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# Dirac Green function for $\delta$ potentials

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**Abstract** – The Green function for a singular one-dimensional $\delta(x)$ potential is explicitly obtained in the relativistic context of the Dirac equation, using Dyson's equation. From it, two bound states are easily found, one for the particle and another for the antiparticle, both depending on the $\delta$ potential intensity. When a second $\delta$ perturbation is introduced at a different point, the problem can also be solved analytically, for the possible bound states. A transcendental equation is obtained, which can be considered as a relativistic generalization of the well-known Lambert equation.



**Introduction.** – Our starting point is the Schrödinger Green function for the one-dimensional non-relativistic Hamiltonian

$$\mathcal{H}_{nr} = -\frac{\hbar^2}{2m}\partial_x^2 - a\delta(x), \tag{1}$$

which was worked out in 1984 by Schulman [1] and subsequently by Blinder [2] and by Grosche [3], who found

$$G_{nr}(x,y) = \frac{1}{2ik}\left[e^{ik|x-y|} - \frac{a}{a+ik}e^{ik(|x|+|y|)}\right], \tag{2}$$

where $\hbar = m = 1$, $E = k^2$ and $a \geq 0$. This Green function is continuous, though its derivative is discontinuous as it must be in order for the second derivative to produce a delta function.

One-dimensional non-relativistic quantum-mechanical point potentials of the type (1) have received renewed attention in different areas of theoretical physics throughout the present century [4,5]. The main reason is probably that they constitute a class of solvable or quasi-solvable potentials easy to deal with and suitable for analysing basic quantum properties, such as bound states, resonances or scattering processes [6]. But this kind of singular point potential is also interesting because they are used to model physical situations, such as point defects in materials [7], different kinds of thin structures [8,9], heterostructures described by an abrupt effective mass change [10], and they also appear in the context of topological insulators [11]. It is worth mentioning that in nanophysics, for example, this type of point potential can be used to model sharply peaked impurities inside one-dimensional quantum dots [12,13], and in scalar Quantum Field Theory on a line they are used to model impurities, to provide external singular backgrounds [14], and also to implement boundary conditions compatible with a scalar theory on an interval [15]. Moreover, point interactions of the form $\delta(x)$, or even $\delta'(x)$, have been used recently to analyze perturbations of a free kinetic Schrödinger Hamiltonian [16], the harmonic oscillator [17–19], a constant electric field [17], the infinite square well [20,21], the conical oscillator [22,23], and even the semi-oscillator, which has been used as a simple toy model potential showing resonance phenomena [24]. Finally, we mention that in black-hole theory and the 't Hooft brick wall model [25] the addition of $\delta(x)$ and $\delta'(x)$ interactions to the Hamiltonian is relevant at least in three different ways: first, it can help in introducing time-dependent boundaries [26], second, the $\delta'(x)$ term is needed when fermions are considered in order to build self-adjoint extensions of a Dirac operator with a $\delta(x)$ potential [27], and third, it can also serve to model membrane mechanisms for certain precise black-hole horizons [26,28].

Curiously, unlike non-relativistic quantum-mechanical point potentials, the relativistic counterparts have not yet attracted much attention. To start making progress in this new direction, in the present work the Green function for

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$$G(0^+,Y) = -\frac{i\,e^{i\eta|Y|}}{2\Delta}\begin{pmatrix} \xi(1-4\alpha^2\tau_Y)-2i\alpha(\tau_Y+1) & \dfrac{2i\alpha(\tau_Y+1)}{\xi}+4\alpha^2-\tau_Y \\ 2i\alpha\xi(\tau_Y+1)+4\alpha^2-\tau_Y & \dfrac{1-4\alpha^2\tau_Y}{\xi}-2i\alpha(\tau_Y+1) \end{pmatrix}, \qquad (11)$$

$$G(0^-,Y) = -\frac{i\,e^{i\eta|Y|}}{2\Delta}\begin{pmatrix} \xi(1+4\alpha^2\tau_Y)+2i\alpha(\tau_Y-1) & \dfrac{2i\alpha(\tau_Y-1)}{\xi}-4\alpha^2-\tau_Y \\ 2i\alpha\xi(\tau_Y-1)-4\alpha^2-\tau_Y & \dfrac{1+4\alpha^2\tau_Y}{\xi}+2i\alpha(\tau_Y-1) \end{pmatrix}, \qquad (12)$$

a one-dimensional relativistic singular Hamiltonian is determined using Dyson's equation, finding the bound states next and analyzing different limits. Afterwards, two such point perturbations are considered simultaneously at two different points. We determine the corresponding wave function and from it, the bound states, which appear as the solutions of a kind of relativistic Lambert equation. From the relativistic Green function, the Feynmann propagator will be determined, but only for the free case.

**One $\delta$ perturbation.** – Let us consider the one-dimensional Dirac Hamiltonian with $V(x) = -a\delta(x)\mathbb{I}$,

$$\mathcal{H} = \mathcal{H}_0 + V(x), \qquad \mathcal{H}_0 = -i\hbar c\sigma_x\partial_x + mc^2\sigma_z\,, \qquad (3)$$

where $\mathbb{I}$ is the $2\times 2$ identity matrix and $\sigma_{x,z}$ are standard Pauli matrices. If $E$ denotes a possible value of the energy, we introduce the dimensionless variables

$$X = \frac{mc}{\hbar}x, \qquad \epsilon = \frac{E}{mc^2}, \qquad A = \frac{a}{\hbar c}, \qquad (4)$$

and the Dirac equation becomes

$$\left(-i\sigma_x\partial_X + \sigma_z - A\delta(X)\mathbb{I}\right)\vec{\psi} = \epsilon\vec{\psi}, \qquad (5)$$

where $\vec{\psi}^T = (\psi_1(X), \psi_2(X))$ is the spinor wave function.

We begin with $G_0(x,y)$, the relativistic free-particle Green function ($A=0$) derived by Fairbairn *et al.* [29] in 1973. Due to the change of variables (4), we will use in the following the dimensionless Green function $G_0(X,Y) := \hbar c G_0(x,y)$, which reads

$$G_0(X,Y) = -\frac{i}{2}\,e^{i\eta|X-Y|}\begin{pmatrix} \xi & \tau_{(X-Y)} \\ \tau_{(X-Y)} & \xi^{-1} \end{pmatrix}, \qquad (6)$$

where

$$\eta = \sqrt{\epsilon^2-1}, \qquad \xi = \sqrt{\frac{\epsilon+1}{\epsilon-1}}, \qquad \tau_{(X-Y)} = \mathrm{sign}(X-Y). \qquad (7)$$

Note that this function is itself discontinuous, due to the presence of the sign function.

From Dyson's equation

$$G(X,Y) = G_0(X,Y) + \int_{-\infty}^{\infty} G_0(X,t)V(t)G(t,Y)\mathrm{d}t, \qquad (8)$$

invoking Kurasov's [30] prescription for the action of the Dirac delta on a discontinuous function $f(y)$,

$$f(y)\delta(y-p) = \frac{f(p^+)+f(p^-)}{2}\delta(y-p), \qquad (9)$$

we have for the singular potential $V(X) = -A\delta(X)$ given in (5)

$$G(X,Y) = G_0(X,Y) \\ -\frac{A}{2}\left[G_0(X,0^+)G(0^+,Y) + G_0(X,0^-)G(0^-,Y)\right]. \qquad (10)$$

By setting $X = 0^{\pm}$ in (6) we are led to a $4\times 4$ matrix problem with solution

*see eqs.* (11) *and* (12) *above*

where $\alpha = A/4$ and $\Delta = (\xi - 2i\alpha)(\xi^{-1} - 2i\alpha)$. When $\alpha = 0 = A$, eqs. (11), (12) give

$$G(0^+,Y)\big|_{a=0} = G(0^-,Y)\big|_{\alpha=0} = \frac{ie^{i\eta|Y|}}{2}\begin{pmatrix} -\xi & \tau_Y \\ \tau_Y & \dfrac{-1}{\xi} \end{pmatrix},$$

which coincides with $G_0(0,Y)$ in (6), because $\tau_{(-Y)} = -\tau_Y$.

Using eqs. (11), (12) and the result of Fairbairn *et al.* [29] given in (6), we get the following explicit expression for the relativistic Green function (10):

*see eq.* (13) *on the next page*

where $\alpha = A/4$. The bound-state conditions are obtained precisely from the poles of this Green function, $\Delta = 0$, that is:

$$\xi_1^2 = -\frac{1}{4\alpha^2}, \qquad \xi_2^2 = -4\alpha^2.$$

Taking into account (7), we get the following values for the bound-state energies

$$\epsilon_1 = \frac{E_1}{mc^2} = 1 - \frac{2A^2}{4+A^2} = 1 - \frac{2a^2}{4\hbar^2c^2+a^2}, \qquad (14)$$

$$\epsilon_2 = \frac{E_2}{mc^2} = -1 + \frac{2A^2}{4+A^2} = -\epsilon_1. \qquad (15)$$

$$G(X,Y) = -i\,\frac{e^{i\eta|X-Y|}}{2}\begin{pmatrix} \xi & \tau_{(X-Y)} \\ \tau_{(X-Y)} & \xi^{-1}\end{pmatrix}$$
$$+\frac{\alpha\, e^{i\eta(|X|+|Y|)}}{\Delta}\begin{pmatrix} \xi(\xi-2i\alpha)+(2i\alpha\xi-1)\tau_X\tau_Y & (2i\alpha-\xi)\tau_Y+(\xi^{-1}-2i\alpha)\tau_X \\ \dfrac{(2i\alpha\xi-1)\tau_Y}{\xi}+(\xi-2i\alpha)\tau_X & \dfrac{(\xi^{-1}-2i\alpha)}{\xi}+(2i\alpha\xi^{-1}-1)\tau_X\tau_Y\end{pmatrix}, \tag{13}$$

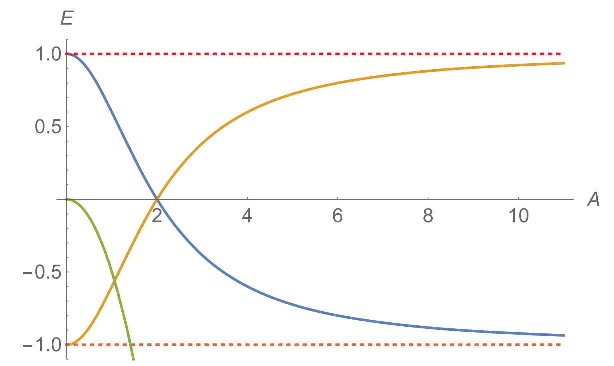


Fig. 1: (Colour online) The particle ($E_1$ in blue) and antiparticle ($E_2$ in yellow) bound-state energies (14), (15) as functions of $A = a/(\hbar c)$, in units of $mc^2$. The non-relativistic limit is obtained at the origin. The non-relativistic energy result $E_{nr}$ in (16) is given by the green parabola branch. The dotted red lines correspond to the particle and antiparticle asymptotic values ($\pm 1$).

The value $\epsilon_1$ corresponds to the *particle* energy and $\epsilon_2$ should be associated to the *antiparticle* energy. The non-relativistic energy $E_{nr}$ is obtained when $c \to \infty$ in $E_1$, indeed

$$E_{nr} = \lim_{c\to\infty}(E_1 - mc^2) = -\frac{ma^2}{2\hbar^2} = -\frac{A^2}{2}\,mc^2, \tag{16}$$

a well-known result [31]. We have plotted in fig. 1 the two energies of the bound states for the particle and antiparticle, given in (14), (15), and also the non-relativistic energy (16), in units of $mc^2$ and as functions of the parameter $A = a/\hbar c$.

It is quite interesting to observe the differences between the relativistic and the non-relativistic behavior of the solutions for the particle case, the blue and green curves in fig. 1: as $a, A \to \infty$ the non-relativistic energy $E_{nr} \to -\infty$, whereas the relativistic energy is bounded and $E_1 \to -mc^2$. This is consistent with the absence of the Klein emission in this system [29].

**Two $\delta$ perturbations.** – In this section we will generalize the quantum relativistic problem analysed in the first section to the atractive potential,

$$V(x) = -a\delta(x)\mathbb{I} - b\delta(x-q)\mathbb{I}, \qquad a,b,q \geq 0. \tag{17}$$

We are interested in determining the spectrum, studying different limits and obtaining the Green function. The approach will be different from the previous section: instead of looking directly for the Green function, we will solve Schrödinger's equation in the three intervals determined by potential (17), and impose appropriate matching conditions at the singular points.

*The spectrum.* For the potential (17) the one-dimensional Dirac Hamiltonian reads

$$\mathcal{H} = -i\hbar c\sigma_x\partial_x + mc^2\sigma_z - [a\delta(x) + b\delta(x-q)]\,\mathbb{I}, \tag{18}$$

where $\mathbb{I}$ is the $2\times 2$ identity matrix and $\sigma_{x,z}$ are Pauli matrices. Making the changes of variables (4) and also

$$Q = \frac{mc}{\hbar}q, \qquad B = \frac{b}{\hbar c}, \tag{19}$$

and calling $\psi_1(X)$ and $\psi_2(X)$ the two components of the spinor on which $\mathcal{H}$ acts, Dirac's equation (18) becomes

$$(1-\epsilon)\psi_1 - [A\delta(X) + B\delta(X-Q)]\,\psi_1 = i\psi_2', \tag{20}$$
$$-(1+\epsilon)\psi_2 - [A\delta(X) + B\delta(X-Q)]\,\psi_2 = i\psi_1', \tag{21}$$

the prime denotes the derivative with respect to $X$.

In the following, first we will find the bound states, corresponding to solutions with energy $-mc^2 < E < mc^2$, that is $-1 < \epsilon < 1$, in the three regions determined by the two singularities of the potential, *i.e.*, $X < 0$, $0 < X < Q$ and $X > Q$; then, we will impose the matching conditions at the singularities $X = 0$ and $X = Q$.

*Solution for* $X < 0$: The square integrable solutions of (20), (21) are

$$\psi_1 = C_1\, e^{X\sqrt{1-\epsilon^2}}, \quad \psi_2 = -i\sqrt{\frac{1-\epsilon}{1+\epsilon}}\,C_1\, e^{X\sqrt{1-\epsilon^2}}, \tag{22}$$

*Solution for* $0 < X < Q$: In this interval the solutions are

$$\psi_1 = C_2\, e^{X\sqrt{1-\epsilon^2}} + D_2\, e^{-X\sqrt{1-\epsilon^2}}, \tag{23}$$
$$\psi_2 = i\sqrt{\frac{1-\epsilon}{1+\epsilon}}\left[-C_2 e^{X\sqrt{1-\epsilon^2}} + D_2 e^{-X\sqrt{1-\epsilon^2}}\right]. \tag{24}$$

*Solution for* $X > Q$: The square integrable solutions are

$$\psi_1 = D_3\, e^{-X\sqrt{1-\epsilon^2}}, \quad \psi_2 = i\sqrt{\frac{1-\epsilon}{1+\epsilon}}\,D_3\, e^{-X\sqrt{1-\epsilon^2}}, \tag{25}$$

*Matching conditions at* $X = 0$: Due to the presence of the Dirac delta term $-A\delta(X)$ in (20), (21), the functions $\psi_1$

$$-e^{Q\sqrt{1-\epsilon^2}}\left(2+B\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right)C_2 - e^{-Q\sqrt{1-\epsilon^2}}\left(2-B\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right)D_2 + e^{-Q\sqrt{1-\epsilon^2}}\left(2+B\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right)D_3 = 0, \tag{32}$$

$$e^{Q\sqrt{1-\epsilon^2}}\left(B-2\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right)C_2 + e^{-Q\sqrt{1-\epsilon^2}}\left(B+2\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right)D_2 + e^{-Q\sqrt{1-\epsilon^2}}\left(B-2\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right)D_3 = 0, \tag{33}$$

$$\begin{vmatrix} A\sqrt{\frac{1-\epsilon}{1+\epsilon}}+2 & A\sqrt{\frac{1-\epsilon}{1+\epsilon}}-2 & -e^{Q\sqrt{1-\epsilon^2}}\left(A\sqrt{\frac{1-\epsilon}{1+\epsilon}}+2\right) & 0 \\ A-2\sqrt{\frac{1-\epsilon}{1+\epsilon}} & A+2\sqrt{\frac{1-\epsilon}{1+\epsilon}} & e^{Q\sqrt{1-\epsilon^2}}\left(A-2\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right) & 0 \\ 0 & -e^{Q\sqrt{1-\epsilon^2}}\left(B\sqrt{\frac{1-\epsilon}{1+\epsilon}}+2\right) & B\sqrt{\frac{1-\epsilon}{1+\epsilon}}-2 & B\sqrt{\frac{1-\epsilon}{1+\epsilon}}+2 \\ 0 & e^{Q\sqrt{1-\epsilon^2}}\left(B-2\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right) & B+2\sqrt{\frac{1-\epsilon}{1+\epsilon}} & B-2\sqrt{\frac{1-\epsilon}{1+\epsilon}} \end{vmatrix} = 0. \tag{34}$$

and $\psi_2$ must have a finite discontinuity at $X=0$, such that

$$i\left(\psi_1(0^+)-\psi_1(0^-)\right) = -A\,\frac{\psi_2(0^+)+\psi_2(0^-)}{2}, \tag{26}$$

$$i\left(\psi_2(0^+)-\psi_2(0^-)\right) = -A\,\frac{\psi_1(0^+)+\psi_1(0^-)}{2}, \tag{27}$$

where, we have taken into account (9). Using (22)–(24) we get

$$\left(A\sqrt{\frac{1-\epsilon}{1+\epsilon}}+2\right)C_1 + \left(A\sqrt{\frac{1-\epsilon}{1+\epsilon}}-2\right)C_2 - \left(A\sqrt{\frac{1-\epsilon}{1+\epsilon}}+2\right)D_2 = 0, \tag{28}$$

$$\left(A-2\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right)C_1 + \left(A+2\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right)C_2 + \left(A-2\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right)D_2 = 0. \tag{29}$$

*Matching conditions at $X=Q$*: Again, from the presence of the Dirac delta term $-B\delta(X-Q)$ in (20), (21), the functions $\psi_1$ and $\psi_2$ must have a finite discontinuity at $X=Q$, such that

$$i\left(\psi_1(Q^+)-\psi_1(Q^-)\right) = -B\,\frac{\psi_2(Q^+)+\psi_2(Q^-)}{2}, \tag{30}$$

$$i\left(\psi_2(Q^+)-\psi_2(Q^-)\right) = -B\,\frac{\psi_1(Q^+)+\psi_1(Q^-)}{2}, \tag{31}$$

Using (23)–(25) we get after some algebra:

*see eqs.* (32) *and* (33) *above*

Equations (28), (29) and (32), (33) form a linear homogeneous system which has non-trivial solution if and only if the following determinant vanishes:

*see eq.* (34) *above*

From here we get

$$(4-A^2)(4-B^2) - 4(A+B)(4-AB)\epsilon\sqrt{1-\epsilon^2} - (4-2A-2B-AB)(4+2A+2B-AB)\epsilon^2 = 16AB\; e^{-2Q\sqrt{1-\epsilon^2}}, \tag{35}$$

which is a generalization of the Lambert equation [32,33] in which a square root in the unknown dimensionless energy $\epsilon$ appears.

Some particular cases of (35) deserve to be considered in detail:

- If $B=0=b$, that is if no singularity is present at $X=Q$, then (35) is simply

  $$4(4-A^2) - (4-2A)(4+2A)\epsilon^2 - 16A\epsilon\sqrt{1-\epsilon^2} = 0, \tag{36}$$

  whose solutions are found to be

  $$\epsilon_1 = \frac{4-A^2}{4+A^2} = 1 - \frac{2A^2}{4+A^2}, \qquad \epsilon_2 = -\epsilon_1, \tag{37}$$

  which, as we already know, correspond to the particle ($\epsilon_1$) and antiparticle ($\epsilon_2$) bound states when only one delta singular potential is present at $X=0$, the results (14), (15) were already found in the previous section.

- If $A=0=a$, the same solution (37) is basically obtained, with $B$ instead of $A$.

- If $A=0=B$, the only solutions are $\epsilon=\pm 1$, and the problem has no bound states at all.

- The most interesting situation appears when $q=Q=0$, that is, when we make the two singularities $X=0$ and $X=Q$ coalesce ($Q\to 0^+$). In this case (35)

$$G(Q^+,Q^+) = G(Q^+,Q^-) \;=\; -\frac{i}{2}\begin{pmatrix} \xi - \dfrac{2i\alpha}{\Delta}e^{2i\eta Q}(\xi^2-1) & 1 + \dfrac{2i\alpha}{\Delta}e^{2i\eta Q}\xi^{-1}(\xi^2-1) \\ 1 + \dfrac{4\alpha^2}{\Delta}e^{2i\eta Q}(\xi^2-1) & \xi^{-1}\left[1 - \dfrac{4\alpha^2}{\Delta}e^{2i\eta Q}(\xi^2-1)\right] \end{pmatrix}, \qquad (41)$$

$$G(Q^-,Q^-) = G(Q^-,Q^+) \;=\; -\frac{i}{2}\begin{pmatrix} \xi - \dfrac{2i\alpha}{\Delta}e^{2i\eta Q}(\xi^2-1) & -1 + \dfrac{2i\alpha}{\Delta}e^{2i\eta Q}\xi^{-1}(\xi^2-1) \\ -1 + \dfrac{4\alpha^2}{\Delta}e^{2i\eta Q}(\xi^2-1) & \xi^{-1}\left[1 - \dfrac{4\alpha^2}{\Delta}e^{2i\eta Q}(\xi^2-1)\right] \end{pmatrix}, \qquad (42)$$

becomes

$$\begin{aligned}&(4-A^2)(4-B^2) - 16AB = \\ &(4-2A-2B-AB)(4+2A+2B-AB)\,\epsilon^2 \\ &+4(A+B)(4-AB)\,\epsilon\,\sqrt{1-\epsilon^2},\end{aligned}$$

whose two solutions are

$$\epsilon_1 = 1 - \frac{8(A+B)^2}{(4+A^2)(4+B^2)}, \qquad \epsilon_2 = -\epsilon_1, \qquad (38)$$

$\epsilon_1$ being the eigenvalue of the particle bound state and $\epsilon_2$ the energy of the antiparticle bound state. This is a simple and elegant solution of the problem, and it can be easily proved that, for any value of $A$ and $B$, the energies of the bound states are such that $-1 < \epsilon_{1,2} < 1$.

Nevertheless, there is something apparently strange in this result: if the two Dirac delta terms of the singular potential $V(x)$ in (17) or $-A\delta(X) - B\delta(X-Q)$, that appear in the Dirac equation (20), (21), coalesce at the origin ($Q \to 0^+$), one should naively expect to obtain a result like (37), with $(A+B)$ instead of $A$. But this is not at all the result of (38). The explanation is simple: if we consider both equations (20), (21), taking the derivative of any of them it is possible to eliminate either $\psi_1$ or $\psi_2$, and we get for the other function a second-order differential equation of Schrödinger type in which the "effective" potential is not just a combination of Dirac deltas such as $-A\delta(X) - B\delta(X-Q)$, but

$$\begin{aligned}V_{eff}(X) \;&=\; c_0\delta(X) + c_1\delta(X-Q) \\ &\quad +d_0\delta'(X) + d_1\delta'(X-Q),\end{aligned}$$

that is, a linear combination of Dirac deltas and their derivatives at $X=0$ and $X=Q$, a problem which was only recently considered in the literature [34]. The special case in which $Q \to 0^+$ was studied in detail by Gadella *et al.* [35], who showed that in this limit the process is not Abelian or additive, contrary to what one could in principle expect. Hence, the results (38) obtained here in a relativistic context are in perfect agreement with those of [35].

*Green function calculation.* To conclude this section, we provide an application to the relativistic bound-state energy of the perturbation $V(X) = -A\delta(X)\mathbb{I}$ in the presence of an additional potential $V_1(X) = -B\delta(X-Q)$, but now using the Green function $G(X,Y)$ evaluated in (13). In this case, the two-component wave function satisfies the integral equation

$$\begin{aligned}\psi(X) \;&=\; \int_{-\infty}^{\infty} G(X,Y)V_1(X)\psi(Y)\mathrm{d}Y \\ &= -\frac{B}{2}G(X,Q^+)\psi(Q^+) - \frac{B}{2}G(X,Q^-)\psi(Q^-), \quad (39)\end{aligned}$$

which, in the usual way, leads to the $4\times 4$ determinant consistency relation[1]

$$\begin{vmatrix} I + \dfrac{B}{2}G(Q^+,Q^+) & \dfrac{B}{2}G(Q^+,Q^-) \\ \dfrac{B}{2}G(Q^-,Q^+) & I + \dfrac{B}{2}G(Q^-,Q^-) \end{vmatrix} = 0. \qquad (40)$$

Since in the present case, by symmetry, we can assume that $Q > 0$, $\tau_X = \tau_Y = 1$, and there are only two independent cases of the Green function: $G(Q^+,Q^+) = G(Q^+,Q^-)$ and $G(Q^-,Q^-) = G(Q^-,Q^+)$, which are

*see eqs.* (41) *and* (42) *above*

By defining, in analogy with $\alpha$, the parameter $\beta = b/4$, the determinant (40) reduces to

*see eq.* (43) *on the next page*

It is the vanishing of this determinant that determines the new energy levels. Observe that as the energy is embeded in both $\xi$ and $\eta$, given by (7), eq. (43) is a transcendental equation that can be considered to generalize the well-known Lambert equation [32,33]. Note that if we set $a = A = 0 = \alpha$, we find the two bound-state solutions (14), (15) with $A$ replaced by $B$ (or, if you prefer, $\alpha$ replaced by $\beta$), as expected.

[1]There is a subtlety in the use of (39). In general this equation has an imhomogeneous term corresponding to the unperturbed wave function. However, the condition for a bound state is that the RHS becomes infinite at the correct energy value, so that the inhomogeneous term can be neglected. Indeed, eq. (40) is the condition for selecting this energy. Also note that $B$ cannot be set to zero without restoring the inhomogeneous term.

$$\left|\begin{array}{cc} 1-2i\beta\xi\left[1-\frac{2i\alpha}{\Delta}e^{2i\eta Q}\frac{(\xi^2-1)}{\xi}\right] & \frac{4\alpha\beta}{\Delta}e^{2i\eta Q}\frac{(\xi^2-1)}{\xi} \\ -\frac{8i\alpha^2\beta}{\Delta}e^{2i\eta Q}(\xi^2-1) & 1-\frac{2i\beta}{\xi}\left[1-\frac{4\alpha^2}{\Delta}e^{2i\eta Q}\frac{(\xi^2-1)}{\xi}\right] \end{array}\right| = 0. \quad (43)$$

In this case we got the spectrum from the wave function $\psi$. If we calculate the new Green function $\mathcal{G}$ (starting from $G$), this determinant would appear in a denominator.

**The Feynman propagator.** – A related quantity is the Feynman propagator $K(x,x';t)$ whose evaluation is generally carried out by the technically demanding procedure of path integration. However, as pointed out by Moshinsky *et al.* [36]

$$K(x,y;t)=\frac{1}{2\hbar ci}\int_{-\infty+ik}^{\infty+ik} e^{-iEt/\hbar}G(x,y;E)\,\mathrm{d}E, \quad (44)$$

where $k>0$, provides a simpler algorithm.

In the free-particle case (6), invoking translational symmetry, we have for $K_0(x,0;t)$:

$$\frac{-1}{4\hbar^2c^2}\int_{-\infty+ik}^{\infty+ik}\mathrm{d}E\; e^{-iEt/\hbar}e^{i\eta|X|}\begin{pmatrix}\xi & \tau_X\\ \tau_X & \xi^{-1}\end{pmatrix}. \quad (45)$$

By introducing the dimensionnless variables $\epsilon=E/mc^2$, $v=mc^2t/\hbar$, and closing the contour into the lower-half $\epsilon$-plane, while avoiding the branch cut $[-1,1]$, one has for $K_0(x,0;t)\equiv K_0(X,0;v)$:

$$\frac{-m}{4\hbar^2}\oint_\gamma \mathrm{d}\epsilon\; e^{-i(v\epsilon-|X|\sqrt{\epsilon^2-1})}\begin{pmatrix}\sqrt{\frac{\epsilon+1}{\epsilon-1}} & \tau_X\\ \tau_X & \sqrt{\frac{\epsilon-1}{\epsilon+1}}\end{pmatrix}, \quad (46)$$

where $\gamma$ is a clockwise loop enclosing the branch cut. This amounts to integrating the discontinuity of the integrand from $u=-1$ to $u=1$. The resulting integrals are Bessel functions, thus reproducing the Jacobson-Schulman [37] free-particle propagator derived by path integration in 1983 ($\hbar=c=1, T=\sqrt{t^2-x^2}$)

$$K_0(x,0;t)=\frac{m}{2}\begin{pmatrix}-\frac{x+t}{T}J_1(mT) & iJ_0(mT)\\ iJ_0(mT) & \frac{x-t}{T}J_1(mT)\end{pmatrix}. \quad (47)$$

A similar procedure may be applied to (13), but due to the complexity of the result, it is not given here.

**Discussion.** – In the present work, within the framework provided by Jackiw [38] and Wodkiewicz [39] for two one-dimensional point potentials of the form $-a\delta(x)$ and $-a\delta(x)-b\delta(x-q)$ the eigenenergies of the bound states for the relativistic Dirac equation were obtained either using the Green function approach or solving the Dirac equation directly. In the second case, when $q\to 0$ an unexpected solution results, which can be understood in terms of an effective potential in which not only $\delta(x)$ but also $\delta'(x)$ singularities are present. These problems may also be approached by the method of self-adjoint extensions, but we have not implemented that here.

The Green function developed here can now be used in various contexts. For example it can be applied as a component in the Green function matching scheme developed in [40] and [41] for obtaining relativistic surface and interface state energies or for dealing with the relativistic analogues of the composite quantum systems studied in [42]. Work in this direction is presently in progress.

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