Algebraic and combinatorial aspects of incidence groups and linear system non-local games arising from graphs

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Many of the results in this thesis were produced in collaboration with William Slofstra, Vincent Russo, and Turner Silverthorne. Some of the computations used in the thesis are due to preliminary investigations done by Turner Silverthorne during an undergraduate research project with William Slofstra. Instances, where these results are used, are explicitly mentioned in the text. Any other work is my own.

Abstract

To every linear binary-constraint system (LinBCS) non-local game, there is an associated algebraic object called the solution group. Cleve, Liu, and Slofstra showed that a LinBCS game has a perfect quantum strategy if and only if an element, denoted by J, is non-trivial in this group. In this work, we restrict to the set of graph-LinBCS games, which arise from $\mathbb{Z}_2$-linear systems $Ax = b$, where A is the incidence matrix of a connected graph, and b is a (non-proper) vertex 2-colouring. In this context, Arkhipov's theorem states that the corresponding graph-LinBCS game has a perfect quantum strategy, and no perfect classical strategy, if and only if the graph is non-planar and the 2-colouring b has odd parity. In addition to efficient methods for detecting quantum and classical strategies for these games, we show that computing the classical value, a problem that is NP-hard for general LinBCS games can be done efficiently. In this work, we describe a graph-LinBCS game by a 2-coloured graph and call the corresponding solution group a graph incidence group. As a consequence of the Robertson-Seymour theorem, we show that every quotient-closed property of a graph incidence group can be expressed by a finite set of forbidden graph minors. Using this result, we recover one direction of Arkhipov's theorem and derive the forbidden graph minors for the graph incidence group properties: finiteness, and abelianness. Lastly, using the representation theory of the graph incidence group, we discuss how our graph minor criteria can be used to deduce information about the perfect strategies for graph-LinBCS games.

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Dedication

for Madeline

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Table of Contents

List of Tables x

List of Figures xi

1 Introduction 1

2 Preliminaries 8

2.1 Graph theory . . . 8

2.1.1 Simple graphs and non-proper colourings . . . 8

2.1.2 Hypergraphs and higher incidence structures . . . 9

2.1.3 Graph minors . . . 10

2.2 Group theory . . . 12

2.2.1 Basic properties of groups . . . 12

2.2.2 Finitely presented groups . . . 14

2.2.3 Products for presented groups . . . 15

2.3 Representation theory . . . 16

2.3.1 Unitary representations of groups . . . 16

2.3.2 A representation of the dihedral group . . . 18

2.3.3 Characters of a representation . . . 18

2.3.4 Representations of groups generated by involutions . . . 21

2.4 Quantum measurements . . . 21

2.4.1 Quantum states 22
2.4.2 Measurement postulates and POVM's 23
2.4.3 Projective measurements 23
2.4.4 Joint measurements on entangled systems 23
2.5 Non-local games 24
2.5.1 Classical strategies for non-local games 24
2.5.2 Quantum strategies for non-local games 25
2.5.3 A non-local game with a quantum advantage 27
2.5.4 Linear binary-constraint systems and operator solutions 28
2.6 Correlation sets for quantum models 30
2.6.1 Decomposition of unitary elements into sets of projections 31

3 Graph incidence groups for graph-LinBCS games 33
3.1 Graph-LinBCS non-local games 33
3.1.1 The space of classical strategies for graph-LinBCS games 34
3.1.2 A deterministic polynomial time algorithm for computing ω for graph-LinBCS games 36
3.1.3 Perfect quantum strategies for graph-LinBCS games 40
3.1.4 The CHSH game is a graph-LinBCS game 41
3.2 The solution group of a graph-LinBCS game 42
3.3 Pictures and relations for graph incidence groups 45
3.3.1 Pictures as weak planar covers 47
3.3.2 Pictures for groups generated by involutions 48
3.3.3 The van Kampen lemma 48
3.4 Graph incidence group pictures and Arkhipov's theorem 50
3.4.1 Graph incidence group pictures for the magic squares game 51

4 Graph minor operations and graph incidence groups **54**

4.1 Graph minor operations for 2-coloured graphs 54

4.2 Proof of main lemma 55

4.2.1 Graph minors and quotient-closed properties 58

4.3 Incidence groups for graphs without vertex disjoint cycles 59

4.3.1 Finiteness and abelianness for graph without disjoint cycles 65

4.4 Proof of main theorems 66

5 Graph incidence group characters and correlations for graph-LinBCS games **68**

5.0.1 Correlations and characters supported on observable conjugacy classes 68

5.1 Observable characters of graph incidence groups 70

5.1.1 Character tables for dihedral groups 72

5.1.2 The correlations for Dih_∞ 72

References **74**

APPENDICES **78**

A Implementing incidence groups in SAGE 78

A.1 Code for the `mp_group()` function 78

A.2 Finding rewriting systems and computing the value of J 82

A.3 Finding deterministic strategies for incidence groups in SAGE . . . 82

B A toolbox for testing higher dimensional correlations 84

C Future work & open questions 85

C.1 Minors for other quotient properties 86

C.2 Planar covers and Negami's conjecture 86

C.3 Quantum advantage for oddly-2-coloured planar graphs 87

C.4 Other planarity criteria for graphs 87

C.5 Graph-LinBCS games and MIP* 87

List of Tables

2.1 The character table of $\mathbb{Z}_2$, the columns index the conjugacy classes via some class representative, while the rows are the labelled by its irreducible representations. 21

4.1 Finiteness $|\cdot|$ and abelianness $[\cdot]$ properties for the (homogenous) graph incidence groups from a selection of graph families that do not contain two vertex disjoint cycles. When the group is finite we give the order of the group. 66

List of Figures

1.1 On the right, transposing the hypergraph of the magic squares game arrangement results in a game $\mathcal{G}(K_{3,3}, b)$, where one gives assignments to the edges. Treating the binary constraints as vertex labels, one can exhibit the game structure as the 2-coloured complete bipartite graph $K_{3,3}$. On the left, the resulting graph from the magic pentagram game $\mathcal{G}(K_5, b)$. We chose the convention that blue vertices represent a 1 constraint and red vertices a 0 constraint, thus both games have odd-colourings. 5

2.1 The graph minor operation of deleting the edge e from G. The dashed line indicates the removed edge from the graph, following the operation. 10

2.2 The graph minor operation contracting the edge d in $E(G)$. Observe that this operation results in the merging of the two end-vertices of $d \in E(G)$. Hence, this minor operation reduces the number of vertices in $V(G)$ by one. 11

2.3 The graph minor operation $G \to G \backslash v$ deleting vertex v from G. The removed vertex and its subsequent edges are indicated by dashed lines in the graph following the vertex deletion. Observe that deleting vertex v, also removes all the edges incident to v. Essentially the vertex-deletion minor operation induces the edge-deletion minor for every edge incident to v. 12

2.4 Four types of graphs that do not contain two disjoint cycles. From left-to-right, K_5, the wheel graph W_5, the bipartite graph $K_{3,4}$, and a cycle with tree branches $\mathcal{R}_n$. 12

3.1 Any 0 or 1 assignment to e_{uv} would be inconsistent with the colouring of G, hence we see that this edge assignment does not give a deterministic strategy for this graph-LinBCS game, since giving Alice u and Bob v may result in an inconsistent labelling of e_{uv} if they were to employ h as their strategy. 35

3.2 Visualization of Algorithm 1. Paths between pairs are labelled by the dashed edges and the single (odd) vertex by the dashed circle. Observe the only loss can occur if the odd, along with one of its two adjacent vertices, is chosen and Alice flips the colour value on the wrong of the two edges. Since the colouring satisfies 9 of the 10 constraint, the classical value of this graph-LinBCS game is $\omega = 0.99$ by Proposition 3.1.7. 40

3.3 The assignment, of the following tensor product of 2×2 Pauli matrices, to the edges of $K_{3,3}$ give a perfect quantum strategy for the magic squares game, where the -1 constraint appears once in the relation to the vertex incident with the bold edges $\{7, 8, 9\}$, all other relations are given the 1 constraint. 41

3.4 The CHSH game as a graph-LinBCS game, an oddly 2-coloured graph, on two vertices with two edges. 42

3.5 Generators of $\Gamma_0(K_4)$ labelling the edges of K_4 , the arrow denotes an arbitrary *simplifying* isomorphism between the graph representation of the group. The local vertex relations on the outer cycle are $\{x_1x_3x_4, x_1x_5x_2, x_2x_6x_3\}$, each generator commutes within the relation, hence we can rewrite $x_1x_3x_4 = 1 \Rightarrow x_1x_3 = x_4$, $x_1x_5x_2 = 1 \Rightarrow x_1x_2 = x_5$, and $x_2x_6x_3 = 1 \Rightarrow x_2x_3 = x_6$. This does not change the group structure; it simply makes inferring the group structure from the graph easier for small graphs. 44

3.6 The addition of a vertex with three edges to the graph K_4 yields the graph $K_4 \cup \{v\}$. With the aid of a computer, one can deduce an isomorphic presentation of the corresponding graph incidence group on fewer generators. The order of $\Gamma_0(K_4)$ is 16, and it is isomorphic to the direct product of cyclic groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_2^4$, and therefore is abelian. 45

3.7 The neighbouring edges, read counter-clockwise about the red vertex (left) encode the commutation relation $x_1x_2x_3$, while the relations of the blue vertex (right) encodes an anti-commutation relation $x_1x_2x_3J$. 45

3.8 Each red vertex of degree 2 translates in the group to a relation on a pair of commuting generating involutions $x_1x_2 = 1 \Rightarrow x_1 = x_2$. Hence, one can simplify the graph incidence group presentation by replacing these vertices with a single edge, and identifying $x_1 \equiv x_2$ in the group. 46

3.9 By simplifying the degree 2-relations in the left graph, we observe that the graph incidence group of that graph is isomorphic to the graph incidence group generated by the graph on the right. 46

3.10 The graph incidence group for the minimal connected graph with 2 vertex-disjoint cycles is $\Gamma_0(C_3 \sqcup C_3) \cong \langle x_1, x_5 : x_1^2, x_5^2 \rangle = \mathbb{Z}_2 * \mathbb{Z}_2$. After a simplifying isomorphism, we observe that the graph incidence group is generated by two non-commuting involutions x_1 and x_5. 47

3.11 Three isotopic versions of the same a picture $\mathcal{P}$. Observe, that $\mathcal{P}$ consisting of 3 points and 5 curves, and how the special point (the grey vertex) is isotopy equivalent to the boundary in the right diagram. 49

3.12 In this case, the initial graph with multiple edges is a closed picture $\mathcal{P}$ with $\mathrm{ch}(\mathcal{P}) = (1,1)$, and $b = (1,0)$. So by the van Kampen lemma $\mathrm{ch}(\mathcal{P}) \cdot b = 1$ and we witness the relation $J = 1$. 50

3.13 An embedding of the oddly 2-coloured $K_{3,3}$ in plane with a special vertex is isotopy equivalent to the $\Gamma(K_{3,3}, b)$-picture with boundary word $\mathrm{bd}(\mathcal{P}) = x_1x_5x_1x_5$. 52

3.14 An embedding of the oddly 2-coloured K_5 in the plane with a special vertex is isotopy equivalent to the $\Gamma(K_5, b)$-picture with boundary word $\mathrm{bd}(\mathcal{P}) = x_1x_5x_1x_5$. 53

4.1 Colour swapping of adjacent vertices is our additional allowed minor operation on 2-coloured graphs. This minor operation induces an isomorphism of the underlying graph incidence groups by Proposition 4.1.1. 55

4.2 Removal of the edge generator $\langle x_e \rangle \in \Gamma(G)$ via the morphism $\phi(x_e) = 1$. . . 56

4.3 Removal of an isolated vertex with trivial relation $r := 1$ via the identity morphism denoted by $\iota(1) = 1$. 57

4.4 The contraction minor creates two relations in the group r^2 and rr'; however, it is easy to see that $r^2 = 1$, and only a single new relation is created in the graph incidence group of the resulting graph. 58

4.5 Contraction of an edge $\langle x_d \rangle \in \Gamma(G)$ via the morphism $\varphi(x_d) = x_e x_g x_f$. Note the $\mathbb{Z}_2$ addition of the vertex colouring in the contraction is $1 + 1 \bmod 2 \equiv 0$. 58

4.6 $K_{3,4} \sqcup \mathcal{C}$-bipartite graph, with a cycle added to the 3-partition. Observe that edges can only be added to the 3-partition, otherwise one creates 2 disjoint cycles in the graph. 60

4.7 A series of simplifying isomorphisms for the incidence group of the *pseudo-tree* on 13 vertices, can be viewed as a recursive leaf decomposition of the branches emitting from the inner 3-vertex cycle. White vertices and dotted edges represent contracted leaves from the *pruning* process. The last two simplifications come from enforcing the *local* vertex relations. 61

4.8 The *homogenous* graph incidence group of the wheel graph on 8 vertices has 8 generators after a simple identification of the *spokes* as products of the outer-cycle edge generators. We give an example where we take two non-adjacent edge generators x_7 and x_2, and show they commute. First consider the elements under *conjugation* by their adjacent outside edge generators (we chose the edge in the direction of the other generator), i.e. $x_7 = x_8x_7x_8$ and $x_2 = x_1x_2x_1$. After conjugating by the outside edge generators, the commutation relation $[x_8x_7x_8, x_1x_2x_1] = 1$ makes the result obvious, since the words x_1, x_1x_2 and x_8, x_7x_8 are incident to a set of shared vertices (marked in grey) they commute by definition $x_7x_2 = (x_8x_7x_8)(x_1x_2x_1) = (x_8x_7)(x_8x_1)(x_2x_1) = (x_1x_2)(x_1x_8)(x_7x_8) = (x_1x_2x_1)(x_8x_7x_8) = x_2x_7$. 63

4.9 Two copies of Dih_4 involutions generating $\Gamma(K_{3,3}, b)$, one generator set is labelled in the dashed edges $\langle a, (ab) | a^2 = (ab)^2 = 1\rangle$, the other two generators are dotted edges $\langle c, (cd) | c^2 = (cd)^2 = 1\rangle$. In this figure the anti-commutation relation is denoted by the blue vertex. 64

4.10 The three copies of Dih_4 involutions found in $\Gamma(K_5, b)$, each labelled by dotted, dashed, and dashed-dotted edges. In this figure the anti-commutation relation is denoted by the blue vertex. 65

Chapter 1

Introduction

Mathematically *quantum states* are unit vectors of a Hilbert space. *Physical* operations on these states are described by the action of certain operators acting on the Hilbert space. Unlike classical states, quantum states can be *entangled* with other quantum states in a variety of ways. In this context, performing *local*-operations on these states can result in systems that are strongly correlated in astounding ways.

Non-local games provide a convenient framework to explore these quantum correlations. In a non-local game, each player receives an input, according to some distribution, and they respond to a verifier, each with an element from some output set. The players win if their combined output satisfies a winning predicate[1]. In a non-local game, players are not permitted to communicate once they have received their input from the verifier. However, the players are allowed to share entangled quantum states. In addition to sharing these quantum states, the players can perform local-operations on these states, including measurements. Interestingly, the additional resource of *entanglement* allows certain non-local games to be won at higher instances than without [1, 11, 8, 29, 13]. The object in studying non-local games is to understand the limitations and nature of entanglement as a general resource. We realize this goal by exploring the space of *quantum strategies* for these games.

A *deterministic* strategy for a non-local game is a collection of functions, where each function maps from the input set to the output set for each player. A more general *classical* strategy consists of a probability distribution over a collection of deterministic strategies. Loosely speaking, a quantum strategy consists of a collection of quantum *measurements*, which the players employ in a predetermined fashion to determine their outputs. In this context, we will see that the space of classical strategies corresponds to the set of classical

[1]This predicate is a boolean function of the outputs returned by the players.

correlations, while quantum strategies correspond to the larger set of quantum *correlations*. We will elaborate on the description of a non-local game and the connection to correlations in Section 2.5.

Because the set of quantum correlations includes the set of classical ones. It is of interest to classify the non-local games where entanglement provides an advantage to the players over any classical strategy. An interesting set of such non-local games are those with a *perfect* quantum strategy, but no perfect deterministic strategy. In the literature, these games have been referred to as *magic*[2], or pseudo-telepathy games [3, 8]. A well-known example of such a game is the magic squares[3] game [30, 31, 34]. In the magic squares game, the best classical strategy wins with probability ω[4] strictly less than 1, meanwhile there is a quantum strategy that always wins, meaning the *entangled value* of the magic squares game is 1. To further explore where the separations between the quantum and classical strategies occur, we restrict to a class of non-local games with more mathematical structure.

A rich class of non-local games are the *binary constraint system* (BCS) games. These non-local games, introduced in [13], consists of a collection of binary variables and a system of binary constraints. Additionally, each of the constraints is an arbitrary boolean function of some subset of the variables. From the description of a BCS, it is clear that deciding if a perfect classical-strategy exists is NP-hard, as the constraints of the game could involve solving instances of 3-SAT. Additionally, it has been shown that even approximating the entangled value of a BCS game is NEXP-hard [24].

In addition to general BCS games, the authors of [13] considered an interesting class of BCS games that they called *parity*-BCS games. In a parity-BCS game, the binary value of each constraint depends only on the *parity* of the supporting variables. Following the language of [5], we have chosen to refer to these parity-BCS games as *Lin*BCS games, since the (parity) constraints in the game can be expressed by a *linear* system $Ax = b$ over $\mathbb{Z}_2$. With these definitions, we denote an instance of a LinBCS game by $\mathcal{G}(A, b)$. The rough idea of a LinBCS non-local game is for two players to convince a verifier that they have a solution[5] x to the system of linear equations $Ax = b$. A nice property held by these LinBCS games is unlike for general BCS games, it is "easy" and efficient way to decide if

[2]Personally, I believe we should leave the word "magic" to the realm of the supernatural.

[3]The idea of the game originates from a particular contextuality scenario that was independently discovered by the duo. It was first considered as a non-local game by Aravind [1].

[4]The classical (resp. entangled) value of a game ω (ω^*) is the maximum (supremum) winning probability obtained with a classical (quantum) strategy.

[5]This, of course, might not be an actual solution of $Ax = b$, in particular when b is not in the range. In fact, this is why determining the classical value of a LinBCS game remains NP-hard.

there is a perfect deterministic strategy[6].

In this thesis, we consider a restricted class of the LinBCS games which we call graph-LinBCS games $\mathcal{G}(G,b)$. These non-local games comprise the set of LinBCS games where each variable appears in two or zero equations. In this case, we can view A as the incidence matrix of a simple graph G. The principal motivation for this thesis is to understand the connections between two seemingly unrelated classifications of perfect quantum-strategies for these non-local games.

Firstly, given any LinBCS game $\mathcal{G}(A,b)$ there is an associated finitely presented group $\Gamma(A,b)$, called the *solution group* [12, 13]. Moreover, the existence of a perfect quantum strategy turns out to be a property of this group. Specifically, the existence of a perfect commuting-operator quantum strategy corresponds to the existence of a central non-trivial involution, typically denoted by J. This J element is vital in constructing an entangled state on the Hilbert space required by any concrete quantum strategy. A general examination of LinBCS games and their solution groups can be found in [12].

To distinguish from the general LinBCS case, we call the solution group of a graph-LinBCS game a *graph incidence group*[7] and denote it by $\Gamma(G,b)$. We now give the main result of [12] adapted for graph-LinBCS games and graph incidence groups.

Theorem 1.0.1 (The Cleve-Mittal-Liu-Slofstra (CMLS) theorem)**.** Given a graph-LinBCS game $\mathcal{G}(G,b)$:

(i) $\mathcal{G}(G,b)$ has a perfect quantum *commuting-operator* strategy if and only if $J \neq 1$ in the graph incidence group $\Gamma(G,b)$.

(ii) $\mathcal{G}(G,b)$ has a perfect quantum *tensor-product* strategy if and only if the graph incidence group $\Gamma(G,b)$ has a finite-dimensional unitary representation π, on a Hilbert space $\mathcal{H}$, with $\pi(J) \neq \mathbb{1}_{\mathcal{H}}$.

Theorem 1.0.1 holds more generally for any LinBCS game and its corresponding solution group. Proof of part (i) can be found in [12], while part (ii) is essentially due to [13]. Furthermore, the representations of $\Gamma(G,b)$ comprise of operator solutions to the LinBCS. Finite-dimensional operator solutions can be turned into tensor-product strategies using a construction found in [12]. Operator solutions to a LinBCS are generalizations of vectors

[6] This is equivalent to deciding if there is a solution to the $\mathbb{Z}_2$-linear system of equations.

[7] More generally, an incidence group could be defined for any incidence structure, not just graphs.

solutions; in the sense that, one-dimensional operator-solutions are vector solutions[8]. We discuss these operator solutions to LinBCS's in Section 2.5.4.

Secondly, concurrent to Cleve and Mittal's introduction of LinBCS (or parity-BCS) games, Alex Arkhipov developed in [3], a geometric characterization of the non-local games that we now refer to as graph-LinBCS games. Arkhipov observed the correspondence between linear systems in which each variable appeared in only two equations and linear systems $Ax = b$ where A is the incidence matrix of a (non-proper) 2-coloured[9] graph. The bijection is as follows: take A to be the incidence matrix of a graph $G = (V, E)$, with columns labelled by the edges E, and rows by the vertices V, and take b to be a non-proper 2-colouring of the vertices $b : V \to \mathbb{Z}_2$.

In recent work, Slofstra studied general LinBCS games where he identified $(n \times m)$-linear system games, $Ax = b$ over $\mathbb{Z}_2$, with vertex decorated hypergraphs [39]. Inspired by Arkhipov's construction, they took A to be in the incidence matrix of a *hypergraph* $\mathcal{H}$ with n vertices, m hyperedges, and took b to be a vertex-decorating function $b : V(\mathcal{H}) \to \mathbb{Z}_2$. With this correspondence, Slofstra showed that one can embed any finitely presented group into a solution group for some LinBCS game. This result allowed Slofstra to give an explicit separation between the commuting-operator and tensor-product strategies by employing the fact that there are finitely presented groups, with the desired $J \neq 1$ element, but which have no *faithful* finite-dimensional representations.

With the above constructions, it is clear that we can describe a graph-LinBCS games by the pair (G, b), where G is a graph, and b is a non-proper 2-colouring of V. Under this identification, Arkhipov observed that the magic squares game is a graph-LinBCS game whose 2-coloured graph appears in Figure 1.1 as the 2-coloured complete bipartite graph on 6 vertices also known as $K_{3,3}$.

Moreover, the well-known magic pentagram game is a graph-LinBCS game on the complete graph K_5, see the left diagram of Figure 1.1. Arkhipov noticed that these two "magic" games corresponded to the excluded graph minors for *planarity*. He then proved that any graph-LinBCS game arising from a planar graph cannot have a perfect non-deterministic strategy! This characterization provides a surprising graph-minors classification of perfect quantum strategies for graph-LinBCS games[10]. We explicitly state Arkhipov's main theorem below.

[8] One-dimensional operator-solutions are unitary representations into $\langle -1 \rangle$. To recover the vector space solution, one can take the isomorphism from $\langle -1 \rangle$ into $\mathbb{Z}_2$. This yields the familiar vector $x \in \mathbb{Z}_2^m$ satisfying $Ax = b$.

[9] Though, Arkhipov never called this a colouring instead he referred to this as a signing of the vertices.

[10] In his work, Arkhipov used the term arrangement to refer to the graph of these LinBCS game.

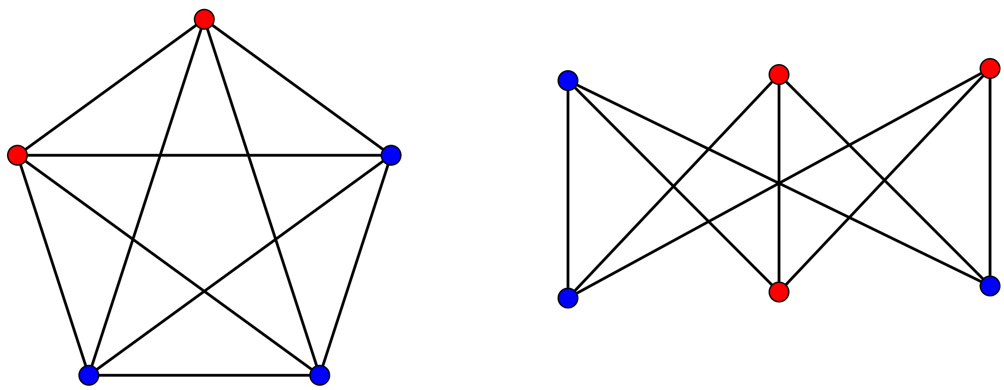

Figure 1.1: On the right, transposing the hypergraph of the magic squares game arrangement results in a game $\mathcal{G}(K_{3,3}, b)$, where one gives assignments to the edges. Treating the binary constraints as vertex labels, one can exhibit the game structure as the 2-coloured complete bipartite graph $K_{3,3}$. On the left, the resulting graph from the magic pentagram game $\mathcal{G}(K_5, b)$. We chose the convention that blue vertices represent a 1 constraint and red vertices a 0 constraint, thus both games have odd-colourings.

Theorem 1.0.2 (Arkhipov's theorem)**.** Let G be a connected graph. Then the corresponding graph-LinBCS game $\mathcal{G}(G, b)$ has a perfect quantum strategy and no perfect classical strategy if and only if the graph G contains $K_{3,3}$ or K_5 as graph minors and the non-proper vertex 2-colouring b has odd parity.

Recalling that a simple graph is non-planar if $|E| > 3|V| - 6$, Arkhipov's theorem seems to suggest that if G is *sufficiently connected* then there exists a perfect quantum strategy. We will see in Section 3.1.1 that if G is connected and b is even[11], then there is always a deterministic[12] strategy to $\mathcal{G}(G, b)$. By this simple fact, we observe that any separation between perfect quantum and classical strategies can only be obtained if the colouring b is odd. The appearance of planarity[13] in Arkhipov's theorem, and the existence of the element J in the graph incidence group can be partially explained using the theory of *pictures*. A picture is a planar diagram which represents relations of a group. Curves or edges in the disk, correspond to generators in the group, while vertices correspond to relations.

The theory of pictures was developed in [39] for the solution groups of LinBCS games

[11]A vector in $\mathbb{Z}_2^n$ is odd (resp. even), if it has odd (even) Hamming weight.

[12]We will see that it is natural to consider deterministic strategies as one-dimensional quantum strategies

[13]In the appendix, we discuss other planarity characterizations that might be worth exploring.

with corresponding hypergraphs. These *pictures* are dual to the van Kampen diagrams developed in [26]. Applied to graph incidence groups, pictures are a useful tool for exploring the connection between attributes of the graph G, and properties of the graph incidence group $\Gamma(G, b)$. Unlike in the more general setting of hypergraphs, for graphs pictures are a type of *weak* graph cover. We examine some connections between the theory of covers for graphs and pictures for graph incidence groups in Section 3.3.1.

A corollary of Arkhipov's theorem is an efficient algorithm for deciding if a graph-LinBCS game has a perfect quantum strategy, given by the linear-time planarity testing of [23]. This result for graph-LinBCS games is contrasted by the same problem in general LinBCS games, whereby, deciding if there is a perfect quantum strategy can be *undecidable* [39].

Recall that planarity is a graph property that is *closed* under the graph minor operations. Graph minor operations play a fundamental role in the structure theory of graphs. For instance, Wagner's theorem states that the forbidden minors for planarity are $K_{3,3}$ and K_5 [44]. This observation motivates a natural question: is there a broader connection between graph minors and solution groups from graph-LinBCS games? If so, are their other properties of graph incidence groups that can be characterized by forbidden minors? This thesis provides an affirmative answer to the above question.

In Chapter 4 we establish a rigorous connection between graph minor operations on 2-coloured graphs and graph incidence group properties. This connection is made precise by our main lemma.

Lemma 1.0.3. If (H, b') is a minor of (G, b), then there is a surjective group homomorphism $\phi : \Gamma(G, b) \to \Gamma(H, b')$.

We prove Lemma 1.0.3 in Section 4.2, by demonstrating that for 2-coloured graphs, there is a natural extension of the graph minor operations, each of which induces a surjective group homomorphism between graph incidence groups. In addition, our main lemma has the following immediate corollary by the Robertson-Seymour theorem [35].

Corollary 1.0.4. Every quotient-closed property of a graph incidence group $\Gamma(G, b)$ is characterized by a finite set of forbidden minors.

Once the connection between quotient-closed properties of graph incidence groups and graph minors was established. It is natural to wonder what the forbidden minors for quotient-closed properties of graph incidence groups are. In Chapter 4, we derive the set of forbidden graph minors for two basic quotient-closed graph incidence group properties: *finiteness* and *abelianness*.

Theorem 1.0.5. The graph incidence group $\Gamma(G, b)$, is a finite group if and only if it does not contain $K_{3,6}$ or two independent vertex-disjoint cycles $\mathcal{C}^{(1)} \sqcup \mathcal{C}^{(2)}$ as graph minors.

Theorem 1.0.6. If b is even, then $\Gamma(G, b)$ is an abelian group if and only if (G, b) does not contain $K_{3,4}$ or two independent vertex-disjoint cycles $\mathcal{C}^{(1)} \sqcup \mathcal{C}^{(2)}$ as minors. If b is odd, then $\Gamma(G, b)$ is an abelian group if and only if (G, b) does not contain any of $K_{3,3}$, K_5 or two independent vertex-disjoint cycles $\mathcal{C}^{(1)} \sqcup \mathcal{C}^{(2)}$ as graph minors.

The proofs of Theorem 1.0.6 and Theorem 1.0.5 rely on Lovasz's classification [28] of graphs that do not contain 2 vertex-disjoint cycles. Another consequence of the Robertson-Seymour theorem is an efficient (runtime $O(|V|^3)$ for simple graphs) algorithm for the problem of detecting a given minor. This implies that once the forbidden minors are known, any quotient-closed property of a graph incidence group can be detected efficiently.

In Chapter 5 we apply our main corollary (Corollary 1.0.4) to investigate the quantum correlations arising from perfect strategies for graph-LinBCS games. In particular, using our graph minor characterizations from Theorem 1.0.6 and Theorem 1.0.5, we make some observations about the *perfect*[14] correlations for abelian and finite graph incidence groups. To achieve this, we make use of the connection between perfect correlations and *characters* of the graph incidence group. Lastly, we investigate the characters of the dihedral groups, as these groups appear to emerge naturally in graph incidence groups.

Many of the results in this thesis were supported by computations using computer algebra systems. In the Appendix, we include a SAGE implementation of a function that takes a 2-coloured graph (or hypergraph) and returns the incidence group. This function allows for testing or as a sandbox for future research. We also include some examples of useful tools in SAGE for examining the perfect classical and quantum correlations for graph-LinBCS games.

[14]Perfect correlations are those that arise from perfect quantum/classical strategies.

Chapter 2

Preliminaries

The subject matter of this thesis requires some background in the theory of graphs, finitely presented groups, representation theory, quantum information and non-local games. In the interest of a self-contained thesis, we provide a brief survey of required definitions and results that are used or referred to later in this work.

2.1 Graph theory

In this section, we present some basic graph theory terminology. In addition, we review the notion of graph minors which will be referred to later in the work. These statements can be found in any standard text on graph theory such as [7].

2.1.1 Simple graphs and non-proper colourings

A *graph* G is an ordered pair of sets (V, E), where $V = \{v_1, v_2, \ldots, v_n\}$ is a set of *vertices* and E is collection of *edges* (distinct two-element subsets of V), $e_{uv} = \{u, v\} \in E$, for $u, v \in V$. Two vertices u, v are *adjacent* if there is an edge $e_{uv} \in E$. Adjacency is an *equivalence relation* on G. For example, if $v \sim u$ for $u, v \in V$, then there exist an edge $e_{uv} \in E$. Similarly, we say an edge e is *incident* to a vertex $v \in V$, if there is element of E containing v. We may write $e \sim v$ to indicate that e is incident with v. More generally, graphs can contain edges with the same endpoint called *loops*, and can contain multiple edges between the same two vertices. A graph is *simple* if it does not contain any loops or multi-edges.

The *degree* of a vertex is the number of edges incident to that vertex. A graph is k-*regular*, if every vertex has degree k, and is *complete* if each vertex has degree $n-1$. The complete graph on n vertices is denoted by K_n. A graph is said to be k-*partite*, if there exists a partition of the vertex set into k disjoint subsets, such that no vertices in the same partition share an edge. A *walk* is a sequence, of possibly repeating, adjacent edges $e_1e_2\ldots e_k$. If in addition to being a walk, each edge is distinct, then the sequence is a *path* of length $n-1$, and if $v_1 = v_n$, then it is a *cycle* $\mathcal{C}$. A graph with no cycles is said to be *acyclic*. A graph is *connected*, if there is a path between every pair of vertices.

A *non-proper* vertex colouring of G is a function from the set of vertices to a finite set of colours $f : V \to \mathbb{Z}_n$. Similarly, a non-proper edge colouring of G is a function $g : E \to \mathbb{Z}_n$. Graph theorist often study *proper* graph colourings; whereby, any pair of adjacent vertices do not share a colour. Given two graphs G and H, the map $\phi : V(G) \to V(H)$ is a *graph homomorphism*, if for every pair $v \sim u$ in $V(G)$, then $\phi(v) \sim \phi(u)$ in $V(H)$. The notion of *onto* and *one-to-one* maps are described as functions between the finite labeled sets $V(G)$ and $V(H)$. An invertible graph homomorphism defines a graph *isomorphism*. Lastly, a graph homomorphism is a *local*-isomorphism, if the cardinality of each *fibre* $\phi^{-1}(v)$ is one for all $v \in V(H)$.

2.1.2 Hypergraphs and higher incidence structures

A natural generalization of a graph is a *hypergraph* $\mathcal{H}$, which consists of vertices V along with a set of *hyperedges* $\mathcal{E}$. Hyperedges are k-element subsets of V, for any $1 \leq k \leq n$. For any hypergraph $\mathcal{H}$ with $|V| = n$, and $|\mathcal{E}| = m$, we define its *incidence matrix* to be the $n \times m$ matrix $A_{v,e}$ with entries

$$(a_{v,e})_{(v,e)=(1,1)}^{(n,m)} = \begin{cases} 1\,, & \text{if } v \text{ is incident to } e \\ 0\,, & \text{otherwise} \end{cases} \quad . \tag{2.1.1}$$

If a $A_{v,e}$ is the incidence matrix of a connected graph, then every column has exactly 2 non-zero entries. More generally, a connected hypergraph is said to be k-*uniform* if every column of its corresponding incidence matrix sums to k. Observe that permuting rows and columns of $A_{v,e}$ is analogous to relabeling the vertices or edges in G $(\mathcal{H})$. From this, the incidence matrix of any disconnected graph $G^{(D)}$, with connected components C, can be written as a direct sum of connected incidence matrices, for each connected component,

$$A_{v,e}^{(D)} = \bigoplus_{c \in C} {A_{v,e}}^{(c)}\,. \tag{2.1.2}$$

Therefore, we can work exclusively with connected graphs, and the above construction determines how properties will carry back to the disconnected case.

2.1.3 Graph minors

Graph *minors* are operations by which one can deconstruct a graph. The operations consist of edge deletion, edge contraction, and vertex deletion. Edge deletion is the simplest operation; if $e \in E(G)$, then the deletion $G \setminus e$ of e is the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$. If $v \in V(G)$, then the deletion $G \setminus v$ of v, results in the graph with vertex set $V(G) \setminus \{v\}$ and edge set $E(G) \setminus \{e \in E(v)\}$, where $E(v) = \{e \in E : e \sim v\}$. Finally, if $e = v_1v_2$ is an edge of G, then the contraction G/e of e results in the graph where the edge e is removed, the vertices v_1 and v_2 are merged into a new vertex v, and there is an edge wv in G/e if and only if there is an edge $wv_i \in E(G)$ for $i = 1, 2$. Edge deletion is depicted in Figure 2.1, edge contraction[1] shown in Figure 2.2, and deletion of an isolated vertex is shown in Figure 2.3. We say a graph $H \preceq G$ is a minor of G (or G

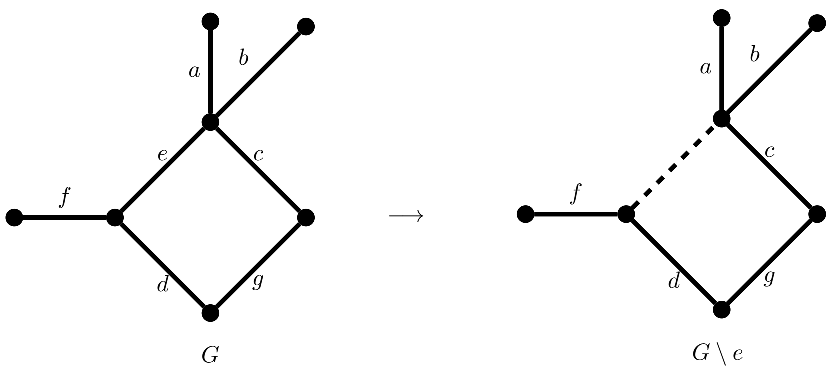

Figure 2.1: The graph minor operation of deleting the edge e from G. The dashed line indicates the removed edge from the graph, following the operation.

contains H as a minor), if there is an appropriate sequence of graph minor operations m_i, taking G to H

$$G \xrightarrow{m_1} G_1 \xrightarrow{m_2} \cdots \xrightarrow{m_{k-1}} G_{k-1} \xrightarrow{m_k} H\,. \tag{2.1.3}$$

Graph *planarity* is the property of a graph having a non-crossing embedding in the 2D-plane. It was famously shown by Wagner that a graph is planar if and only if it does

[1] In the case of simple graphs to avoid creating loops, we consider the more precise *neighbourhood-disjoint* edge contraction, i.e. one must first remove any extraneous shared edges before contracting a desired edge

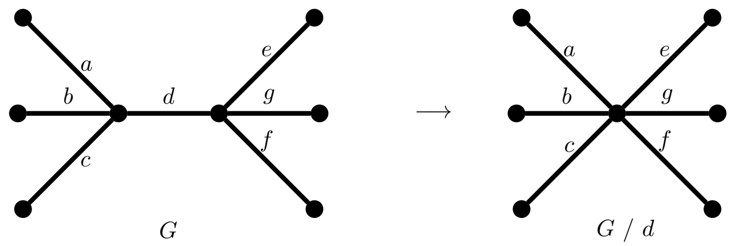

Figure 2.2: The graph minor operation contracting the edge d in $E(G)$. Observe that this operation results in the merging of the two end-vertices of $d \in E(G)$. Hence, this minor operation reduces the number of vertices in $V(G)$ by one.

not contain the complete bipartite graph on six vertices $K_{3,3}$ or the complete graph K_5 as minors [44][2]. Wagner famously conjectured that for every graph minor closed property P, there is a finite list L of graphs, such that a graph satisfies P if and only if it does not contain any minors in L. Graphs in L are called the forbidden minors of property P.

Wagner's conjecture was proved by the celebrated Robertson-Seymour theorem [37], which states more generally that the set of finite graphs is *well-quasi-ordered*[3] with respect to the graph minor operations. The Robertson-Seymour theorem also gives a basis for which the detection of a forbidden minor H can be done in runtime $O(|V|^3)$.

Another graph property we consider in this thesis is not containing two vertex-disjoint cycles $\mathcal{C}^{(1)} \sqcup \mathcal{C}^{(2)}$. The set of graphs (and multi-graphs) without 2 vertex-disjoint cycles were characterized by Lovasz [28]. These graphs include: trees $\mathcal{T}_n$, wheels W_n, the complete graph K_5. As well as bipartite graphs for which $G \backslash \{u, v, w\}$ is edgeless for some set of edges u, v, and w. These include $K_{(2,3)}$, $K_{3,n}$ amongst many other variants of these. Figure 2.4 illustrates an example of a $(3, n)$-bipartite graph variant where a cycle $\mathcal{C}$ has been adjoined to the 3-partition. Observe, that the addition of this cycle has not increased the number of vertex-disjoint cycles in the graph. Therefore, we must consider these graph variants in our analysis of graphs without 2 vertex-disjoint cycles. Moreover, remark that the 3-partition is very important, as it is not hard to find two disjoint cycles in $K_{4,n}$ for any $n \geq 4$. On this note, it was shown in [6] that the runtime of an algorithm for finding two vertex-disjoint cycles is linear in the number of vertices.

[2] Though it had already been shown by Kuratowski that a graph is planar if and only if the graph doesn't contain an edge-subdivision of $K_{3,3}$ or K_5 [27].

[3] In a well-quasi-ordered class, any infinite sequence of elements contains an increasing pair $x_i \leq x_j$, with $i < j$.

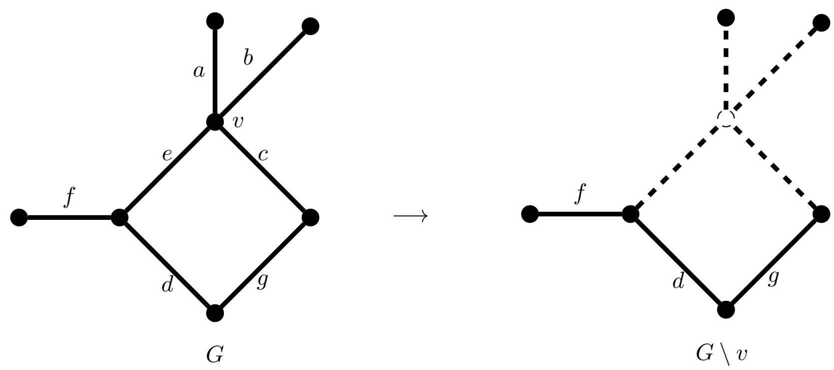

Figure 2.3: The graph minor operation $G \to G\backslash v$ deleting vertex v from G. The removed vertex and its subsequent edges are indicated by dashed lines in the graph following the vertex deletion. Observe that deleting vertex v, also removes all the edges incident to v. Essentially the vertex-deletion minor operation induces the edge-deletion minor for every edge incident to v.

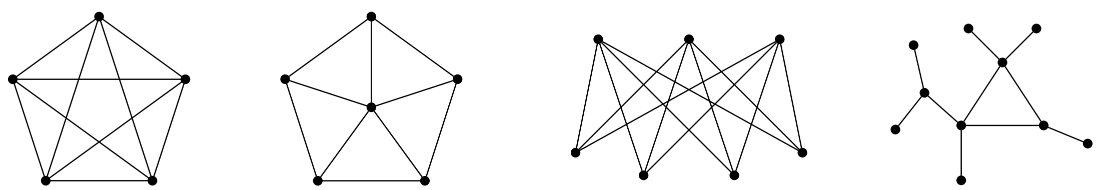

Figure 2.4: Four types of graphs that do not contain two disjoint cycles. From left-to-right, K_5, the wheel graph W_5, the bipartite graph $K_{3,4}$, and a cycle with tree branches $\mathcal{R}_n$.

2.2 Group theory

We present some basic background on relevant group theoretic terms, references include the basic texts [20, 4].

2.2.1 Basic properties of groups

A *group* Γ, consists of a non-empty set S with an associative[4] binary operation $(\cdot)$ such that the following hold:

[4] For all a, b, c an operation is associative if $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(i) There is a unique identity element $1 \in \Gamma$ with respect to the binary operation

(ii) Every element $x \in \Gamma$ has a unique inverse $x^{-1} \in \Gamma$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$

(iii) $\Gamma(S)$ is *closed*[5] with respect to the binary operation.

A subset $\Sigma \subseteq \Gamma$ is called a *subgroup*, if $1 \in \Sigma$, and Σ is closed under the group operation; furthermore, each element of Σ has an inverse in Σ. We say that Σ is a *proper* subgroup, if $\Sigma \neq \{1\}$ and $\Sigma \neq \Gamma$. We say a subgroup $\Lambda \subset \Gamma$ is *normal*, if $x\Gamma x^{-1} \in \Lambda$ for all $x \in \Gamma$, or if every left *coset* $x\Gamma \in \Lambda$ is also a right coset $\Gamma x \in \Lambda$. If Λ is a normal subgroup of Γ, denoted $\Lambda \trianglelefteq \Gamma$, then Γ/Λ is called the *quotient* of Γ by Λ. The *normal closure* of a subgroup R is the smallest normal subgroup $R^\Gamma = \{g^{-1}Rg : \forall g \in \Gamma\}$ containing R.

The *commutator* of $x, y \in \Gamma$ is defined as $[x, y] = xyx^{-1}y^{-1}$. We SAY x and y commute if $[x, y] = 1$. The *centre* of a group Γ consists of the elements that commute with everything in the group. We say a group is *abelian* if $\Gamma \subseteq \mathcal{Z}(\Gamma)$. We say that a group is finite, or has finite *order*, if it has a finite number of elements. A group Γ is *simple*, if it has no (non-trivial) normal subgroups.

Let Γ and Σ be groups. A group *homomorphism* is a map $\vartheta : \Gamma \to \Sigma$ such that $\vartheta(xy) = \vartheta(x)\vartheta(y)$ for all $x, y \in \Gamma$. The *kernel* of a homomorphism $\vartheta : \Gamma \to \Gamma'$ is the set of elements in Γ that are mapped to the identity $1'$ in Γ'. A homomorphism is surjective (or onto) if the kernel is trivial, and injective (or one-to-one) if it is invertible on its image. We now recall the fundamental theorem of group *isomorphisms* as it highlights the interplay between normal subgroups and homomorphisms.

Theorem 2.2.1 (Isomorphism theorem for groups)**.** If Γ and Σ are groups and $\vartheta : \Gamma \to \Sigma$ is a surjective group homomorphism, then there is an isomorphism

$$\varphi : \Gamma/\Lambda \to \Sigma\,, \tag{2.2.1}$$

with $\vartheta = \varphi \circ \varpi$, where $\Lambda = \mathrm{Ker}(\vartheta)$, and $\varpi : \Gamma \to \Gamma/\Lambda$.

Proof. Let $\varphi(\Lambda g) = \varphi \circ \varpi(g) = \vartheta(g)$. To see that φ is well defined; note that, if $\Lambda g = \Lambda h$, then there exist $\lambda \in \Lambda$ such that $g = \lambda h$. Hence, $\vartheta(g) = \vartheta(\lambda h) = \vartheta(\lambda)\theta(h) = \vartheta(h)$, since λ is in the kernel of ϑ. To see that φ is a surjection, consider an element $\varphi(g) \in \Sigma$, since ϑ is surjective there exist a $g \in \Gamma$ such that $\vartheta(g) = \varphi(\Lambda g)$. To see that φ is injective, suppose that $\varphi(\Lambda g) = \varphi(\Lambda h)$, then $\vartheta(g) = \vartheta(h)$, and $\Sigma \ni 1 = \vartheta(g)\vartheta(h)^{-1}$. It follows that, since ϑ is a homomorphism, gh^{-1} is in Λ a normal subgroup. Thus $\Lambda g = \Lambda h$, and φ is injective. Lastly, observe the quotient map $\varpi : \Gamma \to \Gamma/\Lambda$. It follows that φ is a homomorphism, since $\varphi(\Lambda g \Lambda h) = \varphi(\Lambda gh) = \varphi \circ \varpi(gh) = \vartheta(gh) = \vartheta(g)\vartheta(h) = \varphi(\Lambda g)\varphi(\Lambda h)$. $\square$

[5] A set S is closed w.r.t an operation $(\cdot)$, if $c \in S$ whenever $a \cdot b = c$, for $a, b \in S$.

Let Φ and Λ be normal subgroups of Γ such that $\Phi \cap \Lambda = \{1\}$. We write $\Gamma = \Lambda \times \Phi$ and say that Γ is the *direct product* of Φ and Λ. Let Γ and Σ be groups with corresponding central subgroups $\Gamma_Z \subseteq Z(\Gamma)$ and $\Sigma_Z \subseteq Z(\Sigma)$, such that there is an isomorphism of subgroups $\alpha : \Gamma_Z \leftrightarrow \Sigma_Z,$. Then the *central product* is the quotient of the direct product $\Gamma \times \Sigma$ by the normal subgroup $\Lambda = \{(\gamma, \sigma), \gamma \in_Z, \sigma \in \Sigma_Z : \alpha(\gamma)\sigma = 1\}$. We will denote the central product by $\times_\Lambda \Sigma$.

Let Γ be a group and Λ a normal subgroup. There is a natural *projection* map $\pi : \Gamma \to \Gamma/\Lambda$ given by $\pi(x) = x\Lambda$ for $x \in \Gamma$. Furthermore, if $\eta : \Phi \to \Gamma$ is an injective map, and $\pi : \Gamma \to \Lambda$ is a projection, then we say the following sequence is *exact*, and write

$$1 \to \Phi \xrightarrow{\eta} \Gamma \xrightarrow{\pi} \Lambda \to 1\,. \tag{2.2.2}$$

If in addition to the exact sequence, there is a homomorphism $\phi : \Lambda \to \Gamma$ such that $\pi \circ \phi = 1$, then Γ is the semi-direct product of Λ acting on Φ[6] with respect to the map $\varphi : \Lambda \to \mathrm{Aut}(\Phi)$, written $\Gamma = \Lambda \rtimes_\varphi \Phi$. The dihedral group Dih_n is an example of the semi-direct product. In other words, Dih_n is the semi-direct product of $\mathbb{Z}_n$ acting on $\mathbb{Z}_2$. In particular, we have that $\mathrm{Dih}_n \cong \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$, where $\varphi(a) : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_n)$, and $\varphi(a)(z) = z^{(-1)^a}$ acts by *inversion*.

A sequence of subgroups $\Gamma_0 > \Gamma_1 > \ldots > \Gamma_r > 1$ is a composition series of length r, if $\Gamma_{i+1} \trianglelefteq \Gamma_i$ for all i, and every successive quotient Γ_i/Γ_{i+1} is simple. Recursively one can define the *derived* subgroups $\Gamma^{(n)} = [\Gamma^{(n-1)}, \Gamma^{(n-1)}]$, as the group generated by the commutator with $\Gamma^{(0)} = \Gamma$. We call the group $\Gamma^{(1)} = [\Gamma, \Gamma]$ the *commutator* subgroup. From this definition, we obtain the normal *derived series* $\Gamma \triangleleft \Gamma^{(1)} \triangleleft \cdots \triangleleft \Gamma^{(n)}$. A group is *solvable*, if the series terminates with $\Gamma^{(n)} = \langle 1 \rangle$. The commutator subgroup is a normal subgroup. Furthermore, the homomorphism $\Gamma \to \mathscr{Z}(\Gamma)$ induced by the quotient of Γ by the commutator subgroup $\Gamma/\Gamma^{(1)} \cong \mathscr{Z}(\Gamma)$ is called the *abelianization* of Γ, and its centre $\mathscr{Z}(\Gamma)$ is sometimes referred to as the subgroup of abelian invariants.

2.2.2 Finitely presented groups

Every group Γ can be given a *presentation*. A presentation of Γ consists of a set of generators S and relations R, we denote a presentation of Γ by $\langle S : R \rangle$. A group is said to be finitely presented if there exist presentations with S and R finite. For example, the infinite dihedral group has a simple finite presentation,

$$\mathrm{Dih}_\infty = \left\langle x, y : x^2, y^2 \right\rangle. \tag{2.2.3}$$

[6] One might also say that Γ splits over Φ.

The free group $\mathcal{F}(S)$ is the set of all *reduced* words of S. Every group Γ has a presentation due to the following isomorphism of groups,

$$\alpha : \mathcal{F}(S)/R \to \Gamma\,, \tag{2.2.4}$$

where R is the normal closure of the relations defining all words in Γ. An important remark here is that there is no algorithm with bounded runtime, which given a presentation of a group can decide if the order of a group is finite. This is because deciding if a particular element has finite order is related to the following decision problem. Given a word w, as a product of generators in $S(\Gamma)$ determine, if $w = 1 \in \Gamma$. This is called the word problem for finitely presented groups, and it is known to be undecidable [16].

Even with the undecidability result, in practice, we can typically reduce the order of arbitrary words and thus often compute the order of a group. One approach to this is using an algorithm that generates a *rewriting system* for the presented group. A rewriting system (RWS) consists of a set of *rules* for transforming words into other words in the group. We say the rewriting system is *confluent* if its rules reduce words to the identity, and *finite* if it has a finite number of rewriting rules. A finite and confluent rewriting system is called *complete*. Given a complete RWS for a group, one has effectively solved the word problem for that group and therefore one can determine the order of the group. Algorithms for computing RWS's come standard in most computer algebra systems including SAGE or GAP.

2.2.3 Products for presented groups

Given two presented groups $\Gamma_1 = \langle S_1 : R_1 \rangle$ and $\Gamma_2 = \langle S_2 : R_2 \rangle$, the *direct product* can be expressed as

$$\Gamma_1 \times \Gamma_2 = \big\langle S_1 \cup S_2 : R_1 \cup R_2 \cup R_{1,2} \big\rangle \tag{2.2.5}$$

where $R_{1,2}$ consists of the relations enforcing that every element of S_1 commutes with every element in S_2. Similarly, we can define the *free product* of two groups,

$$\Gamma_1 * \Gamma_2 = \big\langle S_1 \cup S_2 : R_1 \cup R_2 \big\rangle\,. \tag{2.2.6}$$

Observe in the free product, that there is no enforced commutation relations between S_1 and S_2.[7]

[7] In the category of groups the direct product is merely the universal *product*, while the free product is the *co-product*.

2.3 Representation theory

We give the standard definitions and facts concerning the representations of finite groups. These facts can be found in the works [38, 9]. In everything that follows we are working over the field of complex numbers $\mathbb{C}$.

2.3.1 Unitary representations of groups

A *representation* of a finite group Γ is a pair $(\varphi, \mathcal{V})$ such that

$$\varphi : \Gamma \mapsto \mathrm{GL}(\mathcal{V})\,, \tag{2.3.1}$$

is a group homomorphism into $\mathrm{GL}(\mathcal{V})$[8]. We are primarily interested in *unitary* representations of a group[9] $\varphi : \Gamma \mapsto \mathcal{U}(\mathcal{V})$. When there is little room for confusion, we may refer to a representation as the space $\mathcal{V}$.

The dimension of a representation is the *rank* of the identity $\varphi(1) \in \mathrm{GL}(\mathcal{V})$. A *sub-representation* $(\vartheta, \mathcal{W})$ is a Γ-invariant subspace of $\mathcal{V}$, such that $\varphi\big|_{\mathcal{W}} = \vartheta$. The subspace of Γ-invariants is described as

$$\mathcal{V}^{\Gamma} = \{\nu \in \mathcal{V} : \varphi(g)\nu = \nu\,,\ \forall\, g \in \Gamma\}\,. \tag{2.3.2}$$

A representation $(\varphi, \mathcal{V})$ is said to be *irreducible*, if $\mathcal{V}$ contains no proper Γ-invariant subspaces. Using the above definitions one can show the following important facts regarding representations. First, if $\mathcal{W}$ is a sub-representation of $\mathcal{V}$, then there exist a *complementary* invariant subspace $\mathcal{W}^{\perp}$ such that $\mathcal{V} \cong \mathcal{W} \oplus \mathcal{W}^{\perp}$; which in turn, implies that every representation is the direct sum of irreducible representations. Remark that if ρ is a unitary representation, then every invariant subspace is also a *reducing* subspace. This fact makes the decomposition of unitary representation very nice.

We now state a fundamental result, which describes the relationship between irreducible representations. Let the space of the Γ-invariant maps between irreducible representations $\mathcal{V}$ and $\mathcal{W}$ be denoted by $\mathrm{Hom}_{\Gamma}(\mathcal{V}, \mathcal{W}) \cong (\mathcal{V}^* \otimes \mathcal{W})^{\Gamma}$. These are also sometimes called Γ-module homomorphism.

Lemma 2.3.1 (Schur's lemma)**.** If $(\rho, \mathcal{V})$ and $(\varphi, \mathcal{W})$ are irreducible representations of Γ and the map $\alpha : \mathcal{V} \mapsto \mathcal{W}$ is in $\mathrm{Hom}_{\Gamma}(\mathcal{V}, \mathcal{W})$, then

[8] The group of invertible linear transformations acting on the vector space $\mathcal{V}$.

[9] The unitary group $\mathcal{U}$ are the elements of $\mathrm{GL}(\mathcal{V})$ with the $*$-involution property $U^* = U^{-1}$.

(i) Either α is an isomorphism, or $\alpha = 0$

(ii) If $\mathcal{V} = \mathcal{W}$, then $\alpha = c \cdot \mathbb{1}$, for some $c \in \mathbb{C}$, and $\mathbb{1}$ is the identity on $\mathcal{V}$.

Proof. Let $\mathcal{K}$ be the kernel of α. Given an $x \in \mathcal{K}$, observe that $(\alpha\rho)x = (\varphi\alpha)x = 0$, hence $\rho x \in \mathcal{K}$ and $\mathcal{K}$ is an invariant subspace of $\mathcal{V}$, but since $\mathcal{V}$ is irreducible either $\mathcal{K} = \mathcal{V}$ or $\mathcal{K} = 0$. Similarly, let $\mathcal{R}$ be the image of α, such that, if $x \in \mathcal{R}$ then $\varphi x \in \mathcal{R}$, and $\mathcal{R}$ is an invariant subspace of $\mathcal{W}$. Now, since $\mathcal{W}$ is also irreducible, observe that $\mathcal{R} = W$ or $\mathcal{R} = 0$. It follows, that either α is an isomorphism or $\alpha = 0$. Now assume $\mathcal{V} = \mathcal{W}$, and let λ be an eigenvalue of α. It follows from (i) that $\alpha - \lambda I \in \mathrm{Hom}(\mathcal{V}, \mathcal{V})$, so $\alpha - \lambda I = 0 \Rightarrow \alpha = \lambda I$, since $\alpha - \lambda I$ being an isomorphism contradicts the fact that we let λ be an eigenvalue of α. $\square$

The main consequence of Schur's lemma is that we can decompose any representation into a direct sum of irreducible components of the form

$$\mathcal{V} = \bigoplus_{j=1}^{k} \mathcal{V}_j^{\oplus m_j}\,, \tag{2.3.3}$$

where each m_j is the *multiplicity* of the *distinct* irreducible representation $\mathcal{V}_j$. Recall that the *dual space* of a vector space $\mathcal{V}$, is the vector space of homomorphisms on that space denoted by $\mathcal{V}^* := \mathrm{Hom}(\mathcal{V}, \mathcal{V})$. Suppose that $(\varphi, \mathcal{V})$ is a representation of Γ, then the *dual* representation of Γ is given by $(\varphi^*, \mathcal{V}^*)$ and

$$(\varphi^*(g)f)(v) = f(\varphi(g^{-1})v)\,, \tag{2.3.4}$$

for all $g \in \Gamma$, $f \in \mathcal{V}^*$ and $v \in \mathcal{V}$. Remark that if $f : \mathcal{V} \mapsto \mathbb{C}$ and $\mathcal{V}$ is a complex inner product space, then the Riesz representation theorem states that, for any $f \in \mathcal{V}^*$ there exist a $u \in \mathcal{V}$ such that $f_u(v) = \langle u, v\rangle_{\mathcal{V}}$.

A finite set S with a left Γ-action, where $\Gamma \mapsto \mathrm{Aut}(S)$ is a homomorphism, defines the *permutation representation* of Γ on S. In this case one, we view $\mathcal{V}$ as a vector space with basis $\{v_s : s \in S\}$, where Γ acts by

$$g\left(\sum_s a_s v_s\right) = \sum_s a_s v_{g(s)}\,. \tag{2.3.5}$$

If we let $\Gamma = S$ in the above definition, we obtain what is known as the *left-regular representation* of Γ on $\mathcal{V}$. Where the dimension of the representation is $|\Gamma|$, and each element of the group is an element of the basis for $\mathcal{V} \cong \mathbb{C}^{|\Gamma|}$.

2.3.2 A representation of the dihedral group

The group Dih_4 has 4 one-dimensional representations and one dimension-2 irreducible representation. A two dimensional unitary representation of Dih_4 with generators $\langle x, y\rangle$ and $xyx = y^{-1}$ is given by mapping the generators,

$$\rho(x) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X \qquad \text{and} \qquad \rho(y) \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = iZ. \tag{2.3.6}$$

Where X, and Z are the 2×2 Pauli matrices. It is easy to check that $\rho(x)$ has order 2, and $\rho(y)$ has order 4. The only remaining relation to verify is $(\rho(x)\rho(y))^2 = \mathbb{1}$. Observe,

$$(\rho(x)\rho(y))^2 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right)^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.3.7}$$

The product of representations is given by the tensor product of the irreducible representations modulo the representation of the centre $(y^2, y^2) \mapsto -\mathbb{1}_2 \otimes -\mathbb{1}_2 = \mathbb{1}_4$. Where the tensor product of matrix representations is described by the *Kronecker product* of the representative matrices.

A set of matrices $\{X_1, \ldots, X_n\}$ are called *Clifford unitaries*, if they are self-adjoint, involutions, that anti-commute $X_iX_j + X_jX_i = 0$. Recall that matrix elements of a representation commute if $X_iX_j - X_jX_i = 0$ for all $i \neq j$. The Pauli group[10] is such, a collection of 16 matrices.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.3.8}$$

and

$$XZ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -ZX \tag{2.3.9}$$

2.3.3 Characters of a representation

The *character* of a representation $(\varphi, \mathcal{V})$ of Γ is the complex valued functional $\chi_{\mathcal{V}} : \Gamma \to \mathbb{C}$, such that $\chi_{\mathcal{V}}(g) = \mathrm{tr}\big(\varphi(g)\big)$. The properties of the trace give us some immediate properties of the character. Firstly, that $\chi_{\mathcal{V}}$ is a class function, (i.e. constant on conjugacy classes), and secondly the degree of a representation is given by the character of the identity $\dim(\mathcal{V}) = \chi_{\mathcal{V}}(1)$. Characters behave nicely, with respect to the sum and product of representations, we state without proof that $\chi_{\mathcal{V}\oplus\mathcal{W}} = \chi_{\mathcal{V}} + \chi_{\mathcal{W}}$, $\chi_{\mathcal{V}\otimes\mathcal{W}} = \chi_{\mathcal{V}} \cdot \chi_{\mathcal{W}}$, and lastly $\chi_{\mathcal{V}^*} = \overline{\chi_{\mathcal{V}}}$.

[10] Abstractly, the Pauli group can be described as the central product of $\mathbb{Z}_4$ with Dih_4.

The utility of characters in representation theory is seen by observing the following construction. Character theory is of particular use in determined the dimensions and multiplicities of irreducible representations. Consider the problem of finding the dimension of the Γ-invariant subspace of $\mathcal{V}$ of a unitary representation. Consider the orthogonal projection onto $\mathcal{V}^\Gamma$ by defining

$$\Pi = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \varphi(g) \tag{2.3.10}$$

We verify that Π is an orthogonal projection. Firstly, it is easy to see that Π is self-adjoint since summing over the elements of $g \in \Gamma$ is the same as summing over the inverse elements $g^{-1} \in \Gamma$ as φ is a unitary representation. Secondly, we show that $\Pi^2 = \Pi$,

$$\Pi^2 = \frac{1}{|\Gamma|^2} \sum_{(g,g') \in \Gamma} \varphi(g)\varphi(g') \tag{2.3.11}$$
$$= \frac{1}{|\Gamma|^2} \sum_{(g,g') \in \Gamma} \varphi(g \cdot g') \tag{2.3.12}$$
$$= \frac{1}{|\Gamma|^2} \sum_{h \in \Gamma} \varphi(h)|\Gamma| \tag{2.3.13}$$
$$= \frac{1}{|\Gamma|} \sum_{h \in \Gamma} \varphi(h) \tag{2.3.14}$$

Now, observe that we can use Π to compute the dimension of the irreducible sub-representations,

$$\dim(\mathcal{V}^\Gamma) = \mathrm{tr}(\Pi) = \frac{1}{|\Gamma|} \sum_{g \in G} \mathrm{tr}(\varphi(g)) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_{\mathcal{V}}(g)\,. \tag{2.3.15}$$

Corollary 2.3.2 (Consequences of Schur's lemma)**.** Let $\mathcal{V} = \bigoplus_{j=1}^{k} m_j \mathcal{V}_j$ be a decomposition of the representation $\mathcal{V}$ into a direct sum of irreducible representations $\mathcal{V}_j$ with multiplicity m_j, then the following non-trivial statements can be obtained

(a) The number of irreducible representations is equal to the number of *conjugacy classes* in Γ

(b) The irreducible characters with respect to the above Hermitian inner product form an *orthonormal basis* for the set of class functions on Γ

(c) A representation $\mathcal{V}$ is irreducible if and only if $(\chi_{\mathcal{V}}, \chi_{\mathcal{V}}) = 1$

(d) $m_j = (\chi_{\mathcal{V}}, \chi_{\mathcal{V}_j})$

(e) $|\Gamma| = \sum_j \dim(\mathcal{V}_j)^2$

We refer the reader to any standard text representation theory for the proofs of the above results such as [38, 9].

There is a hermitian inner-product on the set of characters[11], from which we have the following application of Schur's lemma,

$$(\chi_{\mathcal{V}}, \chi_{\mathcal{W}}) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \overline{\chi_{\mathcal{V}}(g)} \chi_{\mathcal{W}}(g) \tag{2.3.16}$$

$$= \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_{(\mathcal{V}^* \otimes \mathcal{W})^\Gamma}(g) \tag{2.3.17}$$

$$= \dim(\mathrm{Hom}_\Gamma(\mathcal{V}, \mathcal{W})) = \begin{cases} 1 \text{ if } \mathcal{V} \cong \mathcal{W} \\ 0 \text{ if } \mathcal{V} \ncong \mathcal{W} \end{cases} . \tag{2.3.18}$$

If a group Γ decomposes into a product of smaller groups, of which the representations are known, then one can obtain a representation of the whole group with the following fact. If $\mathcal{V}$ and $\mathcal{W}$ are representations of Γ and Σ, then $\mathcal{V} \otimes \mathcal{W}$ is a representation of $\Gamma \times \Sigma$.

Perhaps the most important use of characters is for decomposing a representation into irreducible representations. In particular, the characters can be used to construct a set of orthogonal projections onto the space of irreducible representations. Consider the projection given by

$$\Pi_i = \frac{d_i}{|\Gamma|} \sum_{g \in \Gamma} \chi_i(g)^* \rho_i \,. \tag{2.3.19}$$

We verify that this is indeed a projection onto the ith irreducible representation $\mathcal{V}_i$ of dimension d_i. Since, the characters are orthogonal, the projection onto $\mathcal{V}_j$ is 0 for all $i \neq j$. On the other hand, when $i = j$, we obtain $d_i/|\Gamma|\langle \chi_i | \chi_i \rangle = d_i^2/|\Gamma|$ and therefore

$$\frac{1}{|\Gamma|} \sum_{i \in \mathcal{C}} d_i^2 = 1 \,. \tag{2.3.20}$$

Example 2.3.3. The cyclic group $\mathbb{Z}_2$ is generated by $\langle a \rangle$. The unitary irreducible representation of $\mathbb{Z}_2$ have characters ± 1. The two-by-two character table is shown in Table 2.1.

[11] More generally this inner product is defined on the set of class functions.

	1	a
χ_0	1	1
χ_1	1	-1

Table 2.1: The character table of $\mathbb{Z}_2$, the columns index the conjugacy classes via some class representative, while the rows are the labelled by its irreducible representations.

Example 2.3.4. The dihedral group of order 8 is denoted by Dih_4, is generated by two elements $\langle r, s\rangle$ along with the following relations $R = \{r^4, s^2, srsr\}$. Dih_4 has 5 conjugacy classes, a set of conjugacy class representatives is given by $\{1, sr, sr^2, r, r^2\}$. The 5 irreducible representations of Dih_4 are given in the following *character-table* matrix. Each column correspond to conjugacy classes while the rows correspond to the characters of the irreducible representations. Since the identity forms its own conjugacy class in every group, the first column gives the dimension of the representations $\chi_\rho(1) = \dim(\mathcal{V})$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 2 & 0 & 0 & 0 & -2 \end{pmatrix} \tag{2.3.21}$$

2.3.4 Representations of groups generated by involutions

We denote the group $\mathrm{Inv}\langle S : R\rangle$ to be the finitely presented group generated by *involutions*. Meaning R contains the relation $s^2 = 1$ for all $s \in S$. If π is a finite unitary representation of a group generated by involutions, the resulting unitary elements of $\mathcal{V}$ are elements of the subgroup of *reflections* (self-adjoint unitaries). The characters of each one-dimensional representation of these groups can only take on the values ± 1. While any higher-dimensional characters of these representations can take on integer values.

2.4 Quantum measurements

The notion of a quantum correlation comes from the statistical theory of quantum states and quantum measurements. The information in the proceeding section is taken from [45, 43], and [11] in the context of non-local games.

2.4.1 Quantum states

A pure *quantum state* is a unit vector $|\psi\rangle$ in a complex Hilbert space. Hilbert spaces are *self-dual* (reflexive), which means that there is a one-to-one correspondence between elements of $\mathcal{H}$ and the linear functionals on $\mathcal{H}$ (these are elements of the dual space $\mathcal{H}^*$). This is captured in Dirac's convention of *bra-ket* notation. A *ket* is an element $|\psi\rangle$ of the Hilbert space, and a *bra* $\langle\psi|$ is the corresponding linear functional on the Hilbert space. The idea behind the Dirac notation, is that whenever a functional $\langle\psi|$ is applied to a vector $|\psi\rangle$ one recovers the inner-product $\langle\psi|\psi\rangle$. Whenever, the Hilbert space $\mathcal{H}$ has finite-dimension d we can identify $\mathcal{H} \cong \mathbb{C}^d$, and $\mathcal{B}(\mathcal{H}) \cong M_d(\mathbb{C})$ ($d \times d$ complex matrices). If $\mathcal{H}$ is a Hilbert space with elements $h, h' \in \mathcal{H}$, then we let $\langle h, h'\rangle_{\mathcal{H}}$ denote the inner product on $\mathcal{H}$. This inner product on $\mathcal{H}$ induces the norm $\langle h, h\rangle = \|h\|^2$ under which every Hilbert space is a *complete metric space*.

Two principal features of quantum mechanics are *locality*, and *entanglement*. These features are captured mathematically by the notion of *tensor-products* of Hilbert spaces. For our purposes, the tensor product of Hilbert spaces can be thought of as the unique *bilinear* mapping for which the following identity holds,

$$\langle h_1 \otimes h_1' | h_2 \otimes h_2' \rangle_{\mathcal{H}\otimes\mathcal{H}'} = \langle h_1 | h_2 \rangle_{\mathcal{H}} \cdot \langle h_1' | h_2' \rangle_{\mathcal{H}'} \,, \tag{2.4.1}$$

for $h_1, h_2 \in \mathcal{H}$ and $h_1', h_2' \in \mathcal{H}'$. A quantum state on the tensor product of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is called a *bipartite* state. A *pure* bipartite state is said to be *separable* if it can be expressed as

$$|\psi\rangle \otimes |\phi\rangle \,, \tag{2.4.2}$$

where $|\psi\rangle \in \mathcal{H}_1$ and $|\phi\rangle \in \mathcal{H}_2$ are *local* states of their corresponding Hilbert spaces. Furthermore, we say that a state is *entangled* if it is not *separable*[12]. The canonical maximally entangled bipartite state on two d-dimensional systems isomorphic to $\mathbb{C}^d \otimes \mathbb{C}^d$ is expressed as

$$|\tau\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle \,, \tag{2.4.3}$$

where $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ is the computational basis of $\mathbb{C}^d$. It should be noted that a bipartite state can be maximally entangled with respect to any pair of orthonormal bases. Now that we have defined a quantum state we can discuss the mathematical framework in which we model measurements of quantum states.

[12]The problem of deciding separability for an arbitrary quantum state is NP-hard [19].

2.4.2 Measurement postulates and POVM's

The postulates of quantum measurement can be summarized by the following observation. Let $\{M_i\}_{i=1}^k$ be a collection of matrix elements, and $p = (p_1, \ldots, p_k) \in \mathbb{R}_+^k$ be a probability vector so that $\sum_i^k p_i = 1$. If $|\psi\rangle \in \mathbb{C}^d$ is a quantum state then,

$$1 = \sum_{i=1}^k p_i = \sum_{i=1}^k \|M_i|\psi\rangle\|^2 = \sum_{i=1}^k \langle\psi|M_i^* M_i|\psi\rangle \tag{2.4.4}$$

Note that the above holds if and only if $\sum_{i=1}^k M_i^* M_i = \mathbb{1}$ for every state $|\psi\rangle$. Let $\mathcal{O}_i = M_i^* M_i$, then we refer to the collection of positive operators $\{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$, as a *positive-operator-value measure* or POVM.

2.4.3 Projective measurements

Observe that if the collection of operators are orthogonal projections $\{P_1, \ldots, P_k\}$, such that $P_i = P_i^* = P_i^2$ then,

$$1 = \sum_{i=1}^k p_i = \sum_{i=1}^k \|P_i|\psi\rangle\|^2 = \sum_{i=1}^k \langle\psi|P_i|\psi\rangle\,, \tag{2.4.5}$$

and we have the requirement that $\sum_{i=1}^k P_i = \mathbb{1}$. A set of projections $\{P_i\}_{i=1}^k$ is called a projection valued measure (or PVM). One can show that every outcome achieved by an arbitrary POVM's can be achieved by a suitable choice of PVM[13].

2.4.4 Joint measurements on entangled systems

Consider a quantum state on Alice and Bob's joint state space $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. We say a set of operators $\{X_1, \ldots, X_n\}$ form a *measurement system* if $\sum_i X_i^* X_i = \mathbb{1}_{\mathcal{H}}$. Let Alice and Bob have measurement systems $\{X_i\}_{i=1}^\ell$ such that $\sum_i X_i^* X_i = \mathbb{1}_{\mathcal{H}_A}$ and $\{Y_j\}_{j=1}^\ell$ $\sum_j Y_j^* Y_j = \mathbb{1}_{\mathcal{H}_B}$ respectively. Let p_k^A be the probability that Alice obtains outcome k

$$p_k^A = \langle\psi|(X_k \otimes \mathbb{1}_B)|\psi\rangle \tag{2.4.6}$$

similarly, for Bob we have

$$p_\ell^B = \langle\psi|(\mathbb{1}_A \otimes Y_\ell)|\psi\rangle \tag{2.4.7}$$

[13]This follows from the fact that the PVM's are the extreme points in the space of all POVM's.

and the joint probability is given by

$$p_{k,\ell}^{A,B} = \langle\psi|(X_k \otimes Y_\ell)|\psi\rangle\,. \tag{2.4.8}$$

Remark that for separable states the resulting joint probability is what we would expect from classical definitions. Now that we have established the basic probability statements for quantum states we can move onto the theory of non-local games.

2.5 Non-local games

A two-player *non-local game* $\mathcal{G}$ consists of finite sets of inputs $\mathcal{I}_A$ and $\mathcal{I}_B$ for each player, and finite sets of outputs $\mathcal{O}_A$ and $\mathcal{O}_B$, along with a boolean predicate $\mathcal{V} : \mathcal{I}_A \times \mathcal{I}_A \times \mathcal{O}_A \times \mathcal{O}_B \to \{0,1\}$. Elements of the inputs are given to each player according to a probability distribution $\mathcal{P}$ on $\mathcal{I}_A \times \mathcal{I}_B$. Once a player receives their input they are forbidden from communicating with the other. Upon receiving the input $x \in \mathcal{I}_A$ ($y \in \mathcal{I}_B$ resp.) each player must return an output $a \in \mathcal{O}_A$ (b[14] $\in \mathcal{O}_B$ resp.) to the verifier. The players win if $\mathcal{V}(a,b|x,y) = 1$ and lose if $\mathcal{V}(a,b|x,y) = 0$. With this definition we can now formalize the notion of a *strategy* for these non-local games.

2.5.1 Classical strategies for non-local games

We begin by defining classical strategies. For a two-player non-local game, a *deterministic strategy* is a pair of functions $f : \mathcal{I}_A \to \mathcal{O}_A$ and $g : \mathcal{I}_B \to \mathcal{O}_B$, such that $\mathcal{V}(f(x), g(y)|x,y) = \mathcal{V}(f(x), g(y))$. Furthermore, a deterministic strategy is *perfect* if $\mathcal{V}(f(x), g(y)) = 1$ for all $x, y \in \mathcal{I}_A \times \mathcal{I}_B$. In the case of a more general classical strategy, consider Alice and Bob share a distribution $\mathcal{Z}$ over the joint set of outputs $\mathcal{O}_A \times \mathcal{O}_B$. In this case we obtain the conditional probability,

$$p(a,b|x,y) = \Pr\big(f(x) = a, g(y) = b\ : x, y \in \mathcal{I}_A \times \mathcal{I}_B\big)\,. \tag{2.5.1}$$

Since this is a probability distribution, we have that $p(a,b|x,y) \geq 0$ for all a, b, x, y, and

$$\sum_{a \in \mathcal{O}_A, b \in \mathcal{O}_B} p(a,b|x,y) = 1\,, \tag{2.5.2}$$

[14] We hope from context that it is clear when we use b for the output of a player, and when we use b for the function that 2-colours a graph.

for any input pair $(x, y) \in \mathcal{I}_A \times \mathcal{I}_B$. We model the forbidden communication between the players by placing a *non-signalling* condition on the distribution, which mathematically translates to the conditions

$$\sum_{x \in \mathcal{O}_A} p(a, b|x, y) = \sum_{x \in \mathcal{O}_A} p(a, b|x, y') \quad \text{and} \sum_{y \in \mathcal{O}_B} p(a, b|x, y) = \sum_{y \in \mathcal{O}_B} p(a, b|x', y) \qquad (2.5.3)$$

for all a, b, x', y'. A strategy is said to be *synchronous* if $p(a, b|x, x) = 0$ for all $a \neq b$. In other words, the only non-zero synchronous probability is $p(a, a|x, x)$. The synchronous condition effectively means, that given the same input, the players always agree on their outputs.

Given a game and a distribution over the set of inputs, we can rigorously define the classical *value* of a game by

$$\omega(\mathcal{G}, \mathcal{P}) = \max_{f,g} \left\{ \sum_{a,b \in \mathcal{O}_A \times \mathcal{O}_B} \sum_{x,y \in \mathcal{I}_A \times \mathcal{I}_B} \mathcal{P}(x, y) \mathcal{V}(a, b|x, y) p(a, b|x, y) \right\}, \qquad (2.5.4)$$

where we maximize over all strategy functions $f : \mathcal{I}_A \to \mathcal{O}_A$ and $g : \mathcal{I}_B \to \mathcal{O}_B$. Remark from the above definition that a *perfect* deterministic strategy exists if and only if $\omega(\mathcal{G}) = 1$. Furthermore, since the set of classical strategies form a convex set, the optimal value of ω will be obtained by some deterministic strategy. Hence, we can express the classical value as

$$\omega(\mathcal{G}, \mathcal{P}) = \max_{f,g} \left\{ \sum_{x,y \in \mathcal{I}_A \times \mathcal{I}_B} \mathcal{P}(x, y) \mathcal{V}(f(x), f(y)|x, y) \right\}. \qquad (2.5.5)$$

2.5.2 Quantum strategies for non-local games

Let Alice and Bob be the two players participating in a non-local game $\mathcal{G}$. To simplify things, consider the case where they have the same input and output sets $\mathcal{I}$ and $\mathcal{O}$. Suppose we view the set $\mathcal{O}$ as the *outcomes* of a quantum experiment. Moreover, we can view a quantum measurements as a function (denoted by δ) from the set $\mathcal{I}$ to the set of POVM measurements $\mathcal{M}_{\mathcal{O}}$ with outcomes in $\mathcal{O}$, given by

$$\delta : \mathcal{I} \to \mathcal{M}_{\mathcal{O}} \qquad (2.5.6)$$

$$\delta(x) = X_a\,, \qquad (2.5.7)$$

where each X_a is a *positive* element of $B(\mathcal{H})$. Furthermore, we insist that this function satisfies the following identity,

$$\sum_{a \in \mathcal{O}} \delta(x) = \sum_{a \in \mathcal{O}} X_a = \mathbb{1}_{\mathcal{H}}\,. \qquad (2.5.8)$$

If δ is a measurement with outcomes in $\mathcal{O}$, and the state being measured is $|\psi\rangle \in \mathcal{H}$, then the probability of outcome $a \in \mathcal{O}$ is given by the equation

$$p_a = \langle \psi | \delta(x) | \psi \rangle \,, \tag{2.5.9}$$

for each $a \in \mathcal{O}_A$.

Now, a *quantum strategy* is a pair of functions $\delta_A : \mathcal{I}_A \to \mathcal{M}_{\mathcal{O}_A}$ and $\delta_B : \mathcal{I}_B \to \mathcal{M}_{\mathcal{O}_A}$, along with a quantum state $|\psi\rangle$[15]. Quantum strategies are a generalization of classical strategies in the following way. Assigning elements of $\mathcal{I}$ to one-dimensional elements of $\mathcal{M}_{\mathcal{O}}$ is equivalent to a deterministic strategy since these one-dimensional elements are precisely the elements of $\mathcal{O}$. However, it is well known in quantum information theory that any outcome of a POVM can be attained on by suitable *projection valued measure* or PVM. A PVM is merely a POVM where each element is an orthogonal projection $P_a^2 = P^* = P$. We now give a description of quantum correlations, which can be thought of as analogous to quantum strategies.

A correlation $p(a, b|x, y)$ is called a quantum *commuting-operator* correlation if there exist collections of mutually commuting projections $\{P_a^x\}_{a \in \mathcal{O}_A}$ for every $x \in \mathcal{I}_A$, and $\{Q_b^y\}_{b \in \mathcal{O}_B}$ for every $y \in \mathcal{I}_B$ (i.e. $P_a^x Q_b^y = Q_b^y P_a^x$ for all $(a, b, x, y) \in \mathcal{I}_A \times \mathcal{I}_A \times \mathcal{O}_A \times \mathcal{O}_B$) and a global quantum state $|\psi\rangle \in \mathcal{H}$, such that

$$p(a, b|x, y) = \langle \psi | P_a^x Q_b^y | \psi \rangle \,, \tag{2.5.10}$$

for all $(a, b, x, y) \in \mathcal{I}_A \times \mathcal{I}_A \times \mathcal{O}_A \times \mathcal{O}_B$.

A correlation $p(a, b|x, y)$ is called a quantum *tensor-product* correlation if there exist a projection valued measure $\{P_a^x\}_{a \in \mathcal{O}_A} \in \mathcal{H}_A$ for every $x \in \mathcal{I}_A$, a $\{Q_b^y\}_{b \in \mathcal{O}_B}$ on $\mathcal{H}_B$ for every $y \in \mathcal{I}_B$, and a quantum state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, such that

$$p(a, b|x, y) = \langle \psi | P_a^x \otimes Q_b^y | \psi \rangle \,, \tag{2.5.11}$$

for all $(a, b, x, y) \in \mathcal{I}_A \times \mathcal{I}_A \times \mathcal{O}_A \times \mathcal{O}_B$.

In the tensor product model, we choose the quantum state to be the canonical maximally entangled state[16] on the tensor product of the local Hilbert spaces.

[15] We will see shortly, that specific types of quantum strategies depend on the pre-supposed structure of the Hilbert space containing $|\psi\rangle$. For example, one can consider operators that act only on the local space factors of $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ or as commuting operators of a single *global* Hilbert space $\mathcal{H}$.

[16] In the commuting-operator model, the entangled state comes from defining a *tracial state* on the global Hilbert space using the action of the left-regular representation of the group algebra $\mathbb{C}[\Gamma(G, b)]$. For more details on these matters, we refer the reader to [12] or [43].

We can now define the *entangled* value of a non-local game,

$$\omega^*(\mathcal{G}, \mathcal{P}) = \sup \left\{ \sum_{x,y \in \mathcal{I}_A \times \mathcal{I}_B} \sum_{a,b \in \mathcal{O}_A \times \mathcal{O}_B} \mathcal{P}(x,y) p(a,b|x,y) \mathcal{V}(a,b|x,y) \right\}, \tag{2.5.12}$$

where the supremum[17] is taken over all possible sets of measurements (P_a^x, Q_b^y) obtained from (δ_x, δ_y) and quantum states $|\psi\rangle$. Remark that there is a perfect quantum strategy if $\omega^*(\mathcal{G}) = 1$. A sufficient condition for a strategy to be perfect, is for every losing tuple (a, b, x, y) (i.e. $\mathcal{V}(a, b|x, y) = 0$) we have $p(a, b|x, y) = 0$.

2.5.3 A non-local game with a quantum advantage

The Clauser-Holt-Shimony (CHSH) game [10], is perhaps the simplest concrete game for exhibiting the separation of quantum and classical correlations. In this non-local game, a verifier produces two bits x and y uniformly at random. Alice receives the first bit x and responds with the bit a. Similarly, Bob receives a bit y and responds with a bit b. The players are forbidden from classically communicating the value of their bits with the other player. They win the game if and only if $x \wedge y = a \oplus b$ (i.e. the bitwise "AND" of their outputs equals the bitwise "OR" of their outputs). It is easy to see that the players will win on 3/4 of the inputs if they employ a strategy where $a \oplus b = 0$. It turns out that any pair of classical players cannot do better than this 3/4 winning probability. Surprisingly, there is a quantum strategy that obtains a higher winning probability than the best classical strategy!

Suppose Alice and Bob share a maximally entangled state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. The quantum strategy is given as follows. If Alice receives the bit $x = 0$, she measures her state with the observable A_0, and if she receives $x = 1$ she measures with the observable A_1. In either case she responds to the verifier with the outcome of her measurement. Where her operators are given as

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.5.13}$$

Likewise, Bob has a similar collection of observables, which he performs on his half of the maximally entangled state depending on his input y. If $y = 0$ he applies B_0 and if $y = 1$, then he applies B_1.

$$B_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad B_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \tag{2.5.14}$$

[17] We have a supremum because the dimensions of underlying Hilbert space could be unbounded.

Similarly, he responds to the verifier with the outcome of his measurement.

Observe that the operators $\{A_0, A_1, B_0, B_1\}$ do not form suitable POVM's, so how can they constitute a quantum strategy as outlined above! First, observe that these elements are ± 1-valued hermitian operators. We call these elements binary observables, and we will see shortly that 2-outcome projections are equivalent to these binary observables. In fact, we will make extensive use of this connection in the following sections of this thesis.

Returning to the CHSH game, observe that the outcomes for each player are the values $\langle\psi|A_x|\psi\rangle = \pm 1$ and $\langle\psi|B_y|\psi\rangle = \pm 1$. To recover the $\{0,1\}$-bits for the verifier to compute the predicate on. Alice and Bob merely apply the inverse of the mapping $a \mapsto (-1)^a$ to their measurement outcomes before sending them to the verifier. The astonishing fact is that this quantum strategy wins with probability $\omega^*(CHSH) = \cos^2(\pi/8) \approx 0.85$. Which is substantially better than the optimal classical value of $\omega(CHSH) = 0.75$. Furthermore, it can be shown, that the maximum value attained by a quantum strategy for the CHSH game is bounded above by this value $\frac{1}{2}(1 + \frac{1}{\sqrt{2}}) \approx 0.85$. This bound is known as Tsirelson's bound [42], and thus the outlined quantum strategy above achieves ω^*.

In the case of the CHSH game, we saw how a quantum strategy obtained an advantage over the best classical strategy. In the next sections, we will investigate when we can find *perfect* quantum strategies. In particular, when do we have perfect quantum strategies and no perfect classical ones. To do this we will investigate a very particular subset of non-local games.

2.5.4 Linear binary-constraint systems and operator solutions

A linear binary-constraint system or LinBCS non-local game $\mathcal{G}(A, b)$ consists of an $n \times m$ linear system $Av = b$, over $\mathbb{Z}_2$. The game proceeds in the following steps:

1. Alice receives an index $1 \le x \le n$ corresponding to a row in A, meanwhile Bob received an index $1 \le y \le m$ corresponding to a variable of v. We assume that these inputs are both picked independently and uniformly at random.

2. Alice's returns a function $f : V_x \to \mathbb{Z}_2$, where V_x is the set of non-zero entries in the xth row of A, and Bob returns an assignment to the yth variable of $v_y = g(y) \in \mathbb{Z}_2$

3. The players win if $\sum_{e \in V_x} f(e) = b(x)$, and either $y \notin V_x$ or if $y \in V_x$ and $f(y) = g(y)$.

In other words, given her input x, Alice outputs a $\{0,1\}$-assignment to each non-zero variable in the xth row of A. While Bob, who is given a particular variable in v, outputs a

$\{0,1\}$ assignment to variable $g(y) = v_y$. The winning condition is based on the *consistency* of any overlapping assignments to the variables/row-entries.

We will see that if Alice and Bob have a solution to the linear system of equations, then they have the means of constructing a perfect deterministic strategy for the game. Thus, the interesting instances of these $\mathbb{Z}_2$-linear system non-local games are when b is not in the column space of A. In this case, the players must determine how to minimize their probability of losing using the structure of A and b.

We now consider a type of generalized solution for a linear system of binary constraints.

Definition 2.5.1. An operator solution to a linear system of equations $Ax = b$ over $\mathbb{Z}_2$, is a collection of self-adjoint linear operators $\{X_1, \ldots, X_m\}$, such that the following hold:

(i) $X_j^2 = \mathbb{1}$[18] for all $1 \leq j \leq m$.

(ii) If x_k and x_ℓ appear in the same equation $1 \leq k, \ell \leq m$ (each equation is the dot product of the ith row of A with x) then $X_k X_\ell = X_\ell X_k$ (i.e. the corresponding operators commute).

(iii) For each equation $\sum_k x_k = b_\ell$, $V_k = \{1 \leq j \leq m : a_{\ell j} x_j \neq 0\}$ for all $1 \leq \ell \leq n$, then

$$\prod_k X_k = (-1)^{b_\ell} \mathbb{1} \,. \tag{2.5.15}$$

For operator solutions to linear binary-constraint systems, one can analogously define the d-dimensional *quantum column space* as the subspace of vectors in $\mathbb{Z}_2^n$, such that there is an operator solution of dimension d. The CMLS theorem (Theorem 1.0.1) states that b belongs to the d-dimensional quantum column-space whenever there is a d-dimensional unitary representations of the solution group $\Gamma(A, b)$ with $J \neq \mathbb{1}_{\mathbb{C}^d}$. In addition, the CMLS theorem says that, if one has such a representation, then it can be converted into a corresponding perfect tensor-product strategy for the corresponding non-local game $\mathcal{G}(A, b)$. On the other hand, if $\Gamma(A, b)$ is a solution group with $J \neq 1$, and the representation forms a perfect infinite-dimensional operator-solution, then the CMLS theorem states that there exists some infinite-dimensional commuting-operator strategy for $\mathcal{G}(A, b)$. Furthermore, this infinite-dimensional commuting-operator strategy, is a proper commuting-operator strategy, if it contains an infinite irreducible representation. For more on operators-solutions to LinBCS games and how they can be expressed as quantum strategies see [12, 13].

[18] Remark that if X is self-adjoint $X = X^*$ and self-inverse $X = X^{-1}$ then $X^* = X^{-1}$ is unitary.

2.6 Correlation sets for quantum models

Consider two quantum experimentalist Alice and Bob, who prepare quantum states between separated laboratories. Suppose they each have n experiments, each with m outcomes, to perform on their shared quantum states. If Alice performs experiment x for $1 \leq x \leq n$, and Bob performs experiment y for $1 \leq y \leq n$, according to some distribution $\mathcal{P}$. If they obtain outcomes a and b respectively, then the space of correlations between their joint outcomes forms the space of quantum correlations for the triple $(n, m, \mathcal{P})$. These correlations can be described by real matrices of the form,

$$\big(p(a,b|x,y)\big)_{i=1,j=1}^{n^2,m^2} \in \mathbb{R}_+^{n^2 \times m^2} . \tag{2.6.1}$$

Where $p(a,b|x,y)$ is read, the probability that Alice and Bob return outcomes (a,b), given the inputs (x,y). Observe the similarities of this to the space of *strategies* we discussed in Section 2.5. Thus, the set of all finite dimensional strategies can be expressed as a finite dimensional matrix space, and is equivalent to the space of quantum finite quantum correlations. It came as a surprise to researchers that in infinite-dimensions, the space of correlations depended on the interpretation of *locality* for the corresponding space of operators. We focus on the two important models relevant in this work.

Definition 2.6.1. Two relevant models for quantum correlations are:

(a) The quantum *tensor-product* model C_{qs}, where the measurements are sets of local orthogonal projections $\{P_a^x\}_{a=1}^{m_a} \in \mathcal{B}(\mathcal{H}_A)$, and $\{Q_b^y\}_{b=1}^{m_b} \in \mathcal{B}(\mathcal{H}_B)$, and $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

(b) The quantum *commuting-operator* model C_{qc}, where the measurements are orthogonal projections $\{P_a^x\}_{a=1}^{m_a}, \{Q_b^y\}_{b=1}^{m_b} \in \mathcal{B}(\mathcal{H})$ such that every P_a commutes with every Q_b.

It follows, that we could have alternatively described the set of quantum correlations and thus strategies, in the following way. Let $|\psi\rangle$ be a state on a Hilbert space $\mathcal{H}$, and let $\{P_a^x\}_{a=1}^{m_a}$ and $\{Q_b^y\}_{b=1}^{m_b}$ be *commuting* projective measurements on $\mathcal{H}$ (i.e. $P_a^x Q_b^y = Q_b^y P_a^x$ for all (a,b,x,y)), then a quantum *commuting correlation* is described by

$$p(a,b|x,y) = \langle \psi | P_a^x Q_b^y | \psi \rangle , \tag{2.6.2}$$

for all tuples $(a,b,x,y) \in \mathcal{I}_A \times \mathcal{I}_A \times \mathcal{O}_A \times \mathcal{O}_B$. As strategies, these are called quantum commuting-operator strategies. In this framework existence of the entangled state $|\psi\rangle$ is defined more abstractly, instead of as a canonical maximally entangled state on the bipartite space it is a particular *tracial* state on the algebra of operators from the *global* Hilbert space. When the commuting projections are infinite dimensional, the existence of this tracial state, subtlety depends on the algebraic structure of the measurement operators (i.e. the group of observables Γ that they manifest in). As mentioned in Chapter 1, this tracial state exists if and only if $J \neq 1$. In this case, the existence of nontrivial $|J\rangle$ indicates that there is such an operator on the group Hilbert space $\mathbb{C}[\Gamma]$ given by a non-trivial action on the space.

By definition, one can observe that the tensor-product model is contained in the commuting-operator model. Simply consider the commuting elements $(P_a^x \otimes \mathbb{1}_B)$ and $(\mathbb{1}_A \otimes Q_b^y)$. In finite-dimensions, it can be shown that these models are the same. However, when the Hilbert spaces are infinite it is not so clear. If one considers the limit of finite-dimensional quantum tensor product strategies, as the dimension of $\mathcal{H}_A$ and $\mathcal{H}_B$ go to infinity, we obtain the closure of C_{qs} denoted C_{qa}. This model is known as the approximable C_{qa} correlation model. As we have already established, there are strict separations between the classical and quantum correlations sets C_c and C_q[19]. More generally Tsirelson's problem asks, whether there are other separations in the following chain of correlations sets,

$$C_c \subseteq C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc}\,. \tag{2.6.3}$$

Using a powerful solution group construction, along with the CMLS theorem, Slofstra showed that $C_{qs} \neq C_{qc}$ [39]. The question, of whether or not $C_{qa} = C_{qc}$, is equivalent to the famous Connes embedding conjecture [25, 33, 17], and remains the only open separation in (2.6.3). At this point in time, it has been established that all the other inclusions are proper ($\subsetneq$) [40, 42, 14, 15].

2.6.1 Decomposition of unitary elements into sets of projections

Observe that any collection of projective measurements $\{P_a^x\}_{a=1}^m$ can be thought of as the spectral projections of a m-dimensional unitary

$$U = \sum_{a=1}^{m} e^{2\pi(a-1)/m} P_a\,. \tag{2.6.4}$$

Thus, there is a correspondence between m-outcome PVM's and representations of the group algebra of $\mathbb{C}[\mathbb{Z}_m]$. In the case of LinBCS games, there is a drastic simplification of the general non-local game framework. In particular, the set of outcomes is binary. Since any

[19]These separations are known as Bell inequalities.

value attained by a POVM can be attained through PVM's, we need only consider binary projective measurements. It is convenient to take all POVM's to be binary observables[20] These unitary elements on a Hilbert space form a group generated by involutions.

Definition 2.6.2. The value $t(a,b|x,y)$ is a *Tsirelson correlation* if there exists commuting collections of unitary self-adjoint operators $\{\mathcal{O}_a^x\}_{x=1}^{n_A}$, $\{\mathcal{O}_b^y\}_{y=1}^{n_B}$, with a quantum state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, such that

$$t(a,b|x,y) = \langle\psi|\mathcal{O}_a^x\mathcal{O}_b^y|\psi\rangle\,. \tag{2.6.5}$$

Proposition 2.6.3. Tsirelson's correlations are equivalent to finite dimensional quantum correlations.

Proof. It suffices to show that for every collection of projective measurements, there is a corresponding collection of self-adjoint unitary observables and vice versa. Let P_0, P_1 be a projective measurement, so $P_i^* = P_i = P_i^2$ and $P_0 + P_1 = \mathbb{1}$. If we have a set of observables, such that $A^*A = \mathbb{1}$, and $A^2 = \mathbb{1}$, then we can transfer between projective measurement and unitary observables via $P_0 = \frac{\mathbb{1}+A}{2}$ and $P_1 = \frac{\mathbb{1}-A}{2}$. We observe that $P_i^2 = \frac{1}{4}(2\mathbb{1} \pm 2A) = \frac{\mathbb{1}\pm A}{2}$ and $P_i^* = \frac{(\mathbb{1}\pm A)^*}{2} = \frac{\mathbb{1}^*\pm A^*}{2} = \frac{\mathbb{1}\pm A}{2}$. On the other hand, consider taking $A = P_0 - P_1$. Recall, since $P_0P_1 = 0 = P_1P_0$ the projections have disjoint support, and $A^2 = P_0^2 + P_0P_1 + P_1P_0 + P_1^2 = P_0 + P_1 = \mathbb{1}$. ☐

Proposition 2.6.3 means that given a strategy in terms of 2-outcome projections, one can transform the strategy into one with unitary observables using the fact $t(a,b|x,y) = \langle\psi|\mathcal{O}_a^x\mathcal{O}_b^y|\psi\rangle = 2p(a,b|x,y) - 1$. Thus, for a losing quantum strategy in terms of observables, instead of $\mathcal{V}(a,b|x,y) = 0$, we have $t(a,b|x,y) = -1$. Remark, t is not probability distribution, but it does take on values $\langle -1\rangle \cong (\mathbb{Z}_2, \cdot)$, and this fact will be important in Chapter 5.

[20] Hermitian matrices with ± 1 eigenspaces, are precisely a subset of the unitary matrices known as *reflections*.

Chapter 3

Graph incidence groups for graph-LinBCS games

In this chapter, we establish the definition of a graph-LinBCS game. We begin by defining the classical and quantum strategies for graph-LinBCS games. We then give some examples and discuss the case of some well-known graph-LinBCS games. Following this, we define the solution group for these non-local games, which we call the graph incidence group. Since graphs are the only incidence structure we consider in this thesis, we may sometimes drop the "graph" and write *incidence group*. We then explore some immediate connections between the graph structure and characteristics of the incidence group.

3.1 Graph-LinBCS non-local games

In this section we investigate how the properties of non-local games, discussed in Chapter 2, specialize to graph-LinBCS non-local games. In particular, we explore how the specialization to graphs allows us to make conclusions about strategies and values for these games, that is not possible for generic LinBCS games.

A graph-LinBCS game is a two-player non-local game. Both players have access to a 2-coloured graph (G, b). We consider two general versions of a graph-LinBCS game referred to as v.1 and v.2 respectively. Each non-local game depends on the distribution over the input sets, and therefore should be expressed by the data $\mathcal{G}(G, b, \mathcal{P})$. However, when $\mathcal{P}$ is the uniform distribution, we omit the description (as we did for LinBCS games), and we denote the graph-LinBCS game by $\mathcal{G}(G, b)$.

Definition 3.1.1 (graph-LinBCS game v.1)**.** Given a loop-less graph $G = (V, E)$, with a (non-proper) vertex 2-colouring $b : V \to \mathbb{Z}_2$, the corresponding graph-LinBCS game $\mathcal{G}(G, b)$ is given as follows. For an arbitrary vertex $v \in V$, let $E(v)$ be the subset of edges in E incident to v. The game proceeds in the the following steps:

1. Each player Alice and Bob, receive vertices $u \in V$ and $v \in V$ respectively, according to a distribution $\mathcal{P}$ on $V(G) \times V(G)$.
2. They reply with functions $f : E(u) \to \mathbb{Z}_2$ and $g : E(v) \to \mathbb{Z}_2$ respectively, such that

$$\sum_{e \in E(u)} f(e) = b(u) \quad \text{and} \quad \sum_{e \in E(v)} g(e) = b(v)\,. \tag{3.1.1}$$

3. The players win if $f(e) = g(e)$ for every $e \in E(u) \cap E(v)$.

In the second version of the game we change the set of Bob's inputs, from vertices to edges. The lack of symmetry in this version in the reason we consider the other version to be more natural.

Definition 3.1.2 (graph-LinBCS game v.2)**.** Given a graph $G = (V, E)$, with a (non-proper) vertex 2-colouring $b : V \to \mathbb{Z}_2$, the alternative version graph-LinBCS game $\mathcal{G}(G, b)$ proceeds as follows:

1. Alice receives a vertex $v' \in V$ and Bob receives an edge $e' \in E$, according to some distributions on $V \times E$.
2. Alice replies with a function $f : E(v') \to \mathbb{Z}_2$, such that $\sum_{e \in E(v')} f(e) = b(v')$ and Bob replies with the single edge assignment $g(e') \in \mathbb{Z}_2$,
3. The players lose the game if and only if $e' \in E(v)$ and $f(e') \neq g(e')$.

3.1.1 The space of classical strategies for graph-LinBCS games

For both versions of the graph-LinBCS game a deterministic strategy is a function $h : E \to \mathbb{Z}_2$. To illustrate, we provide an example of a (non-perfect) deterministic strategy in Figure 3.1. Furthermore, observe that the function $h : E \to \mathbb{Z}_2$ is a perfect deterministic strategy if $\sum_{e \in E(v)} h(e) = b(v)$ for all $v \in V$. Moreover, Alice and Bob can employ the strategy given by h by choosing $f(e) = h\big|_{E(v)}(e)$ and $g(e) = h\big|_{E(u)}(e)$ (or $g(e) = h(e)$).

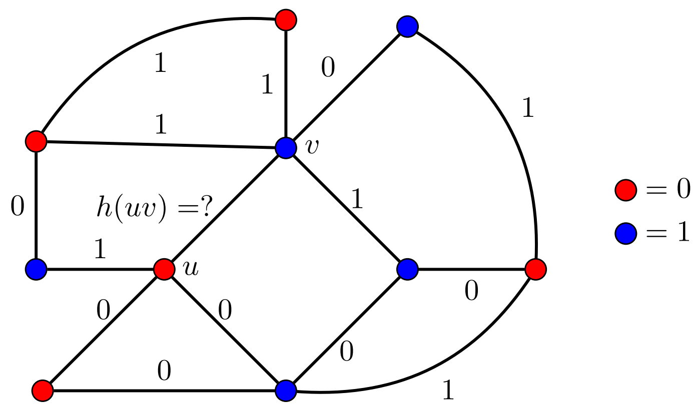

Figure 3.1: Any 0 or 1 assignment to e_{uv} would be inconsistent with the colouring of G, hence we see that this edge assignment does not give a deterministic strategy for this graph-LinBCS game, since giving Alice u and Bob v may result in an inconsistent labelling of e_{uv} if they were to employ h as their strategy.

Proposition 3.1.3. There is a one-to-one correspondence between perfect deterministic strategies for $\mathcal{G}(G, b)$, and solutions of the corresponding linear system $Ax = b$, where A is the incidence matrix of G.

Proof. The function $f : E \to \mathbb{Z}_2$ is a perfect deterministic strategy for $\mathcal{G}(G, b)$ if and only if $\sum_{e \in E(v)} f(e) = b(v)$ for all $v \in V$. Now consider the linear system $Ax = b$, written as

$$\sum_{e \in E} a(v, e)x(e) = b(v)\,, \tag{3.1.2}$$

for each $v \in V$. Observe that $a_{v,e} = 1$ if and only if there is an $e \sim v$, otherwise $a_{v,e} = 0$. Hence we obtain,

$$\sum_{e \in E(v)} x(e) = b(v)\,, \tag{3.1.3}$$

and note that $x : E \to \mathbb{Z}_2$ is a perfect deterministic strategy. The other direction follows from the reverse argument. $\square$

Lemma 3.1.4. Let G be a connected graph. A vector $b \in \mathbb{Z}_2^n$ is in the image of an $(m \times n)$-incidence matrix if and only if it has even parity.

Proof. ($\Rightarrow$) The image of a matrix is the $\mathbb{Z}_2$-span of its columns, since G is connected each column contains 2 non-zero entries. Any linear combination of the columns will contain an even number of non-zero entries. ($\Leftarrow$) Let $b \in \mathbb{Z}_2^n$ have even parity. For each pair of vertices u and v with $b(v) = b(v) = 1$, let $\mathcal{L} = \{P_1, \ldots, P_k\}$ be a collection of paths between the k pairs of vertices. It follows that $\sum_{e \in \mathcal{L}} A(e) = b$, where $A(e)$ is the column of A corresponding to $e \in E(G)$. □

Corollary 3.1.5. If b has even parity, then the graph-LinBCS game $\mathcal{G}(G, b)$ has a perfect deterministic strategy.

The proof follows directly from Proposition 3.1.3 and Lemma 3.1.4.

3.1.2 A deterministic polynomial time algorithm for computing ω for graph-LinBCS games

Graph-LinBCS games are not only a simple class of games with respect to the detection of perfect quantum or classical strategies. In the following section, we show that for graph-LinBCS games, the classical value of $\mathcal{G}(G, b)$ can be computed efficiently. This will follow from the proof of correctness and runtime analysis of the following algorithm, which obtains the optimal classical strategy. For a graph with n vertices, we bound the number of edges allowed in E by $O(poly(|V|))$.

Consider the following algorithm, which given a 2-coloured graph (V_B, V_R, E), generates a deterministic strategy (i.e. a 2-colouring of G) for the graph-LinBCS game $\mathcal{G}(G, b)$.

Algorithm 1 (Optimal deterministic strategy finding for graph-LinBCS games)**.** Let V_B be the set of blue vertices and V_R the set of red vertices in G.
Input: (V_B, V_R, E)

1. Initialize $e \mapsto 0$ for all $e \in E$
2. Partition the vertices in V_B into k pairs of blue vertices, if there is a leftover vertex (when $|V_B|$ is odd), label in v_0.
3. For each pair of blue vertices $\{v, v'\}_{i=1}^{k}$, find a path (using breadth-first search starting at an unpaired vertex) $P_i = e_v, \ldots, e'_v$, denote the length of a path by ℓ.
4. For each path P_i, and every edge in the path $\{e_j\}_{j=1}^{\ell} \in P_i$ if $e = 0$ set $e \mapsto 1$, else if $e = 1$ set $e \mapsto 0$.

Output: A 2-colouring $\eta : E \to \mathbb{Z}_2$ such that $\sum_{e \in E(v)} h(e) = b(v)$ for all $v \neq v_0 \in V$.

Proof of correctness. First, observe that initializing all edges with the value of 0 automatically satisfies all red vertex constraints. We now prove that every path flip in Step (4) between each pair of blue vertices, simultaneously satisfies the constraints of all the blue vertices. Additionally, we need to show that flipping the values along each path does not alter the constraints of other vertices along the paths (i.e. those not at the endpoints). To see that a path flip (step 4) only affects the vertex constraints at the endpoints. Observe, that following a path flip, the parity of any non-endpoint vertex will change on both an incoming and outgoing edge, thus cancelling out. On the other hand, the parity of each endpoint will change on a single edge, which simultaneously satisfies the two blue endpoint constraints. Thus, if there is an even number of blue vertices this colouring satisfies all the constraints of the graph. However, if there is an odd number of blue vertices, then there remains one unsatisfied constraint. Overall the resulting colouring satisfies $n-1$ of the vertex constraints. For the analysis of the runtime let $n = |V|$, and $m = |E|$. The first step takes $O(m)$ operations, secondly there are $|V_B|/2 = k$ pairs with $O(k) = O(n)$, and finding a path between each pair is $O(m+n)$ using BFS. Recall we restricted $O(m) = O(poly(n))$, since $O(\ell) = O(m)$ our runtime is polynomial in the number of vertices. If G is simple, then we observe that our runtime is $O(n^3)$, since in that case $O(m) = O(n^2)$. □

Recall that in version 1 of the non-local game, it is possible that Alice and Bob, receive the same vertex. In this case, they win if and only if they give the same local edge labelling about that vertex. Thus, to optimize their strategy, the players should predetermine their edge pairings by picking agreeing labels for the leftover vertex v_0, when b is odd[1].

We now prove that the strategy given by Algorithm 1 achieves the best classical value ω on the uniform distribution.

Proposition 3.1.6. The deterministic strategy η is optimal, and achieves $\omega(\mathcal{G})$ under the uniform distribution $\mathcal{P}$ on $(u, v) \in V \times V$.

Proof. If b is even, then we obtain a perfect deterministic strategy (see Proposition 3.1.3). Thus we need only show that η is optimal for odd b. So suppose b is odd, by the proof of correctness for the algorithm above, we observe that η loses on a single pair of vertices incident to v_0. Suppose towards contradiction, that there is a strategy η' such that $\omega(\eta') > \omega(\eta)$. Then η' must lose on fewer than a single pair of vertices, but that means η' is a perfect strategy, which is a contradiction. □

Proposition 3.1.7. If G is a connected, oddly 2-coloured graph, and $\mathcal{P}$ is uniform and independent on $V \times V$, then there is a deterministic strategy that wins with probability $\omega(\mathcal{G}(G, b), \mathcal{P}) = 1 - \frac{1}{|V|^2}$.

[1] This also makes their strategy synchronous.

Proof. Assume the uniform distribution on the pairs of vertices. By Corollary 3.1.5 we know that if b is even, then there is a perfect deterministic strategy. Therefore, $\mathcal{V}(e, e', u, v)$ depends only on the odd vertex v_0. The classical value of this game is obtained by η with $\mathcal{P}(u, v) = \frac{1}{|V|^2}$ on $V \times V$. Recall that probabilistic strategies are normalized, such that $p(e, e'|v, v')$ are proper conditional distributions on the outputs,

$$\sum_{e \in E(v)} \sum_{e \in E(v')} p(e, e'|v, v') = 1\,. \tag{3.1.4}$$

Using the fact that this strategy wins on all but one pair of vertices, we observe

$$\begin{aligned}
\omega\big(\mathcal{G}(G, b)\big) &= \sum_{v \in V} \sum_{v' \in V} \mathcal{P}(v, v') \sum_{e \in E(v)} \sum_{e' \in E(v')} \mathcal{V}(e, e'|v, v') p(e, e'|v, v') && (3.1.5)\\
&= \frac{1}{|V|^2} \Bigg(\sum_{v \neq v_0 \in V} \sum_{v' \neq v_0' \in V} \mathcal{V}(f_v(e), g_u(e)) + \sum_{\substack{(v_0, v_0') \\ e = e' \sim v_0}} \mathcal{V}(f_v(e), g_u(e)) \Bigg) && (3.1.6)\\
&= \frac{1}{|V|^2} \Bigg(\sum_{v \neq v_0 \in V} \sum_{v' \neq v_0' \in V} 1 + 0 \Bigg) && (3.1.7)\\
&= \frac{1}{|V|^2} \big(|V|^2 - 1 \big) && (3.1.8)\\
&= 1 - \frac{1}{|V|^2} && (3.1.9)
\end{aligned}$$

Similarly, one can consider the probability the strategy loses. Whereby, the probability that Alice flips the bit on the edge (v_0, v_0') (in an attempt to satisfy her local constraint) when Alice receives v_0 and Bob receives the adjacent edge v_0'.

$$\begin{aligned}
\Pr(\text{Lose}) &= \Pr(\text{Alice receives } v_0) \Pr(\text{Bob receives } v \sim v_0) \Pr(\text{Alice bit-flips the wrong edge}) && (3.1.10)\\
&= \left(\frac{1}{|V|}\right) \left(\frac{\deg(v_0)}{|V|}\right) \left(\frac{1}{\deg(v_0)}\right) && (3.1.11)\\
&= \frac{1}{|V|^2} && (3.1.12)
\end{aligned}$$

$\square$

The same strategy η, is in fact optimal for version 2 of the game, the distribution of the inputs is slightly different.

Proposition 3.1.8. If G is a connected, oddly 2-coloured graph, and $\mathcal{P}$ is independent and uniform on V and E, then there is a deterministic strategy that wins version 2 of the graph-LinBCS game $\mathcal{G}'(G,b)$ with probability $\omega(\mathcal{G}'(G,b)) = 1 - \frac{1}{|V||E|}$.

Proof. For version 2 of the game, we have if $\mathcal{P}(e,v)$ is uniform and independent on V and E. For a given optimal deterministic strategy, if b is odd then there exist some pair $v_0 \in V$ and $e_0 \in E(v)$ such that $f_{v_0}(e_0) \neq g(e_0)$, where $f_v(e)$ is the function $f : E(v) \to \mathbb{Z}_2$, hence

$$\begin{aligned}
\omega\big(\mathcal{G}'(G,b)\big) &= \sum_{v\in V}\sum_{e\in E}\mathcal{P}(v,e)\mathcal{V}(f_v(e),g(e)) && (3.1.13)\\
&= \frac{1}{|V||E|}\Bigg(\sum_{v\neq v_0\in V}\sum_{e\in E}\mathcal{V}(f_v(e),g(e)) + \sum_{e\in E}\mathcal{V}(f_{v_0}(e),g(e))\Bigg) && (3.1.14)\\
&= \frac{1}{|V||E|}\Bigg(\sum_{v\neq v_0\in V}\sum_{e\in E}\mathcal{V}(f_v(e),g(e)) + \sum_{e\neq e_0\in E}\mathcal{V}(f_v(e),g(e)) + \mathcal{V}(f_{v_0}(e_0),g(e_0))\Bigg) && (3.1.15)\\
&= \frac{1}{|V||E|}\big((|V|-1)|E| + (|E|-1) + 0\big) && (3.1.16)\\
&= \frac{1}{|V||E|}\big(|V||E| - |E| + |E| - 1\big) && (3.1.17)\\
&= \frac{|V||E|-1}{|V||E|} && (3.1.18)\\
&= 1 - \frac{1}{|V||E|}\,. && (3.1.19)
\end{aligned}$$

□

We can summarize the above Propositions in the following theorem.

Theorem 3.1.9. If G is a connected 2-coloured graph, then the classical value of a graph-LinBCS game $\mathcal{G}(G,b)$ with uniform distribution on the inputs is given by

(i) $\omega = 1$ if b is even

(ii) $\omega = 1 - \frac{1}{|V|^2}$ (or $\omega = 1 - \frac{1}{|V||E|}$), when b is odd.

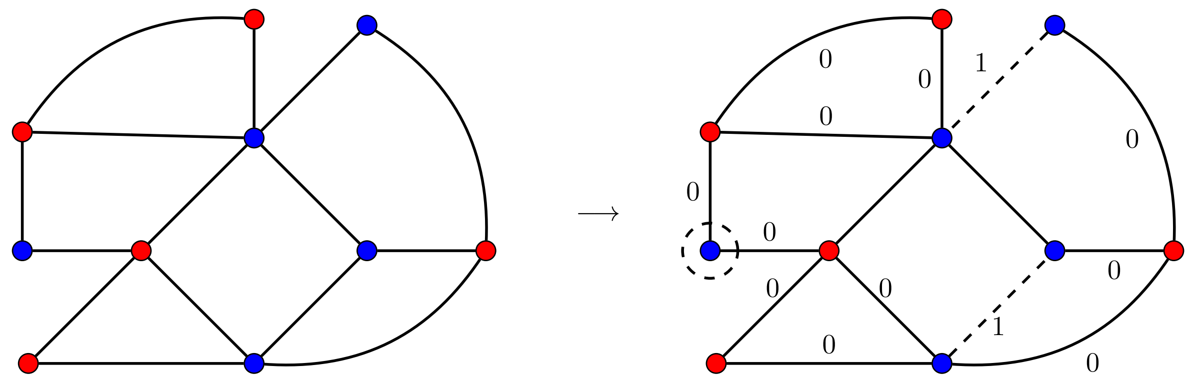

Figure 3.2: Visualization of Algorithm 1. Paths between pairs are labelled by the dashed edges and the single (odd) vertex by the dashed circle. Observe the only loss can occur if the odd, along with one of its two adjacent vertices, is chosen and Alice flips the colour value on the wrong of the two edges. Since the colouring satisfies 9 of the 10 constraint, the classical value of this graph-LinBCS game is $\omega = 0.99$ by Proposition 3.1.7.

3.1.3 Perfect quantum strategies for graph-LinBCS games

Let us now describe how a concrete quantum strategy can be employed for a graph-LinBCS game. Given a game $\mathcal{G}(G, b)$, where Alice and Bob each receive a vertex $u, v \in V$, and respond with 2-colourings of the adjacent edge neighbourhoods $E(v)$ and $E(u)$ respectively.

The two-player begin by preparing $m = O(poly(n))$, d-dimensional maximally entangled quantum states $\{|\psi\rangle_k\}_{k=1}^{m}$, where $m = |E|$ is the number of edges in G. Alice's portion of the quantum strategy then consists of m, d-dimensional observables, each one over the outputs of the game $\mathcal{O}_a^x$, such that her output[2] is determined by the outcome of the measurement on her half of the maximally entangled pair $\langle\psi_A|\mathcal{O}_a^x|\psi_A\rangle = a$. One can show that for a perfect quantum strategy, Bob's strategy observables can be taken to be the *transpose* of Alice's observables[3] $\mathcal{O}_b^y = (\mathcal{O}_a^x)^T$, and his output is given by the measurement outcome $\langle\psi_B|\mathcal{O}_b^y|\psi_B\rangle = b$. Because we can take these observables, to be self-adjoint unitaries, the values taken by a and b are elements of the multiplicative $\mathbb{Z}_2$ group $\langle -1\rangle$. To obtain corresponding the additive $\mathbb{Z}_2 = \{0, 1\}$ edge-colours we map the outcomes $1 \mapsto 0$ and $-1 \mapsto 1$. The entanglement shared between these states ensures that the outputs determined by their joint measurements will be consistent. For example, if Alice and Bob receive

[2] $\mathbb{Z}_2$-assignment to edge a given vertex x.
[3] Thus a perfect quantum strategy can be determined by specifying a single set of observables.

adjacent vertices $u \sim v$, then the outcome of the measurement e_{uv}, will be the same $\langle\psi_A|\mathcal{O}_e^v|\psi_A\rangle = \langle\psi_B|\mathcal{O}_e^u|\psi_B\rangle$.

The magic squares game is a key example here. For the magic squares game there is a set of 9 observables (one for each output), each of which can be represented as tensor products of single-qubit Pauli matrices. The array below encodes an assignment of operators to the 9 edges of graph, which give a perfect quantum strategy for the game.

$$\begin{array}{|c|c|c|}\hline e_1 & e_2 & e_3 \\ \hline e_4 & e_5 & e_6 \\ \hline \boldsymbol{e_7} & \boldsymbol{e_8} & \boldsymbol{e_9} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline I\otimes Z & Z\otimes I & Z\otimes Z \\ \hline X\otimes I & I\otimes X & X\otimes X \\ \hline -X\otimes Z & -Z\otimes X & -XZ\otimes XZ \\ \hline \end{array}. \tag{3.1.20}$$

Figure 3.3: The assignment, of the following tensor product of 2×2 Pauli matrices, to the edges of $K_{3,3}$ give a perfect quantum strategy for the magic squares game, where the -1 constraint appears once in the relation to the vertex incident with the bold edges $\{7, 8, 9\}$, all other relations are given the 1 constraint.

In the typical magic squares game, Alice receives a vertex, according to some distribution $\mathcal{P}$ of V, and Bob receives an incident edge drawn according to some distribution possibly conditioned on $\mathcal{P}$. Under our the slightly different variant of the game, the classical value will be $\frac{35}{36} \approx 0.97$ (or $\frac{17}{18} \approx 0.94$) as we saw above. Commonly, the classical value given for the magic squares is the worst-case probability of $8/9$ where it is assumed that Alice has received the "bad" vertex and Bob will lose if he obtains the "losing" edge with probability $1/9$.

3.1.4 The CHSH game is a graph-LinBCS game

Consider, the variant of the graph-LinBCS game where Bob receives, instead of a vertex, some adjacent edge $e \sim v$ to Alice's vertex v, drawn according to some distribution on the edge neighbourhoods $E(v)$. In this case, the winning condition is simply $f(e) = g(e)$, and there exist a function satisfying $g : E(u) \to \mathbb{Z}_2$, such that $\sum_{e\in E(u)} g(e) = b(u)$, where u is the other endpoint of e.

Consider the 2-coloured graph, with two vertices v_1 and v_2, and two edges e_2 and e_2

pictured in Figure 3.4. As a linear system game it has the form,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{3.1.21}$$

This graph-LinBCS game has no perfect classical strategy by simple linear algebra, and no perfect quantum strategy[4]. The game is analogous to the CHSH game, the vertices are encoded in the bit sent to Alice, and the edge is encoded by the bit sent to Bob. Their output bits specify the edge colouring of e_1 (the vertex relations enforce the colouring of e_2). The winning predicate is based on the consistency of their colouring, just as in the CHSH game outlined in Subsection 2.5.3.

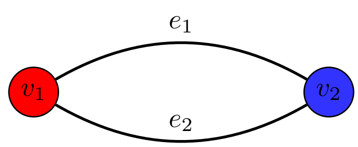

Figure 3.4: The CHSH game as a graph-LinBCS game, an oddly 2-coloured graph, on two vertices with two edges.

The classical value of this graph-LinBCS game is $\omega = 3/4$. However, we know that for the CHSH game, there is a quantum strategy that outperforms the classical one. Therefore, even when connected multi-graphs are planar, if b is not in the range of A, it is still possible to find proper quantum strategies that yield some quantum advantage. Nevertheless, we do not know of any graph-theoretic characterization that gives a quantitative way to measure when this sort of advantage exists.

3.2 The solution group of a graph-LinBCS game

The *solution group* was introduced in [39, 12], and is based on some observations in [13]. The solution group can be associated with any LinBCS game, in particular, they can be defined for graph-LinBCS games. In what follows we assume that G is a loopless graph with $|E| = O(poly(|V|))$.

Definition 3.2.1. Let $G = (V, E)$ be a graph with incidence specified by A, and let $b : V \mapsto \mathbb{Z}_2$ be a non-proper vertex 2-colouring. The *graph incidence group* $\Gamma(G, b)$ associated with

[4]We will see that this fact follows easily from the van Kampen lemma in Subsection 3.3.3.

the graph-LinBCS game pair (G, b), is generated by the set of generators $\{x_e, e \in E\} \cup \{J\}$, and the following relations:

(i) $x_e^2 = 1$ for all $e_i \in E$ and $J^2 = 1$

(ii) $[x_e, J] = 1$ for all $e_i \in E$

(iii) $[x_e, x_e'] = 1$ if there is a vertex v incident to both e and e'

(iv) $\prod_e x_e^{A_{ve}} = J^{b_v}$ for all $v \in V$.

Observe that that the function b only appears in the last relation. Thus, given a graph (with no colouring) we can define the graph incidence group $\Gamma_0(G)$ using the generators $\{x_e, e \in E\}$ and relations (i)-(iii). We say that a relation $r(v) = \prod_{e \in E(v)} x_e J^a$ is odd (resp. even) if a is odd (resp. even). We denote $r(v)^+ = \prod_{e \in E(v)} x_e$ to be the *even* part of the relation r. The group formed by $\mathrm{Inv}\langle E \cup \{J\} : R^+\rangle \cong \mathrm{Inv}\langle E : R^+\rangle \times \mathbb{Z}_2 \cong \Gamma_0(G) \times \mathbb{Z}_2$, and we call $\Gamma_0(G)$ the *homogenous* graph incidence group of G. When depicting relations or isomorphisms of the homogenous graph-incidence group in a figure, we label the vertices black to illustrate that these relationships are independent of the graph colouring.

Definition 3.2.2. A relation $r(v)$ in an graph incidence group is said to be an *anti-commutation* relation, if the relation includes an odd power of J, and is a *commutation* relation, if it does not (which includes when the power of J is even).

Thus, the so-called *odd* relations are equivalently referred to as anti-commuting, and the *even* relations are the same as commuting relations. For example, if a and b are generators, then $ab = ba$ is a commutation relation, and $ab = Jba$ would be an anti-commutation relation. This definition is motivated by the anti-commutator for general algebras. In this case, the J element plays the role of -1. It is not surprising that idea appears intimately in the definition of these groups, as commutation and anti-commutation of observables, arises naturally in the theory of quantum mechanics. We now give an example of a graph incidence group.

Example 3.2.3 (The homogenous graph incidence group $\Gamma_0(K_4)$)**.** To get an idea of the relationship between the graph incidence group and some simple 2-coloured graphs. Let us consider a simple example, the complete graph on 4 vertices. The incidence matrix for the graph K_4 is given by

$$A(K_4) = \begin{array}{c c} & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}. \tag{3.2.1}$$

The corresponding group generated by involutions is the homogenous graph incidence group $\Gamma_0(K_4)$. From Definition 3.2.1 we obtain the group presentation,

$$\Gamma_0(K_4) = \big\langle x_\ell,\ \ell \in \{1,\ldots,6\} : x_1x_4x_3, x_1x_2x_5, x_2x_3x_6, x_4x_5x_6, \tag{3.2.2}$$

$$(x_ix_j)^2 \text{ for all pairs } (i,j) \text{ in } v_k,\ k \in \{1,\ldots,4\}\big\rangle \tag{3.2.3}$$

Figure 3.5 gives the graphical representation of $\Gamma_0(K_4)$, from which we observe that this graph gives rise to an abelian graph incidence group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. As another example of how the simplifying isomorphisms work, see Figure 3.6.

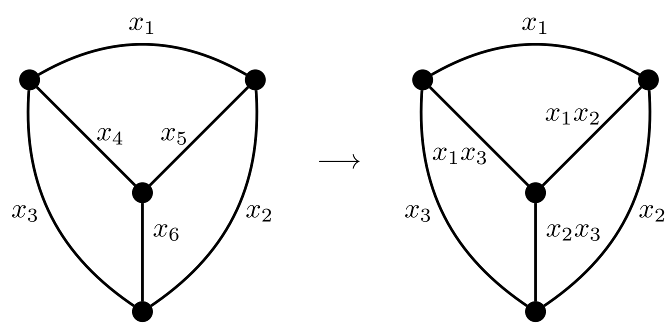

Figure 3.5: Generators of $\Gamma_0(K_4)$ labelling the edges of K_4 , the arrow denotes an arbitrary *simplifying* isomorphism between the graph representation of the group. The local vertex relations on the outer cycle are $\{x_1x_3x_4, x_1x_5x_2, x_2x_6x_3\}$, each generator commutes within the relation, hence we can rewrite $x_1x_3x_4 = 1 \Rightarrow x_1x_3 = x_4$, $x_1x_5x_2 = 1 \Rightarrow x_1x_2 = x_5$, and $x_2x_6x_3 = 1 \Rightarrow x_2x_3 = x_6$. This does not change the group structure; it simply makes inferring the group structure from the graph easier for small graphs.

Given a vertex relation with k-generators, we can rewrite the relation to express one of the generators by the other $k-1$.

$$\prod_{e \in N(v)} x_e = 1 \Rightarrow \prod_{e \in N(v)/\{e'\}} x_e = x_{e'}\,. \tag{3.2.4}$$

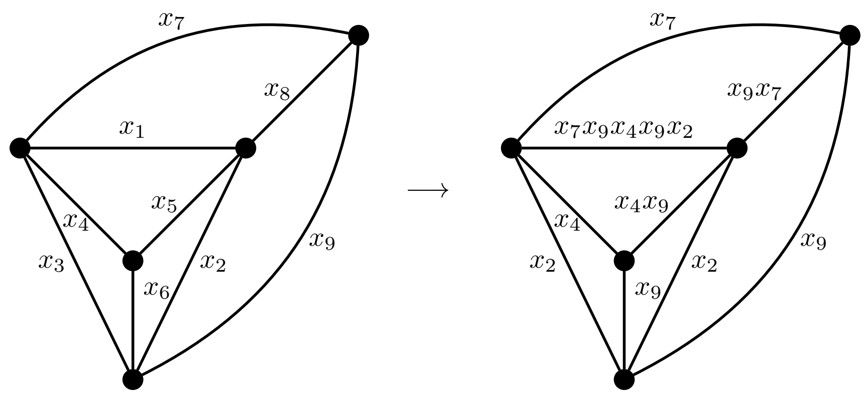

Figure 3.6: The addition of a vertex with three edges to the graph K_4 yields the graph $K_4 \cup \{v\}$. With the aid of a computer, one can deduce an isomorphic presentation of the corresponding graph incidence group on fewer generators. The order of $\Gamma_0(K_4)$ is 16, and it is isomorphic to the direct product of cyclic groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_2^4$, and therefore is abelian.

This generalizes the generator reduction about a vertex that is illustrated in Figure 3.8.

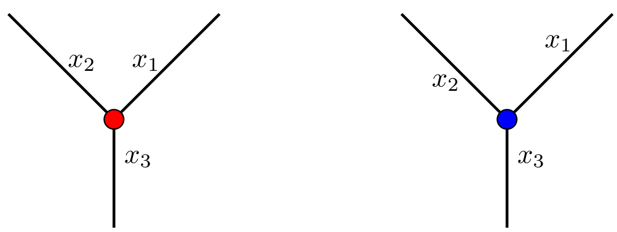

Figure 3.7: The neighbouring edges, read counter-clockwise about the red vertex (left) encode the commutation relation $x_1x_2x_3$, while the relations of the blue vertex (right) encodes an anti-commutation relation $x_1x_2x_3J$.

3.3 Pictures and relations for graph incidence groups

The concept of a *picture* was introduced in [39] to make conclusion about certain relations in the solutions group of LinBCS games. For a hypergraph, a $\mathcal{H}$-picture $\mathcal{P}$ is a planar

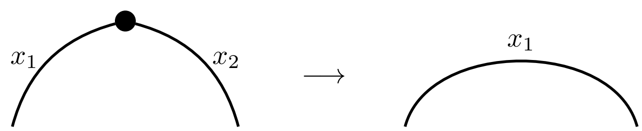

Figure 3.8: Each red vertex of degree 2 translates in the group to a relation on a pair of commuting generating involutions $x_1x_2 = 1 \Rightarrow x_1 = x_2$. Hence, one can simplify the graph incidence group presentation by replacing these vertices with a single edge, and identifying $x_1 \equiv x_2$ in the group.

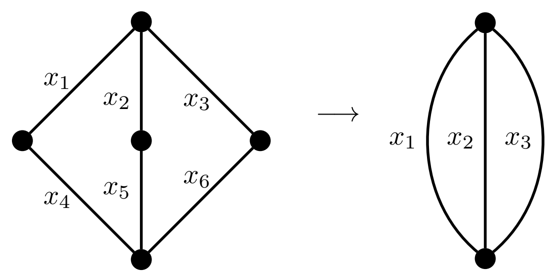

Figure 3.9: By simplifying the degree 2-relations in the left graph, we observe that the graph incidence group of that graph is isomorphic to the graph incidence group generated by the graph on the right.

embedding of $\mathcal{H}$ into the 2D disk. Formally, pictures are dual to van Kampen diagrams [26]. Following the previous definition for hypergraph pictures, we give the following definition for graphs and graph incidence groups.

Definition 3.3.1. A *picture* is denoted by the triple $(V, E, \mathcal{D})$, where

(i) $\mathcal{D}$ is a region, whose boundary consists of a *simple closed curve*[5],

(ii) V is a finite collection of points (or vertices), in $\mathcal{D}$,

(iii) E is a collection of edges (or curves), and

  (a) if the endpoint of a curve is a point p, then the other end connects to either a point or to the boundary

[5]By *curve*, we mean the image of a smooth function γ, from the interval $[0, 1]$, onto the 2D-plane. A curve is *simple* if it only intersects at its endpoints, forming a *closed* curve, otherwise $\gamma(0)$ and $\gamma(1)$ are the endpoints.

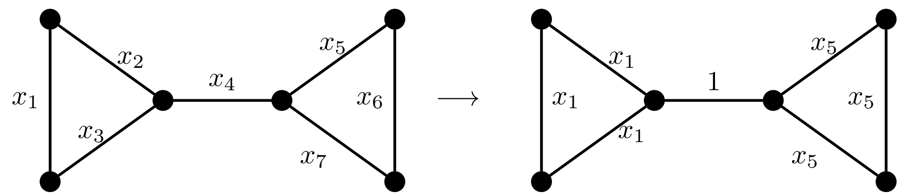

Figure 3.10: The graph incidence group for the minimal connected graph with 2 vertex-disjoint cycles is $\Gamma_0(C_3 \sqcup C_3) \cong \langle x_1, x_5 : x_1^2, x_5^2 \rangle = \mathbb{Z}_2 * \mathbb{Z}_2$. After a simplifying isomorphism, we observe that the graph incidence group is generated by two non-commuting involutions x_1 and x_5.

(b) if one endpoint of a curve is the boundary, then the other must be a point

Curves that share a point are said to be incident, and any curve which has an endpoint along the boundary is said to be incident to the boundary. If there are no curves incident to the boundary, then the picture is said to be *closed*.

3.3.1 Pictures as weak planar covers

Definition 3.3.2. Let G be a graph with incidence specified by $A_{v,e}$, a *G-picture* is a picture $\mathcal{P}$, with a pair of labelling functions given by:

$$\phi_V : V(\mathcal{P}) \to V(G) \qquad \text{and} \qquad \phi_E : E(\mathcal{P}) \to E(G), \tag{3.3.1}$$

such that for all $v \in V(\mathcal{P})$, and $e' \in E(G)$, if we list the edges $e_1, \ldots, e_n$ of $\mathcal{P}$ incident to v with multiplicity then,

$$A_{\phi(v),e'} = |\{1 \leq i \leq n : \phi(e_i) = e'\}|. \tag{3.3.2}$$

Alternatively, a G-picture is a type of *weak* covering of G. Recall, a graph H is *cover* of G, if there is a graph homomorphism $\varphi : H \to G$, such that φ is bijective onto the neighbourhoods of $\varphi(v) \in V(G)$, when restricted to the neighbourhood of $v \in V(H)$. It follows that if φ is a covering, then $|\varphi^{-1}(v)|$ is constant, and called the *fold* number of G. In fact, every planar cover is a *closed* G-picture; however, the converse is not clear and remains the object of further investigation (see the Appendix 5.1.2).

3.3.2 Pictures for groups generated by involutions

Pictures are also particularly useful for diagrammatically representing relations in groups. Let $\Lambda = \mathrm{Inv}\langle S : R\rangle$, then a Λ-*picture* is a picture, where every point corresponds to a relation $r(v) \in R$ such that $e_1, e_2, \ldots, e_k$ is a sequence of curves $e \in \mathcal{P}(E)$ incident to $v \in \mathcal{P}(V)$. Let $s(e_\ell) = x_\ell$, read counterclockwise with multiplicity about a vertex v, the word $s(e_1)s(e_2)\cdots s(e_k)$ is a commutation relation in R^+. Just at the boundary acts a special vertex, the curves incident to the boundary read counterclockwise with multiplicity form relations called cyclic boundary words $\mathrm{bd}(\mathcal{P}) = s(e_1)s(e_2)\cdots s(e_k) \in R^+_{cyc}$. For example, the boundary word in Figure 3.11 is

$$\mathrm{bd}(\mathcal{P}) = s(e_6)s(e_5)s(e_1) \equiv s(e_5)s(e_1)s(e_6) \equiv s(e_1)s(e_6)s(e_5)\,. \tag{3.3.3}$$

If $\mathcal{P}$ is a closed picture, then the boundary equals the empty word, $\mathrm{bd}(\mathcal{P}) = 1$. With these conventions, the define the *sign* of a Λ-picture, to be the parity of the number of vertices in $\mathcal{P}$ whose corresponding relations contain *odd* (anti-commutation) relations. More precisely,

$$\mathrm{sgn}(\mathcal{P}) = \big|\{v \in V(\mathcal{P}) : r(v) \text{ is odd}\}\big| \bmod 2. \tag{3.3.4}$$

3.3.3 The van Kampen lemma

The following lemma attributed to [26] was used by Slofstra in [39] as a geometric tool for investigating relations in graph incidence groups. Given a finitely presented group Λ, generated by involutions, a Λ-picture describes the relations of Λ in the following sense.

Lemma 3.3.3 (van Kampen lemma)**.** Let $\Lambda = \mathrm{Inv}\langle S \cup \{J\} : R\rangle$ be a finitely presented group, let r be a word over S, and let $a \in \mathbb{Z}_2$ (a is even or odd). Then $r = J^a$ if Λ if and only if there is a Λ-picture $\mathcal{P}$ with $\mathrm{bd}(\mathcal{P}) = r$ and $\mathrm{sgn}(\mathcal{P}) = a$.

For more details concerning Lemma 3.3.3 we refer the reader to [39]. Because J is an order-2 central element, it is determined by relations in the graph incidence group Γ. The key idea is that G-pictures are planar diagrams that encode the relations of the graph incidence groups. The following definitions give two important properties of G-pictures. We define them following the work of [39] for hypergraph ($\mathcal{H}$-pictures). The *character*[6] of a G-picture is the vector $\mathrm{ch}(\mathcal{P})_v = |\phi^{-1}(v)| \mod 2 \in \mathbb{Z}_2^n$. With this definition, we can give a more applicable version of the van Kampen lemma for graph incidence groups.

[6]The character of a picture is not to be confused with the character of a representation.

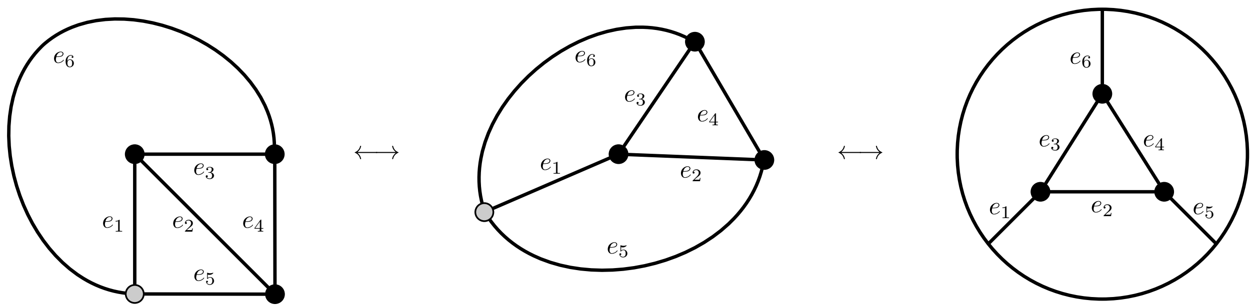

Figure 3.11: Three isotopic versions of the same a picture $\mathcal{P}$. Observe, that $\mathcal{P}$ consisting of 3 points and 5 curves, and how the special point (the grey vertex) is isotopy equivalent to the boundary in the right diagram.

Lemma 3.3.4. (van Kampen lemma for graph incidence groups) Let $\Gamma(G, b)$ be a graph incidence group. Then $x_{e_1} \cdots x_{e_\ell} = J^a$ in $\Gamma(G, b)$ if and only if there is a G-picture $\mathcal{P}$, with $\mathrm{bd}(\mathcal{P}) = e_1 \cdots e_\ell$ and $\mathrm{ch}(\mathcal{P}) \cdot b = a$.

Observe, by Lemma 3.3.4 that if a graph incidence group admits a closed odd-sign picture, then J is trivial. Next, we show how the van Kampen lemma (Lemma 3.3.4) can be used to deduce the existence of quantum strategies for graph-LinBCS games by finding *witnessing* relations of the J element.

Example 3.3.5 (An illustrating application of the van Kampen lemma)**.** Here we repeat the illuminating example given by Slofstra in [39]. Where we apply the van Kampen lemma to a graph with multiple edges and attain the value of J in the graph incidence group. Consider the graph-LinBCS game which arises from the incidence matrix of the graph on 2 vertices with 3 edges between them. $Ax = b$ is realized by

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The corresponding graph incidence group is generated by commuting involutions $\langle x_1, x_2, x_3, J \rangle$, modulo the anti-commutation relation $x_3x_2x_1J$ and the commutation relation $x_1x_2x_3$. Observe that b has odd parity, and the rank of the incidence matrix A is 1. Therefore, the linear system has no solution, and no corresponding perfect deterministic strategy. Through the application of the van Kampen lemma in Figure 3.12 we observe that the game has no perfect quantum strategy (commuting-operator strategy) by observing that J is trivial, by the witnessing picture $\mathcal{P}$. In this particular example we remark that $\mathcal{P} \cong G$. However, this relationship does not appear to hold in general.

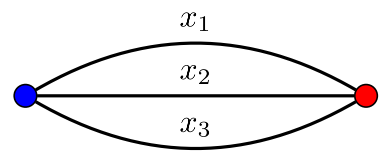

Figure 3.12: In this case, the initial graph with multiple edges is a closed picture $\mathcal{P}$ with $\mathrm{ch}(\mathcal{P}) = (1,1)$, and $b = (1,0)$. So by the van Kampen lemma $\mathrm{ch}(\mathcal{P}) \cdot b = 1$ and we witness the relation $J = 1$.

The van Kampen lemma was used to prove Theorem 1.0.1 [39, 12], establishing the connection between the existence of quantum commuting strategies as a solution group property. Because graph incidence groups are generated by involutions and we generate the group from a graph, we can analyze the weak cover G-picture directly. Typically, one needs to consider the group and then the corresponding picture. However, for graph incidence groups there is a more intimate relationship between the graphs which generate the group, and the relations. We explore these connections in the next section.

3.4 Graph incidence group pictures and Arkhipov's theorem

Recall Arkhipov's theorem 1.0.2 from Chapter 1. We refer the reader to [3] for the proof.

Theorem 3.4.1. Let G be a connected graph, then the graph-LinBCS game $\mathcal{G}(G, b)$ has a perfect quantum strategy and no perfect classical strategy if and only if G is non-planar and b has odd parity.

Interestingly, we can prove the following proposition, which proves one direction of Arkhipov's theorem using the theory of pictures and groups.

Proposition 3.4.2. If G is a planar graph, then the graph-LinBCS game $\mathcal{G}(G, b)$ has no proper quantum strategies.

Proof. If G is planar, then there is a closed Γ-picture $\mathcal{P}$ given by the one-to-one embedding $h^{-1} : G \to \mathcal{P}$. In this case $\mathrm{ch}(\mathcal{P})_v = 1$ for all $v \in V$, and thus $a = |b| \mod 2$. If b is odd, then by the van Kampen lemma (Lemma 3.3.4) we witness a picture of an anti-commutation relation with $J = 1$. So by the CMLS theorem (Theorem 1.0.1) $\mathcal{G}(G, b)$ has a perfect quantum commuting-operator strategy. If b is even, then $\mathcal{G}(G, b)$ has a perfect deterministic strategy by Corollary 3.1.5. □

Unfortunately, we do not know of a way to prove the converse other than by using Arkhipov's proof directly. We give this alternate proof since it demonstrates the connections with the graph incidence group idea.

Proposition 3.4.3. If a graph-LinBCS game $\mathcal{G}(G, b)$ has no proper quantum strategies and b is odd, then G is non-planar.

Proof. If $\mathcal{G}(G, b)$ has no proper quantum strategies, then by Theorem 1.0.1 $J = 1$ in $\Gamma(G, b)$, and by Arkhipov's theorem (G, b) cannot contain the forbidden minors $(\mathcal{F}, b) = \{(K_{3,3}, b^{odd}), (K_5, b^{odd})\}$. Otherwise, there would be $J \neq 1$ preserving pullback $\Gamma(G, b) \leftarrow \Gamma(\mathcal{F}, b)$ giving a contradiction. □

Proposition 3.4.3 is a direct corollary of Arkhipov's theorem, which states that the only games with perfect quantum strategies are those on graphs with $K_{3,3}$ or K_5 as minors and having no perfect classical strategy.

We believe the correspondence of planarity and proper quantum strategies is in the connection between weak graph-coverings and pictures. It is worth noting, that the forbidden graph minors for planarity are minimal elements, in the sense that they could be embedded in the plane by *resolving* a single crossing. Observe, in Figures 3.13 & 3.14 that adding this special boundary vertex allows for a planar embedding.

3.4.1 Graph incidence group pictures for the magic squares game

Let A be the incidence matrix of the graph $K_{3,3}$ with a single blue vertex, the corresponding incidence linear system is

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.4.1}$$

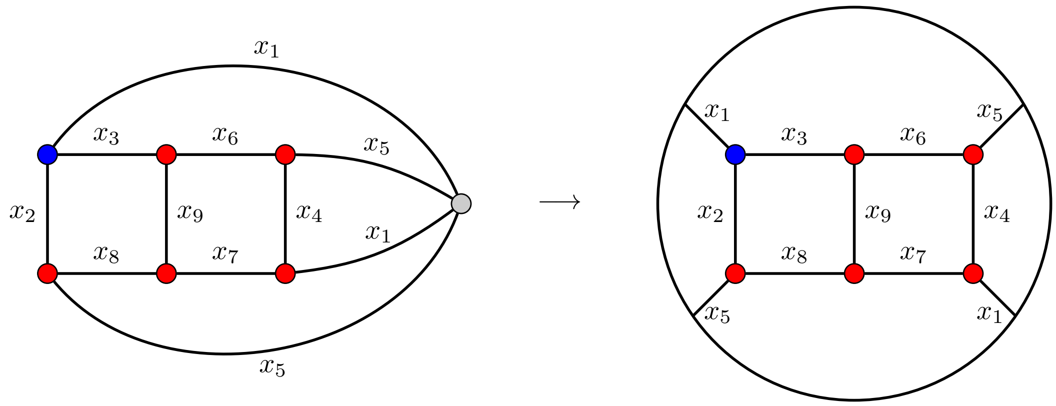

Figure 3.13: An embedding of the oddly 2-coloured $K_{3,3}$ in plane with a special vertex is isotopy equivalent to the $\Gamma(K_{3,3}, b)$-picture with boundary word $\text{bd}(\mathcal{P}) = x_1x_5x_1x_5$.

A graph incidence group of the $K_{3,3}$ graph is given by the linear system in equation (3.4.1), where v_1 is coloured blue (describes an anti-commutation relation), as $\Gamma(K_{3,3}, b)$. In fact, $\Gamma(K_{3,3}, b)$ is an example of a graph incidence group where we obtain a separation between classical and quantum strategies. The graph incidence group of the v_1-distinguished $K_{3,3}$ game has the following presentation,

$$\Gamma(K_{3,3}, b) = \text{Inv}\langle a, b, c, d, e, f, g, h, i, J \; : \; abcJ, def, ghi, adg, beh, cfi\rangle \,. \tag{3.4.2}$$

In the above presentation (3.4.2) we have only listed the vertex-relations. We have omitted the order-2 relations on the generators, the relations that make J central, and the commutation relations amongst pairs of generators that share any vertex relation.

The picture depicted in Figure 3.13 shows that J is non-trivial in $\Gamma(K_{3,3}, b)$. A similar construction can be done for the magic pentagram game, see Figure 3.14. In the next chapter, we explore further connection between graph-minors and graph incidence groups.

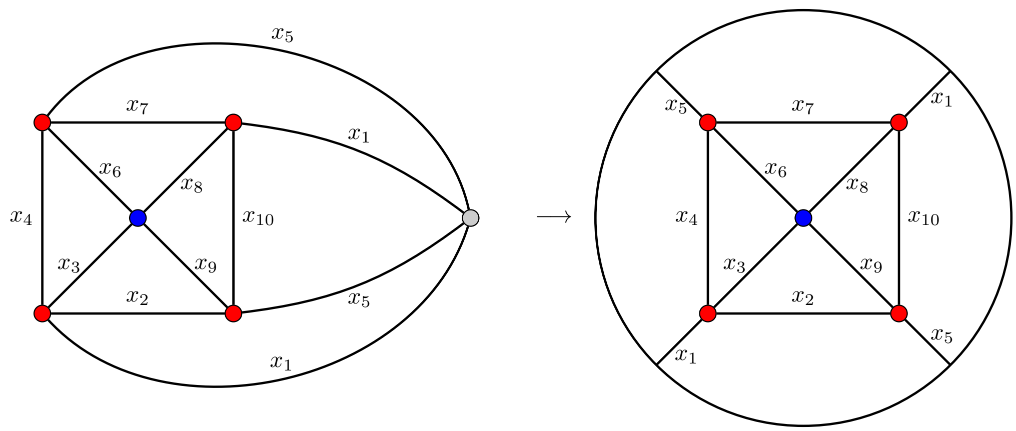

Figure 3.14: An embedding of the oddly 2-coloured K_5 in the plane with a special vertex is isotopy equivalent to the $\Gamma(K_5, b)$-picture with boundary word $\mathrm{bd}(\mathcal{P}) = x_1x_5x_1x_5$.

Chapter 4

Graph minor operations and graph incidence groups

In the first section of this chapter, we describe the graph minor operations for 2-coloured graphs. We then proceed with the proof of the main lemma (Lemma 1.0.3); in which we demonstrate, that each graph minor operation induces a surjective group homomorphism of the corresponding graph incidence groups. We then state the corollary given by the Robertson-Seymour theorem for these incidence-group properties.

In Section 4.3 we derive two graph-minor-closed criteria for graph-LinBCS games. The first characterizes when a graph incidence group is finite, while the second gives the forbidden minors for when the graph incidence group is abelian.

4.1 Graph minor operations for 2-coloured graphs

The following proposition establishes the graph incidence group dependence of perfect deterministic strategies and the parity of b. Recall that we can alternatively describe the parity of a 2-colouring b to be the $\sum_{v\in V} b(v) \bmod 2$.

Proposition 4.1.1. If b and b' are different 2-colourings of a connected graph G of the same parity, then there is an isomorphism $\Gamma(G,b) \cong \Gamma(G,b')$ sending $J_{G,b} \mapsto J_{G,b'}$.

Proof. If b and b' are colourings of the same parity, then $\tilde{b} = b+b'$ is an even parity colouring. If A is the incidence matrix of G then $Ax = \tilde{b}$ has a solution y. The homomorphism $\phi : \Gamma(G,b) \to \Gamma(G,b')$ sending $x_e \mapsto J^{y(v)}x_e$ and $J_{G,b} \mapsto J_{G,b'}$ is the desired isomorphism. $\square$

For 2-coloured graphs we allow the same graph minor operations as in Subsection 2.1.3 with one restriction, while also introducing an additional minor operation. Edge contraction and deletion are both allowed, but vertex deletion is only allowed for vertices with $b(v) = 0$ (even parity constraints). Following the deletion or contraction of a vertex sending $G \to H$, the colouring is updated via the restriction $b \to b|_H$. For edge contraction, if H is the result of contracting an edge e in G with endpoints v_1 and v_2, and v is the vertex identified with v_1 and v_2 in H, then we regard H as the coloured graph (H, b') where $b'(v) = b(v_1) + b(v_2)$ and $b'(w) = b(w)$ for $w \neq v$. The new operation we allow on 2-coloured graphs is illustrated in Figure 4.1, and is called the colour swap minor. Given an edge $e = v_1v_2$, we swap the colours of the incident vertices via the mapping $(\tau(b))(v_i) = b(v_j)$ for $i = 1, 2$. We say that (H, b') is a minor of (G, b) if it is possible to get (H, b') from (G, b) by successively applying these 2-colour graph-minor operations in the sense of (2.1.3).

4.2 Proof of main lemma

In this section, we prove the main lemma (Lemma 1.0.3) introduced at the beginning of Chapter 1. This result gives a correspondence[1] between graph minor operations and surjective homomorphisms between graph incidence groups. To prove Lemma 1.0.3, we show through a series of propositions, that for each 2-coloured graph minor-operation sending $F : (G, b) \to (H, b')$ there is a surjective homomorphism of graph incidence groups $\phi : \Gamma(G, b) \to \Gamma(H, b')$. Unless the colouring is explicitly required in the following sections we may abbreviate $\Gamma(G, b)$ by $\Gamma(G)$ and assume that b is implicit in G.

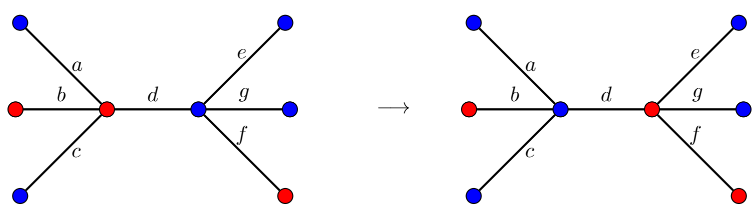

Figure 4.1: Colour swapping of adjacent vertices is our additional allowed minor operation on 2-coloured graphs. This minor operation induces an isomorphism of the underlying graph incidence groups by Proposition 4.1.1.

[1]This connection is, in essence, a functorial relationship between the category of 2-coloured graphs with minor operations and the category of graph incidence groups with surjective group homomorphisms

Proposition 4.2.1. The minor operation of edge deletion $G \mapsto G\backslash e$ is given by the surjective morphism of graph incidence groups $\phi : \Gamma(G) \to \Gamma(G\backslash e)$, with $\mathrm{Ker}(\phi) = \langle x_e \rangle$.

Proof. Consider the image of the graph incidence group under the surjective morphism $\phi(G)$. Observe, that $\phi(\Gamma(G))$ has the same generators as $\Gamma(G\backslash e)$ and every word $w \in \Gamma(G)$ containing x_e is equal to a word $w' \in \Gamma(G\backslash e)$. So, since $w = 1 \cdot w' = w' \cdot 1 = w'$ we are dones. □

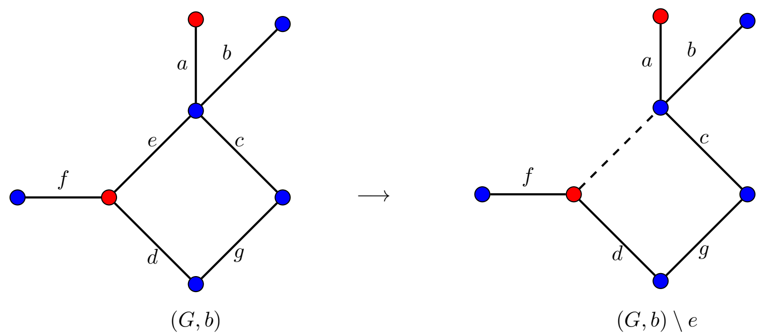

Figure 4.2: Removal of the edge generator $\langle x_e \rangle \in \Gamma(G)$ via the morphism $\phi(x_e) = 1$.

From the surjective morphism for edge deletion we derive the following isomorphism of graph incidence groups:

$$\Gamma(G)/\langle x_e \rangle \cong \Gamma(G\backslash e)\,. \tag{4.2.1}$$

Proposition 4.2.2. The minor operation of isolated vertex deletion $G \mapsto G\backslash\tilde{v}$, is given by the identity morphism $\iota(\Gamma(G)) = \Gamma(G)$.

Proof. Recall that an isolated red vertex corresponds to the trivial relation $r \coloneqq 1$ in the graph incidence group. The map which sends these relations to 1 and has kernel $\langle 1 \rangle$. So it must be the identity up to a permutation of the generators. □

We obtain the following graph incidence group isomorphism for the isolated vertex deletion operation. Let $\tilde{v}$ be an isolated vertex, then

$$\Gamma(G) \cong \Gamma(G\backslash\tilde{v})\,. \tag{4.2.2}$$

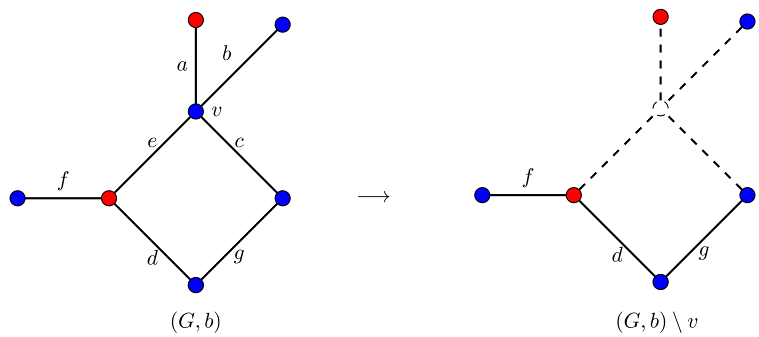

Figure 4.3: Removal of an isolated vertex with trivial relation $r \coloneqq 1$ via the identity morphism denoted by $\iota(1) = 1$.

Proposition 4.2.3. Let R be a normal subgroup relation containing the generator e. The minor operation of edge contraction $G \mapsto G/e$ is given by the surjective morphism $\phi : \Gamma(G) \twoheadrightarrow \Gamma(G/e)$, which sends $\phi : x_e \mapsto r$ and r is one of two relations containing x_e. Furthermore, this map is the identity homomorphism on all other generators.

Proof. Consider the image of the graph incidence group $\phi(\Gamma(G))$. First, observe that the generators of $\Gamma(G/e)$ are mapped, one-to-one, from those in $\phi(\Gamma(G))$. We now show that ϕ sends the relations of $\Gamma(G)$, to those of $\Gamma(G/e)$. We only need to consider the relations $\hat{r} \in \Gamma(G)$, which include e. Choose an arbitrary relation containing x_e, if $\hat{r} = x_e \cdot r$ then $\phi(\hat{r} = \phi(x_e \cdot r) = r^2 = 1$, as each relation consists of commuting involutions. Now because G is a graph $\langle x_e \rangle$ is contained in exactly two relations $\hat{r}$ and $\hat{r}'$. In the image of ϕ, the other vertex relation is $r' \cdot \phi(x_e) = r' \cdot r$. Now, observe that this is the only non-trivial relation of the graph incidence group $\Gamma(G/e)$ that is not found in $\Gamma(G)$. So, ϕ maps the generating set and the relations of $\Gamma(G)$ onto those of $\Gamma(G/e)$. □

From the morphism for vertex contraction we derive the following isomorphism of graph incidence groups,

$$\Gamma(G)/\langle r^2 \rangle \cong \Gamma(G/e)\,. \tag{4.2.3}$$

Proof of lemma 1.0.3. By Propositions 4.2.3, 4.2.2, and 4.2.1 we see that, for every graph minor there is a corresponding surjective homomorphism of graph incidence groups. The isomorphisms described in equations (4.2.1),(4.2.2), and (4.2.3) describe the equivalence of the graph-minor-operations for the graph incidence groups. Since composition of these minor operations and surjective homomorphisms is well defined, the result follows. □

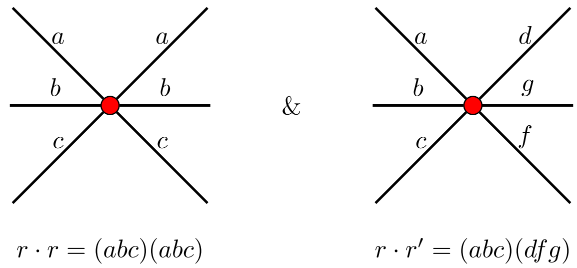

Figure 4.4: The contraction minor creates two relations in the group r^2 and rr'; however, it is easy to see that $r^2 = 1$, and only a single new relation is created in the graph incidence group of the resulting graph.

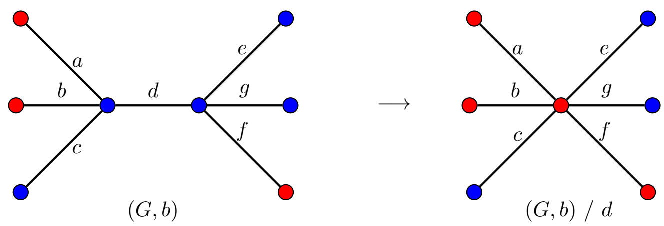

Figure 4.5: Contraction of an edge $\langle x_d \rangle \in \Gamma(G)$ via the morphism $\varphi(x_d) = x_e x_g x_f$. Note the $\mathbb{Z}_2$ addition of the vertex colouring in the contraction is $1 + 1 \bmod 2 \equiv 0$.

By Lemma 1.0.3 we remark that the characterizations of Cleve and Arkhipov are a particular example of this more general correspondence. However, it does not explain why particular graphs arise as minors in the characterization of certain properties.

4.2.1 Graph minors and quotient-closed properties

Through the construction in Section 4.2 we have established a relationship between graph minors and quotients of the graph incidence groups $\Gamma(G)$. Recall that a group property P is *quotient-closed*, if Γ satisfies P, then P is satisfied by every quotient Γ/Ω. Combining the Robertson-Seymour theorem for graph minors and Lemma 1.0.3 we obtain the following important corollary.

Corollary 4.2.4. Every quotient closed property of a graph incidence group can be expressed as a forbidden set of graph minors in the incidence group and vice versa.

In their monumental work on graph minors, Robertson and Seymour discuss a generalization of graph minors to hypergraphs [36]. It is an interesting question of whether one can find appropriate hypergraph minors that are related to solution groups in a similar fashion.

4.3 Incidence groups for graphs without vertex disjoint cycles

From some initial investigations of graph incidence groups, we observed that two vertex-disjoint cycles, for example, seen in the barbell graph of Figure 3.10 have incidence groups with infinite order. By Corollary 4.2.4 we have the following simple proposition.

Proposition 4.3.1. If G contains two vertex-disjoint cycles as a minor, then $\Gamma_0(G)$ is infinite.

Proof. Suppose $\Gamma_0(G)$ is finite, and $\mathcal{C}^{(1)} \sqcup \mathcal{C}^{(2)} = H$ is the minor of G consisting of two vertex-disjoint cycles. Then the mapping $G \to G \backslash H$ removing the 2-disjoint cycles induces some quotient $\Gamma_0(G) \to \Gamma_0(G)/\mathcal{Q}$, and the order of this quotient group is $|\Gamma_0(G)|/|\mathcal{Q}|$ by Lagrange's theorem, but $|\mathcal{Q}| \geq |\Gamma_0(C_2)| = \infty$, which a contradiction, as we assumed $G \neq H$. □

We observe that a necessary condition for graph incidence group finiteness was the exclusion of two cycles in G. Naturally, one could hope that the converse of Proposition 4.3.1 holds, giving a necessary and sufficient characterization of infinite graph incidence groups by the 2 disjoint cycle minor. However, we will see that this is not the case.

Fortunately, the set of graphs that do not contain two vertex-disjoint cycles was characterized by Lovasz in [28]. There are essentially three families of graphs that do not contain two vertex-disjoint cycles: wheel type graphs W_n, tree type graphs $\mathcal{T}_n$, and complete $(3, n)$-partite type graphs $\mathcal{K}_{3,n}$, where the 3 vertex independent set is completed with edges. The set W_n also includes any graph obtained, through graph minor operations, from the wheel graph on n vertices. Another graph without 2 disjoint cycles is given by a tree with an adjoining vertex of degree d, denoted $\mathcal{T}_n \cup \{v_0^d\}$.

An example of a graph in $\mathcal{K}_{3,n}$ is $K_{3,4} \cup \{e_1, e_2, e_3\}$, where the three edges connect the three vertices of the first partition. This graph is shown in Figure 4.6. It also includes,the bipartite variant $K_{3,2} \cup \{e_1, e_2, e_3\} = K_5$ and $K_{3,2} \cup \{e_1, e_2\} = K_5 \setminus e$. More generally, we

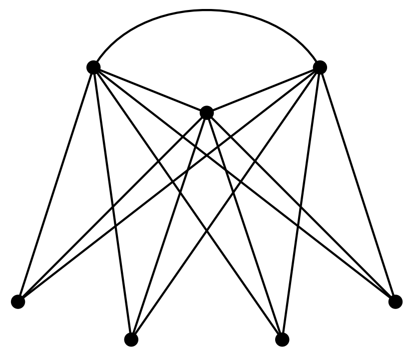

Figure 4.6: $K_{3,4} \sqcup \mathcal{C}$-bipartite graph, with a cycle added to the 3-partition. Observe that edges can only be added to the 3-partition, otherwise one creates 2 disjoint cycles in the graph.

must consider the graph that are comprise of collections (forests) of these W_n or $\mathcal{K}_{3,n}$ type structures with additional acyclic components embedded or attached.

Our strategy will be to consider each class of graphs that do not contain 2 vertex-disjoint cycles, and determined, whether their corresponding incidence groups are finite or not. If we are able to successfully characterize the order of these graph incidence groups, then by Proposition 4.3.1 we will have a full characterization for the graph-minors for finiteness by Lemma 1.0.3. Along the way, we will see that this same approach essentially works for the graph incidence group property of abelianness.

Proposition 4.3.2. If $\Gamma(G, b)$ is the graph incidence group of a 2-coloured graph (G, b), with a distinguished root vertex v_0, then there is an isomorphism of groups $\Gamma(G, b) \cong \Gamma(G, b_{v_0})$, where b_{v_0} is the 2-colouring

$$b_{v_0}(v) = \begin{cases} \sum_{v \in V} b(v) \mod 2, \text{if } v = v_0 \\ 0, \text{otherwise.} \end{cases} \tag{4.3.1}$$

Proof. Recall that by Proposition 4.1.1 there is an isomorphism $\Gamma(G, b) \cong \Gamma(G, b')$. The statement above is the case where $b' = b_{v_0}$. $\square$

Proposition 4.3.3. Let $(\mathcal{T}_n, b)$ be a 2-coloured tree on n vertices, then the graph incidence group $\Gamma(\mathcal{T}_n, b)$ isomorphic to $\langle J^{|b|} : J^2 \rangle$. In either case $\Gamma(\mathcal{T}_n, b)$ is a finite group.

Proof. Root $\mathcal{T}_n$ at some leaf, and denote this vertex by v_0. Consider the set of leaves ℓ of $(\mathcal{T}_n, b_{v_0})$. By Proposition 4.3, it is clear that every $x_\ell = 1 \in \Gamma(\mathcal{T}_n, b_{v_0})$, and so $\Gamma(\mathcal{T}_n, b_{v_0}) \cong \Gamma(\mathcal{T}_n \setminus \ell, b_{v_0})$. Now consider the new set of leaves ℓ'. Again they all describe trivial relations by the same result. Repeating the above argument we observe a chain of group isomorphisms (see an example in Figure 4.7). Since the only non-trivial relation lies at the root vertex, this "pruning" procedure of the leaves can be iterated until one reaches the root vertex at which point we observe that $\Gamma(\mathcal{T}_n, b') \cong \langle J^{|b|} \rangle$. □

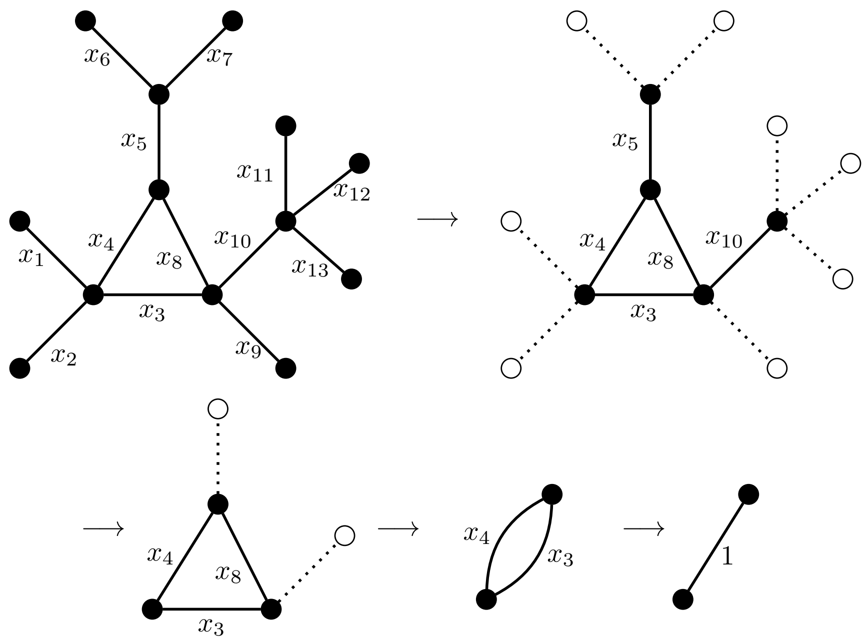

Figure 4.7: A series of simplifying isomorphisms for the incidence group of the *pseudo-tree* on 13 vertices, can be viewed as a recursive leaf decomposition of the branches emitting from the inner 3-vertex cycle. White vertices and dotted edges represent contracted leaves from the *pruning* process. The last two simplifications come from enforcing the *local* vertex relations.

Let $\mathcal{R}_n$ be a rooted cycle with connected acyclic components. By construction, these graphs do not contain any vertex-disjoint cycles, as any additional cycle would have to share the root cycle and thus not be vertex-disjoint.

Proposition 4.3.4. The graph incidence group $\Gamma(\mathcal{R}_n, b)$ is finite.

Proof. We will prove that $\Gamma(\mathcal{R}_n, b_{v_0})$ is a finite quotient of the group $\mathbb{Z}_2^{deg(v_0)}$. Suppose without loss of generality that $deg(v_0) = n$. First observe that we have an isomorphism of groups by replacing every leaf ℓ in $\mathcal{T}_n$ with the edge incident to v_0. Replacing all leaves with the edges incident to v_0 may result in multiple edges from v_0 to the new leaves $\ell' \in \mathcal{T}_n$. To rectify this problem we delete arbitrary edges $\ell'_i v_0$ until there is only one edge $\ell' v_0$. Continue the process of *leaf contraction*, until only edges incident to v_0 remain. At this point we have a set of edges incident to the root vertex. By property (iii) of Definition 3.2.1 the corresponding graph incidence group is abelian, and since the kernel of the edge deletion map is finite we obtain the desired result. $\square$

Figure 4.7 illustrates the *pruning* algorithm described in the proof of proposition 4.3.4. Given a graph that consists of branches coming off a single cycle, the recursive leaf contraction reduces the graph of the homogenous graph incidence group to one for the trivial group. By Lemma 1.0.3, this transformation is a group isomorphism, so every graph incidence group of this graph is the trivial group.

Proposition 4.3.5. The graph incidence group $\Gamma(W_n, b)$ is isomorphic to $\mathbb{Z}_2^n$.

Proof. For any wheel W_n (with n spokes) $|E| = 2n$. There is an isomorphic presentation with $2n$ starting generators of $\Gamma_0(W_n$. For any vertex of degree 3 the relations of an graph incidence group are $e_ie_je_k = 1 \Leftrightarrow e_k = e_ie_j$, and so we can canonically reduce the number of generators to n. Performing the above transformation on all the "spokes", the centre vertex local commutation relations read $[e_ie_{i+1}, e_je_{j+1}] = 1$ for all $1 \leq i, j \leq n$. Since, $e_{i+1}e_ie_{i+1} = e_i$ for any wheel edge, let e_i and e_j be non-adjacent generators on the wheel, now observe that

$$\begin{aligned}
e_ie_j &= (e_{i+1}e_ie_{i+1})e_j && (4.3.2)\\
&= e_{i+1}e_i(e_{i+2}e_{i+1}e_{i+2})e_j && (4.3.3)\\
&= e_{i+1}e_ie_{i+2}e_{i+1}\cdots(e_{j-1}e_{j-2}e_{j-1})e_j && (4.3.4)\\
&= (e_{i+1}e_i)(e_{i+2}e_{i+1})\cdots(e_{j-1}e_{j-2})(e_{j-1}e_j) && (4.3.5)\\
&= (e_{j-1}e_j)\cdots(e_{i+2}e_{i+1})(e_{i+1}e_i) && (4.3.6)\\
&= (e_je_{j-1})\cdots(e_{i+2}e_{i+1})(e_{i+1}e_i) && (4.3.7)\\
&= e_j(e_{j-1}e_{j-1})\cdots(e_{i+2}e_{i+2})(e_{i+1}e_{i+1})e_i && (4.3.8)\\
&= e_je_i\,. && (4.3.9)
\end{aligned}$$

so $[e_i, e_j] = 1$ for every $1 \leq i, j \leq n$, hence $\Gamma(W_n)$ is abelian and has order 2^n. $\square$

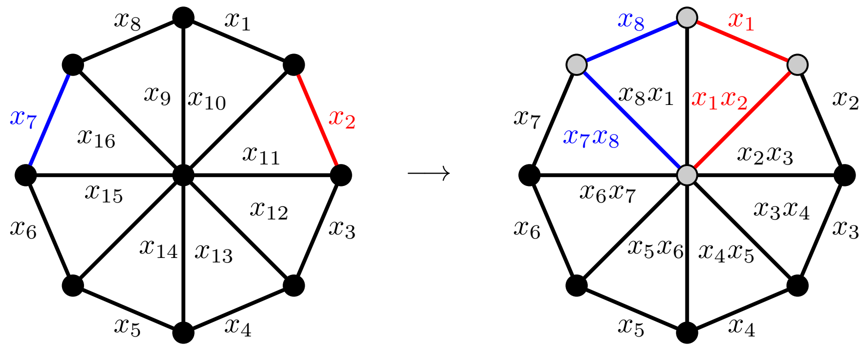

Figure 4.8: The *homogenous* graph incidence group of the wheel graph on 8 vertices has 8 generators after a simple identification of the *spokes* as products of the outer-cycle edge generators. We give an example where we take two non-adjacent edge generators x_7 and x_2, and show they commute. First consider the elements under *conjugation* by their adjacent outside edge generators (we chose the edge in the direction of the other generator), i.e. $x_7 = x_8x_7x_8$ and $x_2 = x_1x_2x_1$. After conjugating by the outside edge generators, the commutation relation $[x_8x_7x_8, x_1x_2x_1] = 1$ makes the result obvious, since the words x_1, x_1x_2 and x_8, x_7x_8 are incident to a set of shared vertices (marked in grey) they commute by definition $x_7x_2 = (x_8x_7x_8)(x_1x_2x_1) = (x_8x_7)(x_8x_1)(x_2x_1) = (x_1x_2)(x_1x_8)(x_7x_8) = (x_1x_2x_1)(x_8x_7x_8) = x_2x_7$.

Figure 4.8 illustrates how the graph incidence group of the wheel graph is isomorphic to the abelian group $\mathbb{Z}_2^8$, by showing that each outside edge generator commutes with any non-adjacent edge generator.

The group corresponding to the oddly-coloured $K_{3,3}$ is the smallest graph incidence group, where J is non-trivial. This group is a quotient of a product of identical dihedral groups. Recall that the dihedral group is given by the semi-direct product, with γ being inversion ($\gamma(a) = a^{-1}$).

$$\mathrm{Dih}_n = \left\langle z_1, z_2 | z_1^2 = z_2^2 = (z_1z_2)^n = 1 \right\rangle \cong \mathbb{Z}_n \rtimes_\gamma \mathbb{Z}_2 \tag{4.3.10}$$

Proposition 4.3.6. The graph incidence group for $\mathcal{G}(K_{3,3}, b)$ with odd b is the quotient of a product of dihedral groups $\Gamma(K_{3,3}, b) \cong (\mathrm{Dih}_4 \times \mathrm{Dih}_4)/\mathcal{Q}_{\mathcal{Z}}$.

Proof. Consider the alternative presentation of $\Gamma(K_{3,3}, b)$ with generating involutions $\langle x_1, x_2, x_4, x_5\rangle$ and commutation relations $\{(x_1x_2)^2 = (x_4x_5)^2 = (x_1x_4)^2 = (x_5x_2)^2 = (x_1x_2x_4x_5)^2 = (x_1x_4x_2x_5)^2 = 1\}$. The dihedral group Dih_4 has presentation $\langle a, b | a^2 = b^4 = 1\rangle$, let

$\langle c, d | c^2 = d^4 = 1\rangle$ denote the presentation of a second copy of Dih_4. Denote by Υ the image of the group $\mathrm{Dih}_4 \times \mathrm{Dih}_4$ under the quotient $\mathcal{Q}_{\mathcal{Z}} = \langle b^2(d^2)^{-1}\rangle$. With the generators $\langle a, b, c, d\rangle$. Υ has relations $\{a^2 = c^2 = d^2b^2 = (d^{-1}b)^2 = (ba)^2 = (dc)^2 = (ac)^2 = adad^{-1} = bcb^{-1}c\}$. Now consider the following map on the generators, $\langle x_1, x_2, x_4, x_5\rangle \mapsto \langle a, c, (cd), (ab)\rangle$. It follows that $x_1x_5 \mapsto b$ and $x_2x_4 \mapsto d$. One can verify that the inverse of this map is well defined and therefore gives the desired isomorphism. It follows that the order of this graph incidence group is 32 $\square$

A particular isomorphism of the graph incidence group $\Upsilon(\Gamma(K_{3,3}, b))$ graph can be seen in Figure 4.9. The incidence group of the 2-coloured K_5 system has a similar structure; in fact, it is isomorphic to a similar quotient of three direct copies of Dih_4. See Figure 4.10 to see the embedded involutions that generate the incidence group of the 2-coloured K_5 systems. Another worthy observation is that the homogenous incidence group $\Gamma_0(K_{3,3})$ is abelian.

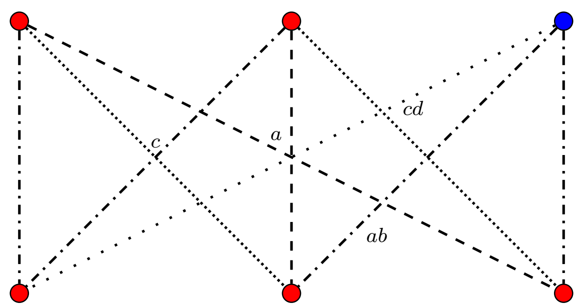

Figure 4.9: Two copies of Dih_4 involutions generating $\Gamma(K_{3,3}, b)$, one generator set is labelled in the dashed edges $\langle a, (ab)|a^2 = (ab)^2 = 1\rangle$, the other two generators are dotted edges $\langle c, (cd)|c^2 = (cd)^2 = 1\rangle$. In this figure the anti-commutation relation is denoted by the blue vertex.

The remaining orders for the graph incidence groups found in Table 4.1 were calculated with Turner Silverthorne using supplementary software. The software makes use of the underlying computer algebra systems GAP [18] and in SageMath [41]. The groups not contained in the above tables are those which contain $K_{3,n}$ as minors, for $n < 6$. These graphs consist of $K_{3,n}$, with the addition of edges connecting the vertices of the 3 vertex independent set. For each n there are 8 such graphs we have computationally verified that the graph incidence groups of these graphs are all finite.

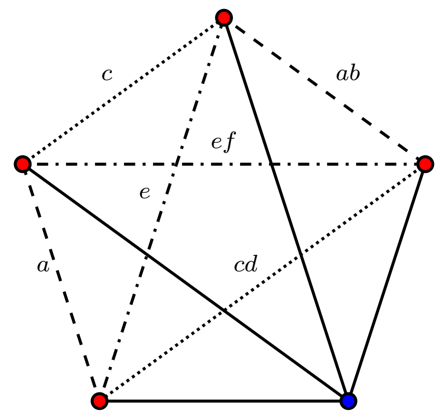

Figure 4.10: The three copies of Dih_4 involutions found in $\Gamma(K_5, b)$, each labelled by dotted, dashed, and dashed-dotted edges. In this figure the anti-commutation relation is denoted by the blue vertex.

4.3.1 Finiteness and abelianness for graph without disjoint cycles

The remaining, possibly infinite graphs are those containing $K_{3,n}$ as a minor, for $n \geq 6$. Evidence that the incidence group $\Gamma(K_{3,6})$ was infinite was given first by the fact that the KBMAG (Knuth-Bendix Algorithm) [22], along with GAP packages failed to compute the order of the group after many hours of computation. By attempting various input reordering's of the group presentation in GAP my collaborator Turner Silverthorne was able to verify that a quotient Λ of $K_{3,6}$ does have infinite order! By our graph minor characterization, this is sufficient to show that $\Gamma_0(K_{3,6})$ is infinite.

Proposition 4.3.7. There is a quotient Λ of $\Gamma(K_{3,6}, b)$ of infinite order. Moreover, this quotient is not contained in $\Gamma(K_{3,5}, b)$.

Proof. Consider the finitely presented group Λ with generators $\langle x_1, x_2, \ldots, x_9\rangle$ and relations $\{x_1x_2x_3 = x_4x_5x_6 = x_7x_8x_9 = (x_1x_4x_7)^2 = (x_2x_5x_8)^2 = (x_3x_6x_9)^2\}$. There is an embedding $\Lambda \hookrightarrow \Gamma(K_{3,6}, b)$ by taking the generators $\langle x_i\rangle$, modulo by their labels $i \equiv 0(\text{mod}9)$, for $1 \leq i \leq 18$. Hence, Λ is a quotient of $\Gamma(K_{3,6}, b)$. Under this mapping one can find a confluent rewriting system for Λ using a computer[2], from which one can certify that the

[2]Particular algorithms are able to detect infinite words in the RWS, in this case, the algorithm outputs that of the group is infinite. The algorithm we implemented in finding rewriting systems and the order of a group was a variant of the Knuth-Bendix completion algorithm known as KBMAG.

Graph	$\lvert\Gamma_0(G)\rvert$	$[\Gamma_0(G,]$	$\lvert\Gamma(G,b)\rvert$	$[\Gamma(G,b)]$
$\mathcal{T}_n$	1	Yes	1	Yes
W_n	2^n	Yes	2^{n+1}	Yes
$K_{3,3}$	16	Yes	32	No
K_5	64	Yes	128	No
$K_{3,4}$	256	No	512	No
$K_{3,5}$	2^{13}	No	2^{14}	No
$K_{3''',5}$	2^{16}	No	2^{17}	No
$K_{3,6}$	∞	No	∞	No

Table 4.1: Finiteness $|\cdot|$ and abelianness $[\cdot]$ properties for the (homogenous) graph incidence groups from a selection of graph families that do not contain two vertex disjoint cycles. When the group is finite we give the order of the group.

quotient has infinite order, it follows that $\Gamma(K_{3,6}, b)$ is infinite. Remark, that this group is not a quotient of the group $\Gamma(K_{3''',5}, b)$, as it does not contain $K_{3,6}$ as a minor. $\square$

4.4 Proof of main theorems

We are now prepared to prove our finiteness criterion for graph incidence groups.

Theorem 4.4.1. Let G be a connected graph. The graph incidence group $\Gamma_0(G)$ is finite if and only if G does not contain the following minors: two disjoint cycles $\mathcal{C}^{(1)} \sqcup \mathcal{C}^{(2)}$ or $K_{3,6}$.

Proof of Theorem 1.0.5. If G contains $\mathcal{C}^{(1)} \sqcup \mathcal{C}^{(2)}$ as a minor, then there is a quotient of $\Gamma_0(G)$ that is isomorphic to the infinite group $\mathbb{Z}_2 * \mathbb{Z}_2$. If G contains $K_{3,6}$, then a quotient of $\Gamma(G)$ isomorphic to the graph incidence group $\Gamma(K_{\widetilde{3,6}})$. In both of these cases it follows that $\Gamma(G)$ is infinite. From this case analysis of the graphs outlined by Lovasz, we are able to conclude that the graph incidence group of every graph not containing $\mathcal{C}^{(1)} \sqcup \mathcal{C}^{(2)}$ or $K_{3,6}$ is finite[3]. $\square$

Theorem 4.4.2. If b is even, then the graph incidence group $\Gamma(G, b)$ is an abelian group if and only if (G, b) does not contain $K_{3,4}$ or two independent vertex-disjoint cycles $\mathcal{C}^{(1)} \sqcup \mathcal{C}^{(2)}$

[3]It is unfortunate that we rely on a computer assisted proof of this fact in the case of $K_{3,6}$, the graph incidence group is infinite.

as minors. If b is odd, then $\Gamma(G, b)$ is an abelian group if and only if (G, b) does not contain any of $K_{3,3}$, K_5, or two independent vertex-disjoint cycles $\mathcal{C}^{(1)} \sqcup \mathcal{C}^{(2)}$ as graph minors.

Proof of Theorem 1.0.6. If G contains $K_{3,4}$ as a minor, by Lemma 1.0.3 $\Gamma(G)_0$ there is a surjective group homomorphism $\Gamma_0(G) \to \Gamma_0(K_{3,4})$, since $\Gamma_0(K_{3,4})$ is non-abelian so must $\Gamma_0(G)$. Similarly, the group $\Gamma_0(\mathcal{C}^{(1)} \sqcup \mathcal{C}^{(2)})$ is non-abelian, and the same argument follows. The other direction follows from our results for the groups $\Gamma_0(G)$ that do not contain 2-vertex disjoint cycles, the groups W_n, $\mathcal{R}_n$, and $\mathcal{T}_n$ are abelian. □

In the next chapter, we will show how these graph minor characterizations can be used to study the space of classical and quantum strategies via the representation theory of these incidence groups.

Chapter 5

Graph incidence group characters and correlations for graph-LinBCS games

In this chapter we investigate the connection between characters[1] of graph incidence groups, and the quantum correlations of graph-LinBCS games introduced in Chapter 3. In particular, we analyze the correlations for games where the graph incidence groups are finite or abelian. These are natural properties to investigate, as we know the forbidden graph minors for these properties, and they are relevant in describing the representations.

We first demonstrate that finite dimensional correlations arising from maximally entangled states correspond to a restriction of the one-dimensional characters of the graph incidence group. While restricted normalized[2] higher-dimensional characters describe correlations from perfect *mixed* strategies, which may be quantum or classical. We then infer a few observations about the space of perfect deterministic and quantum strategies for these graph-LinBCS games from the representation theory.

5.0.1 Correlations and characters supported on observable conjugacy classes

The quantum correlations observed by in a non-local game depend not only on the measurements, but the quantum state they are performed on. We say a correlation is *perfect* if it arises from a perfect (quantum or classical) strategy. Consider a graph-LinBCS game

[1] Here we are referring to characters in the context of unitary representations of the incidence group.

[2] A normalized character is divided by the dimension of the representation.

$\mathcal{G}(G, b)$, and recall from Section 2.6 of Chapter 2 that if $|\psi\rangle$ is maximally entangled, then a perfect correlation in the tensor-product framework is given by sets of projective 2-outcome measurements $\{P_a^x\}_{a=1}^m$ and $\{Q_b^y\}_{b=1}^m$.

$$p(a,b|x,y) = \langle\tau|P_a^x Q_b^y|\tau\rangle \tag{5.0.1}$$

$$= \frac{1}{\sqrt{m}}\sum_{a=1}^m \langle a|\otimes\langle a|(P_a^x \otimes Q_b^y)\frac{1}{\sqrt{m}}\sum_{b=1}^m |b\rangle\otimes|b\rangle \tag{5.0.2}$$

$$= \frac{1}{m}\sum_{a=1}^m\sum_{b=1}^m \langle a|P_a^x|b\rangle\otimes\langle a|Q_b^y|b\rangle \tag{5.0.3}$$

$$= \frac{1}{m}\mathrm{tr}(P_a^x (Q_b^y)^T)\,. \tag{5.0.4}$$

It turns out, that for perfect strategies, Bob must choose his operators to be the *transpose* of Alices. Thus we obtain that these perfect correlations only depend on the elements,

$$= \frac{1}{m}\mathrm{tr}(P_a^x P_b^y)\,. \tag{5.0.5}$$

We now recall the correspondence between binary observables and 2-outcome projective value measures. Under this correspondence, we observe that a perfect correlation is related to the quantity described by the product of two m-dimensional real unitary matrices, via an appropriate linear transformation,

$$\widetilde{\delta} = \frac{1}{m}\mathrm{tr}(\mathcal{O}_x\mathcal{O}_y)\,. \tag{5.0.6}$$

If these unitary matrices form a perfect strategy in the context of Theorem 1.0.1, then they must be unitary representations of the graph incidence group $\Gamma(G, b)$. Since, $\phi : \Gamma(G, b) \to \mathcal{U}(\mathcal{H})$ is a $*$-homomorphism, we observe that

$$\mathcal{O}_x\mathcal{O}_y = \mathcal{O}_{xy} = \phi(xy) = \phi(x)\phi(y)\,. \tag{5.0.7}$$

If the graph incidence group is finite, then any entangled correlation can be thought of as a restriction to a normalized group character, supported on the set of *observable conjugacy classes*. The observable conjugacy classes are those which contain any element whose representations form valid *joint operations* in the game. A joint operation is the resulting transformation of the global quantum state determined by the local operations that Alice and Bobs perform of their half of the shared state. These joint operations are representations of elements in the incidence group, and are determined by the vertex constraints of the game. Given the constraints of a game, one can derive this list of joint operations and therefore determine the set of observable conjugacy classes. Then any character following the restriction to the set of observable conjugacy classes is a possible correlation

For finite groups, the observable conjugacy classes can be enumerated and membership can be determined using a lookup table. Our graph minor criterion implies that this group property can be checked efficiently. From a finite graph-incidence group, computational methods can be used to derive all the *indecomposable* perfect strategies of the game. Additionally, one can deduce the number of perfect (deterministic and quantum) strategies by partitioning the one-dimensional observable characters from the higher dimensional ones. With a description of the incidence group characters and the supporting observable conjugacy classes, one can deduce whether the strategies of higher-dimensional representations are "properly" quantum, by examining if these higher-dimensional strategies are in the probabilistic span of the deterministic characters. With this observation, quantum strategies that can be decomposed into a direct sum of *sub-strategies*, and strategies that cannot be decomposed further form *indecomposable strategies*.

Let δ be a strategy, then by (2.3.3) we have the corresponding statement for strategies,

$$\widetilde{\delta} = \bigoplus_{i \in \mathcal{C}} \widetilde{\delta}_i \tag{5.0.8}$$

where $\widetilde{\delta}_i$ is an indecomposible strategy corresponding to the ith observable conjugacy class $\mathcal{C}_i$ of $\Gamma(G, b)$.

5.1 Observable characters of graph incidence groups

Given the connection between correlations and the representation theory of the graph incidence group. We state several simple facts about the space of entangled correlations for graph-LinBCS games from basic representation theory results.

Recall by Theorem 1.0.1, that a representation of the incidence group gives an operator-solution to a LinBCS game. To see how the operator solution can be transformed into a perfect quantum strategy, we refer the reader to [12]. It is a standard fact in representation theory that $\Gamma(G, b)$ is abelian if and only if all its irreducible representations are one dimensional. Similarly, if $\Gamma(G, b)$ is finite, then all its irreducible unitary representations are finite-dimensional, and consequently, any indecomposable strategy of a finite incidence group is finite dimensional.

Proposition 5.1.1. If $\Gamma(G, b)$ is abelian, then all the indecomposable perfect strategies for $\mathcal{G}(G, b)$ are deterministic.

Proof. If $\Gamma(G, b)$ is abelian, then every representation is a direct sum of 1-dimensional irreducible representations. These irreducible representations correspond to deterministic strategies, through the restriction to the observable conjugacy classes. $\square$

Proposition 5.1.2. If $\Gamma(G, b)$ is non-abelian, and b has even parity, then there are both perfect deterministic and higher-dimensional indecomposable perfect strategies.

Proof. To have an indecomposable perfect higher-dimensional strategy then the incidence group $\Gamma(G, b)$ must have an irreducible representation π of dimension greater than one, such that $\pi(J) \neq \mathbb{1}$. Because $\Gamma(G, b)$ is non-abelian, then it must have some irreducible representation ϕ greater than one. Observe, that if $\phi(J) \neq \mathbb{1}$ then we are done. So suppose $\phi(J) = \mathbb{1}$, we will show that we can always construct a new representation, $\varphi(J) = -\mathbb{1}$ yielding a higher-dimensional perfect strategy. Since b has even parity, then $\Gamma(G, b)$ must also have a one-dimensional irreducible representation ψ with $J \neq 1$. Since J is central and order 2 we can tensor this one-dimensional representation ψ with our higher-dimensional representation ϕ to the obtain a representation $\phi \otimes \psi = \varphi$ of the same dimension as ϕ, but now $\varphi(J) = -\mathbb{1}$. $\square$

Proposition 5.1.3. If $\Gamma(G, b)$ is finite, then $\mathcal{G}(G, b)$ has no perfect infinite-dimensional commuting-operator strategy.

Proof. Towards contradiction, suppose $\mathcal{G}(G, b)$ has an infinite-dimensional commuting-operator strategy. Then the corresponding linear system (A, b) has an infinite-dimensional operator-solution. Since $\Gamma(G, b)$ is finite, any corresponding infinite-dimensional operator-solution is necessarily an infinite direct sum of finite dimensional irreducible representations. Thus by part (ii) of the CMLS theorem (Theorem 1.0.1) each finite dimensional indecomposable component of the operator solution has a corresponding perfect tensor-product strategy. Thus the commuting-operator strategy is not a proper infinite dimensional commuting operator strategy, as it is a direct sum of tensor-product strategies. $\square$

Notice that Propositions 5.1.3, 5.1.2 and 5.1.1 are all characterized by forbidden graph minors by Theorem 1.0.5 and Theorem 1.0.6. Thus, both properties can be easily detected in the graph-LinBCS game data using efficient algorithms on the graphs.

5.1.1 Character tables for dihedral groups

We have seen many examples where the incidence group is obtained from a quotient of a direct product of dihedral groups, such as in the case of $K_{3,3}$ and K_5. We examine its character table along with the structure of its irreducible representations in the context of strategies. The character matrix of Dih_4 is given as follows

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 2 & 0 & 0 & 0 & -2 \end{pmatrix} \tag{5.1.1}$$

Observe that the group has 4 linear representations μ_i and one 2-dimensional representation, depicted by the rows of the matrix. The corresponding candidate quantum correlation is given by the observable conjugacy class vector $\nu = (1, 0, 0, 0, -1)$. Whether this correlation can be achieved by shared randomness is a question of whether ν is a probabilistic span of the μ_i's. Note, however, that the 5th entry of each character is 1 while the corresponding entry in ν is -1. Since there is no way a probabilistic sum of positive numbers can be negative we observe that ν is not in the positive span. That being said, it is not clear that the dihedral groups correspond directly to a particular graph incidence group. However, it does suggest why these graph-LinBCS games do admit proper quantum strategies.

5.1.2 The correlations for Dih_∞

Given a graph consisting of two vertex-disjoint cycles we saw how the corresponding incidence group was $\Gamma_0(\mathcal{C} \sqcup \mathcal{C}) \cong \mathbb{Z}_2 * \mathbb{Z}_2$. This graph incidence group is infinite and non-abelian; however, it is also isomorphic to the infinite dihedral group Dih_∞. Using the representation theory of this group we can make some conclusions about the correlations corresponding to the $\mathcal{C} \sqcup \mathcal{C}$-LinBCS games.

The group Dih_∞ is generated by s (the reflection) and r_α (the rotation by α). It is well-known that the infinite dihedral group has a countably infinite family of 2-dimensional irreducible representations, see chapter 5 of [38]. The correlations that come from these representation can be parametrized by $q_\alpha = (1, \cos(\alpha), 0, 0)$. Where the corresponding observable conjugacy classes are $(1, r_\alpha, s, sr_\alpha)$. A corresponding unitary representation is given by

$$\varphi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \varphi(r_\alpha) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \varphi(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \varphi(sr_\alpha) = \begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix}. \tag{5.1.2}$$

Recall the character matrix for $\mathbb{Z}_2 \times \mathbb{Z}_2$ gives us a basis for the deterministic strategies.

$$\mathcal{D} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \tag{5.1.3}$$

Determining if the correlation is classical is equivalent to showing that for all $-\pi \leq \alpha \leq \pi$, there is a non-negative solution x to $\mathcal{D}x = q_\alpha$.'

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & \cos(\alpha) & 0 & 0 \end{pmatrix} \tag{5.1.4}$$

Using Cramer's rule one can show that

$$x = \frac{1}{4}\big(1 + \cos(\alpha), 1 - \cos(\alpha), 1 - \cos(\alpha), 1 + \cos(\alpha)\big) \tag{5.1.5}$$

is a non-negative solution for all α, since $-1 \leq \cos(\alpha) \leq 1$. Hence, this higher order correlation can be observed by a probabilistic combination of deterministic strategies, so it is a classical correlation.

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APPENDICES

A Implementing incidence groups in SAGE

Below we describe an implementation of a short algorithm that converts a 2-coloured graph (or hypergraph) into the corresponding finitely presented incidence (solution) group. The implementation is given in the freely available open-source computer algebra system SAGE.

Given an object `G` from the class of `graphs` (or `hypergraphs`) one can construct the incidence matrix `A` via `A=G.incidence_matrix()` which returns an object in `sage.matrix.matrix_integer_sparse`. Given a graph the function `mp_group()` outputs the corresponding incidence group. The options for this function include the parity of a 2-colouring `c` which labels vertices $\{0,1\}$, the default is an even, the flag `h` denotes whether to include the J generator. The default (not including J) gives the homogenous incidence group $\Gamma_0(G)$. The flag `p` is a binary option of outputting the group as a permutation group `sage.groups.per_gps`. The group is by default output as a `sage.groups.finitely_presented` (finitely presented group). The final option `s` indicates whether to output the group after attempting to find a simplifying isomorphism which ideally reduces the number of required generators to express the group. Working with an isomorphic copy on fewer generators may help when attempting to calculate the properties of the group. Many algorithms, especially those using GAP seem to perform better on permutation groups over the finitely presented ones. Since, the groups can be very large, determining the simple properties of these groups can be computationally intensive.

A.1 Code for the `mp_group()` function

```
# SAGE code for generating incidence groups from graphs

import sage.all
```

```
# using the SAGE graph database you can easily construct
# the incidence matrix for your favourite graph

# define your favourite graph G, some examples

# G=graphs.CompleteBipartiteGraph(3,4)
# G=graphs.Cubegraph(3)
# G=graphs.Completegraph(5)
# G=graphs.BarbellGraph(3,0)

# options: c='coloured' p='return as permutation group'
# s='return simplified group', h='homogenous group', a='input incidence matrix'
# we have defined the flags to their default setting

c=true
h=false
p=false
s=false
a=false

# given a graph and the above flags the function returns the incidence group
# corresponding to the graph

def mp_group(G,*args):

# compute the incidence matrix (0,1)-matrix A of the graph G or
# hypergraph, if a=true, the input was an incidence matrix
    if a==false:
        A=G.incidence_matrix()
    else:
        A=G

# introduce convenient parameters for the number of rows/columns
    n=A.nrows()
    m=A.ncols()

# for every edge define a list Inv of generators including J
```

```
    Gens=[var('x_%d' % i) for i in range(m)]
    if h==false:
        Gens.append(var('J'))

# generate the free group on the generators
    F=FreeGroup(Gens)

# create a list of the generators who share a vertex

VtxRels=[[i+1 for i in range(len(r)) if list(r)[i]==1] for r in list(A.rows())]

# if c=true insert the J relation into VtxRels
    if h==false and c==true:
        VtxRels[0].append(m+1)

# generate a list of product relations
    ProdRels=[F(p) for p in VtxRels]

# create a list of commuting relations this list also contains the
# involution relations
    ComRels=[]
    for k in VtxRels:
        for i in k:
          for j in k:
             ComRels.append(F([i,j,-i,-j]))

# involution relations
    for i in range(m+1):
        ComRels.append(F([i+1,i+1]))

# append J^2 relation if required
    if h==false:
        for i in range(m):
            ComRels.append(F([i+1,m+1,-i-1,-m-1]))

# combine lists of relations
    Rels=ComRels+ProdRels
```

```
# remove any trivial relations
    while F.one() in Rels:
        Rels.remove(F.one())

# create the group via quotient by Rels
    K=F.quotient(Rels)

# consider a simplified group by computing a
# simplifying isomorphism (not unique)
    L=K.simplified()

# check for "simplification" return options s
    if s==true:
        MPGrp=L
    else:
        MPGrp=K

# check for "permutation group" return options
    if p==true:
        print "incidence group as permutation group"
        return MPGrp.as_permutation_group()
    else:
        print "incidence group as finitely presented group"
        return MPGrp

# Find any info you want from the simplified FPG

# L.structure_description()
# L.order()
# L.abelian_invariants()
# L.center()
# L.conjugacy_class_representatives()

# R=L.rewriting_system()
# R.is_confluent()
# R.make_confluent()

# as a permutation group you can get more info about reps
```

A.2 Finding rewriting systems and computing the value of J

Given a group, SAGE provides simple functions for attempting to solve the word problem. We provide a short example.

```
sage: G=DihedralGroup(8)
sage: G
(Dihedral group of order 16 as a permutation group,)
sage: F=G.as_finitely_presented_group(); F
Finitely presented group < a, b | b^2, (b*a^-1)^2, a^8 >
sage: R=F.rewriting_system();R
Rewriting system of Finitely presented group < a, b | b^2, (b*a^-1)^2, a^8 >
with rules:
    b^2    --->    1
    (b*a^-1)^2    --->    1
    a^8    --->    1
sage: R.is_confluent()
False
sage: R.make_confluent(); R
Rewriting system of Finitely presented group < a, b | b^2, (b*a^-1)^2, a^8 >
with rules:
    b^-1    --->    b
    b*a^-1    --->    a*b
    b*a    --->    a^-1*b
    b^2    --->    1
    a^-4    --->    a^4
    a^5    --->    a^-3
```

In particular, when looking to solve the word problem using a computer algebra system such as SAGE, we can use the `make_confluent()` function, which may not terminate.

A.3 Finding deterministic strategies for incidence groups in SAGE

One can also investigate the space of deterministic strategies using SAGE. Given the incidence matrix of a graph, we can generate the matrix space over $\mathbb{Z}_2$ with the following code. In this example, we look at the graph for K_4.

```
# generate a graph from the SAGE library
G=graphs.CompleteGraph(4)

# compute its incidence matrix
A=G.incidence_matrix()

# create the matrix space
V=MatrixSpace(IntegerModRing(2), A.nrows(), A.ncols())

Full MatrixSpace of 4 by 6 dense matrices over Ring of integers modulo 2

# re-format the incidence matrix as an element of the matrix space
A=V(G.incidence_matrix())

# generate the column space of the incidence matrix
A.column_space()

Vector space of degree 4 and dimension 3 over Ring of integers modulo 2
Basis matrix:
[1 0 0 1]
[0 1 0 1]
[0 0 1 1]

# remark the span of these vectors contains all even vectors

# you can even look at the full image of A
A.image()

Vector space of degree 6 and dimension 3 over Ring of integers modulo 2
Basis matrix:
[1 0 0 1 1 0]
[0 1 0 1 0 1]
[0 0 1 0 1 1]

# remark that the rank of A gives the number of classical strategies
```

The information here coincides with the representation theory of the graph incidence group. We saw earlier that the group $\Gamma_0(K_4)$ is isomorphic to the product of three cyclic groups $\mathbb{Z}_2^3$ and the rank of this group is 3. Furthermore, it contains three one-dimensional representations.

B A toolbox for testing higher dimensional correlations

Suppose we have a game that admits a perfect deterministic strategy, but the corresponding incidence group is non-abelian. In this case, the representation theory states that there are a series of observable one-dimensional representations that coincide with these deterministic strategies. However, whenever a group is non-abelian, it must also admit some irreducible representation of dimension greater than one. This higher-dimensional irreducible representation corresponds to a perfect operator strategy on some higher dimensional space. The question is, are these higher dimensional representations really quantum strategies, or are the correlations they describe also captured by probabilistic combinations of deterministic ones. In the case of finite incidence groups, we can answer this question using character theory. We do not yet have a toolset for dealing with arbitrary infinite-dimensional incidence groups. However, in cases where the character theory of these infinite groups is nice perhaps one can say something.

The canonical example here is $K_{3,4}$ where the corresponding homogenous graph incidence group $\Gamma_0(K_{3,4})$ is non-abelian. Furthermore, this incidence group has order 264, and the number of abelian invariants is 64 (which is also the number of one-dimensional representations irreducible representations). By deduction, there are 12 remaining 4 dimensional irreducible representations to check. This fact can be checked with the following SAGE computations.

```
# setting the homogenous flag to true, ensures the mp_group() function
returns the group without the J element
sage: h=true
sage: H=mp_group(G,h)
incidence group as finitely presented group
sage: H.order()
256
sage: H.abelian_invariants()
(2, 2, 2, 2, 2, 2)
sage: CC=H.conjugacy_classes()
```

```
sage: len(CC)
76
sage: P=H.as_permutation_group()
sage: T=P.character_table()
76 x 76 dense matrix over Cyclotomic Field of order 1 and degree 1
(use the '.str()' method to see the entries)
sage: T.str()
```

The process of recovering the observable conjugacy classes can be obtained by simply searching each conjugacy class for any word in the list of generated or observable elements. The observable elements depend on the game, and in particular the degrees of the vertices. Once found, the unobservable conjugacy classes can be removed from the character table, leaving us with only the correlations possible from the particular game. We call this the *restricted* character table.

We then construct the square matrix consisting of all the deterministic correlations (one-dimensional characters) and the observable conjugacy classes. Then, for each candidate quantum correlation, we check if the solution to the linear system $Cx = q$ is solved by a vector with all non-negative entries. If a positive solution is found, it implies that the candidate correlation lies in the probabilistic span of the deterministic correlations and is therefore classical. Based on our findings we present the following conjecture.

Conjecture B.1. If a graph-LinBCS game has a deterministic strategy, then the correlations from any higher dimensional strategy can be observed by a probabilistic mixture of deterministic strategies.

Evidence for this conjecture is given by the fact that it holds for all finite graph-LinBCS groups. For the infinite dihedral group, since its corresponding graph is planar we know that it has no proper quantum strategies; however, this does not rule out the possibility of higher dimensional correlations. In fact, because the group is not abelian we know that there is at least one higher-dimensional irreducible representation.

C Future work & open questions

Here we list some possible avenues for future study and open problems.

C.1 Minors for other quotient properties

Corollary 1.0.4 says that every quotient-closed property of a graph incidence group has a set of forbidden graph minors. What are the forbidden graph minors for the properties of solvability, simplicity, and amenability?

C.2 Planar covers and Negami's conjecture

The notion of projective embedding and finite covers is the subject of the famous conjecture of Negami [32], see [21] for the current state of the problem. Negami conjectured in 1988 that projective embeddings and finite planar covers are one and the same. In 1986 Negami proved a weaker version of the conjecture for regular coverings. A covering is *regular* if there is a subgroup of $\mathrm{Aut}(H)$ such that $\varphi(u) = \varphi(v)$ if and only if $\sigma(u) = v$ for some automorphism $\sigma \in \mathrm{Aut}(H)$. Nemagi's conjecture is that one can drop the regularity condition from the above theorem. One can immediately see that showing every graph with a cover has a regular cover is not easy since H may not have any automorphisms.

If $\phi : \mathcal{P} \to G$ has the property that for any edge $uv \in \mathcal{P}$ then $h(u) \neq h(v)$, the $\mathcal{P}$ is a planar cover of G. And if Negami's conjecture holds, then $\mathcal{P}$ can be embedded in the projective plane. What is interesting is that the pictures with boundary words describing J in $\Gamma(K_{3,3}, b)$ and $\Gamma(K_5, b)$ have only one crossing.

In [2] it was shown that if G is a planar cover of a non-planar graph H, then the *fold number* is even. A sort of converse to this theorem would be desirable for the theory of pictures for solution groups. Every planar cover of a graph is clearly a picture, however, the converse is believed to be false in general, it would seem natural to prove some sort of result characterizing when pictures had to be covers.

Proposition C.1. If $\mathcal{P}$ is a planar cover of G then, either

$$\mathrm{ch}(\mathcal{P})_v = \begin{cases} 1 \\ 0 \end{cases}, \text{ for all } v \in V\,. \tag{C.1}$$

Proof. Follows directly from the fact that every cover is a closed G-picture of G and $\mathrm{ch}(\mathcal{P})_v$ is the parity of the $|\phi^{-1}(v)|$. □

Is there an angle on Negami's conjecture here, or using this machinery to prove Arkhipov's theorem with pictures? In particular, can we prove Proposition 3.4.3 without Arkhipov's construction?

C.3 Quantum advantage for oddly-2-coloured planar graphs

When the colouring is odd and G is connected and planar we know that there are no classical nor quantum perfect strategies to $\mathcal{G}(G, b)$. However, for a given distribution on the inputs, we can calculate the classical value of the game. In this case, we know by embedding the CHSH game into a graph-LinBCS game that there are quantum strategies that provide some advantage over the optimal classical strategy, in particular, there are graph-LinBCS games for which $\omega < \omega^* < 1$. It would be desirable to relate the quantum advantage $\omega^* - \omega$ to some parameter of the pair (G, b). Perhaps for graph-LinBCS games, there is an efficient semi-definite program for computing the entangled value ω^* of these games?

C.4 Other planarity criteria for graphs

Recall that a simple graph is not-planar if $|E| > 3n - 6$, given that $\omega = 1$ when G is non-planar, it is natural to conjecture that perhaps the higher the connectivity $\kappa(G)$ of the graph the larger the value of ω. There are also other abstract characterizations of graph planarity, such as MacLane's planarity criterion and the Hanani-Tutte theorem, perhaps these are related to the value of ω and ω^*?

C.5 Graph-LinBCS games and MIP*

The complexity class MIP* of multi-prover interactive proofs (MIP), where provers[3] can share arbitrary entanglement is closely related to non-local games. It is a famous result in complexity theory due to Babai, Fortnow, and Lund that MIP=NEXP (non-deterministic exponential time). However, in the case of MIP* with the additional entanglement resource, there is evidence to suggest that NEXP$\subsetneq$MIP*. Given the efficiency at which classical values and potentially quantum values can be found for graph-LinBCS games, perhaps there are consequences of our results in the context of complexity theory.

[3] In the case of a non-local game the players are the provers.