CHARACTERIZATION OF COCYCLE ATTRACTORS FOR NON-AUTONOMOUS REACTION-DIFFUSION EQUATIONS

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Abstract. In this paper we describe in detail the global and cocycle attractors related to non-autonomous differential equations with diffusion. The associated semiflows are strongly monotone which allows us to give a full characterization of the cocycle attractor. In particular, we prove that the flow is persistent in the positive cone, and we study the stability and the set of continuity points of the associated minimal set acting as the global attractor for the skew product semiflow. We illustrate our result with some non-trivial examples showing the richness of the dynamics on this attractor, which in some situations can be even characterized as a pinched set with internal chaotic dynamics in the Li-Yorke sense. We also include the sublinear and concave cases in order to go further in the characterization of the attractors, coping, for instance, a non-autonomous version of the Chafee-Infante equation. In this last case we can show exponentially forwards attraction to the cocycle (pullback) attractors.

1. Introduction

The topological and geometrical description of the global attractor of an infinite-dimensional dynamical system is always a difficult task, so that there is only a small set of examples for which a full characterization of their attractors is available. One of these classical models is the Chafee-Infante equation, for which the attractor consists of an odd number of stationary points (which bifurcate from the origin) and the unstable manifolds joining them (Hale [Hale (1988)]; Henry [Henry (1981)], Chafee-Infante [Chaffe & Infante (1974)]; Robinson [Robinson (2001)]).

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Carvalho et al. [Carvalho *et al.* (2012)] study the asymptotic behaviour of the following non-autonomous version of the Chafee-Infante equation:

$$\begin{cases} u_t = u_{xx} + \lambda u - \beta(t)u^3, \ 0 \leq x \leq \pi \text{ and } t > \tau \\ u(0,t) = u(\pi,t) = 0 \\ u(x,\tau) = \phi(x), \end{cases} \tag{1.1}$$

where $\lambda \in [0,\infty)$ and $\phi \in X := H_0^1(0,\pi)$. It is proved for (1.1) that if $\beta(t)$ is a small non-autonomous perturbation of an autonomous β_0, then the associated pullback attractor can be described in a similar manner as the global attractor for the autonomous case. However, when we want to study the asymptotic dynamics of (1.1) when $\beta(t)$ is not a small non-autonomous perturbation of an underlying autonomous system we are not able to go much further in the description of the structure of the attractor.

In this paper we want to focus on the simplest cases of an infinite dimensional dynamical system, and show the extreme richness of the dynamics. Indeed, we will study non-autonomous scalar parabolic equation with Neumann or Robin boundary conditions in the positive cone of solutions, which will include the Chafee- Infante equation as a particular case. Even in this situation, and for almost-periodic non-autonomous terms, we will find that the attractor in th positive cone can be characterized as a pinched set for which even chaotic behavior holds (see Section 4). In the particular case of the Chafee-Infante equation, we will prove that, in the positive cone, there exists a complete bounded trajectory acting as an exponential forwards attractor (see Theorem 5.1).

2. Basic notions

Let $(P,\sigma,\mathbb{R})$ be a minimal, almost-periodic flow on a compact metric space (P,d_P). We consider an open bounded domain U in $\mathbb{R}^m$, $m \geqslant 1$ with enough regular boundary ∂U. Define the shift operators $\theta_t : P \to P$ as $\theta_t p = p(t+\cdot)$.

The goal of this paper is to investigate the behavior of solutions of the family of reaction-diffusion equations

$$\frac{\partial y}{\partial t} = \Delta y + h(\theta_t p, x)y + g(\theta_t p, x, y), \ \ x \in \bar{U}, t \geqslant 0 \tag{2.1}$$

with boundary condition

$$By := \alpha(x)y + \frac{\partial y}{\partial n} = 0$$

on ∂U. Here, Δ denotes the Laplace operator on U, $\frac{\partial}{\partial n}$ denotes the outward normal derivative on the boundary and the coefficient $\alpha : \partial U \to \mathbb{R}$ is sufficiently regular.

Let $h : P \times \bar{U} \to \mathbb{R}$ be a function with a Lipschitz variation on trajectories of P, that is, there exists $L > 0$ such that

$$|h(\theta_{t_1}p, x) - h(\theta_{t_2}p, x)| \leqslant L|t_1 - t_2|$$

for all $p \in P, x \in \bar{U}, t_1, t_2 \in \mathbb{R}$.

Denote by $g : P \times \bar{U} \times \mathbb{R} \to \mathbb{R}$ a continuous function with continuous first and second derivatives with respect to u. In addition, $g_* \in \{g, \frac{\partial g}{\partial u}\}$ has local Lipschitz variation on the trajectory of P, i.e., there exist $L_r > 0$ such that

$$|g_*(\theta_{t_1}p, x, u) - g_*(\theta_{t_2}p, x, u)| \leqslant L_r|t_1 - t_2|$$

for all $p \in P, x \in \bar{U}, \|u\| \leqslant r, t_1, t_2 \in \mathbb{R}$.

We also assume that

$$g(p, x, 0) = \frac{\partial g}{\partial u}(p, x, 0) = 0 \text{ and } ug(p, x, u) \leqslant 0$$

for all $p \in P, x \in \bar{U}, u \in \mathbb{R}$ and

$$\lim_{|u| \to \infty} \frac{g(p, x, u)}{|u|} = -\infty$$

uniformly on $P \times \bar{U}$.

We consider the Banach space $X := C(\bar{U})$ with norm $|| \cdot ||$ of real and continuous functions on $\bar{U}$, and

$$X_+ = \{z : z(x) \geq 0 \text{ in } U\}$$

and

$$\text{Int}X_+ = \{z : z(x) > 0 \text{ in } U\}.$$

Our Banach space is strongly ordered, i.e., $\text{Int}X_+ \neq \emptyset$ and we can define a strong order relation in X as follows

$$z_1 \leqslant z_2 \Longleftrightarrow z_1 - z_2 \in X_+$$

$$z_1 < z_2 \Longleftrightarrow z_1 - z_2 \in X_+, z_1 \neq z_2$$

$$z_1 \ll z_2 \Longleftrightarrow z_1 - z_2 \in \text{Int}X_+$$

We also consider the differential operator $A_0 z := \Delta z$ defined on

$$D(A_0) := \{z \in C^2(U) \cap C^1(\bar{U}) |\, A_0 z \in C(\bar{U}),\ Bz = 0\}$$

Then A, the closure of A_0 in $C(\bar{U})$, it is the generator of a analytic semigroup $\{T(t)\}_{t \geqslant 0}$ which is strongly continuous, i.e. $T(t)$ is a compact operator for all $t > 0$.

We denote by $\tilde{h} : P \to X$, $\tilde{h}(p)(x) = h(p, x)$ for all $p \in P, x \in \bar{U}$. Similarly, $\tilde{g} : P \times X \to X$ is given by $\tilde{g}(p, z)(x) = g(p, x, z(x))$ for all $p \in P, x \in \bar{U}$.

We can then consider the family of Cauchy problems

$$\begin{cases} u'(t) = Au(t) + \tilde{h}(\theta_t p)u(t) + \tilde{g}(\theta_t p, u(t)),\ t \geqslant 0 \\ u(0) = z \in X \end{cases} \tag{2.2}$$

for each $p \in P$. In this case, there exists a unique mild solution $u(\cdot) := u(\cdot, p, z)$ which satisfies the integral equation

$$u(t) = T(t)z + \int_0^t T(t-s)[\tilde{h}(\theta_s p)u(s) + \tilde{g}(\theta_s p, u(s))]ds \tag{2.3}$$

for all $p \in P, t \geqslant 0$. In this case, $u : R_+ \times \bar{U} \to \mathbb{R}$ is a classic solution of (2.1) (see Smith [Smith (1995)]).

Now we can define an associated skew product semiflow as

$$\begin{aligned} S :\ \mathbb{R}_+ \times P \times X \to &\quad P \times X \\ (t, p, z) \mapsto &\quad (\theta_t p, \varphi(t, p)z) \end{aligned} \tag{2.4}$$

with $\varphi(t, p)z = u(t, p, z)$, which is well defined and continuous for each $p \in P, z \in X$.

Furthermore, by using the compactness of $T(t)$ for $t > 0$ and the variation of constants formula (see (2.3)), and following the arguments of [Travis & Webb (1974)], it is easy to prove that the application flow $S(t)$ is compact for each $t > 0$. More generally, we have:

Theorem 2.1. *Let $0 < s < t$ and B a bounded set in X. Then $C := \overline{S([s,t])(P \times B)}$ is a compact subset of $P \times X$.*

Now we consider the linear part of (2.1)

$$\frac{\partial y}{\partial t} = \Delta y + h(\theta_t p, x)y,\ x \in \bar{U}, t \geq 0 \tag{2.5}$$

with Neumann or Robin boundary conditions. Then, $y \equiv 0$ is a solution of (2.5). In a abstract way, we can represent this problem as

$$\begin{cases} v'(t) = Av(t) + \tilde{h}(\theta_t p)v(t), t \geqslant 0 \\ v(0) = z \in X \end{cases} \tag{2.6}$$

By using this solution, we obtain a linear skew product semiflow

$$\begin{aligned} L: \quad \mathbb{R}_+ \times P \times X \to & \quad P \times X \\ (t, p, z) \mapsto & \quad (\theta_t p, \phi(t, p)z) \end{aligned} \tag{2.7}$$

where $\phi(t, p)z := v(t, p, z)$.

Definition 2.2. *We say that the linear skew product L has exponential dichotomy on P if there are constants $\beta, c > 0$ and a family of projectors $\Pi_p : X \to X, p \in P$ such that*

(1) $\phi(t, p) \circ \Pi_p = \Pi_{\theta_t p} \circ \phi(t, p)\ \ \forall p \in P, t \geqslant 0$

(2) *For each $p \in P$ and $t \geqslant 0$, $\phi(t, p)|_{R_g(\Pi_p)} : R_g(\Pi_p) \to R_g(\Pi_{\theta_t p})$ is an isomorphism. Then, we can define $\phi(-t, p) := (\phi(t, p)|_{R_g(\Pi_p)})^{-1}$.*

(3)
$$\begin{aligned} \|\phi(t, p)(I - \Pi_p)\| \leqslant & \quad ce^{-\beta t}, \quad \forall p \in P, t \geqslant 0 \\ \|\phi(t, p)\Pi_p\| \leqslant & \quad ce^{\beta t}, \quad\ \ \forall p \in P, t \leqslant 0 \end{aligned}$$

The fundamental properties of the exponential dichotomy can be found in [Sacker & Sell (1974)] and [Chow & Leiva (1994)].

Given $\lambda \in \mathbb{R}$, we consider the associated skew product semiflows

$$\begin{aligned} L_\lambda: \quad \mathbb{R}_+ \times P \times X \to & \quad P \times X \\ (t, p, z) \mapsto & \quad (\theta_t p, e^{-\lambda t}\phi(t, p)z). \end{aligned}$$

Definition 2.3. *The Sacker-Sell spectrum is the set*

$$\Sigma(L) := \{\lambda \in \mathbb{R} : L_\lambda \textit{ has no exponential dichotomy } \}$$

The set $\rho := \mathbb{R} \setminus \Sigma(L)$ is called the resolvent of the linear skew product L.

For each $p \in P$, we define the Lyapunov Exponent by

$$\lambda_p := \overline{\lim_{t\to\infty}} \frac{\ln \|\phi(t,p)\|}{t}.$$

We define the Upper Lyapunov Exponent by

$$\lambda_P := \sup_{p\in P} \lambda_p.$$

Following Shen and Yi [Shen & Yi (1998)], we have $\lambda_P = \sup \Sigma(L) < \infty$.

The Sacker-Sell spectrum provides a decomposition of the bundle $P \times X$ in invariant subbundles associated with distinct intervals of $\Sigma(L)$ (see Sacker and Sell [Sacker & Sell (1974)] and Chow and Leiva[Chow & Leiva (1995)]) in which the dynamics of the semiflow becomes more simple.

In our context, the equations (2.1) and (2.5) or abstract versions (2.2) and (2.6) generate strongly monotone semiflows S and L in the sense that

$$u(t,p,z_1) \ll u(t,p,z_2) \text{ and } v(t,p,z_1) \ll v(t,p,z_2)$$

for all $t > 0, p \in P$ and $z_1, z_2 \in X$ with $z_1 < z_2$ (see [Smith (1995)]).

This monotone structure determines an important part of the spectral decomposition of the linear semiflow L, as shown in [Poláčik &Tereščák (1993)] and [Shen & Yi (1998)].

In this paper (P,σ) is uniquely ergodic and then, the continuous spectrum of L can be written as $\Sigma(L) = \{\lambda_P\} \cup \Sigma_1$ with $\sup \Sigma_1 < \lambda_P$ with $\{\lambda_P\}$ the upper Lyapunov exponent defined above.

To study the asymptotic behavior of a non-autonomous differential equation such as (2.1), we need to deal with the following dynamical systems:

a) the *skew-product semiflow* $\{S(t): \ t \geq 0\}$ defined on the product space $P \times X$,
b) the associated *non-autonomous dynamical system* $(\varphi,\theta)_{(X,P)}$ with $\varphi(t,s.p)y_0 = y(t+s,p,y_0)$.

Observe that these dynamical systems can possess associated attractors:

(i) A global attractor $\mathbb{A}$ for the skew-product semiflow $S(t)$,
(ii) a cocycle attractor $\{A(p)\}_{p\in p}$ for the cocycle semiflow φ, (see Definition 2.6, and Kloeden and Rasmussen [Kloeden and Rasmussen (2001)])

In this paper we always assume that the base flow $(P, \theta, \mathbb{R})$ is minimal. We first consider some topological notions.

Definition 2.4. *(i) A minimal set $K \subset P \times X$ is said an automorphic extension of the base P if, for some $p \in P$, $K \cap \Pi_P^{-1}(p)$ is singleton, with Π_P the projection on the first component of $P \times X$. In these conditions we say that the minimal set K is almost-automorphic when the flow on the base P is almost-periodic.*

(ii) A compact invariant set $K \subset P \times X$ is called a pinched set if there exists a residual set $P_0 \subset P$ such that $K \cap \Pi_P^{-1}(p)$ is a singleton for all $p \in P_0$ and $K \cap \Pi_P^{-1}(p)$ is not a singleton for all $p \notin P_0$.

Note that an invariant compact set $K \subset P \times X$ is almost automorphic if it is pinched and minimal.

Suppose that the associated skew product semiflow semigroup $\{S(t) : t \geqslant 0\}$ possesses a global attractor $\mathbb{A}$ on $P \times X$. We know that $\{S(t) : t \geqslant 0\}$ has a global attractor if and only if there exists a compact set $\mathbb{K} \subset P \times X$ such that

$$\lim_{t \to \infty} \operatorname{dist}(\Pi(t)\mathbb{B}, \mathbb{K}) = 0, \tag{2.8}$$

for any bounded subset $\mathbb{B}$ of $P \times X$, where dist denotes the Hausdorff semidistance between sets defined as

$$\operatorname{dist}(A, B) = \sup_{a \in A} \inf_{b \in b} d(a, b).$$

Definition 2.5. *(i) A* non-autonomous set *is a family $\{D(p)\}_{p \in P}$ of subsets of X indexed in p. We say that $\{D(p)\}_{p \in P}$ is an open (closed, compact) non-autonomous set if each fiber $D(p)$ is an open (closed, compact) subset of X.*

(ii) A non-autonomous set $\{D(p)\}_{p \in P}$ is invariant *under the NDS $(\varphi, \theta)_{(X,P)}$ if*

$$\varphi(t, p)D(p) = D(\theta_t p),$$

for all $t \geqslant 0$ and each $p \in P$.

Given a subset $\mathbb{E}$ of $P \times X$ we denote by $E(p) = \{x \in X : (x, p) \in \mathbb{E}\}$ the $p-$section of $\mathbb{E}$; hence

$$\mathbb{E} = \bigcup_{p \in p} \{p\} \times E(p) \tag{2.9}$$

Given a non-autonomous set $\{E(p)\}_{p\in P}$ we denote by $\mathbb{E}$ the set defined by (2.9).

Note that

$$\bigcup_{p\in p} E(p) = \Pi_X \mathbb{E},$$

where we denote by Π_X the projection on the second component in $P \times X$.

Definition 2.6. *Suppose P is compact and invariant and that $\{\theta_t : t \in \mathbb{R}\}$ is a group over P and $\theta_t^{-1} = \theta_{-t}$, for all $t > 0$. A compact non-autonomous set $\{A(p)\}_{p\in P}$ is called a* cocycle attractor *of $(\varphi,\theta)_{(X,P)}$ if*

- **(i)** *$\{A(p)\}_{p\in P}$ is invariant under the NDS $(\varphi,\theta)_{(X,P)}$; i.e., $\varphi(t,p)A(p) = A(\theta_t p)$, for all $t \geqslant 0$.*
- **(ii)** *$\{A(p)\}_{p\in P}$ pullback attracts all bounded subsets $B \subset X$, i.e., for all $p \in P$,*

$$\lim_{t\to+\infty} \operatorname{dist}(\varphi(t,\theta_{-t}p)B, A(p)) = 0.$$

We can now relate the concept of cocycle attractors for $(\varphi,\theta)_{(X,P)}$ with the global attractor for the associated skew–product semiflow $\{S(t) : t \geqslant 0\}$.

The following result can be found, for instance, in Propositions 3.30 and 3.31 in Kloeden and Rasmussen [Kloeden and Rasmussen (2001)], or Theorem 3.4 in Caraballo et al. [Caraballo *et al.* (2013)].

Theorem 2.7. *Let $(\varphi,\theta)_{(X,P)}$ be a non-autonomous dynamical system, where P is compact, and let $\{\Pi(t) : t \geqslant 0\}$ be the associated skew–product semiflow on $P \times X$ with a global attractor $\mathbb{A}$. Then $\{A(p)\}_{p\in P}$ with $A(p) = \{x \in X : (x,p) \in \mathbb{A}\}$ is the cocycle attractor of $(\varphi,\theta)_{(X,P)}$.*

The following result offers a converse (see Proposition 3.31 in [Kloeden and Rasmussen (2001)], or Lemma 16.5 in [Carvalho *et al.* (2013)]).

Theorem 2.8. *Suppose that $\{A(p)\}_{p\in P}$ is the cocycle attractor of $(\varphi,\theta)_{(X,P)}$, and $\{S(t) : t \geqslant 0\}$ is the associated skew–product semiflow. Assume that $\{A(p)\}_{p\in P}$ is uniformly attracting, i.e., there exists $K \subset X$ compact such that, for all $B \subset X$ bounded,*

$$\lim_{t\to+\infty} \sup_{p\in P} \operatorname{dist}(\varphi(t,\theta_{-t}p)B, K) = 0,$$

and that $\bigcup_{p\in P} A(p)$ is precompact in X. Then the set $\mathbb{A}$ associated with $\{A(p)\}_{p\in P}$, given by

$$\mathbb{A} = \bigcup_{p\in P} \{p\} \times A(p),$$

is the global attractor of the semigroup $\{\Pi(t) : t \geqslant 0\}$*.*

3. Cocycle attractors for reaction-diffusion equations

From now on we will write $p.t$ or simply pt for $\theta_t p$, $p \in P$.

We consider the non-autonomous reaction-diffusion equations with the regularity conditions of Section 2,

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta y + h(\theta_t p, x)y + g(\theta_t p, x, y) = \Delta y + G(\theta_t p, x, y) \\ By = 0 \text{ on } \partial U. \end{cases} \tag{3.1}$$

with Neumann ($By = \dfrac{\partial y}{\partial t} = 0$) or Robin ($By = \alpha(x)y + \dfrac{\partial y}{\partial t} = 0$) boundary conditions.

In general $\Sigma(L) = \Sigma_p(L) \cup \Sigma_1(L)$ with $\Sigma_p(L) = [\alpha_p, \lambda_p]$ and $\sup \Sigma_1(L) < \alpha_p$. If (P, σ) is uniquely ergodic, then $\Sigma_p(L) = \{\lambda_p\}$ is a singleton.

The following concepts and results are in [Núñez *et al.* (2010)] and [Núñez *et al.* (2012)].

Definition 3.1. *A Borel map* $a : P \to X$ *such that* $\varphi(t, p)a(p)$ *is defined for any* $t \geqslant 0$ *is said to be*

a) *an equilibrium if* $a(\theta_t p) = \varphi(t, p)a(p)$*, for any* $p \in P$ *and* $t \geqslant 0$*,*

b) *a super-equilibrium if* $a(\theta_t p) \geqslant \varphi(t, p)a(p)$*, for any* $p \in P$ *and* $t \geqslant 0$*,*

c) *a sub-equilibrium if* $a(\theta_t p) \leqslant \varphi(t, p)a(p)$*, for any* $p \in P$ *and* $t \geqslant 0$*.*

Definition 3.2. *A super-equilibrium (resp. sub-equilibrium)* $a : P \to X$ *is semi-continuous if the following holds*

i) $\Gamma_a = \text{closure}_X\{a(p) : p \in P\}$ *is a compact subset in* X*;*

ii) $C_a = \{(p, x) : x \leqslant a(p)\}$ *(resp.* $C_a = \{(p, x) : x \geqslant a(p)\}$*) is a closed subset of* $P \times X$*.*

An equilibrium is semi-continuous if it holds i) and ii) above. We name a semi-equilibrium for a sub-equilibrium or a super-equilibrium. Every semi-continuous semi-equilibrium admits a residual invariant set of continuity points.

Definition 3.3. *A super-equilibrium (resp. sub-equilibrium)* $a : P \to X$ *is strong if there exists* $\delta > 0$ *such that* $a(p \cdot \delta) \gg u(\delta, p, a(p))$ *(resp.* $a(p \cdot \delta) \ll u(\delta, p, a(p))$*) for all* $p \in P$*.*

Proposition 3.4. i) *If $a(\cdot)$ is a semi-continuous super-equilibrium (resp. sub-equilibrium) and there exists $\delta > 0$, $p_0 \in P$ point of continuity of $a(\cdot)$ such that $a(p_0 \cdot \delta) \gg u(\delta, p_0, a(p_0))$ (resp. $\ll$), then $a(\cdot)$ is strong.*

ii) *If $a(\cdot)$ is a strong semi-continuous super-equilibrium (resp. sub-equilibrium), then there exists $e \gg 0$ and $\delta > 0$ such that $u(s,p,a(p)) + e \leqslant a(p \cdot s)$, (resp. $u(s,p,a(p)) - e \geqslant a(p \cdot s)$) for all $p \in P$ and $s \geqslant \delta$.*

Theorem 3.5. *Let $a : P \to X$ be continuous such that for all $p \in P$, the map $a_p : [0,\infty) \times \bar{U} \to \mathbb{R}$ given by $a_p(t,x) := a(pt)x$ is continuously differentiable in $(0,\infty) \times \bar{U}$, twice continuously differentiable with respect to $x \in U$ for all $t > 0$ and satisfies the boundary condition*

$$Ba_p(t,x) = 0 \text{ for all } x \in \partial U, p \in P.$$

Denote by

$$a'(p)(x) := \frac{\partial}{\partial t} a(\theta_t p)(x)|_{t=0} \text{ for all } p \in P, x \in \bar{U}$$

If $a'(p)(x) \geqslant \Delta a(p)(x) + G(p,x,a(p)(x))$ for all $p \in P, t \geqslant 0$, then $a(\cdot)$ is a strong super-equilibrium. Furthermore, if $a'(p)(x_0) > \Delta a(p_0)(x_0) + G(p_0, x_0, a(p_0)(x_0))$ for some $p_0 \in P, x_0 \in U$, then the super-equilibrium is strong.

Proof. The proof is in [Núñez *et al.* (2010)] (Lemma 2.11 (ii)) in the Neumann case. We recall the arguments for Robin boundary conditions. The fact that a is a super-equilibrium is a standard argument by comparison ([Fife & Tang (1981)]). So we have $a(p \cdot s) \geqslant u(s,p,a(p))$, $s \geqslant 0$, $p \in P$. To prove that the equilibrium is strong, we apply the following argument

$$a(pt) \geqslant u(t,p,a(p)), \ \ \forall t \geqslant 0, p \in P.$$

Furthermore, there exist $\epsilon_0 > 0$ (near to 0) with $a(p_0 \cdot \epsilon_0) > u(\epsilon_0, p_0, a(p_0))$. Since the flow is strongly monotone, if $t = t_0 + \epsilon_0, t_0 > 0$, then

$$a(\theta_t p) \geqslant u(t_0, p_0 \cdot \epsilon_0, a(p_0 \cdot \epsilon_0)) \geqslant u(t_0, p_0 \cdot \epsilon, u(\epsilon, p_0, a(p_0))) = u(t, p_0, a(p_0))$$

and the result follows by Proposition 3.4 (i). □

We consider the first eigenvalue $\lambda_0 \geqslant 0$ and the correspondent eigenfunction $e_0 \in \text{Int}X_+, \|e_0\| = 1$. There exists $\delta > 0$ such that $\inf_{x \in \bar{U}} e_0(x) = \delta$.

We choose $r^* > 0$ such that:

- if $r \geqslant r^*\delta$, then $G(p,x,y) > 0 \ \forall p \in P, x \in \bar{U}, y \geqslant r$
- if $r \leqslant r^*\delta$, then $G(p,x,y) < 0 \ \forall p \in P, x \in \bar{U}, y \geqslant r$.

The applications $a : P \to X, p \mapsto a_p(x) = re_0(x) \ \forall x \in \bar{U}$ are:

- strong super-equilibrium if $r \geqslant r^*$
- strong sub-equilibrium if $r \leqslant -r^*$

If $r_1, r_2 \in \mathbb{R}, r_1 \leqslant r_2$, we denote

$$[r_1e_0, r_2e_0] := \{z \in X : r_1e_0 \leqslant z \leqslant r_2e_0\}$$

We consider $C_1 := S(1)(P \times [-r^*e_0, r^*e_0])$ which is a compact subset of $P \times X$.

Proposition 3.6. *C_1 is an absorbing compact set, i e, given $(p,z) \in P \times X$, there exists $t_0 = t_0(p,z)$ such that $S(t,p,z) \in C_1$ for all $t \geqslant t_0$.*

Proof. Is sufficient to prove that the set $P \times [-r^*e_0, r^*e_0]$ is absorbing.

Consider $z = re_0$ with $r \geqslant r^*$. We define

$$L_r := \{r_1 \in [r^*, r] : \exists\, t(r_1) > 0 \text{ such that } u(p,t,re_0) \ll r_1e_0 \ \forall p \in P, t \geqslant t(r_1)\}$$

Since a_r is a strong super-equilibrium, it follows that $r \in L_r$. Moreover, if $r_1 \in L_r$, then $[r_1, r] \subset L_r$. Define $r_2 := \inf L_r$. We will prove by contradiction that $r_2 = r^*$. Suppose $r^* < r_2 \leqslant r$. Then $u(t,p,r_2e_0) \ll r_2\rho_0$ for all $p \in P, t \geqslant 0$ (by strong super-equilibrium properties). Fixed $t_1 > 0$, there exists $\epsilon > 0$ such that $u(t_1,p,r_3e_0) \ll (r_2 - \epsilon)e_0$ for all $r_3 \in [r^*, r^* + \epsilon], p \in P$.

Fix $r_2 + \epsilon$, there exists $t_2 = t(r_2 + \epsilon)$ with $u(t_2,p,r\rho_0) \ll (r_2 + \epsilon)\rho_0$. Thus

$$u(t + t_2, p, re_0) = u(t, \theta_t p_2, u(t_2,p,re_0)) \ll u(t, \theta_t p_2, (r_2+\epsilon)e_0) \ll (r_2 - \epsilon)e_0$$

for all $t \geqslant t_1$. This contradicts the definition of r_2 and then $L_r = [r^*, r]$, i e, for all $x = re_0$, there exists $t(r)$ with $u(t,p,re_0) \ll r^*e_0$, $\forall t \geqslant t(r), p \in P$.

Similarly, for any $x = -re_0, r > 0$, there exists $t(r)$ with $u(t,p,-re_0 \gg -r^*e_0)$ for all $t \geqslant t(r)$.

Finally, for each $z \in X$, there exists $r > 0$ such that $-re_0 \leqslant z \leqslant re_0$, so that the conclusion holds for all $(p,z) \in P \times X$ and the set $P \times [-r^*e_0, r^*e_0]$ is absorbing.

Finally, $C = S(1)(P \times [-r^*e_0, r^*e_0])$ is compact absorbing. ∎

The arguments used in the theorem below are in [Cheban *et al.* (2002)], [Kloeden and Rasmussen (2001)], [Caraballo *et al.* (2013)].

Theorem 3.7. *The non-linear skew product semiflow* (2.4) *generated by* (2.1) *admits a global attractor* $\mathbb{A} = \bigcup_{p\in P}\{p\} \times A(p) \subset P \times B_{r^*}$*. Furthermore, the family* $\{A(p)\}_{p\in P}$ *with* $A(p) := \{z \in X : (p,z) \in \mathbb{A}\}$ *is a cocycle attractor of the non-autonomous system* $(\varphi,\theta)_{(X,P)}$.

We now use the method of construction of the cocycle (pullback) attractor described in section 3 of [Caraballo *et al.* (2013)].

Proposition 3.8. *Let* $r \geqslant r^*$ *and*

$$a_T(p) = u(T, p\cdot(-T), -re_0), \quad \text{and} \quad b_T(p) = u(T, p\cdot(-T), re_0),$$

for each $p \in P$*. Then*

$$a(p) := \lim_{T\to\infty} a_T(p) \quad \text{and} \quad b(p) := \lim_{T\to\infty} b_T(p).$$

are well defined and are semi-equilibrium. Moreover,

$$a(p) = \min\{x \in X : x \in A(p)\} \quad \text{and} \quad b(p) = \max\{x \in X : x \in A(p)\}$$

for each $p \in P$.

Proof. Since S has global attractor $\mathbb{A}$, it follows that

$$\text{dist}_H(S(T)(P \times \{-r^*e_0\}), \mathbb{A}) \to 0 \text{ as } T \to +\infty;$$

and then

$$d(S(T)(p\cdot(-T), -r^*e_0), \mathbb{A}) \to 0 \text{ as } T \to \infty;$$

but $S(T)(p\cdot(-T), -r^*e_0) = (p, u(T, p\cdot(-T), -r^*e_0)) = (p, a_T(p))$ so that

$$d(a_T(p), A(p)) \to 0 \text{ as } T \to \infty;$$

and then, for each sequence $\{T_n\}_{n\in\mathbb{N}}$ with $T_n \overset{n\to\infty}{\longrightarrow} \infty$, there exists a subsequence $\{T_{n_k}\}_{k\in\mathbb{N}}$ with $a_{T_{n_k}} \overset{k\to\infty}{\to} x$ for some $x \in A(p)$. Let $a \in A(p)$. Then there exist $a_k \in A(p\cdot(-T_{n_k}))$ such that $u(T_{n_k}, p\cdot(-T_{n_k}), a_k) = a$ and so

$$a_{T_{n_k}}(p) = u(T_{n_k}, p\cdot(-T_{n_k}), -r^*e_0) \leqslant u(T_{n_k}, p\cdot(-T_{n_k}), a_k) = x$$

so that $x \leqslant a$. Thus $x = \min A(p)$ and $a_T(p) \stackrel{T\to\infty}{\to} a(p) = \min A(p)$ for each $p \in P$.

By monotonicity we have

$$a_T(p) = u(T, p\cdot(-T), -r^* e_0) \leqslant u(T, p\cdot(-T), a(p\cdot(-T))) = a(p)$$

for all $p \in P$. Thus

$$\{(p,x):\ x \geqslant a(p)\} \supseteq \bigcap_{T\geqslant 0}\{(p,x):\ x \geqslant a_T(p)\}$$

On the other hand, if $x \geqslant a_T(p)$ for all $T \in [0,\infty)$, so that $x \geqslant a(p)$. Then

$$\{(p,x):\ x \geqslant a(p)\} = \bigcap_{T\geqslant 0}\{(p,x):\ x \geqslant a_T(p)\}$$

The proof of properties for $b(\cdot)$ is analogous. □

In the following we consider a family of linear equations that are relevant to our paper. Let $\lambda_0 \geqslant 0$ the first eigenvalue and $e_0 \in X$ the eigenfunction for

$$\begin{cases} \Delta u + \lambda u = 0, & x \in U \\ Bu = 0, & x \in \partial U \end{cases} \tag{3.2}$$

We now that $e_0(x) \geqslant \text{const} > 0$ for all $x \in \bar{U}$ and $\|e_0\| = 1$.

Theorem 3.9. *Let $a, b : P \to X$ defined above. Then*

(i) *The functions a, b admit a residual set P_r of points of continuity.*

(ii) *The function b (resp. a) satisfies one of the two conditions:*

  (ii1) *$b(p) = 0$ (resp. $a(p) = 0$) for all $p \in P$ or*

  (ii2) *there exist $\lambda_0 > 0$ such that $b(p) \gg \lambda_0 e_0 \gg 0$ (resp. $a(p) \ll -\lambda_0 e_0$) for all $p \in P$.*

Proof. The proof is similar to Proposition 15 of [Caraballo *et al.*]. By the strong monotonicity, it is easy to prove that $b(p) = 0$ implies $b(\theta_t p) = 0, \forall t \in \mathbb{R}$. Otherwise we would have $b(p) \gg 0$. In fact, $b(p) = u(t, p\cdot(-t), b(p\cdot(-t)))$.

Suppose there exists $p_1 \in P$ with $b(p_1) \gg 0$, i.e., does not satisfy $(ii1)$. We can assume $b(p_1) \geqslant 2\lambda_1 e_0$ for some $\lambda_1 > 0$. By continuity, there exist $r_1 > 0$ such that if $\text{dist}(p, p_1) \leqslant r_1$, then $b(p) \geqslant \lambda_1 e_0 \geqslant 0$.

Since (P,σ) is minimal, there exists $\tau > 0$ such that if $p \in P$, we can find $t = t(p) \in [0,\tau]$ with $p\cdot(-t) \in \overline{B(p_1,r_1)}$. Let $D := [0,\tau]\times\overline{B(p_1,r_1)}\times(\Pi_X(C_1)\cap[\delta e_0, r^* e_0])$ for some $\delta > 0$ and C_1 the compact set defined in Proposition 3.6. Then

$$\begin{aligned} u : D \to &\ \ \mathrm{Int}X_+ \\ (t,p,z) \mapsto &\ \ u(t,p,z) \gg 0 \end{aligned}$$

is continuous and strongly positive. So there exists $\delta_1 > 0$ with $u(t,p,z)(x) \geqslant \delta_1$ for all $(t,p,z,x) \in D\times\bar{U}$. Then there exists $\lambda_0 > 0$ with $u(t,p,z) \geqslant \lambda_0 e_0 \geqslant 0$ for all $(t,p,z) \in D$.

Let $p \in P$ and $t = t(p)$ with $p\cdot(-t) \in \overline{B(p_1,r_1)}$ and then $b(p\cdot(-t)) \in \Pi_X(C_1)\cap[\delta e_0, r^* e_0]$. In fact,

$$b(p\cdot(-t)) = u(1, p\cdot(-t-1), b(p\cdot(-t-1)))$$

with $b(p\cdot(-t-1)) \in P\times[-r^* e_0, r^* e_0]$. Thus,

$$b(p) = u(t, p\cdot(-t), b(p\cdot(-t))) \geqslant \lambda_0 e_0$$

which is the item $(ii2)$.

The result is analogous for $a(\cdot)$. □

Now we will characterize the structure of cocycle attractor for (2.1) as a function of the upper Lyapunov exponent λ_p of the linear equation (2.5). In particular, we analyze the cases when $\lambda_P \neq 0$.

Note that (2.5) is a linearized version of (2.1) on the solution $y \equiv 0$.

Denote by λ_P the upper Lyapunov exponent of the linear semiflow (2.7) generated by (2.5).

Theorem 3.10. *Suppose $\lambda_P < 0$. Then it holds:*

(i) *For all $0 < \epsilon < |\lambda_P|$, there exist C_ϵ such that*

$$\|u(t,p,z)\| \leqslant C_\epsilon e^{(\lambda_P+\epsilon)t}\|z\|,\ \forall t \geqslant 0, p \in P, z \in X.$$

(ii) *The global attractor of the skew product semiflow* (2.4) *is* $\mathbb{A} = P\times\{0\}$.

Proof. Let $z \in X$, then $-|z(x)| \leqslant z(x) \leqslant |z(x)|$ for all $x \in U$. The monotonicity of the semiflow implies $u(t,p,|z|) \geqslant 0$ and furthermore

$$-u(t,p,|z|) \leqslant u(t,p,z) \leqslant u(t,p,|z|)$$

for all $t \geqslant 0, p \in P$. Standard comparison arguments for parabolic equations imply

$$0 \leqslant u(t,p,|z|) \leqslant v(t,p,|z|)$$

for all $t \geqslant 0, p \in P$.

Let $0 < \epsilon < |\lambda_P|$. Then there exist $C_\epsilon > 0$ (see Lemma 3.2 in [Chow & Leiva (1994)]) with

$$\|u(t,p,z)\| \leqslant \|u(t,p,|z|)\| \leqslant \|v(t,p,|z|)\| \leqslant C_\epsilon e^{(\lambda_P+\epsilon)t}\|z\|$$

for all $t \geqslant 0, p \in P$. This proves (i), from which it follows that $\mathbb{A} = P \times \{0\}$. □

Theorem 3.11. *Suppose $\lambda_P > 0$. Then:*

(i) *The semiflow (2.7) is uniformly persistent in the positive cone, i.e., there exist $\lambda_0 > 0$ such that for all $p \in P, z > 0$ (resp. $z < 0$) there exists $t_0 = t_0(p,z) > 0$ with $u(t,p,z) \geqslant \lambda_0 e_0$ (resp. $u(t,p,z) \leqslant -\lambda_0 e_0$) for all $t \geqslant t_0(p,z)$. Moreover, the semiflow S admits a global attractor $\mathbb{A}_+ \subset \mathbb{A} \cap (P \times \mathrm{int}X_+)$. in the positive cone.*

(ii) *Let $b(p) = \max\{x \in A(p)\}, p \in P$ and P_c^1 the residual set of continuity points of b. Then $b(p) \geqslant \lambda_0 e_0$ for all $p \in P$. Let $p_1 \in P_c^1$ and $K_1 := \overline{\{(p_1 \cdot t, b(p_1 \cdot t)) : t \in \mathbb{R}\}}$. Then (K_1, S) is a minimal flow which defines an almost automorphic extension of the base (P,σ) and if L_{K_1} is the linearized semiflow on K_1, then its principal spectrum satisfies $\Sigma_p(L_{K_1}) \cap (-\infty, 0] \neq 0$.*

(iii) *If $\lambda_{K_1} < 0$, then $b : P \to X_+$ is continuous, $K_1 := \{(p, b(p)) : p \in P\}$ and (K_1, S) is a minimal almost periodic flow.*

(iv) *The application $c : P \to X_+$ given by $c(p) := \min\{z \in X : (p,z) \in \mathbb{A}, u(t,p,z) \geqslant \lambda_0 e_0 \ \forall t \geq 0\}$ is well defined and $c(p) \in A(p)$ for each $p \in P$. Moreover, c is a semi-continuous equilibrium and admits a residual set P_c^2 of continuity points. Moreover, $A_+(p) = A(p) \cap [c(p), b(p)]$ for all $p \in P$.*

(v) *Let $p_2 \in P_c^2$ and $K_2 := \overline{\{(p_1 \cdot t, c(p_1 \cdot t)) : t \in \mathbb{R}\}}$. Then (K_2, S) is a minimal flow which defines an extension almost automorphic of the base (P,σ) and if L_{K_2} is the linearized semiflow on K_2, then its principal spectrum satisfies $\Sigma_p(L_{K_2}) \cap (-\infty, 0] \neq 0$.*

(vi) *If $\lambda_{K_2} < 0$, then $c : P \to X_+$ is continuous and (K_2, S) is an almost periodic flow.*

(vii) *If $\lambda_{K_1} < 0, \lambda_{K_2} < 0$ and $K_1 \neq K_2$, then there exists a minimal set $K_3 \subset \cup_{p\in P}\{p\} \times [c(p), b(p)]$ which is unstable on $\mathbb{A}$.*

Remark 3.12. *Similarly there exists a global attractor $\mathbb{A}_-$ for the restriction of the semi flow S to $(P \times \text{int}X_+)$, with a similar characterization as in the above theorem.*

Proof. (i) This item is proved in [Mierczyński & Shen (2004)] (Theorem C) and [Novo *et al.* (2013)] (Theorem 5.6).

(ii) Let $(p, z) \in P \times \text{Int}X_+$. Then there exist $t_0 = t_0(p, z)$ with $u(t, p, z) \geqslant \lambda_0 e_0$ for all $t \geqslant t_0$. Denote by C the omega limit of (p, z). Then $C \subset \mathbb{A}$. There exist $z' \in X_+$ with $(p, z') \in C$ and then $\lambda_0 e_0 \leqslant z' \leqslant b(p)$.

Follows from Theorem 3.9 that the function $b : P \to X_+$ is semi-continuous and admits a residual set P_c^1 of continuity points. Let $p_1 \in P_c^1$, $K := \overline{\{(p_1 \cdot t, b(p_1 \cdot t)) : t \in \mathbb{R}\}}$. The uniqueness properties for the backwards extension of the parabolic equations ([Temam (1988), Henry (1981)]) proves that (K, S) is a minimal semiflow.

Suppose $(p, x) \in K$ with $p \in P_c$ and t_n such that $\theta_{t_n} p_1 \to p$. Then, by continuity, we also have that $b(\theta_{t_n} p_1) \to b(p)$. Thus, $x = b(p)$ and $K_1 \cap \Pi_P^{-1}(p) = \{(p, a(p))\}$. This implies that K_1 are minimal semiflows and sections (in p) are singleton if $p \in P_c$, so that they are almost automorphic extensions of (P, θ).

Finally, let λ_{K_1} the upper Lyapunov exponent of the linear semiflow L_{K_1} given by linearization of (2.1) on the solutions in K_1. If the principal spectrum $\Sigma_p(L_{K_1}) \subset (0, \infty)$, it follows from [Novo *et al.* (2013)] that the semiflow is strongly persistent on K_1, i.e., there exist a minimal set $K' \subset P \times X_+$ and a constant $\lambda_0' > 0$ such that $(p, z') \in K'$ satisfies $b(p) + \lambda_0' e_0 \leqslant z'$. This contradicts the definition of b. Consequently $\lambda_{K_1} \leqslant 0$.

(iii) We now prove that (K_1, S) is an almost automorphic extension of the base (P, σ). Let us prove that it is a copy of the base $K = \{(p, b(p)) : p \in \text{P}\}$ with $b : P \to X$ continuous and moreover that it is exponentially stable. Let us fix $p^1 = (p, z_1) \in K_1$ and denote $p^1 t = S(t, p, z_1)$ for $t \geqslant 0$. We denote $y^1(t, x) = u(t, p, z_1)(x)$ for all $t \geqslant 0$, $x \in \bar{U}$.

The linearized equation through p^1 is defined by

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta y + h_1(p^1 t, x)y + h_2(p^1 t, x)y, \ t \geqslant 0 \\ By = 0, \ on\ \partial U. \end{cases} \tag{3.3}$$

where $h_1(p^1,x) = h(p^1,x)$, $h_2(p^1,x) = \dfrac{\partial g}{\partial y}(p^1,x,z^1(x))$. Its solutions generate a skew product semiflow $S^1 : \mathbb{R}_+ \times K \times X \to K \times X$, $(t,p^1,z) \mapsto (p^1t, u^1(t,p^1,z))$ with $u^1(t,p^1,z) = \phi^1(t,p^1)z$, $\phi^1(t,p^1) \in \mathcal{L}(X)$ for all $(t,p^1) \in \mathbb{R}_+ \times K$.

Given $0 < \lambda < |\lambda_{K_1}|$ there exists $c_1 = c(\lambda)$ with $|\phi(t,p_1)| \leq c_1 e^{-\lambda t}$, for all $p^1 \in K_1$, $t \geq 0$. Consider $(p,z_1) \in K_1$, $(p,z) \in P \times X$. Denote by $y(t,x) = u(t,p,z)(x)$, $y_1(t,x) = u(t,p,z_1)(x)$, for $t \geq 0, x \in U$. We now introduce a new variable $\hat{y}(t,x)$ as $\hat{y}(t,x) = y(t,x) - y_1(t,x)$ which satisfies

$$\begin{cases} \dfrac{\partial \hat{y}}{\partial t} = \Delta \hat{y} + h_1(p^1t,x)\hat{y} + h_2(p^1t,x)\hat{y} + g_1(p^1t,x,\hat{y}),\ t \geqslant 0 \\ B\hat{y} = 0,\ on\ \partial U. \end{cases} \tag{3.4}$$

where

$$g_1(p^1,x,y) = \int_0^1 [\frac{\partial g}{\partial y}(p,x,r_1(x)+\delta y) - \frac{\partial g}{\partial y}(p,x,r_1(x))]yds$$

so that we can write the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + h_1(p^1t)u(t) + h_2(p^1t)u(t) + \hat{g}_1(p^1t,u(t)),\ t \geqslant 0 \\ u(0) = z - z^1\ on\ \partial U. \end{cases} \tag{3.5}$$

Now, by the constants variation formula

$$u(t) = \Phi^1(t,p^1)(z-z^1) + \int_0^t \Phi(t-s,p^1s)\hat{g}_1(p^1s,u(s))ds.$$

Moreover, for each $0 < \epsilon < \lambda$ there exists $\sigma > 0$ such that

$$||\hat{g}_1(p^1,v)|| \leq \epsilon ||v|| \text{ for } ||v|| \leq \sigma.$$

Finally, by Gronwall inequality we get $c_2 > 0$ with

$$|\hat{y}(t,x)| \leq c_2 ||z-z_1|| e^{-(\lambda-\epsilon)t} \text{ for all } t \geq 0,$$

which implies (iii).

(iv) We consider the restriction of semiflow in $P \times \mathrm{Int}X_+$. It follows from Proposition 3.6 and item (i) that

$$D = P \times \{z \in X_+ : \lambda_0 e_0 \leqslant z, \|z\|_\infty \leqslant r^*\}$$

is an absorbing set. Consequently $C_3 := S(1)D$ is a compact absorbing. Thus, it follows from [Cheban *et al.* (2002)] and [Caraballo *et al.* (2013)] the existence of a global attractor in $\mathbb{A} \cap (P \times \mathrm{Int} X_+)$.

To see that $c : P \to X_+$ is semicontinuous equilibrium, just note that $c(p) = \lim_{T\to\infty} c_T(p)$ with $c_T(p) = u(T, p\cdot(-T), \lambda_0 e_0)$ and the proof follows as in Proposition 3.8.

(v) Let $p_2 \in P_c^2$ and $K_2 := \overline{\{(p_2\cdot t, c(p_2\cdot t)) : t \in \mathbb{R}\}}$ which is an almost automorphic extension of the base.

Consider the linearized semiflow on K_2, denoted by L_{K_2} and its principal spectrum $\Sigma_p(L_{K_2})$, then the semiflow is uniformly persistent below K_2. Consequently, as in ii), $\Sigma_p(L_{K_2}) \cap (-\infty, 0] \neq \emptyset$.

(vi) It is analogous to (iii).

(vii) We have $K_1 = \{(p, b(p)) : \ p \in P\}$, $K_2 = \{(p, c(p)) : \ p \in P\}$. Fix $p_0 \in P$. As $\mathbb{A}$ is connected there exists a continuous function $\gamma : \ [0,1] \to \mathbb{A}$ with $\gamma(0) = (p_0, b(p_0)$, $\gamma(1) = (p_0, c(p_0)$. Let

$$I = \{s_0 \in [0,1] \text{ such that } P(\gamma(s)) = K_1 \text{ for all } 0 \le s \le s_0\},$$

where $P(\cdot)$ denotes the omega-limit set. Since K_1 is exponentially stable, it is clear that $0 \in I$, $I \subset [0,1)$ and it is open. Let $\delta = \sup I$. If $P(\delta)) \cup K_1 \neq \emptyset$, then $P(\delta)) = K_1$ and this is true in a neighbourhood of δ, which s a contradiction. In the same way, $P(\delta)) \cup K_1 = \emptyset$. Moreover, there exists a minimal $K_3 \subset P(\delta))$. It is clear that K_3 is unstable in $\mathbb{A}$.

□

4. Examples

We next introduce some examples of dissipative differential equations that illustrate different properties of the global attractor $\mathbb{A}$ in the positive cone. Indeed, we consider the families of dissipative scalar parabolic equations with Neumann boundary conditions

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta y + y g(y), \ x \in [0,1], t \geqslant 0 \\ \dfrac{\partial y}{\partial x}(t,0) = \dfrac{\partial y}{\partial x}(t,1) = 0, \ t \geqslant 0. \end{cases} \tag{4.1}$$

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta y + \epsilon h(\theta_t p) y + y g(\theta_t p, y), \ x \in [0,1], p \in P, t \geqslant 0 \\ \dfrac{\partial y}{\partial x}(t,0) = \dfrac{\partial y}{\partial x}(t,1) = 0, \ t \geqslant 0. \end{cases} \tag{4.2}$$

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta y + y + \mu h(p \cdot (\mu t))y + yg(p \cdot (\mu t), y), \ x \in [0,1], p \in P, t \geqslant 0 \\ \dfrac{\partial y}{\partial x}(t,0) = \dfrac{\partial y}{\partial x}(t,1) = 0, \ t \geqslant 0. \end{cases} \tag{4.3}$$

whose coefficients and non-linear terms are defined in the Examples 1,2 and 3 below.

(ES EN ESTE PUNTO DONDE DEBEMOS PONER, SI ES POSIBLE, EL RESULTADO SOBRE SOLUCIONES ESPACIALMENTE HOMOGENEAS EN EL ATRACTOR)

The global attractors of these reaction-diffusion equations preserve the main dynamical properties of the associated ODE models, which we now pass to study in detail.

4.1. **Example 1: an autonomous equation.** We consider an autonomous ODE

$$\dot{x} = x + xg(x) \tag{4.4}$$

where $g \in C^1(\mathbb{R})$, $g(0) = 0$, $g(x) \leq 0$ for $x \geqslant 0$ and $\lim_{|x| \to \infty} g(x) = -\infty$. Let us fix $0 < x_1 < x_2 < \cdots < x_{2n}$. Define $k(x) = -(x - x_1) \cdots (x - x_{2n})$ and $g(x) = \dfrac{k(x) - x}{x}$ for $x \geqslant x_0$. We also assume a convenient definition of g on $(-\infty, x_0]$ such that all the previous conditions are satisfied.

If $x \geqslant x_0$, (4.4) becomes

$$\dot{x} = -(x - x_1) \cdots (x - x_{2n})$$

The constants $x_1, \ldots, x_{2n}$ are solutions. The attractor in the positive cone has x_{2n} as the upper equilibrium which is asymptotically stable, and x_i, $i = 1, \cdots, 2n-1$ has one stable and one unstable direction, joining the ordered sequence of equilibria. This clarifies the statement of the theorem.

4.2. **Example 2: a pinched cocycle attractor.** We define

$$B(P) := \{d \in C_0(P) : \sup_{t \in \mathbb{R}} \left| \int_0^t d(p \cdot s) \, ds \right| < \infty, \, \forall p \in P\}$$

and $U(P) := C_0(P) \setminus B(P)$.

$B(P), U(P)$ are dense subspaces of $C_0(P)$. $B(P)$ is a first category set in $C_0(P)$ and $U(P)$ is a residual set in $C_0(P)$ (see [Gottschalk & Hedlund (1955)]).

We consider the non-autonomous family of ODEs

$$\dot{x} = x + \epsilon h(\theta_t p)x + xg(\theta_t p, x),\ p \in P \tag{4.5}$$

where ϵ is small, $h \in U(P)$, g is differentiable in the x-component with $g, \dfrac{\partial g}{\partial x} \in C(P \times \mathbb{R})$, $g(p, 0) = 0$, $g(p, x) \leqslant 0$ for every $p \in P$, $x \in \mathbb{R}$ and $\lim_{|x|\to\infty} g(p, x) = -\infty$ uniformly on $p \in P$. Let us fix $x_0 > 0$ and $k \in C^1(\mathbb{R})$ with $k(x) \leqslant 0$ for all $x \in \mathbb{R}$, $k(x) = 0$ for $x \in [-2x_0, 2x_0]$, and $\lim_{x\to\infty} \dfrac{k(x)}{x} = -\infty$.

Let us define $g(p, x) = \dfrac{-\epsilon h(p)x_0 - x + k(x)}{x}$, for $p \in P$, $x \geqslant x_0$, which is negative if ϵ is small enough. We also assume a convenient definition of g on $P \times (-\infty, x_0]$ such that all the previous conditions are satisfied.

If $x \geqslant x_0$, the family (4.5) becomes

$$\dot{x} = \epsilon h(\theta_t p)(x - x_0) + k(x),\ p \in \mathrm{P}. \tag{4.6}$$

The point $x \equiv x_0$ is a constant solution of (4.6) and the structure of $\mathbb{A}$ above x_0 is described in [Caraballo *et al.*]. Indeed, there exists a semicontinuous function $b : P \to [x_0, \infty)$ such that $b(p) = x_0$ for $p \in P_c$ the residual invariant set of continuity points of the map. This fact leads to a pinched set on the attractor in the positive cone of solutions. In addiction $b(p) > x_0$ for every $p \in P_f = P \setminus P_c$ that is an invariant subset of first category. In the case where $m(P_f) = 1$, the set $\mathbb{A} \cup (P \times \{x \geqslant x_0\})$ is chaotic in measure in the sense of Li-Yorke (see [Caraballo *et al.*].)

Finally, for any $h \in U(P)$, if $\mathbb{A} = \cup_{p\in P}\{p\} \times A(p)$ the family of cocycle attractors $\{A(p)\}_{p\in P}$ is not uniform. The elements $p \in P_f$ are such that $A(p)$ is forwards attracting of the process on $P \times \{x \geqslant x_0\}$ (see [Caraballo *et al.*]).

4.3. **Example 3: strange non-chaotic cocycle attractors.** We consider an almost periodic flow (P_1, σ_1) and a concave a quadratic equation

$$x_1' = -x_1^2 + h(\theta_t p)x_1 + k(\theta_t p), p \in P \tag{4.7}$$

with $h, k \in U(P)$ that induces a local skew-product semiflow on $P \times \mathbb{R}$ verify the following properties

(i) $P \times \mathbb{R}$ contains a unique minimal set K that is an almost automorphic extension of (P_1, σ_1). We denote by m_1 the ergodic measure under σ_1 on P_1.

(ii) If $b_1(w) := \sup\{x : (w,x) \in K\}$, $a_1(w) := \inf\{x : (w,x) \in K\}$, then b_0, a_0 are semicontinuous ( ) is a residual invariant set of points of continuity with $m_1(P_{1,s}) = 0$ such that $b_1(w) = a_1(w)$ for every $w \in P_1$ and an invariant subset $P_{1,f} = P_1 \setminus P_{1,s}$ first category with $m_1(P_{1,s}) = 1$ and $b_1(w) > a_1(w)$ for every $w \in P_{1,f}$.

(iii) The relations

$$\int_{P\times\mathbb{R}} f dv_b = \int_P f((w, b_1(w)))dm_1$$

$$\int_{P\times\mathbb{R}} f dv_a = \int_P f((w, a_1(w)))dm_1$$

for every $f \in C(K)$, it defines a ergodic measures μ_a, μ_b on $(K_1, ?)$. In addiction

$$\nu_a = \int_K (-2x_1 + h(w))d\mu_a > 0 > \int_K (-2x_1 + h(w))d\mu_b = \nu_b,$$

that is the flow (K_1, τ_1) is not unique ergodic and the graph $\{(w, b_1(w)) : w \in P\}$ defines a strong chaotic attractor in the terminology of [Glendinning *et al.* (2006)]. Johnson [Johnson (1982)] showed that a quadratic equation (4.7) with these properties can be constructed as the Ricatti equation of a two dimensional Hamiltonian equation uniformly weakly disconjugate with positive upper Lyapunov Exponent $\beta > 0$ and Sacker Sell Spectrum $[-\beta, \beta]$ (see [Jorba *et al.* (2007)])

Examples of such almost periodic hamiltonian systems have been provided by Millionščikov [**?**] and Vinograd [**?**].

Taking a translation in the x_1-component if was necessary we can assume the existence of $x_0 > 0$ such that $x_1 \geqslant x_0$ for every $(0, x_1) \in K$. For each $\mu > 0$, the map $\sigma_1 : \mathbb{R}\times P_1 \to P_1$, $(t, p_1) \mapsto p_1 \cdot (t.\mu)$ define a continuous flow on P_1. We denote by $x_1(t, p, x_0)$ the solution of (4.7) through p with $x_1(0, p, x_0) = x_0$. Then the function $x(t) = x_1(t\mu, p, x_0)$ satisfies

$$x' = -\mu x^2 + \mu h(p \cdot (\mu t))x + k(p \cdot (\mu t)),\ p \in P_1. \tag{4.8}$$

We fix $\mu > 0$ with $\mu|K|_\infty < x_0, \mu|K|_\infty < 1$. Note that (K_1, τ_a) is a minimal set the flow induzed by (4.8).

Now we take a non-autonomous family of ODEs

$$x' = x + \mu h(p \cdot (\mu t))x + xg(p \cdot (\mu t), x),\ p \in P_1 \tag{4.9}$$

where g is differentiable in the x-component with $g, \dfrac{\partial g}{\partial x} \in C(P_1 \times \mathbb{R}),\ g(p,0) = 0$, $g(p,x) \leqslant 0$ for every $(p,x) \in P_1 \times \mathbb{R}$ and $\lim_{|x|\to\infty} g(p,x) = -\infty$ uniformly on $p \in P_1$. In particular, if we take $g(p,x) = \dfrac{-\mu x^2 + \mu k(p) - x}{x}$ for every $x \geqslant x_0$ that we assume that is extended on $P \times (-\infty, x_0]$ satisfying all the previous properties.

If $x \geqslant x_0$, the family (4.9) becomes

$$x' = -\mu x^2 + \mu h(p\cdot(\mu t))x + \mu k(p\cdot(\mu t)),\ p \in P_1$$

In consequence, the minimal set K is in the global attractor $\mathcal{A}_+ = \mathcal{A} \cap (0,\infty)$ which exhibits ingredients of highly complexity several ergodic measure and the Lyapunov Exponents. In addiction, there is $P_2 \subset P_{1,f}$ invariant with $m_1(P_i) = 1$ that if $p \in P_2$, $x_1 > b_1(p)$ and $x(t,p,x_1)$ denote the solution of (4.9) through p with $x(0,p,x_1) = x_1$, then

$$\lim_{t\to\infty}[x(t,p,x_1) - b_1(p\cdot(\mu t))] = 0.$$

This implies that if $\mathcal{A} = \cap_{p\in P}\{p\} \times A(p)$ and $b_3(p) := \sup A(p)$ then this map is semicontinuous and $b_3(p) = b_2(p)$ for every p in an invariant set of complete measure.

The graph $\{(p, b_3(p)) : p \in P\}$ is a strange non-chaotic attractor. We conclude again that the family of pullback attractors $(A(p))_{p\in P}$ is not uniform.

5. The sublinear and concave cases

Suppose that the function $G : P \times \bar{U} \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions of section 1. The following semiflows will be considered:

The function $G : P \times \bar{U} \times \mathbb{R}_+ \to \mathbb{R}$ is sublinear (in the y component) if it satisfies:

$$G(p,x,\lambda y) \geqslant \lambda G(p,x,y),\ \forall p \in P, x \in \bar{U}, y \in \mathbb{R}_+, \lambda \in [0,1] \tag{5.1}$$

The function is strongly sublinear at a point $p_0 \in P$ if

$$G(p_0,x,\lambda y) > \lambda G(p_0,x,y),\ x \in \bar{U}, y > 0, \lambda \in (0,1) \tag{5.2}$$

Following the arguments of [Novo *et al.* (2005)] for parabolic equations of type (2.1) its easy to proof that if G satisfies (5.1), then the semiflow (2.4) generated by the solutions of the differential equation is sublinear in the positive cone, i.e.,

$$u(t, p, \lambda z) \geqslant \lambda u(t, p, z),\ \forall t \geqslant 0, p \in P, z \in X_+, \lambda \in [0, 1] \tag{5.3}$$

Moreover, if G satisfies (5.1) and (5.2), then there exist $t_1 > 0, p_1 \in P$ such that u satisfies

$$u(t_1, p_1, \lambda z_1) > \lambda u(t_1, p_1, z_1),\ \forall z_1 \in X_+, \lambda \in (0, 1) \tag{5.4}$$

Note that if G is sublinear, then it admits a decomposition

$$G(p, x, y) = h(p, x)y + g(p, x, y) \text{ for } y \geqslant 0$$

checking the conditions considered in Section 1.

The function $G : P \times \bar{U} \times \mathbb{R}_+ \to \mathbb{R}$ is concave (in the y component) if satisfies:

$$G(p, x, \lambda y_1 + (1 - \lambda)y_2) \geqslant \lambda G(p, x, y_1) + (1 - \lambda)G(p, x, y_2), \tag{5.5}$$

for all $p \in P, x \in \bar{U}, y_1, y_2 \in \mathbb{R}_+$ and $\lambda \in [0, 1]$.

In these conditions, the semiflow satisfies the concavity condition (see [Novo *et al.* (2005)])

$$u(t, p, \lambda z_1 + (1 - \lambda)z_2) \geqslant \lambda u(t, p, z_1) + (1 - \lambda)u(t, p, z_2), \tag{5.6}$$

for all $p \in P, z_1, z_2 \in X_+, z_1 \leqslant z_2$ and $\lambda \in [0, 1]$

Note that if the vector field G is concave, then it is sublinear and verifying the above properties. We define the function $\tilde{G}$ by

$$\tilde{G}(p, x, y) = -G(p, x, y), y > 0, p \in P, x \in \bar{U}.$$

If $\tilde{G}$ is concave in $P \times \bar{U} \times \mathbb{R}_+$, then G is convex in $P \times \bar{U} \times \mathbb{R}_-$ and the cocycle u gets the same properties.

See also [Zhao (2003)] and [Mierczyński & Shen (2004)].

The following result completes the conclusions to sublinear or concave vector fields.

Theorem 5.1. *Suppose that $\lambda_P > 0$ and let*

$$b(p) = \max\{x \in A(p)\} \textit{ and } c(p) = \min\{x \in A(p) \cap \operatorname{Int} X_+\}$$

for all $p \in P$.

(i) *If G satisfies the condition of sublinearity* (5.1)*, then the functions b, c are continuous equilibria.*

If G also satisfies (5.2)*, then $b = c$, $A(p) \cap \mathrm{Int}X_+ = \{b(p)\}$, for all $p \in P$, and for all $z > 0$, we have that $b(\cdot)$ is forwards attracting, i.e.*

$$\lim_{t\to\infty} \|u(t,p,z) - b(\theta_t p)\| = 0.$$

(ii) *If G satisfies* (5.1) *and $c(p) < b(p)$ for some $p \in P$, then there exist δ_0 with $b(p)-c(p) \geqslant \delta_0 e_0$ for all $p \in P$. The compact invariant set $\mathbb{A}_+ \subset (P \times \mathrm{Int}X_+)$ is uniformly stable and there exists $0 < \rho < 1$ such that $c(p) = \rho b(p)$, $p \in P$ and it holds that*

$$\mathbb{A}_+ = \{(p, \lambda b(p)) : \ p \in P, \ \rho \le \lambda \le 1\}.$$

Moreover,

$$g(p,x,y) = g(p,x,b(p)(x))y \text{ for all } p \in P, x \in \overline{U}, \text{ and } \rho b(p)(x) \le y \le b(p)(x).$$

(see case A_2 of the Theorem 3.8 in [**?**]*).*

(iii) *If G satisfies* (5.5)*, then $b = c$ and*

$$\mathbb{A} \cap (P \times \mathrm{Int}X_+) = K_1 = \{(p, b(p)) : p \in P\}$$

is a compact invariant exponentialy stable set, i.e, $\lambda_{K_1} < 0$ and for all $0 < \epsilon < |\lambda_{K_1}|, \rho > 1$, there exist $c_{\epsilon,\rho} > 0$ such that if $z \in X_+$ and $\frac{1}{\rho} e_0 \leqslant z \leqslant \rho e_0$, then $b(\cdot)$ is exponentially forwards attracting, i.e.,

$$\|u(t,p,z) - b(\theta_t p)\| \leqslant c_{\epsilon,\rho} e^{(\lambda_K - \epsilon)t} \|z - b(p)\|,$$

for all $p \in P, t \geqslant 0$.

Proof. (i) In this case, the semiflow S is sublinear, i.e., satisfies (5.3) in the positive cone. As the semiflow is persistent, the dynamic structure of $\mathbb{A} \cap (P \times \mathrm{Int}X_+)$ is described by one of the A_1 or A_2 cases in Theorem 3.13 in [**?**]

(HACE FALTA ESCRIBIR QUE ES EXACTAMENTE A_1 y A_2.).

If G satisfies (5.2), then u satisfies (5.4) and the dynamics corresponding to case A_1, the compact invariant $\mathbb{A} \cap (P \times \mathrm{Int}X_+) = K_1 = \{(p, b(p)) : p \in P\}$ is asymptotically stable.

(ii) Suppose $c(p) < b(p)$ for some $p \in P$. We now follow the argument of Proposition 3.8 in [Novo *et al.* (2005)]. There exists a continuous and connected family $\{K_s\}_{s\in[0,1]}$ of strongly positive ordered minimals sets with $K_0 = \{(p, c(p)) : p \in P\}$, $K_1 = \{(p, b(p)) : p \in P\}$ and for $0 \leq s_1 < s_2 \leq 1$ we have that $K_{s_1} < K_{s_2}$; so that, for every $(p, z_1) \in K_{s_1}$ there exists $(p, z_2) \in K_{s_2}$ with $z_1 \leq z_2$.

Note that for $0 < \lambda < 1$

$$u(t, p, \lambda b(p)) \geq \lambda u(t, p, b(p)) = \lambda b(pt),$$

i.e., the map $b_\lambda(p) = \lambda b(p)$, $p \in P$ is a continuous sub-equilibrium.

Fix $s \in (0, 1)$ and take

$$J_s = \{\lambda \in [0, 1] : \lambda b(p) \leq z \text{ for every } (p, z) \in K_s\}.$$

Let $\lambda_0 = \sup J_s$. It is obvious that $\lambda_0 \in J_s$, i.e. $\lambda_0 b(p) \leq z$ for every $(p, z) \in K_s$. Note that it does not hold that $\lambda_0 b(p) < z$ for every $(p, z) \in K_s$. Indeed, suppose at some point (p_0, z_0) we have $\lambda_0 b(p_0) < z_0$. Then, by the extensibility of S on minimal sets we can take $z_{-t} = u(-t, p_0, z_0)$ with $(p(-t), z_{-t}) \in K_s$ for $t \in [-1, 0]$. Then $\lambda_0 b(p(-t)) \leq z_{-t}$ and there exists $\epsilon > 0$ such that $\lambda_0 b(p(-t)) < z_{-t}$ for $t \in [0, \epsilon]$. The strong monotonicity of the semi flow implies

$$\lambda b(p) \leq u(t, p(-t)), \lambda b(p(-t)) < u(t, p(-t), z_{-t}) = z_0.$$

As a consequence, there exists $(p_0, z_0) \in K$ with $z_0 = \lambda_0 b(p_0)$. The above argument also implies that $z_s = u(s, p_0, z_0) = \lambda_0 b(ps)$ for all $r \leq 0$.Taking the alpha limit set of (p_0, z_0) we conclude that

$$K_s = \lambda_0 K_1 = \{(p, \lambda_0 b(p)) : p \in P\}.$$

Thus, there exists $\rho > 0$ such that $c(p) = \rho b(p)$ for all $p \in P$. Finally, we conclude that $\{(p, \lambda b(p)) : p \in P\}$ is a minimal set for every $\lambda \in [\rho, 1]$. They define the lamination of minimal sets joining K_1 and K_0.

It now follows from [Novo *et al.* (2005)] that $B = \{(p, \lambda b(p)) : p \in P, \lambda \in [\rho, 1]\}$ is a uniformly stable compact invariant set. We show that it coincides with $\mathbb{A}_+$. It is clear that $B \subset \mathbb{A}_+$. Now let $(p^*, z^*) \in \mathbb{A}$. It has a backwards extension and we consider a minimal set K^* in its alpha-limit set. The above argument proves the existence of

$\rho^* \in [\rho, 1]$ with $K^* = \{(p, \rho^* b(p)) : \ p \in P\}$. For every $\epsilon > 0$ there exists $\delta > 0$, $t^* < 0$ such that

$$||u(t, p^*, z^*) - \rho^* b(p^* t)|| < \delta \text{ for all } t \leq t^*$$

and

$$||u(t, p^*, z^*) - \rho^* b(p^* t)|| \leq \epsilon \text{ for all } t \geq t^*.$$

Thus,

$$||z^* - \rho^* b(p^*)|| \leq \epsilon \text{ for every } \epsilon > 0,$$

so that $z^* = \rho^* b(p^*)$ and $(p^*, z^*) \in B$.

Finally, define

$$g_0(p, x, y) = \frac{g(p, x, y)}{y}.$$

Then g_0 is continuous and negative. Moreover, g is sublinear in the $y-$ component for $y \geq 0$ if and only if g_0 is decreasing in the $y-$ component for $y \geq 0$. By the restriction in $\mathbb{A}_+$ we conclude that

$$g(p, x, \lambda b(p)(x)) = \lambda g(p, x, b(p)(x)) \text{ for every } p \in P, x \in \bar{U}, \lambda \in [\rho, 1].$$

Thus,

$$g_0(p, x, \lambda b(p)(x)) = \lambda g_0(p, x, b(p)(x)) \text{ for every } p \in P, x \in \bar{U}, \lambda \in [\rho, 1]$$

and

$$g(p, x, y) = g_0(p, x, b(p)(x)) \text{ for every } p \in P, x \in \bar{U}, \ \text{ and } \rho b(p)(x) \leq y \leq b(p)(x).$$

(iii) In this case the semiflow induced is concave, i.e., satisfies (2.2). As the flow is uniformly persistent over 0, the dynamic of $\mathbb{A} \cap (P \times \mathrm{Int} X_+)$ is described by case A_1 of [Núñez *et al.* (2012)].

□

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