call w *near-repetitive*, and call xyx a *near-repetition*. If w is not near-repetitive, call w *varied*.

3. **Construction of varied words.** By König's Infinity Lemma, to show that there is an infinite varied word over a finite alphabet S, it suffices to show that there are arbitrarily long varied words over S. Let S be the alphabet $S = \{1, 2, 3, 4, 5\}$. Consider the substitution $f: S^* \longrightarrow S^*$ given by

$$\begin{aligned} f(1) &= 123145213412435 \\ f(2) &= 123154234531425 \\ f(3) &= 123152413425324 \\ f(4) &= 123143254135245 \\ f(5) &= 123153452132534. \end{aligned}$$

We will prove that $f^n(1)$ is varied. To begin, we make some observations concerning f:

OBSERVATION 1. We see that f replaces each letter of S by a string of fifteen letters. Thus if $u \in S^*$, $|f(u)| = 15|u|$. ■

OBSERVATION 2. The images of different letters under f can have a common suffix of length at most 1. That is, suppose that $u, v \in S$ and we have

$$f(u) = UW, f(v) = VW, \ |W| \geq 2.$$

Then $u = v$. ■

One concludes from Observation 2 that f is 1-1.

OBSERVATION 3. The images of different letters under f can have a common prefix of length at most 5. Thus suppose that $u, v \in S$ and we have

$$f(u) = WU'', f(v) = WV'', \ |W| \geq 6.$$

It follows that $u = v$. ■

OBSERVATION 4. The images of different letters under f can have a common subword of length at most 6. In fact, suppose that $u, v \in S$ and we have

$$f(u) = U'WU'', f(v) = V'WV'', \ |W| \geq 7.$$

We must have $U' = V'$, $U'' = V''$, $u = v$. ■

OBSERVATION 5. Call a word w a *suffix-prefix* if we can write $w = uv$ where u is the non-empty suffix of the image of some letter under f, and v is the non-empty prefix of the image of some letter. Note that no non-empty prefix of the image of a letter is the suffix of the image of a letter. Thus if w can be expressed as a suffix-prefix then the words u and v are unique.

The longest instance of a suffix-prefix in the image under f of a letter is 3412 in $f(1)$. Thus if $u, v, w \in S$ and

$$f(u) = U'V''W'U'',\ f(v) = V'V'',\ f(w) = W'W'', \text{ with } W', V'' \neq \epsilon,$$

then $|V''W'| \leq 4$. ∎

Using some of these observations we prove the following lemma.

LEMMA. *Let* $u = u_1u_2\cdots u_m$, $v = v_1v_2\cdots v_n$ *with the* $u_i, v_j \in S$. *Let* $f(u_i) = U_i$, $f(v_i) = V_i$. *Suppose that for some word* w *we can write*

$$f(u) = U_1U_2\cdots U_j'wU_k''U_{k+1}\cdots U_m$$

and

$$f(v) = V_1V_2\cdots V_s'wV_t''V_{t+1}\cdots V_n, \quad |w| \geq 7$$

where

$$U_j = U_j'U_j'',\ U_k = U_k'U_k'',\ V_s = V_s'V_s'',\ V_t = V_t'V_t''.$$

Then

$$|U_j'| \equiv |V_s'| \pmod{15},\ |U_k''| \equiv |V_t''| \pmod{15}.$$

PROOF. By Observation 1, it follows that

$$|U_j'| + |w| + |U_k''| \equiv |V_s'| + |w| + |V_t''| \equiv 0 \pmod{15}.$$

It thus suffices to show that $U_j' \equiv V_s' \pmod{15}$. To do this, we will assume that $|w| = 7$, replacing w by its first 7 letters if necessary. It follows that $k \leq j + 1$, $t \leq s + 1$. We will also assume without loss of generality that $|U_j'|, |V_s'|, |U_k''|, |V_t''| < 15$. The word w is thus a subword of $U = U_jU_{j+1}$ and of $V = V_sV_{s+1}$.

Suppose that w is not a suffix-prefix. Then w must be a subword of either U_j or U_{j+1}. Assume first that w is a subword of U_j. Again, w must be a subword of either V_s or V_{s+1}. If w is a subword of V_s, then Observation 4 implies that $|U_j'| = |V_s'|$, and we are done. Otherwise, w is a prefix of V_{s+1}, and $|V_s'| = 0$. By Observation 4, w is also a prefix of U_j, so that $U_j' = \epsilon = V_s'$. (In this case $j = k$.) A symmetrical argument deals with the possibility that w is a subword of U_{j+1}.

Suppose then that w is a suffix-prefix, $w = U_j''U_{j+1}' = V_s''V_{s+1}'$. It follows from Observation 5 that $U_j' = V_s'$. ∎

THEOREM 1. *For all* $n \in \mathbb{N}$, *the word* $f^n(1)$ *is varied.*

PROOF. We proceed by induction. One checks that $f^1(1) = f(1)$ is varied. Let n be least such that $f^n(1)$ is near repetitive. Let $e = e_1e_2\cdots e_m$ be a subword of $f^{n-1}(1)$ of minimal length such that $f(e)$ contains a near repetition xyx, $|y| \leq |x|$. It is convenient to make two cases:

CASE 1. We have $|x| \leq 6$.

In this case, $|xyx| \leq 18$. It follows that $|e| \leq 3$. Moreover, e is a varied word since it is a subword of $f^{n-1}(1)$. To show the impossibility of this case, it suffices to check that $f(e)$ is varied whenever $e \in S^*$ is varied and $|e| = 3$. Such a word e must consist of three distinct letters, and one checks that the relevant 60 words are varied.

CASE 2. We have $|x| \geq 7$. We may also assume, by our disposition of case 1, that $m \geq 4$.

Let $f(e_i) = E_i$ and write $f(e) = E'_1xyxE''_m = E'_1xE''_jE_{j+1}\cdots E_m = E_1\cdots E'_kxE''_m$, where $E_1 = E'_1E''_1$, $E_j = E'_jE''_j$, $E_k = E'_kE''_k$, $E_m = E'_mE''_m$ and E''_1, E'_j, E''_k, E'_m are non-empty. (We know that E''_1 and E'_m are non-empty by the minimality of $|e|$. Let the others be non-empty by a notational convention.) We must have $j < m$. Otherwise E_2E_3 is a subword of our first occurrence of x, but the second occurence of x is a subword of E_m. This is a contradiction on the length of x. Also, $k < m$. Otherwise the second occurrence of x is a subword of E_m, but $E''_1E_2E_3$ is a subword of xy. This gives the contradiction $30 < |E''_1E_2E_3| \leq |xy| \leq 2|x| \leq 2|E_m| = 30$. Similarly, $1 < j \leq k < m$.

By the lemma, $|E'_1| \equiv |E'_k|$, $|E''_j| \equiv |E''_m|$ (mod 15). Since E''_1, E'_j, E''_k, E'_m are non-empty, the congruence can in fact be replaced by equality. Without loss of generality, we may assume that $|E''_1| \leq 1$. Suppose not. Then $|E''_1| = |E''_k| \geq 2$. Since E''_1 and E''_k are prefixes of x, and have the same length they are equal. It follows from Observation 3 that $e_1 = e_k$.

Write $x = x'x''$ where $|x''| = \max(0, |E'_1| - |y|)$. If $|y| > |E'_1|$, then write $y = \hat{y}y''$ where $|y''| = |E'_1|$. Otherwise, let $\hat{y} = \epsilon$. We see that $f(e)$ contains the near repetition $\hat{x}\hat{y}\hat{x}$, where $\hat{x} = E'_1x'$. If we replace x by $\hat{x}$, and y by $\hat{y}$ in our argument, we get $|E_1| = 0$. (In other words, we extend both the occurrences of our original x by adding a prefix $E'_1 = E'_k$ in front. In the case of the second x, this will shorten y by $|E'_1|$. If $|y|$ is shorter than $|E'_1|$, an amount $|E'_1| - |y|$ is removed from the end of each x, and y disappears.) Similarly, without loss of generality, we may assume that $|E'_m| \leq 5$.

We can write

$$x = E''_1E_2\cdots E'_j = E''_kE_{k+1}\cdots E'_m.$$

In fact, $E''_1 = E''_k$, $E'_j = E'_m$, $E_2E_3\cdots E_{j-1} = E_{k+1}E_{k+2}\cdots E_{m-1}$. Since f is 1-1, we have $e_2\cdots e_{j-1} = e_{k+1}\cdots e_{m-1}$.

Let $a = e_2\cdots e_{j-1} = e_{k+1}\cdots e_{m-1}$, $b = e_j\cdots e_k$. We claim that aba is a near repetition in e; that is, that $|b| \leq |a|$. This will be a contradiction, for e must be varied. If $j = k$ the claim is clearly true. Otherwise,

$$\begin{aligned} |a| = |e_2\cdots e_{j-1}| &= (|E_2\cdots E_{j-1}|)/15 \\ &= \big(|x| - (|E''_1| + |E'_j|)\big)/15 \\ &\geq \big(|x| - (1+5)\big)/15 \\ &= (|x| - 6)/15, \end{aligned}$$

$$\begin{aligned}|b| = |e_j \cdots e_k| &= (|E_j \cdots E_k|)/15 \\ &= \big(|y| + (|E_j'| + |E_k''|)\big)/15 \\ &\leq (|x| + 6)/15.\end{aligned}$$

It follows that $|b| - |a| \leq 12/15$. Since $|a|$ and $|b|$ are integers, we conclude that $|b| \leq |a|$. ■

One discovers quickly that the longest varied words over the alphabet $\{1,2,3\}$ are permutations of 1231. Thus there is no infinite varied word on a 3 letter alphabet. Let $T = \{1,2,3,4\}$, and let $g\colon T^* \longrightarrow T^*$ be given by

$$\begin{aligned}g(1) = {}&123421432413423124321341231421324123421431241321423124\\ &321341231432413421431234132142312413421432412314213243\\ g(2) = {}&123421432413423124321423413243123421324123142134124231\\ &423124132143123413243142134123143213423124321423413243\\ g(3) = {}&123421432413423143213412314213243123413214312413421432\\ &412314213412431423413243123421324123143213412431421324\\ g(4) = {}&123421432413423143213412431423413214312413421432412342\\ &132431423412432134231432413421431241321423412431421324\end{aligned}$$

THEOREM 2. *The word* $g^n(1)$ *is varied for every* $n \in \mathbb{N}$.

This theorem is proved analogously to Theorem 1, with proportionately more checking. We see that g replaces each letter of T by a string of 108 letters. The images of different letters under g can have a common suffix of length at most 13, a common prefix of length at most 24. With similar observations and proceeding as in the previous theorem, one establishes a lemma:

LEMMA. *Let* $u = u_1u_2 \cdots u_m$, $v = v_1v_2 \cdots v_n$ *with the* $u_i, v_j \in S$. *Let* $g(u_i) = U_i$, $g(v_i) = V_i$. *Suppose that for some word* w *we can write*

$$g(u) = U_1U_2 \cdots U_j'wU_k''U_{k+1} \cdots U_m \text{ and } g(v) = V_1V_2 \cdots V_s'wV_t''V_{t+1} \cdots V_n,$$

$|w| \geq 38$ *where*

$$U_j = U_j'U_j'', \; U_k = U_k'U_k'', \; V_s = V_s'V_s'', \; V_t = V_t'V_t''$$

Then

$$|U_j'| \equiv |V_s'| \pmod{108}, \; |U_k''| \equiv |V_t''| \pmod{108}.$$

■

The proof of Theorem 2 is similar to that of Theorem 1. In the final phase, the proof of Theorem 1 depended on an inequality involving the quantities in Observations 1, 2 and 3: $1 + 5 < 15/2$. In Theorem 2, we have the analogous inequality: $13 + 24 < 108/2$.

We have thus answered Edmunds' question in the affirmative, and shown that a four letter alphabet is the smallest on which infinite varied words exist.

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