Journal of Pure and Applied Algebra 216 (2012) 1879–1886

Contents lists available at SciVerse ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Internal structures in n-permutable varieties

Diana Rodelo*

Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, Portugal
Centre for Mathematics of the University of Coimbra, 3001-454 Coimbra, Portugal

ARTICLE INFO

Article history:
Available online 25 February 2012
Communicated by G. Janelidze

MSC: 18C10; 18D35

ABSTRACT

We analyze the notions of reflexive multiplicative graph, internal category and internal groupoid for n-permutable varieties.

Introduction

The property of n-permutability of congruences for a variety of universal algebras $\mathcal{E}$ has been widely studied by several authors over the past years. In [13], Mal'tsev shows that the 2-permutability of congruences for a variety of universal algebras $\mathcal{E}$ holds precisely when the theory of $\mathcal{E}$ admits a ternary operation p such that $p(x, y, y) = x$ and $p(x, x, y) = y$. Such an operation is called a Mal'tsev operation and $\mathcal{E}$ is called a Mal'tsev variety. It has been shown that Mal'tsev varieties satisfy many useful properties. In this work, we are interested in exploring the property stating that the notions of reflexive multiplicative graph, internal category and internal groupoid coincide in a Mal'tsev context (we generalize part of [11]; see also [6] for the Mal'tsev categorical context).

It is natural to ask whether internal structures have such a behavior in the weaker Goursat (3-permutable) context and, more generally, in any n-permutable one. The aim of this paper is to give a positive answer for Goursat varieties and a partially positive answer for n-permutable varieties. The reader interested in this line of research should also see [12] where the congruence modular case is investigated (a common property only of the Mal'tsev and Goursat contexts) and [14] where the internal categories and groupoids are compared in a weak Mal'tsev context.

A variety of universal algebras is called Goursat when the congruences are 3-permutable. In general, for any $n \geq 2$, the n-permutability of the congruences gives rise to the notion of an n-permutable variety. It is well known that such varieties are characterized by the existence of $n-1$ ternary operations, $q_1, \ldots, q_{n-1}$, satisfying certain identities [10]: two Mal'tsev type identities (for the first and last operations q_1 and q_{n-1}) and several other identities comparing each operation q_i with the next one q_{i+1}, $i = 1, \ldots, n-2$. The nature of these identities allows us to adapt the proofs used in the Mal'tsev case [7] to the general n-permutable case.

1. n-permutability

A variety of universal algebras is called a *Mal'tsev* variety when the composition of congruences is 2-permutable: $RS = SR$, for any pair of congruences R and S on a same object. *Goursat* varieties satisfy the strictly weaker 3-permutability condition: $RSR = SRS$. These are the first two members of the infinite sequence of conglomerates of *n-permutable* varieties satisfying the condition $(R, S)_n = (S, R)_n$, where the composition of n alternating factors R and S is denoted by $(R, S)_n = RSRS \cdots$.

* Corresponding author at: Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, Portugal.
E-mail address: drodelo@ualg.pt.

0022-4049/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.jpaa.2012.02.025

It is well known that a variety of universal algebras is n-permutable when its theory $\mathbb{T}$ contains $n-1$ ternary operations satisfying appropriate identities [10]. For the 2-permutability (*Mal'tsev*) property the theory $\mathbb{T}$ contains a ternary (*Mal'tsev*) operation p such that

$$\begin{array}{ll} (\mathbf{M1}) & p(x,y,y)=x, \\ (\mathbf{M2}) & p(x,x,y)=y, \end{array}$$

and for the 3-permutability (*Goursat*) condition it contains two ternary operations r and s satisfying the identities

$$\begin{array}{ll} (\mathbf{G1}) & r(x,y,y)=x, \\ (\mathbf{G2}) & r(x,x,y)=s(x,y,y), \\ (\mathbf{G3}) & s(x,x,y)=y. \end{array}$$

The general n-permutability is characterized by the fact that its theory $\mathbb{T}$ contains $n-1$ ternary operations $q_1,\ldots,q_{n-1}$ such that

$$\begin{array}{ll} (\mathbf{N1}) & q_1(x,y,y)=x, \\ (\mathbf{N2}) & q_i(x,x,y)=q_{i+1}(x,y,y), \quad i=1,\ldots,n-2, \\ (\mathbf{N3}) & q_{n-1}(x,x,y)=y. \end{array}$$

Remark 1.1. The notion of an n-permutable variety has been generalized to categories: a regular category [1] is called an n-permutable category when the composition of equivalence relations on a same object is n-permutable [4].
It would be interesting to know whether the results of this paper extend, or not, to the categorical n-permutable context.

2. Internal structures

Let $\mathcal{C}$ be an arbitrary category with pullbacks. Recall that an (internal) *reflexive graph* (in $\mathcal{C}$) is given by a diagram

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0, \arrow[l, "e" description] \end{tikzcd}$$

where X_0 is called the "object of objects" and X_1 the "object of arrows", such that the domain morphism d, codomain morphism c and identity morphism e satisfy $de=ce=1_{X_0}$. In set theoretical terms, given an object A of X_0 and an arrow $f\colon A\to B$ of X_1, we have $e(A)=1_A$, $d(f)=A$ and $c(f)=B$. If (X_2,π_1,π_2) denotes the pullback of (c,d), then a reflexive graph is called a *reflexive multiplicative graph* when it is equipped with a multiplication (also called composition) m

$$\begin{tikzcd} X_2 \arrow[r, shift left=3, "\pi_1"] \arrow[r, "m" description] \arrow[r, shift right=3, "\pi_2"'] & X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0 \arrow[l, "e" description] \end{tikzcd}$$

such that

$$\begin{array}{ll} (\mathbf{R1}) & m(ed,1_{X_1})=1_{X_1}, \\ (\mathbf{R2}) & m(1_{X_1},ec)=1_{X_1}. \end{array}$$

We call X_2 the "object of composable pairs". In set theoretical terms, the composite of a composable pair $(f\colon A\to B, g\colon B\to C)$ of X_2, is denoted by $m(f,g)=g\circ f$. The identities **(R1)** and **(R2)** just mean that $f\circ 1_A=f=1_B\circ f$.

An *internal category* is a reflexive multiplicative graph such that

$$\begin{array}{ll} (\mathbf{IC1}) & dm=d\pi_1, \\ (\mathbf{IC2}) & cm=c\pi_2, \\ (\mathbf{IC3}) & m(m\times_{X_0}1_{X_1})=m(1_{X_1}\times_{X_0}m). \end{array}$$

So, $g\circ f$ has the same domain as f and the same codomain as g and the composition is associative. Finally, an internal category is called an *internal groupoid* when there exists an inversion morphism $i\colon X_1\to X_1$ such that

$$\begin{array}{ll} (\mathbf{IG1}) & di=c, \\ (\mathbf{IG2}) & ci=d, \\ (\mathbf{IG3}) & m(i,1_{X_1})=ec, \\ (\mathbf{IG4}) & m(1_{X_1},i)=ed. \end{array}$$

We denote the inverse of an arrow $f\colon A\to B$ of X_1 by $i(f)=f^{-1}\colon B\to A$. The identities **(IG3)** and **(IG4)** mean that $f\circ f^{-1}=1_B$ and $f^{-1}\circ f=1_A$.

The above identities express the usual axioms for a category or a groupoid.

Note that, if we consider such internal structures in a variety, then all the morphisms involved must be homomorphisms of algebras, i.e. they must preserve any n-ary operation. Suppose w is a ternary operation (the only arity we are interested in throughout this work), then it must be preserved by the domain:

$$d(\,w(f_1,f_2,f_3)\,)=w(\,d(f_1),d(f_2),d(f_3)\,),$$

for $f_i \in X_1$, $i = 1, 2, 3$; it must be preserved by the composition:

$$w(g_1, g_2, g_3) \circ w(f_1, f_2, f_3) = w(g_1 \circ f_1, g_2 \circ f_2, g_3 \circ f_3),$$

for $(f_i, g_i) \in X_2$, $i = 1, 2, 3$, etc.

We write $\mathrm{Rg}(\mathcal{C})$, $\mathrm{Rmg}(\mathcal{C})$, $\mathrm{Cat}(\mathcal{C})$ and $\mathrm{Gpd}(\mathcal{C})$ for the categories of reflexive graphs, reflexive multiplicative graphs, internal categories and internal groupoids in $\mathcal{C}$, respectively. We obtain a chain of forgetful functors

$$\mathrm{Gpd}(\mathcal{C}) \overset{U}{\hookrightarrow} \mathrm{Cat}(\mathcal{C}) \overset{V}{\hookrightarrow} \mathrm{Rmg}(\mathcal{C}) \overset{W}{\hookrightarrow} \mathrm{Rg}(\mathcal{C}),$$

where U and V are always full. If $\mathcal{C}$ is a regular Mal'tsev category, then W is also full and, moreover, U and V are isomorphisms [6]. It is also known that $\mathrm{Cat}(\mathcal{C})$ is exact [1] and Mal'tsev whenever $\mathcal{C}$ is [7]. In [9] it is shown that $\mathrm{Gpd}(\mathcal{C})$ is a reflective subcategory of $\mathrm{Rg}(\mathcal{C})$ for a regular Goursat category $\mathcal{C}$ with coequalizers and, consequently, $\mathrm{Gpd}(\mathcal{C})$ is also a regular Goursat category.

3. Internal structures in Mal'tsev varieties

We begin by recalling the known case of (internal structures in) Mal'tsev varieties [11], where it is shown that the notions of reflexive multiplicative graph, internal category and internal groupoid coincide. We shall use the same notations as in Section 2. The approach used in the Mal'tsev case will give us some ideas on how to generalize the result to the case of Goursat varieties.

In this section we work in a Mal'tsev variety. So, its theory contains a Mal'tsev operation p.

First we prove that every internal category in a Mal'tsev variety is an internal groupoid. Given an arrow $f: A \to B$ in X_1, we must define its inverse, thus an arrow in the opposite direction. Such an arrow is obtained by applying the Mal'tsev operation p to the triple $(1_A, f, 1_B)$. We define f^{-1} as

$$B \overset{\mathbf{(M2)}}{=} p(A, A, B) \xrightarrow{f^{-1} = p(1_A, f, 1_B)} p(A, B, B) \overset{\mathbf{(M1)}}{=} A.$$

Now, to see that it is actually the inverse of f, we use the fact that the composition preserves p

$$\begin{aligned} f \circ p(1_A, f, 1_B) &\overset{\mathbf{(M1)}}{=} p(f, 1_B, 1_B) \circ p(1_A, f, 1_B) \\ &= p(f \circ 1_A, 1_B \circ f, 1_B \circ 1_B) \\ &= p(f, f, 1_B) \\ &\overset{\mathbf{(M2)}}{=} 1_B; \end{aligned}$$

similarly, $p(1_A, f, 1_B) \circ f = 1_A$.

We have defined an inversion morphism $i: X_1 \to X_1$ by $i = p(ed, 1_{X_1}, ec)$, which is necessarily a homomorphism of algebras. This is a direct consequence of the uniqueness of inverses and the fact that the composition is a homomorphism of algebras.

As a final remark, we point out the fact that the inverse of f could equally be defined with the triple $(1_B, f, 1_A)$, i.e. $p(1_A, f, 1_B) = f^{-1} = p(1_B, f, 1_A)$.

Now we prove that every reflexive multiplicative graph is an internal category and that, moreover, the multiplicative structure is unique. We begin by obtaining a formula for the composite of a composable pair of arrows $(f: A \to B, g: B \to C)$ in X_2, again using the fact that the composition preserves p,

$$\begin{aligned} g \circ f &\overset{\mathbf{(M1),(M2)}}{=} p(g, 1_B, 1_B) \circ p(1_B, 1_B, f) \\ &= p(g \circ 1_B, 1_B \circ 1_B, 1_B \circ f) \\ &= p(g, 1_B, f). \end{aligned}$$

Then, we conclude that $d(g \circ f) = d(p(g, 1_B, f)) = p(d(g), d(1_B), d(f)) = p(B, B, A) = A = d(f)$, since the domain preserves p; similarly, we can show that $c(g \circ f) = c(g)$. The associativity and uniqueness of m follow directly from the above formula for the composite; the proof for the uniqueness is trivial. As for the associativity, given a composable triple of arrows $(f: A \to B, g: B \to C, h: C \to D)$, we have

$$\begin{aligned} h \circ (g \circ f) &= p(h, 1_C, g \circ f) \\ &= p(h \circ 1_C, 1_C \circ 1_C, g \circ f) \\ &= p(h, 1_C, g) \circ p(1_C, 1_C, f) \\ &\overset{\mathbf{(M2)}}{=} (h \circ g) \circ f. \end{aligned}$$

We also remark that the composite of (f, g) could be equally be defined as $p(g, 1_B, f) = g \circ f = p(f, 1_B, g)$.

4. Internal structures in Goursat varieties

After analyzing internal structures in Mal'tsev varieties, we turn our attention to Goursat varieties. In this section we work in a Goursat variety; its theory contains two ternary operations r and s such that the identities **(G1)**–**(G3)** hold. The goal is to get similar results to those obtained in the Mal'tsev case by using similar arguments. The main difficulty in adapting these arguments comes from the fact that now we have two ternary operations r and s, each satisfying only one of the Mal'tsev identities, designated by **(G1)** and **(G3)**, respectively. However, there is also the extra identity **(G2)** providing a link between the two ternary operations r and s which will help us overcome this difficulty.

In a Goursat context we can still prove that the notions of reflexive multiplicative graph, internal category and internal groupoid coincide. Again, we use the same notations as in Section 2.

From this section on, we will not write trivial composites as $1_B \circ 1_B, f \circ 1_A, 1_B \circ f$, etc.

Proposition 4.1. *In a Goursat variety, every internal category is an internal groupoid.*

Proof. Given an arrow $f: A \to B$ in X_1, we must define its inverse $f^{-1}: B \to A$. Now, if we apply both ternary operations r and s to the same triple $(1_A, f, 1_B)$ as used in the Mal'tsev case, then $r(1_A, f, 1_B)$ is a map from $r(A, A, B)$ to $r(A, B, B) = A$, while $s(1_A, f, 1_B)$ is a map from $s(A, A, B) = B$ to $s(A, B, B)$. But, we also have $s(A, B, B) = r(A, A, B)$, thus giving a composable pair. We define f^{-1} as the composite

$$\begin{tikzcd} B \overset{\text{(G3)}}{=} s(A,A,B) \arrow[r, "s(1_A,f,1_B)"] \arrow[rr, bend right, "f^{-1}=r(1_A,f,1_B)\circ s(1_A,f,1_B)"'] & s(A,B,B) \overset{\text{(G2)}}{=} r(A,A,B) \arrow[r, "r(1_A,f,1_B)"] & r(A,B,B) \overset{\text{(G1)}}{=} A. \end{tikzcd}$$

Next we prove the identity **(IG3)** (the proof of **(IG4)** is similar)

$$\begin{aligned} f \circ r(1_A, f, 1_B) \circ s(1_A, f, 1_B) &\overset{\text{(G1)}}{=} r(f, 1_B, 1_B) \circ r(1_A, f, 1_B) \circ s(1_A, f, 1_B) \\ &= r(f, f, 1_B) \circ s(1_A, f, 1_B) \\ &\overset{\text{(G2)}}{=} s(f, 1_B, 1_B) \circ s(1_A, f, 1_B) \\ &= s(f, f, 1_B) \\ &\overset{\text{(G3)}}{=} 1_B. \end{aligned}$$

This defines the inversion morphism $i = m(\, s(ed, 1_{X_1}, ec)\, ,\, r(ed, 1_{X_1}, ec)\,)$, necessarily a homomorphism of algebras as explained in the Mal'tsev case. □

Remark 4.2. As in the Mal'tsev case, the inverse of an arrow $f: A \to B$ in X_1 could equally be defined with the triple $(1_B, f, 1_A)$, that is, $r(1_A, f, 1_B) \circ s(1_A, f, 1_B) = f^{-1} = s(1_B, f, 1_A) \circ r(1_B, f, 1_A)$.

Remark 4.3. Mal'tsev vs. Goursat varieties.
It is well known that a Mal'tsev variety is necessarily a Goursat variety. That is, if the theory contains a Mal'tsev operation p, then it also contains Goursat operations r and s defined, for instance, by $r(x, y, z) = p(x, p(y, z, y), z)$ and $s(x, y, z) = p(x, p(x, y, x), z)$. Given an arrow $f: A \to B$ in a reflexive graph, then the Mal'tsev and Goursat inverse of f is the same

$$\begin{aligned} r(1_A, f, 1_B) \circ s(1_A, f, 1_B) &= p(1_A, p(f, 1_B, f), 1_B) \circ p(1_A, p(1_A, f, 1_A), 1_B) \\ &= p(1_A, p(f, 1_B, f) \circ p(1_A, f, 1_A), 1_B) \\ &= p(1_A, p(f, f, f), 1_B) \\ &= p(1_A, f, 1_B). \end{aligned}$$

Proposition 4.4. *In a Goursat variety, every reflexive multiplicative graph is an internal category. Such a multiplicative structure is unique.*

Proof. For a reflexive multiplicative graph to be an internal category, we just need to prove that the equalities **(IC1)** and **(IC2)** hold, since the associativity and uniqueness of m follow from Corollary 4.4 [3] (see also Remark 4.5). Consider a composable pair of arrows $(f: A \to B, g: B \to C)$ in X_2. By applying the ternary operation r to the same triple $(g, 1_B, f)$ as used in the Mal'tsev case

$$\begin{aligned} r(g, 1_B, f) &= r(g, 1_B, 1_B) \circ r(1_B, 1_B, f) \\ &\overset{\text{(G1),(G2),(G3)}}{=} s(1_B, 1_B, g) \circ s(1_B, f, f) \\ &= s(1_B, f, g \circ f). \end{aligned}$$

They must have the same codomain

$$\begin{aligned} & c(\, r(g, 1_B, f)\,) = c(\, s(1_B, f, g \circ f)\,) \\ \Rightarrow\ & r(c(g), B, B) = s(B, B, c(g \circ f)) \\ \overset{\text{(G1),(G3)}}{\Rightarrow}\ & c(g) = c(g \circ f). \end{aligned}$$

Similarly, $s(g, 1_B, f) = r(g \circ f, g, 1_B)$ and, by applying the domain, we get $d(f) = d(g \circ f)$. $\square$

Remark 4.5. By Corollary 4.4 [3], given a reflexive multiplicative graph that satisfies **(IC1)** and **(IC2)**, the multiplication m is necessarily associative and unique in a Goursat context. We now give a direct proof of these facts which serves as a motivation for the generalization to the n-permutable case.

Let W represent the equalizer of $m(m \times_{X_0} 1_{X_1})$ and $m(1_{X_1} \times_{X_0} m)$. For any composable triple of arrows $(f: A \to B, g: B \to C, h: C \to D)$ in such a reflexive multiplicative graph, we have $(f, g, 1_C), (1_C, 1_C, 1_C), (1_C, 1_C, h) \in W$. By applying r, we conclude that $(\, r(f, 1_C, 1_C), r(g, 1_C, 1_C), r(1_C, 1_C, h)\,) \in W$, thus $(f, g, s(1_C, h, h)) \in W$. So

$$\begin{array}{ll} & s(1_C, h, h) \circ (g \circ f) = (s(1_C, h, h) \circ g) \circ f \\ \overset{\textbf{(G3)}}{\Rightarrow} & s(1_C, h, h) \circ s(1_C, 1_C, g \circ f) = (s(1_C, h, h) \circ s(1_C, 1_C, g)) \circ s(1_C, 1_C, f) \\ \Rightarrow & s(1_C, h, h \circ (g \circ f)) = s(1_C, h, h \circ g) \circ s(1_C, 1_C, f) \\ \Rightarrow & s(1_C, h, h \circ (g \circ f)) = s(1_C, h, (h \circ g) \circ f). \end{array}$$

If we compose both sides with $s(h, 1_D, 1_D)$, we get

$$\begin{array}{ll} & s(h, 1_D, 1_D) \circ s(1_C, h, h \circ (g \circ f)) = s(h, 1_D, 1_D) \circ s(1_C, h, (h \circ g) \circ f) \\ \Rightarrow & s(h, h, h \circ (g \circ f)) = s(h, h, (h \circ g) \circ f) \\ \overset{\textbf{(G3)}}{\Rightarrow} & h \circ (g \circ f) = (h \circ g) \circ f. \end{array}$$

The uniqueness of the multiplicative structure follows a similar argument. Suppose there exists another multiplication m'. Given a composable pair of arrows $(f: A \to B, g: B \to C)$ in X_2, we denote $m'(f, g) = g * f$. Let U be the equalizer of m and m'. We have $(f, 1_B), (1_B, 1_B), (1_B, g) \in U$. By applying r, we conclude that $(\, r(f, 1_B, 1_B)\,,\ r(1_B, 1_B, g)\,) \in U$, i.e. $(\, f, s(1_B, g, g)\,) \in U$. So

$$\begin{array}{ll} & s(1_B, g, g) \circ f = s(1_B, g, g) * f \\ \overset{\textbf{(G3)}}{\Rightarrow} & s(1_B, g, g) \circ s(1_B, 1_B, f) = s(1_B, g, g) * s(1_B, 1_B, f) \\ \Rightarrow & s(1_B, g, g \circ f) = s(1_B, g, g * f). \end{array}$$

Now, if we use m (or m') to compose both sides with $s(g, 1_C, 1_C)$, we get

$$\begin{array}{ll} & s(g, 1_C, 1_C) \circ s(1_B, g, g \circ f) = s(g, 1_C, 1_C) \circ s(1_B, g, g * f) \\ \Rightarrow & s(g, g, g \circ f) = s(g, g, g * f) \\ \overset{\textbf{(G3)}}{\Rightarrow} & g \circ f = g * f. \end{array}$$

5. Internal structures in *n*-permutable varieties

In this section we work in an n-permutable variety, $n \geq 4$. So, its theory contains $n - 1$ ternary operations $q_1, \ldots, q_{n-1}$ such that the identities **(N1)**–**(N3)** hold. In this context, we shall adapt the arguments used for Goursat varieties to compare the notions of reflexive multiplicative graph, internal category and internal groupoid. We use the notations of Section 2 once again. As before, we still have two ternary operations q_1 and q_{n-1}, each satisfying only one of the Mal'tsev identities, designated by **(N1)** and **(N3)**. Now, the main difficulty comes from the existence of the ternary operations $q_2, \ldots, q_{n-2}$ that do not satisfy any kind of Mal'tsev identity. These operations only satisfy the identities **(N2)** that link them with the previous or the next one. In particular, q_2 and q_{n-2} can be associated to (the more manageable) q_1 and q_{n-1} by the identities $q_1(x, x, y) = q_2(x, y, y)$ and $q_{n-2}(x, x, y) = q_{n-1}(x, y, y)$.

Proposition 5.1. *In an n-permutable variety, every internal category is an internal groupoid.*

Proof. The proof of this result is similar to that of Proposition 4.1 done for Goursat varieties. Given an arrow $f: A \to B$ in X_1, we define its inverse $f^{-1}: B \to A$ as the composite

$$B \overset{\textbf{(N3)}}{=} q_{n-1}(A, A, B) \xrightarrow{q_{n-1}(1_A, f, 1_B)} q_{n-1}(A, B, B) \overset{\textbf{(N2)}}{=} q_{n-2}(A, A, B) \xrightarrow{q_{n-2}(1_A, f, 1_B)} \cdots$$

$$\cdots \xrightarrow[q_2(1_A, f, 1_B)]{} q_2(A, B, B) \overset{\textbf{(N2)}}{=} q_1(A, A, B) \xrightarrow[q_1(1_A, f, 1_B)]{} q_1(A, B, B) \overset{\textbf{(N1)}}{=} A,$$

i.e. $f^{-1} = q_1(1_A, f, 1_B) \circ q_2(1_A, f, 1_B) \circ \cdots \circ q_{n-1}(1_A, f, 1_B)$. As for the equality **(IG3)** (the proof of **(IG4)** is similar), we have

$$
\begin{aligned}
& f\circ q_1(1_A,f,1_B)\circ q_2(1_A,f,1_B)\circ\cdots\circ q_{n-1}(1_A,f,1_B)\\
\overset{\textbf{(N1)}}{=}\ & q_1(f,1_B,1_B)\circ q_1(1_A,f,1_B)\circ q_2(1_A,f,1_B)\circ\cdots\circ q_{n-1}(1_A,f,1_B)\\
=\ & q_1(f,f,1_B)\circ q_2(1_A,f,1_B)\circ\cdots\circ q_{n-1}(1_A,f,1_B)\\
\overset{\textbf{(N2)}}{=}\ & q_2(f,1_B,1_B)\circ q_2(1_A,f,1_B)\circ\cdots\circ q_{n-1}(1_A,f,1_B)\\
=\ & q_2(f,f,1_B)\circ\cdots\circ q_{n-1}(1_A,f,1_B)\\
\vdots\ & \\
=\ & q_{n-2}(f,f,1_B)\circ q_{n-1}(1_A,f,1_B)\\
\overset{\textbf{(N2)}}{=}\ & q_{n-1}(f,1_B,1_B)\circ q_{n-1}(1_A,f,1_B)\\
=\ & q_{n-1}(f,f,1_B)\\
\overset{\textbf{(N3)}}{=}\ & 1_B.
\end{aligned}
$$

This defines the inversion morphism (we write 1 instead of 1_{X_1})

$$i=m(q_{n-1}(ed,1,ec),\ldots,m(q_3(ed,1,ec),m(q_2(ed,1,ec),q_1(ed,1,ec)))\cdots),$$

necessarily a homomorphism of algebras as explained in the Mal'tsev case. □

Remark 5.2. As in the Mal'tsev case, the inverse of an arrow $f:A\to B$ in X_1 could equally be defined with the triple $(1_B,f,1_A)$; again $q_1(1_A,f,1_B)\circ q_2(1_A,f,1_B)\circ\cdots\circ q_{n-1}(1_A,f,1_B)=f^{-1}=q_{n-1}(1_B,f,1_A)\circ\cdots\circ q_2(1_B,f,1_A)\circ q_1(1_B,f,1_A)$.

The fact that the notions of internal category and internal groupoid coincide for any n-permutable variety gives us hope that also reflexive multiplicative graphs are internal categories in this context. However, many difficulties were found concerning the proof of the identities **(IC1)** and **(IC2)**. All the tested approaches seemed to fail and it still remains an open question. Nevertheless, if we assume identities **(IC1)** and **(IC2)** to hold, then we can prove the associativity of m and, consequently, that we have an internal category; the uniqueness of the multiplication also holds (Proposition 5.3). In the proof we will repeatedly use the following kind of equalities (to simplify notation, we write 1 instead of 1_X, for any object $X\in X_0$)

$$
\begin{aligned}
& q_1(x,y,z)\circ a \overset{(\mathbf{N_1})}{=} q_1(x,y,z)\circ q_1(a,1,1)=q_1(x\circ a,y,z)\\
& q_i(x,y,z)\circ a = q_i(x,y,z)\circ q_i(a,a,a)=q_i(x\circ a,y\circ a,z\circ a),\quad i=2,\ldots,n-2\\
& q_{n-1}(x,y,z)\circ a \overset{(\mathbf{N_3})}{=} q_{n-1}(x,y,z)\circ q_{n-1}(1,1,a)=q_{n-1}(x,y,z\circ a),
\end{aligned}
\tag{1}
$$

whenever the composite of arrows x,y,z and a of X_1 are defined. Similar equalities hold for composites of the type $a\circ q_i(x,y,z)$, $i=1,\ldots,n-1$.

Proposition 5.3. *In an n-permutable variety, every reflexive multiplicative graph that satisfies* **(IC1)** *and* **(IC2)** *is an internal category. Such a multiplicative structure is unique.*

Proof. Let us prove that m is associative and unique by generalizing the arguments used in Remark 4.5. Let W represent the equalizer of $m(m\times_{X_0}1_{X_1})$ and $m(1_{X_1}\times_{X_0}m)$ and consider any composable triple of arrows $(f:A\to B, g:B\to C, h:C\to D)$ in such a reflexive multiplicative graph. The following triples belong to W

$$
\begin{array}{ll}
(f,g,1_C) & (f,1_B,1_B)\\
(1_C,1_C,1_C) & (1_B,1_B,1_B)\\
(1_C,1_C,h) & (1_B,g,h).
\end{array}
\tag{2}
$$

By applying q_1 to the left triples of (2), we conclude that $(f,g,q_1(1_C,1_C,h))\in W$, then $(f,g,q_2(1_C,h,h))\in W$. Thus

$$
\begin{aligned}
& q_2(1_C,h,h)\circ(g\circ f)=(q_2(1_C,h,h)\circ g)\circ f\\
\overset{(1)}{\Rightarrow}\ & q_2(g\circ f,h\circ(g\circ f),h\circ(g\circ f))=q_2(g,h\circ g,h\circ g)\circ f\\
\overset{(1)}{\Rightarrow}\ & q_2(g\circ f,h\circ(g\circ f),h\circ(g\circ f))=q_2(g\circ f,(h\circ g)\circ f,(h\circ g)\circ f).
\end{aligned}
$$

By composing both sides with $q_2(h,1_D,1_D)$, we get

$$
\begin{aligned}
(\mathrm{L}_1)\quad h\circ(g\circ f) &= q_2(h\circ(g\circ f),(h\circ g)\circ f,(h\circ g)\circ f)\\
&\overset{\textbf{(N2)}}{=} q_1(h\circ(g\circ f),h\circ(g\circ f),(h\circ g)\circ f).
\end{aligned}
$$

Similarly, by applying q_1 to the right triples of (2) we conclude that

$$(\mathrm{R}_1)\quad (h\circ g)\circ f=q_1((h\circ g)\circ f,(h\circ g)\circ f,h\circ(g\circ f)).$$

Next, we apply q_2 to the left triples of (2) to obtain

$$\begin{aligned}
& q_2(1_C, 1_C, h) \circ q_2(g \circ f, 1_C, 1_C) = \\
& = (q_2(1_C, 1_C, h) \circ q_2(g, 1_C, 1_C)) \circ q_2(f, 1_C, 1_C) \\
\overset{(\mathbf{N2})}{\Rightarrow}\ & q_2(1_C, 1_C, h) \circ q_1(g \circ f, g \circ f, 1_C) = \\
& = (q_2(1_C, 1_C, h) \circ q_1(g, g, 1_C)) \circ q_1(f, f, 1_C) \\
\overset{(1)}{\Rightarrow}\ & q_1(\, q_2(g \circ f, g \circ f, h \circ (g \circ f))\, ,\ g \circ f\ ,\ 1_C\,) = \\
& = q_1(\, q_2(g, g, h \circ g), g, 1_C\,) \circ q_1(f, f, 1_C) \\
\overset{(1)}{\Rightarrow}\ & q_1(\, q_2(g \circ f, g \circ f, h \circ (g \circ f))\, ,\ g \circ f\ ,\ 1_C\,) = \\
& = q_1(\, q_2(g \circ f, g \circ f, (h \circ g) \circ f)\, ,\ g \circ f\ ,\ 1_C\,).
\end{aligned}$$

By precomposing both sides with $q_1(1_A, 1_A, g \circ f)$ and using (**N1**), we get

$$q_2(g \circ f, g \circ f, h \circ (g \circ f)) = q_2(g \circ f, g \circ f, (h \circ g) \circ f);$$

composing both sides with $q_2(h, h, 1_D)$ gives

$$(\mathrm{L}_2)\quad h \circ (g \circ f) = q_2(\, h \circ (g \circ f)\, ,\ h \circ (g \circ f)\, ,\ (h \circ g) \circ f\,).$$

Similarly, by applying q_2 to the right triples of (2) we conclude that

$$(\mathrm{R}_2)\quad (h \circ g) \circ f = q_2(\, (h \circ g) \circ f\ ,\ (h \circ g) \circ f\ ,\ h \circ (g \circ f)\,).$$

By applying q_j to the triples of (2), we can prove that

$$\begin{aligned}
(\mathrm{L}_j)\quad & h \circ (g \circ f) = q_j(\, h \circ (g \circ f)\, ,\ h \circ (g \circ f)\, ,\ (h \circ g) \circ f\,) \\
(\mathrm{R}_j)\quad & (h \circ g) \circ f = q_j(\, (h \circ g) \circ f\ ,\ (h \circ g) \circ f\ ,\ h \circ (g \circ f)\,),
\end{aligned}$$

for $2 \leq j \leq m$ and $m = n/2$ (when n is even) or $m = (n-1)/2$ (when n is odd). For $j \geq 3$, to obtain the identity (L_j) we use (R_{j-1}). In fact, by applying q_j to the left triples of (2), we get

$$\begin{aligned}
& q_j(1_C, 1_C, h) \circ q_j(g \circ f, 1_C, 1_C) = \\
& = (q_j(1_C, 1_C, h) \circ q_j(g, 1_C, 1_C)) \circ q_j(f, 1_C, 1_C) \\
\overset{(\mathbf{N2})}{\Rightarrow}\ & q_j(1_C, 1_C, h) \circ q_{j-1}(g \circ f, g \circ f, 1_C) = \\
& = (q_j(1_C, 1_C, h) \circ q_{j-1}(g, g, 1_C)) \circ q_{j-1}(f, f, 1_C) \\
\overset{(1)}{\Rightarrow}\ & q_j(q_{j-1}(g \circ f, g \circ f, 1_C), q_{j-1}(g \circ f, g \circ f, 1_C), q_{j-1}(h \circ (g \circ f), h \circ (g \circ f), h)) = \\
& = q_j(q_{j-1}(g, g, 1_C), q_{j-1}(g, g, 1_C), q_{j-1}(h \circ g, h \circ g, h)) \circ q_{j-1}(f, f, 1_C) \\
\overset{(1)}{\Rightarrow}\ & q_j(q_{j-1}(g \circ f, g \circ f, 1_C), q_{j-1}(g \circ f, g \circ f, 1_C), q_{j-1}(h \circ (g \circ f), h \circ (g \circ f), h)) = \\
& = q_j(q_{j-1}(g \circ f, g \circ f, 1_C), q_{j-1}(g \circ f, g \circ f, 1_C), q_{j-1}((h \circ g) \circ f, (h \circ g) \circ f, h)).
\end{aligned}$$

By composing both sides with $q_j(q_{j-1}(h, h, h \circ (g \circ f)), q_{j-1}(h, h, h \circ (g \circ f)), 1_D)$ we get

$$\begin{aligned}
& q_j(\, h \circ (g \circ f)\, ,\ h \circ (g \circ f)\, ,\ q_{j-1}(h \circ (g \circ f), h \circ (g \circ f), h)\,) = \\
& q_j(\, h \circ (g \circ f)\, ,\ h \circ (g \circ f)\, ,\ q_{j-1}((h \circ g) \circ f, (h \circ g) \circ f, h)\,)
\end{aligned}$$

and precomposing both sides with $q_j(1_A, 1_A, q_{j-1}(1_A, 1_A, g \circ f))$ gives

$$\begin{aligned}
& h \circ (g \circ f) = \\
= \ & q_j(\, h \circ (g \circ f)\, ,\ h \circ (g \circ f)\, ,\ q_{j-1}((h \circ g) \circ f, (h \circ g) \circ f, h \circ (g \circ f))\,) \\
\overset{(\mathrm{R}_{j-1})}{=}\ & q_j(\, h \circ (g \circ f), h \circ (g \circ f), (h \circ g) \circ f\,).
\end{aligned}$$

Similarly, to get (R_j) we use (L_{j-1}).

Using similar arguments, by applying q_k to the triples of (2), we can prove that

$$\begin{aligned}
(\mathrm{L}_k)\quad & h \circ (g \circ f) = q_k(\, (h \circ g) \circ f\ ,\ h \circ (g \circ f)\, ,\ h \circ (g \circ f)\,) \\
(\mathrm{R}_k)\quad & (h \circ g) \circ f = q_k(\, h \circ (g \circ f)\, ,\ (h \circ g) \circ f\ ,\ (h \circ g) \circ f\,),
\end{aligned}$$

for $m + 1 \leq k \leq n - 1$. In this case, we should begin by obtaining directly the identities (L_{n-1}), (R_{n-1}), (L_{n-2}) and (R_{n-2}). Then, to obtain (L_k) we use (R_{k+1}) and to get (R_k) we use (L_{k+1}), for $m + 1 \leq k \leq n - 3$.

Finally, we compare the identities (L_m) and (R_{m+1})

$$\begin{aligned}
h \circ (g \circ f) \ & \overset{(\mathrm{L}_m)}{=}\ q_m(\, h \circ (g \circ f)\, ,\ h \circ (g \circ f)\, ,\ (h \circ g) \circ f\,) \\
& \overset{(\mathbf{N_2})}{=}\ q_{m+1}(\, h \circ (g \circ f)\, ,\ (h \circ g) \circ f\ ,\ (h \circ g) \circ f\,) \\
& \overset{(\mathrm{R}_{m+1})}{=}\ (h \circ g) \circ f.
\end{aligned}$$

The uniqueness of the multiplicative structure follows a similar argument. Suppose there exists another multiplication m'. Given a composable pair of arrows $(f: A \to B, g: B \to C)$ in X_2, we denote $m'(f, g) = g * f$. Let U be the equalizer of m and m'. We have $(f, 1_B), (1_B, 1_B), (1_B, g) \in U$. By applying each q_i, $i = 1, \dots, n-1$, to this triple we get

$$\begin{array}{ll} (\mathrm{M}_j) & g \circ f = q_j(g \circ f, g \circ f, g * f), \quad 1 \le j \le m \\ (\mathrm{M}_k) & g \circ f = q_k(g * f, g \circ f, g \circ f), \quad m+1 \le k \le n-1. \end{array}$$

If we switch the roles of m and m', we obtain

$$\begin{array}{ll} (\mathrm{M'}_j) & g * f = q_j(g * f, g * f, g \circ f), \quad 1 \le j \le m \\ (\mathrm{M'}_k) & g * f = q_k(g \circ f, g * f, g * f), \quad m+1 \le k \le n-1. \end{array}$$

To finish, we compare the equalities (M_m) and $(\mathrm{M'}_{m+1})$

$$\begin{aligned} g \circ f &\overset{(\mathrm{M}_m)}{=} q_m(g \circ f, g \circ f, g * f) \\ &\overset{(\mathbf{N2})}{=} q_{m+1}(g \circ f, g * f, g * f) \\ &\overset{(\mathrm{M'}_{m+1})}{=} g * f. \quad \square \end{aligned}$$

Acknowledgements

The author would like to thank Marino Gran and Tim Van der Linden for many useful suggestions and discussions during the preparation of this work as well as the referee. The research was supported by CMUC and by FCT (Portugal), through the European Program COMPETE/FEDER.

References

[1] M. Barr, Exact Categories, in: Lecture Notes in Math., 236, Springer, 1971, pp. 1–120.
[2] F. Borceux, D. Bourn, Mal'cev, Protomodular, Homological and Semi-Abelian Categories, Kluwer, 2004.
[3] D. Bourn, M. Gran, Categorical aspects of modularity, in: G. Janelidze, B. Pareigis, W. Tholen (Eds.), Galois theory, Hopf Algebras and Semiabelian Categories, in: Fields Institute Communications, 43, Amer. Math. Soc, 2004.
[4] A. Carboni, G.M. Kelly, M.C. Pedicchio, Some remarks on Maltsev and Goursat categories, Appl. Categ. Structures 1 (1993) 385–421.
[5] A. Carboni, J. Lambek, M.C. Pedicchio, Diagram chasing in Malcev categories, J. Pure Appl. Algebra 69 (1991) 271–284.
[6] A. Carboni, M.C. Pedicchio, N. Pirovano, Internal graphs and internal groupoids in Mal'cev categories, CMS Conference Proceedings 13 (1992) 97–109.
[7] M. Gran, Internal categories in Mal'cev categories, J. Pure Appl. Algebra 143 (1999) 221–229.
[8] M. Gran, M.C. Pedicchio, *n*-permutable locally finitely presentable categories, Theory Appl. Categ. 8 (1) (2001) 1–15.
[9] M. Gran, D. Rodelo, A universal construction in Goursat categories, Cah. Topol. Géom. Différ. Catég. XLIX-3 (2008) 196–208.
[10] J. Hagemann, A. Mitschke, On *n*-permutable congruences, Algebra Universalis 3 (1973) 8–12.
[11] G. Janelidze, Internal Categories in Mal'cev Varieties, York University, Toronto (1990) Preprint.
[12] G. Janelidze, M.C. Pedicchio, Internal categories and groupoids in congruence modular varieties, J. Algebra 193 (1997) 552–570.
[13] A.I. Mal'cev, On the general theory of algebraic systems, Mat. Sb. N.S. 35 (1954) 3–20.
[14] N. Martins-Ferreira, T. Van der Linden, Categories vs. Groupoids via Generalised Mal'tsev Properties, University of Coimbra, Coimbra (2010) Preprint.
[15] A. Mitschke, Implication algebras are 3-permutable and 3-distributive, Algebra Universalis 1 (1971) 182–186.
[16] E.T. Schmidt, On *n*-permutable equational classes, Acta Sci. Math. (Szeged) 33 (1972) 29–30.