Integration over the space of functions and Poincaré series: a revision *

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To S.P. Novikov with admiration.

Abstract

Earlier (2000) the authors introduced the notion of the integral with respect to the Euler characteristic over the space of germs of functions on a variety and over its projectivization. This notion permitted to rewrite in new terms known definitions and statements and also appeared to be an effective tool to compute Poincaré series of multi-index filtrations in some situations. However the "classical" (initial) notion can be applied only to multi-index filtrations defined by so-called finitely determined valuations (or order functions). Here we introduce a modified version of the notion of the integral with respect to the Euler characteristic over the projectivization of the space of function germs. This version can be applied in a number of settings where the "classical approach" does not work. We give examples of application of this concept for definitions and computations of the Poincaré series of collections of plane valuations which include valuations not centred at the origin, including equivariant ones.

1 Introduction

The notion of the integral with respect to the Euler characteristic over the space of germs of functions on the complex affine space $\mathbb{C}^n$ at the origin and over its projectivization was introduced in [2]. This notion was inspired by the

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notion of motivic integration over the space of arcs on a variety: [13]. In [2] it was used to rewrite in the corresponding terms the definition of the Poincaré series of a collection of r irreducible plane curve singularities (appearing to coincide with the Alexander polynomial of the corresponding r-component link for $r > 1$ and with the Alexander polynomial divided by $(1-t)$ for $r = 1$). Its natural generalizations for the integral over the space of germs of functions on a variety and over its projectivization was given in [6]. It was found that this notion not only permits to rewrite in new terms known definitions and statements, but also is an effective tool to compute Poincaré series in some situations: see. e.g., [4], [12], [5].

The notion of the integral with respect to the Euler characteristic over the space $\mathcal{O}_{V,0}$ of germs of functions on an analytic variety $(V, 0)$ (or over its projectivization $\mathbb{P}\mathcal{O}_{V,0}$) was based on the idea (came from the theory of motivic integration) to define the notion of the Euler characteristic for so-called cylindric subsets of the space $\mathcal{O}_{V,0}$ or of the space $\mathbb{P}\mathcal{O}_{V,0}$. This means that the condition for a function to belong to the subset is determined by its jet of finite order (depending only on the subset) and is a constructible condition on the space of jets (see Section 2 for details).

Poincaré series in several variables are defined, in particular, for collections of valuations (or of so-called order functions) on the ring of germs of functions on a variety $(V, 0)$. The initial applications ([3, 12]) were for the so-called curve and divisorial valuations on the ring of germs of functions on the complex plane $(\mathbb{C}^2, 0)$ or on a surface singularity ([5]). These valuations are finitely determined, i. e., the fact that the value of a valuation ν of this sort on a function germ f is equal to a particular number k is determined by the jet $j^N f$ of certain order N (depending only on k) of the function f. This means that the set $\{f \in \mathbb{P}\mathcal{O}_{V,0} : \nu(f) = k\}$ (or the set $\{f \in \mathbb{P}\mathcal{O}_{V,0} : \nu_1(f) = k_1, \ldots, \nu_r(f) = k_r\}$ for a collection $\nu_1, \ldots, \nu_r$ of r valuations of this sort) is cylindric in the sense of [2]. In this situation the mentioned concept of the integral with respect to the Euler characteristic over the space $\mathbb{P}\mathcal{O}_{V,0}$ gives sense to expressions of the form

$$\int_{\mathbb{P}\mathcal{O}_{V,0}} \underline{t}^{\underline{\nu}(f)} d\chi, \tag{1}$$

where $\underline{t} = (t_1, \ldots, t_r)$, $\underline{\nu}(f) = (\nu_1(f), \ldots, \nu_r(f))$, $\underline{t}^{\underline{\nu}} = t_1^{\nu_1} \cdot \ldots \cdot t_r^{\nu_r}$, and one can show that this integral is equal to the Poincaré series $P_{\{\nu_i\}}(\underline{t})$ of the collection $\{\nu_i\}$ of valuations. Some properties of the integral with respect to the Euler characteristic (say, the Fubini formula) permits to compute the integral (1) (and thus the Poincaré series) in some cases. This can be also applied to filtrations defined by so-called order functions: a notion less restrictive than the one of a valuation.

Later the study of the Poincaré series of a collection of valuations was extended to general valuations centred at the origin on the ring of germs of functions in two variables: [9]. These valuations are not, in general, finitely determined. Therefore the sets of functions with the fixed values of the valuations are not, in general, cylindric and the concept of the integral with respect to the Euler characteristic from [2] cannot be applied.

Another example when a Poincaré series (or rather its generalization) cannot be written as an integral with respect to the Euler characteristic in the mentioned sense is met when one wants to define an equivariant (say, with respect to an action of a finite group G) version of the Poincaré series: see Section 3. The problem is that the condition that the isotropy subgroup $G_f = \{a \in G : a^*f = f\}$ of a germ f is a fixed subgroup $H \subset G$ is not determined by a finite jet of f. Therefore the set of function germs with the fixed isotropy subgroup is not cylindric.

Here we introduce a modified version of the notion of the integral with respect to the Euler characteristic over the projectivization $\mathbb{P}\mathcal{O}_{V,0}$ of the space of function germs. This version can be applied in some settings where the "classical approach" does not work (in particular, in the described above settings). We show examples of application of this concept for definitions and computations of the Poincaré series of collections of plane order functions which include order functions not centred at the origin, including equivariant ones.

2 Poincaré series of filtrations and the "classical" integral with respect to the Euler characteristic

Let $(V, 0)$ be a germ of a complex analytic variety and let $\mathcal{O}_{V,0}$ be the ring of germs of functions on it. A function $\nu : \mathcal{O}_{V,0} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ is called an *order function* if

1) $\nu(\lambda g) = \nu(g)$ for $\lambda \in \mathbb{C}$, $\lambda \neq 0$;

2) $\nu(g_1 + g_2) \geq \min(\nu(g_1), \nu(g_2))$.

If, besides the conditions 1) and 2), one has $\nu(g_1 g_2) = \nu(g_1) + \nu(g_2)$, the function ν is a valuation on the ring $\mathcal{O}_{V,0}$.

A collection $\{\nu_i : i = 1, \ldots, r\}$ of order functions on $\mathcal{O}_{V,0}$ defines a multi-index filtration on $\mathcal{O}_{V,0}$ by

$$J(\underline{v}) = \{g \in \mathcal{O}_{V,0} : \underline{\nu}(g) \geq \underline{v}\}\ , \qquad (2)$$

where $\underline{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r_{\geq 0}$, $\underline{\nu}(g) = (\nu_1(g), \ldots, \nu_r(g))$ and $\underline{v}' = (v'_1, \ldots, v'_r) \geq \underline{v} = (v_1, \ldots, v_r)$ if and only if $v'_i \geq v_i$ for all $i = 1, \ldots, r$. Equation (2) defines the subspaces $J(\underline{v})$ for all $\underline{v} \in \mathbb{Z}^r$.

The Poincaré series of the filtration $\{J(\underline{v})\}$ (or of the collection $\{\nu_i\}$ of order functions) is defined by ([10]):

$$P_{\{\nu_i\}}(t_1, \ldots, t_r) = \frac{\mathcal{L}(t_1, \ldots, t_r) \cdot \prod_{i=1}^{r}(t_i - 1)}{t_1 \cdot \ldots \cdot t_r - 1}\,, \tag{3}$$

where

$$\mathcal{L}(t_1, \ldots, t_r) := \sum_{\underline{v} \in \mathbb{Z}^r} \dim(J(\underline{v})/J(\underline{v} + \underline{1})) \cdot \underline{t}^{\,\underline{v}}\,,$$

$\underline{1} = (1, 1, \ldots, 1) \in \mathbb{Z}^r$. This definition makes sense if and only if all the quotients $J(\underline{v})/J(\underline{v} + \underline{1})$ are finite-dimensional.

In some cases Equation (3) can be written in terms of the integral with respect to the Euler characteristic over the projectivization $\mathbb{P}\mathcal{O}_{V,0}$ of the space $\mathcal{O}_{V,0}$.

Let $\mathfrak{m}$ be the maximal ideal in $\mathcal{O}_{V,0}$ and let $J^N_{V,0} = \mathcal{O}_{V,0}/\mathfrak{m}^{N+1}$ be the space of N-jets of functions on $(V, 0)$. ($J^N_{V,0}$ is a finite dimensional vector space.) Let $\mathbb{P}\mathcal{O}_{V,0}$ and $\mathbb{P}J^N_{V,0}$ be the projectivizations of $\mathcal{O}_{V,0}$ and of $J^N_{V,0}$ respectively. (We consider $\mathbb{P}\mathcal{O}_{V,0}$ as a set, say, without any topology.) Let $\mathbb{P}^*J^N_{V,0}$ be the union of $\mathbb{P}J^N_{V,0}$ with an additional point $*$. One has the natural maps $\pi_N : \mathbb{P}\mathcal{O}_{V,0} \to \mathbb{P}^*J^N_{V,0}$ and $\pi_{N,M} : \mathbb{P}^*J^N_{V,0} \to \mathbb{P}^*J^M_{V,0}$ for $N \geq M$. (Elements which go to zero under the maps $\mathcal{O}_{V,0} \to J^N_{V,0}$ or $J^N_{V,0} \to J^M_{V,0}$ are sent to the point $*$.) Over $\mathbb{P}J^M_{V,0} \subset \mathbb{P}^*J^M_{V,0}$ the map $\pi_{N,M}$ is a locally trivial fibration whose fibre is a complex affine space.

Definition: A subset $X \subset \mathbb{P}\mathcal{O}_{V,0}$ is called *cylindric* if $X = \pi_N^{-1}(Y)$ for a constructible subset $Y \subset \mathbb{P}J^N_{V,0}$ for a certain N.

Definition: For a cylindric subset $X \subset \mathbb{P}\mathcal{O}_{V,0}$ ($X = \pi_N^{-1}(Y)$, $Y \subset \mathbb{P}J^N_{V,0}$), its *Euler characteristic* $\chi(X)$ is defined as the Euler characteristic $\chi(Y)$ of the (constructible) set Y.

One can see that the Euler characteristic of a cylindric subset is well defined.

The Euler characteristic on the algebra of cylindric subsets of $\mathbb{P}\mathcal{O}_{V,0}$ permits to define in the standard way ([16]) the notion of the integral with respect to the Euler characteristic of a function with values in an abelian group over the projectivization $\mathbb{P}\mathcal{O}_{V,0}$.

An order function $\nu : \mathcal{O}_{N,0} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ is called *finitely determined* if the condition $\nu(f) = v$ is determined by the N-jet of f for certain N (dependent on v).

It was shown that if ν_i, $i = 1, \ldots, r$, are finitely determined order functions, one has

$$P_{\{\nu_i\}}(\underline{t}) = \int_{\mathbb{P}\mathcal{O}_{V,0}} \underline{t}^{\,\underline{\nu}(f)} d\chi \tag{4}$$

(see, e.g., [6]). Equation (4) permits to compute the Poincaré series $P_{\{\nu_i\}}(\underline{t})$ for some valuations (see, e. g., [4, 5]).

3 Integration with respect to the equivariant Euler characteristic

As it was explained in Section 2, Poincaré series of multi-index filtrations are related with the notion of the integral with respect to the Euler characteristic. In a similar way equivariant (with respect to a finite group actions) analogues of the Poincaré series (see, e. g., [7, 8]) are related with integrals with respect to equivariant analogues of the Euler characteristic.

A natural equivariant (with respect to a finite group G action) analogue of the Euler characteristic is the *equivariant Euler characteristic* with values in the Burnside ring $A(G)$ of the group G. The Burnside ring $A(G)$ of G is the Grothendieck ring of finite G-sets. As an abelian group $A(G)$ is freely generated by the classes $[G/H]$ of the G-sets G/H for representatives H of the conjugacy classes $[H] \in \operatorname{Conjsub} G$ of subgroups of G. Let V be a sufficiently nice space (say, a quasi-projective variety) with an action of the group G. The equivariant Euler characteristic $\chi^G(V)$ is defined as

$$\sum_{[H]\in \operatorname{Conjsub} G} \chi\left(\{x \in V : G_x \in [H]\}/G\right)[G/H]\,,$$

where $\operatorname{Conjsub} G$ is the set of conjugacy classes of subgroups of G, $G_x = \{a \in G : ax = x\}$ is the isotropy subgroup of the point x: [15]. The equivariant Euler characteristic $\chi^G(V)$ is an additive invariant on the algebra generated by the G-subvarieties of a G-variety (that is on the algebra of G-invariant constructible subsets) and thus can be used as an analogue of a measure for the corresponding notion of integration with respect to the Euler characteristic.

Let W be a G-variety and let $\varphi : W \to \mathcal{A}$ be a G-invariant function with values in an abelian group $\mathcal{A}$. The integral of the function φ with respect to the equivariant Euler characteristic is defined by

$$\int_W \varphi\, d\chi^G = \sum_{\substack{a\in\mathcal{A}\\ [H]\in \operatorname{Conjsub} G}} a \otimes \chi^G\left(\{x \in W : \varphi(x) = a\}\right) \in \mathcal{A} \otimes_{\mathbb{Z}} A(G)\,.$$

Assume that, besides a G-action, the space V is endowed with a (constructible) function α with values in the group $\mathrm{Hom}\,(G_x, \mathbb{C}^*)$ of one-dimensional representations of the isotropy subgroup G_x ($x \in V \mapsto \alpha_x \in \mathrm{Hom}\,(G_x, \mathbb{C}^*)$) such that, for $a \in G$, one has $\alpha_{ax}(b) = \alpha_x(a^{-1}ba)$, where $b \in G_{ax} = aG_xa^{-1}$.

Example. This takes place, for example, in the following situation. Let a germ $(V, 0)$ of a complex analytic variety be endowed with an action of the group G. The group G acts on the space (a ring) $\mathcal{O}_{V,0}$ of germs of functions on $(V, 0)$, on the space $J^N_{V,0} = \mathcal{O}_{V,0}/\mathfrak{m}^{N+1}$ of N-jets of functions on $(V, 0)$ and on their projectivizations $\mathbb{P}\mathcal{O}_{V,0}$ and $\mathbb{P}J^N_{V,0}$ (the latter being a (finite dimensional) projective space). The class of a function f (in $\mathbb{P}\mathcal{O}_{V,0}$ or in $\mathbb{P}J^N_{V,0}$; we will denote it by f as well) is invariant with respect to the action of its isotropy subgroup G_f. This means that, for $a \in G_f$, one has $a^*f = \alpha f$ with $\alpha \in \mathbb{C}^*$. The factor $\alpha = \alpha_f(a)$ considered as a function of f with values in $\mathrm{Hom}\,(G_x, \mathbb{C}^*)$ possesses the described property.

In such a situation one can define a (refined) versions of the equivariant Euler characteristic with values in a modification $\widetilde{A}(G)$ of the Burnside ring and of the integral of a G-invariant $\mathcal{A}$-valued function which is an element of $\mathcal{A} \otimes_{\mathbb{Z}} \widetilde{A}(G)$.

The ring $\widetilde{A}(G)$ is the Grothendieck group of so-called finite equipped G-sets ([8]). A finite equipped G-set is a pair $\widetilde{X} = (X, \alpha)$, where X is a finite G-set, α associates to each point $x \in X$ a one-dimensional representation α_x of the isotropy subgroup $G_x = \{a \in G : ax = x\}$ of the point x so that, for $a \in G$, $b \in G_{ax} = aG_xa^{-1}$, one has $\alpha_{ax}(b) = \alpha_x(a^{-1}ba)$. (The product of (the classes of) two equipped G-sets $\widetilde{X} = (X, \alpha)$ and $\widetilde{Y} = (Y, \beta)$ is the pair $(X \times Y, \gamma)$, where $\gamma_{(x,y)}(b) = \alpha_x(b)\beta_y(b)$ for $b \in G_{(x,y)} = G_x \cap G_y$.) As an abelian group $\widetilde{A}(G)$ is freely generated by the classes of the irreducible equipped G-sets $[G/H]_\alpha$ for all the conjugacy classes $[H]$ of subgroups of G and for all conjugacy classes $[\alpha]$ of one-dimensional representations of H: a representative of the conjugacy class $[H] \in \mathrm{Conjsub}\, G$. (Two representations α and α' of the subgroup H are conjugate if there exists an element a from the normalizer $N_G(H)$ of the subgroup H such that $\alpha'(b) = \alpha(a^{-1}ba)$ for $b \in H$.) The corresponding function on G/H is defined by $\alpha_{[e]} = \alpha$, where $[e]$ is the class in G/H of the unit element e. (There was a certain inaccuracy at this place in [8] which did not influence the results of the paper.)

The *enhanced equivariant Euler characteristic* of a G-space W with a function α of the described type on it is

$$\widetilde{\chi}^G(W) = \sum_{[H] \in \mathrm{Conjsub}\, G} \chi\left(\{x \in X : G_x \in [H], \alpha_x \in [\alpha]\}/G\right)[G/H]_\alpha \in \widetilde{A}(G)\,.$$

If W is a G-variety endowed with a function α with values in $\mathrm{Hom}\,(G_x, \mathbb{C}^*)$

and $\varphi : W \to \mathcal{A}$ is a G-invariant function with values in an abelian group $\mathcal{A}$, then there is defined the *integral* of φ *with respect to the enhanced equivariant Euler characteristic* $\widetilde{\chi}^G(\cdot)$:

$$\int_W \varphi \, d\widetilde{\chi}^G = \sum_{\substack{a\in\mathcal{A} \\ [H]\in \operatorname{Conjsub} G}} \widetilde{\chi}\left(\{x \in W : \varphi(x) = a,\}\right) a[G/H]_\alpha \, .$$

It is an element of $\mathcal{A} \otimes_{\mathbb{Z}} \widetilde{A}(G)$.

In [8], there was defined an equivariant analogue of the Poincaré series of several (curve and/or divisorial) valuations on the ring $\mathcal{O}_{V,0}$ of germs of functions on a germ $(V,0)$ of a G-variety. Let ν_1, ..., ν_r be curve and/or divisorial valuations on $\mathcal{O}_{V,0}$ and let us consider functions $\widehat{\nu}_1, \ldots, \widehat{\nu}_r$ on $\mathcal{O}_{V,0}$ (with values in $\mathbb{Z} \cup \{+\infty\}$) defined by $\widehat{\nu}_i(f) = \sum\limits_{a\in G} \nu_i(a^*f)$ for $f \in \mathcal{O}_{V,0}$. These are G-invariant functions on $\mathcal{O}_{V,0}$ and on $\mathbb{P}\mathcal{O}_{V,0}$ (neither valuations, no order functions in general). As it was explained above, for each $f \in \mathbb{P}\mathcal{O}_{V,0}$ one has a one-dimensional representation of the isotropy subgroup G_f. The equivariant Poincaré series of the collection $\{\nu_i\}$ of valuations was defined as a sort of integral with respect to $\widetilde{\chi}^G(\cdot)$ over $\mathbb{P}\mathcal{O}_{V,0}$ of the function $\underline{t}^{\underline{\widehat{\nu}}(f)}$ with values in $\mathbb{Z}[[t_1, \ldots, t_r]]$. The usual notion of the integral with respect to the equivariant Euler characteristic (Section 2) cannot be applied in this situation (see the explanation in Section 5). The definition of the corresponding analogue of the notion of the integral with respect to $\widetilde{\chi}^G(\cdot)$ given in [8] was adapted for that particular case. Here we will give a general definition which works in the described situation as well.

There is a natural power structure (see [14]) over the ring $\widetilde{A}(G)$. It gives sense to expressions of the form $(1 + a_1t + a_2t^2 + \ldots)^m$, where a_i and m are elements of $\widetilde{A}(G)$. In particular, for an equipped finite G-set $\widetilde{X} = (X, \alpha)$, the coefficient at t^k in the series $(1-t)^{-[\widetilde{X}]}$ is represented by by the kth symmetric power $S^kX = X^k/S_k$ of the set X with the natural action of the group G and with the corresponding representations of the isotropy subgroups of points described in [8].

4 A revised notion of the integral

Let S_i, $i = 1, \ldots, r$, be a well-ordered semigroup ([1]) with zero and with the cancellation property. We assume that S_i is combinatorially finite, i. e., each element $a \in S_i$ has a finite number of representations as the sum $a_1 + a_2$ of two elements of S_i (see, e. g., [11]). Let $S := \bigoplus\limits_{i=1}^{r} S_i$ and let $\mathbb{Z}[[S]]$ be the ring of power series on the semigroup S, i. e., the set of formal expressions of the form $\sum\limits_{\underline{v}\in S} a_{\underline{v}}\underline{t}^{\underline{v}}$ ($a_{\underline{v}} \in \mathbb{Z}$, $\underline{v} := (v_1, \ldots, v_r)$, $\underline{t}^{\underline{v}} := t_i^{v_1} \cdots t_r^{v_r}$) with the usual ring operations.

Remark. In the definition below of the integral with respect to the Euler characteristic of a function with values in $\mathbb{Z}[[S]]$ the ring structure on $\mathbb{Z}[[S]]$ is formally not necessary. It is sufficient to consider it as an abelian group. The ring structure is used only in applications (e. g., to write an equation like (6) below). The property of $\mathbb{Z}[[S]]$ to be a ring is guaranteed by the fact that each semigroup S_i is well-ordered. Thus for the definition of the integral with respect to the Euler characteristic below it is sufficient to assume that S_i are ordered semigroups.

The ring $\mathbb{Z}[[S]]$ is filtred by the ideals $\mathcal{A}_{\underline{s}^0}$, $\underline{s}^0 \in S$, defined by

$$\mathcal{A}_{\underline{s}^0} = \left\{ \sum_{\underline{s}\in S} a_{\underline{s}}\underline{t}^{\underline{s}} : a_{\underline{s}} = 0 \text{ for } \underline{s} \not\geq \underline{s}^0 \right\} .$$

This filtration defines an obvious topology on $\mathbb{Z}[[S]]$. (For $S = \mathbb{Z}^r_{\geq 0}$, $\mathbb{Z}[[S]] = \mathbb{Z}[[t_1, \ldots, t_r]]$ and the topology under consideration is the I-adic topology, where $I = \langle t_1, \ldots, t_r \rangle$.)

Let $(V, 0)$ be a germ of a complex analytic variety and let $\mathcal{O}_{V,0}$ be the ring of germs of functions on $(V, 0)$. Let $\nu_i : \mathcal{O}_{V,0} \to S_i \cup \{+\infty\}$, $i = 1, \ldots, r$, be an *order function* on $\mathcal{O}_{V,0}$. This means that

- $\nu_i(f_1 + f_2) \geq \min(\nu_i(f_1), \nu_i(f_2))$;
- $\nu_i(\lambda f) = \nu_i(f)$ for $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Let us consider the function $\underline{t}^{\underline{\nu}(f)}$ on the projectivization $\mathbb{P}\mathcal{O}_{V,0}$ with values in $\mathbb{Z}[[S]]$; where we assume that $t_i^{+\infty} = 0$. We want to define the integral of this function with respect to the Euler characteristic (in a way different from that in [2] or [4]).

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{V,0}$ and let $J^N_{V,0} := \mathcal{O}_{V,0}/\mathfrak{m}^{N+1}$ be the space of N-jets of functions on $(V, 0)$. Let $\nu_i^N : J^N_{V,0} \to S_i \cup \{+\infty\}$ be the map (in fact an order function) defined by

$$\nu_i^N(j) = \sup_{f : j^N(f) = j} \nu_i(f) ,$$

where $j \in J^N_{V,0}$, $j^N(f)$ is the N-jet of the function f and $\sup \nu_i(f)$ is assumed to be equal to $\max \nu_i(f)$ if this maximum exists and to $+\infty$ otherwise. The function $\underline{t}^{\underline{\nu}^N(f)}$ is a constructible function on the finite-dimensional projective space $\mathbb{P}J^N_{V,0}$. Therefore the (usual) integral of it with respect to the Euler characteristic

$$\int_{\mathbb{P}J^N_{V,0}} \underline{t}^{\underline{\nu}^N(f)} d\chi \in \mathbb{Z}[[S]]$$

is defined.

Definition: The integral with respect to the Euler characteristic of the function $\underline{t}^{\underline{\nu}(f)}$ over the projectivization $\mathbb{P}\mathcal{O}_{V,0}$ of the space $\mathcal{O}_{V,0}$ is defined by

$$\int_{\mathbb{P}\mathcal{O}_{V,0}} \underline{t}^{\underline{\nu}(f)} d\chi = \lim_{N\to\infty} \int_{\mathbb{P}J^N_{V,0}} \underline{t}^{\underline{\nu}^N(f)} d\chi \in \mathbb{Z}[[S]] \qquad (5)$$

if the limit in the right hand side (with respect to the topology described above) exists.

If this limit does not exist, we regard the function $\underline{t}^{\underline{\nu}(f)}$ as a non-integrable one.

If all the order functions ν_i are finitely determined, the definition (5) of the integral with respect to the Euler characteristic coincides with the classical one and thus its value is the Poincaré series of the collection $\{\nu_i\}$ of order functions. This permits to give the following definition.

Definition: The Poincaré series $P_{\{\nu_i\}}(\underline{t})$ of a collection $\{\nu_i\}$ of order functions (not necessary finitely determined ones) is the element of $\mathbb{Z}[[S]]$ defined as the left hand side of (5) if the integral exists (i. e., if the function $\underline{t}^{\underline{\nu}(f)}$ is integrable).

5 Examples

1. For $f \in \mathcal{O}_{\mathbb{C}^2,0}$, let $\omega_x(f) := \max\{s : x^s | f\}$. The function ω_x is a valuation on $\mathcal{O}_{\mathbb{C}^2,0}$ with values in $\mathbb{Z}_{\geq 0}$. (It is not centred at the origin and is not a finitely determined one.) One can see that

$$\int_{\mathbb{P}J^N_{\mathbb{C}^2,0}} t^{\omega_x^N(f)} d\chi = (N+1) + Nt + \ldots + 2t^{N-1} + t^N \in \mathbb{Z}[[t]] \,.$$

Therefore the function $t^{\omega_x(f)}$ on $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ is not integrable (and the corresponding Poincaré series in the sense of Section 4 is not defined).

2. For $f \in \mathcal{O}_{\mathbb{C}^2,0}$, let $\omega_x(f) := \max\{s : x^s | f\}$, $\omega_y(f) := \max\{s : y^s | f\}$. The functions ω_x and ω_y are valuations on $\mathcal{O}_{\mathbb{C}^2,0}$ with values in $\mathbb{Z}_{\geq 0}$. One can show that

$$\int_{\mathbb{P}J^N_{\mathbb{C}^2,0}} t_1^{\omega_x^N(f)} t_2^{\omega_y^N(f)} d\chi = \sum_{i+j\leq N} t_1^i t_2^j \,.$$

Therefore

$$\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}} t_1^{\omega_x(f)} t_2^{\omega_y(f)} d\chi = \frac{1}{(1-t_1)(1-t_2)} \,. \tag{6}$$

Thus one has that the Poincaré series $P_{\{\omega_x,\omega_y\}}(t_1,t_2)$ of the valuations ω_x and ω_y (in the sense of Section 4) is equal to $\frac{1}{(1-t_1)(1-t_2)}$.

The valuations ω_x and ω_y define the natural graduated ring

$$A = \bigoplus_{(i,j)\in\mathbb{Z}^2_{\geq 0}} A_{ij}, \quad \text{where} \quad A_{ij} := \frac{J(i,j)/J(i+1,j)}{J(i,j+1)/J(i+1,j+1)}.$$

The Poincaré series

$$P_A(t_1,t_2) = \bigoplus_{(i,j)\in\mathbb{Z}^2_{\geq 0}} \dim A_{ij} \cdot t_1^i t_2^j$$

of the graded algebra A is equal to $\frac{1}{(1-t_1)(1-t_2)}$ and thus coincides with the Poincaré series of the valuations ω_x and ω_y defined above.

3. Let $(V,0)$ be a germ of a complex analytic space and let ν_i, $i=1,\ldots,r$, be a curve or a divisorial valuations on the ring $\mathcal{O}_{V,0}$. The valuations ν_i are finitely determined and therefore the integral (5) is equal to the Poincaré series of the collection $\{\nu_i\}$ of valuations in the sense of [10].

4. Let ν_i, $i=1,\ldots,r$, be arbitrary valuations on the ring $\mathcal{O}_{\mathbb{C}^2,0}$ of germ of functions in two variables centred at the origin (see, e. g., [9]; some of them may be not finitely determined). The arguments in the proof of Theorem 3.1 in [9] show that the integral (5) exists and the Poincaré series of the collection $\{\nu_i\}$ of valuations in the sense of Section 4 is equal to the one defined (and computed) in [9].

5. Assume that the complex analytic germ $(V,0)$ is endowed with an action of a finite group G. One has the induced action of G on the ring $\mathcal{O}_{V,0}$ and on the jet-space $J^N_{V,0}$ and the corresponding one-dimensional representation α_f of the isotropy subgroup G_f for $f\in\mathbb{P}\mathcal{O}_{V,0}$ or for $f\in J^N_{V,0}$ described in Section 3. If the order functions ν_i, $i=1,\ldots,r$, (with values in $\mathbb{Z}$) are G-invariant, one has the integrals

$$\int_{\mathbb{P}J^N_{X,0}} \underline{t}^{\underline{\nu}^N(f)} d\widetilde{\chi}^G \in \widetilde{A}(G)[[t_1,\ldots,t_r]] \tag{7}$$

with the ring $\widetilde{A}$ defined in Section 3. (In fact the value of the integrals belong to $\widetilde{A}(G)[t_1,\ldots,t_r] \subset \widetilde{A}(G)[[t_1,\ldots,t_r]]$.) A natural version of the definition (5) is the following one.

Definition: The integral with respect to the enhanced equivariant Euler characteristic of the function $\underline{t}^{\underline{\nu}(f)}$ over the projectivization $\mathbb{P}\mathcal{O}_{V,0}$ is the element of the ring $\widetilde{A}(G)[[t_1,\ldots,t_r]]$ defined by

$$\int_{\mathbb{P}\mathcal{O}_{V,0}} \underline{t}^{\,\underline{\nu}(f)} d\chi^G = \lim_{N\to\infty} \int_{\mathbb{P}J^N_{X,0}} \underline{t}^{\,\underline{\nu}^N(f)} d\widetilde{\chi}^G \tag{8}$$

if the limit in the right hand side (defined by the powers of the ideal $\langle t_1,\ldots,t_r\rangle \subset \widetilde{A}(G)[[t_1,\ldots,t_r]]$) exists.

If this integral exists it is regarded as an equivariant version of the Poincaré series of the collection $\{\nu_i\}$ of order functions.

The construction described in [8] show that if the order functions ν_1, $\ldots$, ν_r are defined by

$$\nu_i(f) := \sum_{a\in G} \upsilon_i(a^* f)\,,$$

where υ_i, $i = 1,\ldots,r$, are curve or/and divisorial valuations on $\mathcal{O}_{V,0}$, the integral defined by (8) exists and is equal to the equivariant Poincaré series $P^G_{\{\upsilon_i\}}(\underline{t})$ (as an element of the ring $\widetilde{A}(G)[[t_1,\ldots,t_r]]$) defined in [8].

6 Poincaré series of some collections of plane order functions not centred at the origin

Let h_i, $i = 1,\ldots,s$, be (non-trivial) germs of functions on $(\mathbb{C}^2, 0)$ such that:

1) each h_i has an isolated critical point at the origin, i. e., the curve $L_i = \{h_i = 0\}$ is reduced;

2) for $i \neq j$, $\gcd(h_i, h_j) = 1$, i. e., $L_i \cap L_j = \{0\}$.

Each h_i defines an order function ω_i on $\mathcal{O}_{\mathbb{C}^2,0}$ by $\omega_i(f) = \max\{s : h_i^s | f\}$.

Remark. The order function ω_i is a valuation if and only if h_i is irreducible. The order function ω_i is not centred at the origin and is not finitely determined.

Let ν_j, $j = 1,\ldots,r$, be curve or/and divisorial valuations on $\mathcal{O}_{\mathbb{C}^2,0}$ such that, if ν_j is a curve valuation and is defined by an (irreducible) curve germ $(C_j, 0) \subset (\mathbb{C}^2, 0)$, then no h_i, $i = 1,\ldots,s$, vanishes on $(C_j, 0)$, i. e., $C_j \not\subset L_i$.

Let us consider the collection $\{\omega_i, \nu_j\}$ of order functions and let $P_{\{\omega_i,\nu_j\}}(\underline{T}, \underline{t})$ be the Poincaré series of this collection in the sense of Section 4. Here $\underline{T} = (T_1,\ldots,T_s)$ are variables corresponding to the order functions ω_1, $\ldots$, ω_s,

$\underline{t} = (t_1, \ldots, t_r)$ are variables corresponding to the valuations $\nu_1, \ldots, \nu_r$. We assume that $r \geq 1$, i. e., that at least one order function from the collection is centred at the origin.

Let $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ be a resolution of the collection $\{\omega_i, \nu_j\}$. This means the following:

1) $\mathcal{X}$ is a smooth complex surface and π is a proper map, $\mathcal{D} = \pi^{-1}(0)$;

2) π is an isomorphism outside of the origin in $\mathbb{C}^2$;

3) the total transform $\pi^{-1}\left(\bigcup_{i=1}^{s} L_i \cup \bigcup_j C_j\right)$ of the union of the curves $L_i = \{h_i = 0\}$ and of the curves C_j for $j \in \{1, \ldots, r\}$ defining curve valuations ν_j is a normal crossing divisor on $\mathcal{X}$;

4) the exceptional divisor $\mathcal{D}$ (who is a normal crossing divisor on $\mathcal{X}$) contains all the (irreducible) components defining the divisorial valuations from the collection $\{\nu_j\}$.

Let $\mathcal{D} = \bigcup_{\sigma \in \Gamma} E_\sigma$ be the representation of the exceptional divisor $\mathcal{D}$ as the union of the irreducible components. (Each E_σ is isomorphic to the projective line $\mathbb{CP}^1$.) For a component E_σ of the exceptional divisor $\mathcal{D}$, $\sigma \in \Gamma$, let γ_σ be a smooth (irreducible) germ of a curve on $\mathcal{X}$ intersecting $\mathcal{D}$ at a point of E_σ and transversal to $\mathcal{D}$ at a smooth point of the total transform $\pi^{-1}\left(\bigcup_{i=1}^{s} L_i \cup \bigcup_j C_j\right)$. Let $\ell_\sigma = \pi(\gamma_\sigma) \subset (\mathbb{C}^2, 0)$ be given by an equation $g_\sigma = 0$, $g_\sigma \in \mathcal{O}_{\mathbb{C}^2,0}$. The curve germ ℓ_σ is called *a curvette* at the component E_σ. Let $m_j^\sigma := \nu_j(g_\sigma)$, $\underline{m}^\sigma := (m_1^\sigma, \ldots, m_r^\sigma)$.

The numbers m_j^σ can be also described in the following way. Let $(E_\sigma \circ E_\delta)$ be the intersection matrix of the components of $\mathcal{D}$. (The diagonal elements $E_\sigma \circ E_\sigma$ are negative; for $\sigma \neq \delta$, the intersection number $E_\sigma \circ E_\delta$ is equal to 1 if the components E_σ and E_δ intersect and is equal to 0 otherwise.) Let $(m_{\sigma\delta}) := -(E_\sigma \circ E_\delta)^{-1}$. The entries of the matrix $(m_{\sigma\delta})$ are positive integers. For $j = 1, \ldots, r$, let $E_{\sigma(j)}$ be either the divisor defining the valuation ν_j (if it is divisorial), or the component of $\mathcal{D}$ intersecting the strict transform $\widetilde{C}_j = \overline{\pi^{-1}(C_j \setminus \{0\})}$ of the curve C_j (if ν_j is a curve valuation). Then one has $m_j^\sigma = m_{\sigma\sigma(j)}$.

Let the curve $L_i = \{h_i = 0\}$ be the union of the irreducible components $L_{ik} = \{h_{i,k} = 0\}$, $k = 1, \ldots, s_i$, and let $E_{\sigma(i,k)}$ be the component of the exceptional divisor $\mathcal{D}$ intersecting the strict transform $\widetilde{L}_{ik} = \overline{\pi^{-1}(L_{ik} \setminus \{0\})}$ of the curve L_{ik}.

For $\sigma \in \Gamma$, let $\overset{\circ}{E}_\sigma$ (respectively $\overset{\bullet}{E}_\sigma$) be the "smooth part" of the component E_σ in the total transform $\pi^{-1}\left(\bigcup\limits_{i=1}^{s} L_i \cup \bigcup\limits_{j} C_j\right)$ (in $\pi^{-1}\left(\bigcup\limits_{j} C_j\right)$ respectively), i. e., the component E_σ itself minus the intersection points of E_σ with the strict transforms $\widetilde{L}_i$ and $\widetilde{C}_j$ (with the strict transforms $\widetilde{C}_j$ respectively) and with all the other components of the exceptional divisor $\mathcal{D}$. Let $\overset{\circ}{\mathcal{D}}= \bigcup_{\sigma\in\Gamma} \overset{\circ}{E}_\sigma$, $\overset{\bullet}{\mathcal{D}}= \bigcup_{\sigma\in\Gamma} \overset{\bullet}{E}_\sigma$.

Theorem 1 *One has*

$$P_{\{\omega_i,\nu_j\}}(\underline{T},\underline{t}) = \prod_{i=1}^{s} \frac{1-\underline{t}^{\underline{\nu}(h_i)}}{1-T_i\underline{t}^{\underline{\nu}(h_i)}} \prod_{\sigma\in\Gamma} \left(1-\underline{t}^{\underline{m}^\sigma}\right)^{-\chi(\overset{\bullet}{E}_\sigma)} . \tag{9}$$

Proof. We shall prove Equation (9) up to a fixed (arbitrary large) degree $\underline{V}$ of $\underline{t}$. Let us assume that the resolution π is such that, for any (non-trivial) function $g \in \mathcal{O}_{\mathbb{C}^2,0}$ with $\underline{\nu}(g) \leq \underline{V}$, the strict transform of the curve $\{g = 0\}$ intersects the exceptional divisor $\mathcal{D}$ only at points of $\overset{\bullet}{\mathcal{D}}$. Such a resolution can be obtained from any one (say, from the minimal one) by a finite number of additional blow-ups at intersection points of the components of the exceptional divisor and/or at intersection points of the exceptional divisor with the strict transforms of the curves C_j (corresponding to the curve valuations in the collection $\{\nu_j\}$). The smooth parts of the components of the exceptional divisor born under these additional blow-ups ($\overset{\circ}{E}_\sigma=\overset{\bullet}{E}_\sigma$ in these cases) have zero Euler characteristics and thus these components do not participate in the right hand side of Equation (9). Therefore it is sufficient to prove the equation (up to terms of degree $\underline{V}$) for this resolution.

Let

$$Y = \prod_{\sigma\in\Gamma}\left(\coprod_{q=0}^{\infty} S^q\, \overset{\circ}{E}_\sigma\right) = \coprod_{\{q_\sigma\}\in\mathbb{Z}_{\geq 0}^{\Gamma}} \left(\prod_{\sigma\in\Gamma} S^{q_\sigma}\, \overset{\circ}{E}_\sigma\right)$$

($S^qX = X^q/S_q$ is the qth symmetric power of the space X) be the configuration space of effective divisors on $\overset{\circ}{\mathcal{D}}$ and let

$$\widehat{Y} = \mathbb{Z}_{\geq 0}^{\sum_{i=1}^{s} s_i} \times \mathbb{Z}_{\geq 0}^{\sum_{i=1}^{s} s_i} \times Y\,.$$

We shall denote the coordinates in the first factor $\mathbb{Z}_{\geq 0}^{\sum_{i=1}^{s} s_i}$ by k_{ij} and the coordinates in the second (identical) factor by n_{ij} ($i = 1,\ldots,s$, $j = 1,\ldots,s_i$).

For a function $f \in \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ with $\underline{v}(f) \leq \underline{V}$, let $I(f) \in \widehat{Y}$ be defined by:

1) k_{ij} is the maximal power of h_{ij} which divides f;

2) n_{ij} is the intersection number of the strict transform of the zero level set of the function $f/\prod_{i,j} h_{ij}^{k_{ij}}$ with the exceptional divisor $\mathcal{D}$ at the point $P_{ij} = \widetilde{L}_{ij} \cap \mathcal{D}$;

3) the component of $I(f)$ in Y is the divisor equal to the intersection of the strict transform of the curve $\{f = 0\}$ with $\overset{\circ}{\mathcal{D}}$.

Let Ψ be the map from the configuration space $\widehat{Y}$ to $\mathbb{Z}^s_{\geq 0} \times \mathbb{Z}^r_{\geq 0}$ defined in the following way. For an element $\widehat{y} = ((k_{ij}), (n_{ij}), y) \in \widehat{Y}$ with $y \in \prod_{\sigma\in\Gamma} S^{q_\sigma} \overset{\circ}{E}_\sigma$, $\Psi(\widehat{y}) = (\underline{M}(\widehat{y}), \underline{m}(\widehat{y}))$, where $\underline{M}(\widehat{y}) = (\min_{1\leq j\leq s_1} k_{1,j}, \ldots, \min_{1\leq j\leq s_s} k_{s,j})$,

$$\underline{m}(\widehat{y}) = \sum_{\sigma\in\Gamma} q_\sigma \underline{m}^\sigma + \sum_{i,j} (k_{i,j} + n_{i,j}) \underline{m}^{\sigma(i,j)} \,.$$

One can see that for $f \in \mathcal{O}_{\mathbb{C}^2,0}$ (with $\underline{\nu}(f) \leq \underline{V}$), one has $(\underline{\omega}(f), \underline{\nu}(f)) = \Psi \circ I(f)$.

Let π_N be the natural map $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0} \to \mathbb{P}^* J^N_{\mathbb{C}^2,0} = \mathbb{P} J^N_{\mathbb{C}^2,0} \cup \{*\}$, where the functions $f \in \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ with the zero N-jet are mapped to the additional point $*$. A function f such that $I(f) = ((k_{ij}), (n_{ij}), y)$ has the form $\prod_{i,j} h_{ij}^{k_{ij}} \cdot g$, where $h_{ij} \not| \, g$. For functions f with $\underline{\nu}(f) \leq \underline{V}$ the order of the product $\prod_{i,j} h_{ij}^{k_{ij}}$ is bounded from above; say, it is $\leq R = R(\underline{V})$. One can see that the N-jet of f determines the $(N - R)$-jet of g. Together with the fact that for a function f with $\underline{v}(f) \leq \underline{V}$ the intersection of the strict transform of the curve $\{f = 0\}$ with the exceptional divisor $\mathcal{D}$ is determined by the N-jet of f with N large enough (this means that the set of functions with the fixed intersection is cylindric) this implies that (for N large enough) on the set $\mathbb{P}J^{N,\underline{V}}_{\mathbb{C}^2,0}$ of N-jets with $\underline{\nu}^N(f) \leq \underline{V}$ one has a well-defined map $I^N : \mathbb{P}J^{N,\underline{V}}_{\mathbb{C}^2,0} \to \widehat{Y}$.

Let us show that for $\overline{n} = (n_{ij}) \neq \overline{0}$ (i. e., if at least one of n_{ij} is different from zero) the preimage under I^N of a point $\widehat{y} = ((k_{ij}), (n_{ij}), y) \in \widehat{Y}$ (with $\underline{m}(\widehat{y}) \leq \underline{V}$) has the Euler characteristic equal to zero. For an effective divisor D on $\overset{\bullet}{\mathcal{D}}$, let $\mathcal{O}^{(D)}$ be the set of functions f with $\widetilde{\{f = 0\}} \cap \overset{\bullet}{\mathcal{D}} = D$, where $\widetilde{\{f = 0\}}$ is the strict transform of the curve $\{f = 0\}$. The image of $\mathcal{O}^{(D)}$ in the projectivization $\mathbb{P}J^N_{\mathbb{C}^2,0}$ of the jet-space is an affine subspace in it: [4, Proposition 2]. One has

$$(I^N)^{-1}(\widehat{y}) = \prod_{i,j} h_{ij}^{k_{ij}} \cdot \left[\pi_N \left(\mathcal{O}^{(y+\sum\limits_{i,j} n_{ij}P_{ij})} \right) \setminus \bigcup_{i',j':n_{i'j'}\neq 0} \pi_N \left(h_{i'j'} \mathcal{O}^{(y+\sum\limits_{i,j} n_{ij}P_{ij} - P_{i'j'})} \right) \right] .$$

(Pay attention that the corresponding subspace of $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ is not cylindric.) The fact that the images under π_N of all the spaces $\mathcal{O}^{(y+\sum\limits_{i,j} n_{ij}P_{ij})}$, $h_{i'j'}\mathcal{O}^{(y+\sum\limits_{i,j} n_{ij}P_{ij}-P_{i'j'})}$ and of all the intersections of the latter ones are affine spaces (non-empty for N large enough) and therefore have the Euler characteristics equal to 1 implies the statement (through the inclusion-exclusion formula).

Therefore the part of the integral (5) over

$$(I^N)^{-1} \left(\mathbb{Z}_{\geq 0}^{\sum s_i} \times \left(\mathbb{Z}_{\geq 0}^{\sum s_i} \setminus \{\overline{0}\} \right) \times Y \right)$$

is equal to zero and one has to consider only the integral over the space $(I^N)^{-1} \left(\mathbb{Z}_{\geq 0}^{\sum s_i} \times \{\overline{0}\} \times Y \right)$. Just as above the preimage under I^N of a point $\widehat{y} = \left((k_{ij}), \overline{0}, y \right) \in \widehat{Y}$ (with $\underline{m}(\widehat{y}) \leq \underline{V}$) has the Euler characteristic equal to 1. Therefore (up to terms of degree $\underline{V}$ in $\underline{t}$) one has

$$P_{\{\omega_i,\nu_j\}}(\underline{T}, \underline{t}) = \int_{\mathbb{Z}_{\geq 0}^{\sum s_i} \times \{\overline{0}\} \times Y} \underline{T}^{\underline{M}(\widehat{y})} \underline{t}^{\underline{m}(\widehat{y})} d\chi =$$

$$= \left(\sum_{(k_{ij}) \in \mathbb{Z}_{\geq 0}^{\sum s_i}} \prod_{i=1}^{s} T_i^{\min\limits_{1\leq j\leq s_i} k_{ij}} \underline{t}^{\sum\limits_j k_{ij}\underline{m}^{\sigma(i,j)}} \right) \times \left(\sum_{\{q_\sigma\} \in \mathbb{Z}_{\geq 0}^{\Gamma}} \prod_{\sigma\in\Gamma} \chi \left(S^{q_\sigma} \overset{\circ}{E} \right) \underline{t}^{\sum\limits_\sigma q_\sigma \underline{m}^\sigma} \right) .$$

According to the Macdonald equation the second factor is equal to $\prod\limits_{\sigma\in\Gamma} \left(1 - \underline{t}^{\underline{m}^\sigma} \right)^{-\chi(\overset{\circ}{E}_\sigma)}$ (see, e.g., [4]). Therefore one has

$$P_{\{\omega_i,\nu_j\}}(\underline{T}, \underline{t}) = \prod_{i=1}^{s} \left(\sum_{(k_{ij}) \in \mathbb{Z}_{\geq 0}^{s_i}} T_i^{\min\limits_{1\leq j\leq s_i} k_{ij}} \underline{t}^{\sum\limits_j k_{ij}\underline{m}^{\sigma(i,j)}} \right) \times \prod_{\sigma\in\Gamma} \left(1 - \underline{t}^{\underline{m}^\sigma} \right)^{-\chi(\overset{\circ}{E}_\sigma)} . \tag{10}$$

To compute the first factor we will use the following statement.

Lemma 1

$$\sum_{\overline{k}=(k_1,\ldots,k_p)\in\mathbb{Z}_{\geq 0}^p} T^{\min\limits_{1\leq i\leq p} k_i} \prod_{j=1}^{p} U_j^{k_j} = (1 - T \prod_{j=1}^{p} U_j)^{-1} (1 - \prod_{j=1}^{p} U_j) \prod_{j=1}^{p} (1 - U_j)^{-1}. \tag{11}$$

A proof is obtained by routine computations.

Applying (11) to (10) one gets

$$P_{\{\omega_i,\nu_j\}}(\underline{T},\underline{t}) = \prod_{i=1}^{s}\left(\left(1-T_i\underline{t}^{\sum_j \underline{m}^{\sigma(i,j)}}\right)^{-1}\times\left(1-\underline{t}^{\sum_j \underline{m}^{\sigma(i,j)}}\right)\times\right.$$
$$\left.\times\prod_{j=1}^{s_i}\left(1-\underline{t}^{\underline{m}^{\sigma(i,j)}}\right)^{-1}\right)\times\prod_{\sigma\in\Gamma}\left(1-\underline{t}^{\underline{m}^{\sigma}}\right)^{-\chi(\overset{\circ}{E}_\sigma)}.$$

One has $\sum\limits_j \underline{m}^{\sigma(i,j)} = \underline{\nu}(h_i)$. Since $\overset{\bullet}{E}_\sigma = \overset{\circ}{E}_\sigma \cup \bigcup\limits_{(i,j):\sigma(i,j)=\sigma}\{p_{ij}\}$, one has

$$\prod_{i=1}^{s}\prod_{j=1}^{s_i}\left(1-\underline{t}^{\underline{m}^{\sigma(i,j)}}\right)^{-1}\times\prod_{\sigma\in\Gamma}\left(1-\underline{t}^{\underline{m}^{\sigma}}\right)^{-\chi(\overset{\circ}{E}_\sigma)} = \prod_{\sigma\in\Gamma}\left(1-\underline{t}^{\underline{m}^{\sigma}}\right)^{-\chi(\overset{\bullet}{E}_\sigma)}.$$

This gives Equation (9). $\square$

Remark. It is not difficult to see that Equation (9) can be written as

$$P_{\{\omega_i,\nu_j\}}(\underline{T},\underline{t}) = \prod_{i=1}^{s}\frac{1-\underline{t}^{\underline{\nu}(h_i)}}{1-T_i\underline{t}^{\underline{\nu}(h_i)}}\times P_{\{\nu_j\}}(\underline{t}),$$

where $P_{\{\nu_j\}}(\underline{t})$ is the Poincaré series of the collection $\{\nu_j\}$ of (finitely determined) valuations defined in the usual way: Equation (3).

7 Equivariant Poincaré series of some collections of plane order functions not centred at the origin

Assume that (the germ of) the complex plane $(\mathbb{C}^2, 0)$ is endowed with a complex analytic action of a finite group G. (Without loss of generality we may assume that the action is linear, that is, it is induced by a representation of the group.) The group G acts on the ring $\mathcal{O}_{\mathbb{C}^2,0}$ as well.

Let $(L_i, 0) = \{h_i = 0\}$, $i = 1, \ldots, s$, be irreducible germs of curves on $(\mathbb{C}^2, 0)$ such that for $i \neq j$ and for any $a \in G$ the curve germs L_i and aL_j do not coincide. The curves $(L_i, 0)$ (or the function germs h_i) define valuations ω_i on $\mathcal{O}_{\mathbb{C}^2,0}$ by $\omega_i(f) = \max\{s : h_i^s | f\}$. These valuations are not centred at the origin and are not finitely determined.

Let ν_j, $j = 1, \ldots, r$, be curve or/and divisorial valuations on $\mathcal{O}_{\mathbb{C}^2,0}$ such that, if ν_j is a curve valuation and is defined by an (irreducible) curve germ

$(C_j, 0) \subset (\mathbb{C}^2, 0)$, then, for any $i = 1, \ldots, s$ and for any $a \in G$, the function h_i does not vanish on $(aC_j, 0)$, i. e., $aC_j \neq L_i$.

Let $\widehat{\omega}_i$ and $\widehat{\nu}_j$ be the functions $\mathcal{O}_{V,0} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ defined by $\widehat{\omega}_i = \sum_{a \in G} a^* \omega_i$ and $\widehat{\nu}_j = \sum_{a \in G} a^* \nu_j$ respectively. The functions $\widehat{w}_i$ and $\widehat{\nu}_j$ are G-invariant order functions on $\mathcal{O}_{V,0}$.

Remark. In the setting of Section 6 the corresponding functions $\widehat{\omega}_i$ would not be, in general, order functions since the curves $(L_i, 0)$ were not assumed to be irreducible.

As usual, for an element $f \in \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ (of the projectivization of the ring of germs of functions), there is defined a representation α_f of the isotropy subgroup G_f of the element f.

Definition: The equivariant Poincaré series of the collection $\{\omega_i, \nu_j\}$ is defined by

$$P^G_{\{\omega_i,\nu_j\}}(\underline{T}, \underline{t}) = \int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}} \underline{T}^{\widehat{\underline{\omega}}(f)} \underline{t}^{\widehat{\underline{\nu}}(f)} d\widetilde{\chi}^G, \tag{12}$$

where $\underline{T} = (T_1, \ldots, T_s)$, $\underline{t} = (t_1, \ldots, t_r)$, the integral in (12) is defined as

$$\lim_{N \to \infty} \int_{\mathbb{P}J^N_{\mathbb{C}^2,0}} \underline{T}^{\widehat{\underline{\omega}}^N(f)} \underline{t}^{\widehat{\underline{\nu}}^N(f)} d\widetilde{\chi}^G \in \widetilde{A}(G)[[\underline{T}, \underline{t}]] .$$

Let $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ be an equivariant resolution of the collection $\{\omega_i, \nu_j\}$. This means that:

1) $\mathcal{X}$ is a smooth complex surface with a G-action and π is a proper G-equivariant map, $\mathcal{D} = \pi^{-1}(0)$;

2) π is an isomorphism outside of the origin in $\mathbb{C}^2$;

3) the total transform

$$\pi^{-1} \left(\bigcup_{i=1}^{s} \bigcup_{a \in G} aL_i \cup \bigcup_{j} \bigcup_{a \in G} aC_j \right) \tag{13}$$

(where $L_i = \{h_i = 0\}$, the union "$\bigcup_j$" is over those $j \in \{1, \ldots, r\}$ for which ν_j is a curve valuation defined by the (irreducible) curve C_j) is a normal crossing divisor on $\mathcal{X}$;

4) the exceptional divisor $\mathcal{D}$ (who is a normal crossing divisor on $\mathcal{X}$) contains all the (irreducible) components defining the divisorial valuations from the collection ν_j.

Let $\overset{\circ}{\mathcal{D}}$ be the "smooth part" of the exceptional divisor $\mathcal{D}$ in the total transform (13), i. e., $\mathcal{D}$ itself without all the intersection points of its components and all the intersection points with the strict transforms of the curves GL_i and GC_j. Let P_i be the point $\widetilde{L}_i \cap \mathcal{D}$, where $\widetilde{L}_i = \overline{\pi^{-1}(L_i \setminus \{0\})}$ is the strict transform of the curve L_i, $i = 1, \ldots, s$, and let $\overset{\bullet}{\mathcal{D}} = \overset{\circ}{\mathcal{D}} \cup \bigcup_i \{P_i\}$ be the smooth part of the exceptional divisor $\mathcal{D}$ in the total transform $\pi^{-1}\left(\bigcup_j \bigcup_{a\in G} aC_j\right)$ of the curve $\bigcup_j GC_j$. For a point $x \in \overset{\bullet}{\mathcal{D}}$, let ℓ_x be a curvette at the point x invariant with respect to the isotropy subgroup G_x of the point x. Here $\ell_x = \pi(\gamma_x)$, where γ_x is a smooth G_x-invariant germ of a curve transversal to $\mathcal{D}$ at the point x. We can assume that ℓ_x is given by an equation $g_x = 0$, where g_x is a G_x-equivariant function germ, i. e., $a^* g_x = \alpha_x(a)$ for $a \in G_x$. (In other words the class of g_x in $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ is G_x-invariant.) It is possible to assume that initially the germs h_i are such that for $x = P_i$ one can take $g_x = h_i$.

Let $\{\widehat{\Xi}\}$ be a stratification of the smooth curve $\widehat{\mathcal{D}} = \overset{\circ}{\mathcal{D}}/G$ such that:

1) each stratum $\widehat{\Xi}$ is connected;

2) for each point x from the pre-image $\Xi = p^{-1}(\widehat{\Xi})$ (p is the quotient map $\overset{\circ}{\mathcal{D}} \to \widehat{\mathcal{D}}$), the conjugacy class of the isotropy subgroup G_x of the point x is the same, i. e., depends only on the stratum $\widehat{\Xi}$.

The latter is equivalent to say that the quotient map $p : \overset{\circ}{\mathcal{D}} \to \widehat{\mathcal{D}}$ is a (non-ramified) covering over each stratum $\widehat{\Xi}$.

For a point $x \in \overset{\bullet}{\mathcal{D}}$, let $\widetilde{X}_x = \widetilde{X}_{h_x}$ be the finite equipped G-set defined as the orbit of h_x in $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$, where h_x is the G_x-equivariant function defining the chosen curvette at the point x with the corresponding representation α_{h_x} (see above). For a fixed stratum $\widehat{\Xi}$, the class $[\widetilde{X}_x] \in \widetilde{A}(G)$ is one and the same for all $x \in \Xi$ and therefore it defines an element $[\widetilde{X}_{\widehat{\Xi}}] \in \widetilde{A}(G)$.

Let

$$\underline{\widehat{m}}^x := \underline{\nu}\left(\prod_{a\in G} h_{ax}\right), \quad \underline{\widehat{M}}^x := \underline{\omega}\left(\prod_{a\in G} h_{ax}\right) \quad (x \in \overset{\bullet}{\mathcal{D}}).$$

For a fixed $\widehat{\Xi}$, the elements $\underline{\widehat{m}}^x \in \mathbb{Z}^r_{\geq 0}$ and $\underline{\widehat{M}}^x \in \mathbb{Z}^s_{\geq 0}$ are the same for all $x \in \Xi$ (in fact they depend only on the component E_σ containing x) and therefore they define elements $\underline{\widehat{m}}^{\widehat{\Xi}}$ and $\underline{\widehat{M}}^{\widehat{\Xi}}$. Let $\underline{\widehat{m}}(i) := \underline{\widehat{m}}^{P_i}$.

Theorem 2 *One has*

$$P_{\{\omega_i,\nu_j\}}(\underline{T},\underline{t}) = \prod_{i=1}^{s} \left(1 - T_i \underline{t}^{\underline{\widehat{m}}(i)}\right)^{-[\widetilde{X}_{P_i}]} \times \prod_{\Xi} \left(1 - \underline{t}^{\underline{\widehat{m}}^{\Xi}}\right)^{-\chi(\widehat{\Xi})[\widetilde{X}_{\Xi}]}, \tag{14}$$

where the exponents of the binomials are understood in the sense of the power structure over the ring $\widetilde{A}(G)$.

Proof. As for Theorem 1, we shall prove Equation (14) up to a fixed (arbitrary large) degree $\underline{V}$ of $\underline{t}$. Let us assume that the (equivariant) resolution π is such that, for any function $g \in \mathcal{O}_{\mathbb{C}^2,0}$ with $\underline{\nu}(g) \le \underline{V}$, the strict transform of the curve $\{g = 0\}$ intersects the exceptional divisor $\mathcal{D}$ only at points of $\overset{\bullet}{\mathcal{D}}$. It is sufficient to prove Equation (14) (up to terms of degree $\underline{V}$) for this resolution.

Let

$$Y = \prod_{\Xi} \left(\bigsqcup_{q=0}^{\infty} S^q \Xi \right) = \bigsqcup_{\{q_\Xi\} \in \mathbb{Z}_{\ge 0}^{\{\Xi\}}} \left(\prod_{\Xi} S^{q_\Xi} \Xi \right)$$

be the configuration space of effective divisors on $\overset{\circ}{\mathcal{D}}$. Each component $\prod_\Xi S^{q_\Xi}\Xi$ of the space Y is an equipped G-variety. Let

$$\widehat{Y} = \left(\prod_{i=1}^{s} \left(\bigsqcup_{q=0}^{\infty} S^q X_{P_i} \right) \right) \times \left(\prod_{i=1}^{s} \left(\bigsqcup_{q=0}^{\infty} S^q X_{P_i} \right) \right) \times Y\,.$$

Each component

$$\left(\prod_{i=1}^{s} S^{q_i} X_{P_i} \right) \times \left(\prod_{i=1}^{s} S^{q'_i} X_{P_i} \right) \times \left(\prod_{\Xi} S^{q_\Xi} \Xi \right)$$

of $\widehat{Y}$ is a finite equipped G-set. Let p_1, p_2, and p_3 be the projections of $\widehat{Y}$ to the first factor $\prod_{i=1}^{s} S^{q_i} X_{P_i}$, to the second factor $\prod_{i=1}^{s} S^{q'_i} X_{P_i}$, and to the third factor Y respectively.

For a function $f \in \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ (with $\underline{\nu}(f) \le \underline{V}$), let $I(f) \in \widehat{Y}$ be defined in the following way:

1) $p_1(I(f)) = \sum\limits_{i=1}^{s} \sum\limits_{a \in G} k_{ia} a^* h_{p_i}$, where k_{ia} is the maximal power of $a^* h_i$ which divides f;

2) $p_2(I(f)) = \sum\limits_{i=1}^{s} \sum\limits_{a \in G} n_{ia} a^* h_{p_i}$, where n_{ia} is the the intersection number of the strict transform of the zero level set of the function $f/(a^* h_i)^{k_{ia}}$ with the exceptional divisor $\mathcal{D}$ at the point aP_i;

3) $p_3(I(f))$ is the divisor on $\overset{\circ}{\mathcal{D}}$ equal to the intersection of the strict transform of the curve $\{f = 0\}$ with $\overset{\circ}{\mathcal{D}}$.

Let Ψ be the map from $\widehat{Y}$ to $\mathbb{Z}^s_{\geq 0} \times \mathbb{Z}^r_{\geq 0}$ defined in the following way: for $\widehat{y} = \left(\sum\limits_{i=1}^{s} \sum\limits_{a\in G} k_{ia} a^* h_{p_i}, \sum\limits_{i=1}^{s} \sum\limits_{a\in G} n_{ia} a^* h_{p_i}, y\right)$, $y \in \prod\limits_{\Xi} S^{q_\Xi}\Xi$, one has $\Psi(\widehat{y}) = (\underline{M}(\widehat{y}), \underline{m}(\widehat{y}))$, where $\underline{M}(\widehat{y}) = \left(\sum\limits_{a\in G} k_{1a}, \ldots, \sum\limits_{a\in G} k_{sa}\right)$, $\underline{m}(\widehat{y}) = \sum_\Xi q_\Xi \cdot \underline{\widehat{m}}^\Xi + \sum_{i=1}^s (k_{ia}+n_{ia})\underline{\widehat{m}}^{P_i}$. For $f \in \mathcal{O}_{\mathbb{C}^2,0}$ (with $\underline{v}(f) \leq \underline{V}$), one has $(\underline{\widehat{\omega}}(f), \underline{\widehat{\nu}}(f)) = \Psi \circ I(f)$.

The arguments of the proof of Theorem 1 give

$$P^G_{\{\omega_i,\nu_j\}}(\underline{T},\underline{t}) = \int_{\left(\prod_{i=1}^s\left(\bigsqcup_{q=0}^\infty S^q X_{P_i}\right)\right)\times\{\overline{0}\}\times Y} \underline{T}^{\underline{M}(\widehat{y})} \underline{t}^{\underline{m}(\widehat{y})} d\widetilde{\chi}^G. \tag{15}$$

It is not difficult to compute the latter integral. One has

$$\begin{aligned}
P^G_{\{\omega_i,\nu_j\}}(\underline{T},\underline{t}) &= \int_{\left(\prod_{i=1}^s\left(\bigsqcup_{q=0}^\infty S^q X_{P_i}\right)\right)\times\{\overline{0}\}\times Y} \underline{T}^{\underline{M}(\widehat{y})} \underline{t}^{\underline{m}(\widehat{y})} d\widetilde{\chi}^G \\
&= \left(\sum_{\{k_i\}\in\mathbb{Z}^s_{\geq 0}} \prod_{i=1}^s [S^{k_i}\widetilde{X}_{P_i}] T_i^{k_i} \underline{t}^{\sum k_i \underline{\widehat{m}}(i)}\right) \times \left(\sum_{q_\Xi\in\mathbb{Z}^{\{\Xi\}}_{ge0}} \prod_\Xi [S^{q_\Xi}\Xi] \underline{t}^{\sum_\Xi q_\Xi \underline{\widehat{m}}^\Xi}\right) \\
&= \left(\prod_{i=1}^s \sum_{k=0}^\infty [S^k \widetilde{X}_{P_i}] T_i^k \underline{t}^{k\underline{\widehat{m}}(i)}\right) \times \left(\prod_\Xi \sum_{q=0}^\infty [S^q \Xi] \underline{t}^{q\underline{\widehat{m}}^\Xi}\right).
\end{aligned}$$

Using the Macdonald formula one gets

$$\begin{aligned}
P^G_{\{\omega_i,\nu_j\}}(\underline{T},\underline{t}) &= \prod_{i=1}^s \left(1 - T_i \underline{t}^{\underline{\widehat{m}}(i)}\right)^{[\widetilde{X}_{P_i}]} \times \prod_\Xi \left(1 - \underline{t}^{\underline{\widehat{m}}^\Xi}\right)^{[\Xi]} \\
&= \prod_{i=1}^s \left(1 - T_i \underline{t}^{\underline{\widehat{m}}(i)}\right)^{[\widetilde{X}_{P_i}]} \times \prod_\Xi \left(1 - \underline{t}^{\underline{\widehat{m}}^\Xi}\right)^{\chi(\widehat{\Xi})[\widetilde{X}_\Xi]}.
\end{aligned}$$

□

Remark. The end of the proof of Theorem 2 is somewhat shorter than that of Theorem 1 since here the curves L_i are assumed to be irreducible. If, in the equivariant setting, these curves are not irreducible, to compute the Poincaré series one needs an equivariant analogue of Lemma 1. However this analogue is not clear. (Moreover, in some sense it does not exist: the corresponding infinite sum cannot be expressed as a finite product of polynomials (say, binomials) with exponents from the ring $\widetilde{A}(G)$ understood in the sense of the power structure over $\widetilde{A}(G)$.)

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