On Amenability Properties of Some Closed Ideals of $\mathcal{B}(X)$

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Abstract

In this dissertation, we investigate the relationship between the geometric properties of Banach spaces and some amenability properties of Banach algebra of operators on a Banach space. In doing this, we first carried out an extensive review on the general theory of Banach algebras and some closed ideals of $B(X)$, the Banach algebra of bounded linear operators on a Banach space X. Thereafter, various geometric properties of Banach spaces are studied and examples given. Finally, the notions of amenability in general Banach algebras are examined and characterized. This paves a way for our main focus in this study, where we investigate how the geometric properties of Banach spaces affect the amenability properties of some closed ideals of $B(X)$.

Declaration

I, undersigned, here by declare that the work contained in this dissertation is my original work, and that any work done by others or myself has been acknowledged and referenced accordingly.

This work was done under the guidance of Dr. O.T. Mewomo, at the University of Kwa-Zulu Natal, Durban, South Africa.

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In my capacity as supervisor of the candidate's Masters dissertation, I certify that the above statements are true to the best of my knowledge.

Supervisor:..

Date:..

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1 Introduction and Preliminaries

1.1 Brief Introduction

The notion of amenability in Banach algebras was first introduced in 1972 by Johnson in his Memoir of the American Mathematical Society. He showed that the group algebra $L^1(G)$ is amenable as a Banach algebra if and only if the locally compact group G is amenable as a group. This is equivalent to saying that $L^1(G)$ has vanishing first order Hoschschild cohomology with coefficients in dual Banach $L^1(G)$-bimodules. Consequently, he called Banach algebras satisfying this cohomological triviality condition amenable, see [25].

Since 1972, amenability has played an important role in the study of Banach algebras and has penetrated other branches of mathematics, such as Von Neumann algebra, operator spaces and even differential geometry. Amenability is in many situations a finiteness condition. This means that amenable algebras or amenable groups have properties that may be interpreted as weak versions of properties of finite-dimensional semisimple algebra.

The definition of amenabilty in Banach algebra given by Johnson in 1972 was too strong to allow for the development of a rich general theory, at the same time too weak to include a variety of interesting examples, see Mewomo [31]. Thus, several notions of amenabilty such a *weak amenability, approximate amenability, pseudo-amenability, character amenability* and many others were introduced to deal with the restrictiveness of the definition of amenability given by Johnson in 1972.

In this dissertation, we study the notions of amenability, weak amenabilty and approximate amenability for some closed operators ideals of the Banach algebra $\mathcal{B}(X)$ of bounded linear operators on Banach space X. In particular, we consider the algebras of approximable operators and compact operators ($\mathcal{A}(X)$ and $\mathcal{K}(X)$ respectively) on the Banach space X. This study was begun by Johnson, where he showed that $\mathcal{K}(X)$ is amenable if $X = \ell_p$, $1 < p < \infty$, or $X = C[0,1]$. Further progress was made later on by Gronbeak, Willis and Johnson [22] as relevant geometric propeties of Banach spaces were understood better, such as the approximation property.

This chapter reviews some basic definitions and concepts in Banach algebra that will be used throughout this study such as *tensor products* and *direct sums* which are vital for some later results. Chapter 2 looks at some classes of closed ideals of the Banach algebra $\mathcal{B}(X)$ of bounded linear operators on a Banach space X. Chapter 3 explores some geometric properties of

Banach spaces and how they affect the structure of operator algebras and its closed ideals. Chapter 4 focuses on some important notions of amenability in Banach algebras. Some of the hereditary properties, characterizations and general results of these notions of amenability in Banach algebras are also explored. Finally, chapter 5 mainly focuses on these notions of amenability on approximable and compact operators in relation to the geometric properties of the underlying Banach spaces.

1.2 Preliminaries

Definition 1.2.1. A *linear space* X is a set of vectors over a scalar field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) with two defined operations of vectors addition, and scalar multiplication such that the following conditions hold:

(a) $\forall x, y, z \in X$

- (i) $x + y = y + x$;
- (ii) $x + (y + z) = (x + y) + z$;
- (iii) $\exists$ a zero vector 0 such that $x + 0 = x$;
- (iv) for each $x \in X$ $\exists$ a unique $-x$ such that $x + (-x) = 0$.

(b) $\forall x, y \in X$ and $\alpha, \beta \in \mathbb{F}$

- (i) $1 \cdot x = x$;
- (ii) $(\alpha + \beta)x = \alpha x + \beta x$;
- (iii) $\alpha(\beta x) = (\alpha\beta x)$;
- (iv) $\alpha(x + y) = \alpha x + \beta x$.

Definition 1.2.2. A *norm* on a linear space X is a map $\|\cdot\| : X \to \mathbb{R}$ such that the following conditions hold:
$\forall x, y \in X$ and $\forall \alpha \in \mathbb{R}$,

(a) $\|x\| \geq 0$,

(b) $\|x\| = 0 \Leftrightarrow x = 0$,

(c) $\|\alpha x\| = |\alpha|\|x\|$,

(d) $\|x + y\| \leq \|x\| + \|y\|$.

If $\|.\|$ satisfies (a), (c) and (d) only, then $\|.\|$ is called a *semi-norm*.

A linear space X equipped with a norm is called *normed linear space* written as $(X, \|\cdot\|)$.

Definition 1.2.3. Let X be a normed space. A *sequence* in X is a mapping $x : \mathbb{N} \to X$ defined by

$$x(n) = x_n, \qquad \forall n \in \mathbb{N},$$

often denoted by (x_n) or $\{x_n\}_{n=1}^{\infty}$.

Definition 1.2.4. A sequence (x_n) in a normed space X is said to *converge* to $x \in X$ if $\forall \epsilon > 0, \exists\ N(\epsilon) \in \mathbb{N}$ such that

$$\|x_n - x\| < \epsilon, \qquad \text{whenever } n \geq N(\epsilon).$$

Often written as $\lim_{n\to\infty} x_n = x$.

Definition 1.2.5. A sequence (x_n) in a normed space X is said to be a *Cauchy sequence* in X if $\forall \epsilon > 0, \exists\ N(\epsilon) \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \epsilon, \qquad \text{whenever } m, n \geq N(\epsilon).$$

Definition 1.2.6. A normed space X is said to be *complete* if every Cauchy sequence in X converges in X. A complete normed space is called a *Banach space*.

Example 1.2.7. (a) $L^p(I) = \left\{g : I \to \mathbb{R} \mid \int_I |g(x)|^p dx < \infty\right\}$, the set of all Lebesgue integrable functions is complete under the L^p-norm,

$$\|g\|_p = \left(\int_I |g(x)|^p dx\right)^{1/p}.$$

(b) The set $C[0,1] = \{f : [0,1] \to \mathbb{F} : f \text{ is continuous}\}$ is a Banach space under supremum norm

$$\|f\| = \sup\{|f(x)| : x \in [0,1]\}.$$

However it is not complete under the L^p-norm for $1 \leq p < \infty$.

Definition 1.2.8. A *partially ordered set* or *poset* is a set $\mathscr{D}$ equipped with an order relation $\leq$ satisfying,

(a) $a \leq a \qquad \forall a \in \mathscr{D}$;

(b) if $a \leq b$ and $b \leq a$ then $a = b \qquad \forall a, b \in \mathscr{D}$;

(c) if $a \leq b$ and $b \leq a$ then $a \leq c \qquad \forall a, b, c \in \mathscr{D}$.

Definition 1.2.9. Let $\mathscr{D}$ be a partially ordered set. The set $\mathscr{D}$ is called a *directed set* if for all $a, b \in \mathscr{D}$, there exists $c \in \mathscr{D}$ such that $a \leq c$ and $b \leq c$.

Definition 1.2.10. Let $\mathscr{D}$ be a directed set and X any set. A *net* in X is the map $\lambda : \mathscr{D} \to X$.

Definition 1.2.11. Let $\mathscr{D}$ be a poset. An element $\alpha \in \mathscr{D}$ is said to a *maximal element* if whenever $\alpha \leq \beta$ $(\beta \in \mathscr{D})$, then $\alpha = \beta$.

Example 1.2.12. A sequence is a special case of a net with the natural numbers $\mathbb{N}$ as the directed set.

Theorem 1.2.13. *(Zorn's Lemma)*
If each chain in a non-empty partially ordered set $\mathscr{D}$ has an upper bound, then $\mathscr{D}$ has a maximal element.

Definition 1.2.14. Let $X \neq \emptyset$ be any set. A *filter* $\mathscr{F}$ is a non-empty collection of subsets of X such that

(a) $\emptyset \notin \mathscr{F}$,

(b) If $A, B \in \mathscr{F}$, then $A \cap B \in \mathscr{F}$,

(c) If $A \in \mathscr{F}$ and $A \subseteq B$, then $B \in \mathscr{F}$.

If the filters are partialy ordered by set inclusion(i.e $\subseteq$) then Zorn's lemma ensures existence of a maximal filter called an *ultrafilter.*

Theorem 1.2.15. *[11] A filter $\mathcal{U}$ on a set X is an ultrafilter if and only if for each $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.*

Definition 1.2.16. A *linear map* between two linear spaces X and Y is a map $T : X \to Y$ such that

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty, \quad \forall x, y \in X \text{ and } \alpha, \beta \in \mathbb{F}.$$

A linear map is often referred to as a *linear operator.*

Definition 1.2.17. Let $T : X \to Y$ be a linear operator between linear spaces X and Y. For all $x \in X$

(a) If $Y = \mathbb{F}$, then T is said to be a *linear functional.*

(b) The set $\text{range}(T) = \{Tx : x \in X\}$ is called the *range* of T.

(c) The set $\ker(T) = \{x \in X : Tx = 0\}$ is called the *kernel* or *nullspace* of T.

(d) T is *injective* if $Tx = Ty$ then $x = y$, *surjective* if $\text{range}(T) = Y$ and T is a *bijection* if T is both injective and surjective.

(e) If $Y = X$ and $Tx = x$, then T is called the *identity operator*, denoted as Id_X.

Definition 1.2.18. A linear map $T : X \to Y$ between normed spaces X and Y is said to be *bounded* if there exists $M \geq 0$ such that

$$\|Tx\|_Y \leq M\|x\|_X, \quad \forall \quad x \in X.$$

If no such constant exists, then T is said to be *unbounded.*

Definition 1.2.19. If T is a bounded linear operator from a normed space X to a normed space Y. Then we define the *operator norm* to be the number

$$\begin{aligned}\|T\| &= \inf\{M : \|Tx\|_Y \leq M\|x\|_X, \ \forall x \in X\} \\ &= \sup\{\|Tx\| : \|x\| \leq 1\} \\ &= \sup\{\|Tx\| : \|x\| = 1\} \\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} : \|x\| \neq 0\right\}.\end{aligned}$$

Let $\mathcal{B}(X,Y) = \{T : X \to Y \mid T \text{ is linear and bounded}\}$, define the operations on $\mathcal{B}(X,Y)$ as follows:

(a) Addition by $(T_1 + T_2)x = T_1x + T_2x \qquad \forall T_1, T_2 \in \mathcal{B}(X,Y),\ x \in X;$

(b) Scalar multiplication by $(\alpha T)x = \alpha Tx \qquad \forall T \in \mathcal{B}(X,Y),\ \alpha \in \mathbb{F}.$

Then $\mathcal{B}(X, Y)$ is a linear space and thus is a normed space equipped with the operator norm.

If $X = Y$, we write $\mathcal{B}(X, X) = \mathcal{B}(X)$. Moreover if $T \in \mathcal{B}(X)$ then T is said be a *linear transformation*.

Proposition 1.2.20. *Let X be a normed space and Y a Banach Space, then $\mathcal{B}(X, Y)$ is a Banach space.*

Proof. Suppose that Y be a Banach space. Let (T_n) be a Cauchy sequence in $\mathcal{B}(X, Y)$. Then for all $\epsilon > 0$, there exist $N(\epsilon) \in \mathbb{N}$ such that $\|T_n - T_m\| < \epsilon$ whenever $m, n \geq N(\epsilon)$. Choose an arbitrary $x \in X$. Then whenever $m, n \geq N(\epsilon)$,

$$\begin{aligned} \|T_n x - T_m x\| &= \|(T_n - T_m)x\| \\ &\leq \|T_n - T_m\| \|x\| \\ &< \epsilon \|x\|. \end{aligned} \tag{1}$$

It follows that $(T_n x)$ is Cauchy sequence in Y. But since Y is complete $(T_n x)$ converges to some Tx in Y, i.e

$$\lim_{n\to\infty} T_n x = Tx \tag{2}$$

where $T : X \to Y$. Now to show $\lim_{n\to\infty} T_n x = T$ and that $T \in B(X, Y)$.

By (1) and linearity of T_n, we have for any $x, y \in X$ and $\alpha, \beta \in \mathbb{F}$

$$\begin{aligned} T(\alpha x + \beta y) &= \lim_{n\to\infty} T_n(\alpha x + \beta y) \\ &= \lim_{n\to\infty} (\alpha T_n x + \beta T_n y) \\ &= \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n y \\ &= \alpha Tx + \beta Ty. \end{aligned}$$

Hence T is linear.

From (2), we have

$$\begin{aligned} Tx - T_m x &= (\lim_{n\to\infty} T_n x) - T_m x \qquad \forall x \in X, m \in \mathbb{N} \\ &= \lim_{n\to\infty} (T_n x - T_m x) \end{aligned}$$

and so

$$\|Tx - T_m x\| = \lim_{n\to\infty} \|T_n x - T_m x\|.$$

But then by (1)

$$\|Tx - T_m x\| \leq \epsilon \|x\| \qquad \forall x \in X \text{ and } m \geq N \tag{3}$$

Now from (3) and any $x \in X$,

$$\begin{aligned} \|Tx\| &= \|Tx - T_N x + T_N x\| \\ &\leq \|Tx - T_N x\| + \|T_N x\| \\ &\leq \epsilon \|x\| + \|T_N\| \|x\| \\ &= (\epsilon + \|T_N\|)\|x\|. \end{aligned}$$

Hence T is bounded i.e $T \in \mathcal{B}(X, Y)$.

Finally, inequality (3) gives

$$\|T - T_m\| = \sup\{\|Tx - T_m x\| : \|x\| \leq 1\} \leq \epsilon \quad \forall m \geq N.$$

i.e $\lim_{n\to\infty} T_n = T$. Hence $\mathcal{B}(X, Y)$ is a Banach space. □

Definition 1.2.21. Let X and Y be normed spaces.

(a) A linear operator $T \in \mathcal{B}(X, Y)$ is said to be an *isomorphism* if there exist $S \in \mathcal{B}(Y, X)$ such that $ST = Id_X$ and $TS = Id_Y$, by composition. Thus X and Y are said to be *isomorphic* to each other, written as $X \cong Y$.

(b) A linear operator $T : X \to Y$ not necessarily surjective is said to be an *isometry* if

$$\|Tx\| = \|x\|, \qquad \forall x \in X.$$

Definition 1.2.22. A *dual space* X' of a normed space X is defined to be the set

$$X' \stackrel{\text{def}}{=} \{\mu : X \to \mathbb{F} : \mu \text{ is a continuous linear functional}\}.$$

Remark 1.2.23. We note $X' = \mathcal{B}(X, \mathbb{F})$. The *bidual* is the dual of the dual space X' of X i.e $X'' = (X')'$. Furthermore higher duals of X can be defined as $X''' = (X'')'$, $X'''' = (X''')'$, ...etc.

Remark 1.2.24. From the definition of dual spaces and Proposition 1.2.20, we have that dual spaces are Banach spaces.

Theorem 1.2.25. *If X'' is a bidual of a normed space X, then there's a natural (canonical) embedding map $\imath : X \to X''$ defined by*

$$\imath(x)(\mu) = \langle \mu, x \rangle = \mu(x) \qquad \forall x \in X, \mu \in X',$$

with the following properties:

(a) $\imath$ is linear,

(b) $\imath$ is an isometry.

Definition 1.2.26. Let X be a normed space. For each $\mu \in X'$, the functional

$$\rho_\mu(x) = |\langle \mu, x \rangle|, \qquad x \in X.$$

is a seminorm on X. The topology induced by the family of seminorms $\{\rho_\mu\}_{\mu \in X'}$ is said to be the *weak topology* on X, denoted as $\sigma(X, X')$.

A sequence (x_n) in X is said to *converge weakly* to $x \in X$, denoted as $x_n \xrightarrow{w} x$, if

$$\lim_{n\to\infty} \langle \mu, x_n \rangle = \langle \mu, x \rangle, \qquad \forall \mu \in X'.$$

Definition 1.2.27. Let X be a normed space. The topology induced by the family of seminorms $\{\rho_{\imath(x)}\}_{x \in X}$ is said to be the *weak* topology* on X', denoted as $\sigma(X', X)$.

A sequence (μ_n) in X' is said to *converge weakly** to $\mu \in X'$, denoted as $\mu_n \xrightarrow{w^*} \mu$ if

$$\lim_{n\to\infty} \langle \mu_n, x \rangle = \langle \mu, x \rangle, \qquad \forall x \in X.$$

Proposition 1.2.28. *Let X be a normed space.*

(a) Every convergent sequence in X is weakly convergent.

(b) The limit of a weakly convergent sequence in X is unique

Proof. (a) Let (x_n) be a sequence in X such that $x_n \to x \in X$. Since $\|x_n - x\| \to 0$, we have

$$|\mu(x_n) - \mu(x)| \le |\mu(x_n - x)| \le \|\mu\| \|x_n - x\| \to 0 \quad \forall \mu \in X'.$$

Thus completing proof, however the converse is not always true.

(b) Let (x_n) be a sequence such that $x_n \overset{w}{\to} x$ and $x_n \overset{w}{\to} y$. This implies that there's a sequence of scalars $(\mu(x_n))$ such that $\mu(x_n) \to \mu(x)$ and $\mu(x_n) \to \mu(y)$. Hence $\mu(x) = \mu(y)$. This implies that $\langle \mu, x - y\rangle = \langle \mu, 0\rangle$.Therefore $x = y$.

□

Remark 1.2.29. In general, weak convergence implies weak* convergence but the converse is not always true. However, the notions are equivalent if the normed space X is reflexive as we shall see in the next proposition.

Definition 1.2.30. Let X be Banach space. The *closed unit ball* of X, denoted by B_X is defined to be the set

$$B_X = \{x \in X : \|x\| \leq 1\}.$$

Lemma 1.2.31. *(*Goldstine's Theorem*) Let X be a Banach space and $\bar{B}_X$ be a closed unit ball identified as a subset of X'' under the canonical embedding. Then B_X is weak* dense in $B_{X''} = \{x'' \in X : \|x''\| \leq 1\}$.*

Theorem 1.2.32. *(*Banach Alaoglu*) Let X be a Banach space. Then $B_{X'}$ is compact in the weak*-topology.*

Proposition 1.2.33. *Let X be a Banach space. The the following are equivalent:*

(a) X is reflexive;

(b) X'' is reflexive;

(c) The weak topology on X'' equals the weak topology on X';*

(d) The unit ball of X is compact in its weak topolgy;

(e) The unit ball of X' is compact in its weak topolgy.

1.3 Banach Algebras

1.3.1 Definitions with Examples

Definition 1.3.1. A vector space $\mathcal{A}$ over a scalar field $\mathbb{F}$ is said to be an *algebra*, if $\mathcal{A}$ has a multiplication operation called an *algebra product*,

$$(\mathcal{A}, \mathcal{A}) \to \mathcal{A}$$
$$(a, b) \to ab, \quad a, b \in A,$$

such that the following conditions hold for all $a, b, c \in \mathcal{A}$ and for all $\alpha \in \mathbb{F}$,

(a) $a(bc) = (ab)c$;

(b) $a(b + c) = ab + ac$;

(c) $(a + b)c = ac + bc$;

(d) $(\alpha a)b = a(\alpha b) = \alpha(ab)$.

Definition 1.3.2. An algebra $\mathcal{A}$ is said be *commutative* if the algebra product is abelian.

Definition 1.3.3. A *normed algebra* is an algebra $\mathcal{A}$ equipped with a norm $\|.\|$ which is *sub-multiplicative*, that is:

$$\|ab\| \leq \|a\|\|b\|, \qquad \forall a, b \in \mathcal{A}.$$

Moreover if $(\mathcal{A}, \|.\|)$ is a Banach space, then $\mathcal{A}$ is called a *Banach algebra*.

Example 1.3.4. Let X be a compact Hausdorff space. Then the Banach space

$$C(X) = \{f : X \to \mathbb{F} : f \text{ is continuous}\}$$

is a commutative Banach algebra with the constant function 1 as a unit, pointwise multiplication, and the norm

$$\|f\| = \sup_{x \in X} |f(x)|, \qquad f \in C(X).$$

In general if X is a topological space, then the Banach space

$$C_b(X) = \{f : X \to \mathbb{F} : f \text{ is continuous and bounded}\}$$

is a Banach algebra called the *function algebra*.

Example 1.3.5. Let X be Banach space. Then $\mathcal{B}(X)$ equipped with the operator norm and composition product becomes a Banach algebra called the *operator algerba*.

Example 1.3.6. Let G be a locally compact group and λ a left Haar measure. Then $L^1(G) = \left\{f : G \to \mathbb{R} : f \text{ is measurable and } \int_G |f(s)| d\lambda(s) < \infty\right\}$, the set of Borel measurable functions on G equipped the norm

$$\|f\|_1 = \int_G |f(s)| d\lambda(s)$$

and the convolution product $(f,g) \mapsto f * g$ defined by

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)d\lambda(t) \quad (f, g \in L^1(G), s, t \in G)$$

is a Banach algebra. Indeed for $f, f \in L^1(G)$,

$$\begin{aligned}
\|f * g\|_1 &= \int_G |(f * g)(s)| d\lambda(s) \\
&= \int_G \left| \int_G f(t)g(t^{-1}s)d\lambda(t) \right| d\lambda(s) \\
&\leq \int_G \int_G \left| f(t)g(t^{-1}s) \right| d\lambda(t)d\lambda(s) \\
&= \int_G \int_G |f(t)||g(t^{-1}s)| d\lambda(s)d\lambda(t) \\
&= \int_G |f(t)| \left(\int_G g(s)ds \right) d\lambda(t) \\
&= \|f\|_1 \|g\|_1.
\end{aligned}$$

In fact, it is commutative if and only if G is abelian. Also unital if and only if G is discrete. $L^1(G)$ is called a *group algebra* of G.

Example 1.3.7. Let G be a locally comapact abelian group and ω a *weight* function on G. Then $L^1(G, \omega)$ is the space of complex-valued functions $f : G \to \mathbb{C}$, called a *Beurling algebra*, for which

$$\|f\|_\omega = \int_G |f(s)|\omega(s)d\lambda(s) < \infty,$$

where λ is the left Haar measure on G. $L^1(G, \omega)$ is Banach algebra with the convolution product

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)d\lambda(t) \quad (f, g \in L^1(G, \omega), s, t \in G).$$

We note that if $\omega = 1$ then $L^1(G, \omega) = L^1(G)$.

Example 1.3.8. Let G be a locally compact group. The *measure algebra* of G is the space of bounded regular measures on G denoted by $M(G)$. It is well known that $M(G)$ can be identified as the dual of $C_0(G)$ the Banach space of continuous complex-valued functions on G which vanish at infinity.

1.4 Some Basic Concepts in Banach Algebra

1.4.1 Ideal, Quotient algebra and Homomorphism

Definition 1.4.1. Let $\mathcal{A}$ be an algebra.

(a) $\mathcal{A}$ is said to be *unital* if there exists $e_{\mathcal{A}} \in \mathcal{A}$, called a *unit*, such that $e_{\mathcal{A}}a = ae_{\mathcal{A}} = a$ for all $a \in \mathcal{A}$.

(b) If $\mathcal{A}$ is a unital algebra, an element $a \in \mathcal{A}$ is said to be *invertible* if there exists $b \in \mathcal{A}$ such that $ab = ba = e_{\mathcal{A}}$.
Let $Inv\mathcal{A}$ be the set of all invertible elements of $\mathcal{A}$, then $Inv\mathcal{A}$ becomes a group with the mulplication product.

Remark 1.4.2. For a unital Banach algebra $\mathcal{A}$, we have that:

(a) The unit element $e_{\mathcal{A}}$ in $\mathcal{A}$ is uniquely determined, for if $e'_{\mathcal{A}}$ is another unit, we have $e_{\mathcal{A}} = e_{\mathcal{A}}e'_{\mathcal{A}} = e'_{\mathcal{A}}$.

(a) If $a \in Inv\mathcal{A}$ then the inverse b is unique. In this case we write $b = a^{-1}$.

Definition 1.4.3. A unital algebra $\mathcal{A}$ is called a *division algebra* if every non-zero element in $\mathcal{A}$ is invertible.

Definition 1.4.4. Let $\mathcal{A}$ be a Banach algebra (not necessarily unital), then the Cartesian product

$$\tilde{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{A} \times \mathbb{F}$$

with the multiplication

$$(a, \alpha)(b, \beta) = (ab + \beta a + \alpha b, \alpha\beta), \quad a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{F}$$

is an algebra with unit $(0, 1)$. $\tilde{\mathcal{A}}$ becomes an unital Banach algebra equipped with the norm $\|(a, \alpha)\| = \|a\| + |\alpha|$, $(a \in \mathcal{A}, \alpha \in \mathbb{F})$. $\tilde{\mathcal{A}}$ is called the *unitization* of $\mathcal{A}$.

Remark 1.4.5. $\tilde{\mathcal{A}}$ is commutative if and only if $\mathcal{A}$ is commutative.

Definition 1.4.6. Let $\mathcal{A}$ be an algebra and $\mathcal{M} \subseteq \mathcal{A}$ a linear subspace. Then $\mathcal{M}$ is said to be a *subalgebra* if for each $a, b \in \mathcal{M}$, we have $ab \in \mathcal{M}$. In particular if $\mathcal{A}$ is a Banach algebra then $\mathcal{M}$ is called a *subBanach algebra*.

Definition 1.4.7. Let $\mathcal{A}$ be an algebra and $\mathcal{I}$ a subalgebra of $\mathcal{A}$, then $\mathcal{I}$ is said to be a

(a) *left ideal* if $ab \in \mathcal{I}, \quad \forall a \in \mathcal{A}, \forall b \in \mathcal{I}$;

(b) *right ideal* if $ba \in \mathcal{I}, \quad \forall a \in \mathcal{A}, \forall b \in \mathcal{I}$;

(c) *two-sided ideal* or simply an *ideal* if $\mathcal{I}$ is both left and right ideal in $\mathcal{A}$.

Definition 1.4.8. If $\{0\} \subsetneq \mathcal{I} \subsetneq \mathcal{A}$, then $\mathcal{I}$ is called a *proper ideal.* $\{0\}$ and $\mathcal{A}$ itself are called *trivial ideals.*

Proposition 1.4.9. *[11] If $\mathcal{I}$ is a proper ideal in a unital Banach algebra, then the closure $\bar{\mathcal{I}}$ of $\mathcal{I}$ is also a proper ideal.*

Definition 1.4.10. Let $\mathcal{I}$ be an ideal of an algebra $\mathcal{A}$, then the algebra $\mathcal{A}/\mathcal{I} = \{a + I : a \in A\}$ is said to a *quotient algebra* of $\mathcal{A}$ by $\mathcal{I}$.

Definition 1.4.11. Let $\mathcal{A}$ be an algebra. A linear space X is called a $\mathcal{A}$-module if there exists bilinear maps

$$\mathcal{A} \times X \to X, (a, x) = ax \quad \text{and} \quad X \times \mathcal{A} \to X, (x, a) = xa$$

such that for each $a, b \in \mathcal{A}$ and $x \in X$, the following holds:

(a) $(ab)x = a(bx)$

(b) $x(ab) = (xa)b$

(c) $a(xb) = (ax)b$

Proposition 1.4.12. *[11] Let $\mathcal{A}$ be a unital Banach algebra, then the product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous.*

Proof. Let $a, a_1, b, b_1 \in \mathcal{A}$, then we have

$$\begin{aligned}
\|ab - a_1b_1\| &= \|ab - ab_1 + ab_1 - a_1b_1\| \\
&= \|a(b - b_1) + (a - a_1)b_1\| \\
&\leq \|a\|\|b - b_1\| + \|a - a_1\|\|b_1\|.
\end{aligned}$$

So if $a_n \times b_n \to a \times b$ as $n \to \infty$ then $a_nb_n \to ab$ as $n \to \infty$. □

Proposition 1.4.13. *[11] Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. If $\|a\| < 1$ then $1 - a \in Inv\mathcal{A}$ with*

$$\|(1 - a)^{-1}\| \leq \frac{1}{1 - \|a\|}.$$

Proof. Let $x_n = \sum_{i=0}^{n} a^i$. For some $\epsilon > 0$, let $N(\epsilon) \geq 0$ such that $\|a\|^{N(\epsilon)+1} < \epsilon(1 - \|a\|)$. Now for $N \leq m < n$, we have

$$\|x_n - x_m\| \leq \sum_{i=m+1}^{n} \|a\|^i \leq \frac{\|a\|^{m+1}}{1 - \|a\|} < \epsilon.$$

Hence (x_n) is a Cauchy sequence, and $\lim_{n\to\infty} x_n = \sum_{i=0}^{\infty} a^i$ exists. So by continuity of the multiplication and letting $x = \sum_{i=0}^{\infty} a^i$

$$(1 - a)x = (1 - a)\sum_{i=0}^{\infty} a^i = \sum_{i=0}^{\infty} a^i - \sum_{i=0}^{\infty} a^{i+1} = 1,$$

and by similar argument $x(1 - a) = 1$. Finally, we also have

$$\|x\| \leq \sum_{i=0}^{\infty} \|a\|^i = \frac{1}{1 - \|a\|},$$

which gives a norm bound on the inverse. □

Corollary 1.4.14. *[11] Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$ such that $\|e - a\| \leq 1$, then $a \in \mathrm{Inv}\mathcal{A}$ with*

$$\|a^{-1}\| \leq \frac{1}{1 - \|1 - a\|}.$$

Proof. This follows from a simple application of the previous proposition to $1 - (1 - a)$. □

Proposition 1.4.15. *[11] If $\mathcal{A}$ is a unital Banach algebra, then* $\mathrm{Inv}\mathcal{A}$ *is an open set.*

Proof. Let $a \in \mathrm{Inv}\mathcal{A}$. We need to to show that $\mathrm{Inv}\mathcal{A}$ contain an open set, in particular $B(a) \stackrel{\text{def}}{=} \{b \in \mathrm{Inv}\mathcal{A} : \|a - b\| < \|a^{-1}\|^{-1}\} \subseteq \mathrm{Inv}\mathcal{A}$.

Let $x \in B_\epsilon(a)$, then we have

$$\begin{aligned}\|e - a^{-1}x\| &= \|a^{-1}a - a^{-1}x\| \\ &= \|a^{-1}(a - x)\| \\ &\leq \|a^{-1}\|\|a - x\| \\ &< 1.\end{aligned}$$

Hence $a^{-1}x \in \mathrm{Inv}\mathcal{A}$. It follows that $a(a^{-1}x) = x \in \mathrm{Inv}\mathcal{A}$. Therefore $B_\epsilon(a) \subseteq \mathrm{Inv}\mathcal{A}$. □

Proposition 1.4.16. *[11] Let $\mathcal{A}$ be a unital algebra, then the map on Inv$\mathcal{A}$ defined by $a \to a^{-1}$ is continuous.*

Proof. Let $a, b \in \text{Inv}\mathcal{A}$ such that $\|a-b\| < \frac{1}{2\|a^{-1}\|}$.
We then have

$$\begin{aligned}\|e - a^{-1}b\| &= \|a^{-1}a - a^{-1}b\| \\ &= \|a^{-1}(a-b)\| \\ &\leq \|a^{-1}\|\|a-b\| \\ &< \|a^{-1}\|\frac{1}{2\|a^{-1}\|} \\ &= \frac{1}{2},\end{aligned}$$

and so the inverse of $a^{-1}b$ has a bound by the previous proposition. In particular

$$\begin{aligned}\|b^{-1}\| &= \|b^{-1}aa^{-1}\| \\ &\leq \|b^{-1}a\|a^{-1}\| \\ &= \|(a^{-1}b)^{-1}\|\|a^{-1}\| \\ &\leq \frac{1}{1-\|1-a^{-1}b\|}\|a^{-1}\| \\ &\leq 2\|a^{-1}\|.\end{aligned}$$

Hence it follows that

$$\begin{aligned}\|a^{-1} - b^{-1}\| &= \|a^{-1}bb^{-1} - a^{-1}ab^{-1}\| \\ &= \|a^{-1}(b-a)b^{-1}\| \\ &\leq \|a^{-1}\|\|b-a\|\|b^{-1}\| \\ &\leq 2\|a^{-1}\|^2\|b-a\|.\end{aligned}$$

So then if a sequence $(a_n) \subset \text{Inv}\mathcal{A}$ converges to a as n becomes large. We have that

$$\|a^{-1} - a_n^{-1}\| \leq 2\|a^{-1}\|^2\|a - a_n^{-1}\| \to 0 \text{ as } n \to \infty,$$

thus we have continuity. □

Definition 1.4.17. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map between algebra $\mathcal{A}$ and $\mathcal{B}$. Then ϕ is said to be a *homomorphism* if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$. If ϕ is a bijection then it is called an *isomorphism*.

Proposition 1.4.18. *[11] Let $\mathcal{A}$ be a unital Banach algebra and $\phi : \mathcal{A} \to \mathbb{C}$ a homomorphism. Then for all $a \in \mathcal{A}$ we have $|\phi(a)| \leq \|a\|$.*

Theorem 1.4.19. (Gelfand-Mazur). *Let $\mathcal{A}$ be a Banach algebra such every non-zero element is invertible ($\mathcal{A}$ is a division algebra). Then there exists a unique isometric isomorphism of $\mathcal{A}$ onto $\mathbb{C}$ i.e $\mathcal{A} \cong \mathbb{C}$.*

Proposition 1.4.20. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism, where $\mathcal{A}$ and $\mathcal{B}$ are commutative Banach algebra with identity and $\mathcal{B}$ semi-simple. Then ϕ is continuous.*

1.4.2 Spectrum of a Banach algebra

Definition 1.4.21. Let $\mathcal{A}$ be a complex unital Banach algebra and $a \in \mathcal{A}$. The *spectrum* of a is the set $\sigma_{\mathcal{A}}(a) \subset \mathbb{C}$ defined by

$$\sigma_{\mathcal{A}}(a) \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} : \lambda e - a \notin \text{Inv}\mathcal{A}\}.$$

The *resolvent* set of a is defined to be the set

$$\rho_{\mathcal{A}}(a) \stackrel{\text{def}}{=} \mathbb{C} \setminus \sigma_{\mathcal{A}}(a).$$

If $\sigma_{\mathcal{A}}(a)$ is non-empty, then the *spectral* radius of a is the set

$$r_A(a) \stackrel{\text{def}}{=} \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(a)\}.$$

Example 1.4.22. Let $\mathcal{A} = C(X)$, where X is a compact Hausdorff space. For $f \in \mathcal{A}$, the spectrum $\sigma(f)$ is the range of f.

Example 1.4.23. Let $\mathcal{A} = C_b(X)$, X any arbitrary topological space. If $f \in \mathcal{A}$, then the spectrum of f is the closure of its range.

Example 1.4.24. Let $\mathcal{A} = \mathbb{C}$. If $z \in \mathcal{A}$, then $\sigma(z) = z$.

Proposition 1.4.25. *Let $\mathcal{A}$ be a complex unital algebra and $a \in Inv\mathcal{A}$.*

(a) $\sigma_{\mathcal{A}}(a^{-1}) = \sigma_{\mathcal{A}}(a)^{-1}$.

(b) $\lambda \in \sigma_{\mathcal{A}}(a)$ *if and only if* $\lambda \in \sigma_{\mathcal{A}}(a^{-1})$.

(c) If $\lambda \in \sigma_{\mathcal{A}}(ab)$, *then* $\lambda \in \sigma_{\mathcal{A}}(ba)$.

(d) If $a \in Inv\mathcal{A}$ *then* $\sigma_{\mathcal{A}}(ab) = \sigma_{\mathcal{A}}(ba)$.

Proposition 1.4.26. *Let* $\mathcal{A}$ *be a complex unital algebra and* $a \in \mathcal{A}$. *Then* $\sigma_{\mathcal{A}}(a)$ *is non-empty compact subset of* $\mathbb{C}$ *with* $r_{\mathcal{A}}(a) = \|a\|$.

Proposition 1.4.27. (Spectral radius formula). *Let* $\mathcal{A}$ *be a complex unital algebra and* $a \in \mathcal{A}$, *then* $\|r_{\mathcal{A}}(a)\| \leq \|a\|$ *and*

$$r_{\mathcal{A}}(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|a^n\|^{\frac{1}{n}}.$$

Proposition 1.4.28. *Let* $\mathcal{A}$ *be a unital Banach algebra. Then we have the following:*

(a) $r_{\mathcal{A}}(ab) = r_{\mathcal{A}}(ba)$;

(b) If $ab = ba$ *then* $r_{\mathcal{A}}(a+b) \leq r_{\mathcal{A}}(a) + r_{\mathcal{A}}(b)$ *and* $r_{\mathcal{A}}(ab) \leq r_{\mathcal{A}}(a)r_{\mathcal{A}}(b)$.

1.4.3 Direct sums of algebras

Definition 1.4.29. (Finite case) Let $(\mathcal{A}_i)_{i=1}^n$ be a family of Banach algebras. The *direct sum* of $\mathcal{A}_i$'s is defined by

$$\mathcal{A} \stackrel{\text{def}}{=} \oplus_{i=1}^n \mathcal{A}_i$$

, with multiplication and norm defined as follows:
Let $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathcal{A}$. Then

$$x \cdot y = (x_1 y_1, x_2 y_2, ..., x_n y_n),$$
$$\|x\| = \max\{\|x_i\| : 1 \leq i \leq n\}.$$

Definition 1.4.30. (Infinite case) Let $(\mathcal{A}_\lambda)_{\lambda\in\Lambda}$ be a family of Banach algebras. Let

$$\mathcal{A} = \oplus_{\lambda\in\Lambda}\mathcal{A}_\lambda := \{(x_\lambda)_{\lambda\in\Lambda} : \sup_\lambda \|x_\lambda\| < \infty\}.$$

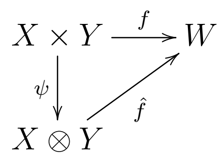
For $x = (x_\lambda), y = (y_\lambda) \in \mathcal{A}$, the multiplication and norm are defined as follows:

$$x \cdot y = (x_\lambda \cdot y_\lambda)_{\lambda\in\Lambda}$$
$$\|x\| = sup_\lambda \|(x_\lambda)\|.$$

Remark 1.4.31. The direct sums are Banach algebras and unital if each (A_i) is unital, and commutative properties hold.

1.4.4 Tensor Products and Tensor Norms

Definition 1.4.32. Let X and Y be two linear spaces. The *tensor product* of X and Y is a linear space $X \otimes Y$ together with a *canonical bilinear map* $\psi : X \times Y \to X \otimes Y$ such that the following universal property holds:
If W is a linear space and $f : X \times Y \to W$ is an arbitrary bilinear map then there exists a unique a map $\hat{f} : X \otimes Y \to W$ such that $f = \hat{f} \circ \psi$, that is to say



commutes.
If $z \in U \otimes V$, then

$$z = \sum_{i=1}^{n} x_i \otimes y_i, \qquad n \in \mathbb{N}$$

where $(x_i)_{i=1}^{n} \subseteq X$ and $(y_i)_{i=1}^{n} \subseteq Y$. This representation is not unique.

Definition 1.4.33. Let X and Y be two Banach spaces. A norm α on the algebraic tensor product $X \otimes Y$ is said to be a *crossnorm* if

$$\alpha(x \otimes y) = \|x\|\|y\|, \qquad \forall x \in X, y \in Y.$$

Moreover if we have for every $\mu \in X'$ and $\lambda \in Y'$. The linear functional $\mu \otimes \lambda$ on $X \otimes Y$ is bounded and $\|\mu \otimes \lambda\|_\alpha \leq \|\mu\|\|\lambda\|$. Then α is called a *reasonable crossnorm*.

Definition 1.4.34. A crossnorm α on $X \otimes Y$ is said to be a *uniform crossnorm* if

$$\alpha((T_1 \otimes T_2)z) \leq \|T_1\|\|T_2\|\alpha(z)$$

for all $T_1 \in B(X), T_2 \in B(Y)$ and $z \in X \otimes Y$.

Definition 1.4.35. A uniform crossnorm α on $X \otimes Y$ is said be *finitely generated* if for $z \in X \otimes Y$, then

$$\alpha(z) = \inf \{\alpha(x) : A \in \mathrm{FIN}(X), B \in \mathrm{FIN}(Y), x \in X \otimes Y\}.$$

A finitely generated uniform crossnorm is called a *tensor norm*. FIN(X) denotes all finite dimensional subspaces of X in the sense of [11].

Definition 1.4.36. Let X and Y be two Banach spaces and $z = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$.

(a) The *injective tensor norm* on $X \otimes Y$ is defined by

$$\varepsilon(z) = \sup \left\{ \left| \sum_{i=1}^{n} \langle \mu, x_i \rangle \langle \lambda, y_i \rangle \right| : \mu \in X', \lambda \in Y' \right\}.$$

(b) The *projective tensor norm* on $X \otimes Y$ is defined by

$$\pi(z) = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : z = \sum_{i=1}^{n} x_i \otimes y_i, x_i \in X, y_i \in Y \right\}.$$

Remark 1.4.37. (a) The projective tensor norm is the *greatest* crossnorm on $X \otimes Y$. The injective tensor is *least reasonable* cross norm on on $X \otimes Y$. Both norms are submultiplicative.

(b) The above defined tensor norms are both finitely generated and uniform crossnorms.

(c) The completion of $X \otimes Y$ with respect to the injective tensor norm ε is denoted by $X \check{\otimes}_\varepsilon Y$ or in short $X \check{\otimes} Y$ and is called *injective tensor product.*

(d) The completion of $X \otimes Y$ with respect to the projective tensor norm π is denoted by $X \widehat{\otimes}_\pi Y$ or in short $X \widehat{\otimes} Y$ and is called the *projective tensor product.*

(e) If $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras then $\mathcal{A} \widehat{\otimes} \mathcal{B}$ and $\mathcal{A} \check{\otimes} \mathcal{B}$ are Banach algebras under their respective norms.

1.4.5 Bounded Approximate Identities

Definition 1.4.38. Let $\mathcal{A}$ be a Banach algebra and $(e_\alpha)_{\alpha \in \mathscr{D}}$ a net in $\mathcal{A}$. Then;

(a) $(e_\alpha)_{\alpha \in \mathscr{D}}$ is said be a *left approximate identity* if we have

$$e_\alpha a \to a \quad a \in A.$$

Similarly a *right approximate identity* if $a e_\alpha \to a \quad a \in A$.

(b) $(e_\alpha)_{\alpha\in\mathscr{D}}$ is said to be a *bounded left/right approximate identity* if $(e_\alpha)_{\alpha\in\mathscr{D}}$ is left/right approximate identity and norm bounded. That is, there exists a constant $c > 0$ such that $\|e_\alpha\| \leq c$ for $\alpha \in \mathscr{D}$.

(c) If $(e_\alpha)_{\alpha\in\mathscr{D}}$ is both (bounded) left and right approximate identity then it is simply called an *(bounded) approximate identity.*

Proposition 1.4.39. *Let $\mathcal{A}$ be a Banach algebra. Suppose that $(e_\alpha)_{\alpha\in\mathscr{D}}$ is bounded left approximate identity for $\mathcal{A}$ and $(f_\beta)_{\beta\in\mathscr{D}'}$ is bounded right approximate identity for $\mathcal{A}$ with bounds $M > 0$ and $N > 0$ respectively. Then the net*

$$(e_\alpha + f_\beta - e_\alpha f_\beta)_{(\alpha,\beta)\in\mathscr{D}\times\mathscr{D}'}$$

is a bounded approximate identity for $\mathcal{A}$.

Proof. Let $g_{(\alpha,\beta)} = e_\alpha + f_\beta - e_\alpha f_\beta$. Then for each $a \in \mathcal{A}$,

$$\begin{aligned}
\|g_{(\alpha,\beta)}a - a\| &= \|(e_\alpha + f_\beta - e_\alpha f_\beta)a - a\| \\
&= \|e_\alpha a + f_\beta a - e_\alpha f_\beta a - a\| \\
&= \|(e_\alpha a - a) - f_\beta(a - a_\alpha a)\| \\
&\leq \|(e_\alpha a - a)\| + \|f_\beta(a - e_\alpha a)\| \\
&\leq \|(e_\alpha a - a)\| + N\|(a - e_\alpha a)\| \to 0.
\end{aligned}$$

By similar argument and for all $a \in \mathcal{A}$.

$$\begin{aligned}
\|ag_{(\alpha,\beta)} - a\| &= \|(af_\beta - a) - e_\alpha(af_\beta - a)\| \\
&\leq \|(af_\beta - a)\| + \|e_\alpha(af_\beta - a)\| \\
&\leq \|(af_\beta - a)\| + M\|(af_\beta - a)\| \to 0.
\end{aligned}$$

Finally, we show boundedness. For each $(\alpha,\beta) \in \mathscr{D}\times\mathscr{D}'$, we have

$$\begin{aligned}
\|g_{(\alpha,\beta)}\| &= \|e_\alpha + f_\beta - e_\alpha f_\beta\| \\
&\leq \|e_\alpha\| + \|f_\beta\| + \|e_\alpha f_\beta\| \\
&\leq M + N + MN.
\end{aligned}$$

Thus completing proof. □

Theorem 1.4.40. *[6] Let $\mathcal{A}$ be a Banach algebra with a bounded net (e_α) such that $e_\alpha a \xrightarrow{w} a$ for all $a \in \mathcal{A}$ (respectively $ae_\alpha \xrightarrow{w} a$ for all $a \in \mathcal{A}$). Then $\mathcal{A}$ has bounded left (respectively right) approximate identity.*

Remark 1.4.41. Let $\mathcal{A}$ Banach algebra. Then $\mathcal{A}$ has *left* (respectively *right*) *approximate unit* of bound $m \geq 0$ if for each $a \in \mathcal{A}$ and $\epsilon > 0$, there exists $u \in \mathcal{A}$ such that

$$\|a - ua\| < \epsilon \quad (\text{respectively } \|a - au\| < \epsilon).$$

It follows that if $\mathcal{A}$ has left (respectively right) approximate identity of bound m then $\mathcal{A}$ has a left (respectively right) approximate unit of bound m. In the next result, we show that the converse holds.

Proposition 1.4.42. *[36] Let $\mathcal{A}$ be a Banach algebra with left (respectively right) approximate unit of bound $m \geq 0$. Then $\mathcal{A}$ has left (respectively right) approximate identity of bound m.*

Theorem 1.4.43. *[6] Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with bounded left(respectively right) approximate identities $(e_\alpha)_{\alpha \in \mathscr{D}}$ and $(f_\beta)_{\beta \in \mathscr{D}'}$, respectively. Then $(e_\alpha \otimes f_\beta)_{(\alpha,\beta) \in \mathscr{D} \times \mathscr{D}'}$ is a (bounded) left approximate identity for the projective tensor product $\mathcal{A} \widehat{\otimes} \mathcal{B}$.*

Proof. Let $z = \sum_{i=1}^{n} a_i \otimes b_i \in \mathcal{A}\widehat{\otimes}\mathcal{B}, a_i \in \mathcal{A}, b_i \in \mathcal{B}, i = 1, 2, ..., n$, we have

$$
\begin{aligned}
\pi((e_\alpha \otimes f_\beta)z - z) &= \|(e_\alpha \otimes f_\beta)z - z\|_p \\
&= \left\|(e_\alpha \otimes f_\beta)\sum_{i=1}^{n} a_i \otimes b_i - \sum_{i=1}^{n} a_i \otimes b_i\right\|_p \\
&= \left\|\sum_{i=1}^{n} a_i \otimes b_i(e_\alpha \otimes f_\beta) - \sum_{i=1}^{n} a_i \otimes b_i\right\|_p \\
&= \left\|\sum_{i=1}^{n} e_\alpha a_i \otimes f_\beta b_i - \sum_{i=1}^{n} [(a_i - e_\alpha a_i) \otimes b_i + e_\alpha a_i \otimes b_i]\right\|_p \\
&= \left\|\sum_{i=1}^{n} e_\alpha a_i \otimes f_\beta b_i - \left[\sum_{i=1}^{n}(a_i - e_\alpha a_i) \otimes b_i + \sum_{i=1}^{n} e_\alpha a_i \otimes b_i\right]\right\|_p \\
&= \left\|\sum_{i=1}^{n}[(e_\alpha a_i \otimes f_\beta b_i) - (e_\alpha a_i \otimes b_i)] + \sum_{i=1}^{n}(a_i - e_\alpha a_i) \otimes b_i\right\|_p \\
&= \left\|\sum_{i=1}^{n}(e_\alpha a_i \otimes f_\beta b_i) - \sum_{i=1}^{n}(e_\alpha a_i \otimes b_i) + \sum_{i=1}^{n}(a_i - e_\alpha a_i) \otimes b_i\right\|_p \\
&= \left\|\sum_{i=1}^{n} e_\alpha a_i(f_\beta b_i - b_i) + \sum_{i=1}^{n}(a_i - e_\alpha a_i) \otimes b_i\right\|_p \\
&= \left\|\sum_{i=1}^{n}(e_\alpha a_i - a_\alpha + a_\alpha) \otimes (f_\beta b_i - b_i) + \sum_{i=1}^{n}(a_i - e_\alpha a_i) \otimes b_i\right\|_p \\
&= \left\|\sum_{i=1}^{n}(e_\alpha a_i - a_\alpha) \otimes (f_\beta b_i - b_i) + \sum_{i=1}^{n} a_\alpha \otimes (f_\beta b_i - b_i) + \sum_{i=1}^{n}(a_i - e_\alpha a_i) \otimes b_i\right\|_p \\
&= \left\|\sum_{i=1}^{n}(e_\alpha a_i - a_\alpha) \otimes (f_\beta b_i - b_i) + \sum_{i=1}^{n} a_\alpha \otimes (f_\beta b_i - b_i) - \sum_{i=1}^{n}(e_\alpha a_i - a_i) \otimes b_i\right\|_p \\
&\leq \left\|\sum_{i=1}^{n}(e_\alpha a_i - a_\alpha) \otimes (f_\beta b_i - b_i)\right\| + \left\|\sum_{i=1}^{n} a_\alpha \otimes (f_\beta b_i - b_i)\right\| \\
&+ \left\|\sum_{i=1}^{n}(e_\alpha a_i - a_i) \otimes b_i\right\|_p \\
&\leq \inf\left\{\sum_{i=1}^{n} \|e_\alpha a_i - a_\alpha\|\|f_\beta b_i - b_i\|\right\} + \inf\left\{\sum_{i=1}^{n} \|a_i\|\|\|f_\beta b_i - b_i\|\right\} \\
&+ \inf\left\{\sum_{i=1}^{n} \|e_\alpha a_i - a_i\|\|b_i\|\right\} \\
&\to 0.
\end{aligned}
$$

Since $\|e_\alpha a_i - a_i\| \to 0$ and $\|f_\beta b_i - b_i\| \to 0$. Thus showing that $(e_\alpha \otimes f_\beta)_{(\alpha,\beta)\in\mathscr{D}\times\mathscr{D}'}$ is left approximate identity. By similar argument, it can be shown that $(e_\alpha \otimes f_\beta)_{(\alpha,\beta)\in\mathscr{D}\times\mathscr{D}'}$ is also a right approximate identity.
Suppose (e_α) and (f_β) are bounded by M and N, respectively. Then we have that

$$\pi(e_\alpha \otimes f_\beta) = \|e_\alpha \otimes f_\beta\|_p = \inf\{\|e_\alpha\|\|f_\beta\|\} \leq \inf\{MN\} \leq MN.$$

Thus completing proof. □

Definition 1.4.44. Let X be a Banach space and $\mathcal{A}$ a Banach algebra. X is said to be a

(a) *Banach left $\mathcal{A}$-module* if X is a left $\mathcal{A}$-module and there exists a constant $k > 0$, such that

$$\|ax\| \leq k\|a\|\|x\| \quad (a \in \mathcal{A}, x \in X).$$

(b) *Banach right $\mathcal{A}$-module* if X is a right $\mathcal{A}$-module and there exists a constant $k > 0$, such that

$$\|ax\| \leq k\|a\|\|x\| \quad (a \in \mathcal{A}, x \in X).$$

(c) *Banach $\mathcal{A}$-bimodule* if X is both left and right Banach $\mathcal{A}$-module.

Example 1.4.45. If X is a Banach $\mathcal{A}$-module, then its topological dual X' is a Banach $\mathcal{A}$-module under the module actions

$$(a \cdot \mu)(x) = \mu(xa) \quad \text{and} \quad (\mu \cdot a)(x) = \mu(ax) \quad (a \in \mathcal{A}, x \in X, \mu \in X');$$

X' is said to be a *dual module* of X. Indeed for $x \in X$ and $\mu \in X'$, we have

$$\begin{aligned}\langle x, a \cdot (\mu \cdot b)\rangle &= \langle x \cdot a, \mu \cdot b\rangle \\ &= \langle b \cdot (x \cdot a), \mu\rangle \\ &= \langle (b \cdot x) \cdot a, \mu\rangle \\ &= \langle b \cdot x, a \cdot \mu\rangle \\ &= \langle x, (a \cdot \mu) \cdot b\rangle.\end{aligned}$$

Also, we have

$$\begin{aligned}\|a \cdot \mu\| = \sup_{\|x\|\leq 1}|\langle x, a \cdot \mu\rangle| &= \sup_{\|x\|\leq 1}|\langle x \cdot a, \mu\rangle| \\ &\leq \sup_{\|x\|\leq 1} k\|x\|\|a\|\|\mu\| \\ &= k\|a\|\|\mu\| \quad (a \in \mathcal{A}, \mu \in X').\end{aligned}$$

Similarly, we have $\|\mu \cdot a\| \leq k\|a\|\|\mu\|$ for all $a \in \mathcal{A}, \mu \in X.'$

Example 1.4.46. If $\mathcal{A}$ is a Banach algebra and X a Banach space such that $X = \mathcal{A}$. Then X is a Banach $\mathcal{A}$-bimodule with the module actions defined by

$$a \cdot x = ax \quad \text{and} \quad x \cdot a = xa \quad (a, x \in \mathcal{A}).$$

Such module actions on $X = A$ are called the canonical Banach $\mathcal{A}$-bimodule actions on $\mathcal{A}$.

Example 1.4.47. Let $\mathcal{A}$ be a Banach algebra and X a Banach space. Then the *zero left (right) action* of $\mathcal{A}$ on X is definded by

$$a \cdot x = 0 \quad (x \cdot a = 0) \quad (a, x \in \mathcal{A}).$$

If X is a Banach $\mathcal{A}$-bimodule, then if we replace one (or both) module actions by the zero action, X will remain a Banach $\mathcal{A}$-bimdule.

Definition 1.4.48. Let $\mathcal{A}$ be a Banach algebra and let X and Y be a Banach $\mathcal{A}$-bimodules. If a homomorphism $\phi : X \to Y$ preserves module multiplication, that is

$$\phi(ax) = a\phi(x), \quad \phi(xa) = \phi(x)a, \quad (a \in \mathcal{A}, x \in X),$$

then ϕ is said to be a *bimodule homomorphism.*

Theorem 1.4.49. (Cohen-Hewitt Factorisation Theorem) *Let $\mathcal{A}$ be a Banach algebra with bounded left (respectively right) approximate identity, and let X be a Banach $\mathcal{A}$-bimoduke. Then*

$$\mathcal{A} \cdot X = \{ax : a \in \mathcal{A}, x \in X\} \quad (\textit{respectively } X \cdot \mathcal{A} = \{ax : a \in \mathcal{A}, x \in X\})$$

is a closed submodule of X.

Corollary 1.4.50. (Cohen's Factorisation Theorem) *Let $\mathcal{A}$ be a Banach algebra with bounded approximation (e_α) and suppose that (e_α) is also a bounded left approximate identity for the Banach $\mathcal{A}$-bimodule X. Then $\mathcal{A}{\cdot}X = X$.*

Remark 1.4.51. If $\mathcal{A}$ is a Banach algebra with bounded approximate identity, then by Cohen's Factorisation Theorem, $\mathcal{A}^2 = \mathcal{A}$.

Definition 1.4.52. Let $\mathcal{A}$ be a Banach algebra. Then a Banach $\mathcal{A}$-bimodule X is said to be *neo-unital* if

$$X = \mathcal{A} \cdot X \cdot \mathcal{A} = \{axb : ab \in \mathcal{A}, x \in X\}.$$

Furthermore X is said be *essential* if

$$X = \overline{\mathcal{A} \cdot X \cdot \mathcal{A}}.$$

Remark 1.4.53. It follows from the above definition that if $\mathcal{A}$ is a Banach algebra with bounded approximation identity (e_α), and X is a neo-unital Banach $\mathcal{A}$-bimodule. Then (e_α) is also a bounded approximate identity for X. Moreover if $\mathcal{A}$ has unit $e_{\mathcal{A}}$, then

$$e_{\mathcal{A}} \cdot x = x = x \cdot e_{\mathcal{A}}.$$

Proposition 1.4.54. *[36] Let $\mathcal{A}$ be a Banach algebra, $\mathcal{I}$ a closed ideal of $\mathcal{A}$ with bounded approximate identity, and let X be a neo-unital Banach $\mathcal{I}$-bimodule. Then $\mathcal{I}$ can be made into a Banach $\mathcal{A}$-bimodule canonically.*

2 Ideals of $\mathcal{B}(X)$

2.1 Closed Ideals of $\mathcal{B}(X)$

Definition 2.1.1. An *operator ideal* $\mathcal{I}$ is an assignment, to each pair of Banach spaces X and Y, a subspace $\mathcal{I}(X,Y) \subseteq \mathcal{B}(X,Y)$ such that:

(a) $\mathcal{I}(X,Y)$ is Banach space equipped with some norm u on $\mathcal{I}(X,Y)$;

(b) $\mathcal{F}(X,Y) \subseteq \mathcal{I}(X,Y)$, $\mu \otimes y \in \mathcal{F}(X,Y)$ $(\mu \in X', y \in Y)$, we have $u(X \otimes Y) = \|\mu\|\|y\|$;

(c) for Banach spaces X_0 and Y_0, $S \in \mathcal{I}(X,Y)$, $T \in \mathcal{B}(X_0,X)$, and $R \in \mathcal{B}(Y,Y_0)$, $RST \in \mathcal{I}(X_0,Y_0)$, and $i(RST) \leq \|R\|i(S)\|T\|$.

Remark 2.1.2. If $\mathcal{I}(X,Y)$ is always a closed subspace of $\mathcal{B}(X,Y)$, then $\mathcal{I}$ is said to be a *closed operator ideal*.

2.1.1 Finite Rank Operators

Definition 2.1.3. Let X and Y be Banach spaces. Then $T \in \mathcal{B}(X,Y)$ is said to be a *finite-rank operator* if T has finite dimensional range.

Every finite rank operator $T \in \mathcal{B}(X,Y)$ can be written in the form

$$T = \sum_{i=1}^{n} \mu_i \otimes y_i, \qquad (\mu_i)_{i=1}^{n} \subseteq X', (y_i)_{i=1}^{n} \subseteq Y.$$

The collection of finite-rank operators in $\mathcal{B}(X,Y)$ will denoted by $\mathcal{F}(X,Y)$. We write $\mathcal{F}(X)$ instead of $\mathcal{F}(X,X)$.

Remark 2.1.4. If T is as above, then the canonical trace, trace : $X' \otimes X \to \mathbb{C}$ is defined by

$$\operatorname{trace}(T) = \sum_{i=1}^{n} \langle \mu_i, y_i \rangle$$

with the following propeties

(a) $\operatorname{trace}(ST) = \operatorname{trace}(TS) \quad T \in \mathcal{F}(X,Y), S \in \mathcal{B}(Y,X)$

(b) $|\operatorname{trace}(T)| \leq n\|T\|$ whenever $\operatorname{rank}(T) \leq n$.

2.1.2 Approximable Operators

Definition 2.1.5. An operator $T \in \mathcal{B}(X,Y)$ is said to be *approximable* if there exists a sequence of finite-rank operators $(T_n) \subset \mathcal{B}(X,Y)$ such that $\lim_{n\to\infty} \|T - T_n\| = 0$.

The collection of approximable operators is denoted by $\mathcal{A}(X,Y)$. We write $\mathcal{A}(X)$ instead of $\mathcal{A}(X,X)$.

Since by definition $\mathcal{A}(X)$ contains all limits of $\mathcal{F}(X)$, we have $\mathcal{A}(X) = \bar{\mathcal{F}}(X) \cong X'\check{\otimes}X$.

Theorem 2.1.6. *[32] $\mathcal{A}(X)$ is the smallest closed operator ideal of $\mathcal{B}(X)$.*

2.1.3 Compact Operators

Definition 2.1.7. An operator $T \in \mathcal{B}(X,Y)$ is *compact* if the image of the unit ball of X has a compact closure in Y.

Equivalently:

Definition 2.1.8. An operator $T \in \mathcal{B}(X,Y)$ is said to be *compact* if every bounded sequence (x_n), the sequence (Tx_n) has a convergent subsequence in Y.

Remark 2.1.9. If $\dim(Y) < \infty$ and $E \subset Y$, then the closure of E is compact if and only if E is bounded, hence all operators in $\mathcal{B}(X,Y)$ are compact. In general all finite-rank operators are compact.

The collection of compact operator in $\mathcal{B}(X,Y)$ is denoted by $\mathcal{K}(X,Y)$. We write $\mathcal{K}(X)$ instead of $\mathcal{K}(X,X)$.

Theorem 2.1.10. *[32] $\mathcal{K}(X)$ is a closed operator ideal of $\mathcal{B}(X)$.*

Proof. Suppose that $R, S \in \mathcal{K}(X)$, $T \in \mathcal{B}(X)$, and $\alpha \in \mathbb{F}$. We show that $R+S, \alpha R, RT, TR \in \mathcal{K}(X)$.

Let (x_n) be a bounded sequence in X. Since R is compact, (Rx_n) has a convergent subsequence (Rx_{n_i}). It follows that (αRx_{n_i}) is a convergent subsequence of (αRx_n). Hence $\alpha R \in \mathcal{K}(X)$.

Also, since (x_{n_i}) is convergent subsequent of (x_n) and B is compact, we have that (Sx_{n_i}) has a convergent subsequence $(Sx_{n_{i_j}})$. It follows that $([R+S]x_{n_{i_j}})$ is convergent subsequence of $([R+S]x_{n_i})$, hence $R+S$ is compact.

Furthermore, we have T is bounded so that (TRx_n) has a convergent subsequence (TRx_{n_i}), showing that TR is compact. Finally we have that (Tx_n) is bounded since (x_n) is bounded and that $\|Tx_n\| \leq \|T\|\|x_n\|$. And since $R \in \mathcal{K}(X)$, (TRx_n) has a convergent subsequence, thus TR is compact. Hence we have shown $\mathcal{K}(X)$ is an operator ideal. The proof of $\mathcal{K}(X)$ is closed is omitted. □

Remark 2.1.11. Let X be a Banach space. Then $\mathcal{K}(X) = \mathcal{B}(X)$ if and only if X is finite dimensional.

Proof. If $\dim(X) < \infty$, then $\mathcal{B}(X) = \mathcal{F}(X) \subseteq \mathcal{K}(X) \subseteq \mathcal{B}(X)$.

Converse if $\mathcal{K}(X) = \mathcal{B}(X)$, then $I \in \mathcal{K}(X)$ and $I(\bar{B}_X) = I(B_X) = B_X$ is compact. In particular, X is finite dimensional. □

2.1.4 Weakly Compact Operators

Definition 2.1.12. An operator $T \in \mathcal{B}(X, Y)$ is said to be *weakly compact* if the closed unit ball $\mathcal{B}_X \subset X$ is mapped into closure of $T(\mathcal{B}_X) \subset Y$ which is compact in it's weak topolgy.

The collection of weakly compact operators in $\mathcal{B}(X, Y)$ is denoted by $\mathcal{W}(X, Y)$.
We write $\mathcal{W}(X)$ instead of $\mathcal{W}(X, X)$.

Theorem 2.1.13. *[32] $\mathcal{W}(X)$ is a closed operator ideal of $\mathcal{B}(X)$.*

Remark 2.1.14. Let X be reflexive Banach space and $T \in \mathcal{B}(X)$ then $T \in \mathcal{W}(X)$. i.e $\mathcal{B}(X) = \mathcal{W}(X)$.

2.1.5 Completely Continuous Operators

Definition 2.1.15. An operator $T \in \mathcal{B}(X, Y)$ is said to be *completely continuous* if every weakly convergence sequence (x_n) in X is mapped into a convergent sequence (Tx_n) in Y.

The collection of completely continuous operators on $\mathcal{B}(X,Y)$ is denoted by $\mathcal{CC}(X,Y)$.
We write $\mathcal{CC}(X)$ instead of $\mathcal{CC}(X,X)$.

Theorem 2.1.16. *[32] $\mathcal{CC}(X)$ is closed operator ideal of $\mathcal{B}(X)$.*

Proposition 2.1.17. *Let T be an operator in $\mathcal{B}(X)$*

(a) If $T \in \mathcal{K}(X)$ then $T \in \mathcal{CC}(X)$. i.e $\mathcal{K}(X) \subset \mathcal{CC}(X)$.

(b) If $T \in \mathcal{CC}(X)$ and X is a reflexive Banach space, then $T \in \mathcal{K}(X)$. ie $\mathcal{K}(X) = \mathcal{CC}(X)$.

Proof. See [8] pg 173. □

2.1.6 Nuclear and Integral Operators

Definition 2.1.18. An operator $T \in \mathcal{B}(X,Y)$ is said to be *nuclear* if it has the form

$$T = \sum_{i=1}^{\infty} \mu_i \otimes y_i \qquad (\mu_i) \subset X', (y_i) \subset Y,$$

such that $\sum_{i=1}^{\infty} \|\mu_i\|\|y_i\| < \infty$.

The collection of all nuclear operators in $\mathcal{B}(X,Y)$ is denoted by $\mathcal{N}(X,Y)$. $\mathcal{N}(X,Y)$ becomes a Banach space equipped with a norm called *nuclear norm* defined by

$$\eta(T) \stackrel{\text{def}}{=} \inf\left\{\sum_{i=1}^{\infty} \|\mu_i\|\|y_i\| : T = \sum_{i=1}^{\infty} \mu_i \otimes y_i\right\}.$$

We write $\mathcal{N}(X)$ instead of $\mathcal{N}(X,X)$.

Theorem 2.1.19. *[32] $\mathcal{N}(X)$ is an operator ideal of $\mathcal{B}(X)$.*

Proof. Since $Id_{\mathbb{F}} = 1 \otimes 1$, we have that $Id_{\mathbb{F}}$ and $\eta(Id_{\mathbb{F}}) \leq 1$.
Let $Id_{\mathbb{F}} = \sum_{i=1}^{\infty} \alpha_i \otimes y_i$ be an arbitrary nuclear representation. Then $\sum_{i=1}^{\infty} \alpha_i y_i = 1$. It follows that $\sum_{i=1}^{\infty} |\alpha_i y_i| \geq 1$ and $\eta(Id_{\mathbb{F}}) \geq 1$.

Now take $T_1, T_2 ... \in \mathcal{N}(X,Y)$ such that $\sum_{i=1}^{\infty} \eta(T_i) < \infty$. For a given $\epsilon > 0$, let

$$T_n = \sum_{i=1}^{\infty} \mu_{n_i} \otimes y_{n_i}$$

be a nuclear representation with

$$\sum_{i=1}^{\infty} \|\mu_{n_i}\| \|y_{n_i}\| \leq (1+\epsilon)\eta(T_n).$$

Then

$$T := \sum_{n=1}^{\infty} T_n = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \mu_{n_i} \otimes y_{n_i}$$

and

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \|\mu_{n_i}\| \|y_{n_i}\| \leq (1+\epsilon) \sum_{n=1}^{\infty} \eta(T_n)$$

implies $T \in \mathcal{N}(X,Y)$ and $\alpha(T) \leq (1+\epsilon)\sum_{n=1}^{\infty} \eta(T_n)$.

Finally let $S = \sum_{i=1}^{\infty} \mu_i \otimes y_i \in \mathcal{N}(X,Y))$ and $\epsilon > 0$ such that

$$\sum_{i=1}^{\infty} \|\mu_i\| \|y_i\| \leq (1+\epsilon)\eta(S).$$

If $R \in \mathcal{B}(Y, Y_0)$ and $T \in \mathcal{B}(X_0, X)$ then we have

$$RST = \sum_{i=1}^{\infty} T'\mu_i \otimes Ry_i$$

and

$$\sum_{i=1}^{\infty} \|T'\mu_i\| \|Ry_i\| \leq (1+\epsilon)\|R\|\alpha(S)\|T\|.$$

Hence $RST \in \mathcal{N}(X_0, Y_0)$ and $\alpha(RST) \leq (1+\epsilon)\|R\|\alpha(S)\|T\|$.

□

Proposition 2.1.20. *[32] All nuclear operators are compact.*

Definition 2.1.21. An operator $T \in \mathcal{B}(X,Y)$ is said to be *integral operator* if there exists a constant $\sigma \geq 0$ such that

$$|\mathrm{trace}(ST)| \leq \sigma\|S\| \qquad \forall S \in \mathcal{F}(Y,X).$$

The *integral norm* is defined by

$$\iota(T) \stackrel{\text{def}}{=} \inf\{\sigma : |\text{trace}(ST)| \leq \sigma\|S\| \,\forall S \in \mathcal{F}(Y,X)\}\,.$$

The collection of integral operators on $\mathcal{B}(X,Y)$ is denoted by $\mathcal{J}(X,Y)$. We write $\mathcal{J}(X)$ instead of $\mathcal{J}(X,X)$.

Theorem 2.1.22. *[32] $\mathcal{J}(X)$ is an operator ideal of $\mathcal{B}(X)$.*

Proof. First we note $\iota(Id_{\mathbb{F}}) \geq 1$. Now since $Id_{\mathbb{F}} = 1 \otimes 1$, we have that

$$|\text{trace}(Id_{\mathbb{F}}S)| = |\langle 1, S(1)\rangle| \leq \|S\|, \qquad S \in \mathcal{F}(\mathbb{F}).$$

Hence $Id_{\mathbb{F}} \in \mathcal{F}$ and $\iota(Id_{\mathbb{F}}) \leq 1$.

Now take $T_1, T_2, .. \in \mathcal{F}(X,Y)$ such that $\sum_{n=1}^{\infty} \iota(T_n)$ and let $T = \sum_{n=1}^{\infty} T_n$. Then for every $S = \sum_{i=1}^{k} \lambda_i \otimes x_i \in \mathcal{F}(X,Y)$, we have

$$\begin{aligned}|\text{trace}(TS)| &\leq \left|\sum_{i=1}^{k} \langle \lambda_i, Tx_i\rangle\right| \\ &\leq \sum_{n=1}^{\infty}\left|\sum_{i=1}^{k} \langle \lambda_i, T_n x_i\rangle\right| \\ &= \sum_{n=1}^{\infty} |\text{trace}(T_n S)| \\ &\leq \sum_{n=1}^{\infty} \iota(T_n)\|S\|.\end{aligned}$$

Hence $T \in \mathcal{J}(X,Y)$ and $\iota(T) \leq \sum_{n=1}^{\infty} \iota(T_n)$.

Finally if $R \in \mathcal{B}(Y,Y_0)$, $S \in \mathcal{J}(X,Y)$, $T \in \mathcal{B}(X_0,X)$, and $U \in \mathcal{F}(Y_0,X_0)$ then

$$\begin{aligned}|\text{trace}(RSTU)| &= |\text{trace} STUR| \\ &\leq \iota(S)\|TUR\| \\ &\leq \|R\|\iota(S)\|T\|\|U\|.\end{aligned}$$

Hence $RST \in \mathcal{J}(X_0,Y_0)$ and $\iota(RST) \leq \|R\|\iota(S)\|T\|$. □

3 Geometric Properties of Banach Spaces

3.1 Reflexive and Separable Banach Spaces

Definition 3.1.1. Let X be a Banach space. Then X is said to be reflexive if the canonical embedding $\imath : X \to X''$ is a bijection. Thus X and X'' are isomorphic.

Theorem 3.1.2. *Let X be a Banach space. Then the following are equivalent:*

(a) X is reflexive.

(b) The closed unit ball $\bar{B}_X$ is weakly compact.

(c) Every bounded sequence in X has a weakly convergent subsequence.

Corollary 3.1.3. *Let X be a Banach space, and K be a closed bounded convex non-empty subset of X. Then K is weakly compact.*

Proposition 3.1.4. *Let X be a Banach space and let Y be a closed subspace of X. Then Y is reflexive.*

Corollary 3.1.5. *Let X be a Banach space. Then X is reflexive if and only if X' is reflexive.*

Proof. Let X be reflexive. We show that X' is reflexive. By Theorem 3.1.2 is suffice to show $B_{X'}$ is compact in the weak topology of X'. Now by Banach Alaoglu theorem, $B_{X'}$ is compact in the weak*-topology of X'. Since X is relfexive , the weak*-topology of X' equals the weak-topology of X'. It follows that $B_{X'}$ is compact in the weak topology of X'. Hence by Thereom 3.1.2 X' is reflexive.

The converse can also easily be shown. □

Example 3.1.6. Let $X = H$ be a Hilbert space. Then $H \cong H' \cong H''$ and hence X is reflexive.

Example 3.1.7. For $1 < p < \infty, L^p$ and ℓ_p are reflexive Banach spaces.

Theorem 3.1.8. *A Banach space X is reflexive if and only if there exists $x \in X$ such that $\mu(x) = 1$, for every $\mu \in S_{X'}$.*

Definition 3.1.9. A normed space X is said to be separable if and only if it contains a countable subset.

Theorem 3.1.10. *Let X be a Banach space such that X' is separable. Then X is separable.*

Corollary 3.1.11. *A Banach space X is reflexive and seprable if and only if X' is reflexive and separable.*

Theorem 3.1.12. *Let X be a separable Banach space. Every bounded sequence in X' has weak*-converging subsequence.*

3.2 Approximation Properties

Definition 3.2.1. Let X be a Banach space. X is said to have the *approximation property* if for every compact $K \subseteq X$ and every $\epsilon > 0$, there exists an operator $T \in \mathcal{F}(X)$ such that

$$\|x - Tx\| \leq \epsilon, \qquad \forall x \in K.$$

Proposition 3.2.2. *[32] Let X be a Banach space. Then the following are equivalent:*

(a) *X has the approximation property.*

(b) *$\mathcal{A}(Y,X) = \mathcal{K}(Y,X)$. i.e $\mathcal{F}(Y,X)$ is dense in $\mathcal{K}(Y,X)$ in the norm topology, for every Banach space Y. In particular, $\mathcal{A}(X) = \mathcal{K}(X)$.*

(c) *The canonical map $\jmath_\pi : Y\widehat{\otimes}X \to Y\check{\otimes}X$ is injective. In particular if $Y = X$, $\jmath_\pi$ is injective.*

(d) *The surjective map $J_\pi : Y'\widehat{\otimes}X \to \mathcal{N}(Y,X)$ is injective. In particular if $Y = X$, J_π is injective.*

(e) *The trace functional $tr : X' \otimes X \to \mathbb{F}$ admits a continuous extension to $\mathcal{N}(X)$.*

Remark 3.2.3. [32] It follows from the approximation property that $\mathcal{N}(X) \cong X'\widehat{\otimes}X$, the projective tensor product.

Definition 3.2.4. Let X be a Banach space. X is said to have the *bounded approximation property* if for every compact $K \subseteq X$ and every $\epsilon > 0$, there exists an operator $T \in \mathcal{F}(X)$ such that $\|T\| \leq c$, $c \geq 0$ and

$$\|x - Tx\| \leq \epsilon, \qquad \forall x \in K.$$

If $c = 1$, then X is said to have the *metric approximation property.*

Remark 3.2.5. In general the bounded approximation property implies approximation property. But the converse is not always true.

Remark 3.2.6. If X' has the bounded approximation property, then so does X.

3.3 Schauder Basis

Definition 3.3.1. Let X be a Banach space. A sequence $(x_n) \subset X$ is said to be *Schauder basis*, or simply a *basis* if for every $x \in X$, there's a unique sequence of scalars $(\alpha_n) \subset \mathbb{F}$ such that

$$x = \sum_{n=1}^{\infty} \alpha_n x_n.$$

Theorem 3.3.2. *Let (x_n) be a sequence in a Banach space X. Then (x_n) is a basis for X if and only if:*

(a) (x_n) is non-zero;

(b) the linear span of (x_n) is dense on X; and

(c) there exists a constant $K > 0$ such that for every sequence $(\alpha_n) \subset \mathbb{F}$ and each $n, m \in \mathbb{N}$, we have that

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq K \left\| \sum_{i=1}^{n+m} \alpha_i x_i \right\|.$$

The smallest constant K such that (c) above holds is called a *basis constant.*

Remark 3.3.3. Any basis in a Banach space X is linearly independent, particularly X is infinite-dimensional and separable. However not all separable Banach spaces have a basis. See [28] for details.

Remark 3.3.4. If a Banach space X has basis, then X has the bounded approximation property.

Example 3.3.5. Let $X = \ell_p$ $(1 \leq p < \infty)$ or c_0. For $n \in \mathbb{N}$ let $e_n = (0, ..., 0, 1, 0, ...) \subset \mathbb{F}$ (i.e e_n has a 1 in the nth position and zero's elsewhere). Then (e_n) is a basis for X, often called *standard-unit vector basis* and has a basis constant 1.

However, (e_n) is not a basis for ℓ_∞, and indeed ℓ_∞ has no basis since it is not separable.

Definition 3.3.6. A sequence (x_n) in Banach space X is said to be a *basic sequence* if it is a basis for its closed linear span.

Every Banach space contains a basic sequence; that is, every Banach space contains a closed infinite-dimensional subspace with a basis.

In many cases, basis for a Banach space are not of much use, as the convergence properties are rather weak. A more useful notion is that of an *unconditional basis.*

Definition 3.3.7. A basis (x_n) in a Banach space X is said to be *unconditional* if whenever the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges, it converges unconditionally.

The standard-unit vector basis is an example of an unconditional basis.

Definition 3.3.8. Let (x_n) be a basis for a Banach space X. We say that (x_n) is *boundedly complete* if whenever (α_n) is a sequence of scalars such that

$$\sup_N \left\| \sum_n^N \alpha_n x_n \right\| < \infty,$$

then the series $\sum_n^{\infty} \alpha_n x_n$ converges.

Definition 3.3.9. Let (x_n) be a basis for a Banach space X. We say that (x_n) is shrinking if the biorthogonal functionals $(x_n^{'})$ is a basis for X'.

We recall that for a Banach space X with its dual space X', $(x_n, x'_n) \subset X \times X'$ is called a biorthogonal system if $x'_n(x_n) = 1$ and $x'_n(x_m) = 0$, for $m \neq n$. We write $[x'_n]$ for the closure of the linear span on (x'_n).

Theorem 3.3.10. *Let (x_n) be a basis for a Banach space X with the biorthogonal functionals (x'_n). Then the following are equivalent:*

(a) (x_n) *is shrinking for* X.

(b) (x'_n) *is a boundedly complete basis for* $[x'_n]$.

(c) $[x'_n] = X'$.

Theorem 3.3.11. *[30] Let X be a Banach space. Then X has a shrinking basis if one of the following holds:*

(a) X' *has a basis;*

(b) X *has a basis and* X' *is separable and has the bounded approximation property.*

Theorem 3.3.12. *[30] Let X be a Banach space with a basis (x_n). Then X is reflexive if and only if (x_n) is boundedly complete and shrinking.*

Proposition 3.3.13. *[30] Either every Banach space X has the approximation property or there is a Banach space Y such that*

(a) Y *has a basis, and*

(b) Y' *is separable but does not have the approximation and hence does not have a basis.*

Proof. See [30] Theorem 2.3. □

Proposition 3.3.14. *[30] A separable Banach space has the bounded approximation property if and only if it is isomorphic to a complemented subspace of Banach space with a basis.*

Definition 3.3.15. Let E be a Banach space with unconditional basis $\mathbf{e} = (e_i)$, and let (X_i) be a sequence of Banach spaces with respectives norm, $\|\cdot\|_i$. Then the space

$$(\oplus_i X_i)_{\mathbf{e}} \stackrel{\text{def}}{=} \left\{ (x_i) \in \prod_i X_i : \sum_i \|x_i\|_i e_i \text{ converges in } E \right\}$$

is a Banach space endowed with norm

$$\|(x_i)\| = \left\| \sum_i \|x_i\|_i e_i \right\|.$$

Furthermore if the **e** is in addition shrinking, then its topological dual can be isometrically identified with $(\oplus_i X'_i)_{\mathbf{e}'}$, where $\mathbf{e}'$ is the uncondtional basis of E' formed by the biorthogonal functionals associated with **e**. If **e** is a standard-unit vector basis for ℓ_p, $1 \leq p \leq \infty$, the space $(\oplus_i X_i)_{\mathbf{e}}$ can be written as $(\oplus_i X_i)_p$.

For a Banach space E the with unconditional basis $\mathbf{e} = (e_i)$, the closed linear span of **e** will be denoted by E^m. If X is Banach space such that $X_i = X$, $1 \leq p \leq \infty$, and $X_i = \{0\}$ for $i > m$. Then $(\oplus_i X_i)_{\mathbf{e}}$ will be denoted by $E^m(X)$ or by $E(X)$ if $X_i = X$ for all i. In particular, the ℓ_p-sum of countably infinitely many (respectively n) copies of X is denoted by $\ell_p^n(X)$ (respectively $\ell_p(X)$).

3.4 Property (P_λ) and Property (Q_λ)

Definition 3.4.1. Let X be a Banach space, and let $\lambda \geq 1$.
(i) X is said to have property (P_λ) if, for each finite-dimensional subspace $F \subset X$, there is a finite-dimensional subspace E of X containing F such that there is a projection of norm not greater than λ onto E (i.e there is a projection $P : X \to E$ with $|| P || \leq \lambda$).
(ii) X is said to have property (Q_λ) if, for each closed subspace F of X with finite codimension (i.e $\dim(X/F) < \infty$), there is a closed subspace E of F with finite codimension in X, such that the canonical surjection
$\pi_E : X \to X/E$ has a bounded right inverse of norm not greater than λ (i.e there exists $\pi : X/E \to X$ such that π is bounded and $\pi_E \pi = I_{X/E}$ with
$|| \pi_E || \leq \lambda$).

Example 3.4.2. Every Hilbert space has properties (P_1) and (Q_1).

Definition 3.4.3. A Banach space X is said to be an $\mathcal{L}_{p,\lambda}$- space
$(1 \leq \lambda < \infty, 1 \leq p \leq \infty)$ if for every finite-dimensional subspace F of X, there is a finite-dimensional subspace E of X containing F and such that

$$d(F, l_p^n) \leq \lambda \ \text{ where } \ n = \dim(F).$$

Then a Banach space X is an $\mathcal{L}_p$- space if it is an $\mathcal{L}_{p,\lambda}$- space for some λ. Some examples of $\mathcal{L}_p$-space are l_p and $L_p(0,1)$. See Lindenstrauss and Pelczyski (1968).

Example 3.4.4. If X is an $\mathcal{L}_p$- space with $1 \leq p \leq \infty$, then X has property (P_λ) for some $\lambda \geq 1$, see [29].

3.5 Other Geometric Properties

Definition 3.5.1. A Banach space X is said to be *uniformly convex* if for any $\epsilon \in (0,2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that whenever $x, y \in S_X = \{x \in X : \|x\| = 1\}$ (unit sphere) and $\|x - y\| > \epsilon$ implies that

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

To prove the next theorem, we require the use of the following lemma.

Lemma 3.5.2. *[7] Let $X = L_p$ or ℓ_p and $x, y \in X$. Then for $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, the following inequalities hold:*

(i) $\|x+y\|^q + \|x-y\|^q \leq 2(\|x\|^p + \|y\|^p)^{q-1} \quad (1 \leq p \leq 2);$

(ii) $\|x+y\|^p + \|x-y\|^p \leq 2(\|x\|^q + \|y\|^q)^{p-1} \quad (p \geq 2);$

(iii) $\|x+y\|^p + \|x-y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p) \quad (p \geq 2).$

Example 3.5.3. The Banach spaces $X = L_p$, $1 < p < \infty$, are uniformly convex.

Proof. We show that for any $0 < \epsilon \leq 2$, there exists $\delta = \delta(\epsilon) > 0$ such that if $f, g \in \bar{B}_X$ and $\|f - g\| \geq \epsilon$, Then

$$\left\| \frac{f+g}{2} \right\| \leq 1 - \delta.$$

We consider the following two cases:

<u>Case 1</u> For $p \in (1,2]$, by previous lemma we have

$$\|f+g\|^q + \|f-g\|^q \leq 2(\|f\|^p + \|g\|^p)^{q-1}$$

$$\begin{aligned}\left\|\frac{f+g}{2}\right\|^q + \left\|\frac{f-g}{2}\right\|^q &\le 2^{-(q-1)}(\|f\|^p + \|g\|^p)^{q-1}\\ &\le 2^{-(q-1)}2^{q-1} \quad \text{since} f,g \in \bar{B}_X\\ &= 1.\end{aligned}$$

It follows,

$$\left\|\frac{f+g}{2}\right\|^q \le 1 - \left\|\frac{f-g}{2}\right\|^q \le 1 - \left(\frac{\epsilon}{2}\right)^q,$$

so that

$$\left\|\frac{f+g}{2}\right\| \le \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{\frac{1}{q}} = 1 - \left(1 - \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{\frac{1}{q}}\right) = 1 - \delta,$$

where $\delta = \left(1 - \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{\frac{1}{q}}\right) > 0$. Thus proving first case.

Case 2 For $p \ge 2$, by previous lemma, we have

$$\|f+g\|^p + \|f-g\|^p \le 2(\|f\|^q + \|g\|^q)^{p-1}$$

$$\begin{aligned}\left\|\frac{f+g}{2}\right\|^p + \left\|\frac{f-g}{2}\right\|^p &\le 2^{-(p-1)}(\|f\|^q + \|g\|^q)^{p-1}\\ &\le 2^{-(p-1)}2^{p-1} \quad \text{since} f,g \in \bar{B}_X\\ &= 1.\end{aligned}$$

It follows,

$$\left\|\frac{f+g}{2}\right\|^p \le 1 - \left\|\frac{f-g}{2}\right\|^p \le 1 - \left(\frac{\epsilon}{2}\right)^p,$$

so that

$$\left\|\frac{f+g}{2}\right\| \le \left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}} = 1 - \left(1 - \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{\frac{1}{q}}\right) = 1 - \delta,$$

where $\delta = \left(1 - \left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}\right) > 0$. Thus proving second case.Therefore the L_p $(1 < p < \infty)$ are uniformly convex.

□

Example 3.5.4. The sequence spaces $\ell_p, 1 < p < \infty$, are uniformly convex.

Proof. Let $x, y \in \ell_p$, such that $x, y \in \bar{B}_X$ and $\|x - y\| \geq \epsilon$ where $0 < \epsilon \leq 2$. Thus we have two cases:

Case 1 For $p \in (1, 2]$, by previous lemma, we have

$$\left\|\frac{x+y}{2}\right\|^q + \left\|\frac{x-y}{2}\right\|^q \leq 1.$$

It follows,

$$\left\|\frac{x+y}{2}\right\|^q \leq 1 - \left\|\frac{x-y}{2}\right\|^q \leq 1 - \left(\frac{\epsilon}{2}\right)^q$$

so that

$$\left\|\frac{x+y}{2}\right\| \leq \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{\frac{1}{q}} = 1 - \left(1 - \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{\frac{1}{q}}\right) = 1 - \delta,$$

where $\delta = \left(1 - \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{\frac{1}{q}}\right)$. Hence $\ell_p, p \in (1, 2]$ is uniformly convex.

Case 2 For $p \geq 2$, by previous lemma we have

$$\left\|\frac{x+y}{2}\right\|^p + \left\|\frac{x-y}{2}\right\|^p \leq 1.$$

It follows,

$$\left\|\frac{x+y}{2}\right\|^p \leq 1 - \left\|\frac{x-y}{2}\right\|^p \leq 1 - \left(\frac{\epsilon}{2}\right)^p$$

so that

$$\left\|\frac{x+y}{2}\right\| \leq \left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}} = 1 - \left(1 - \left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}\right) = 1 - \delta,$$

where $\delta = \left(1 - \left(1 - \left(\frac{\epsilon}{2}\right)^p\right)^{\frac{1}{p}}\right)$. Thus proving $X = \ell_p, p \geq 2$ are uniformly convex. Hence from Case 1 and Case 2 we have $X = \ell_p, (1 < p < \infty)$ are uniformly convex.

□

Example 3.5.5. The Banach spaces $X = \ell_1(2)$ or $\ell_\infty(2)$ equipped with the norms $\|x\| = |x_1| + |x_2|$ and $\|x\| = \max\limits_{i=1,2} |x_i|$ respectively where $x_1, x_2 \in X$ are not uniformly convex.

Example 3.5.6. The Banach space $X = L_1[0,1]$, space of measurable functions such that $\int_0^1 |f(t)|dt < \infty$ with norm $\|f\| = \int_0^1 |f(t)|dt$ is not uniformly convex.

Theorem 3.5.7. *A Banach space X is uniformly convex whenever $\delta_X(\epsilon) > 0$ for $0 < \epsilon \le 2$.*

Theorem 3.5.8. *Let X be uniformly convex Banach space. Then for $\epsilon > 0$, $d > 0$ and for any arbitrary vectors x and y in X with $\|x\| \le d, \|y\| \le d, \|x-y\| \ge \epsilon$, there exists a $\delta > 0$ such that*

$$\left\|\frac{x+y}{2}\right\| \le \left|1 - \delta\left(\frac{\epsilon}{d}\right)\right| d.$$

Proof. For arbitrary vectors x and y in X, let $z_1 = \frac{x}{d}, z_2 = \frac{y}{d}$ and $\bar{\epsilon} = \frac{\epsilon}{d}$. Since both $\epsilon > 0$ and $d > 0$, then $\bar{\epsilon} > 0$. It follows from $\|z_1\| = \left\|\frac{x}{d}\right\| \le 1$ and $\|z_2\| = \left\|\frac{y}{d}\right\| \le 1$ that $z_1, z_2 \in B_X$, and

$$\|z_1 - z_2\| = \left\|\frac{x}{d} - \frac{y}{d}\right\| = \frac{1}{d}\|x-y\| \ge \frac{\epsilon}{d} = \bar{\epsilon}.$$

Now by uniform convexity of X, we have for some $\delta > 0$,

$$\left\|\frac{z_1+z_2}{2}\right\| \le 1 - \delta(\bar{\epsilon}),$$

which implies that

$$\left\|\frac{1}{d}\left(\frac{x+y}{2}\right)\right\| \le 1 - \delta(\bar{\epsilon}).$$

So that

$$\left\|\frac{x+y}{2}\right\| \le (1 - \delta(\bar{\epsilon}))\, d.$$

□

Proposition 3.5.9. *Let X be a uniformly convex Banach space and let $\alpha \in (0,1)$ and $\epsilon > 0$. Then for any $d > 0$, if $x, y \in X$ are such that $\|x\| \le d, \|y\| \le d, \|x-y\| \ge \epsilon$, then there exists $\delta = \delta(\frac{\epsilon}{d}) > 0$ such that*

$$\|\alpha x + (1-\alpha)y\| \le (1-2\delta)\left(\frac{\epsilon}{d}\right)\min\{\alpha, 1-\alpha\}.$$

Theorem 3.5.10. (Milman-Pettis' Theorem) *Let X be a uniformly convex Banach space. Then X is reflexive*

Proof. Suppose X is a uniformly convex space. Let $(x_n) \subseteq S_X$ be a bounded sequence in X such that for $\mu \in S_{X'}$, $\mu(x_n) \to 1$. Suppose (x_n) is not a Cauchy sequence, so there exists a $\epsilon > 0$ and two distinct subsequences (x_{n_i}) and (x_{n_j}) such that $\|x_{n_i} - x_{n_j}\| \geq \epsilon$. Since X is uniformly convex, we have that there exists $\delta > 0$ such that $\left\|\frac{x_{n_i}-x_{n_j}}{2}\right\| \leq 1-\delta$. Now

$$\begin{aligned}\left|\mu\left(\frac{x_{n_i}-x_{n_j}}{2}\right)\right| &\leq \|\mu\|\left\|\frac{x_{n_i}-x_{n_j}}{2}\right\| \\ &\leq \|\mu\|\|1-\delta\| \\ &= 1-\delta,\end{aligned}$$

where $\|\mu\| = 1$ and $\mu(x_n)\| \to 1$, thus a contradiction.
It follows that (x_n) is Cauchy sequence and there exists point x in S_X such that $x_n \to x$. Since,

$$\|x\| = \|\lim_{n\to\infty} x_n\| = \lim_{n\to\infty} \|x_n\| = 1,$$

then $x \in S_X$ and $\mu(x) = 1$. Hence by Theroem 3.1.8, we have that X is reflexive. □

Definition 3.5.11. A Banach space X is said to be *strictly convex* if for all $x, y \in X$, $x, y \neq 0$ and $\|x + y\| = \|x\| = \|y\|$ imply $y = kx, \quad k > 0$.

Equivalently, a Banach space is said to strictly convex if whenever $x, y \in S_X$ and $x \neq y$, we have

$$\left\|\frac{x+y}{2}\right\| < 1.$$

Theorem 3.5.12. *[7] Let X be a uniformly convex Banach space. Then X is strictly convex.*

Definition 3.5.13. A Banach space X is said to have the *Kadec-Klee property* if for every sequence (x_n) in X, converges weakly to $x \in X$, where $\|x_n\| \to \|x\|$, then (x_n) converges strongly to x i.e $\|x_n - x\| \to 0$.

Theorem 3.5.14. *[7] Let X be a uniformly convex Banach space. Then X has the Kadec-Klee property.*

Proof. Suppose X is a uniformly convex space. Let (x_n) be a sequence in X such that (x_n) converges weak to $x \in X$ and $\|x_n\| \to \|x\|$. If $x = 0$, then $\lim\limits_{n\to\infty} \|x_n\| = 0$ so that $x \to 0$. Hence $x_n \to x$. Now suppose $x \neq 0$, we show that $x_n \to x$. Assume the contrary that $\lim\limits_{n\to\infty}(x_n) \neq 0$. That is,

$$\frac{x_n}{\|x_n\|} \nrightarrow \frac{x}{\|x\|}.$$

Then for any $\epsilon > 0$, there exists a subsequence $\left(\frac{x_{n_i}}{\|x_{n_i}\|}\right)$ of $\left(\frac{x}{\|x\|}\right)$ such that

$$\left\|\frac{x_{n_i}}{\|x_{n_i}\|} - \frac{x}{\|x\|}\right\| \geq \epsilon.$$

By uniform convexity of X, there exists $\delta(\epsilon) > 0$ such that

$$\frac{1}{2}\left\|\frac{x_{n_i}}{\|x_{n_i}\|} - \frac{x}{\|x\|}\right\| \leq 1 - \delta(\epsilon).$$

Now since $x_n \xrightarrow{w} x \in X$, we have

$$1 = \left\|\frac{x}{x_n}\right\| \leq \lim_{n\to\infty} \inf \left\|\frac{x_{n_i}}{\|x_{n_i}\|} - \frac{x}{\|x\|}\right\| \leq 1 - \delta,$$

hence a contradiction. Thus $x_n \to x \in X$. Therefore X has Kadec-Klee property. □

Definition 3.5.15. Let $\mathcal{A}$ be a Banach algebra and let N be a closed linear span of the elements of the form $ab \otimes c - a \otimes bc$, for $ab, c \in \mathcal{A}$. Then define

$$\mathcal{A} \underset{\mathcal{A}}{\hat{\otimes}} \mathcal{A} = \mathcal{A} \hat{\otimes} \mathcal{A}/N.$$

Since $N \subseteq \text{Ker}\pi_{\mathcal{A}}$, then $\pi_{\mathcal{A}}$ induces a map $\mathcal{A}\hat{\otimes}_{\mathcal{A}}\mathcal{A} \to \mathcal{A}$. Then $\mathcal{A}$ is said to be *self-induced* if

$$\mathcal{A} \cong \mathcal{A} \underset{\mathcal{A}}{\hat{\otimes}} \mathcal{A}.$$

That is $\text{Ker}\pi_{\mathcal{A}} = N$.

Definition 3.5.16. Let X be a Banach space, Then the Banach algebra $\mathcal{A}(X)$ *factors approximately through* Y if

$$\mathcal{A}(X) \cong \mathcal{A}(Y, X) \underset{\mathcal{A}(Y)}{\hat{\otimes}} \mathcal{A}(X, Y).$$

Lemma 3.5.17. *[21] Suppose that $\mathcal{A}$ is a self-induced Banach algbera. Let $D : \mathcal{A} \to \mathcal{A}'$ be a bounded derivation, and let $\mathcal{B}$ be a Banach algebra such that $\mathcal{A}$ is an ideal in $\mathcal{B}$. Then D has an extension $\tilde{D} : \mathcal{B} \to \mathcal{A}'$ which is a bounded derivation.*

Example 3.5.18. Let $\mathcal{A}$ be a Banach algebra, X a Banach space with approximation property and the map

$$\mathcal{A}(X)\hat{\otimes}\mathcal{A}(X) \to \mathcal{A}(X)$$

is surjective.Then $\mathcal{A}(X)$ is self-induced.

Proposition 3.5.19. *[21] Let X be a Banach space. The X has approximation property if and only if $\mathcal{N}(X)$ is self-induced.*

Proposition 3.5.20. *[21] Let X be a Banach space. Then $(a) \Rightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$, where:*

(a) $\mathcal{A}(X) = \mathcal{N}(X)$;

(b) $\mathcal{A}(X) \subseteq \mathcal{I}(X)$;

(c) for any $S, T \in \mathcal{F}(X)$ there exists a positive constant c such that

$$|trace(ST)| \leq c\|S\|\|T\|;$$

(d) the multiplication $\mathcal{A}(X)\hat{\otimes}\mathcal{A}(X) \to \mathcal{A}(X)$ maps onto $\mathcal{N}(X)$.

4 Some Amenability Properties in Banach Algebras

4.1 Amenability in Banach Algebras

4.1.1 Definitions with Examples

Definition 4.1.1. Let $\mathcal{A}$ be an algebra and X an $\mathcal{A}$-bimodule. A (continuous) *derivation* from $\mathcal{A}$ to X is a (bounded) linear map $D : \mathcal{A} \to X$ that satisfies the identity

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}).$$

Example 4.1.2. The map $\delta_X : a \to a \cdot x - x \cdot a$ is a continuous derivation. Derivations of this form are called *inner derivations*.

Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. Then $\mathcal{Z}^1(\mathcal{A}, X)$ is the space of all continuous derivations from $\mathcal{A}$ into X, $\mathcal{N}^1(\mathcal{A}, X)$ is the space of all inner derivations from $\mathcal{A}$ into X, and the the first cohomology group of $\mathcal{A}$ with coefficients in X is the quotient space

$$\mathcal{H}^1(\mathcal{A}, X) = \mathcal{Z}^1(\mathcal{A}, X)/\mathcal{N}^1(\mathcal{A}, X).$$

Definition 4.1.3. Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is said to be *amenable* if, every continuous derivation $D : A \to X'$ is inner, for every Banach $\mathcal{A}$-bimodule X. Equivalently, the Banach algebra $\mathcal{A}$ is *amenable* if $\mathcal{H}^1(\mathcal{A}, X') = \{0\}$.

4.1.2 Some Characterizations

Let $\mathcal{A}$ be a Banach algebra. Then the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule under the module actions defined by

$$a \cdot (b \otimes c) = ab \otimes c \quad \text{and} \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, \in \mathcal{A}).$$

Definition 4.1.4. Let $\mathcal{A}$ be a Banach algebra. A *bounded approximate diagonal* for $\mathcal{A}$ is a bounded net (d_λ) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that

$$a \cdot d_\lambda - d_\lambda \cdot a \to 0 \quad \text{and} \quad a\pi(d_\lambda) \to a \quad (a \in \mathcal{A}),$$

where π denotes the product map $\pi : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$, defined by $\pi(a \otimes b) = ab$ $(a, b \in \mathcal{A})$, which is an $\mathcal{A}$-bimodule homomorphism with respect to the module structure on $\mathcal{A}\hat{\otimes}\mathcal{A}$. The dual map is given by

$$\pi' : \mathcal{A}' \to (\mathcal{A}\hat{\otimes}\mathcal{A})' \quad \pi'(\mu)(a \otimes b) = \mu(ab) \quad (\mu \in \mathcal{A}', a, b \in \mathcal{A}).$$

Remark 4.1.5. We note that if (d_λ) is an approximate diagonal, then since $\pi(d_\lambda) \to a$ implies that $(\pi(d_\lambda))_\lambda$ is a right approximate identity for $\mathcal{A}$. Indeed, its also a left approximate identity since

$$\begin{aligned} \|\pi(d_\lambda)a - a\| &= \|\pi(d_\lambda)a - a\pi(d_\lambda) + a\pi(d_\lambda) - a\| \\ &\leq \|\pi(d_\lambda)a - a\pi(d_\lambda)\| + \|a\pi(d_\lambda) - a\| \\ &= \|\pi(a \cdot d_\lambda - d_\lambda \cdot a)\| + \|a\pi(d_\lambda) - a\| \\ &\to 0. \end{aligned}$$

Theorem 4.1.6. *[31] Let $\mathcal{A}$ be Banach algebra. Then the following is equivalent:*

(a) $\mathcal{A}$ is amenable.

(b) $\mathcal{A}$ has a bounded approximate diagonal.

If the Banach algebra $\mathcal{A}$ has an approximate diagonal of bound K, then $\mathcal{A}$ is said to be K-*amenable*. The smallest such K is called the *amenability constant of* $\mathcal{A}$.

Definition 4.1.7. Let $\mathcal{A}$ be a Banach algebra. A *virtual diagonal* for $\mathcal{A}$ is an element $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})''$such that

$$a \cdot M = M \cdot a \quad \text{and} \quad \pi''(M)a = a \quad (a \in \mathcal{A}).$$

Theorem 4.1.8. *[31] Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:*

(a) $\mathcal{A}$ is amenable;

(b) $\mathcal{A}$ has a virtual diagonal.

Thus the following Theorem follows

Corollary 4.1.9. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ has a virtual diagonal if and only if it has a bounded approximate diagonal, thus $\mathcal{A}$ is amenable.*

Proposition 4.1.10. *[6] Let $\mathcal{A}$ be an amenable Banach algebra. Then $\mathcal{A}$ has a bounded approximate identity.*

Proof. Assume that $\mathcal{A}$ is an amenable Banach algebra. We show that $\mathcal{A}$ has a bounded approximate identity. Let X be a Banach space such that $X = \mathcal{A}$ and with left and right bimodule actions defined by

$$a \cdot x = ax \quad x \cdot a \quad (a \in \mathcal{A}, x \in X).$$

Now let $\imath : \mathcal{A} \to X''$ be the canonical embedding of $\mathcal{A}$ into its second dual. Then we have for $a, b \in \mathcal{A}$ and $\mu \in X'$,

$$\begin{aligned} \langle \mu, a \cdot \imath(b) \rangle + \langle \mu, \imath(a) \cdot b \rangle &= \langle \mu, a \cdot \imath(b) + \imath(a) \cdot b \rangle \\ &= \langle \mu, a \cdot \imath(b) \rangle = \langle \mu \cdot a, \imath(b) \rangle \\ &= \langle b, \mu \cdot a \rangle = \langle ab, \mu \rangle \\ &= \langle \mu, \imath(ab) \rangle, \end{aligned}$$

showing that $\imath$ is a derivation. Since $\mathcal{A}$ is amenable, there exists $\phi \in X''$ such that

$$\imath(a) = a \cdot \phi - \phi \cdot a = a \cdot \phi \quad (a \in \mathcal{A}).$$

Goldstines's theorem implies that there exists a bounded net $(e_\alpha)_\alpha$ in $\mathcal{A}$ such that $(\imath(e_\alpha))_\alpha$ converges to ϕ in the weak*-topology on X''. Therefore $\imath(a \cdot e_\alpha) \overset{w^*}{\to} a \cdot \phi = \imath(a)$ for all $a \in \mathcal{A}$, or equivalently $a \cdot e_\alpha \overset{w}{\to} a$ for all $a \in \mathcal{A}$. By Theorem 1.4.40, $\mathcal{A}$ has a bounded right approximate identity for $\mathcal{A}$. By similar argument and defining module operations on $\mathcal{A}$ through

$$a \cdot x = 0 \quad \text{and} \quad x \cdot a = xa \quad (a \in \mathcal{A}, x \in X),$$

we obtain a bounded left approximate identity for $\mathcal{A}$. □

Definition 4.1.11. Let X, Y and Z be Banach $\mathcal{A}$-bimodules and $f : X \to Y$, $g : Y \to Z$ be Banach $\mathcal{A}$-module homomorphisms.

(a) The sequence $\sum : 0 \to X \to Y \to Z$ is said to be *exact* if f is an injection, Im$g = Z$ and Imf=Ker$g$.

(b) The exact sequence $\sum$ is *admissable* if there exists a bounded linear map $F : Y \to X$ such that $Ff = I_X$.

(c) The exact sequence $\sum$ *splits* if there exists a Banach $\mathcal{A}$-module homomorphism $F : Y \to X$ such that $Ff = I_X$.

If K is the kernel of π, and $\mathcal{A}$ has bounded approximate identity, then the sequence

$$\Pi : 0 \to K \to \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A} \to 0$$

is exact as a sequence of $\mathcal{A}$-bimodules, this is also true for the dual sequence

$$\Pi' : 0 \to \mathcal{A}' \to (\mathcal{A}\hat{\otimes}\mathcal{A})' \to K' \to 0$$

where

$$i : K \to \mathcal{A}\hat{\otimes}\mathcal{A}, \quad \pi : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}.$$

Theorem 4.1.12. *[9] Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is amenable if and only if:*

(a) $\mathcal{A}$ has bounded approximate identity and

(b) the exact sequence Π' of $\mathcal{A}$-bimodule splits.

Proposition 4.1.13. *[9] Let $\mathcal{A}$ be an amenable Banach algebra and let*

$$\sum : 0 \to X' \to Y \to Z \to 0$$

be an admisassable short exact sequence of left or right Banach $\mathcal{A}$-modules with X' a dual Banach $\mathcal{A}$-module of X. Then $\sum$ splits

4.1.3 Some Hereditary Properties

Proposition 4.1.14. *[31] Let $\mathcal{A}$ and $\mathcal{B}$ be a Banach algebra. Suppose that $\mathcal{A}$ is amenable and there exists a continuous homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ with dense range. Then $\mathcal{B}$ is amenable.*

Proof. Let X be a Banach $\mathcal{B}$-bimodule, and let $D : \mathcal{B} \to X'$ be a bounded derivation. If we define the multiplications on X by

$$ax = \phi(a)x, \quad xa = x\phi(a) \quad (a \in \mathcal{A}, x \in X),$$

and by the continuity of ϕ, then X becomes a Banach $\mathcal{A}$-bimodule. Now $D \circ \phi : \mathcal{A} \to X'$ is bounded derivation. Indeed for $a, b \in \mathcal{A}$, we have

$$\begin{aligned} D(\phi(ab)) &= D(\phi(a)\phi(b)) \\ &= D(\phi(a))\phi(b) + \phi(a)D(\phi(b)) \\ &= D(\phi(a))b + aD(\phi(b)), \end{aligned}$$

and
$$\|D(\phi(a))\| < \|D\|\|\phi(a)\| = \|D\|\|\phi\|\|a\|.$$
It follows by the assumption that $\mathcal{A}$ is amenable that $D \circ \phi$ is inner, that is, there exists $\mu \in X'$ such that

$$D(\phi(a)) = a \cdot \mu - \mu \cdot a \quad (a \in \mathcal{A}).$$

By module multiplication and continuity of D, for $b \in \mathcal{B}$, $b_n \to b$ for some (b_n) in $\phi(\mathcal{A})$, we have

$$\begin{aligned} D(b) &= D(\lim_n b_n) \\ &= \lim_n D(b_n) \\ &= \lim_n (b_n \cdot \mu - \mu \cdot b_n) \\ &= \lim_n (b_n \cdot \mu) - \lim_n (\mu \cdot b_n) \\ &= b \cdot \mu - \mu \cdot b. \end{aligned}$$

□

Proposition 4.1.15. *[31] Let $\mathcal{A}$ and $\mathcal{B}$ be amenable Banach algebras, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is amenable.*

Proposition 4.1.16. *[31] Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is amenable if and only if $\tilde{\mathcal{A}}$ is amenable.*

Theorem 4.1.17. *[22] Let $\mathcal{A}$ be an amenable Banach algebra and let $\mathcal{I}$ be a closed, left ideal of $\mathcal{A}$ with a bounded approximate identity. Then $\mathcal{I}$ is amenable.*

Proof. For convenience we define a new product on $\mathcal{A}\hat{\otimes}\mathcal{A}$ by

$$(a \otimes b) \bullet (c \otimes d) = ac \otimes bd \quad (a, b, c, d \in \mathcal{A}).$$

Let $(d_\lambda)_{\lambda \in \mathscr{A}}$ be a bounded approximate diagonal for $\mathcal{A}$ and $(e_\alpha)_{\alpha \in \mathscr{D}}$ be a bounded approximate identity for $\mathcal{I}$. Then

$$p_{\lambda\alpha\beta} = d_\lambda \bullet (e_\alpha \otimes e_\beta) \quad (\lambda \in \mathscr{A}, \alpha, \beta \in \mathscr{D})$$

belongs to a bounded subset of $\mathcal{I}\hat{\otimes}\mathcal{I}$, since (d_λ) and (e_α) are bounded nets and $\mathcal{I}$ is a left ideal. Now for each $a \in \mathcal{A}$, we have

$$\begin{aligned}\limsup_\beta \|a \cdot p_{\lambda\alpha\beta} - p_{\lambda\alpha\beta} \cdot a\| &= \limsup_\beta \|(a \otimes 1) \bullet d_\lambda \bullet (e_\alpha \otimes e_\beta) - d_\lambda \bullet (e_\alpha \otimes e_\beta) \bullet (1 \otimes a)\| \\ &= \limsup_\beta \|((a \otimes 1) \bullet d_\lambda - d_\lambda \bullet (1 \otimes a)) \bullet (e_\alpha \otimes e_\beta)\| \\ &= \limsup_\beta \|(a \cdot d_\lambda - d_\lambda \cdot a) \bullet (e_\alpha \otimes e_\beta)\|,\end{aligned}$$

using $\lim_\beta(e_\beta c - c e_\beta) = 0$.
Since (d_λ) is a bounded approximate diagonal and (e_α) is bounded and the inequality

$$\|(a \cdot d_\lambda - d_\lambda \cdot a) \bullet e_\alpha \otimes e_\beta\| \leq \|(a \cdot d_\lambda - d_\lambda \cdot a)\|\|e_\alpha\|\|e_\beta\|.$$

It follows that

$$\lim_\lambda \limsup_\alpha \limsup_\beta \|(a \cdot d_\lambda - d_\lambda \cdot a) \bullet e_\alpha \otimes e_\beta\| = 0,$$

and thus

$$\lim_\lambda \limsup_\alpha \limsup_\beta \|a \cdot p_{\lambda\alpha\beta} - p_{\lambda\alpha\beta} \cdot a\| = 0.$$

Moreover, since (e_α) is a left approximate identity for $\mathcal{I}$ and $\mathcal{I}$ a left ideal, and also (d_λ) being a bounded approximate diagonal. We have,

$$\begin{aligned}\lim_\lambda \lim_\alpha \lim_\beta \pi(p_{\lambda\alpha\beta})c &= \lim_\lambda \lim_\alpha \lim_\beta \pi_(d_\lambda \bullet (e_\alpha \otimes e_\beta))a \\ &= \lim_\lambda \lim_\alpha \pi(d_\lambda \bullet (e_\alpha \otimes 1))c \\ &= \lim_\lambda \pi(d_\lambda)a \\ &= a,\end{aligned}$$

Hence a bounded net which is a bounded approximate diagonal for $\mathcal{I}$ may be chosen from $\{p_{\lambda\alpha\beta} | \lambda \in \mathscr{A}; \alpha, \beta \in \mathscr{D}\}$. Therefore by Theorem 4.1.6 $\mathcal{I}$ is amenable.

□

Proposition 4.1.18. *[36] Let $\mathcal{A}$ be a Banach algebra, $\mathcal{I}$ a closed ideal of $\mathcal{A}$ with bounded approximate identity, and let $D \in \mathcal{Z}^1(\mathcal{I}, X')$ where X is a neo-unital Banach $\mathcal{I}$-bimodule. Then there exists a unique extension $\hat{D} \in \mathcal{Z}^1(\mathcal{A}, X')$ of D to $\mathcal{A}$.*

Proposition 4.1.19. *[36] Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity, and let X be a Banach $\mathcal{A}$-bimudule with trivial left or right module action. Then $\mathcal{H}^1(\mathcal{A}, X') = \{0\}$.*

Proposition 4.1.20. *[36] Let $\mathcal{A}$ be a Banach algebra with bounded approximate identity. Suppose that $\mathcal{H}^1(\mathcal{A}, X') = \{0\}$ for all neo-unital Banach $\mathcal{A}$-bimodule X. Then $\mathcal{A}$ is amenable.*

Proposition 4.1.21. *[34] Let $\mathcal{A}$ be an amenable Banach algebra, and let $\mathcal{I}$ be a closed ideal of $\mathcal{A}$. Then the following are equivalent:*

(a) $\mathcal{I}$ is amenable;

(b) $\mathcal{I}$ has a bounded approximate identity;

(c) $\mathcal{I}$ is weakly complemented (that is, its annilhater $\mathcal{I}^{\perp}$ is complemented in $\mathcal{A}'$).

Corollary 4.1.22. *Let $\mathcal{A}$ be an amenable Banach algebra, and let $\mathcal{I}$ be a closed ideal of $\mathcal{A}$ with finite dimension or codimension. Then $\mathcal{I}$ is amenable.*

Corollary 4.1.23. *Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{I}$ be a closed ideal of $\mathcal{A}$ with a bounded approximate identity. Then $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ and $\mathcal{A}/\mathcal{I}$ are both amenable.*

4.1.4 Some General Results

Proposition 4.1.24. *If G is an abelian group, then G is amenable.*

Example 4.1.25. Let G be a group. Then the following are equivalent:

(a) The discrete group algebra $\ell^1(G)$ is amenable.

(b) G is amenable.

Corollary 4.1.26. *[6] The algebra $\mathbb{C}$ is amenable with the usual product and norm.*

Proof. Let $\mathcal{A} = \ell'(0)$. Define the map $\phi : \mathcal{A} \to \mathbb{C}$ by

$$\phi(a) = f(0),$$

which is clearly a isomorphism. Also since $|\phi(f)| = |f(0)| = \|f\|$, for all $f \in \mathcal{A}'$, then ϕ is an isometry. Then the result follows by Proposition 4.1.14. $\square$

Theorem 4.1.27. *Let G be a locally compact group. Then the following are equivalent:*

(a) G is amenable as a group;

(b) $L^1(G)$ is an amenable Banach algebra.

Example 4.1.28. The Banach algebra $L^1(\mathbb{R})$ is amenable.

Theorem 4.1.29. *A C^*-algebra is amenable if and only if it is nuclear.*

4.2 Weak Amenability in Banach Algebras

4.2.1 Definitions with Examples

Definition 4.2.1. The Banach algebra $\mathcal{A}$ is said to be *weakly amenable* if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}') = \{0\}$. For $n \in \mathbb{N}$, $\mathcal{A}$ is *n-weakly amenable* if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^n) = \{0\}$.

Examples 4.2.2. The group algebra $L^1(G)$ is weakly amenable for any locally compact group G.

Example 4.2.3. Every C^*-algebra is weakly amenable.

4.2.2 Some Characterizations and Hereditary Properties

Proposition 4.2.4. *[19] Suppose $\mathcal{A}$ and $\mathcal{B}$ are commutative Banach algebras, and $\phi : \mathcal{A} \to \mathcal{B}$ is a continuous homomorphism with dense range. If $\mathcal{A}$ is weakly amenable, then $\mathcal{B}$ is also weakly amenable.*

Proposition 4.2.5. *Let $\mathcal{A}$ be Banach algebra. If $\mathcal{I}$is a closed ideal of $\mathcal{A}$ such that both $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are weakly amenable. Then $\mathcal{A}$ is weakly amenable.*

Proposition 4.2.6. *[19] Suppose $\mathcal{A}$ is a weakly amenable Banach algebra and $\mathcal{I}$ is a closed ideal in $\mathcal{A}$. Then $\mathcal{I}$ is weakly amenable if and only if it is essential.*

Corollary 4.2.7. *[10] Let $\mathcal{A}$ be a weakly amenable Banach algebra. Then*

(a) $\mathcal{A}^2$ is dense in $\mathcal{A}$;

(b) there are no non-zero point derivations on $\mathcal{A}$.

Proposition 4.2.8. *[19] Suppose there is a short exact sequence*

$$\sum : 0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0$$

of commutative Banach algebras and bounded homomorphisms, i.e. $\mathcal{A}$ is an extension of $\mathcal{B}$ by $\mathcal{I}$. If $\mathcal{I}$ and $\mathcal{B}$ are weakly amenable, then $\mathcal{A}$ is also wealy amenable.

Proposition 4.2.9. *[19] Let $\mathcal{A}$ and $\mathcal{B}$ be weakly amenable Banach algebras, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is weakly amenable.*

Proposition 4.2.10. *[19] Let $(\mathcal{A}_\lambda)_{\lambda\in\Lambda}$ be a family of closed subalgebras of a commutative Banach algebra $\mathcal{A}$ such that $\bigcup \mathcal{A}_\lambda$ is dense in $\mathcal{A}$. Then if for each $\lambda \in \Lambda$, $\mathcal{A}_\lambda$ is weakly amenable, then $\mathcal{A}$ is weakly amenable.*

4.3 Approximate Amenability in Banach Algebras

4.3.1 Definitions with Examples

Definition 4.3.1. Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. A derivation $D : \mathcal{A} \to X$ is said to be *approximately inner* if there exists a net (ξ_λ) in X such that

$$D(a) = \lim_\lambda (a \cdot \xi_\lambda - \xi_\lambda \cdot a) \quad (a \in \mathcal{A}),$$

the limit being in the norm.

Definition 4.3.2. Let $\mathcal{A}$ be a Banach algebra. $\mathcal{A}$ is said to be *approximately amenable* if every continuous derivation $D : \mathcal{A} \to X'$ is approximately inner, for every Banach $\mathcal{A}$-bimodule X.

Example 4.3.3. $M(G)$ is approximately amenable if and only if G is discrete and amenable.

Example 4.3.4. $L^1(G)$ is approximately amenable if and only if G is amenable.

Example 4.3.5. $L^1(G)''$ is approximately amenable if and only if G is finite.

4.3.2 Some Characterizations

Theorem 4.3.6. *[15] Let $\mathcal{A}$ be a Banach algebra. $\mathcal{A}$ is approximately amenable if and only if either of the following conditions hold:*

(a) there is a net (M_v) in $(\tilde{\mathcal{A}}\hat{\otimes}\tilde{\mathcal{A}})''$ such that for each $a \in \tilde{\mathcal{A}}$,

$$a \cdot M_v - M_v \cdot a \to 0 \quad \text{and} \quad \pi''(M_v) \to e;$$

(b) there is a net (M_v') in $(\tilde{\mathcal{A}}\hat{\otimes}\tilde{\mathcal{A}})''$ such that for each $a \in \tilde{\mathcal{A}}$,

$$a \cdot M_v' - M_v' \cdot a \to 0 \quad \text{and} \quad \pi''(M_v) = e.$$

Corollary 4.3.7. *[15] Let $\mathcal{A}$ be a Banach algebra. $\mathcal{A}$ is approximately amenable if and only if there exist nets (M_v'') in $(\tilde{\mathcal{A}}\hat{\otimes}\tilde{\mathcal{A}})''$, $(a_v), (b_v)$ in $\mathcal{A}''$, such that for each $a \in \mathcal{A}$,*

(a) $a \cdot M_v'' - M_v'' \cdot a + a_v \otimes a - a \otimes b_v \to 0;$

(b) $a \cdot a_v \to a, \quad b_v \cdot a \to a;$

(c) $\pi''(M_v'') \cdot a - a_v \cdot a - b_v \cdot a \to 0.$

Proposition 4.3.8. *[15] Suppose that $\mathcal{A}$ is a Banach algebra with bounded approximate identity. Then $\mathcal{A}$ is approximately amenable if and only if every derivation into the dual of any neo-unital bimodule is approximately inner.*

Proposition 4.3.9. *[15] Let $\mathcal{A}$ be a Banach algebra. $\mathcal{A}$ is approximately amenbale if and only if $\tilde{\mathcal{A}}$ is approximately amenable.*

Proposition 4.3.10. *[15] Let $\mathcal{A}$ be a Banach algebra. Suppose that $\mathcal{A}''$ is approximately amenable. Then $\mathcal{A}$ is approximately amenable.*

4.3.3 Some Hereditary Properties

Proposition 4.3.11. *Suppose $\mathcal{A}$ is an approximately amenable Banach algebra. Then $\mathcal{A}$ has left and right approximate identities. In particular $\mathcal{A}^2$ is dense in $\mathcal{A}$.*

Proposition 4.3.12. *[15] Suppose that $\mathcal{A}$ is approximately amenable, $\mathcal{B}$ another Banach algebra, and $\phi : \mathcal{A} \to \mathcal{B}$ is a continuous epimorphism. Then $\mathcal{B}$ is approximately amenable.*

Corollary 4.3.13. *[15] Suppose that $\mathcal{A}$ is approximately amenable and $\mathcal{I}$ a closed ideal of $\mathcal{A}$. Then $\mathcal{A}/\mathcal{I}$ is approximately amenable. If $\mathcal{I}$ is amenable and $\mathcal{A}/\mathcal{I}$ is approximately amenable, then $\mathcal{A}$ is approximmately amenable.*

Proposition 4.3.14. *[15] Suppose $\mathcal{A}$ is approximately amenable, and $\mathcal{I}$ is a closed ideal ideal with a bounded approximate identity. Then $\mathcal{I}$ is approximately amenable.*

Proposition 4.3.15. *[15] Suppose that $\mathcal{A}$ is approximately amenable and has a bounded approximate identity, and $\mathcal{B}$ is an amenable Banach algebra. Then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is approximately amenable. Furthermore , if $\mathcal{A}$ is not amenable, and $\mathcal{B}$ admits a non-zero character, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is not amenable.*

Proposition 4.3.16. *[15] Suppose that $\mathcal{A}$ and $\mathcal{B}$ are aproximately amenable Banach algebras and each having a bounded approximate identity. Then $\mathcal{A}\oplus\mathcal{B}$ is approximately amenable.*

Theorem 4.3.17. *[15] Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ is amenable, and*

$$\sum : 0 \to X' \to Y \to Z \to 0$$

is an admissable short exact sequence of left $\mathcal{A}$-modules. Then $\sum$ approximately splits.

Corollary 4.3.18. *[15] Suppose $\mathcal{A}$ is an approximately amenable Banach algebra and let $\mathcal{I}$ be weakly complemented left ideal of $\mathcal{A}$. Then $\mathcal{I}$ has a right approximate identity. In particular, $\mathcal{I}^2$ is dense in $\mathcal{I}$.*

5 Amenability Properties in Operator Algebras

Over the years, various authors have considered the amenability properties of Banach algebras constructed over locally compact groups, semigroups and $C^*-algebras$ (see [20], [14], [13], [23], [24], [18], and so on). A class of Banach algebras where this philosophy is yet to be fully explored is the algebras of operators on a Banach space.

An important active area of research and open question in connection with the amenability properties of Banach algebras of operators on a Banach space is: What is the property - described in Banach space theoretic terms - which characterizes those Banach spaces X for which the Banach algebra of operators on a Banach space (e.g. $A(X)$, $K(X)$) is amenable or possesses some other amenability properties? We try to study and explore this question in this chapter. Thus, in this chapter, we study the relation between the geometry properties of the Banach space and the amenability properties of operator algebras. That is, how the geometric properties of the Banach space affect the amenability properties of the operator algebras. In particular, we study the weak amenability and amenability of $A(X)$ and $K(X)$, the Banach algebra of approximable operators and compact operators on a Banach space X respectively, with relation to the geometry of the Banach space X. In particular, we consider Banach spaces with properties bounded approximation property and property ($\mathbb{A}$) which was introduced in [22].

5.1 Amenability of Operator Algebras

In this section, we study the amenability of Banach algebras of operators on a Banach space X. This was pioneered by Johnson (1972), where it was shown that $K(X)$ is amenable if X is l_p, $1 < p < \infty$ or C[0,1], see [25].

The central problem for the amenability of Banach algebras of operators on a Banach space X is the following: for which Banach space X precisely is the Banach algebras of operators on a Banach space X amenable? or what are the intrinsic properties of the Banach space X which is equivalent to amenability of the Banach algebras of operators on a Banach space? Gronbaek, Johnson and Willis in [22] give examples to show that amenability of $A(X)$ is not equivalent to X or X^* having the bounded approximation property. They formulated a symmetrized approximation property, called property ($\mathbb{A}$) such that if X has property (A), then $A(X)$ is amenable. Our main reference for this section is [22].

5.1.1 Approximable Operators

Definition 5.1.1. Let X and Y be Banach spaces and let α be a cross norm on $X \otimes Y$. The completion of $X \otimes Y$, the space $X \otimes_\alpha Y$ is said to be a *tight tensor product* of X and Y, if the following conditions hold:

(a) There exists $K > 0$ such that for all $S \in \mathcal{A}(X), T \in \mathcal{A}(Y)$, we have

$$\|S \otimes T\|_\alpha \leq K\|S\|\|T\|$$

for the operator

$$(S \otimes T)x \otimes y = Sx \otimes Ty \quad (x \in X, y \in Y).$$

(b) $\text{span}\{S \otimes T : S \in \mathcal{A}(X), T \in \mathcal{A}(Y)\}$ is dense in $\mathcal{A}(X \otimes_\alpha Y)$.

Gronbaek, Johnson and Willis [22] in the next result give conditions for tightness for some important tensor products, in particular, the injective and the projective tensor products.

Theorem 5.1.2. *[22] Let X and Y be Banach spaces, let [0,1] be the unit interval, and let $(\Omega, \sum, \mu)$ be a σ-finite measure space. Then*

- *The following are equivalent:*
  (i) $X \check{\otimes} Y$ is tight for all X,
  (ii) $C([0,1], Y)$ is a tight tensor product of $C[0,1]$ and Y,
  (iii) Y^* has Radon-Nikodym Property.
- *$X \hat{\otimes} Y$ is tight if and only if $A(Y, X^*) = B(Y, X^*)$.*
- *$L_p(\mu, X), 1 \leq p < \infty$ is a tight tensor product of $L_p(\mu)$ and X if and only if X^* has Randon-Nikodynm Property with respect to μ.*

Also, Gronbaek, Johnson and Willis [22] showed that the amenability of $A(X)$ and $A(Y)$ imply the amenability of $A(Z)$ where Z is the completion of $X \otimes Y$ in some crossnorm topology. This result is stated in the next Theorem.

Theorem 5.1.3. *[22] Let X and Y be Banach spaces. Suppose that $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ are both amenable and that $X \otimes_\alpha Y$ is a tight tensor product. Then $\mathcal{A}(X \otimes_\alpha Y)$ is amenable.*

Definition 5.1.4. A finite biorthogonal system is a set

$$\{(x_i, \mu_j) : x_i \in X, \mu_j \in X', i, j = 1, 2, ..., n\} \subset X \times X'$$

such that

$$\mu_j(x_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \quad i, j = 1, 2, ..., n.$$

.

Remark 5.1.5. If $\{(x_i, \mu_j) : x_i \in X, \mu_j \in X', i, j = 1, 2, ..., n\}$ is a finite biorthogonal system, then

$$\Phi : \mathbb{M}_n \to \mathcal{A}(X)$$
$$(a_{ij}) \mapsto \sum_{i,j=1}^{n} a_{ij} x_i \otimes \mu_j$$

is an algebra homomorphism. Indeed, for all $a = (a_{ij}), b = (b_{ij}) \in \mathbb{M}_n$ and $x \in X$, we have that $ab = \sum_k a_{ik} b_{kj}$ and

$$\begin{aligned} [\Phi(a) \circ \Phi(b)](x) &= \left[\sum_{i,k} a_{ik} x_i \otimes \mu_k\right] \circ \left[\sum_{l,j} b_{lj} x_l \otimes \mu_j\right](x) \\ &= \sum_{i,k} a_{ik} x_i \left(\sum_{l,j} b_{lj} \mu_j(x) x_l\right) \mu_k \\ &= \sum_{i,k} x_i \left(\sum_{j} a_{ik} b_{kj} \mu_j(x)\right) \\ &= \left[\sum_{i,j} \left(\sum_{k} a_{ik} b_{kj}\right) x_i \otimes \mu_j\right](x) \\ &= \Phi\left(\sum_{k} a_{ik} b_{kj}\right)(x). \end{aligned}$$

Thus $\Phi(ab) = \Phi(a)\Phi(b)$.

Definition 5.1.6. A Banach space X is said to have *property* $(\mathbb{A})$ if there is a net of finite biorthogonal systems

$$\{(x_{i,\lambda}, \mu_{j,\lambda}) : x_{i,\lambda} \in X, \mu_{j,\lambda} \in X', i, j = 1, 2, ..., n_\lambda\} \quad (\lambda \in \Lambda)$$

and corresponding homomorphism

$$\Phi_\lambda : M_{n_\lambda}(\mathbb{C}) \to \mathcal{A}(X) \quad (\lambda \in \Lambda)$$

such that

(a) $\Phi_\lambda(I_{n_\lambda}) \xrightarrow{s} Id_X$;

(b) $\Phi_\lambda(I_{n_\lambda})^a \xrightarrow{s} Id_{X'}$;

(c) for each λ, there is a finite group G_λ of invertible elements of M_{n_λ} spanning M_{n_λ} such that

$$\sup_\lambda \max_{g \in G_\lambda} \|\Phi_\lambda(g)\| < \infty.$$

For convenience $\Phi_\lambda(I_{n_\lambda})$ will be denoted by P_λ. Condition (a) and (b) imply that (P_λ) is a bounded approximate identity for $\mathcal{A}(X)$ and that X has a bounded approximate identity.

Proposition 5.1.7. *[5] Let X be a Banach space such that $\mathcal{A}(X)$ has approximate diagonal with bound K and let $(P_\lambda)_\lambda$ be a bounded net of projection of $\mathcal{A}(X)$ such that*

$$P_\lambda \xrightarrow{s} Id_X \text{ and } P_\lambda^a \xrightarrow{s} Id_{X'}.$$

Then $\mathcal{A}(X)$ has an approximate diagonal $(d_\lambda)_\lambda$ with the following properties:

(a) $\limsup_\lambda \kappa(d_\lambda) \leq \alpha K$, where $\alpha = \limsup_\lambda \|P_\lambda\|$ and κ is the projective tensor norm;

(b) $\pi(d_\lambda) = P_\lambda$;

(c) $W \cdot d_\lambda = d_\lambda \cdot W$ for every $W \in P_\lambda \mathcal{A}(X) P_\lambda$;

(d) For each $\lambda \in \Lambda$, there is $\omega = \omega(\lambda) \in \Lambda$ such that

$$d_\lambda \in \mathcal{A}(X)P_\omega \otimes P_\omega \mathcal{A}(X).$$

Theorem 5.1.8. *[22] Let X be Banach space. If X' has property $(\mathbb{A})$, then X has property $(\mathbb{A})$.*

Corollary 5.1.9. *[22] There exists an infinite-dimensional Banach space X such that $\mathcal{B}(X)$ is amenable.*

Examples 5.1.10. (a) Let (Ω, Σ, μ) be a measurable set, then $L^p(\mu)$, $1 \leq p \leq \infty$, has property $(\mathbb{A})$.

(b) $C(K)$ has property $(\mathbb{A})$ for each compact Hausdorff space K.

c Every infinite-dimensional Hilbert space also has property $(\mathbb{A})$.

Theorem 5.1.11. *[22] Let X be a Banach space with a subsymmetric and shrinking basis. Then X has property $(\mathbb{A})$.*

Theorem 5.1.12. *[22] Let X be a Banach space which has property $(\mathbb{A})$. Then $\mathcal{A}(X)$ is amenable.*

Proposition 5.1.13. *[22] Let X be a Banach space such at $\mathcal{A}(X')$ is amenable. Then $\mathcal{A}(X)$ is amenable.*

Definition 5.1.14. Let X be a Banach space. Then X is said to be an $\mathfrak{L}_p$-space $(1 \leq p \leq \infty)$ if it has a net (X_λ) of finite-dimensional Banach spaces, directed by inclusion, whose union is dense in X and such that

$$\sup_\lambda d(X_\lambda, \ell_p^{n_\lambda}) < \infty \text{ where } n_\lambda = \dim(X_\lambda).$$

Example 5.1.15. Let $X = \ell_p$ or $L_p(0,1)$. Then X is an $\mathfrak{L}_p$-space

Remark 5.1.16. Let $X = \mathfrak{L}_p$. Then

(a) X has the approximation property.

(b) X' has bounded approximation property and so $\mathcal{A}(X)$ has a bounded approximate identity.

Theorem 5.1.17. *[22] Let $X = \mathfrak{L}_p$ for $1 \leq p < \infty$. Then $\mathcal{A}(X)$ is amenable.*

The following is a generalization of [[22] Theorem 6.9], given in [5].

Corollary 5.1.18. *[5] Let X be an $\mathfrak{L}_p$-space $(1 \leq p \leq \infty)$ and let Y be an $\mathfrak{L}_q$-space $(1 \leq q \leq \infty)$. Then $\mathcal{A}(X \oplus Y)$ is amenable if and only if one of the following hold:*

(a) $p = q$,

(b) $p = 2$ *and* $1 < q < \infty$,

(c) $q = 2$ *and* $1 < p < \infty$.

Proof. See [5] Corollary 4.6. □

5.1.2 Compact Operators

In this section, we give some results on amenability in Banach algebra $\mathcal{K}(X)$ of compact operators on a Banach space X. Our main reference in this section are [6] and [22].

Theorem 5.1.19. *[22] Let X be a Banach space with property $(\mathbb{A})$, then $\mathcal{K}(X)$ is amenable.*

Proposition 5.1.20. *[22] Let X be a Banach space. If $\mathcal{K}(X')$ is amenable and has a bounded approximate identity, then $\mathcal{K}(X)$ is amenable.*

Proposition 5.1.21. *[22] Let X be a Banach space. If $\mathcal{K}(X)$ is amenable and $\mathcal{K}(X')$ has a bounded approximate identity, then $\mathcal{K}(X')$ is amenable.*

Corollary 5.1.22. *[22] Let $X = \mathfrak{L}_p$ $(1 \leq p \leq \infty)$. Then $\mathcal{K}(X)$ is amenable.*

Proof. Since X has the approximation property, then $\mathcal{A}(X) = \mathcal{K}(X)$. Now by Theorem 5.1.17, $\mathcal{A}(X)$ is amenable and so $\mathcal{K}(X)$ is amenable. □

Theorem 5.1.23. *[22] Let X be a Banach space. Then $\mathcal{K}(X)$ is amenable if and only if $\mathcal{K}(X \oplus \mathbb{C})$ is amenable.*

Theorem 5.1.24. *[6] Let $X = \mathcal{H}$ be a Hilbert space. Then $\mathcal{K}(X) \oplus \mathbb{C}I$ is amenable*

Corollary 5.1.25. *[6] Let $X = \mathcal{H}$ be a separable Hilbert space. Then $\mathcal{K}(X)$ is amenable.*

Theorem 5.1.26. *[6] Let $X = \mathcal{H}$ be a Hilbert space. Then $\mathcal{K}(X)$ is amenable.*

5.2 Weak Amenability of Operator Algebras

This section surveys weak amenability of approximable and compact operators. These results are extensively in [3] and [21]. One interesting result studied in [3] is that bounded approximation property is neither necessary nor sufficient for the weak amenability of Banach algebra $\mathcal{A}(X)$ of approximable operators on a Banach space X and thus a necessary was introduced in [3].

5.2.1 Approximable Operators

Theorem 5.2.1. *[21] Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is not self-induced. Then $\mathcal{A}(X)$ is weakly amenable if and only if the following hold:*

(a) $\mathcal{A}(X) = \mathcal{N}(X)$

(b) $dim(K) = 1$, *where K kernel of the trace* $trace : X\hat{\otimes}X' \to \mathcal{N}(X)$.

Proposition 5.2.2. *[10] Let X be a reflexive Banach space such that $X = \ell_p(Y)$ $1 < p < \infty$, for some Banach space Y. Then $\mathcal{W}(X)$is weakly amenable. In particular, if Y is reflexive then $\mathcal{B}(X)$ is weakly amenable.*

Corollary 5.2.3. *[10] Suppose that the hypotheses of the above theorem hold, and that in addition X has the approximation property. Then $\mathcal{A}(X)$ is weakly amenable.*

Proposition 5.2.4. *[3] Let X be a Banach space, and let $\mathcal{A}$ be a dense subalgebra of $X'\hat{\otimes}X$. Then $\mathcal{A}(X)$ is weakly amenable if and only if, whenever $T \in \mathcal{B}(X')$ is such that*

$$|trace(T(RS - SR)') \leq K\|R\|\|S\| \quad (R, S \in \mathcal{A})$$

for some constant K, there exists $\lambda \in \mathbb{C}$ and a constant L such that

$$\|trace((T - \lambda)W')| \leq L\|W\| \quad (W \in \mathcal{A}).$$

Corollary 5.2.5. *[3] Let X be a reflexive Banach space, and let $\bar{\mathcal{A}} = X'\hat{\otimes}X$. Then $\mathcal{A}(X)$ is weakly amenable if and only if, whenver $T \in \mathcal{B}(X)$ is such that*

$$|trace(T(RS - SR)) \leq K\|R\|\|S\| \quad (R, S \in \mathcal{A})$$

for some constant K, there exists $\lambda \in \mathbb{C}$ and a constant L such that

$$\|trace((T - \lambda)W)| \leq L\|W\| \quad (W \in \mathcal{A}).$$

Definition 5.2.6. Let $\mathcal{A}$ and $\mathcal{B}$ be seld-induced Banach algebra. Then $\mathcal{A}$ and $\mathcal{B}$ are said to be *M-equivalent*, denoted by $\mathcal{A} \underset{M}{\sim} \mathcal{B}$, if there are bimodules ${}_{\mathcal{A}}\mathcal{U}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{V}_{\mathcal{A}}$ and blanced pairings

$$[\cdot] : {}_{\mathcal{A}}\mathcal{U}\underset{\mathcal{B}}{\hat{\otimes}}\mathcal{V}_{\mathcal{A}} \to \mathcal{A} \quad \text{and} \quad [\cdot] : {}_{\mathcal{B}}\mathcal{V}\underset{\mathcal{A}}{\hat{\otimes}}\mathcal{U}_{\mathcal{B}} \to \mathcal{B},$$

which are bimodule isomorphisms satisfying

$$[u\underset{\mathcal{B}}{\hat{\otimes}}v] \cdot u' = u \cdot [v\underset{\mathcal{A}}{\hat{\otimes}}u'] \quad \text{and} \quad [v\underset{\mathcal{A}}{\hat{\otimes}}u] \cdot v' = v \cdot [u\underset{\mathcal{B}}{\hat{\otimes}}v'] \quad u, u' \in {}_{\mathcal{A}}\mathcal{U}_{\mathcal{B}}, v, v' \in {}_{\mathcal{B}}\mathcal{V}_{\mathcal{A}}.$$

Proposition 5.2.7. *[21] Let X and Y be a Banach space. Suppose $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ are seld-induced Banach algebras, and that $\mathcal{A}(X) \underset{M}{\sim} \mathcal{A}(Y)$. If X has the bounded approximation property, then there exists an injection*

$$\mathcal{H}^1(\mathcal{A}(Y), \mathcal{A}(Y)') \to \mathcal{H}^1(\mathcal{A}(X), \mathcal{A}(X)').$$

In particular if X has the bounded approximation property and $\mathcal{A}(X)$ is weakly amenable, then $\mathcal{A}(Y)$ is weakly amenable.

Proposition 5.2.8. *[21] Let X be a Banach space. Then $\mathcal{A}(\ell_p(X))$ is weakly amenable if and only if $\mathcal{A}(\ell_p(X))$ is self-induced.*

Proposition 5.2.9. *[21] Let X be a Banach space with bounded approximation property. Then $\mathcal{A}(L_p(\mu, X))$ is weakly amenable.*

Definition 5.2.10. Let X_1 and X_2 be Banach spaces, and let $X = X_1 \oplus X_2$ be the direct sum. For $k = 1, 2$, the kth *canonical projection (embedding)* are denoted by

$$\gamma_k : X \to X_k \quad (\imath_k : X_k \to X).$$

Then for any $T \in \mathcal{B}(X)$ and $\tilde{T} \in \mathcal{B}(X')$, let

$$T_{kj} = \gamma_k T \imath_j \quad \text{and} \quad \tilde{T}_{kj} = \gamma_k' \tilde{T} \imath_j'.$$

Definition 5.2.11. Let X be a Banach space, then for each $T \in \mathcal{B}(X')$ consider the maps

$$l_T : W \mapsto \text{trace}(TW') \quad (W \in \mathcal{F}(X))$$

and

$$b_T : (R, S) \mapsto \text{trace}(T(RS - SR)') \quad ((R, S) \in \mathcal{F}(X) \times \mathcal{F}(X)).$$

Then define set Δ_X by

$$\Delta_X \stackrel{def}{=} \{T \in \mathcal{B}(X') : b_T \text{ is bounded}\}.$$

Theorem 5.2.12. *[3] Let X_1 and X_2 be infinite-dimensional Banach spaces, and let $X = X_1 \oplus X_2$. Suppose at least one of the following conditions hold:*

(a) X_1' has the bounded approximation property;

(b) X_2' *has the bounded approximation property;*

(c) X *has the bounded approximation property;*

Then, for each $T \in \Delta_X$*, theere exists a constant* K_T *such that*

$$|l_T(\imath_1 W_{12}\gamma_2 + \imath_2 W_{21}\gamma_1)| \leq K_T\|W\| \quad (W \in \mathcal{F}(X)).$$

Proposition 5.2.13. *[3] Suppose that the hypotheses of Theorem 5.2.12 is satisfied. If* $\mathcal{A}(X_1)$ *is weakly amenable and if there is a* $c > 0$ *such that*

$$\|W\|_{X_1} \leq c\|W\| \quad (W \in \mathcal{F}(X_2)).$$

Then $\mathcal{A}(X)$ *is weakly amenable.*

Proof. See [3]. □

Remark 5.2.14. The above result gives a necessary condition for the weakly amenable of Banach algebra of approximable operators on a Banach space X.

Definition 5.2.15. Let J be a Banach space, and let (G_n) be a sequence of finite-dimensional Banach spaces such that for each $i \in \mathbb{N}$, the set $\{n \in \mathbb{N} : G_n \cong G_i\}$ is infinite. Then J is said to be a *Johnson space* if it can be written as the ℓ_p-sum of the sequence (G_n), that is, $J = (\oplus_{n=1}^{\infty} G_n)_p \quad (p = 0, 1 \leq p < \infty)$.

Example 5.2.16. If $X = \ell_p$ $(1 \leq p \leq \infty)$ or c_0. Then X is a Johnson space.

Remark 5.2.17. If $T \in \mathcal{A}(X, Y)$, then there exist a $T_1 \in \mathcal{A}(X, J_p)$ and $T_2 \in \mathcal{A}(J_p, Y)$ such that $T = T_1 \circ T_2$. i.e. every approximable operators factors approximately through J_p.

Definition 5.2.18. Let $J = (\oplus G_n)_p$ be a Johnson space. Then a Banach space X is called a J*-space* if there exists $\lambda \geq 1$ such that for every finite dimensional subspace Y of X, there is subspace G of X such that the Banach -Mazur distance $d(G, G_i) \leq \lambda$ for some $i \in \mathbb{N}$.

Example 5.2.19. Any Banach space is a J_p space for every $1 \leq p \leq \infty$ and $p = 0$.

Theorem 5.2.20. *[3] Let $J = (\oplus_1^\infty G_n)_p$ be a Johnson space, and let X be a J-space. Then $\mathcal{A}(J)$ is weakly amenable, and $\mathcal{A}(X)$ factors approximately through $\mathcal{A}(J)$.*

Theorem 5.2.21. *[3] Let $J = (\oplus_1^\infty G_n)_p$ be a Johnson space, and let X be a J-space. Then $\mathcal{A}(X \oplus J)$ is weakly amenable.*

In general, for every Banach space X, The Banach algebra $\mathcal{A}(X \oplus J)$ is weakly amenable.

Theorem 5.2.22. *[3] Let X be a Banach space and $J_\infty = (\oplus_1^\infty G_n)_\infty$. Then the Banach algebra $\mathcal{A}(X \oplus J_\infty)$ is weakly amenable.*

The following result shows that the Banach space X having the approximation property is not necessary for $\mathcal{A}(X)$ to be weakly amenable.

Theorem 5.2.23. *[3] There are Banach spaces without the approximation property such that $\mathcal{A}(X)$ is weakly amenable.*

Proof. Since every complemented subspace of a Banach space Y will have the (bounded) apprixmation property, if whenever Y has the (bounded) approximation property. Thus the Banach space $X \oplus J_p$ does not have the (bounded) approximation property, if whenever X does not have the (bounded) approximation property. However by Theorem 5.2.21 is always wealy amenable. □

Theorem 5.2.24. *[3] Let $X = X_1 \oplus X_2$ be the direct sum of Banach spaces. Suppose the hypothese of Theorem 5.2.12 is satisfied. If $\mathcal{A}(X_1)$ and $\mathcal{A}(X_2)$ is weakly amenable, then $\mathcal{A}(X)$ is weakly amenable.*

Corollary 5.2.25. *[3] Let $X_i = \mathfrak{L}_{p_i}$, $1 \leq p_i \leq \infty \quad (1 \leq i \leq n)$. Then $\mathcal{A}(\oplus_{i=1}^n X_i)$ is weakly amenable.*

Proof. The desired results follows by using Theorem 5.2.24 and induction. □

Corollary 5.2.26. *[3] Let X be a Banach space with bounded approximation property. If $\mathcal{A}(X)$ is weakly amenable, then $\mathcal{A}(\ell_p^n(X)) \quad (1 \leq p \leq \infty, n \in \mathbb{N})$ is weakly amenable.*

Proposition 5.2.27. *[3] Let X be a Banach space. If $\mathcal{A}(X')$ is weakly amenable, then $\mathcal{A}(X)$ is weakly amenable.*

Proposition 5.2.28. *[3] Let X be a Banach space such that $\mathcal{A}$ is weakly amenable, and $\imath : X \to X''$ the canonical embedding. If the following conditions hold:*

(a) $\imath(X)$ *is complemented in* X''*, in particular, if* X *is a dual space.*

(b) $\|W\|_X \leq c\|W\| \quad (W \in \mathcal{F}(X'')$ *for some constant* $c > 0$*, in particular, if* X'' *has the bounded approximation property.*

Then $\mathcal{A}(X'')$ is weakly amenable. Consequently by previous proposition $\mathcal{A}'(X')$ is weakly amenable.

Corollary 5.2.29. *[3] The algebra $\mathcal{A}(J_\infty)$ is weakly amenable.*

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