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Order of Authors:	Jose Manuel Diego-Mantecón, PhD
	Teresa F. Blanco, PhD
	José Benito Búa Ares, PhD
	Pablo González Sequeiros, PhD
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Graphical Abstract

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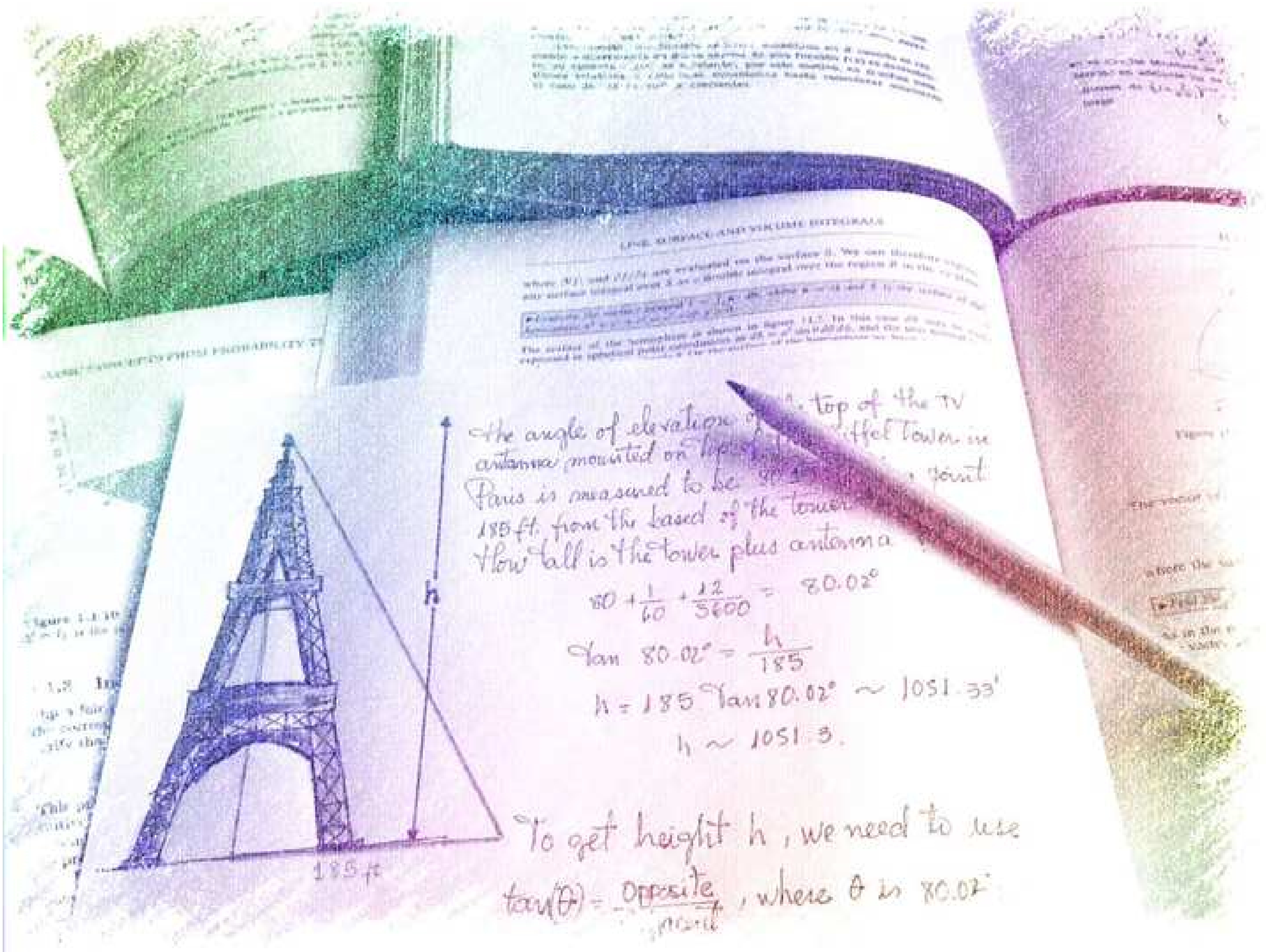

Is the relationship between art and mathematics addressed thoroughly in Spanish secondary school textbooks?

Jose Manuel Diego-Mantecón

Universidad de Cantabria, MATESCO, Facultad de Ciencias

Avda. de los Castros 48, 39005 Santander (Spain) - diegojm@unican.es

Teresa F. Blanco

Universidade de Santiago de Compostela, Departamento de Didácticas Aplicadas, Facultade de Ciencias da Educación

Avda. Xoan XXIII s/n, 15782 Santiago de Compostela (Spain) - teref.blanco@usc.es

José Benito Búa Ares

Xunta de Galicia, I.E.S. Sánchez Cantón

Avda. Raíña Vitoria s/n, 36001 Pontevedra (Spain) - jbenitobua@gmail.com

Pablo González Sequeiros

Universidade de Santiago de Compostela, Departamento de Didácticas Aplicadas, Facultade de Formación do Profesorado

Avda. Ramón Ferreiro s/n, 27001 Lugo (Spain) - pablo.gonzalez.sequeiros@usc.es

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Abstract

This article discusses a review of the relationship between art and mathematics in Spanish secondary school mathematics textbooks. The art-mathematics connection identified in the textbooks was analyzed under six dimensions: (1) art for ornamental purposes; (2) art in calculation and measurement; (3) art to master concepts; (4) art to use technological resources in mathematics; (5) mathematical analysis of art; and (6) creating art with mathematics. Dimensions 1, 2 and 3 clearly prevailed over dimensions 4, 5 and 6, which called for more active participation and analytical reflection. Most of the activities attempted to *illustrate* the mathematics-art connection with real-world examples, but rarely entailed verifying a hypothesis or assumption nor did they encourage critical thinking for analyzing and creating art with mathematical or technological tools.

Keywords: art, mathematics, textbooks, secondary education, curriculum

1. Introduction

Art and mathematics, two expressions of human creativity, share many distinctive traits, such as the friction between creation and technique, intuition and rigour, certainty and uncertainty. The relationship between the two disciplines is complex and can be broached from at least two perspectives [5, 11]. For the artist, artistic creation prevails over mathematics, being considered little more than a tool to generate forms and constructions. It often supplies geometric concepts and procedures, which, given their properties, are technically useful. The artistic creations in which mathematics is used are closely related to mathematical notions and properties. For the mathematician, whilst a mathematical notion may on occasion give rise to something that could be regarded as an artistic creation, that outcome is no more than a consequence of the mathematical notion [6, 9].

Its ‘beauty’ might be said to lie in the mathematical notion itself, the embodiment and sensory perceptibility of which are secondary.

As a result of European recommendations for educators, the relationship between art and mathematics has steadily acquired greater importance nowadays [7, 27]. The question is whether that relationship has found its way onto real world curricula. On the grounds of reference frameworks for Spanish curricula, the answer would appear to be affirmative [20, 26, 31]. Nonetheless, curricular trends and guidelines do not always translate into classroom practice [30]. In fact, the discordance between the intended and implemented mathematics curriculum has been widely reported across different educational contexts [21, 32]. Although there is a large number of studies addressing the relationship between art and mathematics from both a research and educational perspective, see, for example, [6, 18, 28, 29], as far as we know, there is a lack of studies focusing on how this art-mathematics relationship is established in the school lessons.

In the Spanish context, school lessons are strongly influenced by textbooks. These are basic resources for teachers, not only as instructional support, but also as a substantive component in lesson planning [1, 14, 19, 24, 33]. Textbooks may therefore be used as an indicator of the content addressed in the classroom. An analysis of the relationship between art and mathematics may be conducted under that premise, seeking to ascertain the extent to which the interaction between them is addressed in the classroom. This article broaches the question by specifically analyzing the presence of art in secondary school mathematics textbooks in Spain. For this purpose, what we mean by art is limited to visual arts and music. This approach is justified from a curricular point of view and excludes literature because of its independence from the other two disciplines, in the Spanish scholar itineraries.

2. Today's methodological and curricular trends

The European Union's recommendations for increasing the number of students enrolling scientific-technological subjects encourage the implementation of STEAM activities [10]. These activities integrate all scientific areas (Science and Mathematics) and their most prominent applications in modern society (Engineering, Technology and Art) [8, 12]. Interdisciplinarity plays an essential role in these activities, deliberately eschewing a traditional educational approach in which knowledge appears to be divided into unconnected subjects [16, 25].

The priorities defined by the EU have had visible consequences for Spain's official compulsory secondary education (Spanish abbreviation, ESO) and baccalaureate curricula [20]. The new approaches to learning and assessment aim to promote student basic competences such as linguistic and STEAM skills. The methodology is primarily active and participative, furthering interdisciplinary investigative learning based on problem solving, teamwork and interactive groups [13, 15]. A number of skills must be integrated in planning and conducting teaching and learning activities to attain the ultimate goal of enabling students to apply their academic learning to other contexts [4, 17]. Information and communication technologies are routinely used in learning activities, in particular in information searches and analyzes, as well as in the presentation of the research work conducted [23]. In the specific case addressed here, mathematics and art, the Spanish curriculum for secondary school students (12-16 years old) establishes an explicit connection between the two [20].

On the one hand, the mathematics curriculum highlights two main aspects of this subject [20]. The first is that mathematics is indispensable for simplifying, abstracting, interpreting, expressing and understanding social realities, including artistic reality and creativity. Secondly, it is an instrumental tool to progressing in the knowledge acquisition of other disciplines, functioning as a driver of culture and civilization development. In particular, in the Geometry section of the curriculum, the evaluation criteria refer explicitly to that artistic perspective, both for students who plan to pursue their schooling further and for those who do not [20]. Stress is placed, for instance, on Thales' theorem and the standard formulas for obtaining distance, area and volume using artistic expressions such as painting or architecture as real-world examples. Students are asked to recognize, apply and analyze in-plane transformations of works of art, patterns present in nature, and familiar objects designs. The curriculum also makes explicit mention of the identification of centres, axes and planes of symmetry in nature, art and human constructions.

On the other hand, in the art curriculum, mathematical competence contributes to the development of the visual arts and music subjects [20]. In the case of the visual arts subject, students are asked to use a variety of types of logical reasoning and spatial visualization to explain and describe artistic and technical characteristics. Mathematics is applied to analyze and describe the proportions and canons used in art. Students are also taught to recognize the geometric shapes used in art. Space and form, so important to artistic creation, will help students coding and de-coding visual information in works of art, allowing the recognition of patterns, properties, positions, and representations of objects. In the Music subject, mathematical competence is required for reading and interpreting the singularities of music scores. Mathematics reinforces mental capacities such as understanding and logical structuring, enabling students to make personal decisions for creating and improvising music.

In Spain, curricular decisions weigh heavily on the design and formulation of textbooks, one of teachers' most fundamental resources. Their importance and influence on classroom activity are attested by the amount of research conducted on the resource in recent years [1, 2, 3, 22]. The foregoing supports the relevance of analyzing the connection between art and mathematics in Spanish textbooks, and exploring to what extent the European curricular decisions are applied in the Spanish context. Due to the limitations of the study, this article focuses solely on the presence of art in mathematics textbooks. So, the following question raises here: What is the role of art in the mathematics textbooks of the Spanish secondary school?

3. Methods

This section describes the sample selection as well as the data analysis strategy employed for the analysis of the selected textbooks.

Sample selection

The Spanish textbooks selected for this study were the ones most often used in secondary schools. The textbooks analyzed cover all four years of compulsory secondary education, i.e., used by students 12 to 16 years of age. They were released between 2007 and 2017 by eight publishers: Anaya, Bruño, Edebe, Marea Verde, Marfil, Oxford, SM, and Vector. This was a purposive sampling selecting specific textbooks used by two universities in the north of Spain, to which these authors have access. The aforementioned universities made this selection, as a sample of the most common publishers of textbooks in Spain. These books are used in both universities for training pre-service teachers in their PGCE (Postgraduate Certificate in Education) courses. The PGCE is a one-year higher education course in Spain, which provides training in order to allow graduates to become teachers in secondary education. Please note that this is not a complete sample of all the textbooks in Spain, and thus results cannot be generalized to the whole range of textbooks in the country.

Data Analysis strategy: Art-Mathematics dimensions

The textbooks were analyzed by experts in mathematics education. They search for art allusions in different parts of the textbook chapters (cover page, introduction, theory, activities, problems and so on), and different subject areas (algebra, geometry, functions and statistics). In this way, a textual and pictorial analysis was conducted all throughout the textbooks. In a preliminary data-driving analysis of the textbooks, several art-mathematics relationships were identified and categorized into the following six dimensions: (1) art for ornamental purposes (i.e., to make textbooks more appealing); (2) art as a context for calculation and measurement; (3) art as a context for mastering concepts; (4) art as a context for using technological resources in mathematics; (5) mathematical analysis of works of art; and (6) art created with mathematics.

In the first dimension, art for ornamental purposes where no explicit mention is made of mathematics, the artistic element is used only to adorn the written text, by way of decoration or motivation. Specific mention of the text adorned is often absent and the mathematical idea that evokes the artistic image is normally imperceptible for the non-expert reader. In the second dimension, art as a context for calculation and measurement, the artistic element affords a backdrop for the activity, but is not actually necessary to conduct it or understand the concept. It is an excuse for calculating and measuring, with no reference made to the history or formulation of the work of art. In the third dimension, art as a context for mastering concepts, the artistic object is used to exemplify a mathematical concept or property, seeking to enhance its comprehension or awareness of its significance. The mathematical object is not used to analyze the work of art; it involves no research or analysis.

In the fourth dimension, art as a context for applying technological resources to mathematics, art is used as a means or context for applying technological tools (such as computers, mathematical software or apps) to analyze or verify mathematical concepts and properties. In the fifth dimension, art as a context to analyze and explore mathematics, works of art are analyzed using mathematical techniques. The aim is to analyze specific aspects of the work: discovering, for instance, how it was generated or verifying a hypothesis on the mathematical procedures involved. Here the work of art is used not as a support to illustrate concepts or properties, but as the object of research and experimentation. In the sixth dimension, art created with mathematics, mathematics is used to create works or compositions with an artistic component. In other words, the aim is to create a product, original or otherwise, by applying mathematical concepts and properties.

These aforementioned six art-mathematics dimensions, that will be further illustrated in the next section, were employed as a framework for analyzing the role of art in the selected mathematics textbooks. Important, these six art-mathematics dimensions need not be understood as hermetic compartments. Any given activity or exercise may involve several dimensions at the same time. The dimensions may on occasion overlap or even entail an inclusive approach. An activity characterized under dimension 5 (mathematical analysis), for instance, normally calls for mastering concepts (dimension 3) as well as calculating and measuring (dimension 2).

4. Results of textbook analysis

A total of 146 allusions to art were identified in the textbooks and classified under the following art forms: Architecture, Sculpture, Painting and Drawing, Tiling and Friezes, music and, Design of Familiar Objects and Patterns in Nature. The percentages of the art-maths dimensions for each art form are presented in Table 1, as a synthesis of the results.

Specific details for each art form are provided below.

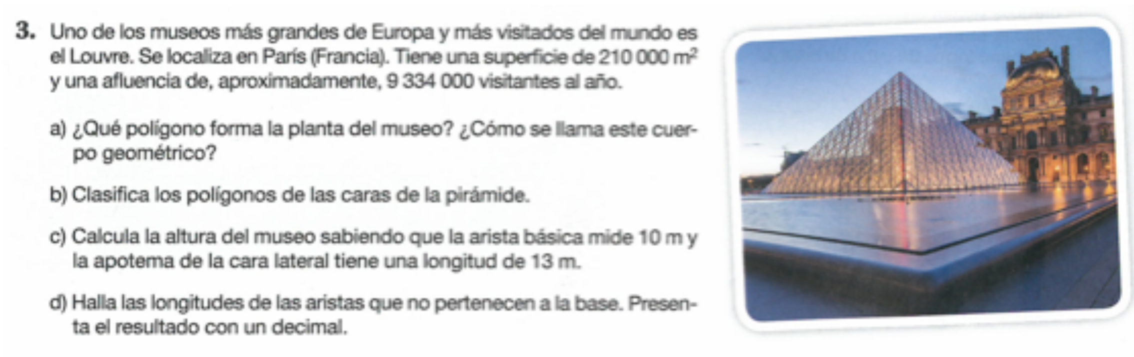
Art forms

Art-Maths Dimensions	Architecture	Sculpture	Painting	Tiling/ Friezes	Music	Design/ Nature	Total %
Ornamental	34.38 %	50%	55.56 %	54.55 %	50 %	53.13%	45.21%
Calculation/ Measurement	37.5 %	37.5%	7.41 %	9.09 %	50 %	15.63%	25.34%
Mastery Concepts	20.31 %	25%	25.93 %	9.09 %	25 %	31.25%	23.29%
Art and technology	10.94 %	-	-	18.18%	-	6.25%	7.53%
Analysis of works of art	4.69 %	-	11.11 %	36.36%	-	12.5%	9.59%
Art Creation	-	-	3.70 %	9.09%	-	-	1.37%
Total number of allusions	64	8	27	11	4	32	146

Table 1. Percentages of each Art-Mathematics dimension per each art form

Architecture

A total of 64 allusions to architecture were identified. Dimension 1 (*architecture for ornamental purposes*) accounted for 34.3 % of the allusions. Photos of structures such as the Eiffel Tower or the Parthenon were normally found adorning the first page in the introductions to chapters on geometry. Architecture appeared as a context for *calculation and measurement* (dimension 2) in 37.5 % of the cases. Pictures of historic pyramids such as at Cheops or modern structures such as the Louvre Museum at Paris were used primarily to calculate areas and volumes. Architecture provided the context for *mastering concepts* (dimension 3) in 20.3 % of the activities. As in the preceding dimension, pictures of modern or historic structures were used in activities in which students were asked to identify geometric bodies and figures. Dimensions 2 and 3 were often combined. In the following activity (Figure 1), for instance, students were asked to identify and classify the polygons forming the Louvre Museum pyramid depicted in the photo. They were also asked to calculate building distances, such as its height. Please note that the activity in Figure 1 is the original taken from the textbook with the Spanish text. An English translation of this activity, and others presented throughout this article, can be found in Supplementary Material A.



3. Uno de los museos más grandes de Europa y más visitados del mundo es el Louvre. Se localiza en París (Francia). Tiene una superficie de 210 000 m² y una afluencia de, aproximadamente, 9 334 000 visitantes al año.

a) ¿Qué polígono forma la planta del museo? ¿Cómo se llama este cuerpo geométrico?

b) Clasifica los polígonos de las caras de la pirámide.

c) Calcula la altura del museo sabiendo que la arista básica mide 10 m y la apotema de la cara lateral tiene una longitud de 13 m.

d) Halla las longitudes de las aristas que no pertenecen a la base. Presenta el resultado con un decimal.

Figure 1. Louvre Museum activity drawn from Edebe 2º ESO (2016, p.254)

Just 10.9 % of activities involved dimension 4 (*art and technology*). An even smaller percentage (4.69 %) involved dimension 5 (*analysis of works of art*), with only three activities identified in all the textbooks. In one of them (Figure 2), students were presented with a drawing of the Parthenon at Athens and asked to analyze its relationship to the golden ratio. Not a single reference was identified that could be construed to induce students to create art with mathematics (dimension 6).

Figure 2. Athens Parthenon activity drawn from Marfil 4 º ESO (2008, p.267)

The activities involving architecture were mostly exercises in geometry, although a few references appeared in lessons on functions and graphs, particularly where works construction entailed the use of curves. Further details about the allusions to this form of art (Architecture) are provided in Table S1 in Supplementary Material B. This table includes a categorization of the 'works of art', 'mathematical concepts' and 'art-maths dimensions' identified per textbook.

Sculpture

Eight allusions to sculpture were detected. 50 % of them were used with *ornamental purposes* (dimension 1), and the 37.5 % used as a context for *calculation and measurement* (dimension 2). Sculpture as a context for *mastering concepts* (dimension 3) appeared in 25 % of the activities. The exercises involving both dimensions 2 and 3 were primarily algebraic and geometric. The former focused on solving equations and the latter on the recognition of three-dimensional bodies (polyhedra and solids of revolution), polyhedral composition, calculation of volumes and areas, symmetries and scales, and change of units. For instance, in the following activity (Figure 3), the 'Atomium' (which depicts the crystalline structure of iron) was used primarily in connection with dimensions 2 and 3, asking students to calculate scales and identify the geometric bodies


1. El Atomium fue construido en 1958 en la ciudad de Bruselas (Bélgica). Representa la estructura cristalina del hierro: un cubo con una esfera en cada uno de sus vértices y una esfera central, unidas todas ellas mediante cilindros. Está aumentada ciento sesenta y cinco mil millones de veces respecto a la estructura cristalina del hierro. El monumento tiene 102 m de altura y el diámetro de las esferas es de 18 m.

a) Determina, a partir del razonamiento geométrico, el número de cilindros y esferas del Atomium.

b) El tubo vertical central es un ascensor que sube a sus visitantes a 5 m/s hasta la esfera más alta. En ella se ubican un mirador y un restaurante. ¿En cuánto tiempo el ascensor lleva a cabo este recorrido?

c) Calcula el diámetro, en nanómetros, de un átomo de hierro si sabemos que 1 nm = 10^{-9} m.

d) Minimundus es un parque de miniaturas situado en Klagenfurt (Austria). En él se encuentra una réplica del Atomium a escala 1:25. ¿Cuál es la altura, en metros, de la réplica?

Figure 3. Atomium activity drawn from Edebe 2º ESO (2016, p.254)

constituting the structure. As Table S2 shows, in Supplementary Material B, allusions to this form of art (sculpture) were not addressed in the rest of the dimensions for any of the textbooks.

Painting and Drawing

Twenty-seven allusions were found to pictorial art by authors such as Kandinsky, Leonardo Da Vinci, Paul Klee and Escher, to name a few. Painting or drawing was used for ornamental purposes in 55.5 % of the cases analyzed. A smaller percentage (25.9 %) of activities involved dimension 3 (*mastering concepts*). Students were normally asked to identify and classify plane geometric figures on drawings or paintings, by authors such as Kandinsky or Paul Klee. For instance, the activity in Figure 4 illustrates *In Blue* Kandinsky painting, asking to indicate and classify the existing quadrilaterals. Prints such as Escher's 'Drawing Hands' (Figure 5) were used to enhance the mastery of the notions of symmetry.

Figure 4. In Blue Kandinsky painting activity drawn from Edebe 1º ESO (2007, p.207)

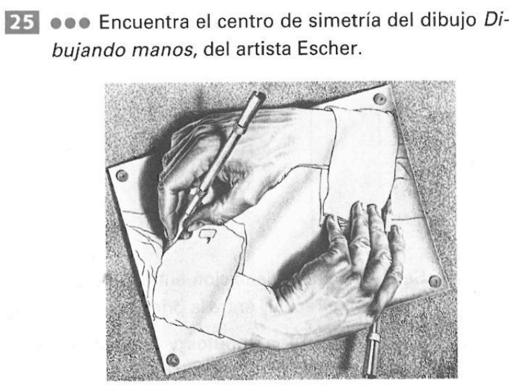


Figure 5. Escher's Drawing Hands activity drawn from Vector 1º ESO (2010, p.234)

Dimensions 2 (*calculation and measurement*) and 5 (*analysis*) informed 7.4 % and 11.1% of the activities respectively. Dimension 2 was found in solving equations activities and dimension 5 in tasks requiring the analysis of proportions and symmetries. One example of the latter was the reproduction of Leonardo Da Vinci's 'Vitruvian Man' canon, in which students were asked to verify whether the same proportions were present in other values (Figure 6).

14.8. Número de oro y arte

Aquí tienes algunos ejemplos de la utilización que el ser humano ha hecho a lo largo de la historia del número de oro. Esperamos que te sirvan como punto de partida para que realices una investigación de búsqueda de datos sobre el tema.

(1) Cánones

A lo largo de la historia han existido distintos modelos o cánones propuestos como perfectos para el cuerpo humano. El dibujo siguiente de Leonardo da Vinci está basado en el canon propuesto por el arquitecto romano Marco Vitrubio. Según este canon, debía de cumplirse la siguiente relación:

$$\frac{\text{Altura total}}{\text{Altura hasta el ombligo}} = \frac{\text{Altura hasta el ombligo}}{\text{Distancia ombligo-cabeza}}$$

a) Si denominas "a" a la altura total y "m" a la altura hasta el ombligo, ¿cómo escribirías la anterior relación?
b) Prueba para distintos valores de "a" y "m". ¿Qué relación puede existir entre ambas medidas?

Figure 6: Leonardo Da Vinci's 'Vitruvian Man' activity drawn from Marfil 4 º ESO (2008, p.266)

Dimension 6 (*art created with mathematics*) appeared in a single activity in a second year ESO textbook published by Oxford (2016, p.248). In that activity (Figure 7), which reproduced Escher's dodecagon, students were asked to find more information about the painter's oeuvre and build their own dodecahedra.

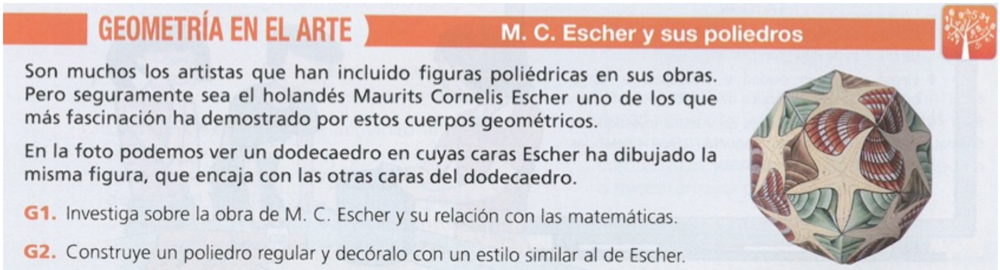


Figure 7. Escher's dodecahedron activity drawn from Oxford 2º ESO (2016, p.248)

Further information about the allusions to this form of art (Painting and drawing) are provided in Table S3 in Supplementary Material B. This table includes a categorization of the 'paintings and drawings', 'mathematical concepts' and 'art-maths dimensions' identified per textbook.

Tiling and Friezes

Eleven allusions to tiling and friezes were identified, normally through illustrations of artistic buildings such as the Alhambra in Granada and the Mosque-Cathedral of Cordoba. Escher's works were also often present. In-plane movement and the identification of plane geometric figures were the types of content most frequently appearing in conjunction with this art form.

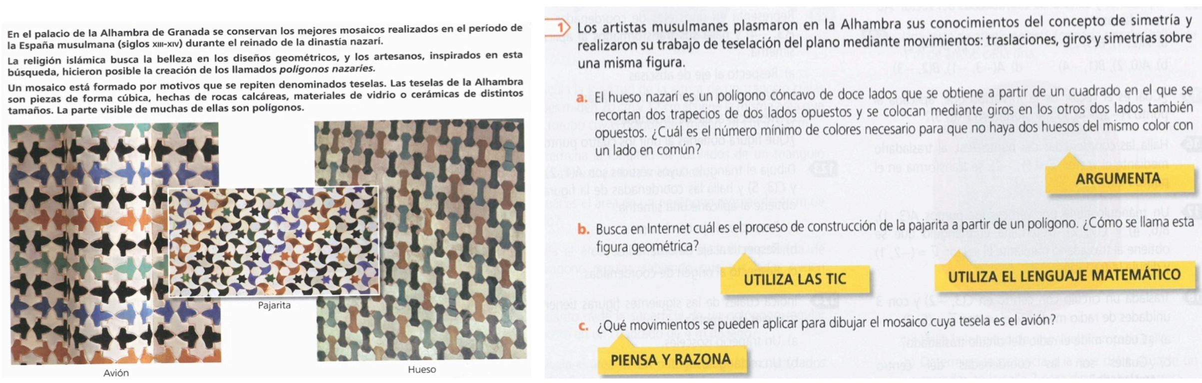


Figure 8. Alhambra tiling activity drawn from Oxford 3º ESO (2015, p.140)

In 54.5 % of the cases, tiling and friezes appeared for *ornamental purposes* (dimension 1), primarily in geometry lessons. Dimension 5 (*analysis*) also accounted for a substantial percentage (36.3 %) of activities, where students were asked to identify the geometric piece (tile) and the movements (symmetry, translation and rotation) from which a tiling or frieze is generated. The activity in Figure 8, for instance, asks about the tile and the movements generating an Alhambra tiling.

Dimension 4 (*art and technology*) was present in 18.1% of the activities related to Tiling and friezes. Several exercises were based solely on seeking information on the internet to understand how tile such as the ones used in the Alhambra tiling are constructed. A reduced number of activities occasionally propose the use of GeoGebra software to study homothety and similarity in tiling designs, as the one exemplified in Figure 9.

Figure 9. GeoGebra activity to study homothety and similarity drawn from Edebe 2º ESO (2016, p.223)

Dimensions 2 (*calculation and measurement*), 3 (*mastering concepts*) and 6 (*art creation*) accounted for 9 % each of the occurrences. Dimension 2 was observed in activities related to the measurement of angles and percentages. Dimension 3 was associated with exercises in which students were invited to identify plane figures and recognize in-plane movements (symmetry, rotation, and translation). Dimension 6 (*art creation*) was observed in only one activity, involving the use of GeoGebra to draw a polygon and generate from it a tiling by applying different in-plane movements.

Extra information about the allusions to this form of art (Tiling and friezes), including the categorization of 'works of art per textbook', 'mathematical concepts', and 'art-maths dimensions' identified, are provided in Table S4 in Supplementary Material B.

Music

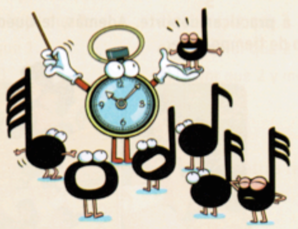
A scant four allusions to the relationship between mathematics and music were found. They appeared only in lessons on fractions, with references to the relationships established between the two by the Pythagoreans. The only dimensions detected were 1 (*art for ornamental purposes*), 2 (*calculation and measurement*) and 3 (*mastering concepts*). Dimension 1 appeared twice, with photographs of musical instruments used as page decoration, with no mention of mathematics. The other two allusions involved dimensions 2 and 3. In one, music and mathematics appeared jointly in an activity dealing work with musical fractions (duration of notes), as exemplified in the following activity (Figure 10) drawn from a textbook published by SM for fourth year ESO (2017, p.26).

PROBLEMA RESUELTO **Fracciones musicales**

En la composición e interpretación de las piezas musicales es muy importante la duración de cada una de las notas musicales.

Según la duración, existen notas redondas, blancas, negras, corcheas, semicorcheas, fusas y semifusas. Tomando como unidad la duración de una nota negra, el resto tiene los siguientes tiempos:

Redonda	Blanca	Negra	Corchea	Semicorchea	Fusa	Semifusa
4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$



1. ¿Cuántas semicorcheas hay en una blanca? ¿A cuántas blancas equivale una fusa?
2. En música se utiliza un símbolo, el puntillo, que incrementa la duración de la nota un 50 %. ¿A cuántas fusas equivaldrá una blanca con puntillo?
3. Si se coloca un doble puntillo, se aumenta la duración de la nota un puntillo y la mitad de un puntillo. ¿Qué fracción de una nota sin puntillo es esa misma nota con doble puntillo?

Figure 10. Musical fractions activity drawn from SM 4º ESO (2017, p.26)

Further details about the allusions to this form of art (Music) are provided in Table S5 in Supplementary Material B. This table includes a categorization of the 'music elements', 'mathematical concepts' and 'art-maths dimensions' identified per textbook.

Design of Familiar Objects and Patterns in Nature

Thirty-two allusions to designs relating to mathematics were identified. Dimension 1 (*ornamental purposes*) appeared in 53 %, being most commonly depicted illustrations credit cards, logotypes, Nautilus shells and animal hides, with no explicit reference to mathematics. Occasionally, the illustrations are accompanied with a text related to a mathematical idea, but without explicitly treating or mastering such a concept or idea, as is the case with the designs in Figure 11, drawn from Edebe 2ºESO (2017, p.194).

Figure 11. Examples of familiar object related to the golden ratio, drawn from Edebe 2ºESO (2017, p.194)

Figure 12. Butterfly symmetry activity drawn from SM 4º ESO (2017, p.26)

Dimension 3 (*mastering concepts*) was found in 31.2 %. These cases normally revolved around industrial design or patterns in nature to illustrate a mathematical concept or property, such as the golden ratio, the golden spiral, in-plane movement, symmetry, sequences, parabolas, and plane/solid figures. In the activity of the butterfly in Figure 12, for instance, students were asked to determine whether it exhibited axial symmetry.

Dimension 2 (*calculation and measurement*) appeared in 15.6 %. In these activities students measured or calculated the size of familiar or natural objects in exercises revolving around the golden number. In the activity depicted in Figure 13, and drawn from a textbook published by Marfil, students are asked to measure the sides of a credit card and determine whether it conforms to the golden ratio.

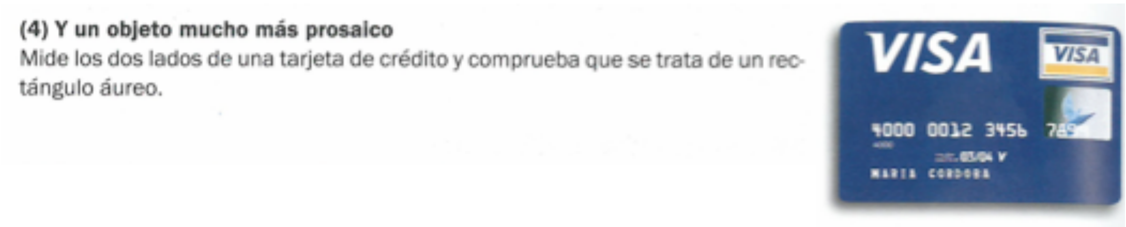


Figure 13. Golden ratio credit card activity drawn from Marfil 4 º ESO (2008, p.268)

Dimension 5 (*analysis*) was found in 12.5 %. The activities proposed involved exploring concepts associated with symmetry and analyzing familiar or natural objects in terms of the golden number. In the activity showed in Figure 14, students were asked to describe how Dürer's spiral was constructed.

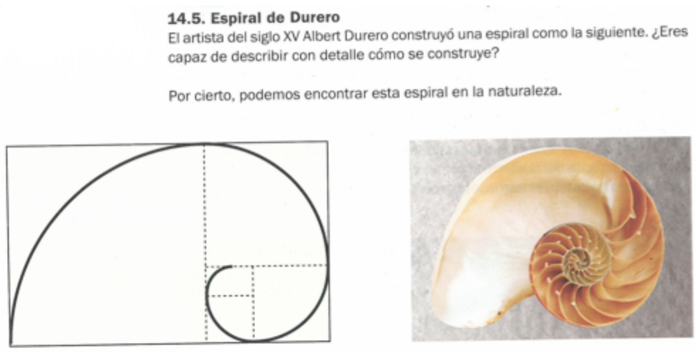


Figure 14. Dürer's spiral activity drawn from Marfil 4 º ESO (2008, p.265)

Dimension 4 (*art and technology*) was present in 6.25 % of the activities, although it was confined to Internet searches on natural elements exhibiting the golden number or animal hides simulating tiling, as the activity in Figure 15. Dimension 6 (*art creation*) was not represented in any of the activities proposed in the textbooks analyzed. Further details about the allusions to this form of art

Figure 15. Animal hides and tilings activity drawn from Edebe 2º ESO (2016, p.223)

(*design/nature*) are provided in Table S6 in Supplementary Material B. This table includes a categorization of the 'identified everyday objects and nature elements' per textbook, the 'mathematical concepts' involved, and 'the art-maths dimensions' comprised.

It is noteworthy that the majority of the allusions to art in the Spanish mathematics textbooks incorporate pictures (e.g. drawings and images). Few references to art without pictures were found. Just some exercises asked for the area or volume of a famous construction, providing its dimensions without the support of a picture.

5. Discussion

The following is a discussion of the most prominent results for the six dimensions analyzed: (1) art for ornamental purposes; (2) art as a context for calculation and measurement; (3) art as a context for mastering concepts; (4) art as a context for using technological resources in mathematics; (5) mathematical analysis of works of art; and (6) art created with mathematics.

Figure 16. Mona Lisa Illustration drawn from Anaya 4º ESO (2008, p.169)

Dimension 1, art as ornament, was present in the textbooks in a large proportion with a 45.2 % of the artistic works used for purely decorative purposes. The majority of those were found on covers or title pages with no text, the underlying mathematical idea was difficult to interpret or infer. Works of art were also reproduced in the margins of pages bearing theoretical introductions to a mathematical concept, with no explicit reference in the text to the piece or in the legend to the title or author. One example can be found in a textbook section on similarity, published by Anaya (2008, p.169), reproducing the 'Mona Lisa' with no reference to the illustration in the text (see Figure 16). The relevance of the work was not clearly explained, nor was any obvious connection to mathematics established. The percentage of dimension 1 occurrences was fairly evenly distributed across the six art forms (architecture, sculpture, painting, tiling and friezes, music and design of familiar objects and patterns in nature).

Dimension 2, art as a context for calculation and measurement, accounted for 25.3 % of the allusions. In the activities proposed students were usually asked to calculate areas or volumes, and to measure works of art from pictures. The purpose was to provide a real-world context to express a mathematical idea or concept, but the works were unnecessary from the standpoint of solving the mathematical problem. In the activity in Figure 1, for instance, involving the merely procedural calculation of the size of the Louvre Museum pyramid, the picture itself was unnecessary: applying the respective algorithms to the data given would have sufficed. Any other non-artistic quadrangular pyramid would have served the same purpose.

Other activities called for measuring a work of art picture or a familiar object design with a ruler. The activity associated with Figure 13 (in an early section), for instance, entailed verifying whether a credit card represent the golden ratio. Measuring with a ruler would afford students an approximate value, which would not be sufficient to determine whether the object actually conforms the golden ratio. If they choose to measure the card with a calliper, the value obtained (1.58515851...) would be nearly the golden number, but would not be the actual card ratio (1.585772508 given by the international standard ISO/IEC 7810:2003). If a technological tool such as GeoGebra were to be used to measure the card (see Figure 17), the value found, 1.60197648, would be closer to the

Figure 17. Credit card measured with GeoGebra Software (own elaboration)

golden number but farther away from the actual value. More than proving the conformity of the object to the golden ratio, this exercise aimed to teach students the importance of measuring with different tools, bearing in mind the characteristic errors of each, and the notion of approximation.

Dimension 3, art as a context for mastering concepts, was present in 23.3 % of the cases identified. This dimension was found in all six art forms (architecture, sculpture, painting, tiling and friezes, music, and design of familiar objects and patterns in nature). In particular, it was widely found in activities related to the design of familiar and natural objects, and only rarely in activities related to tiling and friezes. Table 2 lists the mathematical concepts that explicitly appear in relation to art and the ratio of occurrence of each. The same concept was often found in several works of art and in different art forms. Likewise, in a single work of art different concepts were also integrated.

Mathematical Concept	**Number of Allusions**
Geometry figures and measurements	42
Golden ratio	23
Symmetry	16
Geometric transformation	13
Similarity	11
Functions	11
Arithmetic	11
Proportionality	6
Equations	4
Scales	2
Sequences	2

Table 2: Mathematical Concepts associated to the Art forms

It is worth noting that concepts such as infinity, limits, continuity and perspective, often present in the works of art, were not explicitly addressed. They could have easily been integrated in some of the proposed activities. To cite one example, the activity in Figure 18 presents the Escher's 'Smaller and Smaller' print, and asks to identify the in-plane movement and tile generating the tiling. This print could have also been used to work the notions of infinity and limits.

Figure 18. 'Smaller and Smaller' print activity drawn from Marfil 4º ESO (2008, p.310)

In some cases, the activities addressed concepts too superficially. Students were normally asked to identify a curve in a picture on the sole grounds of observation. For instance, the activity in Figure 14 (in a previous section) aimed to teach the golden ratio by establishing a visual relationship between a Nautilus picture and Dürer's spiral. Greater benefit could have been derived from this activity if it had enlarged upon the concept and analyzed other types of spirals (e.g. Algorithmic, or Archimedean) that might provide a better fit to the Nautilus. That would have also afforded the opportunity to address the notion of approximation.

Similar examples were found in the lessons on functions and graphs, where normally the aim was to work with parabolas by identifying pictures of curves by mere observation. In some cases, the curve shown in the picture does not match the graph of a function. In others, such as in Figure 19, the symmetry of a curve was difficult to perceive because the photo was not taken perpendicular to the line of sight.

Figure 19. Identifying the function related to the curve, activity drawn from Oxford 3 º ESO (2015, p.250)

Dimension 4, art as a context for using technology in mathematics, accounted for a much lower percentage (7.5 %) of occurrences than dimensions 1, 2 and 3. It appeared only in connection with three art forms: architecture, tiling and friezes, and design of familiar objects and patterns in nature. Most of the activities focused on internet searches for information: on the tile generating the Alhambra tiling (as exemplified earlier in the activity of Figure 8); on tiling-like patterns found in nature (as illustrated in Figure 15); and on natural objects conforming to the golden ratio (as in Dürer's spiral activity in Figure 14). Art was seldom used as a backdrop for applying computer software to mathematics. Only a few activities were observed in which students were expected to use GeoGebra to draw a polygon and build a tiling or practise with homothety and similarity in tiling design (as was the case with the Figure 9 activity).

No activities were found in which the use of GeoGebra or other educational software proved to be useful for measuring distances, building structures, verifying properties or mastering concepts that would be suited to exercises involving works of art. Activities could have been, for instance, proposed to use GeoGebra to build a golden rectangle to determine whether a credit card was exactly such a rectangle, as demonstrated in Figure 20.

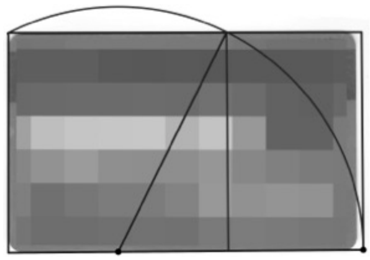


Figure 20. Use of GeoGebra to verify whether or not a credit card is a golden rectangle (own elaboration)

Dimension 5, mathematical analysis of works of art, accounted for just 9.6 % of cases. Although this dimension was present in architecture, painting and design of familiar objects and patterns in nature, it was most visible in tiling and friezes. The activities in question focused on determining which geometric transformations (translation, rotation or symmetry) led to a tiling or frieze, and to identify the shape from which a given tile is generated. An example of the latter is the activity described earlier in Figure 8, and drawn from Oxford 3º ESO (2015, p.140).

Barring the activities with a certain analytical content described in the results, no others called for an in-depth analysis of works of art from the mathematical standpoint. Most activities related mathematics and art very superficially. Figure 11, described earlier, is an example of it: the Twitter logotype appears as an example of a golden ratio design, without asking students to undertake any critical analysis on this relationship. This is unexpected considering the Spanish curriculum highlights analysis as a main competence. A deeper investigation or exploration would have revealed that the sketches for the geometric construction of this logotype were created from arcs

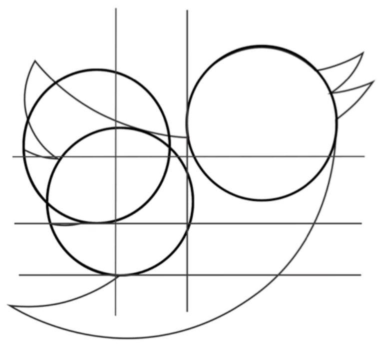


Figure 21. Sketch for the geometric construction of Twitter logotype, retrieved October 1st 2017 from https://dribbble.com/twitter/projects/105370-Brand

of a circumference (Figure 21); a feature of no minor interest from the perspective of the art-mathematics relationship. That explanation would also have entailed studying the application of mathematics to the creation of the logotype.

Textbooks were also observed to propose activities relating Leonardo Da Vinci's Vitruvian Man to the golden ratio, in most cases with superficial analyzes. That work is a canon on the proportions of the human body, of particular significance in art. In an educational context it could be used to propose more analytical activities, such as:

- studying which canon represents the models (normally a wooden figure) used by each artist for their works.
- measuring classroom students' bodies and generating data tables for statistical processing to establish relationships between different dimensions, and compare the results with existing canons and anthropometric measurement tables.

Such activities would analytically combine art, proportions and statistics and involve minor research that could be very useful to students.

Dimension 6, art created with mathematics, was the dimension with the lowest presence (1.4 %). As suggested in the results section, this dimension was found only in connection with pictorial art and the generation of polygons to create tiling and friezes, as well as constructing polyhedrons decorated with artistic paintings, as proposed in Escher's dodecahedron activity (Figure 7).

One activity using pictorial art (drawings or prints) could have included creating compositions with geometric elements inspired for instance by the pieces earlier mention of Kandinsky or Escher. Works such as Dalí's 'Atomic Leda'— formulated around the Pythagorean star (see Figure 22)— could have been given a similar use, taking the artist's idea as the point of departure. For instance, a picture could be drawn using Dürer's shell curves.

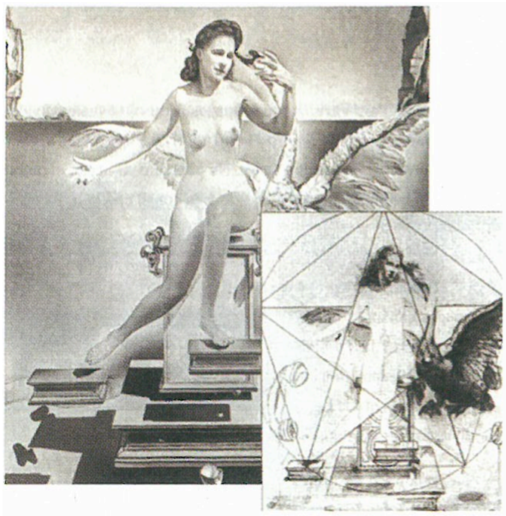


Figure 22. Atomic Leda drawn from Marfil, (2008 p. 267)

Additional exercises involving sculpture might have included clay reproductions for the canons used by artists, or an original creation to be compared to existing canons. Scale models of architectural structures or sculptures could also be built. As noted in the results section, in several activities students were asked to recognize arcs and parabolas in certain illustrations. That could be taken further, using mathematics to reproduce architectural elements on a different scale. One activity, for instance, might involve making a scale model of the 'Atomium' described in earlier sections to help master geometric and arithmetic concepts.

The only activities proposed in connection with music were related to the duration of musical notes (measurement). That could have been enhanced by teaching students to use the value of the musical notes to create bars. Each student might, for instance, create a melody with six bars and four beats per bar, using certain musical notes (minim, crotchet, quaver, semiquaver) and their rests. A melody could be composed by simulating Mozart's 'Musical Dice Game': assigning each bar a number from one to six and throwing a die six times to determine the order of the bars and the number of times they are repeated in the melody. This activity would involve working with fractions (the sum of the duration of each note and its rest should be one), probability (the likelihood of a bar coming up on the die) and combinations (analysis of all the possible melodies that could be generated).

As noted earlier in the section on design, most activities sought to establish connections between designs and the golden ratio, and to identify axes of symmetry. It would have been useful if in connection with this art form each student had been invited to create their own logotype from geometric elements and constructions only. Such an exercise would also involve applying other dimensions such as mastering concepts or calculation, with or without technology.

6. Conclusions

This study analyzed the relationship between art and mathematics in the Spanish education system and more specifically in textbooks for secondary school students aged 12 to 16. The relationship between art and mathematics was assessed on the grounds of six parameters: (1) art for ornamental purposes; (2) art as a context for calculation and measurement; (3) art as a context for mastering concepts; (4) art as a context for using technological resources in mathematics; (5) mathematical analysis of art; and (6) art created with mathematics.

The 146 allusions identified in the textbooks, and classified in different art forms (architecture, sculpture, etc.), were analyzed in terms of the above dimensions to study the relationship between art and mathematics. Architecture, with over 64 references, prevailed over the other art forms, followed by Design of Familiar Objects and Patterns in Nature with 32, and Painting and Drawing with 27. Fewer references were found to Tiling and Friezes (11), Sculpture (8) and music (4). The Eiffel Tower, the pyramids at Cheops and the Parthenon were the architectural works most often cited. In painting, two works authored by Leonardo Da Vinci predominated: ‘Mona Lisa’ and ‘Vitruvian Man’. The tiling and friezes most frequently found were those in the Alhambra at Granada, the Mosque at Cordoba and several of Escher’s works. The designs most often reproduced were credit cards and the Nautilus shell. No sculpture stood out over the others and no specific piece of music was cited.

The distribution of the identified activities by dimension revealed a clear predominance of ‘art as an ornamental element’ (dimension 1) in mathematics textbooks. Over 45.2 % of the activities were classified under this dimension, accounting for over 50 % of the occurrences in five of the art forms. The percentage of activities under dimensions 4 (art and technology), 5 (analysis) and 6 (art creation), which involved a more active role of the student, was under 13 %. Dimension 4 (art and technology) appeared in only some of the art forms and was generally associated with information searches. Dynamic geometry software was present in a few activities involving the study of tiling and friezes. Dimension 5 (analysis) was also found primarily in connection with that art form. With only two activities, the presence of dimension 6 (art creation) was negligible.

In terms of content, nearly 28.8 % of the activities involving artistic elements entailed working with and measuring (distance, area, volume) geometric figures. That was followed by exercises on the golden ratio (15.75 %), symmetry (10.96 %) and geometric transformations (8.9 %). With the exception of the golden ratio, those are the contents explicitly recommended in the Spanish curricular guidelines for working with art as a context in mathematics. Barely any of the activities relating to the golden section prompted experimentation or reflection. They seldom involved verifying a hypothesis or assumption or working with the errors implicit in measurement. Nor did they induce a critical attitude toward the wide dissemination of this topic, which is not always dealt with as rigorously as required. Generally speaking, very few activities were found in which technological resources were applied to explore the art-mathematics connection. The proliferation in recent years of the use of dynamic geometry software for the study of the area seems to have had little effect on textbooks.

On the whole, in most of the activities mathematics were neither significant to the works of art illustrated nor used to analyze them. They were merely an underlying medium. This study shows that the attempts made to illustrate the mathematics-art connection in today's world do not suffice. The solution is not to adorn mathematics activities with hosts of examples of the presence of mathematics in the real world, but to furnish good examples that support the mathematical knowledge being taught in classrooms and liable to be used to analyze and create art with mathematical tools.

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Is the relationship between art and mathematics addressed thoroughly in Spanish secondary school textbooks?

Jose Manuel Diego-Mantecón

Universidad de Cantabria, MATESCO, Facultad de Ciencias

Avda. de los Castros 48, 39005 Santander (Spain) - diegojm@unican.es

Teresa F. Blanco

Universidade de Santiago de Compostela, Departamento de Didácticas Aplicadas, Facultade de Ciencias da Educación

Avda. Xoan XXIII s/n, 15782 Santiago de Compostela (Spain) - teref.blanco@usc.es

José Benito Búa Ares

Xunta de Galicia, I.E.S. Sánchez Cantón

Avda. Raíña Vitoria s/n, 36001 Pontevedra (Spain) - jbenitobua@gmail.com

Pablo González Sequeiros

Universidade de Santiago de Compostela, Departamento de Didácticas Aplicadas, Facultade de Formación do Profesorado

Avda. Ramón Ferreiro s/n, 27001 Lugo (Spain) - pablo.gonzalez.sequeiros@usc.es

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Abstract

This article discusses a review of the relationship between art and mathematics in Spanish secondary school mathematics textbooks. The art-mathematics connection identified in the textbooks was analyzed under six dimensions: (1) art for ornamental purposes; (2) art in calculation and measurement; (3) art to master concepts; (4) art to use technological resources in mathematics; (5) mathematical analysis of art; and (6) creating art with mathematics. Dimensions 1, 2 and 3 clearly prevailed over dimensions 4, 5 and 6, which called for more active participation and analytical reflection. Most of the activities attempted to *illustrate* the mathematics-art connection with real-world examples, but rarely entailed verifying a hypothesis or assumption nor did they encourage critical thinking for analyzing and creating art with mathematical or technological tools.

Keywords: art, mathematics, textbooks, secondary education, curriculum

1. Introduction

Art and mathematics, two expressions of human creativity, share many distinctive traits, such as the friction between creation and technique, intuition and rigour, certainty and uncertainty. The relationship between the two disciplines is complex and can be broached from at least two perspectives [5, 11]. For the artist, artistic creation prevails over mathematics, being considered little more than a tool to generate forms and constructions. It often supplies geometric concepts and procedures, which, given their properties, are technically useful. The artistic creations in which mathematics is used are closely related to mathematical notions and properties. For the mathematician, whilst a mathematical notion may on occasion give rise to something that could be regarded as an artistic creation, that outcome is no more than a consequence of the mathematical notion [6, 9]. Its 'beauty' might be said to lie in the mathematical notion itself, the embodiment and sensory perceptibility of which are secondary.

As a result of European recommendations for educators, the relationship between art and mathematics has steadily acquired greater importance nowadays [7, 27]. The question is whether that relationship has found its way onto real world curricula. On the grounds of reference frameworks for Spanish curricula, the answer would appear to be affirmative [20, 26, 31]. Nonetheless, curricular trends and guidelines do not always translate into classroom practice [30]. In fact, the discordance between the intended and implemented mathematics curriculum has been widely reported across different educational contexts [21, 32]. Although there is a large number of studies addressing the relationship between art and mathematics from both a research and educational perspective, see, for example, [6, 18, 28, 29], as far as we know, there is a lack of studies focusing on how this art-mathematics relationship is established in the school lessons.

In the Spanish context, school lessons are strongly influenced by textbooks. These are basic resources for teachers, not only as instructional support, but also as a substantive component in lesson planning [1, 14, 19, 24, 33]. Textbooks may therefore be used as an indicator of the content addressed in the classroom. An analysis of the relationship between art and mathematics may be conducted under that premise, seeking to ascertain the extent to which the interaction between them is addressed in the classroom. This article broaches the question by specifically analyzing the presence of art in secondary school mathematics textbooks in Spain. For this purpose, what we mean by art is limited to visual arts and music. This approach is justified from a curricular point of view and excludes literature because of its independence from the other two disciplines, in the Spanish scholar itineraries.

2. Today's methodological and curricular trends

The European Union's recommendations for increasing the number of students enrolling scientific-technological subjects encourage the implementation of STEAM activities [10]. These activities integrate all scientific areas (Science and Mathematics) and their most prominent applications in modern society (Engineering, Technology and Art) [8, 12]. Interdisciplinarity plays an essential role in these activities, deliberately eschewing a traditional educational approach in which knowledge appears to be divided into unconnected subjects [16, 25].

The priorities defined by the EU have had visible consequences for Spain's official compulsory secondary education (Spanish abbreviation, ESO) and baccalaureate curricula [20]. The new approaches to learning and assessment aim to promote student basic competences such as linguistic and STEAM skills. The methodology is primarily active and participative, furthering interdisciplinary investigative learning based on problem solving, teamwork and interactive groups [13, 15]. A number of skills must be integrated in planning and conducting teaching and learning activities to attain the ultimate goal of enabling students to apply their academic learning to other contexts [4, 17]. Information and communication technologies are routinely used in learning activities, in particular in information searches and analyzes, as well as in the presentation of the research work conducted [23]. In the specific case addressed here, mathematics and art, the Spanish curriculum for secondary school students (12-16 years old) establishes an explicit connection between the two [20].

On the one hand, the mathematics curriculum highlights two main aspects of this subject [20]. The first is that mathematics is indispensable for simplifying, abstracting, interpreting, expressing and understanding social realities, including artistic reality and creativity. Secondly, it is an instrumental tool to progressing in the knowledge acquisition of other disciplines, functioning as a driver of culture and civilization development. In particular, in the Geometry section of the curriculum, the evaluation criteria refer explicitly to that artistic perspective, both for students who plan to pursue their schooling further and for those who do not [20]. Stress is placed, for instance, on Thales' theorem and the standard formulas for obtaining distance, area and volume using artistic expressions such as painting or architecture as real-world examples. Students are asked to recognize, apply and analyze in-plane transformations of works of art, patterns present in nature, and familiar objects designs. The curriculum also makes explicit mention of the identification of centres, axes and planes of symmetry in nature, art and human constructions.

On the other hand, in the art curriculum, mathematical competence contributes to the development of the visual arts and music subjects [20]. In the case of the visual arts subject, students are asked to use a variety of types of logical reasoning and spatial visualization to explain and describe artistic and technical characteristics. Mathematics is applied to analyze and describe the proportions and canons used in art. Students are also taught to recognize the geometric shapes used in art. Space and form, so important to artistic creation, will help students coding and de-coding visual information in works of art, allowing the recognition of patterns, properties, positions, and representations of objects. In the Music subject, mathematical competence is required for reading and interpreting the singularities of music scores. Mathematics reinforces mental capacities such as understanding and logical structuring, enabling students to make personal decisions for creating and improvising music.

In Spain, curricular decisions weigh heavily on the design and formulation of textbooks, one of teachers' most fundamental resources. Their importance and influence on classroom activity are attested by the amount of research conducted on the resource in recent years [1, 2, 3, 22]. The foregoing supports the relevance of analyzing the connection between art and mathematics in Spanish textbooks, and exploring to what extent the European curricular decisions are applied in the Spanish context. Due to the limitations of the study, this article focuses solely on the presence of art in mathematics textbooks. So, the following question raises here: What is the role of art in the mathematics textbooks of the Spanish secondary school?

3. Methods

This section describes the sample selection as well as the data analysis strategy employed for the analysis of the selected textbooks.

Sample selection

The Spanish textbooks selected for this study were the ones most often used in secondary schools. The textbooks analyzed cover all four years of compulsory secondary education, i.e., used by students 12 to 16 years of age. They were released between 2007 and 2017 by eight publishers: Anaya, Bruño, Edebe, Marea Verde, Marfil, Oxford, SM, and Vector. This was a purposive sampling selecting specific textbooks used by two universities in the north of Spain, to which these authors have access. The aforementioned universities made this selection, as a sample of the most common publishers of textbooks in Spain. These books are used in both universities for training pre-service teachers in their PGCE (Postgraduate Certificate in Education) courses. The PGCE is a one-year higher education course in Spain, which provides training in order to allow graduates to become teachers in secondary education. Please note that this is not a complete sample of all the textbooks in Spain, and thus results cannot be generalized to the whole range of textbooks in the country.

Data Analysis strategy: Art-Mathematics dimensions

The textbooks were analyzed by experts in mathematics education. They search for art allusions in different parts of the textbook chapters (cover page, introduction, theory, activities, problems and so on), and different subject areas (algebra, geometry, functions and statistics). In this way, a textual and pictorial analysis was conducted all throughout the textbooks. In a preliminary data-driving analysis of the textbooks, several art-mathematics relationships were identified and categorized into the following six dimensions: (1) art for ornamental purposes (i.e., to make textbooks more appealing); (2) art as a context for calculation and measurement; (3) art as a context for mastering concepts; (4) art as a context for using technological resources in mathematics; (5) mathematical analysis of works of art; and (6) art created with mathematics.

In the first dimension, art for ornamental purposes where no explicit mention is made of mathematics, the artistic element is used only to adorn the written text, by way of decoration or motivation. Specific mention of the text adorned is often absent and the mathematical idea that evokes the artistic image is normally imperceptible for the non-expert reader. In the second dimension, art as a context for calculation and measurement, the artistic element affords a backdrop for the activity, but is not actually necessary to conduct it or understand the concept. It is an excuse for calculating and measuring, with no reference made to the history or formulation of the work of art. In the third dimension, art as a context for mastering concepts, the artistic object is used to exemplify a mathematical concept or property, seeking to enhance its comprehension or awareness of its significance. The mathematical object is not used to analyze the work of art; it involves no research or analysis.

In the fourth dimension, art as a context for applying technological resources to mathematics, art is used as a means or context for applying technological tools (such as computers, mathematical software or apps) to analyze or verify mathematical concepts and properties. In the fifth dimension, art as a context to analyze and explore mathematics, works of art are analyzed using mathematical techniques. The aim is to analyze specific aspects of the work: discovering, for instance, how it was generated or verifying a hypothesis on the mathematical procedures involved. Here the work of art is used not as a support to illustrate concepts or properties, but as the object of research and experimentation. In the sixth dimension, art created with mathematics, mathematics is used to create works or compositions with an artistic component. In other words, the aim is to create a product, original or otherwise, by applying mathematical concepts and properties.

These aforementioned six art-mathematics dimensions, that will be further illustrated in the next section, were employed as a framework for analyzing the role of art in the selected mathematics textbooks. Important, these six art-mathematics dimensions need not be understood as hermetic compartments. Any given activity or exercise may involve several dimensions at the same time. The dimensions may on occasion overlap or even entail an inclusive approach. An activity characterized under dimension 5 (mathematical analysis), for instance, normally calls for mastering concepts (dimension 3) as well as calculating and measuring (dimension 2).

4. Results of textbook analysis

A total of 146 allusions to art were identified in the textbooks and classified under the following art forms: Architecture, Sculpture, Painting and Drawing, Tiling and Friezes, music and, Design of Familiar Objects and Patterns in Nature. The percentages of the art-maths dimensions for each art form are presented in Table 1, as a synthesis of the results. Specific details for each art form are provided below.

Art-Maths Dimensions	Art forms: Architecture	Sculpture	Painting	Tiling/ Friezes	Music	Design/ Nature	Total %
Ornamental	34.38 %	50%	55.56 %	54.55 %	50 %	53.13%	45.21%
Calculation/ Measurement	37.5 %	37.5%	7.41 %	9.09 %	50 %	15.63%	25.34%
Mastery Concepts	20.31 %	25%	25.93 %	9.09 %	25 %	31.25%	23.29%
Art and technology	10.94 %	-	-	18.18%	-	6.25%	7.53%
Analysis of works of art	4.69 %	-	11.11 %	36.36%	-	12.5%	9.59%
Art Creation	-	-	3.70 %	9.09%	-	-	1.37%
Total number of allusions	64	8	27	11	4	32	146

Table 1. Percentages of each Art-Mathematics dimension per each art form

Architecture

A total of 64 allusions to architecture were identified. Dimension 1 (*architecture for ornamental purposes*) accounted for 34.3 % of the allusions. Photos of structures such as the Eiffel Tower or the Parthenon were normally found adorning the first page in the introductions to chapters on geometry. Architecture appeared as a context for *calculation and measurement* (dimension 2) in 37.5 % of the cases. Pictures of historic pyramids such as at Cheops or modern structures such as the Louvre Museum at Paris were used primarily to calculate areas and volumes. Architecture provided the context for *mastering concepts* (dimension 3) in 20.3 % of the activities. As in the preceding dimension, pictures of modern or historic structures were used in activities in which students were asked to identify geometric bodies and figures. Dimensions 2 and 3 were often combined. In the following activity (Figure 1), for instance, students were asked to identify and classify the polygons forming the Louvre Museum pyramid depicted in the photo. They were also asked to calculate building distances, such as its height. Please note that the activity in Figure 1 is the original taken from the textbook with the Spanish text. An English translation of this activity, and others presented throughout this article, can be found in Supplementary Material A.

3. Uno de los museos más grandes de Europa y más visitados del mundo es el Louvre. Se localiza en París (Francia). Tiene una superficie de 210 000 m^2 y una afluencia de, aproximadamente, 9 334 000 visitantes al año.

a) ¿Qué polígono forma la planta del museo? ¿Cómo se llama este cuerpo geométrico?

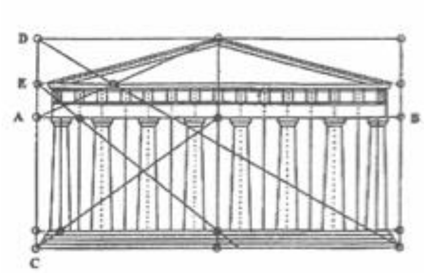
b) Clasifica los polígonos de las caras de la pirámide.

c) Calcula la altura del museo sabiendo que la arista básica mide 10 m y la apotema de la cara lateral tiene una longitud de 13 m.

d) Halla las longitudes de las aristas que no pertenecen a la base. Presenta el resultado con un decimal.

Figure 1. Louvre Museum activity drawn from Edebe 2º ESO (2016, p.254)

Just 10.9 % of activities involved dimension 4 (*art and technology*). An even smaller percentage (4.69 %) involved dimension 5 (*analysis of works of art*), with only three activities identified in all the textbooks. In one of them (Figure 2), students were presented with a drawing of the Parthenon at Athens and asked to analyze its relationship to the golden ratio. Not a single reference was identified that could be construed to induce students to create art with mathematics (dimension 6).



• Si mides algunas longitudes en este croquis del Partenón, probablemente encontrarás más de una vez el número de oro.

Figure 2. Athens Parthenon activity drawn from Marfil 4 º ESO (2008, p.267)

The activities involving architecture were mostly exercises in geometry, although a few references appeared in lessons on functions and graphs, particularly where works construction entailed the use of curves. Further details about the allusions to this form of art (Architecture) are provided in Table S1 in Supplementary Material B. This table includes a categorization of the ‘works of art’, ‘mathematical concepts’ and ‘art-maths dimensions’ identified per textbook.

Sculpture

Eight allusions to sculpture were detected. 50 % of them were used with *ornamental purposes* (dimension 1), and the 37.5 % used as a context for *calculation and measurement* (dimension 2). Sculpture as a context for *mastering concepts* (dimension 3) appeared in 25 % of the activities. The exercises involving both dimensions 2 and 3 were primarily algebraic and geometric. The former focused on solving equations and the latter on the recognition of three-dimensional bodies (polyhedra and solids of revolution), polyhedral composition, calculation of volumes and areas, symmetries and scales, and change of units. For instance, in the following activity (Figure 3), the ‘Atomium’ (which depicts the crystalline structure of iron) was used primarily in connection with dimensions 2 and 3, asking students to calculate scales and identify the geometric bodies


1. El Atomium fue construido en 1958 en la ciudad de Bruselas (Bélgica). Representa la estructura cristalina del hierro: un cubo con una esfera en cada uno de sus vértices y una esfera central, unidas todas ellas mediante cilindros. Está aumentada ciento sesenta y cinco mil millones de veces respecto a la estructura cristalina del hierro. El monumento tiene 102 m de altura y el diámetro de las esferas es de 18 m.

a) Determina, a partir del razonamiento geométrico, el número de cilindros y esferas del Atomium.

b) El tubo vertical central es un ascensor que sube a sus visitantes a 5 m/s hasta la esfera más alta. En ella se ubican un mirador y un restaurante. ¿En cuánto tiempo el ascensor lleva a cabo este recorrido?

c) Calcula el diámetro, en nanómetros, de un átomo de hierro si sabemos que 1 nm = 10^{-9} m.

d) Minimundus es un parque de miniaturas situado en Klagenfurt (Austria). En él se encuentra una réplica del Atomium a escala 1:25. ¿Cuál es la altura, en metros, de la réplica?

Figure 3. Atomium activity drawn from Edebe 2º ESO (2016, p.254)

constituting the structure. As Table S2 shows, in Supplementary Material B, allusions to this form of art (sculpture) were not addressed in the rest of the dimensions for any of the textbooks.

Painting and Drawing

Twenty-seven allusions were found to pictorial art by authors such as Kandinsky, Leonardo Da Vinci, Paul Klee and Escher, to name a few. Painting or drawing was used for ornamental purposes in 55.5 % of the cases analyzed. A smaller percentage (25.9 %) of activities involved dimension 3 (*mastering concepts*). Students were normally asked to identify and classify plane geometric figures on drawings or paintings, by authors such as Kandinsky or Paul Klee. For instance, the activity in Figure 4 illustrates *In Blue* Kandinsky painting, asking to indicate and classify the existing quadrilaterals. Prints such as Escher's 'Drawing Hands' (Figure 5) were used to enhance the mastery of the notions of symmetry.

Figure 4. In Blue Kandinsky painting activity drawn from Edebe 1º ESO (2007, p.207)

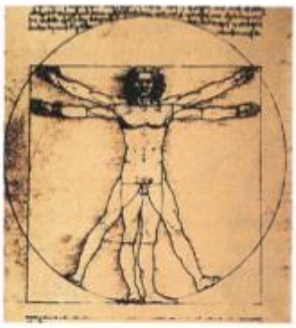
Figure 5. Escher's Drawing Hands activity drawn from Vector 1º ESO (2010, p.234)

Dimensions 2 (*calculation and measurement*) and 5 (*analysis*) informed 7.4 % and 11.1% of the activities respectively. Dimension 2 was found in solving equations activities and dimension 5 in tasks requiring the analysis of proportions and symmetries. One example of the latter was the reproduction of Leonardo Da Vinci's 'Vitruvian Man' canon, in which students were asked to verify whether the same proportions were present in other values (Figure 6).

14.8. Número de oro y arte

Aquí tienes algunos ejemplos de la utilización que el ser humano ha hecho a lo largo de la historia del número de oro. Esperamos que te sirvan como punto de partida para que realices una investigación de búsqueda de datos sobre el tema.

(1) Cánones



A lo largo de la historia han existido distintos modelos o cánones propuestos como perfectos para el cuerpo humano. El dibujo siguiente de Leonardo da Vinci está basado en el canon propuesto por el arquitecto romano Marco Vitrubio. Según este canon, debía de cumplirse la siguiente relación:

$$\frac{\text{Altura total}}{\text{Altura hasta el ombligo}} = \frac{\text{Altura hasta el ombligo}}{\text{Distancia ombligo-cabeza}}$$

a) Si denominas "a" a la altura total y "m" a la altura hasta el ombligo, ¿cómo escribirías la anterior relación?
b) Prueba para distintos valores de "a" y "m". ¿Qué relación puede existir entre ambas medidas?

Figure 6: Leonardo Da Vinci's 'Vitruvian Man' activity drawn from Marfil 4 º ESO (2008, p.266)

Dimension 6 (*art created with mathematics*) appeared in a single activity in a second year ESO textbook published by Oxford (2016, p.248). In that activity (Figure 7), which reproduced Escher's dodecagon, students were asked to find more information about the painter's oeuvre and build their own dodecahedra.

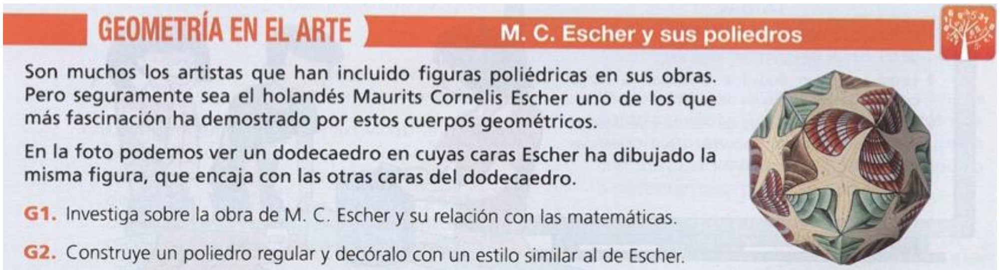


Figure 7. Escher's dodecahedron activity drawn from Oxford 2º ESO (2016, p.248)

Further information about the allusions to this form of art (Painting and drawing) are provided in Table S3 in Supplementary Material B. This table includes a categorization of the 'paintings and drawings', 'mathematical concepts' and 'art-maths dimensions' identified per textbook.

Tiling and Friezes

Eleven allusions to tiling and friezes were identified, normally through illustrations of artistic buildings such as the Alhambra in Granada and the Mosque-Cathedral of Cordoba. Escher's works were also often present. In-plane movement and the identification of plane geometric figures were the types of content most frequently appearing in conjunction with this art form.

In 54.5 % of the cases, tiling and friezes appeared for *ornamental purposes* (dimension 1),

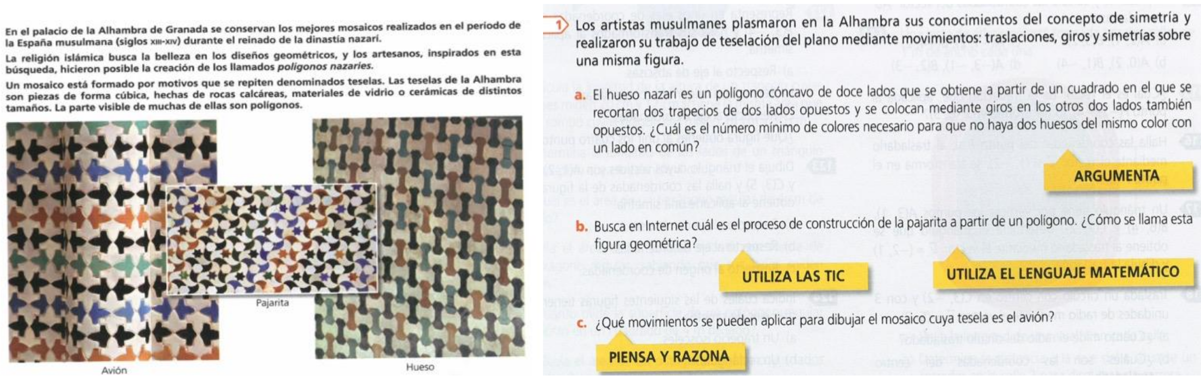


Figure 8. Alhambra tiling activity drawn from Oxford 3º ESO (2015, p.140)

primarily in geometry lessons. Dimension 5 (*analysis*) also accounted for a substantial percentage

(36.3 %) of activities, where students were asked to identify the geometric piece (tile) and the movements (symmetry, translation and rotation) from which a tiling or frieze is generated. The activity in Figure 8, for instance, asks about the tile and the movements generating an Alhambra tiling.

Dimension 4 (*art and technology*) was present in 18.1% of the activities related to Tiling and friezes. Several exercises were based solely on seeking information on the internet to understand how tile such as the ones used in the Alhambra tiling are constructed. A reduced number of activities occasionally propose the use of GeoGebra software to study homothety and similarity in tiling designs, as the one exemplified in Figure 9.

Figure 9. GeoGebra activity to study homothety and similarity drawn from Edebe 2º ESO (2016, p.223)

Dimensions 2 (*calculation and measurement*), 3 (*mastering concepts*) and 6 (*art creation*) accounted for 9 % each of the occurrences. Dimension 2 was observed in activities related to the measurement of angles and percentages. Dimension 3 was associated with exercises in which students were invited to identify plane figures and recognize in-plane movements (symmetry, rotation, and translation). Dimension 6 (*art creation*) was observed in only one activity, involving the use of GeoGebra to draw a polygon and generate from it a tiling by applying different in-plane movements.

Extra information about the allusions to this form of art (Tiling and friezes), including the categorization of 'works of art per textbook', 'mathematical concepts', and 'art-maths dimensions' identified, are provided in Table S4 in Supplementary Material B.

Music

A scant four allusions to the relationship between mathematics and music were found. They appeared only in lessons on fractions, with references to the relationships established between the two by the Pythagoreans. The only dimensions detected were 1 (*art for ornamental purposes*), 2 (*calculation and measurement*) and 3 (*mastering concepts*). Dimension 1 appeared twice, with photographs of musical instruments used as page decoration, with no mention of mathematics. The other two allusions involved dimensions 2 and 3. In one, music and mathematics appeared jointly in an activity dealing work with musical fractions (duration of notes), as exemplified in the following activity (Figure 10) drawn from a textbook published by SM for fourth year ESO (2017, p.26).

Further details about the allusions to this form of art (Music) are provided in Table S5 in

PROBLEMA RESUELTO **Fracciones musicales**

En la composición e interpretación de las piezas musicales es muy importante la duración de cada una de las notas musicales.

Según la duración, existen notas redondas, blancas, negras, corcheas, semicorcheas, fusas y semifusas. Tomando como unidad la duración de una nota negra, el resto tiene los siguientes tiempos:

Redonda	Blanca	Negra	Corchea	Semicorchea	Fusa	Semifusa
4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

1. ¿Cuántas semicorcheas hay en una blanca? ¿A cuántas blancas equivale una fusa?
2. En música se utiliza un símbolo, el puntillo, que incrementa la duración de la nota un 50 %. ¿A cuántas fusas equivaldrá una blanca con puntillo?
3. Si se coloca un doble puntillo, se aumenta la duración de la nota un puntillo y la mitad de un puntillo. ¿Qué fracción de una nota sin puntillo es esa misma nota con doble puntillo?

Figure 10. Musical fractions activity drawn from SM 4º ESO (2017, p.26)

Supplementary Material B. This table includes a categorization of the 'music elements', 'mathematical concepts' and 'art-maths dimensions' identified per textbook.

Design of Familiar Objects and Patterns in Nature

Thirty-two allusions to designs relating to mathematics were identified. Dimension 1 (*ornamental purposes*) appeared in 53 %, being most commonly depicted illustrations credit cards, logotypes, Nautilus shells and animal hides, with no explicit reference to mathematics. Occasionally, the illustrations are accompanied with a text related to a mathematical idea, but without explicitly treating or mastering such a concept or idea, as is the case with the designs in Figure 11, drawn from Edebe 2ºESO (2017, p.194).

Figure 11. Examples of familiar object related to the golden ratio, drawn from Edebe 2ºESO (2017, p.194)

Figure 12. Butterfly symmetry activity drawn from SM 4º ESO (2017, p.26)

Dimension 3 (*mastering concepts*) was found in 31.2 %. These cases normally revolved around industrial design or patterns in nature to illustrate a mathematical concept or property, such as the golden ratio, the golden spiral, in-plane movement, symmetry, sequences, parabolas, and plane/solid figures. In the activity of the butterfly in Figure 12, for instance, students were asked to determine whether it exhibited axial symmetry.

Dimension 2 (*calculation and measurement*) appeared in 15.6 %. In these activities students measured or calculated the size of familiar or natural objects in exercises revolving around the golden number. In the activity depicted in Figure 13, and drawn from a textbook published by Marfil, students are asked to measure the sides of a credit card and determine whether it conforms to the golden ratio.

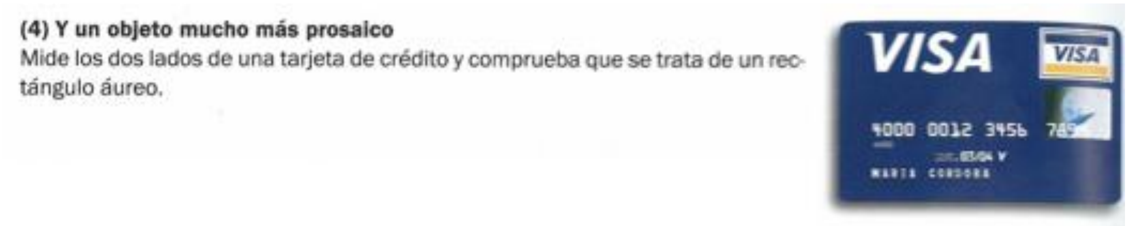


Figure 13. Golden ratio credit card activity drawn from Marfil 4 º ESO (2008, p.268)

Dimension 5 (*analysis*) was found in 12.5 %. The activities proposed involved exploring concepts associated with symmetry and analyzing familiar or natural objects in terms of the golden number. In the activity showed in Figure 14, students were asked to describe how Dürer's spiral was constructed.

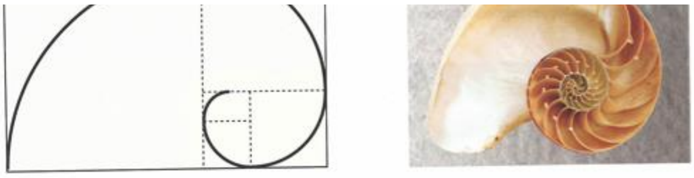


Figure 14. Dürer's spiral activity drawn from Marfil 4 º ESO (2008, p.265)

Dimension 4 (*art and technology*) was present in 6.25 % of the activities, although it was confined to Internet searches on natural elements exhibiting the golden number or animal hides simulating tiling, as the activity in Figure 15. Dimension 6 (*art creation*) was not represented in any of the activities proposed in the textbooks analyzed. Further details about the allusions to this form of art (*design/nature*) are provided in Table S6 in Supplementary Material B. This table includes a categorization of the ‘identified everyday objects and nature elements’ per textbook, the ‘mathematical concepts’ involved, and ‘the art-maths dimensions’ comprised.

Figure 15. Animal hides and tilings activity drawn from Edebe 2º ESO (2016, p.223)

It is noteworthy that the majority of the allusions to art in the Spanish mathematics textbooks incorporate pictures (e.g. drawings and images). Few references to art without pictures were found. Just some exercises asked for the area or volume of a famous construction, providing its dimensions without the support of a picture.

5. Discussion

The following is a discussion of the most prominent results for the six dimensions analyzed: (1) art for ornamental purposes; (2) art as a context for calculation and measurement; (3) art as a context for mastering concepts; (4) art as a context for using technological resources in mathematics; (5) mathematical analysis of works of art; and (6) art created with mathematics.

Dimension 1, art as ornament, was present in the textbooks in a large proportion with a 45.2 % of the artistic works used for purely decorative purposes. The majority of those were found on covers or title pages with no text, the underlying mathematical idea was difficult to interpret or infer. Works of art were also reproduced in the margins of pages bearing theoretical introductions to a mathematical concept, with no explicit reference in the text to the piece or in the legend to the title or author. One example can be found in a textbook section on similarity, published by Anaya (2008, p.169), reproducing the 'Mona Lisa' with no reference to the illustration in the text (see Figure 16). The relevance of the work was not clearly explained, nor was any obvious connection to mathematics established. The percentage of dimension 1 occurrences was fairly evenly distributed across the six art forms (architecture, sculpture, painting, tiling and friezes, music and design of familiar objects and patterns in nature).

Figure 16. Mona Lisa Illustration drawn from Anaya 4º ESO (2008, p.169)

Dimension 2, art as a context for calculation and measurement, accounted for 25.3 % of the allusions. In the activities proposed students were usually asked to calculate areas or volumes, and to measure works of art from pictures. The purpose was to provide a real-world context to express a mathematical idea or concept, but the works were unnecessary from the standpoint of solving the mathematical problem. In the activity in Figure 1, for instance, involving the merely procedural calculation of the size of the Louvre Museum pyramid, the picture itself was unnecessary: applying the respective algorithms to the data given would have sufficed. Any other non-artistic quadrangular pyramid would have served the same purpose.

Other activities called for measuring a work of art picture or a familiar object design with a ruler. The activity associated with Figure 13 (in an early section), for instance, entailed verifying whether a credit card represent the golden ratio. Measuring with a ruler would afford students an approximate value, which would not be sufficient to determine whether the object actually conforms the golden ratio. If they choose to measure the card with a calliper, the value obtained (1.58515851...) would be nearly the golden number, but would not be the actual card ratio (1.585772508 given by the international standard ISO/IEC 7810:2003). If a technological tool such as GeoGebra were to be used to measure the card (see Figure 17), the value found, 1.60197648, would be closer to the golden number but farther away from the actual value. More than proving the conformity of the object to the golden ratio, this exercise aimed to teach students the importance of measuring with different tools, bearing in mind the characteristic errors of each, and the notion of approximation.

Dimension 3, art as a context for mastering concepts, was present in 23.3 % of the cases identified. This dimension was found in all six art forms (architecture, sculpture, painting, tiling and friezes, music, and design of familiar objects and patterns in nature). In particular, it was widely found in activities related to the design of familiar and natural objects, and only rarely in activities related to tiling and friezes. Table 2 lists the mathematical concepts that explicitly appear in relation to art and the ratio of occurrence of each. The same concept was often

Figure 17. Credit card measured with GeoGebra Software (own elaboration)

found in several works of art and in different art forms. Likewise, in a single work of art different concepts were also integrated.

Mathematical Concept	Number of Allusions
Geometry figures and measurements	42
Golden ratio	23
Symmetry	16
Geometric transformation	13
Similarity	11
Functions	11
Arithmetic	11
Proportionality	6
Equations	4
Scales	2
Sequences	2

Table 2: Mathematical Concepts associated to the Art forms

It is worth noting that concepts such as infinity, limits, continuity and perspective, often present in the works of art, were not explicitly addressed. They could have easily been integrated in some of the proposed activities. To cite one example, the activity in Figure 18 presents the Escher's 'Smaller and Smaller' print, and asks to identify the in-plane movement and tile generating the tiling. This print could have also been used to work the notions of infinity and limits.

Figure 18. 'Smaller and Smaller' print activity drawn from Marfil 4º ESO (2008, p.310)

In some cases, the activities addressed concepts too superficially. Students were normally asked to identify a curve in a picture on the sole grounds of observation. For instance, the activity in Figure 14 (in a previous section) aimed to teach the golden ratio by establishing a visual relationship between a Nautilus picture and Dürer's spiral. Greater benefit could have been derived from this activity if it had enlarged upon the concept and analyzed other types of spirals (e.g. Algorithmic, or Archimedean) that might provide a better fit to the Nautilus. That would have also afforded the opportunity to address the notion of approximation.

Similar examples were found in the lessons on functions and graphs, where normally the aim was to work with parabolas by identifying pictures of curves by mere observation. In some cases, the curve shown in the picture does not match the graph of a function. In others, such as in Figure 19, the symmetry of a curve was difficult to perceive because the photo was not taken perpendicular to the line of sight.

Figure 19. Identifying the function related to the curve, activity drawn from Oxford 3 º ESO (2015, p.250)

Dimension 4, art as a context for using technology in mathematics, accounted for a much lower percentage (7.5 %) of occurrences than dimensions 1, 2 and 3. It appeared only in connection with three art forms: architecture, tiling and friezes, and design of familiar objects and patterns in nature. Most of the activities focused on internet searches for information: on the tile generating the Alhambra tiling (as exemplified earlier in the activity of Figure 8); on tiling-like patterns found in nature (as illustrated in Figure 15); and on natural objects conforming to the golden ratio (as in Dürer's spiral activity in Figure 14). Art was seldom used as a backdrop for applying computer software to mathematics. Only a few activities were observed in which students were expected to use GeoGebra to draw a polygon and build a tiling or practise with homothety and similarity in tiling design (as was the case with the Figure 9 activity).

No activities were found in which the use of GeoGebra or other educational software proved to be useful for measuring distances, building structures, verifying properties or mastering concepts that would be suited to exercises involving works of art. Activities could have been, for instance, proposed to use GeoGebra to build a golden rectangle to determine whether a credit card was exactly such a rectangle, as demonstrated in Figure 20.

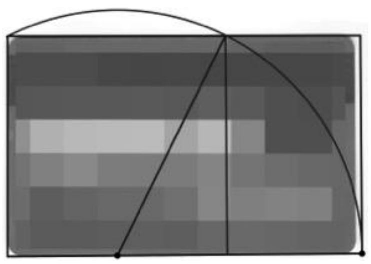


Figure 20. Use of GeoGebra to verify whether or not a credit card is a golden rectangle (own elaboration)

Dimension 5, mathematical analysis of works of art, accounted for just 9.6 % of cases. Although this dimension was present in architecture, painting and design of familiar objects and patterns in nature, it was most visible in tiling and friezes. The activities in question focused on determining which geometric transformations (translation, rotation or symmetry) led to a tiling or frieze, and to identify the shape from which a given tile is generated. An example of the latter is the activity described earlier in Figure 8, and drawn from Oxford 3º ESO (2015, p.140).

Barring the activities with a certain analytical content described in the results, no others called for an in-depth analysis of works of art from the mathematical standpoint. Most activities related mathematics and art very superficially. Figure 11, described earlier, is an example of it: the Twitter logotype appears as an example of a golden ratio design, without asking students to undertake any critical analysis on this relationship. This is unexpected considering the Spanish curriculum highlights analysis as a main competence. A deeper investigation or exploration would have revealed that the sketches for the geometric construction of this logotype were created from arcs of a circumference (Figure 21); a feature of no minor interest from the perspective of the art-mathematics relationship. That explanation would also have entailed studying the application of mathematics to the creation of the logotype.

Textbooks were also observed to propose activities relating Leonardo Da Vinci's Vitruvian Man to the golden ratio, in most cases with superficial analyzes. That work is a canon on the proportions of the human body, of particular significance in art. In an educational context it could be used to propose more analytical activities, such as:

- studying which canon represents the models (normally a wooden figure) used by each artist for their works.
- measuring classroom students' bodies and generating data tables for statistical processing to establish relationships between different dimensions, and compare the results with existing canons and anthropometric measurement tables.

Such activities would analytically combine art, proportions and statistics and involve minor research that could be very useful to students.

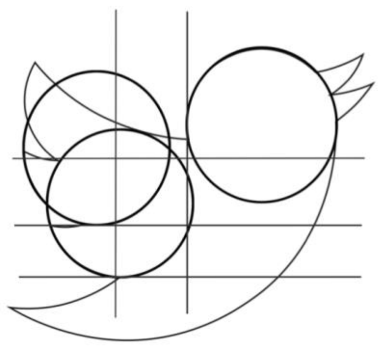


Figure 21. Sketch for the geometric construction of Twitter logotype, retrieved October 1st 2017 from https://dribbble.com/twitter/projects/105370-Brand

Dimension 6, art created with mathematics, was the dimension with the lowest presence (1.4 %). As suggested in the results section, this dimension was found only in connection with pictorial art and the generation of polygons to create tiling and friezes, as well as constructing polyhedrons decorated with artistic paintings, as proposed in Escher's dodecahedron activity (Figure 7).

One activity using pictorial art (drawings or prints) could have included creating compositions with geometric elements inspired for instance by the pieces earlier mention of Kandinsky or Escher. Works such as Dalí's 'Atomic Leda'— formulated around the Pythagorean star (see Figure 22)— could have been given a similar use, taking the artist's idea as the point of departure. For instance, a picture could be drawn using Dürer's shell curves.

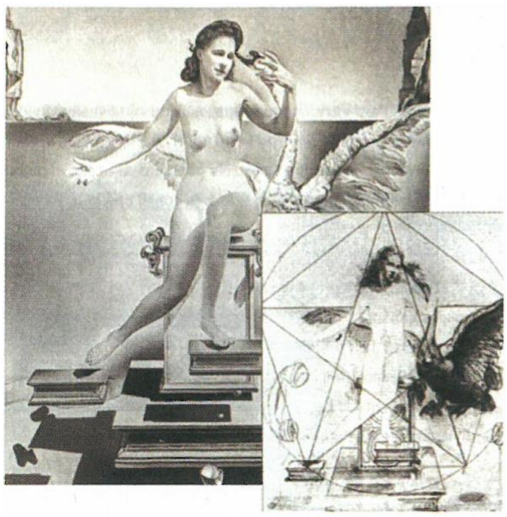


Figure 22. Atomic Leda drawn from Marfil, (2008 p. 267)

Additional exercises involving sculpture might have included clay reproductions for the canons used by artists, or an original creation to be compared to existing canons. Scale models of architectural structures or sculptures could also be built. As noted in the results section, in several activities students were asked to recognize arcs and parabolas in certain illustrations. That could be taken further, using mathematics to reproduce architectural elements on a different scale. One activity, for instance, might involve making a scale model of the 'Atomium' described in earlier sections to help master geometric and arithmetic concepts.

The only activities proposed in connection with music were related to the duration of musical notes (measurement). That could have been enhanced by teaching students to use the value of the musical notes to create bars. Each student might, for instance, create a melody with six bars and four beats per bar, using certain musical notes (minim, crotchet, quaver, semiquaver) and their rests. A melody could be composed by simulating Mozart's 'Musical Dice Game': assigning each bar a number from one to six and throwing a die six times to determine the order of the bars and the number of times they are repeated in the melody. This activity would involve working with fractions (the sum of the duration of each note and its rest should be one), probability (the likelihood of a bar coming up on the die) and combinations (analysis of all the possible melodies that could be generated).

As noted earlier in the section on design, most activities sought to establish connections between designs and the golden ratio, and to identify axes of symmetry. It would have been useful if in connection with this art form each student had been invited to create their own logotype from geometric elements and constructions only. Such an exercise would also involve applying other dimensions such as mastering concepts or calculation, with or without technology.

6. Conclusions

This study analyzed the relationship between art and mathematics in the Spanish education system and more specifically in textbooks for secondary school students aged 12 to 16. The relationship between art and mathematics was assessed on the grounds of six parameters: (1) art for ornamental purposes; (2) art as a context for calculation and measurement; (3) art as a context for mastering concepts; (4) art as a context for using technological resources in mathematics; (5) mathematical analysis of art; and (6) art created with mathematics.

The 146 allusions identified in the textbooks, and classified in different art forms (architecture, sculpture, etc.), were analyzed in terms of the above dimensions to study the relationship between art and mathematics. Architecture, with over 64 references, prevailed over the other art forms, followed by Design of Familiar Objects and Patterns in Nature with 32, and Painting and Drawing with 27. Fewer references were found to Tiling and Friezes (11), Sculpture (8) and music (4). The Eiffel Tower, the pyramids at Cheops and the Parthenon were the architectural works most often cited. In painting, two works authored by Leonardo Da Vinci predominated: 'Mona Lisa' and 'Vitruvian Man'. The tiling and friezes most frequently found were those in the Alhambra at Granada, the Mosque at Cordoba and several of Escher's works. The designs most often reproduced were credit cards and the Nautilus shell. No sculpture stood out over the others and no specific piece of music was cited.

The distribution of the identified activities by dimension revealed a clear predominance of 'art as an ornamental element' (dimension 1) in mathematics textbooks. Over 45.2 % of the activities were classified under this dimension, accounting for over 50 % of the occurrences in five of the art forms. The percentage of activities under dimensions 4 (art and technology), 5 (analysis) and 6 (art creation), which involved a more active role of the student, was under 13 %. Dimension 4 (art and technology) appeared in only some of the art forms and was generally associated with information searches. Dynamic geometry software was present in a few activities involving the study of tiling and friezes. Dimension 5 (analysis) was also found primarily in connection with that art form. With only two activities, the presence of dimension 6 (art creation) was negligible.

In terms of content, nearly 28.8 % of the activities involving artistic elements entailed working with and measuring (distance, area, volume) geometric figures. That was followed by exercises on the golden ratio (15.75 %), symmetry (10.96 %) and geometric transformations (8.9 %). With the exception of the golden ratio, those are the contents explicitly recommended in the Spanish curricular guidelines for working with art as a context in mathematics. Barely any of the activities relating to the golden section prompted experimentation or reflection. They seldom involved verifying a hypothesis or assumption or working with the errors implicit in measurement. Nor did they induce a critical attitude toward the wide dissemination of this topic, which is not always dealt with as rigorously as required. Generally speaking, very few activities were found in which technological resources were applied to explore the art-mathematics connection. The proliferation in recent years of the use of dynamic geometry software for the study of the area seems to have had little effect on textbooks.

On the whole, in most of the activities mathematics were neither significant to the works of art illustrated nor used to analyze them. They were merely an underlying medium. This study shows that the attempts made to illustrate the mathematics-art connection in today's world do not suffice. The solution is not to adorn mathematics activities with hosts of examples of the presence of mathematics in the real world, but to furnish good examples that support the mathematical knowledge being taught in classrooms and liable to be used to analyze and create art with mathematical tools.

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Is the relationship between art and mathematics addressed thoroughly in Spanish secondary school textbooks?

Jose Manuel Diego-Mantecón

Universidad de Cantabria, MATESCO, Facultad de Ciencias

Avda. de los Castros 48, 39005 Santander (Spain) - diegojm@unican.es

Teresa F. Blanco

Universidade de Santiago de Compostela, Departamento de Didácticas Aplicadas, Facultade de Ciencias da Educación

Avda. Xoan XXIII s/n, 15782 Santiago de Compostela (Spain) - teref.blanco@usc.es

José Benito Búa Ares

Xunta de Galicia, I.E.S. Sánchez Cantón

Avda. Raíña Vitoria s/n, 36001 Pontevedra (Spain) - jbenitobua@gmail.com

Pablo González Sequeiros

Universidade de Santiago de Compostela, Departamento de Didácticas Aplicadas, Facultade de Formación do Profesorado

Avda. Ramón Ferreiro s/n, 27001 Lugo (Spain) - pablo.gonzalez.sequeiros@usc.es

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Abstract

This article discusses a review of the relationship between art and mathematics in Spanish secondary school mathematics textbooks. The art-mathematics connection identified in the textbooks was analyzed under six dimensions: (1) art for ornamental purposes; (2) art in calculation and measurement; (3) art to master concepts; (4) art to use technological resources in mathematics; (5) mathematical analysis of art; and (6) creating art with mathematics. Dimensions 1, 2 and 3 clearly prevailed over dimensions 4, 5 and 6, which called for more active participation and analytical reflection. Most of the activities attempted to *illustrate* the mathematics-art connection with real-world examples, but rarely entailed verifying a hypothesis or assumption nor did they encourage critical thinking for analyzing and creating art with mathematical or technological tools.

Keywords: art, mathematics, textbooks, secondary education, curriculum

1. Introduction

Art and mathematics, two expressions of human creativity, share many distinctive traits, such as the friction between creation and technique, intuition and rigour, certainty and uncertainty. The relationship between the two disciplines is complex and can be broached from at least two perspectives [5, 11]. For the artist, artistic creation prevails over mathematics, being considered little more than a tool to generate forms and constructions. It often supplies geometric concepts and procedures, which, given their properties, are technically useful. The artistic creations in which mathematics is used are closely related to mathematical notions and properties. For the mathematician, whilst a mathematical notion may on occasion give rise to something that could be regarded as an artistic creation, that outcome is no more than a consequence of the mathematical notion [6, 9]. Its 'beauty' might be said to lie in the mathematical notion itself, the embodiment and sensory perceptibility of which are secondary.

As a result of European recommendations for educators, the relationship between art and mathematics has steadily acquired greater importance nowadays [7, 27]. The question is whether that relationship has found its way onto real world curricula. On the grounds of reference frameworks for Spanish curricula, the answer would appear to be affirmative [20, 26, 31]. Nonetheless, curricular trends and guidelines do not always translate into classroom practice [30]. In fact, the discordance between the intended and implemented mathematics curriculum has been widely reported across different educational contexts [21, 32]. Although there is a large number of studies addressing the relationship between art and mathematics from both a research and educational perspective, see, for example, [6, 18, 28, 29], as far as we know, there is a lack of studies focusing on how this art-mathematics relationship is established in the school lessons.

In the Spanish context, school lessons are strongly influenced by textbooks. These are basic resources for teachers, not only as instructional support, but also as a substantive component in lesson planning [1, 14, 19, 24, 33]. Textbooks may therefore be used as an indicator of the content addressed in the classroom. An analysis of the relationship between art and mathematics may be conducted under that premise, seeking to ascertain the extent to which the interaction between them is addressed in the classroom. This article broaches the question by specifically analyzing the presence of art in secondary school mathematics textbooks in Spain. For this purpose, what we mean by art is limited to visual arts and music. This approach is justified from a curricular point of view and excludes literature because of its independence from the other two disciplines, in the Spanish scholar itineraries.

2. Today's methodological and curricular trends

The European Union's recommendations for increasing the number of students enrolling scientific-technological subjects encourage the implementation of STEAM activities [10]. These activities integrate all scientific areas (Science and Mathematics) and their most prominent applications in modern society (Engineering, Technology and Art) [8, 12]. Interdisciplinarity plays an essential role in these activities, deliberately eschewing a traditional educational approach in which knowledge appears to be divided into unconnected subjects [16, 25].

The priorities defined by the EU have had visible consequences for Spain’s official compulsory secondary education (Spanish abbreviation, ESO) and baccalaureate curricula [20]. The new approaches to learning and assessment aim to promote student basic competences such as linguistic and STEAM skills. The methodology is primarily active and participative, furthering interdisciplinary investigative learning based on problem solving, teamwork and interactive groups [13, 15]. A number of skills must be integrated in planning and conducting teaching and learning activities to attain the ultimate goal of enabling students to apply their academic learning to other contexts [4, 17]. Information and communication technologies are routinely used in learning activities, in particular in information searches and analyzes, as well as in the presentation of the research work conducted [23]. In the specific case addressed here, mathematics and art, the Spanish curriculum for secondary school students (12-16 years old) establishes an explicit connection between the two [20].

On the one hand, the mathematics curriculum highlights two main aspects of this subject [20]. The first is that mathematics is indispensable for simplifying, abstracting, interpreting, expressing and understanding social realities, including artistic reality and creativity. Secondly, it is an instrumental tool to progressing in the knowledge acquisition of other disciplines, functioning as a driver of culture and civilization development. In particular, in the Geometry section of the curriculum, the evaluation criteria refer explicitly to that artistic perspective, both for students who plan to pursue their schooling further and for those who do not [20]. Stress is placed, for instance, on Thales’ theorem and the standard formulas for obtaining distance, area and volume using artistic expressions such as painting or architecture as real-world examples. Students are asked to recognize, apply and analyze in-plane transformations of works of art, patterns present in nature, and familiar objects designs. The curriculum also makes explicit mention of the identification of centres, axes and planes of symmetry in nature, art and human constructions.

On the other hand, in the art curriculum, mathematical competence contributes to the development of the visual arts and music subjects [20]. In the case of the visual arts subject, students are asked to use a variety of types of logical reasoning and spatial visualization to explain and describe artistic and technical characteristics. Mathematics is applied to analyze and describe the proportions and canons used in art. Students are also taught to recognize the geometric shapes used in art. Space and form, so important to artistic creation, will help students coding and de-coding visual information in works of art, allowing the recognition of patterns, properties, positions, and representations of objects. In the Music subject, mathematical competence is required for reading and interpreting the singularities of music scores. Mathematics reinforces mental capacities such as understanding and logical structuring, enabling students to make personal decisions for creating and improvising music.

In Spain, curricular decisions weigh heavily on the design and formulation of textbooks, one of teachers' most fundamental resources. Their importance and influence on classroom activity are attested by the amount of research conducted on the resource in recent years [1, 2, 3, 22]. The foregoing supports the relevance of analyzing the connection between art and mathematics in Spanish textbooks, and exploring to what extent the European curricular decisions are applied in the Spanish context. Due to the limitations of the study, this article focuses solely on the presence of art in mathematics textbooks. So, the following question raises here: What is the role of art in the mathematics textbooks of the Spanish secondary school?

3. Methods

This section describes the sample selection as well as the data analysis strategy employed for the analysis of the selected textbooks.

Sample selection

The Spanish textbooks selected for this study were the ones most often used in secondary schools. The textbooks analyzed cover all four years of compulsory secondary education, i.e., used by students 12 to 16 years of age. They were released between 2007 and 2017 by eight publishers:

Anaya, Bruño, Edebe, Marea Verde, Marfil, Oxford, SM, and Vector. This was a purposive sampling selecting specific textbooks used by two universities in the north of Spain, to which these authors have access. The aforementioned universities made this selection, as a sample of the most common publishers of textbooks in Spain. These books are used in both universities for training pre-service teachers in their PGCE (Postgraduate Certificate in Education) courses. The PGCE is a one-year higher education course in Spain, which provides training in order to allow graduates to become teachers in secondary education. Please note that this is not a complete sample of all the textbooks in Spain, and thus results cannot be generalized to the whole range of textbooks in the country.

Data Analysis strategy: Art-Mathematics dimensions

The textbooks were analyzed by experts in mathematics education. They search for art allusions in different parts of the textbook chapters (cover page, introduction, theory, activities, problems and so on), and different subject areas (algebra, geometry, functions and statistics). In this way, a textual and pictorial analysis was conducted all throughout the textbooks. In a preliminary data-driving analysis of the textbooks, several art-mathematics relationships were identified and categorized into the following six dimensions: (1) art for ornamental purposes (i.e., to make textbooks more appealing); (2) art as a context for calculation and measurement; (3) art as a context for mastering concepts; (4) art as a context for using technological resources in mathematics; (5) mathematical analysis of works of art; and (6) art created with mathematics.

In the first dimension, art for ornamental purposes where no explicit mention is made of mathematics, the artistic element is used only to adorn the written text, by way of decoration or motivation. Specific mention of the text adorned is often absent and the mathematical idea that evokes the artistic image is normally imperceptible for the non-expert reader. In the second dimension, art as a context for calculation and measurement, the artistic element affords a backdrop for the activity, but is not actually necessary to conduct it or understand the concept. It is an excuse for calculating and measuring, with no reference made to the history or formulation of the work of art. In the third dimension, art as a context for mastering concepts, the artistic object is used to exemplify a mathematical concept or property, seeking to enhance its comprehension or awareness of its significance. The mathematical object is not used to analyze the work of art; it involves no research or analysis.

In the fourth dimension, art as a context for applying technological resources to mathematics, art is used as a means or context for applying technological tools (such as computers, mathematical software or apps) to analyze or verify mathematical concepts and properties. In the fifth dimension, art as a context to analyze and explore mathematics, works of art are analyzed using mathematical techniques. The aim is to analyze specific aspects of the work: discovering, for instance, how it was generated or verifying a hypothesis on the mathematical procedures involved. Here the work of art is used not as a support to illustrate concepts or properties, but as the object of research and experimentation. In the sixth dimension, art created with mathematics, mathematics is used to create works or compositions with an artistic component. In other words, the aim is to create a product, original or otherwise, by applying mathematical concepts and properties.

These aforementioned six art-mathematics dimensions, that will be further illustrated in the next section, were employed as a framework for analyzing the role of art in the selected mathematics textbooks. Important, these six art-mathematics dimensions need not be understood as hermetic compartments. Any given activity or exercise may involve several dimensions at the same time. The dimensions may on occasion overlap or even entail an inclusive approach. An activity characterized under dimension 5 (mathematical analysis), for instance, normally calls for mastering concepts (dimension 3) as well as calculating and measuring (dimension 2).

4. Results of textbook analysis

A total of 146 allusions to art were identified in the textbooks and classified under the following art forms: Architecture, Sculpture, Painting and Drawing, Tiling and Friezes, music and, Design of Familiar Objects and Patterns in Nature. The percentages of the art-maths dimensions for each art form are presented in Table 1, as a synthesis of the results. Specific details for each art form are provided below.

Art forms

Art-Maths Dimensions	Architecture	Sculpture	Painting	Tiling/ Friezes	Music	Design/ Nature	Total %
Ornamental	34.38 %	50%	55.56 %	54.55 %	50 %	53.13%	45.21%
Calculation/ Measurement	37.5 %	37.5%	7.41 %	9.09 %	50 %	15.63%	25.34%
Mastery Concepts	20.31 %	25%	25.93 %	9.09 %	25 %	31.25%	23.29%
Art and technology	10.94 %	-	-	18.18%	-	6.25%	7.53%
Analysis of works of art	4.69 %	-	11.11 %	36.36%	-	12.5%	9.59%
Art Creation	-	-	3.70 %	9.09%	-	-	1.37%
Total number of allusions	64	8	27	11	4	32	146

Table 1. Percentages of each Art-Mathematics dimension per each art form

Architecture

A total of 64 allusions to architecture were identified. Dimension 1 (*architecture for ornamental purposes*) accounted for 34.3 % of the allusions. Photos of structures such as the Eiffel Tower or the Parthenon were normally found adorning the first page in the introductions to chapters on geometry. Architecture appeared as a context for *calculation and measurement* (dimension 2) in 37.5 % of the cases. Pictures of historic pyramids such as at Cheops or modern structures such as the Louvre Museum at Paris were used primarily to calculate areas and volumes. Architecture provided the context for *mastering concepts* (dimension 3) in 20.3 % of the activities. As in the preceding dimension, pictures of modern or historic structures were used in activities in which students were asked to identify geometric bodies and figures. Dimensions 2 and 3 were often combined. In the following activity (Figure 1), for instance, students were asked to identify and classify the polygons forming the Louvre Museum pyramid depicted in the photo. They were also asked to calculate building distances, such as its height. Please note that the activity in Figure 1 is the original taken from the textbook with the Spanish text. An English translation of this activity, and others presented throughout this article, can be found in Supplementary Material A.

Just 10.9 % of activities involved dimension 4 (*art and technology*). An even smaller percentage (4.69 %) involved dimension 5 (*analysis of works of art*), with only three activities identified in all the textbooks. In one of them (Figure 2), students were presented with a drawing of the Parthenon at Athens and asked to analyze its relationship to the golden ratio. Not a single reference was identified that could be construed to induce students to create art with mathematics (dimension 6).

The activities involving architecture were mostly exercises in geometry, although a few references appeared in lessons on functions and graphs, particularly where works construction entailed the use of curves. Further details about the allusions to this form of art (Architecture) are provided in Table S1 in Supplementary Material B. This table includes a categorization of the ‘works of art’, ‘mathematical concepts’ and ‘art-maths dimensions’ identified per textbook.

Sculpture

Eight allusions to sculpture were detected. 50 % of them were used with *ornamental purposes* (dimension 1), and the 37.5 % used as a context for *calculation and measurement* (dimension 2). Sculpture as a context for *mastering concepts* (dimension 3) appeared in 25 % of the activities. The exercises involving both dimensions 2 and 3 were primarily algebraic and geometric. The former focused on solving equations and the latter on the recognition of three-dimensional bodies (polyhedra and solids of revolution), polyhedral composition, calculation of volumes and areas, symmetries and scales, and change of units. For instance, in the following activity (Figure 3), the 'Atomium' (which depicts the crystalline structure of iron) was used primarily in connection with dimensions 2 and 3, asking students to calculate scales and identify the geometric bodies constituting the structure. As Table S2 shows, in Supplementary Material B, allusions to this form of art (sculpture) were not addressed in the rest of the dimensions for any of the textbooks.

Painting and Drawing

Twenty-seven allusions were found to pictorial art by authors such as Kandinsky, Leonardo Da Vinci, Paul Klee and Escher, to name a few. Painting or drawing was used for ornamental purposes in 55.5 % of the cases analyzed. A smaller percentage (25.9 %) of activities involved dimension 3 (*mastering concepts*). Students were normally asked to identify and classify plane geometric figures on drawings or paintings, by authors such as Kandinsky or Paul Klee. For instance, the activity in Figure 4 illustrates *In Blue* Kandinsky painting, asking to indicate and classify the existing quadrilaterals. Prints such as Escher's 'Drawing Hands' (Figure 5) were used to enhance the mastery of the notions of symmetry.

Dimensions 2 (*calculation and measurement*) and 5 (*analysis*) informed 7.4 % and 11.1% of the activities respectively. Dimension 2 was found in solving equations activities and dimension 5 in tasks requiring the analysis of proportions and symmetries. One example of the latter was the reproduction of Leonardo Da Vinci's 'Vitruvian Man' canon, in which students were asked to verify whether the same proportions were present in other values (Figure 6).

Dimension 6 (*art created with mathematics*) appeared in a single activity in a second year ESO textbook published by Oxford (2016, p.248). In that activity (Figure 7), which reproduced Escher's dodecagon, students were asked to find more information about the painter's oeuvre and build their own dodecahedra.

Further information about the allusions to this form of art (Painting and drawing) are provided in Table S3 in Supplementary Material B. This table includes a categorization of the ‘paintings and drawings’, ‘mathematical concepts’ and ‘art-maths dimensions’ identified per textbook.

Tiling and Friezes

Eleven allusions to tiling and friezes were identified, normally through illustrations of artistic buildings such as the Alhambra in Granada and the Mosque-Cathedral of Cordoba. Escher’s works were also often present. In-plane movement and the identification of plane geometric figures were the types of content most frequently appearing in conjunction with this art form.

In 54.5 % of the cases, tiling and friezes appeared for *ornamental purposes* (dimension 1), primarily in geometry lessons. Dimension 5 (*analysis*) also accounted for a substantial percentage (36.3 %) of activities, where students were asked to identify the geometric piece (tile) and the movements (symmetry, translation and rotation) from which a tiling or frieze is generated. The activity in Figure 8, for instance, asks about the tile and the movements generating an Alhambra tiling.

Dimension 4 (*art and technology*) was present in 18.1% of the activities related to Tiling and friezes. Several exercises were based solely on seeking information on the internet to understand how tile such as the ones used in the Alhambra tiling are constructed. A reduced number of activities occasionally propose the use of GeoGebra software to study homothety and similarity in tiling designs, as the one exemplified in Figure 9.

Dimensions 2 (*calculation and measurement*), 3 (*mastering concepts*) and 6 (*art creation*) accounted for 9 % each of the occurrences. Dimension 2 was observed in activities related to the measurement of angles and percentages. Dimension 3 was associated with exercises in which students were invited to identify plane figures and recognize in-plane movements (symmetry, rotation, and translation). Dimension 6 (*art creation*) was observed in only one activity, involving the use of GeoGebra to draw a polygon and generate from it a tiling by applying different in-plane movements.

Extra information about the allusions to this form of art (Tiling and friezes), including the categorization of 'works of art per textbook', 'mathematical concepts', and 'art-maths dimensions' identified, are provided in Table S4 in Supplementary Material B.

Music

A scant four allusions to the relationship between mathematics and music were found. They appeared only in lessons on fractions, with references to the relationships established between the two by the Pythagoreans. The only dimensions detected were 1 (*art for ornamental purposes*), 2 (*calculation and measurement*) and 3 (*mastering concepts*). Dimension 1 appeared twice, with photographs of musical instruments used as page decoration, with no mention of mathematics. The other two allusions involved dimensions 2 and 3. In one, music and mathematics appeared jointly in an activity dealing work with musical fractions (duration of notes), as exemplified in the following activity (Figure 10) drawn from a textbook published by SM for fourth year ESO (2017, p.26).

Further details about the allusions to this form of art (Music) are provided in Table S5 in Supplementary Material B. This table includes a categorization of the 'music elements', 'mathematical concepts' and 'art-maths dimensions' identified per textbook.

Design of Familiar Objects and Patterns in Nature

Thirty-two allusions to designs relating to mathematics were identified. Dimension 1 (*ornamental purposes*) appeared in 53 %, being most commonly depicted illustrations credit cards, logotypes, Nautilus shells and animal hides, with no explicit reference to mathematics. Occasionally, the illustrations are accompanied with a text related to a mathematical idea, but without explicitly treating or mastering such a concept or idea, as is the case with the designs in Figure 11, drawn from Edebe 2ºESO (2017, p.194).

Dimension 3 (*mastering concepts*) was found in 31.2 %. These cases normally revolved around industrial design or patterns in nature to illustrate a mathematical concept or property, such as the golden ratio, the golden spiral, in-plane movement, symmetry, sequences, parabolas, and plane/solid figures. In the activity of the butterfly in Figure 12, for instance, students were asked to determine whether it exhibited axial symmetry.

Dimension 2 (*calculation and measurement*) appeared in 15.6 %. In these activities students measured or calculated the size of familiar or natural objects in exercises revolving around the golden number. In the activity depicted in Figure 13, and drawn from a textbook published by Marfil, students are asked to measure the sides of a credit card and determine whether it conforms to the golden ratio.

Dimension 5 (*analysis*) was found in 12.5 %. The activities proposed involved exploring concepts associated with symmetry and analyzing familiar or natural objects in terms of the golden number. In the activity showed in Figure 14, students were asked to describe how Dürer's spiral was constructed.

Dimension 4 (*art and technology*) was present in 6.25 % of the activities, although it was confined to Internet searches on natural elements exhibiting the golden number or animal hides simulating tiling, as the activity in Figure 15. Dimension 6 (*art creation*) was not represented in any of the activities proposed in the textbooks analyzed. Further details about the allusions to this form of art (*design/nature*) are provided in Table S6 in Supplementary Material B. This table includes a categorization of the 'identified everyday objects and nature elements' per textbook, the 'mathematical concepts' involved, and 'the art-maths dimensions' comprised.

It is noteworthy that the majority of the allusions to art in the Spanish mathematics textbooks incorporate pictures (e.g. drawings and images). Few references to art without pictures were found. Just some exercises asked for the area or volume of a famous construction, providing its dimensions without the support of a picture.

5. Discussion

The following is a discussion of the most prominent results for the six dimensions analyzed: (1) art for ornamental purposes; (2) art as a context for calculation and measurement; (3) art as a context for mastering concepts; (4) art as a context for using technological resources in mathematics; (5) mathematical analysis of works of art; and (6) art created with mathematics.

Dimension 1, art as ornament, was present in the textbooks in a large proportion with a 45.2 % of the artistic works used for purely decorative purposes. The majority of those were found on covers or title pages with no text, the underlying mathematical idea was difficult to interpret or infer. Works of art were also reproduced in the margins of pages bearing theoretical introductions to a mathematical concept, with no explicit reference in the text to the piece or in the legend to the title or author. One example can be found in a textbook section on similarity, published by Anaya (2008, p.169), reproducing the 'Mona Lisa' with no reference to the illustration in the text (see Figure 16). The relevance of the work was not clearly explained, nor was any obvious connection to mathematics established. The percentage of dimension 1 occurrences was fairly evenly distributed across the six art forms (architecture, sculpture, painting, tiling and friezes, music and design of familiar objects and patterns in nature).

Dimension 2, art as a context for calculation and measurement, accounted for 25.3 % of the allusions. In the activities proposed students were usually asked to calculate areas or volumes, and to measure works of art from pictures. The purpose was to provide a real-world context to express a mathematical idea or concept, but the works were unnecessary from the standpoint of solving the mathematical problem. In the activity in Figure 1, for instance, involving the merely procedural calculation of the size of the Louvre Museum pyramid, the picture itself was unnecessary: applying the respective algorithms to the data given would have sufficed. Any other non-artistic quadrangular pyramid would have served the same purpose.

Other activities called for measuring a work of art picture or a familiar object design with a ruler. The activity associated with Figure 13 (in an early section), for instance, entailed verifying whether a credit card represent the golden ratio. Measuring with a ruler would afford students an approximate value, which would not be sufficient to determine whether the object actually conforms the golden ratio. If they choose to measure the card with a calliper, the value obtained (1.58515851...) would be nearly the golden number, but would not be the actual card ratio (1.585772508 given by the international standard ISO/IEC 7810:2003). If a technological tool such as GeoGebra were to be used to measure the card (see Figure 17), the value found, 1.60197648, would be closer to the golden number but farther away from the actual value. More than proving the conformity of the object to the golden ratio, this exercise aimed to teach students the importance of measuring with different tools, bearing in mind the characteristic errors of each, and the notion of approximation.

Dimension 3, art as a context for mastering concepts, was present in 23.3 % of the cases identified. This dimension was found in all six art forms (architecture, sculpture, painting, tiling and friezes, music, and design of familiar objects and patterns in nature). In particular, it was widely found in activities related to the design of familiar and natural objects, and only rarely in activities related to tiling and friezes. Table 2 lists the mathematical concepts that explicitly appear in relation to art and the ratio of occurrence of each. The same concept was often found in several works of art and in different art forms. Likewise, in a single work of art different concepts were also integrated.

Mathematical Concept	**Number of Allusions**
Geometry figures and measurements	42
Golden ratio	23
Symmetry	16

Geometric transformation	13
Similarity	11
Functions	11
Arithmetic	11
Proportionality	6
Equations	4
Scales	2
Sequences	2

Table 2: Mathematical Concepts associated to the Art forms

It is worth noting that concepts such as infinity, limits, continuity and perspective, often present in the works of art, were not explicitly addressed. They could have easily been integrated in some of the proposed activities. To cite one example, the activity in Figure 18 presents the Escher's 'Smaller and Smaller' print, and asks to identify the in-plane movement and tile generating the tiling. This print could have also been used to work the notions of infinity and limits.

In some cases, the activities addressed concepts too superficially. Students were normally asked to identify a curve in a picture on the sole grounds of observation. For instance, the activity in Figure 14 (in a previous section) aimed to teach the golden ratio by establishing a visual relationship between a Nautilus picture and Dürer's spiral. Greater benefit could have been derived from this activity if it had enlarged upon the concept and analyzed other types of spirals (e.g. Algorithmic, or Archimedean) that might provide a better fit to the Nautilus. That would have also afforded the opportunity to address the notion of approximation.

Similar examples were found in the lessons on functions and graphs, where normally the aim was to work with parabolas by identifying pictures of curves by mere observation. In some cases, the curve shown in the picture does not match the graph of a function. In others, such as in Figure 19, the symmetry of a curve was difficult to perceive because the photo was not taken perpendicular to the line of sight.

Dimension 4, art as a context for using technology in mathematics, accounted for a much lower percentage (7.5 %) of occurrences than dimensions 1, 2 and 3. It appeared only in connection with three art forms: architecture, tiling and friezes, and design of familiar objects and patterns in nature. Most of the activities focused on internet searches for information: on the tile generating the Alhambra tiling (as exemplified earlier in the activity of Figure 8); on tiling-like patterns found in nature (as illustrated in Figure 15); and on natural objects conforming to the golden ratio (as in Dürer's spiral activity in Figure 14). Art was seldom used as a backdrop for applying computer software to mathematics. Only a few activities were observed in which students were expected to use GeoGebra to draw a polygon and build a tiling or practise with homothety and similarity in tiling design (as was the case with the Figure 9 activity).

No activities were found in which the use of GeoGebra or other educational software proved to be useful for measuring distances, building structures, verifying properties or mastering concepts that would be suited to exercises involving works of art. Activities could have been, for instance, proposed to use GeoGebra to build a golden rectangle to determine whether a credit card was exactly such a rectangle, as demonstrated in Figure 20.

Dimension 5, mathematical analysis of works of art, accounted for just 9.6 % of cases. Although this dimension was present in architecture, painting and design of familiar objects and patterns in nature, it was most visible in tiling and friezes. The activities in question focused on determining which geometric transformations (translation, rotation or symmetry) led to a tiling or frieze, and to identify the shape from which a given tile is generated. An example of the latter is the activity described earlier in Figure 8, and drawn from Oxford 3º ESO (2015, p.140).

Barring the activities with a certain analytical content described in the results, no others called for an in-depth analysis of works of art from the mathematical standpoint. Most activities related mathematics and art very superficially. Figure 11, described earlier, is an example of it: the Twitter logotype appears as an example of a golden ratio design, without asking students to undertake any critical analysis on this relationship. This is unexpected considering the Spanish curriculum highlights analysis as a main competence. A deeper investigation or exploration would have revealed that the sketches for the geometric construction of this logotype were created from arcs of a circumference (Figure 21); a feature of no minor interest from the perspective of the art-mathematics relationship. That explanation would also have entailed studying the application of mathematics to the creation of the logotype.

Textbooks were also observed to propose activities relating Leonardo Da Vinci's Vitruvian Man to the golden ratio, in most cases with superficial analyzes. That work is a canon on the proportions of the human body, of particular significance in art. In an educational context it could be used to propose more analytical activities, such as:

- studying which canon represents the models (normally a wooden figure) used by each artist for their works.
- measuring classroom students' bodies and generating data tables for statistical processing to establish relationships between different dimensions, and compare the results with existing canons and anthropometric measurement tables.

Such activities would analytically combine art, proportions and statistics and involve minor research that could be very useful to students.

Dimension 6, art created with mathematics, was the dimension with the lowest presence (1.4 %). As suggested in the results section, this dimension was found only in connection with pictorial art and the generation of polygons to create tiling and friezes, as well as constructing polyhedrons decorated with artistic paintings, as proposed in Escher's dodecahedron activity (Figure 7).

One activity using pictorial art (drawings or prints) could have included creating compositions with geometric elements inspired for instance by the pieces earlier mention of Kandinsky or Escher. Works such as Dalí's 'Atomic Leda'— formulated around the Pythagorean star (see Figure 22)— could have been given a similar use, taking the artist's idea as the point of departure. For instance, a picture could be drawn using Dürer's shell curves.

Additional exercises involving sculpture might have included clay reproductions for the canons used by artists, or an original creation to be compared to existing canons. Scale models of architectural structures or sculptures could also be built. As noted in the results section, in several activities students were asked to recognize arcs and parabolas in certain illustrations. That could be taken further, using mathematics to reproduce architectural elements on a different scale. One activity, for instance, might involve making a scale model of the 'Atomium' described in earlier sections to help master geometric and arithmetic concepts.

The only activities proposed in connection with music were related to the duration of musical notes (measurement). That could have been enhanced by teaching students to use the value of the musical notes to create bars. Each student might, for instance, create a melody with six bars and four beats per bar, using certain musical notes (minim, crotchet, quaver, semiquaver) and their rests. A melody could be composed by simulating Mozart's 'Musical Dice Game': assigning each bar a number from one to six and throwing a die six times to determine the order of the bars and the number of times they are repeated in the melody. This activity would involve working with fractions (the sum of the duration of each note and its rest should be one), probability (the likelihood of a bar coming up on the die) and combinations (analysis of all the possible melodies that could be generated).

As noted earlier in the section on design, most activities sought to establish connections between designs and the golden ratio, and to identify axes of symmetry. It would have been useful if in connection with this art form each student had been invited to create their own logotype from geometric elements and constructions only. Such an exercise would also involve applying other dimensions such as mastering concepts or calculation, with or without technology.

6. Conclusions

This study analyzed the relationship between art and mathematics in the Spanish education system and more specifically in textbooks for secondary school students aged 12 to 16. The relationship between art and mathematics was assessed on the grounds of six parameters: (1) art for ornamental purposes; (2) art as a context for calculation and measurement; (3) art as a context for mastering concepts; (4) art as a context for using technological resources in mathematics; (5) mathematical analysis of art; and (6) art created with mathematics.

The 146 allusions identified in the textbooks, and classified in different art forms (architecture, sculpture, etc.), were analyzed in terms of the above dimensions to study the relationship between art and mathematics. Architecture, with over 64 references, prevailed over the other art forms, followed by Design of Familiar Objects and Patterns in Nature with 32, and Painting and Drawing with 27. Fewer references were found to Tiling and Friezes (11), Sculpture (8) and music (4). The Eiffel Tower, the pyramids at Cheops and the Parthenon were the architectural works most often cited. In painting, two works authored by Leonardo Da Vinci predominated: 'Mona Lisa' and 'Vitruvian Man'. The tiling and friezes most frequently found were those in the Alhambra at Granada, the Mosque at Cordoba and several of Escher's works. The designs most often reproduced were credit cards and the Nautilus shell. No sculpture stood out over the others and no specific piece of music was cited.

The distribution of the identified activities by dimension revealed a clear predominance of 'art as an ornamental element' (dimension 1) in mathematics textbooks. Over 45.2 % of the activities were classified under this dimension, accounting for over 50 % of the occurrences in five of the art forms. The percentage of activities under dimensions 4 (art and technology), 5 (analysis) and 6 (art creation), which involved a more active role of the student, was under 13 %. Dimension 4 (art and technology) appeared in only some of the art forms and was generally associated with information searches. Dynamic geometry software was present in a few activities involving the study of tiling and friezes. Dimension 5 (analysis) was also found primarily in connection with that art form. With only two activities, the presence of dimension 6 (art creation) was negligible.

In terms of content, nearly 28.8 % of the activities involving artistic elements entailed working with and measuring (distance, area, volume) geometric figures. That was followed by exercises on the golden ratio (15.75 %), symmetry (10.96 %) and geometric transformations

(8.9 %). With the exception of the golden ratio, those are the contents explicitly recommended in the Spanish curricular guidelines for working with art as a context in mathematics. Barely any of the activities relating to the golden section prompted experimentation or reflection. They seldom involved verifying a hypothesis or assumption or working with the errors implicit in measurement. Nor did they induce a critical attitude toward the wide dissemination of this topic, which is not always dealt with as rigorously as required. Generally speaking, very few activities were found in which technological resources were applied to explore the art-mathematics connection. The proliferation in recent years of the use of dynamic geometry software for the study of the area seems to have had little effect on textbooks.

On the whole, in most of the activities mathematics were neither significant to the works of art illustrated nor used to analyze them. They were merely an underlying medium. This study shows that the attempts made to illustrate the mathematics-art connection in today's world do not suffice. The solution is not to adorn mathematics activities with hosts of examples of the presence of mathematics in the real world, but to furnish good examples that support the mathematical knowledge being taught in classrooms and liable to be used to analyze and create art with mathematical tools.

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Figure 1. Louvre Museum activity drawn from Edebe 2º ESO (2016, p.254)

Figure 2. Athens Parthenon activity drawn from Marfil 4 º ESO (2008, p.267)

Figure 3. Atomium activity drawn from Edebe 2º ESO (2016, p.254)

Figure 4. In Blue Kandinsky painting activity drawn from Edebe 1º ESO (2007, p.207)

Figure 5. Escher's Drawing Hands activity drawn from Vector 1º ESO (2010, p.234)

Figure 6: Leonardo Da Vinci's 'Vitruvian Man' activity drawn from Marfil 4 º ESO (2008, p.266)

Figure 7. Escher's dodecahedron activity drawn from Oxford 2º ESO (2016, p.248)

Figure 8. Alhambra tiling activity drawn from Oxford 3º ESO (2015, p.140)

Figure 9. GeoGebra activity to study homothety and similarity drawn from Edebe 2º ESO (2016, p.223)

Figure 10. Musical fractions activity drawn from SM 4º ESO (2017, p.26)

Figure 11. Examples of familiar object related to the golden ratio, drawn from Edebe 2ºESO (2017, p.194)

Figure 12. Butterfly symmetry activity drawn from SM 4º ESO (2017, p.26)

Figure 13. Golden ratio credit card activity drawn from Marfil 4 º ESO (2008, p.268)

Figure 14. Dürer's spiral activity drawn from Marfil 4 º ESO (2008, p.265)

Figure 15. Animal hides and tilings activity drawn from Edebe 2º ESO (2016, p.223)

Figure 16. Mona Lisa Illustration drawn from Anaya 4º ESO (2008, p.169)

Figure 17. Credit card measured with GeoGebra Software (own elaboration)

Figure 18. ‘Smaller and Smaller’ print activity drawn from Marfil 4º ESO (2008, p.310)

Figure 19. Identifying the function related to the curve, activity drawn from Oxford 3 º ESO (2015, p.250)

Figure 20. Use of GeoGebra to verify whether or not a credit card is a golden rectangle (own elaboration)

Figure 21. Sketch for the geometric construction of Twitter logotype, retrieved October 1st 2017 from https://dribbble.com/twitter/projects/105370-Brand

Figure 22. Atomic Leda drawn from Marfil, (2008 p. 267)

Figure - separate figures for export only

3. Uno de los museos más grandes de Europa y más visitados del mundo es el Louvre. Se localiza en París (Francia). Tiene una superficie de 210 000 m^2 y una afluencia de, aproximadamente, 9 334 000 visitantes al año.

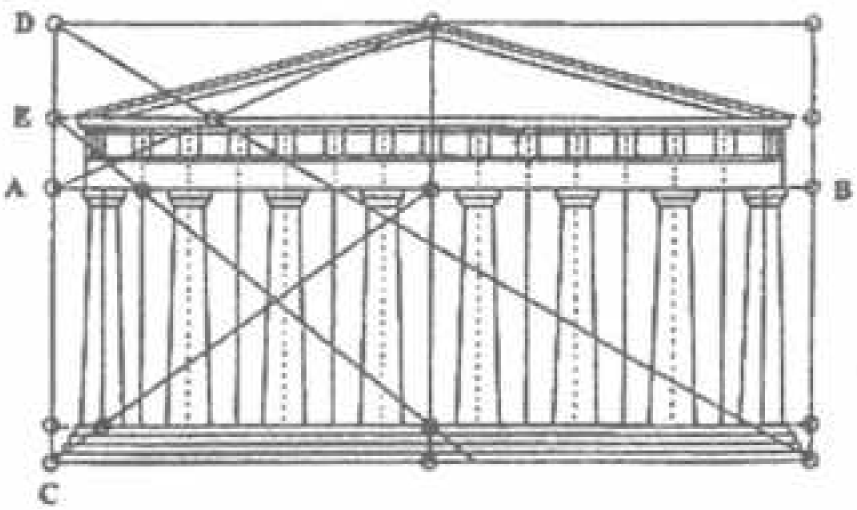
a) ¿Qué polígono forma la planta del museo? ¿Cómo se llama este cuerpo geométrico?

b) Clasifica los polígonos de las caras de la pirámide.

c) Calcula la altura del museo sabiendo que la arista básica mide 10 m y la apotema de la cara lateral tiene una longitud de 13 m.

d) Halla las longitudes de las aristas que no pertenecen a la base. Presenta el resultado con un decimal.

Figure - separate figures for export only



- Si mides algunas longitudes en este croquis del Partenón, probablemente encontrarás más de una vez el número de oro.

Figure - separate figures for export only

1. El Atomium fue construido en 1958 en la ciudad de Bruselas (Bélgica). Representa la estructura cristalina del hierro: un cubo con una esfera en cada uno de sus vértices y una esfera central, unidas todas ellas mediante cilindros. Está aumentada ciento sesenta y cinco mil millones de veces respecto a la estructura cristalina del hierro. El monumento tiene 102 m de altura y el diámetro de las esferas es de 18 m.

a) Determina, a partir del razonamiento geométrico, el número de cilindros y esferas del Atomium.

b) El tubo vertical central es un ascensor que sube a sus visitantes a 5 m/s hasta la esfera más alta. En ella se ubican un mirador y un restaurante. ¿En cuánto tiempo el ascensor lleva a cabo este recorrido?

c) Calcula el diámetro, en nanómetros, de un átomo de hierro si sabemos que 1 nm = 10^{-9} m.

d) Minimundus es un parque de miniaturas situado en Klagenfurt (Austria). En él se encuentra una réplica del Atomium a escala 1:25. ¿Cuál es la altura, en metros, de la réplica?

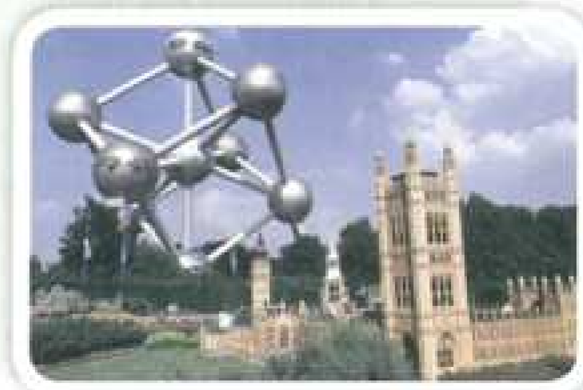
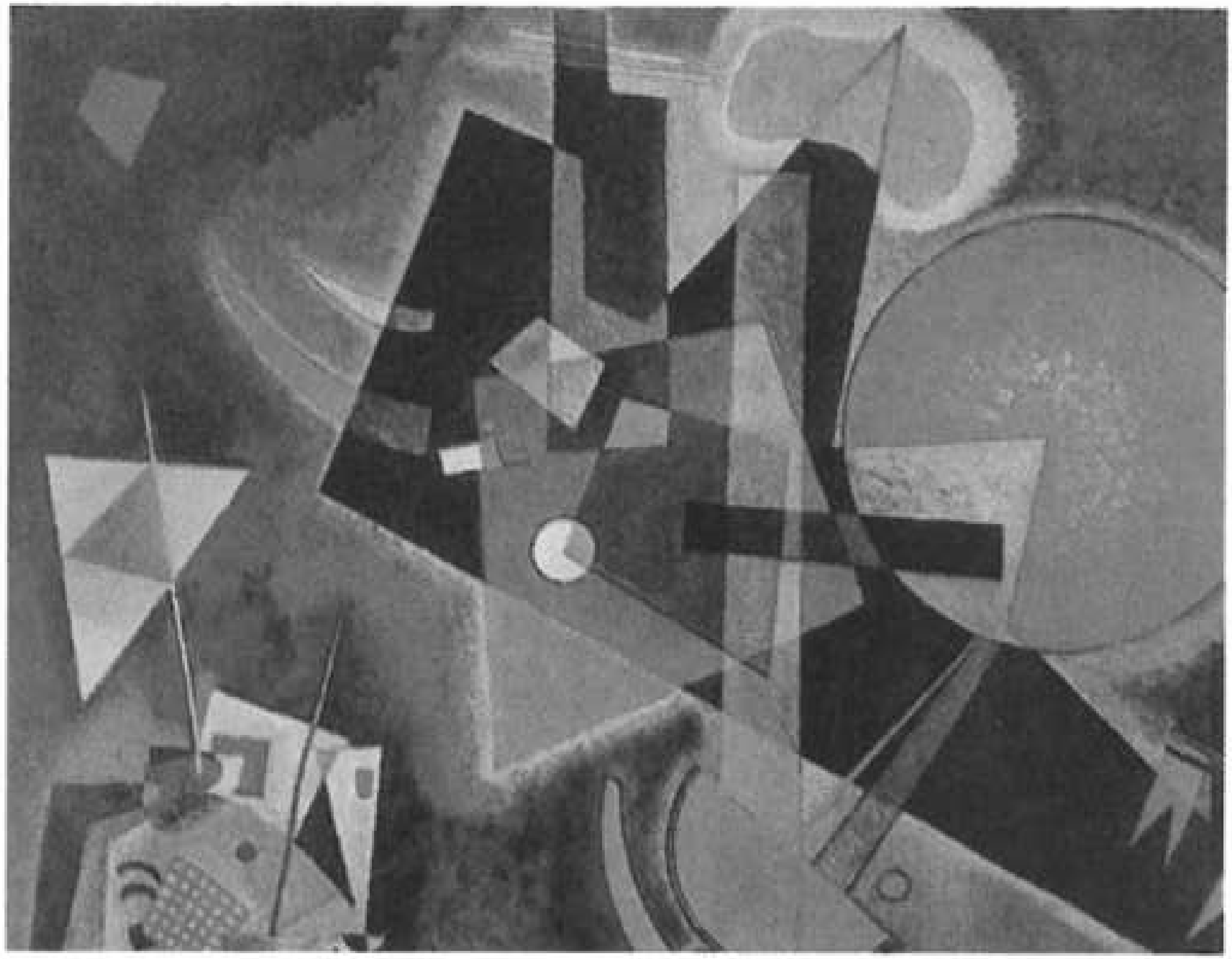


Figure - separate figures for export only

80 Señala los diferentes cuadriláteros que aparecen en la obra *En azul*, de Kandinsky, y clasifícalos.



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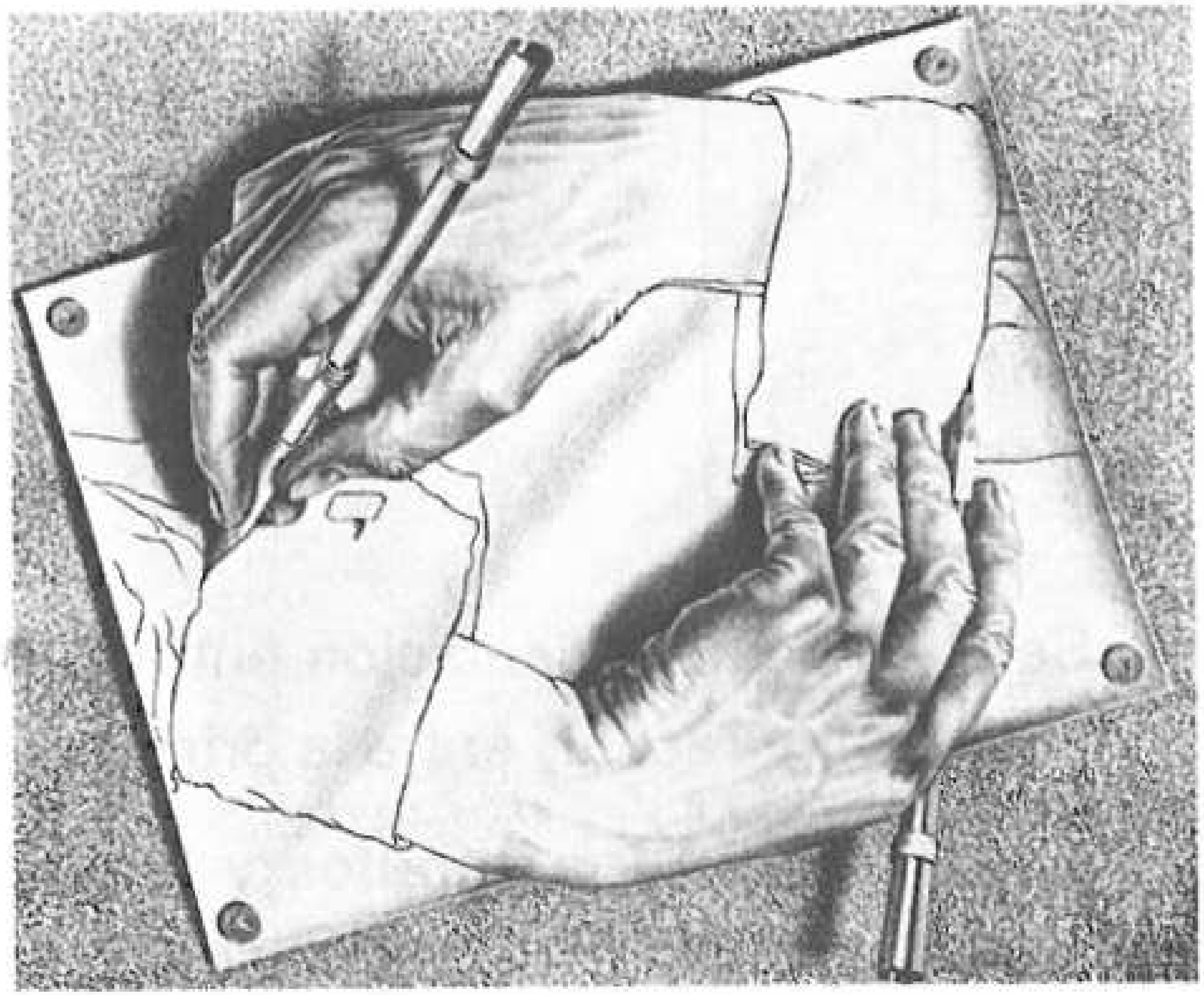
25 ●●● Encuentra el centro de simetría del dibujo *Dibujando manos,* del artista Escher.



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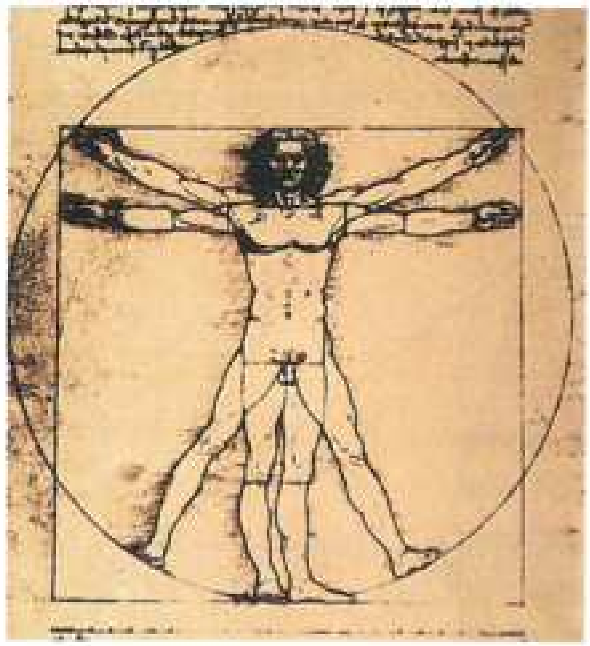
14.8. Número de oro y arte

Aquí tienes algunos ejemplos de la utilización que el ser humano ha hecho a lo largo de la historia del número de oro. Esperamos que te sirvan como punto de partida para que realices una investigación de búsqueda de datos sobre el tema.

(1) Cánones



A lo largo de la historia han existido distintos modelos o cánones propuestos como perfectos para el cuerpo humano. El dibujo siguiente de Leonardo da Vinci está basado en el canon propuesto por el arquitecto romano Marco Vitrubio. Según este canon, debía de cumplirse la siguiente relación:

$$\frac{\text{Altura total}}{\text{Altura hasta el ombligo}} = \frac{\text{Altura hasta el ombligo}}{\text{Distancia ombligo-cabeza}}$$

a) Si denominas "a" a la altura total y "m" a la altura hasta el ombligo, ¿cómo escribirías la anterior relación?

b) Prueba para distintos valores de "a" y "m". ¿Qué relación puede existir entre ambas medidas?

Figure - separate figures for export only

GEOMETRÍA EN EL ARTE

M. C. Escher y sus poliedros

Son muchos los artistas que han incluido figuras poliédricas en sus obras. Pero seguramente sea el holandés Maurits Cornelis Escher uno de los que más fascinación ha demostrado por estos cuerpos geométricos.

En la foto podemos ver un dodecaedro en cuyas caras Escher ha dibujado la misma figura, que encaja con las otras caras del dodecaedro.

G1. Investiga sobre la obra de M. C. Escher y su relación con las matemáticas.

G2. Construye un poliedro regular y decóralo con un estilo similar al de Escher.

Figure - separate figures for export only

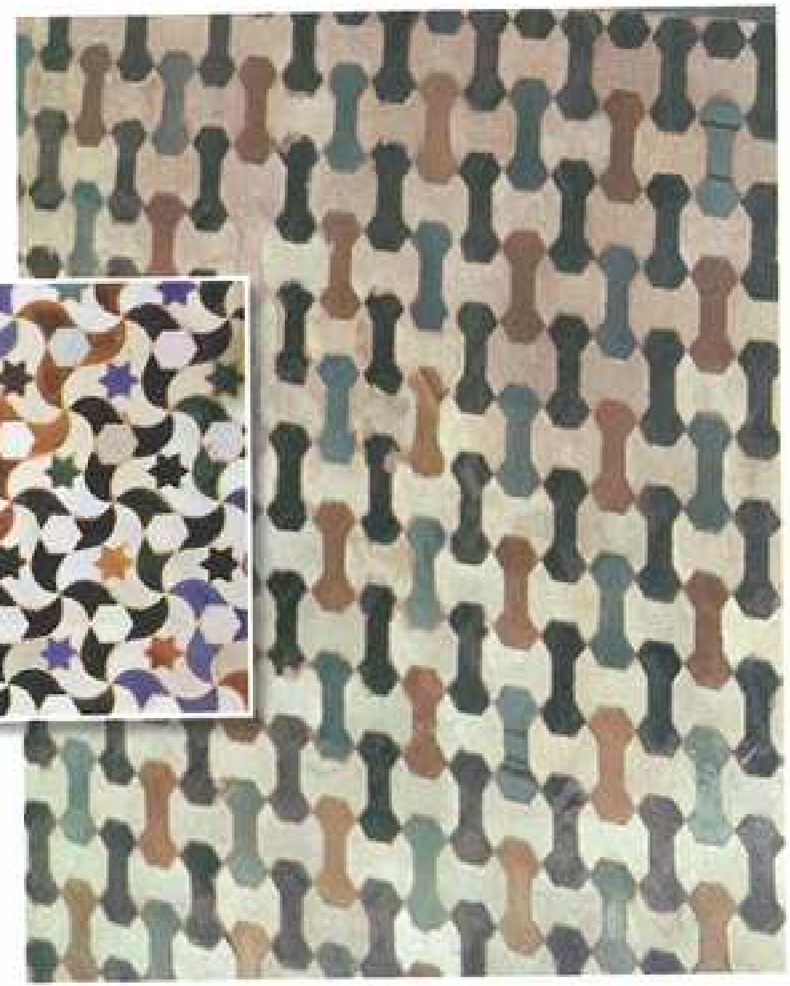
En el palacio de la Alhambra de Granada se conservan los mejores mosaicos realizados en el período de la España musulmana (siglos XIII-XIV) durante el reinado de la dinastía nazarí.

La religión islámica busca la belleza en los diseños geométricos, y los artesanos, inspirados en esta búsqueda, hicieron posible la creación de los llamados *polígonos nazaríes*.

Un mosaico está formado por motivos que se repiten denominados teselas. Las teselas de la Alhambra son piezas de forma cúbica, hechas de rocas calcáreas, materiales de vidrio o cerámicas de distintos tamaños. La parte visible de muchas de ellas son polígonos.

Avión

Pajarita



Hueso

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1 Los artistas musulmanes plasmaron en la Alhambra sus conocimientos del concepto de simetría y realizaron su trabajo de teselación del plano mediante movimientos: traslaciones, giros y simetrías sobre una misma figura.

a. El hueso nazarí es un polígono cóncavo de doce lados que se obtiene a partir de un cuadrado en el que se recortan dos trapecios de dos lados opuestos y se colocan mediante giros en los otros dos lados también opuestos. ¿Cuál es el número mínimo de colores necesario para que no haya dos huesos del mismo color con un lado en común?

ARGUMENTA

b. Busca en Internet cuál es el proceso de construcción de la pajarita a partir de un polígono. ¿Cómo se llama esta figura geométrica?

UTILIZA LAS TIC

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c. ¿Qué movimientos se pueden aplicar para dibujar el mosaico cuya tesela es el avión?

PIENSA Y RAZONA

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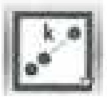
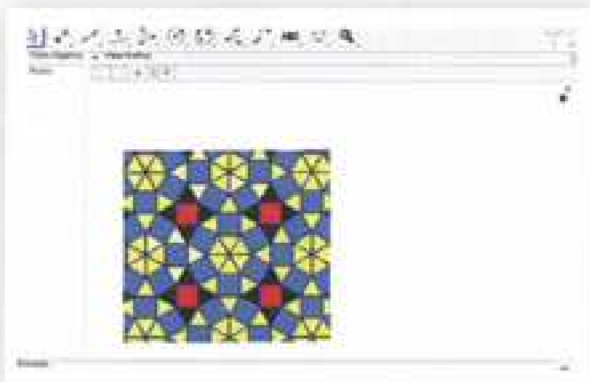
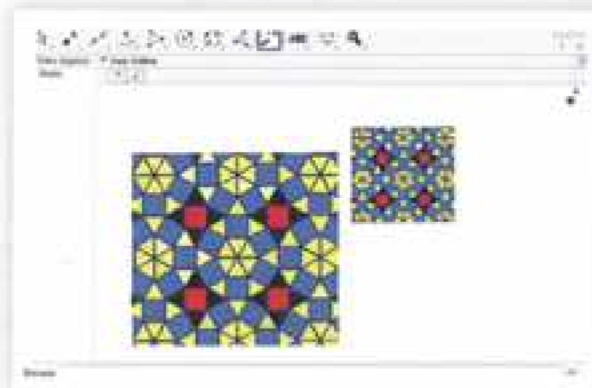
El programa GeoGebra permite construir figuras semejantes con la herramienta *Homotecia* . Observa cómo podemos obtener una imagen semejante a una original.

- Dibuja un polígono con la herramienta *Polígono* o inserta una imagen en la *Vista Gráfica*. Incluye también un punto que será el centro de la homotecia.



- Como resultado visualizaréis dos figuras semejantes con la razón de semejanza (factor de escala) que hayáis insertado en el cuadro de diálogo.



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PROBLEMA RESUELTO

Fracciones musicales

En la composición e interpretación de las piezas musicales es muy importante la duración de cada una de las notas musicales.

Según la duración, existen notas redondas, blancas, negras, corcheas, semicorcheas, fusas y semifusas. Tomando como unidad la duración de una nota negra, el resto tiene los siguientes tiempos:

Redonda	Blanca	Negra	Corchea	Semicorchea	Fusa	Semifusa
4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

1. ¿Cuántas semicorcheas hay en una blanca? ¿A cuántas blancas equivale una fusa?
2. En música se utiliza un símbolo, el puntillo, que incrementa la duración de la nota un 50 %. ¿A cuántas fusas equivaldrá una blanca con puntillo?
3. Si se coloca un doble puntillo, se aumenta la duración de la nota un puntillo y la mitad de un puntillo. ¿Qué fracción de una nota sin puntillo es esa misma nota con doble puntillo?

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Planteamiento del problema

En nuestra vida cotidiana, es habitual encontrar elementos que presentan proporciones geométricas cuyo valor aproximado es el número áureo: $\phi = \frac{1+\sqrt{5}}{2} = 1{,}618\ldots$. Estas imágenes son ilustrativas:

Figure - separate figures for export only

45 Dibuja en tu cuaderno el contorno de una mariposa y explica si posee simetría especular.

Figure - separate figures for export only

(4) Y un objeto mucho más prosaico
Mide los dos lados de una tarjeta de crédito y comprueba que se trata de un rectángulo áureo.

Figure - separate figures for export only

14.5. Espiral de Durero

El artista del siglo XV Albert Durero construyó una espiral como la siguiente. ¿Eres capaz de describir con detalle cómo se construye?

Por cierto, podemos encontrar esta espiral en la naturaleza.

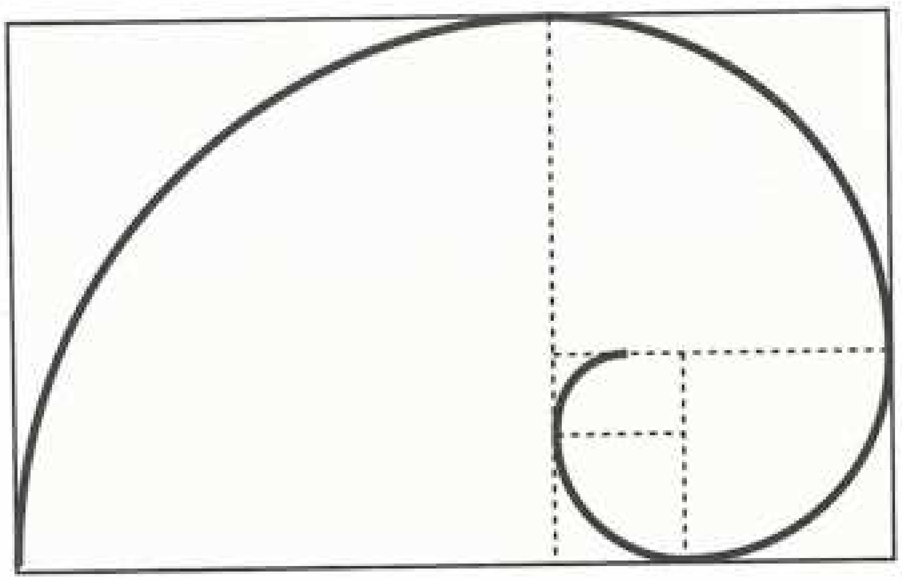

Figure - separate figures for export only

En las manchas de la piel de muchos animales también se esconden mosaicos geométricos: en las jirafas, en las serpientes, etc.

- Buscad más ejemplos de mosaicos en la naturaleza y cread una presentación en la que para cada diapositiva incluyáis un título, una imagen y una descripción sobre el mosaico observado en la ilustración.

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Seguramente, nunca has visto este cuadro. Sin embargo, lo conoces perfectamente por medio de sus reproducciones.

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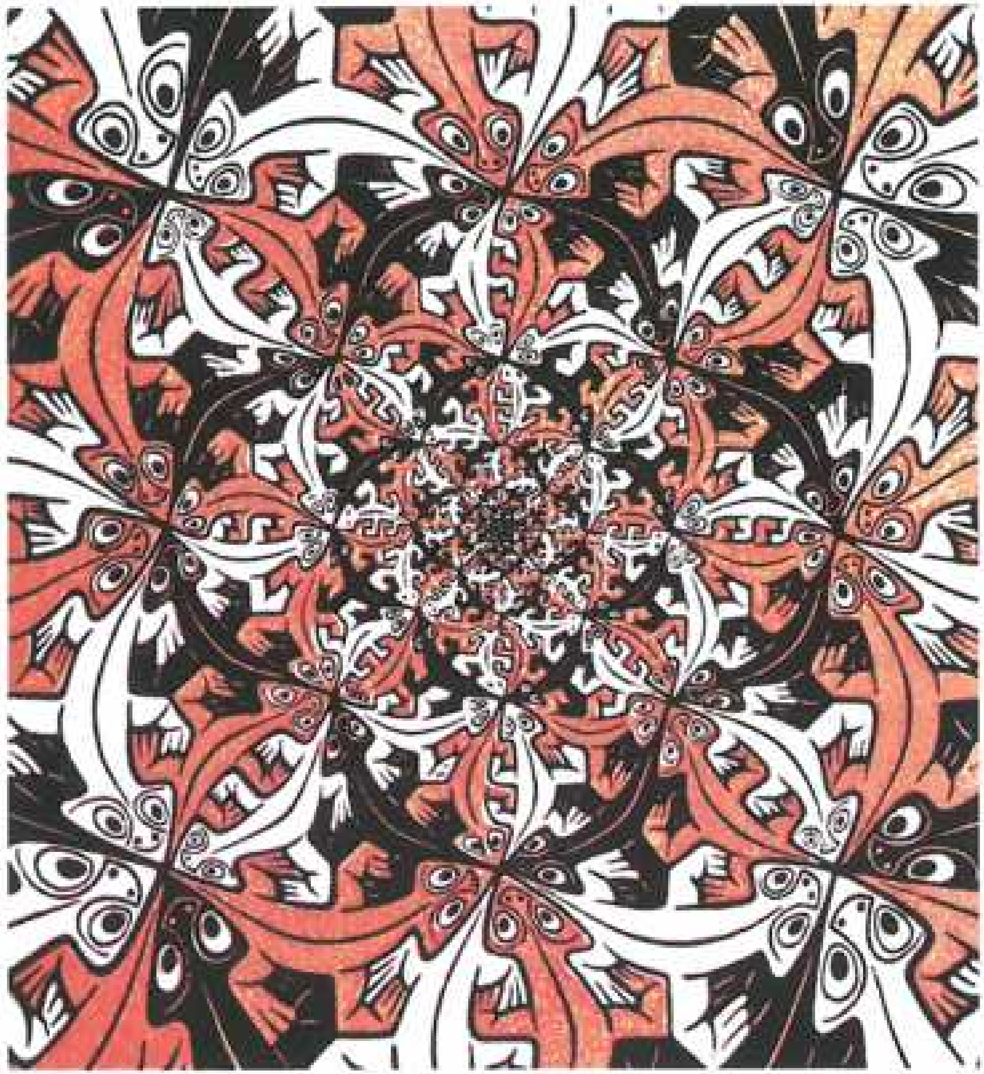
7.4. Más y más pequeño

El nombre de esta actividad es el título del grabado de M. C. Escher que puedes ver a continuación:



- Busca motivos mínimos que lo generan utilizando los movimientos que conoces. Descríbelos.

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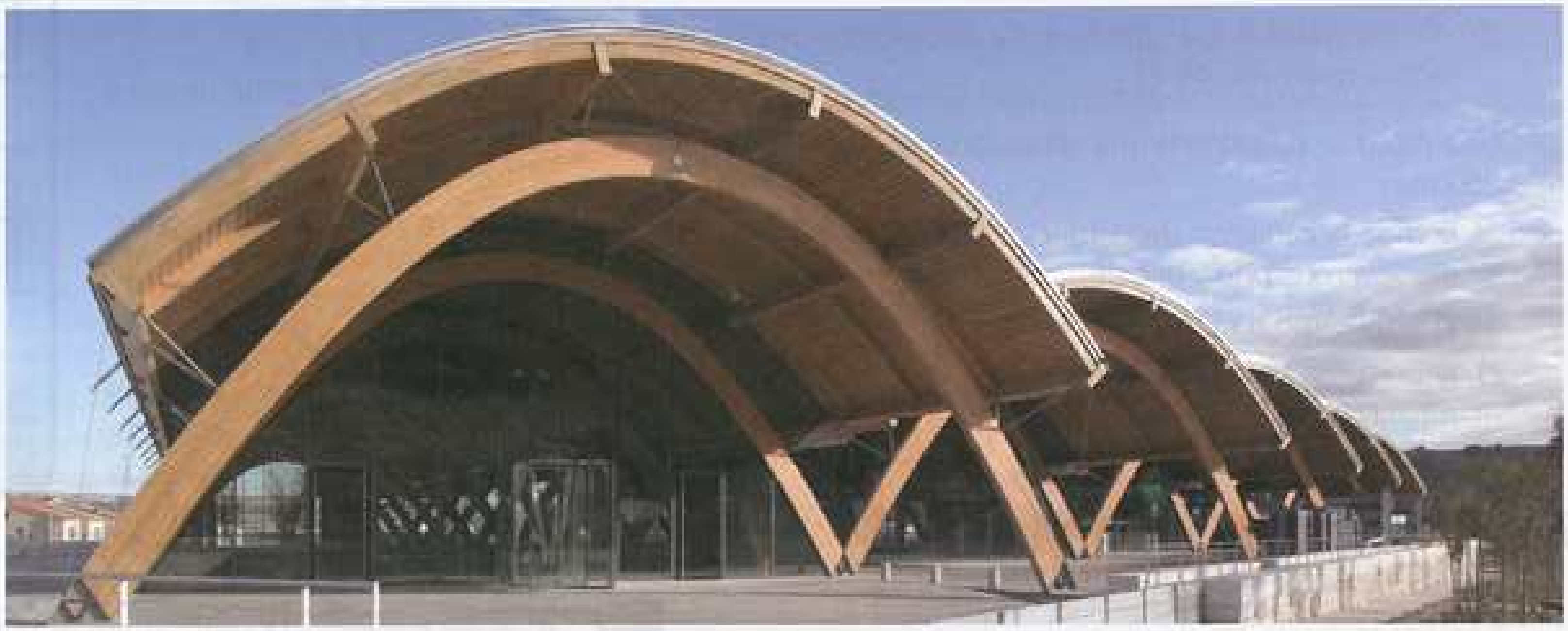
COMPRENDE

1 Observa esta construcción.



a. ¿Qué tipo de función tiene como representación gráfica una curva como la que puedes ver sobre la puerta?

UTILIZA EL LENGUAJE MATEMÁTICO

b. ¿Qué datos necesitarías para representar dicha curva en un dibujo a escala sobre unos ejes de coordenadas?

PIENSA Y RAZONA

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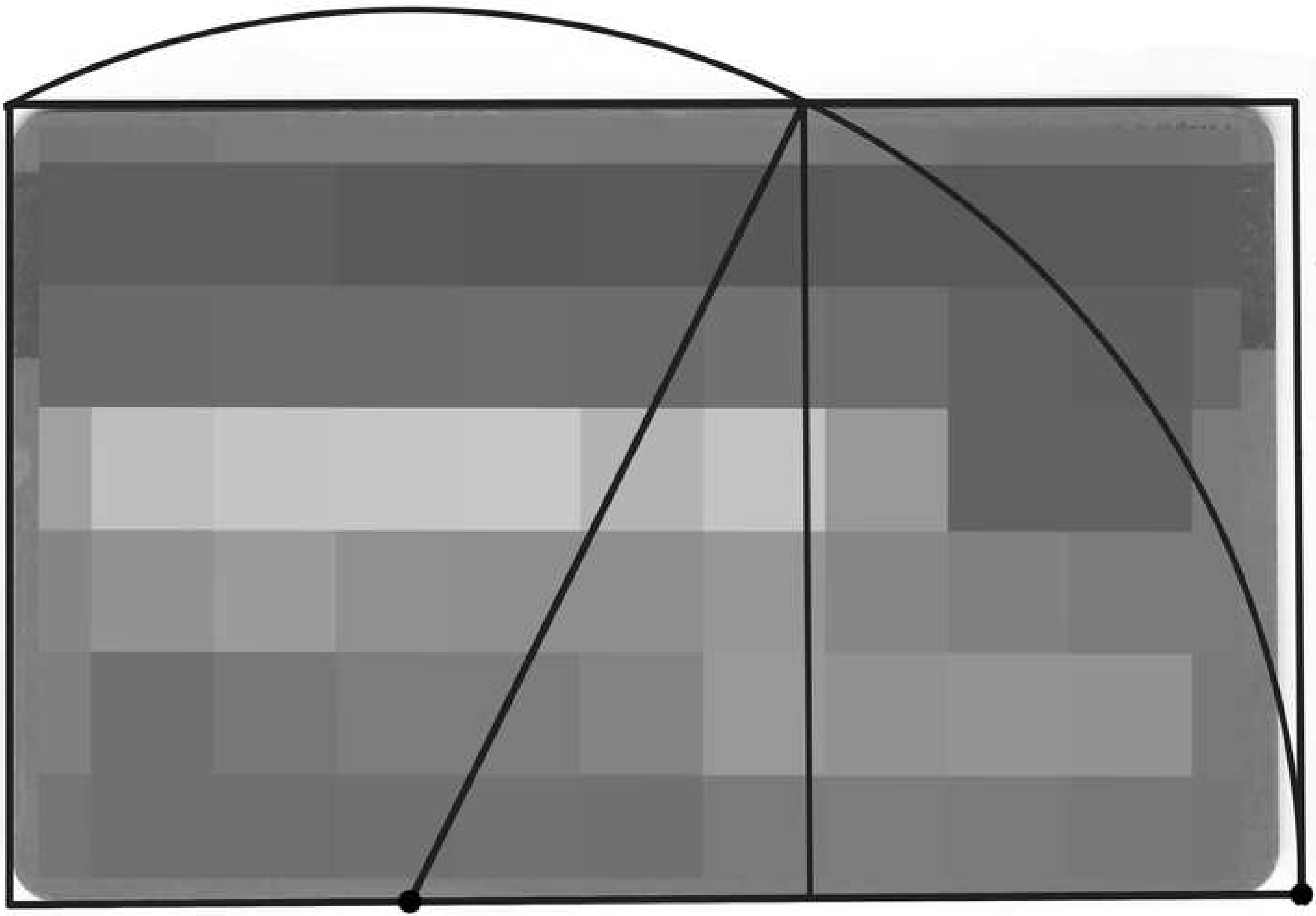


Figure - separate figures for export only

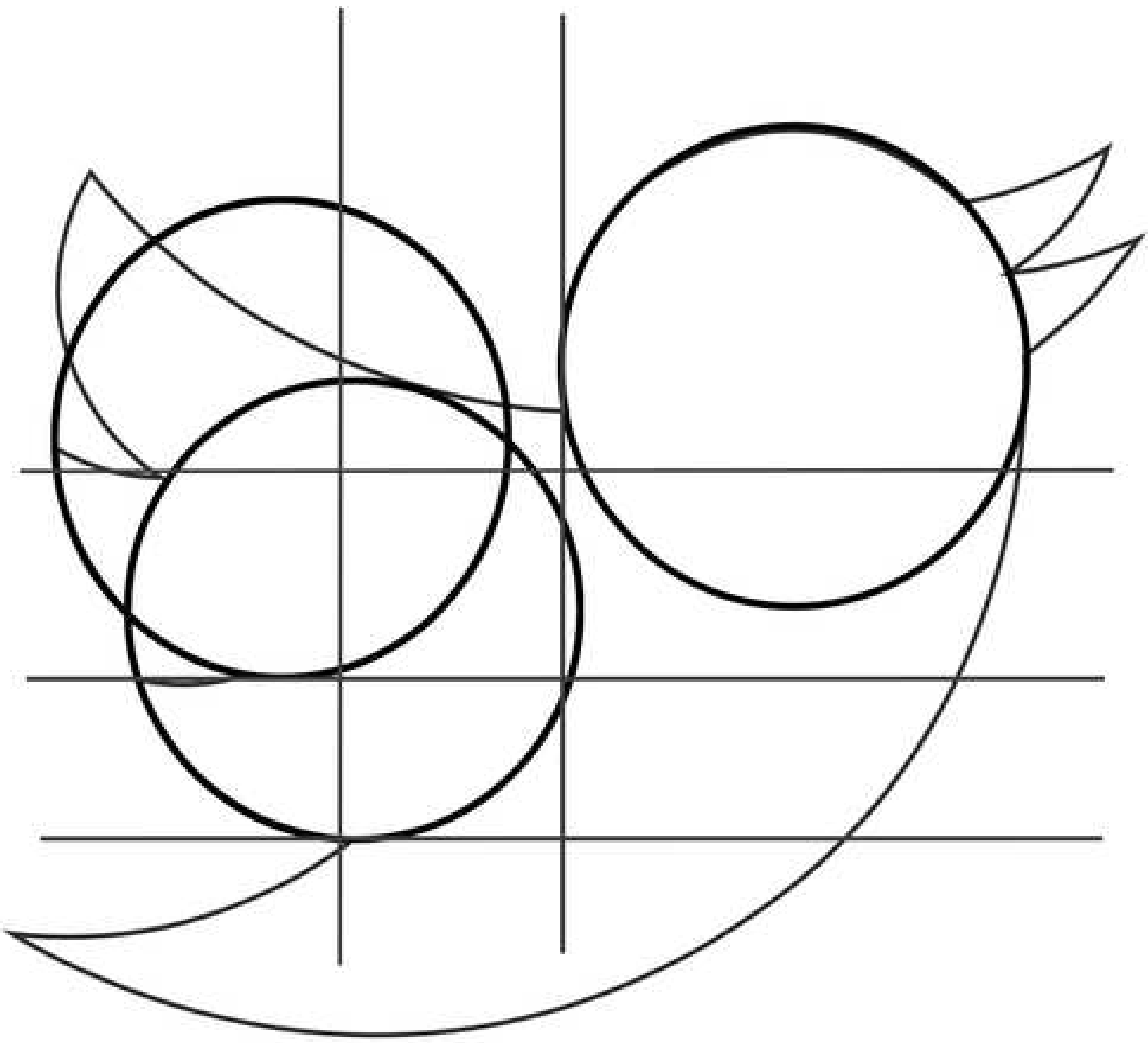
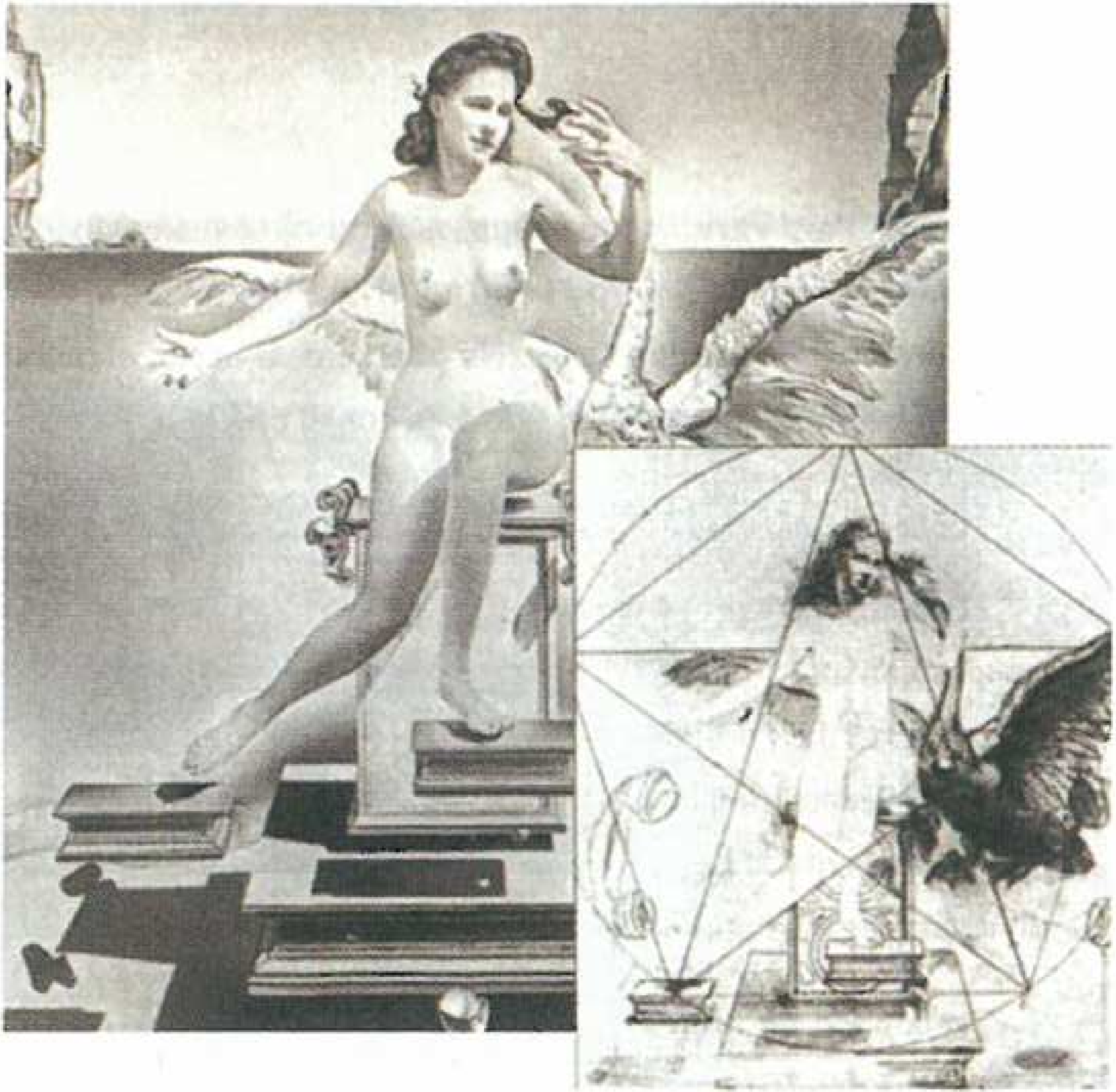


Figure - separate figures for export only



Is the relationship between art and mathematics addressed thoroughly in Spanish secondary school textbooks?

Supplementary Material A: Transcriptions

Image 1. Louvre Museum activity drawn from Edebe 2º ESO (2016, p.254)

3. One of the largest museums in Europe and most visited around the world is Louvre. It is located in Paris (France). It has an area of 210,000m^2 and an influx of, approximately, 9,334,000 visitors per year.

a) What polygon represents the base of this museum? What is the name of this geometric shape?
b) Classify the polygons of the pyramid faces.
c) Calculate the museum height knowing that the base edge measures 10m and the apothem of the lateral face has a length of 13m.
d) Calculate the length of the edges that do not belong to the base. Give the result with one decimal digit.

Image 2. Athens Parthenon activity drawn from Marfil 4 º ESO (2008, p.267)

If you measure some lengths of this Parthenon sketch, you will find more than one time the gold number.

Image 3. Atomium activity drawn from Edebe 2º ESO (2016, p.254)

Atonium was built in 1958 in Brussels (Belgic). It represents the crystal structure of iron: a cube with a sphere in each vertex and a central sphere, joined by cylinders. It is magnified one hundred sixty-five billion times respect of the crystal structure of iron. The monument is 102 meters tall and the diameter of spheres is 18 meters.

a) Calculate, using a geometric reasoning, the number of cylinders and spheres.
b) The central vertical tube is a lift having a speed of 5m/s until the top sphere. In this sphere, there is an overlook and a restaurant. How long does the lift take on this route?
c) Calculate the diameter, in nanometers, of an iron atom knowing that 1nm=10^{-9}m.
d) Minimundus is a park with miniatures located in Klagenburt (Austria). A replica of the Atomium (scale 1:25) is at that park. What is the replica height in meters?

Image 4. In Blue Kandinsky painting activity drawn from Edebe 1º ESO (2007, p.207)

Mark the different quadrilaterals that appear in the Kandinsky's painting, *In blue,* and classify them.

Image 5. Escher's Drawing Hands activity drawn from Vector 1º ESO (2010, p.234)

Find the symmetry center of the Escher's print *Drawing Hands.*

Image 6: Leonardo Da Vinci's 'Vitruvian Man' activity drawn from Marfil 4 º ESO (2008, p.266)

14.8 Golden number and Arts
Here you can see some examples of the use that people humans have made of golden number throughout history. We hope this will help you as starting point for a review of the topic.

(1) Canons

Throughout history, different models or canons have been proposed as perfect for the human body. The Leonardo da Vinci's drawing is based on the canon proposed by the Roman architect Marcus Vitruvius. According to this canon, it must be satisfied the following relation:

$$\frac{\text{total height}}{\text{height until navel}} = \frac{\text{height until navel}}{\text{length from nevel to head}}$$

a) If you call "a" the total height and "m" the height until navel, how would you express the above relation?
b) Give different values to "a" and "m". What relationship can be established between both lengths?

Image 7. Escher's dodecahedron activity drawn from Oxford 2º ESO (2016, p.248)7.4.

Many artists have included polyhedral figures in their works. Maybe, the Dutch Maurits Cornelis Escher is one of the artists who has shown the greatest fascination by these figures.

In the picture, we can see a dodecahedron where Escher has drawn on any face a same figure that fits with the other faces of the dodecahedron.

G1. Search for information on M. C. Escher's work and its relationship with mathematics.
G2. Make a regular polyhedron and decorate it with a similar style to Escher.

Image 8. Alhambra tiling activity drawn from Oxford 3º ESO (2015, p.140)

The Alhambra Palace in Granada preserves the best tilings made in the Muslim Spain period (XIII-XIV) during the reign of the Nasrid dynasty.
Islamic religion searches for beauty through geometric designs, and craftsmen, inspired by this fact, allowed creation of “Nasrid polygons”.

A tiling consists of tiles that are repeated. Alhambra tiles are pieces with cubic form, made of calcareous rocks, glass materials, or ceramics. The visible part of some tiles is a polygon.

Image 9. GeoGebra activity to study homothety and similarity drawn from Edebe 2º ESO (2016, p.223)

Geogebra software allows building similar figures with the tool “Homothety”.

Observe how we can obtain a similar image from the original.

Draw a polygon with the tool “Polygon” or insert an image in the “Graphics View”. Include also a point that will be the center of the homothety.
As result, you will see two similar figures with ratio of magnification that you have inserted in dialog box.

Image 10. Musical fractions activity drawn from SM 4º ESO (2017, p.26)

Musical fractions

In the composition and interpretation of the musical pieces, the notes value is very important.

According to the duration, there are semibreves, minims, crotchets, quavers, semiquavers, demisemiquavers and hemidemisemiquavers. Using as unit the quaver duration, the rest has the following times:

Table

1. How many semiquavers are in a minim? How many minims is a demisemiquaver?
2. In music, the symbol dotted increases the note value by 50%. How many demisemiquavers is a dotted minim?
3. If you put a double dotted, the note value increases a dotted and a half dotted. What fraction of a note without dotted is that note with double dotted?

Image 11. Examples of familiar object related to the golden ratio, drawn from Edebe 2ºESO (2017, p.194)

In our daily life, it is common to find elements presenting geometric proportions whose approximate value is the gold number: $\phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$ These images are illustrative:

Image 12. Butterfly symmetry activity drawn from SM 4º ESO (2017, p.26)

Draw in your notebook a butterfly outline and explain if there is mirror symmetry.

Image 13. Golden ratio credit card activity drawn from Marfil 4 º ESO (2008, p.268)

(4) And an object much more prosaic

Measure two sides of a credit card and verify that it is a golden rectangle.

Image 14. Dürer's spiral activity drawn from Marfil 4 º ESO (2008, p.265)

14.5 Dürer's spiral

The artist Albrecht Dürer (c. XV) made a spiral as the following one. Are you able to describe in detail how it is constructed?

Certainly, we can find this spiral in the nature.

Image 15. Animal hides and tilings activity drawn from Edebe 2º ESO (2016, p.223)

In the skin spots of some animals, there are hidden geometric tilings. For example, giraffes, snakes…

Image 16. Mona Lisa Illustration drawn from Anaya º ESO (2008, p.169)

Maybe you have never seen the original painting. Nonetheless, surely you know it from its reproductions.

Image 18. ‘Smaller and Smaller’ print activity drawn from Marfil 4º ESO (2008, p.310)

7.4 Smaller and Smaller
The name of this activity is the title of the M.C. Escher’s print that you can see below.

Search for minimum tiles generating it, employing the transformations you know. Describe them.

Image 19. Identifying the function related to the curve, activity drawn from Oxford 3 º ESO (2015, p.250)

Observe this construction.

a) What type of function has as graphic representation a curve like the one you can see over the door?

b) What data would you need to represent that curve in a scale drawing on coordinate axes?

Is the relationship between art and mathematics addressed thoroughly in Spanish secondary school textbooks?

Supplementary Material B: Allusions of art

Table S1. Architecture allusions

Building	Textbook	Mathematical concept		Dimension	Percentage in the category (%)
Parthenon of Athens	Anaya 4A-2008-p63	Golden number	Decimal numbers	(1)	7.81
	Anaya 4A-2008-p169 Bruño3-2015-p66		Similarity	(1)	
	Bruño3-2015-p66		Fibonacci sequence	(1)	
	Marfil4-2008-p267		Proportionality	(2)	
	Vector1-2010-p119	Proportionality		(1)	
Great Pyramid of Giza (Egypt)	Bruño3-2015-p176 Oxford3EA-2015-p150	Thales' Theorem		(2)	12.5
	Oxford2-2016-p211			(2)(5)	
	Marfil4-2008-p267	Pythagorean Theorem		(2)	
	Marfil4-2008-p267 SM1-2017-p280	Polygons and polyhedra. Elements, area and volume		(2)	
	Oxford3EA-2015-p180			(2)	
				(3)	
	SM2-2017-p203	Similarity		(3)	
Pyramids of Meroë (Egypt)	Oxford3EA-2015-p170	Polyhedra. Pyramids. Types		(1)	1.56
Pyramid of Kukulcan (Mexico))	Oxford3EA-2015-p170	Trunk of a pyramid		(1)	3.13
	Oxford3EA-2015-p180			(2)	

Tombs of Myra's Lycian Necropolis	Marfil4-2008-p267	Golden number	(1)	1.56
The Giralda of Seville	Bruño3-2011-p207	Thales' Theorem	(2)	1.56
St. Peter's Basilica	Edebé1-2007-p192	Polygons and angles	(2)	1.56
Le Corbusier's designs of houses	Marfil4-2008-p268	Golden number	(1)	1.56
Kunstmuseum Stuttgart	Edebé2-2016-p253	Solid figures. Polyhedra	(3) (4)	1.56
Louvre Pyramid	Edebé2-2016-p254	Polygons and solid figures Pythagorean Theorem	(2)	4.68
	SM4A-2017-p133		(3)	
	Oxford3EA-2015-p171	Trigonometry. Resolution of triangles	(2)	
Luxor Hotel (Las Vegas)	Marfil4-2008-p244	Area, volume and angle measure	(2) (3)	1.56
The Sagrada Famila (Barcelona)	Vector1-2010-p229	Symmetry	(3)	1.56
Eiffel Tower	Oxford2-2016p209	Scales	(2)	6.25
	Vector1-2010-p229	Symmetry	(3)	
	SM1-2017-p200	Measurement	(2)	
	SM1-2017-p204	Similarity	(1)	
Epcot Center at Disney World Orlando	Oxford2-2016p270	Geodesic domes	(2) (5)	1.56
Torre del Oro (Seville)	Vector1-2010-p229	Symmetry	(3)	1.56
Gate of Europe (Madrid)	Oxford3EA-2015-p207	Prisms. Area and volume	(1)	3.13
	Oxford3EA-2015-p210		(2) (3)	
Circular towers in Costa Plana	Oxford3EA-2015-p208	Circular figures. Area	(1)	3.13
	Oxford3EA-2015-p208		(2)(3)	
Church of Our Lady of Peace (Madrid)	Oxford3EA-2015-p211	Pyramids. Types. Lateral area	(2)	1.56
Air Force Academy	Vector1-2010-p181	Triangles	(1)	1.56

(Colorado)		Quadrilaterals		
Bridge. Not referenced	Oxford3EA-2015-p250	Functions and graphics	(1)	1.56
Court of the Myrtles in the Alhambra	Vector1-2010-chapcover	Symmetry	(1)	1.56
Taj Mahal	Vector1-2010-p227	Symmetry	(1)	1.56
Arches. Not referenced	Oxford3EA-2015-p250	Functions and graphics	(5)	3.13
Triangles. Not referenced	Vector1-2010-p183	Triangles	(1)	1.56
Skyscrapers and Pyramids. Not referenced	Vector1-2010-p194	Polyhedra	(3)	1.56
Spheres and cones. Not referenced	Vector1-2010-p195	Solids of revolution	(3)	1.56
Monument. Not referenced	SM1-2017-p8	Roman numerals	(1)	1.56
Maracanã Stadium (Rio de Janeiro)	SM1-2017-p26	Divisibility	(2)	1.56
The Alamillo Bridge (Seville)	SM1-2017-p220	Lines and angles	(3)	1.56
Triangulated cover. Not referenced	SM1-2017-p224	Triangles	(1)	1.56
Photograph of a bridge. Not referenced	SM1-2017-p228	Triangles	(1)	1.56
Plaza Mayor (Madrid)	SM1-2017-p262	Lengths and areas	(2)	1.56
Buildings. Not referenced	SM1-2017-p268	Polyhedra	(1)	1.56
Triple helicoidal staircase in Museum of the Galician People (Santiago de Compostela)	SM3A-2017-cover	Book cover	(1)	1.56
Roller coaster	SM4A-2017-cover	Book cover	(1)	3.13
	SM4A-2017-chapcover	Equations	(2)	
Study of the golden ratio in different geometric constructions	MV4A-2014-p21-22	Golden ratio	(4)	9.38

Ev-K2-CNR Pyramid International Laboratory-Observatory (Nepal)	SM4B-2017-p163	Areas and volumes	(2)	1.56
Hayden Planetarium (New York)	SM4B-2017-p169	Areas and volumes	(2)	1.56

Table S2. Sculpture allusions

Sculpture	Textbook	Mathematical concept	Dimension	Percentage in the category (%)
The Thinker by Rodin	Anaya 4A-2008-p169 Anaya 4B-2008-p128	Similarity	(1)	25
Not referenced	Anaya 4B-2008-chapcover	Equations	(2)	12.5
The Last Judgment by Lemaire	Bruño3-2015-p82	Equations	(3)	12.5
Atomium	Edebé2-2016-p254	Solid figures Scales Conversion of units Time to travel a distance	(2) (3)	12.5
Obelisk of Hatshepsut, Temple of Amon (Karnak)	Oxford3EA-2015-p174	Polyhedrons. Composition. Volume and area	(2)	12.5
Not referenced	SM4A-2017-p165	Symmetry of functions	(1)	12.5
The Great Tree and Eye by Anish Kapoor, Guggenheim Museum (Bilbao)	SM4B-2017-p156	Areas and volumes	(1)	12.5

Table S3. Painting allusions

Painting	Textbook	Mathematical concept		Dimension	Percentage in the category (%)
Melencolia I by Alberto Durero	Anaya1-2008-p94	Magic squares		(2) (3)	3.70
The Vitruvian Man by Leonardo da Vinci	Anaya1-2008-p94	Symmetry		(1)	22.22
	Vector1-2010-p229			(3)	
	Anaya4A-2008-p63	Golden number	Decimal numbers	(5)	
	Marfil4-2008-p266 Oxford3EA-2015-p160		Proportionality	(5)	
	Marfil4-2008-p242	Irrational numbers		(1)	
	SM3A-2017-p126	Proportionality		(5)	
The Mona Lisa by Leonardo da Vinci	Anaya4A-2008-p169 Anaya 4B-2008-p128	Similarity		(1)	11.11
	Bruño3-2015-p66	Similarity. Golden rectangle		(1)	
	Marfil4-2008-p242	Irrational numbers		(1)	
Eros by Paul Klee	Edebé1-2007-p194	Polygons. Triangles		(1)	3.70
In Blue by Kandinsky	Edebé1-2007-p207	Classification of quadrilaterals		(3)	3.70
Sunflowers by Van Gogh	Edebé2-2016-p112	Equations		(2)	3.70

Atomic Leda by Dalí	Marfil4-2008-p267	Golden ratio	(1)	3.70
The open window by Juan Gris	Oxford3EA-2015-p182	Lines, curves and polygons	(3)	3.70
The Reservoir, Horta de Ebro by Picasso	Oxford3EA-2015-p182	Lines, curves and polygons	(3)	3.70
Dodecahedron by Escher	Oxford2-2016-p248	Polyhedra	(6)	3.70
Unspecified work	Vector1-2010-chapcover	Proportionality	(1)	3.70
Village on rocks by Paul Klee	Vector1-2010-p189	Plane figures. Polygons	(1)	3.70
The Disck by Fernand Leger	Vector1-2010-p213	Circular planar figures	(1)	3.70
Circle Limit III by Escher	Vector1-2010-p225	Central symmetry	(1)	3.70
Drawing hands by Escher	Vector1-2010-p34	Symmetry	(3)	3.70
Tomb painting of the pharaoh Ramses I	Vector1-2010-p237	Cartesian coordinates	(1)	3.70
Cave of the Hands, Prehistoric rock paintings	SM1-2017-p6	Natural numbers	(1)	3.70
Composition VII by Kandinsky	SM1-2017-p206	Points, lines and angles	(1)	3.70
Tableau II by Mondrian	SM1-2017-p229	Plane figures	(1)	3.70
Not referenced	SM1-2017-p237	Polygons	(3)	3.70

Table S4. Tilings and Friezes allusions

Work	Textbook	Mathematical concept	Dimension	Percentage in the category (%)
Alhambra of Granada	Bruño3-2015-p190 Edebé1-2007-p193 Marfil1-2007-p158	Rigid motions Friezes and tilings	(1)	18.18
	Oxford3EA-2015-p140		(4) (5)	
Mosque-Cathedral of Córdoba	Bruño3-2015-p190	Rigid motions Friezes and tilings	(1)	9.09
Search for works of art	Edebé2-2016-p222	Polygons Tilings Percentages	(2) (3) (4)	9.09
Sun and moon by Escher	Marfil1-2007-p158	Tilings	(1)	9.09
Roman (The Hall of Animals) and Byzantine (The Court of Justinian) tilings	Marfil1-2007-p158	Tilings	(1)	9.09
Swans by Escher	Marfil1-2007-p159	Tilings	(5) (6)	9.09
Wood engraving print by Escher	Marfil4-2008-p310 Oxford3EA-2015-p142	Transformations Tilings	(5)	9.09
Facade inspired by a work by Escher. Not referenced	SM3A-2017-p172	Transformations Tilings	(5)	9.09
Alhambra tiling. Not referenced	SM3A-2017-p176	Transformations Rotations	(1)	9.09
Taj Mahal tiling	SM3A-2017-p176	Perimeter and area	(1)	9.09

Table S5. Music allusions

Elements of music	Textbook	Mathematical concept	Dimension	Percentage in the category (%)
Note value	Edebé1-2007-p97	Fractions	(2) (3)	25
Photograph of string instrument	SM2-2017-158	Functions	(1)	25
Photograph of saxophone and score	SM3A-2017-p6	Fractions	(1)	25
Musical Fractions	SM4B-2017-p26	Fractions	(2)	25

Table S6. Design of Familiar Objects and Patterns in Nature allusions

Everyday/ in nature objects	Textbook	Mathematical concept	Dimension	Percentage in the category

				(%)
Credit card	Anaya4A-2008-p63	Decimal numbers	(1)	3.13
	Anaya 4A-2008-p169 Anaya4A-2008-p183 Anaya 4B-2008-p145 Edebé1-2007-p194	Golden rectangle	(2) (3)	3.13
	Marfil4-2008-p268		(5)	3.13
Identity document	Anaya 4A-2008-p179	Golden rectangle Homothecy	(2) (3)	3.13
Nautilus shell	Bruño3-2015-p66	Similarity Golden rectangle Golden spiral	(1)	6.25
	Marfil4-2008-p265	Curves Durero spiral	(5)	
Butterfly	Vector1-2010-p229	Symmetry	(1)	9.38
	Bruño3-2011-p227		(3)	
	SM3A-2017-p237	Symmetry of functions	(1)	
Flowers	Oxford3EA-2015-p134	Symmetry	(1)	3.13
Pineapple	Marfil4-2008-p268	Fibonacci sequence Golden number	(1)	3.13
Abdomen of the Honey Bee	Marfil4-2008-p268	Fibonacci sequence Golden number	(1)	3.13
Animal skin stains pattern (snakes and giraffes)	Edebé2-2016-p223	Tilings	(2) (3) (4)	3.13
Honeycombs	Edebé2-2016-p223	Tilings	(2) (3) (4)	12.5
	SM1-2017-p226	Polygons	(1)	
	SM1-2017-bookcover		(1)	
	SM3A-2017-p228	Sequences	(2)	
Snow crystal	Vector1-2010-p229	Symmetry	(1)	3.13
Twitter bird icon	Edebé1-2007-p194	Golden ratio	(1)	3.13
Mitsubishi logo	Vector1-2010-p206	Quadrilaterals Rhombus	(1)	6.25

	Vector1-2010-p234	Symmetry	(3)	
Traffic signals	Vector1-2010-p234	Symmetry	(3)	6.25
	SM1-2017-p235	Symmetry	(3)	
Mobile phone	Edebé1-2007-p194	Golden ratio	(1)	3.13
Wheel design	Marfil1-2007-p163	Star polygons	(3)	3.13
Necklaces	Marfil4-2008-p133	Catenary	(3)	3.13
Starfish	SM3A-2017-p180	Symmetry	(1)	3.13
Kepler's Platonic solid model of the solar system	SM3A-2017-p190	Polyhedra	(5)	3.13
Spiral aloe	SM3B-2017-bookcover		(1)	3.13
3D animation software	SM4A-2017-p32	Algebraic expressions Polynomials	(1)	3.13
Clothoid used in rail engineering	SM4A-2017-p183	Functions	(5)	3.13
Hurricane	SM4B-2017-p14	Irrational numbers Golden number	(1)	3.13