Mathematical Foundations of Time Asymmetric Quantum Mechanics

M Gadella

Departamento de Física Teórica, Atómica y Optica. Facultad de Ciencias, Universidad de Valladolid, Paseo Belén 7. 47011 Valladolid, Spain

E-mail: `manuelgadella1@gmail.com`

Abstract. We review the mathematical tools that are suitable for a formulation of time asymmetry in quantum mechanics. In particular, Hardy functions on a half plane and rigged Hilbert spaces constructed with a subclass of Hardy functions. This time asymmetry often appears in quantum scattering and, in particular, in resonance scattering. We review the construction of Gamow vectors, often considered Gamow states for resonances. A brief summary of the fundamental ideas of time asymmetric quantum mechanics is presented in a last section.

1. Introduction

In the framework of standard non-relativistic quantum mechanics, the time evolution governed by a self adjoint Hamiltonian H is given by a group of unitary operators on a Hilbert space, depending on the parameter time t. In this case, t may reach all possible real values, i.e., $-\infty < t < \infty$. However, not all quantum dynamical processes are governed by a single Hamiltonian. Let us think in scattering, which intuitively and roughly speaking works as follows: Imagine a quantum particle. In the remote past, its quantum state is prepared as free. This means that its time evolution is given by some sort of free Hamiltonian H_0. At some spatial localization, the particle enters into an interaction region, where is subject of some forces that we shall assume that come from the existence of a potential V. Then, inside the interaction region, time evolution of its state is determined by a Hamiltonian $H = H_0 + V$, which will transform this state. Eventually, the particle will abandon the interaction region and evolve again with H_0 to be detected in the far future. In this case, time evolution is given by a Hamiltonian pair $\{H_0, H\}$. Although it is not necessary, we gain in intuition if we think on the interaction region as a bounded domain in space.

A particularly interesting situation will occur when the particle spends in the interaction a time which is much larger than the time it would stay if the interaction would be switched off. In this case, we say that a *metastable state* or *quantum resonance* has been produced. Thus, we may think on resonances as the result of a capture of some particle by a center of forces and its release or emission. The whole process is called *resonance scattering*, while the process of emission is the *decay*.

In atomic or nuclear physics, it is obvious the existence of quantum unstable states. A quantum particle (electron, alpha, etc.) is emitted spontaneously from an atom or nucleus and this is the observed situation. We may describe the situation in terms of resonance scattering, in

which we ignore the capture process and identify the unstable state with the quantum resonance. Here, we are solely interested in the process of decay.

In fact, capture and decay are usually quite different process, which require different conditions. While the capture requires some conditions in the preparation of the incoming state, the emission or decay is spontaneous. Therefore, capture (or preparation of the metastable state) and decay are not mutually symmetric and they are not time reversal of each other.

This type of asymmetry can also be observed in other quantum scattering processes. There are quantum scattering processes, which are very common but instead their time reversal processes are very improbable. A typical example has been discussed in [1].

All these situations give an idea of time asymmetry in quantum mechanics. Related notions are *quantum irreversibility*, which describes the non-invariance with respect to the time reversal operation and the existence of a *time arrow*, which distinguishes between past and future in a unique way [2].

Since we usually observe decay and not capture, we need a formalism that describes the decay of quantum unstable particles or resonances. A comment is in order here. In the past physicists have distinguished between resonances and decaying states [3]. Resonances were characterized by a bump in the cross section with width Γ, which was the measurable quantity. Then, a lifetime can be defined as $\tau := \hbar/\Gamma$. Decaying states were defined by its mean life. Measurements of mean life and cross section are independent, so that resonances and decaying states could not be fully identified until precision in both types of measurements were good enough to establish the accuracy of the formula $\tau = \hbar/\Gamma$. In the relativistic case this is not possible in general since the mean lives of decaying particles are times below the minimal time interval which is possible to measure. Yet, lifetimes of relativistic particles are often determined after the measurement of the cross section by the above formula $\tau := \hbar/\Gamma$. It is sometimes claimed that this relation can be fixed in the context of time asymmetric quantum mechanics [4].

Resonances may also be reasonably modeled by an interaction between a bound state and an external field, as is in the celebrated Friedrichs model [5–7]. Yet, the Friedrichs model may be described as a resonance scattering, in which scattering matrix, Møller wave operators, etc. are well defined [8].

Along this paper, we are going to review a mathematical formalism for the theory of quantum resonances, which are typical irreversible quantum processes, based in the use of Hardy functions on a half plane. Hardy functions are complex analytic functions, which properties we review in the sequel. *Hardy functions seem to be the correct mathematical tool to describe irreversibility in Quantum mechanics.* For other descriptions, which do not exclude Hardy functions, see [9, 10].

2. Rigged Hilbert spaces

A rigged Hilbert space or Gelfand triplet is a tern of vector spaces [11–15]

$$\Phi \subset \mathcal{H} \subset \Phi^\times \,, \tag{1}$$

where:

i) $\mathcal{H}$ is an infinite dimensional (separable) Hilbert space.

ii) Φ is a *dense* subspace of $\mathcal{H}$. Dense means that for any vector $\varphi \in \mathcal{H}$, there is a sequence of vectors, $\{\varphi_n\}$ in Φ such that $\varphi_n \longmapsto \varphi$ in terms of the norm topology on $\mathcal{H}$. This means that any $\varphi \in \mathcal{H}$ can be approached by vectors in Φ with arbitrary precision. In addition, Φ has its own topology, which is *stronger* than the topology Φ inherits from $\mathcal{H}$. This means, in particular, that all convergent sequences in Φ are also convergent sequences in $\mathcal{H}$, but the converse is not true.

iii) In order to define $\Phi^\times$, let us consider the set of mappings $F : \Phi \longmapsto \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers, such that: a) F is antilinear on Φ, i.e., for any $\varphi, \psi \in \Phi$ and any

pair $\alpha, \beta \in \mathbb{C}$, we have that $F(\alpha\varphi + \beta\psi) = \alpha^* F(\varphi) + \beta^* F(\psi)$, where the star denotes complex conjugation, and b) F is continuous, which in particular means that if $\varphi_n \longmapsto \varphi$ in Φ, then, $F(\varphi_n) \longmapsto F(\varphi)$ in $\mathbb{C}$. These mappings form a linear space that we denote as $\Phi^\times$. $F(\varphi)$ is the complex number resulting of applying F to the vector φ. For our purposes, we should use the Dirac notation: $F(\varphi) = \langle \varphi | F \rangle$.

Then, to any $\varphi \in \mathcal{H}$, we may associate a unique $F_\varphi \in \Phi^\times$, defined as $\langle \psi | F_\varphi \rangle := \langle \psi | \varphi \rangle$, for all $\psi \in \Phi$. After identification of F_φ with φ (that mathematicians often call an *abus de langage*, like that, in French), we conclude that $\mathcal{H} \subset \Phi^\times$. We do not want to enter in the discussion of the possible topologies on $\Phi^\times$.

Rigged Hilbert spaces have been used to (among other purposes):

i) Give a rigorous setting to the Dirac formulation of Quantum Mechanics [12, 14–16].

ii) Give a precise meaning to Gamow vectors or vector states for resonances [17–19].

iii) With the use of rigged Hilbert space of Hardy functions, provide a mathematical support for the time asymmetric quantum mechanics [20–23].

Let us briefly discuss some features related with rigged Hilbert spaces (RHS). Let A be a linear operator on $\mathcal{H}$ reduced by Φ. This means that for any $\varphi \in \Phi$, we have that $A\varphi \in \Phi$, or $A\Phi \subset \Phi$, i.e., Φ is *invariant* under the action of A. Then, A may be extended to a linear operator on $\Phi^\times$ by means of the *duality formula*:

$$\langle A\varphi | F \rangle = \langle \varphi | AF \rangle, \qquad \forall \varphi \in \Phi, \quad \forall F \in \Phi^\times. \tag{2}$$

It has been proved that if A is a self adjoint operator on an infinite dimensional (separable) Hilbert space $\mathcal{H}$, there is always a dense subspace Φ, in the domain of A (the space of vectors in which A acts) such that Φ reduces A [13]. This is one of the ingredients towards a rigorous Dirac formulation of quantum mechanics.

From a RHS, we may construct infinitely many others which are, in some sense, equivalent to the original one. Let U be an arbitrary unitary operator on $\mathcal{H}$, or furthermore, U might be a unitary operator between to Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$, i.e., $U : \mathcal{H} \longmapsto \mathcal{G}$. We know that unitary mappings transport topological properties from $\mathcal{H}$ into $\mathcal{G}$. Moreover, there are some other important properties which are preserved relative to operators. In particular, if A is self adjoint in $\mathcal{H}$, its transformed by U on $\mathcal{G}$, UAU^{-1} is self adjoint on $\mathcal{G}$. Thus, if we have a RHS as in (1) and $U : \mathcal{H} \longmapsto \mathcal{G}$ is unitary, the triplet

$$U\Phi \subset \mathcal{G} \subset (U\Phi)^\times \tag{3}$$

is a new RHS with the same properties than the original one. The topology on Φ is transported to $U\Phi$ by U, so that Φ and $U\Phi$ have the same topological properties [18].

3. Hardy functions on a half plane

Let us consider the open upper half plane, $\mathbb{C}^+$, of the complex plane, defined as the set of complex numbers with positive imaginary axis, i.e., $\mathbb{C}^+ := \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$. By $\mathbb{C}$ we always mean the field of complex numbers. A *Hardy function* [18, 19, 24–26] $f(z)$ on the upper half plane is a complex analytic function on $\mathbb{C}^+$ such that

$$\sup_{\alpha > 0} \int_{-\infty}^{\infty} |f(x + i\alpha)|^2 \, dx < K < \infty. \tag{4}$$

This means that for any positive value of α, the function $f(x + i\alpha)$ is square integrable and that all integrals in (4) for all values of $\alpha > 0$ are bounded by the same finite constant K. One

important consequence is that the function on the real line $\mathbb{R}$ defined as $f(x) := \lim_{\alpha \to 0} f(x+i\alpha)$ is also square integrable and

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < K. \tag{5}$$

It is customary to call $f(x)$ the function of the boundary values of $f(z)$. The function $f(x)$ is *uniquely determined* almost elsewhere, which means that it may not be defined on a set of zero Lebesgue measure. The converse of this result is true: *if we know* that a square integrable function on the real line, $f(x)$, gives the boundary values of a Hardy function on the upper half plane, $f(z)$, then we can recover all values of $f(z)$ for $z \in \mathbb{C}^+$. Due to a theorem by Titchmarsh [24, 25], for all $z \in \mathbb{C}^+$, we have that

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x) \, dx}{x - z}, \tag{6}$$

where $f(x)$ is the boundary function of $f(x)$. The proof of this result relies in the Cauchy theorem.

Let us call $\mathcal{H}_+^2$ the set of Hardy functions on the upper half plane. These functions have the following properties:

i) We may identify any function $f(z)$ in $\mathcal{H}_+^2$ with its boundary function $f(x)$. Since $f(x)$ is square integrable, then $f(x) \in L^2(\mathbb{R})$, where $L^2(\mathbb{R})$ is the Hilbert space of all complex square integrable functions on the real line. Thus, $\mathcal{H}_+^2 \subset L^2(\mathbb{R})$.

ii) $\mathcal{H}_+^2$ is a linear space, which is a subspace of $L^2(\mathbb{R})$.

iii) According to the *Titchmarsh theorem*, we may recover the values of a function in $\mathcal{H}_+^2$ from its boundary function, *provided that we know that it is the boundary function of a Hardy function on the upper half plane*. Then, how we can say that a square integrable function is the boundary function of one in $\mathcal{H}_+^2$? The answer was given by Paley and Wiener in a celebrated result [24–26]: The Fourier transform of a square integrable function $f(x)$ on the real line is given by

$$\mathcal{F}(f) := \widehat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{-ikx} \, dx \tag{7}$$

and it is also a square integrable function with the same norm. We say that $f(x)$ is supported on an interval Δ of the real line, either finite or infinite, if its is zero outside Δ.

The *Paley-Wiener theorem* establishes that $f(x) \in \mathcal{H}_+^2$ if and only if $f(x)$ is the Fourier transform of a function supported on the negative semi axis $\mathbb{R}^- \equiv (-\infty, 0]$.

Thus, in order to recognize whether a function, $f(x)$, is in $\mathcal{H}_+^2$, we take its inverse Fourier transform. If the function vanish outside $\mathbb{R}^-$, then it is in $\mathcal{H}_+^2$, otherwise it is not.

iv) The Paley-Wiener theorem is a quite useful tool to construct *all* functions in $\mathcal{H}_+^2$. We only need to consider the space $L^2(\mathbb{R}^-)$ of all square integrable functions on $\mathbb{R}^-$ and take their Fourier transforms. Then, we write that

$$\mathcal{F}[L^2(\mathbb{R}^-)] = \mathcal{H}_+^2. \tag{8}$$

The Fourier transform is a unitary operation on a Hilbert space. Since $L^2(\mathbb{R}^-)$ is a subspace of $L^2(\mathbb{R})$, which is also a Hilbert space, then $\mathcal{H}_+^2$ is a Hilbert subspace of $L^2(\mathbb{R})$.

v) The values of a Hardy function on the upper half plane not only may be recovered by its boundary function on the real line, but also from its boundary values on the positive semi axis $\mathbb{R}^+ \equiv [0, \infty)$ as shown by a result by van Winter [27]. This discussion requires the Mellin transform. Let $f(x)$ be a square integrable function on $\mathbb{R}^+$. Its Mellin transform is given by

$$\mathcal{M}(f)(s) = f_M(s) := \frac{1}{(2\pi)^{1/2}} \int_0^{\infty} f(x) \, x^{is-1/2} \, dx, \qquad s \in \mathbb{R}. \tag{9}$$

The Mellin transform of a square integrable function is also square integrable. The van Winter theorem states that a function $f(x) \in L^2(\mathbb{R}^+)$ is the boundary function on the positive semi axis $\mathbb{R}^+$ of a Hardy function on the upper half plane if and only if its Mellin transform $f_M(s)$ satisfies

$$\int_{-\infty}^{\infty} (1 + e^{2\pi s}) |f_M(s)|^2 \, ds < \infty \,. \tag{10}$$

The values of $f(x)$ for $z = re^{i\theta}$, $0 < r \leq \infty$ and $0 < \theta \leq 2\pi$ are given by:

$$f(z) = \frac{1}{(2\pi)^{1/2}} \int_{\infty}^{\infty} f_M(s) \, (re^{i\theta})^{-is-1/2} \, ds \,. \tag{11}$$

Hardy functions in the lower half plane $\mathbb{C}^- := \{ z \in \mathbb{C} \ | \ \operatorname{Im} z < 0 \}$ are defined analogously. The space of Hardy functions in the lower half plane is denoted by $\mathcal{H}_-^2$. Functions in $\mathcal{H}_-^2$ have similar properties than functions in $\mathcal{H}_+^2$ with some minor differences. In particular, (6) needs of a minus sign right after the equal sign and in iii.) and iv.) we have to replace Fourier transforms supported in the negative semi axis by Fourier transforms of functions on the positive semi axis. There are some additional properties concerning functions in $\mathcal{H}_\pm^2$:

vi) The spaces $\mathcal{H}_\pm^2$ are subspaces of $L^2(\mathbb{R})$, which are also Hilbert spaces. Moreover, as a consequence of the Paley-Wiener theorem, each function in $\mathcal{H}_\pm^2$ is orthogonal to each function in $\mathcal{H}_\mp^2$:

$$\int_{-\infty}^{\infty} f_\pm^*(x) \, g_\mp(x) \, dx = 0 \,, \qquad f_\pm(x) \in \mathcal{H}_\pm^2 \,, \quad g_\mp(x) \in \mathcal{H}_\mp^2 \,. \tag{12}$$

In addition:

$$L^2(\mathbb{R}) = \mathcal{H}_+^2 \oplus \mathcal{H}_-^2 \,, \tag{13}$$

where the sign $\oplus$ means orthogonal direct sum.

vii) The complex conjugate $f^*(x)$ of a function $f(x) \in \mathcal{H}_\pm^2$ is in $\mathcal{H}_\mp^2$, $f^*(x) \in \mathcal{H}_\mp^2$. Furthermore, $[f(z^*)]^* = f(z)$, where the star always denotes complex conjugation.

viii.) Any function $f(z)$ in $\mathcal{H}_\pm^2$ has the following asymptotic behavior for large values of $|z|$:

$$|f(z)| \approx |z|^{-1/2} \,. \tag{14}$$

These are the most relevant properties of Hardy functions on a half plane. Next, we are going to refine the space of Hardy functions.

3.1. Smooth Hardy functions

The Schwartz space $\mathcal{S}$ is the set of all complex functions of a real variable, $f(x)$ satisfying the following properties:

i) Any function $f(x) \in \mathcal{S}$ is continuous and has continuous derivatives of any order at all points.

ii) Any function $f(x) \in \mathcal{S}$, as well as any of its derivatives, goes to zero at the infinity faster than the inverse of any polynomial. i.e.,

$$\lim_{|x| \to \infty} x^n \frac{d^m f(x)}{dx^m} = 0 \,, \qquad n, m = 0, 1, 2, \dots . \tag{15}$$

Functions in $\mathcal{S}$ are called the *Schwartz functions* and have the following properties [28]:

i) $\mathcal{S}$ is a vector space over the field of complex numbers.

ii) Any function in $\mathcal{S}$ is square integrable, so that $\mathcal{S} \subset L^2(\mathbb{R})$. Furthermore, $\mathcal{S}$ is dense in $L^2(\mathbb{R})$.

iii) $\mathcal{S}$ can be endowed with a metric (there exists a distance between vectors) topology, which is *stronger* than the norm topology inherited from $L^2(\mathbb{R})$. This means that the triplet of spaces

$$\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times \tag{16}$$

is a RHS. The space $\mathcal{S}^\times$ is equivalent to the space of tempered distributions, with the only (very minor) difference that elements on $\mathcal{S}^\times$ are *antilinear* mappings on $\mathcal{S}$, while tempered distributions are *linear* mappings on $\mathcal{S}$.

iv) Let $[a, b]$ any interval in the real line $\mathbb{R}$, either finite or infinite. Let $\mathcal{S}[a, b]$ the space of functions in $\mathcal{S}$ *supported* on $[a, b]$. One may prove that

$$S[a, b] \subset L^2[a, b] \subset S^\times[a, b], \tag{17}$$

where $L^2[a, b]$ is the Hilbert space of complex square integrable functions on the interval $[a, b]$, is again a RHS. Recall that $[a, b]$ could be either finite or infinite, as for instance $[0, \infty)$. In particular, $S[a, b]$ *is always dense in* $L^2[a, b]$.

v) The Fourier transform of a Schwartz function is also a Schwartz function. Furthermore, the Fourier transform is an one to one onto mapping from $\mathcal{S}$ onto itself that preserve the topological structre on $\mathcal{S}$.

Now, we are in the situation to construct the spaces of smooth Hardy functions. For that purpose, we shall use the Paley-Wiener theorem as an essential ingredient.

Let us consider the Schwartz spaces $\mathcal{S}(\mathbb{R}^\mp)$ and the space of Fourier transforms of functions in $\mathcal{S}(\mathbb{R}^\mp)$, $\mathcal{F}[\mathcal{S}(\mathbb{R}^\mp)]$. These spaces have the following properties:

i) After the Paley-Wiener theorem, $\mathcal{F}[\mathcal{S}(\mathbb{R}^\mp)] \subset \mathcal{H}_\pm^2$.

ii) The Fourier transform of a Schwartz function is also another Schwartz function and this operation is bijective (one to one and onto), so that

$$\mathcal{F}[\mathcal{S}(\mathbb{R}^\mp)] \equiv \mathcal{S} \cap \mathcal{H}_\pm^2. \tag{18}$$

iii) The spaces $\mathcal{S}^\mp$ are dense in $L^2(\mathbb{R}^\mp)$. Since the Fourier transform is unitary, then, $\mathcal{S} \cap \mathcal{H}_\pm^2$ is dense in $\mathcal{H}_\pm^2$ with the norm topology. In addition due to the above comments on unitary mappings on RHS, the triplets:

$$\mathcal{S} \cap \mathcal{H}_\pm^2 \subset \mathcal{H}_\pm^2 \subset (\mathcal{S} \cap \mathcal{H}_\pm^2)^\times \tag{19}$$

are well defined RHS.

iv) Let us consider the space of restrictions to $\mathbb{R}^+$ of functions in $\mathcal{H}_\pm$ that we shall denote here as $\mathcal{H}_\pm^2\big|_{\mathbb{R}^+}$. After the van Winter theorem, there exists one to one onto mappings (and therefore invertible) $\theta_\pm$:

$$\theta_\pm : \mathcal{H}_\pm^2 \longmapsto \mathcal{H}_\pm^2\big|_{\mathbb{R}^+}, \qquad \theta_\pm : \mathcal{S} \cap \mathcal{H}_\pm^2 \longmapsto \mathcal{S} \cap \mathcal{H}_\pm^2\big|_{\mathbb{R}^+}. \tag{20}$$

Van Winter has also proved that $\mathcal{H}_\pm^2\big|_{\mathbb{R}^+}$ is dense in $L^2(\mathbb{R}^+)$. From here, we can also prove that $\mathcal{S} \cap \mathcal{H}_\pm^2\big|_{\mathbb{R}^+}$ is also dense in $L^2(\mathbb{R}^+)$. The topology on $\mathcal{H}_\pm^2\big|_{\mathbb{R}^+}$ can be transported by $\theta_\pm$ to $\mathcal{S} \cap \mathcal{H}_\pm^2\big|_{\mathbb{R}^+}$, so that the triplets

$$\mathcal{S} \cap \mathcal{H}_\pm^2\big|_{\mathbb{R}^+} \subset L^2(\mathbb{R}^+) \subset (\mathcal{S} \cap \mathcal{H}_\pm^2\big|_{\mathbb{R}^+})^\times \tag{21}$$

are new RHS.

It is important to point out that, although functions in $\mathcal{S} \cap \mathcal{H}_\pm^2\big|_{\mathbb{R}^+}$ can be uniquely extended to the negative semi axis $\mathbb{R}^-$ (and to a half plane), *as functions in* $\mathcal{S} \cap \mathcal{H}_\pm^2\big|_{\mathbb{R}^+}$, *we are considering their values on the positive semi axis* $\mathbb{R}^+$ *only*.

4. Gamow states and their mathematical construction

Once the metastable or decaying state or resonance (all these names represent the same object) has been prepared, it starts to decay. We may use an origin of times $t = 0$ at which the preparation of the decaying state is complete and starts to decay [3] (Although we shall stick to this notion of the origin of times, this determination is somehow ambiguous [29]). Then, assume that the state is represented at $t = 0$ by the state vector ψ. The non-decay probability at time $t > 0$ is

$$P(t) = |\langle \psi | e^{-itH} \psi \rangle|^2 \,, \tag{22}$$

where H is the interacting or total Hamiltonian, where $\{H_0, H\}$ is the Hamiltonian pair, responsible for the resonance scattering.

Assume that ψ is a normalized vector in a Hilbert space (in the subspace of scattering states) and that the Hamiltonian H is semi bounded, i.e., that its spectrum has a lower bound. This is a condition that have most of known quantum Hamiltonians. Then, $P(t)$ goes to zero as $t \longmapsto \infty$ [30]. The vector ψ denotes a decaying state if the non-decay probability is exponential function of the type $e^{-\alpha t}$, $\alpha > 0$. This is because the observed decay rate is exponential [3].

However, under the above conditions imposed to ψ and H, $P(t)$ cannot be exponential. It could be approximately exponential for almost all times within the range of observation. However, it is far from being exponential for very short (Zeno era) and very long (Khalfin region) times [30]. Both deviations of the exponential regime have been reported to be found experimentally [31, 32], but they are difficult to detect, the Zeno era because it is too short and the Khalfin region because it remains very little amount of undecayed material. Thus within a reasonable degree of accuracy, we may assume that resonances decay exponentially at all times. But then, *this exponential decay cannot be produced by a normalizable state vector*. The vector state that decays exponentially for all $t > 0$ is called the (decaying) Gamow state and it cannot be normalized with the usual L^2 norm.

It is well know that (non relativistic) resonances are very often associated to pairs of poles of the analytic continuation, $S(p)$, to the complex plane of the scattering operator (or scattering matrix), in the momentum representation, located symmetrically with respect to the negative imaginary axis. If we shift to the energy representation, the lower half plane is transformed into the second sheet of the two sheeted Riemann surface corresponding to the transformation $p = \sqrt{2mE}$. Then, resonances are characterized by pairs of poles of the analytic continuation of $S(E)$ to the second sheet, at values $z_R = E_R - i\Gamma/2$ and its complex conjugate, z_R^*. Here, E_R is the resonant energy and Γ the width, which gives the mean life [3]. There are some other mathematical and physical definitions of resonances [3] that not always coincide [30]. The study of the relations of between these definitions needs to be completed.

We are now in the position of defining the *decaying Gamow vector*, ψ^D, as an eigenvector of the total Hamiltonian H with the complex eigenvalue $z_R = E_R - i\Gamma/2$, i.e., $H\psi^D = z_R \psi^D$ and the *growing Gamow vector*, ψ^G, as an eigenvector of H with eigenvalue $z_R^* = E_R + i\Gamma/2$, $H\psi^G = z_R^* \psi^G$. Gamow vectors ψ^D and ψ^G cannot be normalizable as they are eigenvectors of a self adjoint Hamiltonian with complex eigenvalues. This means that Gamow vectors do not belong to the Hilbert space in which H is defined as a self adjoint operator. The advantage of this definition is that the decaying Gamow vector decays exponentially for all positive values of time (we should clarify this point later):

$$e^{-itH}\psi^D = e^{-itE_R} e^{-t\Gamma/2} \psi^D \,. \tag{23}$$

The fact is that Gamow vectors belong to the antiduals of two, in principle different, RHS. Now, we proceed to explain the idea of their mathematical construction.

In order to do it, we reduce the hypothesis to a minimum, further generalizations can be constructed without serious difficulties. Assume that H_0 and H have simple continuous spectrum

equal to $\mathbb{R}^+ = [0, \infty)$, so that both have only scattering states. Assume that these operators are defined on certain Hilbert space $\mathcal{H}$. Then, according to a spectral theorem [28] pp. 226-227, there exists a unitary operator $U : \mathcal{H} \longmapsto L^2(\mathbb{R}^+)$ such that H is transformed into the multiplication operator on $L^2(\mathbb{R}^+)$. To understand this, let us consider $\psi(E) \in L^2(\mathbb{R}^+)$ such that $E\,\psi(E) \in L^2(\mathbb{R}^+)$ (we use the argument E to mean energy). On the space of functions with this property, let us define the multiplication operator $\mathcal{E}$ as $\mathcal{E}\psi(E) := E\psi(E)$. Then, $UHU^{-1} = \mathcal{E}$. Note that U *diagonalizes* H by this operation.

Concerning $\mathcal{E}$, the following properties are relevant:

i) The operator $\mathcal{E}$ leaves the spaces $\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}$ invariant, so that for any function $\psi_\pm(E) \in \mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}$, we have that $\mathcal{E}\psi_\pm(E) = E\psi_\pm(E) \in \mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}$.

ii) The operator $\mathcal{E}$ can be extended to the dual $(\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+})^\times$ by means of the duality formula: If $\psi_\pm(E) \in \mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}$ and $F_\pm \in (\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+})^\times$, then,

$$\langle \mathcal{E}\,\psi_\pm(E)|F_\pm\rangle = \langle\psi_\pm(E)|\mathcal{E}\,F_\pm\rangle\,, \tag{24}$$

so that $\mathcal{E}$ acts on all vectors in $(\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+})^\times$.

Next, we construct a new RHS as follows: Let U be the unitary operator which *diagonalizes* H as above ($UHU^{-1} = \mathcal{E}$). If $\Phi_\pm := U^{-1}[\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}]$, we consider these new RHS given by $\Phi_\pm \subset L^2(\mathbb{R}^+) \subset \Phi^\times_\pm$. Then,

$$UH\Phi_\pm = UHU^{-1}[\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}] = \mathcal{E}[\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}] \subset \mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}$$

$$\Longrightarrow H\Phi_\pm \subset U^{-1}[\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}] = \Phi_\pm\,, \tag{25}$$

so that $\Phi_\pm$ reduce H ($H\Phi_\pm \subset \Phi_\pm$). By the duality formula, $\langle H\varphi_\pm|F_\pm\rangle = \langle\varphi_\pm|H\,F_\pm\rangle$, valid for all $\varphi \in \Phi_\pm$ and all $F_\pm \in \Phi^\times_\pm$, we extend H into the antiduals $\Phi^\times_\pm$.

Then, we may obtain some rigorous results that we list in the sequel. These results have been proved in some references like [18, 19]:

i) Assume that $z_R = E_R - i\Gamma/2$ is a resonance pole of the S matrix associated to the Hamiltonian pair $\{H_0, H\}$. Then, there exists a unique vector (which is a generalized function) $\delta_{z_R} \in (\mathcal{S} \cap \mathcal{H}^2_+\big|_{\mathbb{R}^+})^\times$ such that $\mathcal{E}\delta_{z_R} = z_R\,\delta_{z_R}$, i.e., δ_{z_R} is an eigenvector of $\mathcal{E}$ with eigenvalue z_R. This eigenvector lies in the dual $(\mathcal{S} \cap \mathcal{H}^2_+\big|_{\mathbb{R}^+})^\times$ and cannot be in $L^2(\mathbb{R}^+)$ due to the Hermiticity of $\mathcal{E}$. Analogously, there exists $\delta_{z^*_R} \in (\mathcal{S} \cap \mathcal{H}^2_-\big|_{\mathbb{R}^+})^\times$ with $\mathcal{E}\delta_{z^*_R} = z^*_R\,\delta_{z^*_R}$. The star always means complex conjugation.

ii) Let us define the *decaying Gamow vector* as $\psi^D := U^{-1}\delta_{z_R} \in \Phi^\times_+$. Then,

$$\mathcal{E}\delta_{z_R} = z_R\,\delta_{z_R} \Longrightarrow U^{-1}\mathcal{E}UU^{-1}\,\delta_{z_R} = z_R\,U^{-1}\delta_{z_R} \Longrightarrow H\psi^D = z_R\,\psi^D\,. \tag{26}$$

The latter identity makes sense in $\Phi^\times_+$. Analogously, we define the *growing Gamow vector* as $\psi^G := U^{-1}\delta_{z^*_R} \in \Phi^\times_-$. Then, we have that $H\psi^G = z^*_R\,\psi^G$, equation valid in $\Phi^\times_-$. Thus Gamow vectors are defined as the eigenvectors of the total Hamiltonian H with eigenvalues given by resonance poles of the S matrix in the energy representation.

iii) Our structures have been constructed using Hardy spaces on a half plane. Hardy spaces on a half plane split the unitary group given by $e^{-it\mathcal{E}}$, $t \in \mathbb{R}$ into two semigroups. In fact [18, 19]:

If $t \geq 0$, $e^{it\mathcal{E}}\mathcal{H}^2_+ \subset \mathcal{H}^2_+$ and also $e^{it\mathcal{E}}[\mathcal{S} \cap \mathcal{H}^2_+\big|_{\mathbb{R}^+}] \subset \mathcal{S} \cap \mathcal{H}^2_+\big|_{\mathbb{R}^+}$. However, for any $t_0 < 0$, there exists a function $\phi_+(E) \in \mathcal{S} \cap \mathcal{H}^2_+\big|_{\mathbb{R}^+}$, such that $e^{it_0\mathcal{E}}\phi_+(E) \notin \mathcal{S} \cap \mathcal{H}^2_+\big|_{\mathbb{R}^+}$.

If $t \leq 0$, $e^{it\mathcal{E}}\mathcal{H}^2_- \subset \mathcal{H}^2_-$ and also $e^{it\mathcal{E}}[\mathcal{S} \cap \mathcal{H}^2_-\big|_{\mathbb{R}^+}] \subset \mathcal{S} \cap \mathcal{H}^2_-\big|_{\mathbb{R}^+}$. However, for any $t_0 > 0$, there exists a function $\phi_-(E) \in \mathcal{S} \cap \mathcal{H}^2_-\big|_{\mathbb{R}^+}$, such that $e^{it_0\mathcal{E}}\phi_-(E) \notin \mathcal{S} \cap \mathcal{H}^2_-\big|_{\mathbb{R}^+}$.

vi) These ideas can be immediately carried to $\Phi_{\pm}$, since $U^{-1} e^{it\mathcal{E}} U = e^{itH}$. Thus:

If $t \geq 0$, $e^{itH}\Phi_+ \subset \Phi_+$. Furthermore for any $t_0 < 0$, there exists a vector $\varphi_+ \in \Phi_+$ such that $e^{it_0 H}\varphi_+ \notin \Phi_+$.

If $t \leq 0$, $e^{itH}\Phi_- \subset \Phi_-$. Furthermore for any $t_0 > 0$, there exists a vector $\varphi_- \in \Phi_-$ such that $e^{it_0 H}\varphi_- \notin \Phi_-$.

v) Let $\Phi \subset \mathcal{H} \subset \Phi^{\times}$ be a RHS and U a unitary operator on $\mathcal{H}$ such that $U^{\dagger}\Phi \subset \Phi$, where $U^{\dagger}$ is the adjoint of U. Then, U can be extended to an operator on $\Phi^{\times}$ by duality: If $\varphi \in \Phi$ and $F \in \Phi^{\times}$ are chosen arbitrarily, then, $\langle U^{\dagger}\varphi|F\rangle = \langle\varphi|UF\rangle$. This equation defines the action of U on $\Phi^{\times}$ in a unique way. In consequence:

$$e^{-itH}\Phi_+^{\times} \subset \Phi_+^{\times}, \text{if and only if } t > 0 \, ; \qquad e^{-itH}\Phi_-^{\times} \subset \Phi_-^{\times}, \text{if and only if } t < 0 \, . \qquad (27)$$

Due to the properties of Hardy functions the evolution group e^{-itH} has split into two semigroups, one for positive values of time and the other for negative values of time.

On the other hand, we may reconstruct our spaces in such a way that relations (27) be valid for all values of time [33].

vi) Now, it comes the essential property of Gamow states, which has been proved in [18, 19]: For $t \geq 0$, the time evolution of the decaying Gamow state is given by

$$e^{-itH} \psi^D = e^{-itE_R} e^{-t\Gamma/2} \psi^D \, . \qquad (28)$$

This means that the decaying Gamow state *has an exact exponential decay for $t \geq 0$*. Due to the use of Hardy functions in the definition of our spaces, time evolution is not defined for ψ^D at times $t < 0$.

For $t \leq 0$, the time evolution of the growing Gamow state is given by

$$e^{-itH} \psi^G = e^{-itE_R} e^{+t\Gamma/2} \psi^G \, . \qquad (29)$$

This means that the decaying Gamow state *has an exact exponential grow for $t \leq 0$* or an *exact exponential decay to the past*. As for ψ^D, time evolution is not defined for ψ^G at times $t > 0$.

vii) The growing Gamow vector as well as all *growing* process that takes place in $\Phi_-^{\times}$ for $t < 0$ should not be confused with the process of capture, preparation or creation of the resonance. In fact, it is the time reversal of the decaying process: something that *describes the same in the reverse direction of time*. In particular, if T is the time operator, $T\Phi_{\pm}^{\times} = \Phi_{\mp}^{\times}$ and $T\psi^D = \psi^G$ and also $T\psi^G = \psi^G$. Nevertheless, another interpretation is possible. This interpretation is the founding stone of Time Asymmetric Quantum Mechanics.

5. Time Asymmetric Quantum mechanics

As we have just pointed out in the previous section, we may reinterpret $\Phi_+^{\times}$ in a different way than being the time reversal of $\Phi_-^{\times}$. This interpretation was proposed by A. Bohm and collaborators in a series of papers [4, 20–23]. Thus, *the notion of time asymmetric quantum mechanics (TAQM) comes from the idea according to which the processes described by vectors in the space $\Phi_-^{\times}$ are not related with the time reversal of the decay as described by vectors in $\Phi_+^{\times}$.*

We have already mentioned in the Introduction that irreversibility in quantum mechanics include resonances but it goes beyond than resonances. In particular, it must consider many scattering situations in which the probability of the process in one direction of time is much smaller than the probability of its time reversal [1, 2].

In order to introduce a formulation of TAQM, we need to add to the standard formulation of Quantum Mechanics a new axiom to the existing ones. This new axiom is called the **Hardy space axiom**. Since TAQM manifest itself on scattering processes, resonate or not, it is

reasonable to present it in this context and using a causality principle. This causality principle states than in any scattering experiment, first comes the preparation and then the registration.

We may assume that in our scattering experiment, the scattering Møller wave operators exists [34]. Let us denote these operators Ω_{IN} and Ω_{OUT} for the incoming and outgoing Møller operator respectively. The relation between the free and perturbed scattering states is given by these operators [34], so that we define

$$\Phi_{IN} := \Omega_{IN}^{-1}\Phi_- \qquad ; \qquad \Phi_{OUT} := \Omega_{OUT}^{-1}\Phi_+ \,. \tag{30}$$

Under the simplest conditions (no bound states, no singular spectrum, if any of these elements were present, we have to use the Hilbert space of scattering states [18]), Φ_{IN} and Φ_{OUT} are *dense* in the Hilbert space $\mathcal{H}$. These spaces define two new RHS [18, 19]

$$\Phi_{IN} \subset \mathcal{H} \subset \Phi_{IN}^\times \qquad ; \qquad \Phi_{OUT} \subset \mathcal{H} \subset \Phi_{OUT}^\times \,. \tag{31}$$

Due to the construction of the RHS (31) and the properties of the Møller operators, we have the following result:

$$\text{For } t < 0 \,, \quad e^{-itH_0}\Phi_{IN}^\times \subset \Phi_{IN}^\times \,; \quad ; \quad \text{for } t > 0 \,, \quad e^{itH_0}\Phi_{OUT} \subset \Phi_{OUT} \,. \tag{32}$$

To prove the second relation in (32), take $t > 0$ and $\phi_{out} \in \Phi_{OUT}$. We have:

$$\phi_{out}(t) = e^{itH_0}\phi_{out} = \Omega_{OUT}^{-1} e^{itH}\Omega_{OUT}\,\Omega_{OUT}^{-1}\,\phi_+ = \Omega_{OUT}^{-1} e^{itH}\,\phi_+ \in \Phi_{OUT} \,, \quad t > 0 \,. \tag{33}$$

To prove the first relation in (32) we proceed analogously with Φ_{IN} and then, apply the duality formula $\langle e^{itH_0}\varphi|\psi_{in}\rangle = \langle\varphi|e^{-itH_0}\psi_{in}\rangle$ valid for all $\varphi \in \Phi_{IN}$.

In order to introduce the *Hardy space axiom*, we need to do it on the preparation and registration in an scattering experiment separately.

i) *Preparation.* Before the scattering, we have to prepare states of particles to be scattered. This is done with some sort of device called the *preparation apparatus*, which produce the *incoming state* ψ_{in}, which after preparation and before entering into the interaction region, evolves freely.

Then, for incoming states the *Hardy space axiom* is stated as follows:

The incoming state ψ_{in} belongs to a space of incoming free states, $\Phi_{IN}^\times$

Due to the use of Hardy functions in the construction of Φ_{IN}, and therefore of $\Phi_{IN}^\times$, time evolution is defined for times $t < 0$ only, on all vectors in $\Phi_{IN}^\times$. This, for any $t < 0$, $\psi_{in}(t) = e^{-itH_0}\psi_{in}(0)$, where $\psi_{in}(0) := \psi_{in}$.

ii) *Registration.* The unstable quantum state decays after the particle leaves the interaction region. Decaying products (we use this word for general scattering experiments, not only for resonances) are registered in a registration apparatus. This registration apparatus lies outside the interaction region so that it detects free states. As a matter of fact, the registration apparatus detects the projection into the region it covers of the total outgoing state. This projection is given by a vector $\phi_{out}(0) = \phi_{out}$. And this vector is *observed*, so that we *should look at the operator $|\phi_{out}\rangle\langle\phi_{out}|$ as an observable*.

Then, for the registration part the *Hardy space axiom* is stated as follows:

The vector ϕ_{out} that defines the observable $|\phi_{out}\rangle\langle\phi_{out}|$, lies in Φ_{OUT}, $\phi_{out} \in \Phi_{OUT}$.

Thus, according to (32), $\phi_{out}(t) = e^{itH_0}\phi_{out}$, $t > 0$. Observables must evolve according to the Heisenberg rule of time evolution, something which is obvious here, since

$$|\phi_{out}(t)\rangle\langle\phi_{out}(t)| = e^{itH_0}\,|\phi_{out}\rangle\langle\phi_{out}|\,e^{-itH_0} \,. \tag{34}$$

Another way of seeing that the registered observable should evolve in time as an observable comes from the Born probability of measuring $|\phi_{out}(t)\rangle\langle\phi_{out}(t)|$ in the *arbitrary* (pure) state given by the density operator $\rho(t) := |\psi(t)\rangle\langle\psi(t)|$ given by ($t > 0$, $\psi(0) = \psi$):

$$\mathcal{P}_{\rho(t)}(|\phi_{out}\rangle\langle\phi_{out}|) = |\langle\phi_{out}|\ \psi_{in}(t)\rangle|^2 = \mathrm{Tr}\{\ |\phi_{out}\rangle\langle\phi_{out}|[e^{-iH_0 t}|\psi\rangle\langle\psi|e^{iH_0 t}]\}$$

$$= \mathrm{Tr}\{e^{itH_0}|\phi_{out}\rangle\langle\phi_{out}|e^{-iH_0 t}\} = \mathcal{P}_\rho(|\phi_{out}(t)\rangle\langle\phi_{out}(t)|)\,. \qquad (35)$$

Thus, $|\phi_{out}\rangle\langle\phi_{out}|$ evolves for $t > 0$ as an observable. Thus, we have complete consistency.

In consequence: states are vectors $\psi_{in} \in \Phi_{IN}^\times$ or density operators $|\psi_{in}\rangle\langle\psi_{in}|$, the latter acting as operators from Φ_{IN} into $\Phi_{IN}^\times$ (this space is commonly denoted as $\mathcal{L}(\Phi_{IN}, \Phi_{IN}^\times)$). Observables are operators $|\phi_{out}\rangle\langle\phi_{out}|$ on Φ_{OUT}. The time reverse operation transforms $\Phi_\pm^\times$ into $\Phi_\mp^\times$ and from here one deduces that it transforms $\Phi_{IN}^\times$ onto $\Phi_{OUT}^\times$ and vice versa. However, it is not true that it transforms $\Phi_{OUT}^\times$ onto either Φ_{IN} or $\mathcal{L}(\Phi_{IN}, \Phi_{IN}^\times)$. Also, time reversal of observables are not states.

Finally, let us re-state the **Hardy class axiom** as:

We take as space of in-states $\Phi_{IN}^\times$. Any vector of $\Phi_{IN}^\times$ can be represented by a smooth Hardy function on the lower half plane. Analogously, we take as space of observables the operators formed by dyads $|\phi_{out}\rangle\langle\phi_{out}|$ of vectors ϕ_{out} in Φ_{OUT}. These vectors can be represented by smooth Hardy functions on the lower half plane.

In conclusion, we have summarized the essential mathematical formalism that yields to a presentation of the TAQM. We have skipped many details, in particular those concerning to scattering and resonance scattering. The final description of TAQM, as given in the present paper, is very short so that we encourage the interested reader a careful study of the original papers listed in our references.

Acknowledgements

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