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NULL CONTROLLABLE SETS AND REACHABLE SETS FOR NONAUTONOMOUS LINEAR CONTROL SYSTEMS

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Abstract. Under the assumption of lack of uniform controllability for a family of time-dependent linear control systems, we study the dimension, topological structure and other dynamical properties of the sets of null controllable points and of the sets of reachable points. In particular, when the space of null controllable vectors has constant dimension for all the systems of the family, we find a closed invariant subbundle where the uniform null controllability holds. Finally, we associate a family of linear Hamiltonian systems to the control family and assume that it has an exponential dichotomy in order to relate the space of null controllable vectors to one of the Lagrange planes of the continuous hyperbolic splitting.

1. Introduction.

In this paper we study a time-dependent linear control system of the form

$$\mathbf{x}' = A_0(t)\,\mathbf{x} + B_0(t)\,\mathbf{u}(t)\,, \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector and $\mathbf{u}(t) \in \mathbb{R}^m$ is a control vector. The functions A_0, B_0 are assumed to be bounded and uniformly continuous on $\mathbb{R}$, with values in the sets of real matrices of the appropriate dimensions.

We are concerned with the null controllability properties of the nonautonomous problem (1.1). Among the main questions that we pose, we point out two: to describe the dimension, topological structure and other dynamical characteristics of the null controllable set (composed of those states $\mathbf{x}_0 \in \mathbb{R}^n$ for which there exists a suitable control $\mathbf{u}$ steering $\mathbf{x}_0$ to $\mathbf{0}$); and to do the same with the T-reachable sets (i.e., the sets of states $\mathbf{x}_0 \in \mathbb{R}^n$ for which there exists a suitable control $\mathbf{u}$ steering $\mathbf{0}$ to $\mathbf{x}_0$ in time T).

When dealing with a nonautonomous problem, it is frequent to embed it into a family which describes a flow, and which therefore allows one to make use of tools coming from the topological dynamics and the ergodic theory. By means of the

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standard Bebutov procedure of the hull construction, the system (1.1) becomes a particular one of the family

$$\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u}(t)\,, \quad \omega \in \Omega\,, \tag{1.2}$$

where Ω is a compact metric space, $\sigma\colon \mathbb{R}\times\Omega \to \Omega,\ (t,\omega)\mapsto \omega{\cdot}t$ is a continuous flow, and the matrix-valued functions $A\colon \Omega \to \mathbb{M}_{n\times n}(\mathbb{R})$ and $B\colon \Omega \to \mathbb{M}_{n\times m}(\mathbb{R})$ are continuous. Hence, each of the systems of this family is given by the evaluation of A and B along one of the orbits of the flow (Ω,σ). We will analyze the null controllability properties of the whole family, from where one can derive those of the initial system by means of an obvious "restriction" process. The dynamical and ergodic properties of the two flows that the linear family

$$\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x}\,, \quad \omega \in \Omega \tag{1.3}$$

induces on the linear bundle and on the Grassmannian bundles will be one of the main tools in our analysis. We will also make use of another fundamental tool: often, the controllability properties of the family (1.2) are closely related to those of the family of time-reversed control systems

$$\mathbf{x}' = -A(\omega{\cdot}(-t))\,\mathbf{x} - B(\omega{\cdot}(-t))\,\mathbf{u}(t)\,, \quad \omega \in \Omega\,, \tag{1.4}$$

for which the coefficient functions are evaluated along the orbits of the new time-reversed flow $\sigma^-\colon \mathbb{R}\times\Omega\to\Omega,\ (t,\omega)\mapsto\omega{\cdot}(-t)$. It will be clear in what follows that the analysis of the systems (1.4) provides valid information on the family (1.2). As a matter of fact, part of the results will be formulated for the family (1.4), and later translated to the case of (1.2).

Of course, the simplest situation corresponds to the case where the family of control systems (1.2) is uniformly null controllable (i.e., all its systems are null controllable), which is equivalent to say that the family of time-reversed control systems (1.4) is uniformly null controllable (see Fabbri *et al.* [5]). If this is the case, then the set of null controllable points of the system (1.2) coincides with $\mathbb{R}^n$ for all $\omega\in\Omega$, and the same happens with the T-reachable sets if the time T is large enough. Therefore, in this paper we will always consider the case of existence of at least one system of the family (1.2) which is not null controllable.

We will now describe the structure of the paper and give a brief summary of our results. In Section 2 we recall the main notions of topological dynamics and ergodic theory which will be required in the rest of the paper. In particular, the linear and Grassmannian flows induced by (1.3) and the Lagrangian flow induced by a family of linear Hamiltonian systems are introduced here. The concepts of exponential dichotomy and rotation number are also given.

Now we summarize the contents of Section 3, on which we give precise definitions of all the concepts involved. For a fixed $\omega\in\Omega$, we represent by $E(\omega)$ and $\widetilde{E}(\omega)$ the sets of null controllable points for the systems (1.4) and (1.2), respectively. We will prove that $E(\omega)$ is a vector space which coincides with the T-reachable set of the system of (1.2) corresponding to $\omega{\cdot}(-T)$ for T large enough. When ω varies in Ω, the sets $E(\omega)$ present some properties of semi-invariance under the flow induced by (1.3) on the linear bundle. We will also prove that $\dim E(\omega)$ is a lower semicontinuous function which turns out to be constant on the minimal subsets of Ω, and which attains its minimum value on one of these minimal subsets; moreover, $\dim E(\omega)$ is locally constant on the residual subset of its continuity points, on which it attains its maximum value. In addition, if Ω_0 is a compact invariant subset of Ω on which $\dim E(\omega)$ is constant (which is the case if Ω_0 is minimal), then $E(\omega)=\widetilde{E}(\omega)$

for all $\omega \in \Omega_0$, and the set $\{(\omega, \mathbf{x}) \mid \omega \in \Omega_0, \mathbf{x} \in E(\omega)\} \subseteq \Omega_0 \times \mathbb{R}^n$ defines a closed invariant subbundle for the linear flow with the following property: it is the greatest subset of $\Omega_0 \times \mathbb{R}^n$ on which the families (1.2) and (1.4) are uniformly null controllable.

Before describing the second group of results, which are the core of Section 4, we need some preliminary information. It is frequent to associate a family of quadratic functionals

$$\mathcal{Q}_\omega(t, \mathbf{x}, \mathbf{u}) = \frac{1}{2}\left(\langle \mathbf{x}, G(\omega{\cdot}t)\,\mathbf{x}\rangle + 2\,\langle \mathbf{x}, g(\omega{\cdot}t)\,\mathbf{u}\rangle + \langle \mathbf{u}, R(\omega{\cdot}t)\,\mathbf{u}\rangle\right), \quad \omega \in \Omega \qquad (1.5)$$

to the control family (1.2), where $G\colon \Omega \to \mathbb{M}_{n\times n}(\mathbb{R})$, $g\colon \Omega \to \mathbb{M}_{n\times m}(\mathbb{R})$ and $R\colon \Omega \to \mathbb{M}_{m\times m}(\mathbb{R})$ are continuous, G and R are symmetric, and $R > 0$. Here, as usual, $\langle\,,\,\rangle$ represents the Euclidean inner product in $\mathbb{R}^m$ and $\mathbb{R}^n$. If the pair $(\mathbf{x}(t), \mathbf{u}(t))$ solves the system (1.2) for a point $\omega \in \Omega$, then $\int_{t_1}^{t_2} \mathcal{Q}_\omega(t, \mathbf{x}(t), \mathbf{u}(t))\,dt$ often represents the amount of "supply" (meaning in general energy) which has to be delivered to the system in order to transfer it from its state in time t_1 to its state in time t_2. For this reason $\mathcal{Q}_\omega$ is called supply rate or power function. We will refer to the pair given by (1.2) and (1.5) as a linear-quadratic (or LQ, for short) control problem.

Many question involving LQ control problems have been extensively analyzed during the last decades. One of the more classical is that of fixing a point $\omega \in \Omega$ and a initial state $\mathbf{x}_0 \in \mathbb{R}^n$, and finding, among all the L^2-pairs $(\mathbf{x}(t), \mathbf{u}(t))$ which solve (1.2) and satisfy $\mathbf{x}(0) = \mathbf{x}_0$, that for which the quantity $\int_0^\infty \mathcal{Q}_\omega(t, \mathbf{x}(t), \mathbf{u}(t))\,dt$ is minimum. Another classical problem is to determine conditions ensuring the (normal or strict) dissipativity of the LQ problems as well as the existence of (normal or strong) storage functions. Roughly speaking, a dissipative system requires energy coming from the environment to move from its equilibrium position to another one, and the storage function, if it exists, bounds from below the energy that the system requires to pass from the state of minimum storage to a given state.

A fundamental tool for the analysis of these two problems is the description of the dynamics induced by the family of linear Hamiltonian systems associated to the LQ problem given by

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}\,, \quad \omega \in \Omega\,, \qquad (1.6)$$

where $\mathbf{z} = \left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{y}\end{smallmatrix}\right]$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and

$$H(\omega) = \begin{bmatrix} A(\omega) - B(\omega)R^{-1}(\omega)g^T(\omega) & B(\omega)R^{-1}(\omega)B^T(\omega) \\ G(\omega) - g(\omega)R^{-1}(\omega)g^T(\omega) & -A^T(\omega) + g(\omega)R^{-1}(\omega)B^T(\omega) \end{bmatrix}.$$

The family (4.3) was firstly associated to the family of infinite-horizon minimization problems by means of the Pontryagin Maximum Principle. This association requires an additional condition of uniform stabilization for the systems (1.2). When this condition holds, the minimization problem is solvable if and only if the family (1.6) has an exponential dichotomy over Ω (that is, it satisfies the so-called Frequency Condition) and none of the Lagrange planes $l^+(\omega)$ associated to the solutions which are bounded at $+\infty$ lies in the vertical Maslov cycle (that is, the family satisfies the so-called Nonoscillation Condition; or, in other words, for each $\omega \in \Omega$, $l^+(\omega)$ has a basis whose elements are the columns vectors of a $2n \times n$ matrix $\left[\begin{smallmatrix} L_1^+(\omega) \\ L_2^+(\omega)\end{smallmatrix}\right]$ with $\det L_1^+(\omega) \neq 0$ for all $\omega \in \Omega$). This result was previously proved by Yakubovich [24, 25] in the periodic case and extended by Fabbri *et al.* [4] to the general nonautonomous case. The interested reader can find in Chapter 7 of Johnson *et al.* [10] an exhaustive description of these results. Similar conditions are used in the description of the dissipativity of the LQ problems in the papers [26, 5, 9, 8] and in Chapter 8 of [10].

It is obvious that we can simply define (1.6) from (1.2) and (1.5), without any extra assumption on stabilization. This is what we do in this paper. Recall that we consider the case of existence of systems of the family (1.4) which are not null controllable. With the focus put on obtaining results which involve as many situations as possible, we will impose the Frequency Condition, but not the Nonoscillation Condition. Roughly speaking, our results relate the set of null controllable points for the system (1.4) to the image space $\operatorname{Im} L_1^-(\omega)$, for each $\omega \in \Omega$. Here $\left[\begin{smallmatrix} L_1^-(\omega) \\ L_2^-(\omega) \end{smallmatrix}\right]$ is any matrix representing the Lagrange plane $l^-(\omega)$ of the solutions which are bounded at $-\infty$. Clearly, different quadratic forms $\mathcal{Q}_\omega$ will give rise to families (1.6) with or without exponential dichotomy and, even in the case of existence, with different properties of the Lagrange planes $l^-(\omega)$. Therefore, if the only purpose is to know as much as possible of the null controllable sets, then we can play with the choice of the supply rate: changing it may allow us to go deeper in the analysis.

The main results of Section 4 are now summarized. Under the Frequency Condition and the additional assumption of the existence of an ergodic measure m_0 on Ω with full topological support and for which the rotation number of the family (1.6) is zero, we prove that the set $E(\omega)$ is a vector subspace of $\operatorname{Im} L_1^-(\omega)$. (Note that the space $\operatorname{Im} L_1^-(\omega)$ and $\dim \operatorname{Ker} L_1^-(\omega)$ are independent of the basis chosen for $l^-(\omega)$.) We check that the vector space $\operatorname{Im} L_1^-(\omega)$ has properties of semi-invariance under the linear flow induced by (1.6), and that $\dim L_1^-(\omega)$ is a lower semicontinuous function with analogous properties to those previously obtained for $\dim E(\omega)$. Finally, we assume that the dimension of $\operatorname{Im} L_1^-(\omega)$ is constant on Ω and show that the Lagrange subbundle $L^- = \{(\omega, \mathbf{z}) \mid \omega \in \Omega,\, \mathbf{z} \in l^-(\omega)\}$ determined by the exponential dichotomy contains a closed invariant subbundle whose sections intersect the vertical Lagrange plane $l_v \equiv \left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$; and we also assume that $\dim E(\omega)$ is constant on Ω and prove that L^- contains another closed invariant subbundle such that the first components of the vectors of its sections belong to $E(\omega)$ for all $\omega \in \Omega$. As pointed out before, similar results can be obtained for the vector space $\widetilde{E}(\omega)$ of the null controllable points for (1.2), which due to the time-reversion is related to the vector space $\operatorname{Im} L_1^+(\omega)$.

The setting that we consider throughout the paper, that is, the existence of systems of the family (1.2) which are not null controllable, is closely related to the existence of abnormal systems of the family. It is proved in Johnson *et al.* [8] that there are minimal subsets $\Omega^* \subseteq \Omega$ for which all the systems (1.6) are abnormal (i.e., they have solutions of the form $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t) \end{smallmatrix}\right]$ for $t \in \mathbb{R}$), and such that at least one of the associated Lagrange planes $l^\pm(\omega)$ lies on the vertical Maslov cycle $\mathcal{C}$, defined in Subsection 2.2.1, for all $\omega \in \Omega^*$. A more precise description of this connection will also be included at the beginning of Section 4.

2. **Preliminaries.**

This section begins by recalling some basic concepts and properties on topological dynamics and ergodic theory, which are discussed in Ellis [2] and Cornfeld *et al.* [1]. Let Ω be a complete metric space. A *continuous flow* on Ω is a continuous map $\sigma\colon \mathbb{R} \times \Omega \to \Omega,\ (t, \omega) \mapsto \sigma(t, \omega)$ such that $\sigma_0 = \mathrm{Id}$ and $\sigma_{s+t} = \sigma_t \circ \sigma_s$ for each $s, t \in \mathbb{R}$, where $\sigma_t(\omega) = \sigma(t, \omega)$.

Let (Ω, σ) be a continuous flow. The σ-*orbit* of a point $\omega \in \Omega$ is the set $\{\sigma_t(\omega) \mid t \in \mathbb{R}\}$. Restricting the time to $t \geq 0$ or $t \leq 0$ leads to the definition of *forward* or *backward* σ-semiorbit. A subset $\Omega_1 \subseteq \Omega$ is *σ-invariant* (resp. *positively σ-invariant* or *negatively σ-invariant*) if $\sigma_t(\Omega_1) = \Omega_1$ for every $t \in \mathbb{R}$ (resp. $t \geq 0$ or $t \leq 0$). A σ-invariant subset $\Omega_1 \subseteq \Omega$ is *minimal* if it is compact and does not contain properly any other compact σ-invariant set; or, equivalently, if the two semiorbits of any of its elements are dense in it. The continuous flow (Ω, σ) is *recurrent* or *minimal* if Ω itself is minimal.

If the forward semiorbit of a point $\omega_0 \in \Omega$ is relatively compact, its *omega-limit set* $\mathcal{O}(\omega_0)$ is given by those points $\omega \in \Omega$ such that $\omega = \lim_{m\to\infty} \sigma(t_m, \omega_0)$ for some sequence $(t_m) \uparrow \infty$. This set is nonempty, compact, connected and positively σ-invariant. The definition and properties of an alpha-limit set $\mathcal{A}(\omega_0)$ are analogous, working now with sequences $(t_m) \downarrow -\infty$.

The summary of the most basic notions required in the present paper is completed with some definitions concerning the measures on Ω. Let m be a normalized Borel measure on Ω; i.e. a finite regular measure defined on the Borel subsets of Ω and with $m(\Omega) = 1$. The measure m is *σ-invariant* if $m(\sigma_t(\Omega_1)) = m(\Omega_1)$ for every Borel subset $\Omega_1 \subseteq \Omega$ and every $t \in \mathbb{R}$. If, in addition, $m(\Omega_1) = 0$ or $m(\Omega_1) = 1$ for every σ-invariant subset $\Omega_1 \subseteq \Omega$, then the measure m is *σ-ergodic*. The measure m is *concentrated* on $\Omega_1 \subseteq \Omega$ if $m(\Omega_1) = 1$. The *topological support* of m, $\operatorname{Supp} m$, is the complement of the largest open set $O \subseteq \Omega$ for which $m(O) = 0$.

From now on Ω will indicate a compact metric space and $\sigma\colon \mathbb{R} \times \Omega \to \Omega$ a continuous flow, and we will represent $\omega{\cdot}t = \sigma(t, \omega)$.

2.1. **Linear Flow.**

Consider the family of linear systems

$$\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x}\,, \quad \omega \in \Omega\,, \tag{2.1}$$

where A is a continuous real $n \times n$ matrix-valued function on Ω. We use the notation $(2.1)_\omega$ to refer to the system of the family (2.1) corresponding to the point $\omega \in \Omega$; and we will do the same with the remaining families of systems appearing in the paper. Denote by $U_A(t, \omega)$ the fundamental matrix solution of $(2.1)_\omega$ with $U_A(0, \omega) = I_n$, which is globally defined and nonsingular, and jointly continuous on $\mathbb{R} \times \Omega$. By the uniqueness of solutions,

$$U_A(t + s, \omega) = U_A(t, \omega{\cdot}s)\, U_A(s, \omega)\,, \tag{2.2}$$

and hence the map

$$\tau_A\colon \mathbb{R} \times \Omega \times \mathbb{R}^n \to \Omega \times \mathbb{R}^n\,, \quad (t, \omega, \mathbf{z}) \mapsto (\omega{\cdot}t, U_A(t, \omega)\,\mathbf{z}) \tag{2.3}$$

defines a continuous flow on $\Omega \times \mathbb{R}^n$, which is of linear *skew-product* type: it preserves the flow on Ω, which can be considered as the *base* of the bundle $\Omega \times \mathbb{R}^n$; and it is linear on the fiber component.

Frequently, a family of this type comes from a single nonautonomous linear system $\mathbf{z}' = A_0(t)\,\mathbf{z}$ by means of the well-known Bebutov-type construction: if A_0 is bounded and uniformly continuous on $\mathbb{R}$, then its hull Ω, defined by $\Omega = \operatorname{cls}\{A_t \mid t \in \mathbb{R}\}$ is a compact metric space and the time-translation defines a continuous flow σ on it. Here $A_t(s) = A_0(t + s)$ and the closure is taken in the compact-open topology of the set of bounded and uniformly continuous $n \times n$ matrix-valued functions. The base space Ω can be understood as the space in which the nonautonomous law varies with respect to time. Under additional recurrence properties on A_0, the base flow is minimal. This is the case if A_0 is almost periodic or almost automorphic.

Weaker conditions on A_0 may provide a non minimal hull, which can contain different minimal subsets. In some of these cases, the solutions of the different linear systems of the family show a significatively different qualitative behavior.

The same Bebutov procedure can be carried out for the pair of matrix-valued functions (A_0, B_0) giving rise to the initial control system (1.1) in order to include it in the family (1.2).

2.1.1. ***The Grassmannian flows.*** Let W be a d-dimensional linear subspace of $\mathbb{R}^n$. Given $k \in \{0, 1, \ldots, d\}$, let $\mathcal{G}_k(W)$ represent the set of the k-dimensional subspaces of W. The set $\mathcal{G}_k(W)$ can be identified with the homogeneous space of left cosets $\mathrm{GL}(d, \mathbb{R})/\widetilde{\mathcal{H}}$, where $\mathrm{GL}(d, \mathbb{R}) = \{A \in \mathbb{M}_{d\times d}(\mathbb{R}) \mid \det A \neq 0\}$ and $\widetilde{\mathcal{H}}$ is the closed Lie subgroup of $\mathrm{GL}(d, \mathbb{R})$ given by the matrices of the form $\left[\begin{smallmatrix} B & * \\ 0 & C \end{smallmatrix}\right]$ for $B \in \mathrm{GL}(k, \mathbb{R})$ and $C \in \mathrm{GL}(d-k, \mathbb{R})$. Here $*$ represents any $k \times (d-k)$ matrix and 0 represents the zero $(d-k) \times k$ matrix. With this identification, which provides $\mathcal{G}_k(W)$ with a differentiable structure, $\mathcal{G}_k(W)$ is the *Grassmannian manifold of the k-dimensional linear subspaces of W*. The set $\mathcal{G}_k(W)$ is a compact and connected manifold, which agrees with the real projective line if $k = 1$. We refer the reader to Matsushima [14] for the proofs of these properties.

Then the family (2.1) defines a continuous flow

$$\tau_k \colon \mathbb{R} \times \Omega \times \mathcal{G}_k(\mathbb{R}^n) \to \Omega \times \mathcal{G}_k(\mathbb{R}^n)\,, \quad (t, \omega, w) \mapsto (\omega{\cdot}t, U_A(t, \omega){\cdot}w)\,, \tag{2.4}$$

where $U_A(t, \omega){\cdot}w = \{U_A(t, \omega)\,\mathbf{z} \mid \mathbf{z} \in w\}$: note that $\dim U_A(t, \omega){\cdot}w = \dim w$, since $U_A(t, \omega)$ determines an isomorphism of $\mathbb{R}^n$ for any $t \in \mathbb{R}$ and $\omega \in \Omega$.

2.2. **Linear Hamiltonian systems.** Consider now the family of linear Hamiltonian systems

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z} = \begin{bmatrix} H_1(\omega{\cdot}t) & H_3(\omega{\cdot}t) \\ H_2(\omega{\cdot}t) & -H_1^T(\omega{\cdot}t) \end{bmatrix} \mathbf{z}\,, \quad \omega \in \Omega\,, \tag{2.5}$$

where H is a continuous real $2n{\times}2n$ matrix-valued function on Ω and H_2 and H_3 are $n{\times}n$ symmetric matrices. Let $U(t, \omega) = \left[\begin{smallmatrix} U_1(t,\omega) & U_3(t,\omega) \\ U_2(t,\omega) & U_4(t,\omega) \end{smallmatrix}\right]$ represent the fundamental matrix solution of the system $(2.5)_\omega$ with $U(0, \omega) = I_{2n}$, which is globally defined. Then, as before, the map

$$\tau_H \colon \mathbb{R} \times \Omega \times \mathbb{R}^{2n} \to \Omega \times \mathbb{R}^{2n}\,, \quad (t, \omega, \mathbf{z}) \mapsto (\omega{\cdot}t, U(t, \omega)\,\mathbf{z}) \tag{2.6}$$

defines a continuous skew-product flow on $\Omega \times \mathbb{R}^{2n}$. The symplectic nature of the matrix U provides this flow with some additional properties, which we now describe.

2.2.1. ***The Lagrangian flow.*** Recall that two vectors $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{R}^{2n}$ are *isotropic* if $\mathbf{z}^T J \mathbf{w} = 0$, where $J = \left[\begin{smallmatrix} 0_n & -I_n \\ I_n & 0_n \end{smallmatrix}\right]$. Any linear subspace $l \subset \mathbb{R}^{2n}$ whose vectors are pairwise isotropic satisfies $\dim l \le n$, since l is contained in the Euclidean subspace orthogonal to $J{\cdot}l = \{J\,\mathbf{z} \mid \mathbf{z} \in l\}$. An n-dimensional linear subspace $l \subset \mathbb{R}^{2n}$ is a (*real*) *Lagrange plane* if $\mathbf{z}^T J \mathbf{w} = 0$ for all $\mathbf{z}$ and $\mathbf{w}$ in l. The space $\mathcal{L}_\mathbb{R}$ of all real Lagrange planes of $\mathbb{R}^{2n}$ is a compact orientable manifold of dimension $n(n+1)/2$: see [14] and Mishchenko *et al.* [15]. An element l of $\mathcal{L}_\mathbb{R}$ can be represented by a $2n \times n$ real matrix $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ of range n with $L_1^T L_2 = L_2^T L_1$. The representation means that the column vectors form a basis of the Lagrange subspace. Therefore, $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right]$ represent the same Lagrange plane if and only if there exists a nonsingular $n \times n$ real matrix Q such that $L_1 = G_1 Q$ and $L_2 = G_2 Q$. We will write $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ in what follows.

The matrix-valued function H belongs to the symplectic Lie algebra $\mathfrak{sp}(n,\mathbb{R}) = \{G \in \mathbb{M}_{2n\times 2n}(\mathbb{R}) \mid G^T J + JG = 0_{2n}\}$, which implies that, for each $t \in \mathbb{R}$ and $\omega \in \Omega$, $U(t,\omega)$ lies in the symplectic group $\mathrm{Sp}(n,\mathbb{R}) = \{V \in \mathbb{M}_{2n\times 2n}(\mathbb{R}) \mid V^T JV = J\}$: the derivative of $U^T(t,\omega)\, J\, U(t,\omega)$ is the $2n\times 2n$ zero matrix 0_{2n} for all $\omega \in \Omega$ and $t \in \mathbb{R}$. As a consequence of this fact, the vector space $U(t,\omega)\cdot l = \{U(t,\omega)\,\mathbf{z} \mid \mathbf{z} \in l\}$ is a new Lagrange plane for all $t \in \Omega$ and $\omega \in \Omega$: it has dimension n since $U(t,\omega)$ defines an isomorphism on $\mathbb{R}^{2n}$; and if $\mathbf{z}, \mathbf{w} \in l$ then $\mathbf{z}^T U^T(t,\omega)\, JU(t,\omega)\,\mathbf{w} = \mathbf{z}^T J\,\mathbf{w} = 0$. This property implies that the map

$$\tau\colon \mathbb{R}\times\Omega\times\mathcal{L}_{\mathbb{R}} \to \Omega\times\mathcal{L}_{\mathbb{R}}\,, \quad (t,\omega,l) \mapsto (\omega{\cdot}t, U(t,\omega){\cdot}l) \tag{2.7}$$

defines a continuous skew-product flow on $\Omega\times\mathcal{L}_{\mathbb{R}}$. In addition, if $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$, then $U(t,\omega)\cdot l \equiv U(t,\omega)\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} U_1(t,\omega)\,L_1 + U_3(t,\omega)\,L_2 \\ U_2(t,\omega)\,L_1 + U_4(t,\omega)\,L_2 \end{smallmatrix}\right]$.

Consider the open and dense subset $\mathcal{D}$ of $\mathcal{L}_{\mathbb{R}}$ defined by

$$\mathcal{D} = \left\{ l \in \mathcal{L}_{\mathbb{R}} \mid l \equiv \begin{bmatrix} I_n \\ M \end{bmatrix} \right\}.$$

Obviously, each $l \in \mathcal{D}$ admits a unique representation of the form $\left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$, and the $n\times n$ matrix M has to be symmetric. In addition, $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$ belongs to $\mathcal{D}$ if and only if $\det L_1 \neq 0$, in which case $l \equiv \left[\begin{smallmatrix} I_n \\ M \end{smallmatrix}\right]$ for $M = L_2 L_1^{-1}$. Note that $\mathcal{D}$ is the complement in $\mathcal{L}_{\mathbb{R}}$ of the vertical Maslov cycle $\mathcal{C}$ defined as

$$\mathcal{C} = \{ l \in \mathcal{L}_{\mathbb{R}} \mid \dim(l \cap l_v) \geq 1 \}\,, \tag{2.8}$$

where l_v is the Lagrange plane generated by the n last coordinate vectors: $l_v \equiv \left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$.

2.2.2. ***Exponential dichotomy.*** Now we recall the definition of exponential dichotomy (see, e.g., Chapter 1 of [10] for more details). The Euclidean norm in $\mathbb{R}^{2n}$ is fixed and represented by $\|{\cdot}\|$.

Definition 2.1. The family of systems (2.5) has an *exponential dichotomy over* Ω if there exist two positive constants η and β and a splitting $\Omega\times\mathbb{R}^{2n} = L^+ \oplus L^-$ of the real bundle into the Whitney sum of two closed subbundles which are invariant under the flow τ_H given on $\Omega\times\mathbb{R}^{2n}$ by (2.6), with the following properties:

(1) $\|U(t,\omega)\,\mathbf{z}\| \leq \eta e^{-\beta t}\|\mathbf{z}\|$ for every $t \geq 0$ and $(\omega,\mathbf{z}) \in L^+$, and
(2) $\|U(t,\omega)\,\mathbf{z}\| \leq \eta e^{\beta t}\|\mathbf{z}\|$ for every $t \leq 0$ and $(\omega,\mathbf{z}) \in L^-$.

Proposition 2.2. *Suppose that the family (2.5) has an exponential dichotomy over Ω and let $\Omega\times\mathbb{R}^{2n} = L^+ \oplus L^-$ be the corresponding decomposition. Then, this decomposition is unique and, for each $\omega \in \Omega$, the fibers*

$$l^{\pm}(\omega) = \{\mathbf{z} \mid (\omega,\mathbf{z}) \in L^{\pm}\}$$

are real Lagrange planes which vary continuously with respect to ω. In particular, the subbundles $L^{\pm}$ are globally n-dimensional. In addition, $U(t,\omega)\cdot l^{\pm}(\omega) = l^{\pm}(\omega{\cdot}t)$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$. That is, the sets $\mathcal{L}^{\pm} = \{(\omega, l^{\pm}(\omega)) \mid \omega \in \Omega\} \subset \Omega\times\mathcal{L}_{\mathbb{R}}$ are τ-invariant, where τ is given by (2.7); that is, $\tau(t,\omega,l^{\pm}(\omega)) = (\omega{\cdot}t, l^{\pm}(\omega{\cdot}t))$.

Remark 2.3. In general it is not possible to ensure the existence of continuous functions $\left[\begin{smallmatrix} L_1^{\pm} \\ L_2^{\pm} \end{smallmatrix}\right] : \Omega \to \mathbb{M}_{2n\times n}(\mathbb{R})$ such that $l^{\pm}(\omega) \equiv \left[\begin{smallmatrix} L_1^{\pm}(\omega) \\ L_2^{\pm}(\omega) \end{smallmatrix}\right] \in \mathcal{L}_{\mathbb{R}}$ for all $\omega \in \Omega$. However, once a point $\omega \in \Omega$ is fixed, we can choose a local continuous representation in a neighborhood of ω (see, e.g., Subsection 1.2.3 of [10]).

2.2.3. ***Rotation number.*** The section is completed by recalling the definition of the rotation number of the family (2.5) with respect to a given σ-ergodic measure m_0 on Ω. This concept will be used in Section 4, in which will just work with families of Hamiltonian systems $\mathbf{z}' = \left[\begin{smallmatrix} H_1 & H_3 \\ H_2 & -H_1^T \end{smallmatrix}\right] \mathbf{z}$ for which H_3 takes positive semi-definite values ($H_3 \geq 0$). This fact allows us to choose, among the many equivalent definition of the rotation number (see Chapter 2 of [10], which contains and extend the previous results of [16], [3] and [8]), that based on the characteristics of the so-called proper focal points, which is valid just for those Hamiltonian systems with $H_3 \geq 0$. This definition involves several concepts and properties which will be useful later.

Take a *conjoined basis* for the system $(2.5)_\omega$; i.e., a $2n \times n$ matrix solution $U(t,\omega)\cdot l \equiv U(t,\omega) \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{smallmatrix}\right]$ with initial data $l \equiv \left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right] \in \mathcal{L}_\mathbb{R}$. A point $t_0 \in \mathbb{R}$ is a *focal* or *vertical point for* $\left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{smallmatrix}\right]$ if $\det L_1(t_0,\omega) = 0$, which means that this solution intersects the vertical Maslov cycle $\mathcal{C}$ (defined by (2.8)) at t_0. Among these points, the so-called *proper* focal points, defined below, are fundamental in the analysis of the oscillatory properties of the Hamiltonian systems when $H_3 \geq 0$. Under this hypothesis, it is shown in Lemma 2.34 of [10] and in Theorem 3 of [13] that, given any interval $[a,b] \subset \mathbb{R}$, there exists a finite number of points $a = t_1 < \ldots < t_p = b$ such that $\operatorname{Ker} L_1(t,\omega)$ is constant on (t_j, t_{j+1}) for $j = 1, \ldots, p-1$. In particular, the map $t \mapsto \operatorname{Ker} L_1(t,\omega)$ is piecewise constant. That is,

$$\operatorname{Ker}(s_1,\omega) = \operatorname{Ker} L_1(s_2,\omega) \subseteq \operatorname{Ker} L_1(t_j,\omega) \cap \operatorname{Ker} L_1(t_{j+1},\omega) \tag{2.9}$$

for all $s_1, s_2 \in (t_j, t_{j+1})$. The last contention is a trivial consequence of the piecewise constant character. All this justifies the equivalence stated in the next definition (see, e.g., Definition 1.1 of Wahrheit [23]).

Definition 2.4. A point $t_0 \in \mathbb{R}$ is a *proper focal point for* $\left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{smallmatrix}\right]$ if

$$\operatorname{Ker} L_1(t_0^-,\omega) \subsetneq \operatorname{Ker} L_1(t_0,\omega)\,,$$

where $\operatorname{Ker} L_1(t_0^-,\omega)$ denotes the left-hand limit of the constant kernel of $L_1(t,\omega)$ at the point t_0. Or equivalently, if

$$m(t_0) = \dim \operatorname{Ker} L_1(t_0,\omega) - \dim \operatorname{Ker} L_1(t_0^-,\omega) \geq 1\,.$$

In this case, $m(t_0)$ is the *multiplicity* of the proper focal point t_0.

The next result is proved in Section 3.1 of [8].

Theorem 2.5. *Let m_0 be a σ-ergodic measure on Ω. Given $(\omega, l) \in \Omega \times \mathcal{L}_\mathbb{R}$, consider the conjoined basis $\left[\begin{smallmatrix} L_1(t,\omega) \\ L_2(t,\omega) \end{smallmatrix}\right] \equiv U(t,\omega)\cdot l$, and denote by $J_{\omega,l}(t)$ the set of its proper focal points contained in the interval $(0,t]$ and by $m(t^*)$ the multiplicity of $t^* \in J_{\omega,l}(t)$. Then there exists an $\alpha(m_0) \geq 0$ such that*

$$\alpha(m_0) = \lim_{t\to\infty} \frac{\pi}{t} \sum_{t^*\in J_{\omega,l}(t)} m(t^*)$$

for m_0-almost every $\omega \in \Omega$ and all $l \in \mathcal{L}_\mathbb{R}$.

Definition 2.6. Assume that $H_3 \geq 0$. The *rotation number of the family* (2.5) *with respect to the σ-ergodic measure m_0* is the quantity $\alpha(m_0)$ given in Theorem 2.5.

Note that the rotation number is zero in the particular case in which for all the points ω in a subset of Ω with positive measure m_0 there exists a conjoined basis of $(2.5)_\omega$ for which the set of proper focal points is upper-bounded.

3. **Null controllability and reachable sets.** Let Ω be a compact metric space with a continuous flow $\sigma(\omega,t)=\omega{\cdot}t$. Consider the family of time-dependent linear control systems

$$\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u}(t)\,, \quad \omega\in\Omega\,, \tag{3.1}$$

where the functions $A\colon\Omega\to\mathbb{M}_{n\times n}(\mathbb{R})$ and $B\colon\Omega\to\mathbb{M}_{n\times m}(\mathbb{R})$ are continuous and $\mathbf{u}(t)\in\mathbb{R}^m$ is a control vector. Any control function $\mathbf{u}\colon[t_1,t_2]\to\mathbb{R}^m$ appearing throughout the paper will be integrable. Recall that $U_A(t,\omega)$ is the fundamental matrix solution of the system $\mathbf{x}'=A(\omega{\cdot}t)\,\mathbf{x}$ satisfying $U_A(0,\omega)=I_n$. Denote by $\mathbf{x}(t,\omega,\mathbf{x}_0,\mathbf{u})$ the solution of the system $(3.1)_\omega$ for control $\mathbf{u}$ satisfying $\mathbf{x}(0,\omega,\mathbf{x}_0,\mathbf{u})=\mathbf{x}_0$; i.e.,

$$\mathbf{x}(t,\omega,\mathbf{x}_0,\mathbf{u}) = U_A(t,\omega)\,\mathbf{x}_0 + U_A(t,\omega)\int_0^t U_A^{-1}(s,\omega)\,B(\omega{\cdot}s)\,\mathbf{u}(s)\,ds\,. \tag{3.2}$$

Definition 3.1. Fix $\omega\in\Omega$. A control $\mathbf{u}\colon[0,t_0]\to\mathbb{R}^m$ *steers* $\mathbf{x}_0$ *to* $\mathbf{0}$ *in time* t_0 *for the system* $(3.1)_\omega$ if $\mathbf{x}(t_0,\omega,\mathbf{x}_0,\mathbf{u})=\mathbf{0}$. In this case, $\mathbf{x}_0$ is *null controllable for the system* $(3.1)_\omega$ *in time* t_0. The point $\mathbf{x}_0$ is *null controllable for the system* $(3.1)_\omega$ if so is for some time $t_0\ge 0$. If every $\mathbf{x}_0\in\mathbb{R}^n$ is null controllable for $(3.1)_\omega$, the system itself is *null controllable*.

Definition 3.2. The family (3.1) is *uniformly null controllable* if there exists a common time $t_0>0$ such that every $\mathbf{x}_0\in\mathbb{R}^n$ can be steered to $\mathbf{0}$ in time t_0 for all the systems of the family.

Remark 3.3. The family (3.1) is uniformly null controllable if and only if each minimal subset of Ω contains at least one point ω_1 such that the corresponding system $(3.1)_{\omega_1}$ is null controllable. The proof of this property appears in Johnson and Nerurkar [6]. In particular, the family is uniformly null controllable if and only if all its systems are null controllable.

Next consider the time-reversed linear control family

$$\mathbf{x}' = -A(\omega{\cdot}(-t))\,\mathbf{x} - B(\omega{\cdot}(-t))\,\mathbf{u}(t)\,, \quad \omega\in\Omega\,. \tag{3.3}$$

It is easy to check that the time-reversed map $\sigma^-(t,\omega)=\sigma(-t,\omega)=\omega{\cdot}(-t)$ also defines a continuous flow on Ω. In the time-reversed control problems (3.3), the matrices A and B are evaluated along the orbits of this new flow. The fundamental matrix solution of $\mathbf{x}'=-A(\omega{\cdot}(-t))\,\mathbf{x}$ with value I_n at $t=0$ is $U_A(-t,\omega)$. We will denote by $\widetilde{\mathbf{x}}(t,\omega,\mathbf{x}_0,\mathbf{u})$ the solution of the system $(3.3)_\omega$ for control $\mathbf{u}$ satisfying $\widetilde{\mathbf{x}}(t,\omega,\mathbf{x}_0,\mathbf{u})=\mathbf{x}_0$. For each point $\omega\in\Omega$ we consider the set of points which are null controllable for the time-reversed linear control system $(3.3)_\omega$, that is,

$$E(\omega)=\{\mathbf{x}_0\in\mathbb{R}^n \mid \mathbf{x}_0 \text{ is null controllable for } (3.3)_\omega\}\,. \tag{3.4}$$

According to Remark 3.3, the family of control systems (3.1) is uniformly null controllable (which according to Proposition 2.5 of [5] is equivalent to say that the family of time-reversed control systems (3.3) is uniformly null controllable) if and only if the set $E(\omega)$ is equal to $\mathbb{R}^n$ for all $\omega\in\Omega$. The situation which is interesting for the purposes of this paper is the case in which the family of control systems (3.1) is not uniformly null controllable, or equivalently, $E(\omega)\subsetneq\mathbb{R}^n$ for some $\omega\in\Omega$.

The set $E(\omega)$ can be related to the set of those points of $\mathbb{R}^n$ which can be reached from $\mathbf{0}$ at time T for the system $(3.1)_{\omega\cdot(-T)}$, that is,

$$E_T(\omega) = \{\mathbf{x}(T,\omega\cdot(-T),\mathbf{0},\mathbf{u}) \mid \mathbf{u}\colon [0,T]\to\mathbb{R}^m \text{ is a control}\}\,, \tag{3.5}$$

as the following result proves.

Proposition 3.4. *Fix a point $\omega\in\Omega$. Then,*

(i) $\mathbf{x}_0\in E_T(\omega)$ *if and only if there exists a control* $\mathbf{u}\colon[0,T]\to\mathbb{R}^m$ *such that*

$$\mathbf{x}_0=\int_0^T U_A^{-1}(-s,\omega)\,B(\omega\cdot(-s))\,\mathbf{u}(s)\,ds\,. \tag{3.6}$$

(ii) $E_T(\omega)$ *is a vector space, and* $E_{T_1}(\omega)\subseteq E_{T_2}(\omega)$ *whenever* $0\le T_1\le T_2$.

(iii) $E(\omega)=\cup_{T\ge 0}E_T(\omega)$*, and there exists a minimum time* $T(\omega)\ge 0$ *such that* $E(\omega)=E_T(\omega)$ *for every* $T>T(\omega)$*. In particular,* $E(\omega)$ *is a vector space and*

$$E(\omega)=\left\{\int_0^T U_A^{-1}(-s,\omega)\,B(\omega\cdot(-s))\,\mathbf{u}(s)\,ds \mid \mathbf{u}\colon[0,T]\to\mathbb{R}^m \text{ is a control}\right\}$$

for all $T>T(\omega)$.

(iv) $\mathbf{x}(t,\omega,\mathbf{x}_0,\mathbf{u})\in E(\omega\cdot t)$ *for every point* $\mathbf{x}_0\in E(\omega)$*, every time* $t\ge 0$ *and every control* $\mathbf{u}\colon[0,t]\to\mathbb{R}^m$.

Proof. (i) By definition, $\mathbf{x}_0\in E_T(\omega)$ if and only if there exists a control $\mathbf{u}\colon[0,T]\to\mathbb{R}^m$ such that $\mathbf{x}_0=\mathbf{x}(T,\omega\cdot(-T),\mathbf{0},\mathbf{u})$, which according to (3.2) and the equality $U_A(s-T,\omega)\,U_A(T,\omega\cdot(-T))=U_A(s,\omega\cdot(-T))$ (deduced from (2.2)) is equivalent to say that

$$\mathbf{x}_0=\int_0^T U_A^{-1}(s-T,\omega)\,B((\omega\cdot(-T))\cdot s)\,\mathbf{u}(s)\,ds=\int_0^T U_A^{-1}(-r,\omega)\,B(\omega\cdot(-r))\,\widetilde{\mathbf{u}}(r)\,dr$$

for $\widetilde{\mathbf{u}}(t)=\mathbf{u}(T-t)$. Then (i) is proved.

(ii) It is immediate to check that the set $E_T(\omega)$ is a vector subspace of $\mathbb{R}^n$: if $\mathbf{x}_1$, $\mathbf{x}_2\in E_T(\omega)$ for controls $\mathbf{u}_1$ and $\mathbf{u}_2$, then $\lambda_1\,\mathbf{x}_1+\lambda_2\,\mathbf{x}_2\in E_T(\omega)$ for control $\lambda_1\mathbf{u}_1+\lambda_2\mathbf{u}_2$. Now we take $0\le T_1<T_2$ and claim that $E_{T_1}(\omega)\subseteq E_{T_2}(\omega)$. To prove this, we take $\mathbf{x}_1\in E_{T_1}(\omega)$, use (i) to write $\mathbf{x}_1=\int_0^{T_1}U_A^{-1}(-s,\omega)\,B(\omega\cdot(-s))\,\mathbf{u}_1(s)\,ds$ for a control $\mathbf{u}_1\colon[0,T_1]\to\mathbb{R}^m$, define $\mathbf{u}_2\colon[0,T_2]\to\mathbb{R}^m$ by concatenating $\mathbf{u}_1$ on $[0,T_1]$ with $\mathbf{0}$ on $[T_1,T_2]$, observe that $\mathbf{x}_1=\int_0^{T_2}U_A^{-1}(-s,\omega)\,B(\omega\cdot(-s))\,\mathbf{u}_2(s)\,ds$, and deduce from (i) that $\mathbf{x}_1\in E_{T_2}(\omega)$, as asserted.

(iii) By definition, $\mathbf{x}_0\in E(\omega)$ if and only if there exist a nonnegative time $T\ge 0$ and a control $\mathbf{u}\colon[0,T]\to\mathbb{R}^m$ such that $\widetilde{\mathbf{x}}(T,\omega,\mathbf{x}_0,\mathbf{u})=\mathbf{0}$, which according to (3.2) is equivalent to say that

$$\mathbf{0}=U_A(-T,\omega)\left[\mathbf{x}_0-\int_0^T U_A^{-1}(-s,\omega)\,B(\omega\cdot(-s))\,\mathbf{u}(s)\,ds\right].$$

In other words, and according to (i), $\mathbf{x}_0\in E(\omega)$ if and only if there exists a $T\ge 0$ such that $\mathbf{x}_0\in E_T(\omega)$. This proves the first equality in (iii), which in turn makes it immediate to deduce from (ii) that $E(\omega)$ is a vector space too. In addition, since $E_T(\omega)$ is nondecreasing in T and has a bounded dimension, there exists a minimum $T(\omega)$ such that $E(\omega)=E_T(\omega)$ for every $T>T(\omega)$, which proves the remaining assertions.

(iv) According to (iii), if $\mathbf{x}_0 \in E(\omega)$ then there exist a time $T \geq 0$ and a control $\widetilde{\mathbf{u}} : [0,T] \to \mathbb{R}^m$ such that $\mathbf{x}_0 = \int_0^T U_A^{-1}(-s,\omega)\, B(\omega{\cdot}(-s))\, \widetilde{\mathbf{u}}(s)\, ds$. Therefore, given a time $t \geq 0$ and a control $\mathbf{u}\colon [0,t] \to \mathbb{R}^m$, we obtain from (3.2) that

$$\begin{aligned}
\mathbf{x}(t,\omega,\mathbf{x}_0,\mathbf{u}) = U_A(t,\omega) \bigg[&\int_0^T U_A^{-1}(-s,\omega)\, B(\omega{\cdot}(-s))\, \widetilde{\mathbf{u}}(s)\, ds \\
&+ \int_0^t U_A^{-1}(s,\omega)\, B(\omega{\cdot}s)\, \mathbf{u}(s)\, ds \bigg] \\
= U_A(t,\omega) \bigg[&\int_t^{T+t} U_A^{-1}(t-r,\omega)\, B(\omega{\cdot}(t-r))\, \widetilde{\mathbf{u}}(r-t)\, dr \\
&+ \int_0^t U_A^{-1}(t-r,\omega)\, B(\omega{\cdot}(t-r))\, \mathbf{u}(t-r)\, dr \bigg] \\
= \int_0^{t+T} &U_A^{-1}(-r,\omega{\cdot}t)\, B((\omega{\cdot}t)(-r))\, \widehat{\mathbf{u}}(r)\, dr
\end{aligned}$$

for the control $\widehat{\mathbf{u}}\colon [0,t+T] \to \mathbb{R}^m$ defined by

$$\widehat{\mathbf{u}}(r) = \begin{cases} \mathbf{u}(t-r) & \text{if } 0 \leq r \leq t\,, \\ \widetilde{\mathbf{u}}(r-t) & \text{if } t < r \leq t+T\,. \end{cases}$$

Properties (i) and (iii) show that $\mathbf{x}(t,\omega,\mathbf{x}_0,\mathbf{u}) \in E_{t+T}(\omega{\cdot}t) \subseteq E(\omega{\cdot}t)$. □

Next we consider the integer-valued map $d_E\colon \Omega \to \{0,\dots,n\}$, $\omega \mapsto \dim E(\omega)$.

Proposition 3.5. (i) *If $t \geq 0$, then $U_A(t,\omega){\cdot}E(\omega) \subseteq E(\omega{\cdot}t)$ for each $\omega \in \Omega$.*

(ii) *If $t_1 \leq t_2$, then $d_E(\omega{\cdot}t_1) \leq d_E(\omega{\cdot}t_2)$, i.e., $t \mapsto d_E(\omega{\cdot}t)$ is a nondecreasing map for each fixed $\omega \in \Omega$.*

(iii) *For each $\omega \in \Omega$ there are times $\alpha(\omega)$ and $\beta(\omega)$ such that $d_E(\omega{\cdot}t) = d_E(\omega{\cdot}\alpha(\omega))$ for each $t \leq \alpha(\omega)$ and $d_E(\omega{\cdot}t) = d_E(\omega{\cdot}\beta(\omega))$ for each $t \geq \beta(\omega)$.*

(iv) *The map d_E is lower semicontinuous.*

Proof. (i) Take $\mathbf{x}_0 \in E(\omega)$ and $T > T(\omega)$, where $T(\omega)$ is given by Proposition 3.4(iii). This result ensures that

$$U_A(t,\omega)\, \mathbf{x}_0 = U_A(t,\omega) \int_0^T U_A^{-1}(-s,\omega)\, B(\omega{\cdot}(-s))\, \mathbf{u}(s)\, ds$$

for some control $\mathbf{u}\colon [0,T] \to \mathbb{R}^m$. It follows from (2.2) that $U_A(t,\omega)\, U_A^{-1}(-s,\omega) = U_A^{-1}(-s-t,\omega{\cdot}t)$. Hence, if we define $\widehat{\mathbf{u}} : [0,T+t] \to \mathbb{R}^m$ by

$$\widehat{\mathbf{u}}(r) = \begin{cases} \mathbf{0} & \text{if } 0 \leq r \leq t\,, \\ \mathbf{u}(r-t) & \text{if } t < r \leq t+T\,, \end{cases}$$

we deduce that

$$\begin{aligned}
U_A(t,\omega)\, \mathbf{x}_0 &= \int_0^T U_A^{-1}(-s-t,\omega{\cdot}t)\, B(\omega{\cdot}(-s))\, \mathbf{u}(s)\, ds \\
&= \int_0^{T+t} U_A^{-1}(-r,\omega{\cdot}t)\, B(\omega{\cdot}(t-r))\, \widehat{\mathbf{u}}(r)\, dr\,,
\end{aligned}$$

that is, $U_A(t,\omega)\, \mathbf{x}_0 \in E_{t+T}(\omega{\cdot}t)$. Proposition 3.4(iii) shows that $U_A(t,\omega)\, \mathbf{x}_0 \in E(\omega{\cdot}t)$, and this proves (i).

(ii) This property follows from (i) because $U_A(t_2-t_1,\omega{\cdot}t_1){\cdot}E(\omega{\cdot}t_1)\subseteq E(\omega{\cdot}t_2)$ and all the matrices $U_A(t,\omega)$ are non singular.

(iii) Since $0\le d_E(\omega{\cdot}t)\le n$ for all $t\in\mathbb{R}$, the assertion is a consequence of (ii).

(iv) We must check that $d_E(\omega_0)\le\liminf_{\omega\to\omega_0} d_E(\omega)$ for each $\omega_0\in\Omega$. Let (ω_k) be a sequence with $\lim_{k\to\infty}\omega_k=\omega_0$ and $d_E(\omega_0)=d_0>0$ (for $d_0=0$ it is obvious). Fix $T>T(\omega_0)$. We take a basis $\{\mathbf{x}_1,\dots,\mathbf{x}_{d_0}\}$ of $E(\omega_0)$ and, for each $j\in\{1,\dots,d_0\}$, a control $\mathbf{u}_j\colon[0,T]\to\mathbb{R}^m$ such that $\mathbf{x}_j=\mathbf{x}(T,\omega_0{\cdot}(-T),\mathbf{0},\mathbf{u}_j)$. We denote by $X(\omega_0)$ the $n\times d_0$ matrix with columns $\mathbf{x}_1,\dots,\mathbf{x}_{d_0}$ and by $X(\omega_k)$ the $n\times d_0$ matrix with columns $\mathbf{x}(T,\omega_k{\cdot}(-T),\mathbf{0},\mathbf{u}_j)$ for $j=1,\dots,d_0$, which according to Proposition 3.4(iii) are d_0 vectors of $E(\omega_k)$. Since $\lim_{k\to\infty}X(\omega_k)=X(\omega_0)$ and the rank map on matrices is lower semicontinuous, there exists a k_0 such that $\operatorname{rank}X(\omega_k)\ge\operatorname{rank}X(\omega_0)=d_0$ for each $k\ge k_0$; that is, the columns of $X(\omega_k)$ are d_0 linearly independent vectors of $E(\omega_k)$ and, consequently, $d_E(\omega_k)\ge d_E(\omega_0)$. This finishes the proof. □

Next we consider the maps

$$\begin{aligned} d_E^-\colon\Omega &\to \{0,\dots,n\} \\ \omega &\mapsto d_E(\omega{\cdot}\alpha(\omega))\,, \end{aligned}\qquad\qquad \begin{aligned} d_E^+\colon\Omega &\to \{0,\dots,n\} \\ \omega &\mapsto d_E(\omega{\cdot}\beta(\omega))\,, \end{aligned}$$

where $\alpha(\omega)$ and $\beta(\omega)$ satisfy the conditions of Proposition 3.5(iii), and the quantities

$$d_E^M=\max_{\omega\in\Omega}d_E(\omega)\,,\quad d_E^m=\min_{\omega\in\Omega}d_E(\omega)\,.$$

The following result collects several properties of these functions and quantities.

Theorem 3.6. (i) *For each $\omega\in\Omega$, $d_E^-(\omega)\le d_E(\omega)\le d_E^+(\omega)$.*

(ii) *The functions d_E^+ and d_E^- are invariant on Ω, that is, $d_E^+(\omega{\cdot}t)=d_E^+(\omega)$ and $d_E^-(\omega{\cdot}t)=d_E^-(\omega)$ for all $\omega\in\Omega$ and all $t\in\mathbb{R}$.*

(iii) *If $\omega_1\in\mathcal{O}(\omega)$, then $d_E^+(\omega_1)\le d_E^+(\omega)$, and if $\omega_1\in\mathcal{A}(\omega)$, then $d_E^+(\omega_1)\le d_E^-(\omega)$.*

(iv) *If m_0 is a σ-ergodic measure on Ω, then d_E^+, d_E^- and d_E are constant and coincide for m_0-a.e. $\omega\in\Omega$.*

(v) *If $\Omega^*\subseteq\Omega$ is minimal, then there exists a constant d such that $d_E^+(\omega)=d_E^-(\omega)=d_E(\omega)=d$ and $E(\omega{\cdot}t)=U_A(t,\omega){\cdot}E(\omega)$ for all $\omega\in\Omega^*$ and $t\in\mathbb{R}$.*

(vi) *There exists a minimal subset $\Omega^*\subseteq\Omega$ such that $d_E(\omega)=d_E^+(\omega)=d_E^-(\omega)=d_E^m$ for each $\omega\in\Omega^*$.*

(vii) *The set of continuity points for d_E is an open residual subset $\Omega_c\subseteq\Omega$ on which d is locally constant, and it satisfies*

$$\{\omega\in\Omega\mid d_E(\omega)=d_E^M\}\subseteq\Omega_c\,.$$

(viii) *If there exists a point $\omega_0\in\Omega$ with $\mathcal{A}(\omega_0)=\Omega$ then $d_E^-(\omega_0)=d_E^+(\omega_0)=d_E(\omega_0)=d_E^M$. In particular, d is continuous at ω_0 and*

$$\{\omega\in\Omega\mid d_E(\omega)=d_E^M\}=\Omega_c\,.$$

(ix) *If m_0 is a σ-ergodic measure on Ω with $\operatorname{Supp}m_0=\Omega$, then $m_0(\Omega_c)=1$ and $\Omega_c=\{\omega\in\Omega\mid d_E(\omega)=d_E^M\}$.*

(x) *Let $\{\omega_k\}$ be a sequence of Ω such that $\lim_{k\to\infty}\omega_k=\omega\in\Omega$ and $d_E(\omega_k)=d_E(\omega)=d>0$ for each k. Then $\lim_{k\to\infty}E(\omega_k)=E(\omega)$ in the Grassmannian manifold $\mathcal{G}_d(\mathbb{R}^n)$.*

(xi) *Let $\omega_0\in\Omega$ satisfy $d_E(\omega_0{\cdot}t)=d>0$ for all $t\ge0$ and consider the omega-limit set $\mathcal{O}(\omega_0,E(\omega_0))$ for the flow τ_k defined on $\Omega\times\mathcal{G}_d(\mathbb{R}^n)$ by (2.4). If $\omega_1\in\mathcal{O}(\omega_0)$, then $E(\omega_1)\subseteq E_1$ for each $(\omega_1,E_1)\in\mathcal{O}(\omega_0,E(\omega_0))$.*

(xii) *Let* $\omega_0 \in \Omega$ *satisfy* $d_E(\omega_0{\cdot}t) = d > 0$ *for all* $t \le 0$ *and consider the alpha-limit set* $\mathcal{A}(\omega_0, E(\omega_0))$ *for the flow* τ_k *defined on* $\Omega \times \mathcal{G}_d(\mathbb{R}^n)$ *by* (2.4). *If* $\omega_1 \in \mathcal{A}(\omega_0)$, *then* $E(\omega_1) \subseteq E_1$ *for each* $(\omega_1, E_1) \in \mathcal{A}(\omega_0, E(\omega_0))$.

Proof. Recall that Proposition 3.4(iii) associates a time $T(\omega)$ to each point $\omega \in \Omega$.

(i) It is an easy consequence of the definitions and Proposition 3.5(ii)&(iii).

(ii) From the definitions of $d_E^{\pm}(\omega)$ and Proposition 3.5(ii), we have that $d_E^+(\omega) = \max_{s\in\mathbb{R}} d_E(\omega{\cdot}s)$ and $d_E^-(\omega) = \min_{s\in\mathbb{R}} d_E(\omega{\cdot}s)$, from where the statements follow.

(iii) We will only prove that $d_E^+(\omega_1) \le d_E^+(\omega)$ if $\omega_1 \in \mathcal{O}(\omega)$, because the other inequality is proved in an analogous way. First we prove that $d_E(\omega_1) \le d_E^+(\omega)$. If $d_E(\omega_1) = 0$ the inequality is obvious, so let us assume that $d_1 = d_E(\omega_1) \ge 1$, take $T > T(\omega_1)$ and consider an $n \times d_1$ matrix $X(\omega_1)$ of rank d_1 with columns $\mathbf{x}(T, \omega_1{\cdot}(-T), \mathbf{0}, \mathbf{u}_j)$ for some controls $\mathbf{u}_j \colon [0,T] \to \mathbb{R}^m$ for $j = 1, \ldots, d_1$. Since $\omega_1 \in \mathcal{O}(\omega)$, we take a sequence $(t_k) \uparrow \infty$ with $\omega_1 = \lim_{k\to\infty} \omega{\cdot}t_k$ and the $n \times d_1$ matrix $X(\omega{\cdot}t_k)$ with columns $\mathbf{x}(T, \omega{\cdot}(t_k - T), \mathbf{0}, \mathbf{u}_j)$ for $j = 1, \ldots, d_1$ which are d_0 vectors of $E(\omega{\cdot}t_k)$. Since $\lim_{k\to\infty} X(\omega{\cdot}t_k) = X(\omega_1)$, the lower semicontinuity of the rank function on matrices provides a k_0 such that $\operatorname{rank} X(\omega{\cdot}t_k) \ge \operatorname{rank} X(\omega_1) = d_1$ for each $k \ge k_0$; that is, the columns are d_1 linearly independent vectors of $E(\omega{\cdot}t_k)$. Consequently, if k is large enough, then $d_E^+(\omega) = d_E(\omega{\cdot}\beta(\omega)) = d_E(\omega{\cdot}t_k) \ge d_E(\omega_1)$, as claimed. Finally, since $d_E^+(\omega_1) = d_E(\omega_1{\cdot}\beta(\omega_1))$ and $\omega_1{\cdot}\beta(\omega_1) \in \mathcal{O}(\omega)$, we conclude that $d_E^+(\omega_1) \le d_E^+(\omega)$, as stated.

(iv) From (ii) we deduce that d_E^+ and d_E^- are constant for m_0-a.e. In addition, the Poincaré Recurrence Theorem (see [1]) ensures that there exists a subset Ω_0 of full measure such that $\omega \in \mathcal{O}(\omega) \cap \mathcal{A}(\omega)$ for $\omega \in \Omega_0$, and hence the coincidence of d_E, d_E^+ and d_E^- on a set of full measure follows from (iii).

(v) The constant character of d_E, d_E^+ and d_E^- on Ω^* follows immediately from (iii), (i), and the minimal character of Ω^*. In particular, $d_E(\omega) = d_E(\omega{\cdot}t)$ for $\omega \in \Omega^*$ and $t \in \mathbb{R}$. Hence Proposition 3.5(i) ensures that $U_A(t,\omega){\cdot}E(\omega) = E(\omega{\cdot}t)$ because both spaces have the same dimension.

(vi) Take ω_0 with $d_E^m = d_E(\omega_0)$ and a minimal subset $\Omega^* \subseteq \mathcal{A}(\omega_0)$. From (iii) and (i) we deduce that $d_E^+(\omega) \le d_E^-(\omega_0) \le d_E(\omega_0) = d_E^m \le d_E(\omega) \le d_E^+(\omega)$, and we conclude from (v) that $d_E(\omega) = d_E^+(\omega) = d_E^-(\omega) = d_E^m$ for each $\omega \in \Omega^*$.

(vii) Proposition 3.5(iv) states that d_E is lower semicontinuous. Consequently, the set $\Omega_c \subseteq \Omega$ of its continuity points is a residual set which is necessarily open because d_E only takes integer values. Hence, for each $\omega \in \Omega_c$ there exists an open ball $B(\omega, \delta_\omega) \subset \Omega_c$ on which d_E is constant, that is, d_E is locally constant on Ω_c. Finally, we check that each $\omega \in \Omega$ with $d_E(\omega) = d_E^M$ is a continuity point. Let (ω_k) be a sequence such that $\lim_{k\to\infty} \omega_k = \omega$. From the definition of d_E^M and the lower semicontinuity of d_E, we deduce that

$$d_E^M = d_E(\omega) \le \liminf_{k\to\infty} d_E(\omega_k) \le \limsup_{k\to\infty} d_E(\omega_k) \le d_E^M\,,$$

and hence $\lim_{k\to\infty} d_E(\omega_k) = d_E(\omega)$, as claimed.

(viii) It follows from (iii) and (i) that $d_E^+(\omega_0) \le d_E^-(\omega_0) \le d_E(\omega_0) \le d_E^+(\omega_0)$, and therefore the three values coincide. In addition, $d_E(\omega) \le d_E^+(\omega) \le d_E^-(\omega_0)$ for all $\omega \in \Omega$, and taking the maximum in $\omega \in \Omega$ we deduce that $d_E^M \le d_E^-(\omega_0) \le d_E^M$, which proves the first part of the statement. Note also that, since $d_E^+(\omega_0) = \max_{s\in\mathbb{R}} d_E(\omega_0{\cdot}s)$, $d_E^-(\omega_0) = \min_{s\in\mathbb{R}} d_E(\omega_0{\cdot}s)$, and they coincide with $d_E(\omega_0)$, we deduce that d_E is constant along the orbit of ω_0. Next we check that $\Omega_c \subseteq \{\omega \in \Omega \mid d_E(\omega) = d_E^M\}$ to prove the second part of the statement. We take $\omega \in \Omega_c$.

Since $\omega \in \mathcal{A}(\omega_0)$, there exists a sequence $(t_k) \downarrow -\infty$ with $\lim_{k\to\infty} \omega_0{\cdot}t_k = \omega$. Consequently,

$$d_E(\omega) = \lim_{k\to\infty} d_E(\omega_0{\cdot}t_k) = d_E(\omega_0) = d_E^M\,,$$

as stated.

(ix) Since $\operatorname{Supp} m_0 = \Omega$, there exists a subset $\Omega_1 \subseteq \Omega$ with full measure such that $\mathcal{A}(\omega_1) = \Omega$ for all $\omega_1 \in \Omega_1$ (see, e.g., Proposition 1.12 of [10]). Property (viii) proves that $\Omega_1 \subseteq \Omega_c$ (so that $m(\Omega_c) = 1$) as well as the last equality in (ix).

(x) Fix $T > T(\omega)$ and take an $n \times d$ matrix $X(\omega)$ of rank d with columns $\mathbf{x}(T, \omega{\cdot}(-T), \mathbf{0}, \mathbf{u}_j)$ for some controls $\mathbf{u}_j \colon [0,T] \to \mathbb{R}^m$ and $j = 1, \ldots, d$, that is, the columns of $X(\omega)$ are a basis of the d-dimensional subspace $E(\omega)$. Let $X(\omega_k)$ be the $n\times d$ matrix with columns $\mathbf{x}(T, \omega_k{\cdot}(-T), \mathbf{0}, \mathbf{u}_j)$ for $j = 1, \ldots, d$, which, according to Proposition 3.4(iv), are d vectors of $E(\omega_k)$. Since $\lim_{k\to\infty} X(\omega_k) = X(\omega)$, the lower semicontinuity of the rank provides a k_0 such that $\operatorname{rank} X(\omega_k) \geq \operatorname{rank} X(\omega) = d$ for each $k \geq k_0$; that is, the columns of $X(\omega_k)$ are d linearly independent vectors of $E(\omega_k)$, and hence, by hypothesis, they form a basis of $E(\omega_k)$. This implies that $\lim_{k\to\infty} E(\omega_k) = E(\omega)$ in $\mathcal{G}_d(\mathbb{R}^n)$ (see, e.g., Proposition 1.25 of [10]).

(xi) Take $(\omega_1, E_1) \in \mathcal{O}(\omega_0, E(\omega_0))$. Then there exists a sequence $(t_k) \uparrow \infty$ with $t_k \geq 0$, $\lim_{k\to\infty} \omega_0{\cdot}t_k = \omega_1$ and $\lim_{k\to\infty} E(\omega_0{\cdot}t_k) = E_1$ in $\mathcal{G}_d(\mathbb{R}^n)$. Note that Proposition 3.5(i) ensures that $U_A(t_k, \omega){\cdot}E(\omega_0) \subseteq E(\omega_0{\cdot}t_k)$, so that they coincide because they have the same dimension. If $d_E(\omega_1) = d$, as in (x) we can check that $\lim_{k\to\infty} E(\omega_0{\cdot}t_k) = E(\omega_1)$ in $\mathcal{G}_d(\mathbb{R}^n)$ and hence that $E(\omega_1) = E_1$. Thus, assume that $d_E(\omega_1) = d_1 < d$, fix $T > T(\omega_1)$ and take an $n \times d_1$ matrix $X(\omega_1)$ of rank d_1 with columns $\mathbf{x}(T, \omega_1{\cdot}(-T), \mathbf{0}, \mathbf{u}_j)$ for some controls $\mathbf{u}_j \colon [0,T] \to \mathbb{R}^m$ and $j = 1, \ldots, d$, that is, the columns of $X(\omega_1)$ form a basis for the d_1-dimensional subspace $E(\omega_1)$. As before, $\lim_{k\to\infty} X(\omega_0{\cdot}t_k) = X(\omega_1)$, where the columns of $X(\omega_0{\cdot}t_k)$ are $\mathbf{x}(T, \omega_0{\cdot}(t_k{-}T), \mathbf{0}, \mathbf{u}_j)$, and there exists a k_0 such that $\operatorname{rank} X(\omega_0{\cdot}t_k) \geq \operatorname{rank} X(\omega_1) = d_1$ for each $k \geq k_0$; that is, the columns are d_1 linearly independent vectors of $E(\omega_0{\cdot}t_k)$ converging to a basis of $E(\omega_1)$. This together with $\lim_{k\to\infty} E(\omega_0{\cdot}t_k) = E_1$ in $\mathcal{G}_d(\mathbb{R}^n)$ shows that $E(\omega_1) \subseteq E_1$.

(xii) The proof is analogous to that of (xi). □

Next we consider the *null controllable set for the control system* $(3.1)_\omega$,

$$\widetilde{E}(\omega) = \{\mathbf{x}_0 \in \mathbb{R}^n \mid \mathbf{x}_0 \text{ is null controllable for } (3.1)_\omega\}\,.$$

The following result shows that $\widetilde{E}(\omega)$ can be related to the *reachable set for the system* $(3.1)_\omega$ *from* $\mathbf{0}$ *at time* T defined by

$$F_T(\omega) = \{\mathbf{x}(T, \omega, \mathbf{0}, \mathbf{u}) \mid \mathbf{u} \colon [0,T] \to \mathbb{R}^m \text{ is a control}\}\,. \tag{3.7}$$

Note that $F_T(\omega) = E_T(\omega{\cdot}T)$ (see (3.5)).

Proposition 3.7. *Fix a point* $\omega \in \Omega$*, then*

(i) $\mathbf{x}_0 \in \widetilde{E}(\omega)$ *if and only if there exists a time* $T \geq 0$ *such that*

$$\mathbf{x}_0 = \int_0^T U_A^{-1}(s, \omega)\, B(\omega{\cdot}s)\, \mathbf{u}(s)\, ds \tag{3.8}$$

for some control $\mathbf{u} \colon [0,T] \to \mathbb{R}^m$*. Moreover,* $F_T(\omega) \subseteq U_A(T, \omega){\cdot}\widetilde{E}(\omega)$ *and hence* $\dim F_T(\omega) \leq \dim \widetilde{E}(\omega)$ *for each* $T \geq 0$*.*

(ii) *There exists a minimum time* $\widetilde{T}(\omega)\geq 0$ *such that, for every* $T>\widetilde{T}(\omega)$,

$$\widetilde{E}(\omega)=\left\{\int_0^T U_A^{-1}(s,\omega)\,B(\omega{\cdot}s)\,\mathbf{u}(s)\,ds \mid \mathbf{u}\colon[0,T]\to\mathbb{R}^m \text{ is a control}\right\}.$$

In addition, if $T>\widetilde{T}(\omega)$, *then* $F_T(\omega)=U_A(T,\omega){\cdot}\widetilde{E}(\omega)$, *or equivalently,* $\widetilde{E}(\omega)=U_A(-T,\omega{\cdot}T){\cdot}F_T(\omega)$, *and hence* $\dim F_T(\omega)=\dim\widetilde{E}(\omega)$.

Proof. (i) Repeating the proof of Proposition 3.4(iii) we conclude that $\mathbf{x}_0\in\widetilde{E}(\omega)$ if and only if there exist a time $T\geq 0$ and a control $\mathbf{u}\colon[0,T]\to\mathbb{R}^m$ such that (3.8) holds. From this fact, (3.2) and (2.2), we deduce that $F_T(\omega)\subseteq U_A(T,\omega){\cdot}\widetilde{E}(\omega)$ for each $T\geq 0$, which proves the last statements in (i).

(ii) Proposition 3.4(iii) applied to $(3.1)_\omega$ instead of $(3.3)_\omega$ provides a nonnegative time $\widetilde{T}(\omega)\geq 0$ satisfying the characterization stated for $\widetilde{E}(\omega)$. Consequently, if $T>\widetilde{T}(\omega)$ and $\mathbf{x}_0\in\widetilde{E}(\omega)$, there exists a control $\mathbf{u}\colon[0,T]\to\mathbb{R}^m$ such that $\mathbf{x}_0=\int_0^T U_A^{-1}(s,\omega)\,B(\omega{\cdot}s)\,\mathbf{u}(s)\,ds$. Therefore,

$$U_A(T,\omega)\,\mathbf{x}_0=U_A(T,\omega)\int_0^T U_A^{-1}(s,\omega)\,B(\omega{\cdot}s)\,\mathbf{u}(s)\,ds=\mathbf{x}(T,\omega,\mathbf{0},\mathbf{u})\in F_T(\omega)\,,$$

that is, $U_A(T,\omega){\cdot}\widetilde{E}(\omega)\subseteq F_T(\omega)$, which together with $F_T(\omega)\subseteq U_A(T,\omega){\cdot}\widetilde{E}(\omega)$, shown in (i), finishes the proof. □

Proposition 3.8. *Fix* $\omega\in\Omega$. *Let* $T(\omega)$ *and* $\widetilde{T}(\omega)$ *be the nonnegative times for* $E(\omega)$ *and* $\widetilde{E}(\omega)$ *provided by* Proposition 3.4(iii) *and* Proposition 3.7(ii) *respectively. Then,*

(i) $E(\omega)=F_T(\omega{\cdot}(-T))$ *and hence* $\dim E(\omega)\leq\dim\widetilde{E}(\omega{\cdot}(-T))$ *if* $T>T(\omega)$.
(ii) $F_T(\omega)=E_T(\omega{\cdot}T)$ *and hence* $\dim\widetilde{E}(\omega)\leq\dim E(\omega{\cdot}T)$ *if* $T>\widetilde{T}(\omega)$.

Proof. (i) From Proposition 3.4(iii) and definition (3.7) we deduce that

$$E(\omega)=E_T(\omega)=F_T(\omega{\cdot}(-T))$$

whenever $T>T(\omega)$, and hence we conclude from Proposition 3.7(i) that

$$\dim E(\omega)=\dim F_T(\omega{\cdot}(-T))\leq\dim\widetilde{E}(\omega{\cdot}(-T))$$

whenever $T>T(\omega)$, as stated.

(ii) From (3.5), (3.7) and Proposition 3.4(iii), we deduce that $F_T(\omega)=E_T(\omega{\cdot}T)\subseteq E(\omega{\cdot}T)$ for all $T\in\mathbb{R}$. Consequently, from Proposition 3.7(ii) we conclude that $\dim\widetilde{E}(\omega)=\dim F_T(\omega)\leq\dim E(\omega{\cdot}T)$ whenever $T>\widetilde{T}(\omega)$, as claimed. □

The next result characterizes some cases in which the sets $\widetilde{E}(\omega)$ and $E(\omega)$ coincide, or in other words, the null controllable points for $(3.1)_\omega$ coincide with the null controllable points for the time-reversed system $(3.3)_\omega$.

Theorem 3.9. *Fix* $\omega\in\Omega$. *Let* $T(\omega)$ *and* $\widetilde{T}(\omega)$ *be the nonnegative times for* $E(\omega)$ *and* $\widetilde{E}(\omega)$ *provided by* Proposition 3.4(iii) *and* Proposition 3.7(ii) *respectively.*

(i) *If* $\dim E(\omega{\cdot}t)$ *and* $\dim\widetilde{E}(\omega{\cdot}t)$ *are constant for all* $t\in\mathbb{R}$, *then* $E(\omega)=\widetilde{E}(\omega)$.
(ii) *If there exists a constant* $\widehat{T}$ *such that* $T(\omega{\cdot}t)\leq\widehat{T}$ *and* $\widetilde{T}(\omega{\cdot}t)\leq\widehat{T}$ *for each* $t\in\mathbb{R}$, *then* $E(\omega)=\widetilde{E}(\omega)$.

Proof. (i) First, bearing Proposition 3.8 in mind and the fact that $\dim E(\omega{\cdot}t)$ and $\dim \widetilde{E}(\omega{\cdot}t)$ are constant for all $t\in\mathbb{R}$, we deduce that $\dim E(\omega)=\dim\widetilde{E}(\omega)$. Next, we know from Proposition 3.5(i) that $U_A(t,\omega){\cdot}E(\omega)\subseteq E(\omega{\cdot}t)$ for each $t\ge 0$ and, since they have the same dimension, they coincide; i.e., $U_A(t,\omega){\cdot}E(\omega)=E(\omega{\cdot}t)$; or equivalently, $E(\omega)=U_A(-t,\omega{\cdot}t){\cdot}E(\omega{\cdot}t)$ for each $t\ge 0$. Moreover, Proposition 3.7(ii) ensures that, if $T>\widetilde{T}(\omega)$, then $\widetilde{E}(\omega)=U_A(-T,\omega{\cdot}T){\cdot}F_T(\omega)$ and Propositions 3.8(ii) and 3.4(iii) yield $F_T(\omega)\subseteq E(\omega{\cdot}T)$, that is,

$$\widetilde{E}(\omega)=U_A(-T,\omega{\cdot}T){\cdot}F_T(\omega)\subseteq U_A(-T,\omega{\cdot}T){\cdot}E(\omega{\cdot}T)=E(\omega)\,.$$

Since $\dim E(\omega)=\dim\widetilde{E}(\omega)$, we conclude that $E(\omega)=\widetilde{E}(\omega)$, as stated.

(ii) Proposition 3.5(ii) and the corresponding result for $\widetilde{E}$ (with the time reversed flow $(t,\omega)\mapsto\omega{\cdot}(-t)$) prove that $\dim E(\omega{\cdot}t)$ is nondecreasing in t and $\dim\widetilde{E}(\omega{\cdot}t)$ is nonincreasing in t. From Proposition 3.8, since $\widehat{T}>\widetilde{T}(\omega)$ and $\widehat{T}>T(\omega)$, we obtain

$$\dim E(\omega{\cdot}t)\le\dim\widetilde{E}(\omega{\cdot}(t-\widehat{T}))\le\dim E(\omega{\cdot}t)\,,$$

that is, $\dim E(\omega{\cdot}t)=\dim\widetilde{E}(\omega{\cdot}(t-\widehat{T}))$ for each $t\in\mathbb{R}$, which together with the monotonicity of the dimensions implies that $\dim E(\omega{\cdot}t)$ and $\dim\widetilde{E}(\omega{\cdot}t)$ are constant, and the result follows from (i). □

The following result provides conditions under which the hypotheses of Theorem 3.9(ii) hold.

Proposition 3.10. *Let $T(\omega)$ and $\widetilde{T}(\omega)$ be the nonnegative times for $E(\omega)$ and $\widetilde{E}(\omega)$ provided by* Proposition 3.4(iii) *and* Proposition 3.7(ii) *respectively.*

(i) *If $\dim E(\omega)$ is constant on Ω, then*
- $\sup_{\omega\in\Omega}T(\omega)<\infty$*;*
- $\dim\widetilde{E}(\omega)$ *is also constant on Ω, and $\sup_{\omega\in\Omega}\widetilde{T}(\omega)<\infty$.*

(ii) *If $\Omega^*\subseteq\Omega$ is minimal, then $\sup_{\omega\in\Omega^*}T(\omega)<\infty$ and $\sup_{\omega\in\Omega^*}\widetilde{T}(\omega)<\infty$.*

Proof. (i) The result is obvious if $\dim E(\omega)=0$, so let us assume that $d_E(\omega)=d>0$ for all $\omega\in\Omega$. We fix $\omega\in\Omega$ and a time $T_\omega>T(\omega)$. Let $X(\omega)$ be an $n\times d$ matrix of rank d with columns $\mathbf{x}(T_\omega,\omega{\cdot}(-T_\omega),\mathbf{0},\mathbf{u}_j)$ for some controls $\mathbf{u}_j\colon[0,T]\to\mathbb{R}^m$ and $j=1,\dots,d$, that is, the columns of $X(\omega)$ are a basis of the d-dimensional subspace $E(\omega)$. For each $\widehat{\omega}\in\Omega$, let $X(\widehat{\omega})$ be the $n\times d$ matrix with columns $\mathbf{x}(T_\omega,\widehat{\omega}{\cdot}(-T_\omega),\mathbf{0},\mathbf{u}_j)$ for $j=1,\dots,d$, which by Proposition 3.4(iii) are d vectors of $E(\widehat{\omega})$. The semicontinuity of the rank provides an $\varepsilon(\omega)>0$ such that $\operatorname{rank}X(\widehat{\omega})=d$ for each $\widehat{\omega}$ in the open ball $B(\omega,\varepsilon(\omega))$. This fact and $\dim E(\widehat{\omega})=d$ ensure that $E(\widehat{\omega})=E_{T_\omega}(\widehat{\omega})$, and hence Proposition 3.4(iii) guarantees that $T_\omega>T(\widehat{\omega})$, for each point $\widehat{\omega}\in B(\omega,\varepsilon(\omega))$. Therefore, from the equality $\Omega=\cup_{\omega\in\Omega}B(\omega,\varepsilon(\omega))$ and the compactness of Ω, we obtain a finite number of points $\omega_1,\dots,\omega_k$ such that $\Omega=\cup_{i=1}^kB(\omega_i,\varepsilon(\omega_i))$. Hence

$$\sup_{\omega\in\Omega}T(\omega)\le\sup_{i=1,\dots,k}T_{\omega_i}=T^*<\infty\,,$$

which proves the first assertion.

To prove the second one, we deduce from Proposition 3.8(ii) that $\dim\widetilde{E}(\omega)\le d$ for all $\omega\in\Omega$. Hence, Proposition 3.8(i) ensures that if $\omega\in\Omega$ and $T>T^*\ge T(\omega)$, then

$$d=\dim E(\omega)\le\dim\widetilde{E}(\omega{\cdot}(-T))\le d\,,$$

which proves that $\dim \widetilde{E}(\omega) = d$ for all $\omega \in \Omega$. Now the same argument as above completes the proof of (i).

(ii) This assertion follows from (i), Theorem 3.6(v) and the corresponding result for $\widetilde{E}(\omega)$ showing that $\dim \widetilde{E}(\omega)$ is constant on each minimal subset. □

Finally, as a consequence of the previous results, we prove that in some cases, even if the family (3.1) is not uniformly null controllable, there exists a subbundle of $\Omega \times \mathbb{R}^n$, which could be the trivial one $\Omega \times \{\mathbf{0}\}$, on which the uniform null controllability holds, in the sense explained in the following statements. Note that the set Ω can be replaced for any of its σ-invariant subsets on which the main hypothesis (that is, the constant character of $\dim E(\omega)$) holds.

Theorem 3.11. *Consider the subsets of $\Omega \times \mathbb{R}^n$ given by*

$$E = \{(\omega, \mathbf{x}) \mid \omega \in \Omega\,,\ \mathbf{x} \in E(\omega)\} \quad \text{and} \quad \widetilde{E} = \{(\omega, \mathbf{x}) \mid \omega \in \Omega\,,\ \mathbf{x} \in \widetilde{E}(\omega)\}\,,$$

and suppose that $\dim E(\omega)$ is constant on Ω. Then $E = \widetilde{E}$ is a closed τ_A-invariant subbundle of $\Omega \times \mathbb{R}^n$, and there exists a positive time $T_0 > 0$ such that, if $(\omega, \mathbf{x}_0) \in E$, then $\mathbf{x}_0$ is null controllable in time T_0 for the systems $(3.1)_\omega$ and $(3.3)_\omega$. In particular, this happens if Ω is minimal.

Proof. The first two assertions follow Proposition 3.4(iii) ($E(\omega)$ is a vector subspace for each $\omega \in \Omega$), Proposition 3.10(i) and Theorem 3.9(i) ($\dim \widetilde{E}(\omega)$ is also constant, so that $E(\omega) = \widetilde{E}(\omega)$ for each $\omega \in \Omega$ and hence $E = \widetilde{E}$), Proposition 3.10(ii) (for the existence of T_0), Proposition 3.4(iv) (for the invariance) and Theorem 3.6(x) (for the closed character). The last statement follows from Theorem 3.6(v). □

4. **Null controllability and exponential dichotomy.** Let Ω be a compact metric space with a continuous flow $\sigma(\omega, t) = \omega{\cdot}t$. As it was explained in the Introduction, in this section we consider a family of time-dependent linear control systems

$$\mathbf{x}' = A(\omega{\cdot}t)\,\mathbf{x} + B(\omega{\cdot}t)\,\mathbf{u}(t)\,, \quad \omega \in \Omega\,, \tag{4.1}$$

where $A\colon \Omega \to \mathbb{M}_{n\times n}(\mathbb{R})$ and $B\colon \Omega \to \mathbb{M}_{n\times m}(\mathbb{R})$ are continuous, together with a family of time-dependent quadratic forms

$$\mathcal{Q}_\omega(t, \mathbf{x}, \mathbf{u}) = \frac{1}{2}\left(\langle \mathbf{x}, G(\omega{\cdot}t)\,\mathbf{x}\rangle + 2\,\langle \mathbf{x}, g(\omega{\cdot}t)\,\mathbf{u}\rangle + \langle \mathbf{u}, R(\omega{\cdot}t)\,\mathbf{u}\rangle\right)\,, \quad \omega \in \Omega\,, \tag{4.2}$$

where $G\colon \Omega \to \mathbb{M}_{n\times n}(\mathbb{R})$, $g\colon \Omega \to \mathbb{M}_{n\times m}(\mathbb{R})$ and $R\colon \Omega \to \mathbb{M}_{m\times m}(\mathbb{R})$ are continuous, G and R are symmetric, and $R > 0$. We will also consider the family of linear Hamiltonian systems defined from the LQ problems by

$$\mathbf{z}' = H(\omega{\cdot}t)\,\mathbf{z}\,, \quad \omega \in \Omega\,, \tag{4.3}$$

where $\mathbf{z} = \left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{y}\end{smallmatrix}\right]$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and

$$H(\omega) = \begin{bmatrix} A(\omega) - B(\omega)R^{-1}(\omega)g^T(\omega) & B(\omega)R^{-1}(\omega)B^T(\omega) \\ G(\omega) - g(\omega)R^{-1}(\omega)g^T(\omega) & -A^T(\omega) + g(\omega)R^{-1}(\omega)B^T(\omega) \end{bmatrix}.$$

The goal of this section is to relate the dynamical properties of the linear and Lagrangian flows induced by the family (4.3) to the null controllable sets $E(\omega)$ studied in the previous section. Recall that we can play with the choice of the supply rate in order to obtain Hamiltonian families (4.3) with different properties and hence to obtain a sharper analysis of the null controllability properties of (4.1). We refer the reader to Kratz [13] for a similar idea to study the reachable sets in the case in which B is symmetric, $g \equiv 0$ and $R \equiv B$.

Before stating the results, it is convenient to explain the relation between the lack of uniform null controllability and the presence of abnormal systems in the family (4.3). The interested reader can find in Reid [17] and Kratz [12] a previous analysis of abnormal systems, and in Reid [18] and Šepitka-Šimon Hilscher [19, 20] a generalization of the concept and theory of principal solutions suitable for systems of this type.

Recall that $U(t,\omega)$ is the fundamental matrix solution of the system (4.3) for $\omega\in\Omega$ with $U(0,\omega)=I_{2n}$. We consider the linear subspaces

$$\begin{aligned}\Lambda^+(\omega)&=\left\{\mathbf{z}_0\in\mathbb{R}^{2n}\mid U(t,\omega)\,\mathbf{z}_0=\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]\text{ for } t \text{ in a positive half-line}\right\},\\ \Lambda^-(\omega)&=\left\{\mathbf{z}_0\in\mathbb{R}^{2n}\mid U(t,\omega)\,\mathbf{z}_0=\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]\text{ for } t \text{ in a negative half-line}\right\},\\ \Lambda(\omega)&=\left\{\mathbf{z}_0\in\mathbb{R}^{2n}\mid U(t,\omega)\,\mathbf{z}_0=\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]\text{ for } t\in\mathbb{R}\right\},\end{aligned}$$

and define $d(\omega)=\dim\Lambda(\omega)$ and $d^{\pm}(\omega)=\dim\Lambda^{\pm}(\omega)$. A complete analysis of the functions d, d^+ and d^- is carried out in [8].

Definition 4.1. The *index of abnormality* of the system $(4.3)_\omega$ is $d(\omega)$, and its *index of abnormality at* $\pm\infty$ is $d^{\pm}(\omega)$. The linear Hamiltonian system $(4.3)_\omega$ is *abnormal* (resp. *abnormal at* $+\infty$ or *abnormal at* $-\infty$) if there exists a nonzero solution of the form $\mathbf{z}(t,\omega)=\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]$ in $\mathbb{R}$ (resp. in a positive half-line or in a negative half-line), which is equivalent to say that $d(\omega)>0$ (resp. $d^+(\omega)>0$ or $d^-(\omega)>0$).

Proposition 4.2. (i) *The family* (4.1) *is uniformly null controllable if and only if none of the systems of the family* (4.3) *is abnormal at* $+\infty$.

(ii) *Suppose that the family* (4.1) *is not uniformly null controllable. Then there exists at least a minimal subset of* Ω *for which all the systems are abnormal.*

(iii) *Let* $\left[\begin{smallmatrix}L_1\\ L_2\end{smallmatrix}\right]=\left[\begin{smallmatrix}L_1(t,\omega)\\ L_2(t,\omega)\end{smallmatrix}\right]$ *be a conjoined basis for* $(4.3)_\omega$. *If* $\operatorname{Ker}L_1(t,\omega)$ *is constant and nonzero on a positive half-line* (a,∞), *then the system* $(4.3)_\omega$ *is abnormal at* $+\infty$.

Proof. (i) The proof of this property can be found in [10], Corollary 7.35.

(ii) It follows from (i) and Definition 4.1 that there exists at least a point $\omega\in\Omega$ for which $d^+(\omega)>0$. Theorem 3.1(ii) of [8] ensures that $d(\omega_1)>0$ for each $\omega_1\in\mathcal{O}(\omega)$ (the omega-limit set of ω), which ensures (ii).

(iii) If $\mathbf{x}_0\in\operatorname{Ker}L_1(t,\omega)$, then the solution $\left[\begin{smallmatrix}L_1(t,\omega)\\ L_2(t,\omega)\end{smallmatrix}\right]\mathbf{x}_0$ of $(4.3)_\omega$ takes the form $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ on (a,∞), which proves (iii). □

Next, for a fixed $(\omega,l)\in\Omega\times\mathcal{L}_{\mathbb{R}}$, we define

$$d^+(\omega,l)=\dim(\Lambda^+(\omega)\cap l)\,,\quad d^-(\omega,l)=\dim(\Lambda^-(\omega)\cap l)\,,\quad d(\omega,l)=\dim(\Lambda(\omega)\cap l)\,.$$

That is, these quantities measure the number of independent solutions of $(4.3)_\omega$ with initial data in the subspace l which take the form $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t,\omega)\end{smallmatrix}\right]$ on a positive half-line, on a negative half-line and on the full line, respectively.

Now assume that the family (4.1) is not uniformly null controllable (which is the setting we are interested in) and that the family (4.3) admits an exponential dichotomy over Ω (which will be one of the main hypotheses for the results of this section). Let $\Omega\times\mathbb{R}^{2n}=L^+\oplus L^-$ be the corresponding decomposition with associated Lagrange planes $l^{\pm}(\omega)=\{\mathbf{z}\mid(\omega,\mathbf{z})\in L^{\pm}\}$ (see subsection 2.2.2). We define the functions

$$\widetilde{d}^{\pm}\colon\Omega\to\{0,\ldots,n\},\quad\omega\mapsto\widetilde{d}^{\pm}(\omega)=\dim(\Lambda(\omega)\cap l^{\pm}(\omega))\,.$$

Let $\Omega^* \subseteq \Omega$ be one of the minimal sets for which all the systems are abnormal (see Proposition 4.2(ii)). Proposition 3.3 and Theorem 3.4 of [8], which relate $\widetilde{d}^{\pm}(\omega)$ to $d(\omega)$, prove in particular that the functions d, $\widetilde{d}+$ and $\widetilde{d}^-$ are constant on Ω^*, where $d = \widetilde{d}^+ + \widetilde{d}^-$. Consequently, the number of linearly independent solutions of $(4.3)_\omega$ of the form $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2(t,\omega) \end{smallmatrix}\right]$ can be calculated in terms of the number of linearly independent solutions of this form which have initial data in the subspaces $l^+(\omega)$ and $l^-(\omega)$; or, in other words, which are bounded as t goes to $\pm\infty$. And this number is the same for all the elements of Ω^*. In addition, since $d > 0$ on Ω^*, then either $\widetilde{d}^+$ or $\widetilde{d}^-$ (or both of them) is strictly positive on Ω^*. That is, at least one of the associated Lagrange planes $l^{\pm}(\omega)$ lies on the vertical Maslov cycle $\mathcal{C}$ defined by (2.8) simultaneously for all $\omega \in \Omega^*$.

Now we are almost in a good position to begin with the analysis of the null controllable sets $E(\omega)$ for the time-reversed control family, defined by (3.4). In addition to the exponential dichotomy hypothesis, the analysis relies on the existence a σ-ergodic measure with full support for which the rotation number is zero. That is, we will work under these conditions:

Hypotheses 4.3. The family (4.3) has an exponential dichotomy over Ω, there exists a σ-ergodic measure m_0 on Ω with full topological support, and the rotation number of the family (4.3) with respect to m_0 is $\alpha(m_0) = 0$.

To formulate the first result we need to define the sets (4.5), which in turn requires the following perturbation result. Its proof is given in [8] (Theorem 4.18) and in [10] (Theorem 5.73).

Theorem 4.4. *Suppose that* Hypotheses 4.3 *hold. Then there exists a $\rho > 0$, such that, for each $\varepsilon \in (0, \rho)$, the family*

$$\mathbf{z}' = H_\varepsilon(\omega{\cdot}t)\,\mathbf{z} = \left(H(\omega{\cdot}t) + \varepsilon \begin{bmatrix} 0_n & I_n \\ 0_n & 0_n \end{bmatrix} \right) \mathbf{z} \tag{4.4}$$

has an exponential dichotomy over Ω. Moreover, the Lagrange planes $l_\varepsilon^{\pm}(\omega)$ lie in $\mathcal{D} = \mathcal{L}_\mathbb{R} - \mathcal{C}$ and if we represent $l_\varepsilon^{\pm}(\omega) \equiv \left[\begin{smallmatrix} I_n \\ M_\varepsilon^{\pm}(\omega) \end{smallmatrix}\right] \in \mathcal{L}_\mathbb{R}$ for each $\omega \in \Omega$, then we have

$$M_{\varepsilon_1}^+(\omega) \le M_{\varepsilon_2}^+(\omega) < M_{\varepsilon_2}^-(\omega) \le M_{\varepsilon_1}^-(\omega)$$

whenever $0 < \varepsilon_1 \le \varepsilon_2 < \rho$.

Under Hypotheses 4.3 and with the notation established in Theorem 4.4 define, for each $\omega \in \Omega$, the set

$$P(\omega) = \left\{ \mathbf{x}_0 \in \mathbb{R}^n \mid \lim_{\varepsilon \to 0^+} \mathbf{x}_0^T\, M_\varepsilon^-(\omega)\, \mathbf{x}_0 < \infty \right\}. \tag{4.5}$$

The following result shows that, if $l^-(\omega) \equiv \left[\begin{smallmatrix} L_1^-(\omega) \\ L_2^-(\omega) \end{smallmatrix}\right]$, then $P(\omega) = \operatorname{Im} L_1^-(\omega)$, and that $E(\omega)$ is a vector subspace of $P(\omega)$. It also proves that $P(\omega)$ has properties of semi-invariance under the linear flow induced by (4.3), and that $\dim P(\omega)$ is a lower semicontinuous function with analogous properties to those previously obtained for $\dim E(\omega)$.

Note that, in particular, $P(\omega) = \mathbb{R}^n$ (which happens in the particular case that the uniform null controllability property holds, since in this case $E(\omega) = \mathbb{R}^n$) if and only if $M^-(\omega)$ exists (which is equivalent to say that $L_1^-(\omega)$ is nonsingular). So, the interesting case for the next result is that of non global existence of M^-, since otherwise its points (ii)-(v) do not provide any valuable information. Nevertheless, even in the case of global existence of M^-, the null controllable set $E(\omega)$ can be a proper subspace of $P(\omega)=\mathbb{R}^n$.

Theorem 4.5. *Suppose that* Hypotheses 4.3 *hold, and let* $l^\pm(\omega)\equiv\left[\begin{smallmatrix}L_1^\pm(\omega)\\ L_2^\pm(\omega)\end{smallmatrix}\right]\in\mathcal{L}_\mathbb{R}$ *be the Lagrange planes provided by the exponential dichotomy. Then,*

(i) $P(\omega)=\operatorname{Im}L_1^-(\omega)$ *for each* $\omega\in\Omega$*. In particular,* $P(\omega)$ *is a vector space.*

Define the map $d_P\colon\Omega\to\{0,\dots,n\}$, $\omega\mapsto\dim P(\omega)=\operatorname{rank}L_1^-(\omega)$*. Then,*

(ii) $E(\omega)\subseteq P(\omega)$ *and hence,* $d_E(\omega)\le d_P(\omega)=\operatorname{rank}L_1^-(\omega)$ *for each* $\omega\in\Omega$*.*

(iii) *If* $\mathbf{x}_0\in P(\omega)$*, then* $\mathbf{x}(t,\omega,\mathbf{x}_0,\mathbf{u})\in P(\omega{\cdot}t)$ *for any* $t\ge 0$ *and any control* $\mathbf{u}\colon[0,t]\to\mathbb{R}^m$*. In particular,* $U_A(t,\omega){\cdot}P(\omega)\subseteq P(\omega{\cdot}t)$ *for each* $\omega\in\Omega$ *and* $t\ge 0$*.*

(iv) *If* $t_1\le t_2$*, then* $d_P(\omega{\cdot}t_1)\le d_P(\omega{\cdot}t_2)$ *for each* $\omega\in\Omega$*.*

(v) *Fix* $\omega\in\Omega$ *and take for each* $t\in\mathbb{R}$ *the representation*

$$l^-(\omega{\cdot}t)=U(t,\omega){\cdot}l^-(\omega)\equiv U(t,\omega)\begin{bmatrix}L_1^-(\omega)\\ L_2^-(\omega)\end{bmatrix}=\begin{bmatrix}L_1^-(t,\omega)\\ L_2^-(t,\omega)\end{bmatrix},$$

so that $P(\omega{\cdot}t)=\operatorname{Im}L_1^-(t,\omega)$*. Then* $\operatorname{Ker}L_1^-(t_2,\omega)\subseteq\operatorname{Ker}L_1^-(t_1,\omega)$ *if* $t_1\le t_2$*, and there exist constants* $t_-(\omega)$ *and* $t_+(\omega)$ *such that*

(a) the map $t\mapsto d_P(\omega{\cdot}t)$ *is constant on* $(-\infty,t_-(\omega)]$ *and on* $[t_+(\omega),\infty)$*, and hence so is the map* $t\mapsto\operatorname{Ker}L_1^-(t,\omega)$*;*

(b) $d^+(\omega,l^-(\omega))=\dim\operatorname{Ker}L_1^-(t_-(\omega),\omega)=n-d_P(\omega{\cdot}t_-(\omega))$*;*

(c) $\widetilde{d}^-(\omega)=d(\omega,l^-(\omega))=d^+(\omega,l^-(\omega))=\dim\operatorname{Ker}L_1^-(t_+(\omega),\omega)=n-d_P(\omega{\cdot}t_+(\omega))$*.*

(vi) *The map* $d_P\colon\Omega\to\{0,\dots,n\}$, $\omega\mapsto d_P(\omega)$ *is lower semicontinuous.*

Proof. (i) First note that $\operatorname{Im}L_1^-(\omega)$ is independent of the representation chosen for $l^-(\omega)$ (see Subsection 2.2.1). Since, by the robustness of the exponential dichotomy (see, e.g., Theorem 6 of [22]), $l_\varepsilon^-(\omega)$ tends to $l^-(\omega)$ as $\varepsilon\downarrow 0$, there exist nonsingular matrices $P_\varepsilon(\omega)$ such that $\left[\begin{smallmatrix}P_\varepsilon(\omega)\\ M_\varepsilon^-(\omega)\,P_\varepsilon(\omega)\end{smallmatrix}\right]$ tends to $\left[\begin{smallmatrix}L_1^-(\omega)\\ L_2^-(\omega)\end{smallmatrix}\right]$ as $\varepsilon\downarrow 0$. Moreover, we deduce from Theorem 4.4 that $M_\varepsilon^-(\omega)=M_\varepsilon^-(\omega)\,P_\varepsilon(\omega)\,P_\varepsilon^{-1}(\omega)$ decreases as ε decreases. Hence, Theorem 1 of Kratz [11] proves that

$$\lim_{\varepsilon\to 0^+}(L_1^-(\omega))^T\,M_\varepsilon^-(\omega)\,L_1^-(\omega)=(L_1^-(\omega))^T\,L_2^-(\omega)$$

and

$$\lim_{\varepsilon\to 0^+}\mathbf{x}_0^T\,M_\varepsilon^-(\omega)\,\mathbf{x}_0=\infty\quad\text{for all }\mathbf{x}_0\notin\operatorname{Im}L_1^-(\omega)\,,$$

from where it is immediate to check that $\mathbf{x}_0\in\operatorname{Im}L_1^-(\omega)$ if and only if $\mathbf{x}_0\in P(\omega)$, as stated.

(ii) Proposition 3.4(iii) states that, if $\mathbf{x}_0\in E(\omega)$, then there exists a $T>0$ such that $\mathbf{x}_0=\mathbf{x}(T,\omega{\cdot}(-T),\mathbf{0},\mathbf{u})$ for some control $\mathbf{u}\colon[0,T]\to\mathbb{R}^m$. Therefore, reasoning as in Theorem 4.19 of [8], we can check that

$$\int_0^T\mathcal{Q}_{\omega\cdot(-T)}(s,\mathbf{x}(s,\omega{\cdot}(-T),\mathbf{0},\mathbf{u}),\mathbf{u}(s))\,ds\ge V_\varepsilon^{\omega\cdot(-T)}(T,\mathbf{x}(T,\omega{\cdot}(-T),\mathbf{0},\mathbf{u}))\\ -V_\varepsilon^{\omega\cdot(-T)}(0,\mathbf{x}(0,\omega{\cdot}(-T),\mathbf{0},\mathbf{u}))=V_\varepsilon^{\omega\cdot(-T)}(T,\mathbf{x}_0)\,,$$

where

$$V_\varepsilon^{\omega}(t,\mathbf{x}_0)=\frac{1}{2}\,\mathbf{x}_0^T\,M_\varepsilon^-(\omega{\cdot}t)\,\mathbf{x}_0\,;$$

that is,

$$2\int_0^T \mathcal{Q}_{\omega\cdot(-T)}(s,\mathbf{x}(s,\omega\cdot(-T),\mathbf{0},\mathbf{u}),\mathbf{u}(s))\,ds \ge \mathbf{x}_0^T\,M_\varepsilon^-(\omega)\,\mathbf{x}_0\,.$$

Hence, $\lim_{\varepsilon\to 0^+}\mathbf{x}_0^T\,M_\varepsilon^-(\omega)\,\mathbf{x}_0<\infty$ and $\mathbf{x}_0\in\operatorname{Im}L_1^-(\omega)$, as claimed.

(iii) As before, if $\mathbf{x}_0\in P(\omega)$ and $\mathbf{u}\colon[0,t]\to\mathbb{R}^m$ is any control,

$$\begin{aligned}\int_0^t \mathcal{Q}_\omega(s,\mathbf{x}(s,\omega,\mathbf{x}_0,\mathbf{u}),\mathbf{u}(s))\,ds \ge \frac{1}{2}\,\mathbf{x}^T(t,\omega,\mathbf{x}_0,\mathbf{u})\,M_\varepsilon^-(\omega{\cdot}t)\,\mathbf{x}(t,\omega,\mathbf{x}_0,\mathbf{u})\\ -\frac{1}{2}\,\mathbf{x}_0^T\,M_\varepsilon^-(\omega)\,\mathbf{x}_0\,,\end{aligned}$$

and hence,

$$\begin{aligned}\mathbf{x}^T(t,\omega,\mathbf{x}_0,\mathbf{u})\,M_\varepsilon^-(\omega{\cdot}t)\,\mathbf{x}(t,\omega,\mathbf{x}_0,\mathbf{u})\le 2\int_0^t \mathcal{Q}_\omega(s,\mathbf{x}(s,\omega,\mathbf{x}_0,\mathbf{u}),\mathbf{u}(s))\,ds\\ +\mathbf{x}_0^T\,M_\varepsilon^-(\omega)\,\mathbf{x}_0\,,\end{aligned}$$

from where we deduce that $\mathbf{x}(t,\omega,\mathbf{x}_0,\mathbf{u})\in P(\omega{\cdot}t)$, as claimed. Taking now $\mathbf{u}\equiv\mathbf{0}$ we see that $U_A(t,\omega)\,\mathbf{x}_0=\mathbf{x}(t,\omega,\mathbf{x}_0,\mathbf{0})\in P(\omega{\cdot}t)$, and hence $U_A(t,\omega){\cdot}P(\omega)\subseteq P(\omega{\cdot}t)$, as stated.

(iv) This property follows from (iii) because $U_A(t_2-t_1,\omega{\cdot}t_1){\cdot}P(\omega{\cdot}t_1)\subseteq P(\omega{\cdot}t_2)$ and all the matrices $U_A(t,\omega)$ are nonsingular.

(v) It follows from (iv) that the map $t\mapsto\dim\operatorname{Ker}L_1^-(t,\omega)=n-d_P(\omega{\cdot}t)$ is nonincreasing, which together with relation (2.9) shows that $\operatorname{Ker}L_1^-(t,\omega)$ is nonincreasing in t for the order given by the contention of vector spaces. This proves the first assertion in (v). (Incidentally, note that this proves that any conjoined basis for $(4.3)_\omega$ representing the Lagrange planes $l^-(\omega{\cdot}t)$ has no proper focal points: see Definition 2.4). It also follows from (iv) the existence of a negative half-line $(-\infty,t_-(\omega)]$ on which the map $t\mapsto d_P(\omega{\cdot}t)$ attains its minimum value and of a positive half-line $[t_+(\omega),\infty)$ on which it attains its maximum value. This fact together again with (2.9) ensures that the map $t\mapsto\operatorname{Ker}L_1^-(t,\omega)$ is constant on $(-\infty,t_-(\omega)]$ and on $[t_+(\omega),\infty)$. So (a) is proved. Properties (b) and (c) follow easily from (a), the definitions of $d^\pm(\omega,l^-(\omega))$ and $d(\omega,l^-(\omega))$, Proposition 4.2(iii), and, in the case of (c), the fact that $\operatorname{Ker}L_1^-(t_+(\omega),\omega)\subseteq\operatorname{Ker}L_1^-(t,\omega)$ for all $t\in\mathbb{R}$.

(vi) Since $d_P(\omega)=\operatorname{rank}L_1^-(\omega)$, the statement is a consequence of Remark 2.3 and the lower semicontinuity of the rank function on matrices. □

Remark 4.6. Note that, as pointed out in the proof of the previous point (v), under Hypotheses 4.3 and with the notation established in the preceding result, the conjoined basis $\left[\begin{smallmatrix}L_1^-(t,\omega)\\ L_2^-(t,\omega)\end{smallmatrix}\right]$ of (4.3) has no proper focal points. Therefore, an application of the Sturmian separation theory for linear Hamiltonian systems without controllability (see Theorem 1.5 of Šimon Hilscher [21]) shows that any other conjoined basis of (4.3) has at most n proper focal points. In particular, Hypotheses 4.3 ensures that all the systems of the family (4.3) are nonoscillatory at $+\infty$: see, e.g., [8] and [7].

Always under Hypotheses 4.3, we consider the maps

$$\begin{array}{rcl} d_P^-\colon\Omega & \to & \{0,\dots,n\}\\ \omega & \mapsto & d_P(\omega{\cdot}t_-(\omega))\,,\end{array}\qquad \begin{array}{rcl} d_P^+\colon\Omega & \to & \{0,\dots,n\}\\ \omega & \mapsto & d_P(\omega{\cdot}t_+(\omega))\,,\end{array}$$

where $d_P(\omega) = \dim P(\omega)$ and $t_-(\omega)$ and $t_+(\omega)$ satisfy the conditions of Theorem 4.5(v). Next consider the quantities

$$d_P^M = \max_{\omega\in\Omega} d_P(\omega)\,, \qquad d_P^m = \min_{\omega\in\Omega} d_P(\omega)\,.$$

The following result collects several properties of these functions and quantities.

Theorem 4.7. *Suppose that* Hypotheses 4.3 *hold. Then,*

(i) *all the statements* (i)-(xii) *of* Theorem 3.6 *are valid if we change the functions and quantities d_E, $d_E^\pm$ and d_E^M by d_P, $d_P^\pm$ and d_P^M, and the spaces $E(\omega)$ by $P(\omega)$.*

(ii) *If $d_P(\omega) = d_P$ is constant on Ω, then*

$$P = \{(\omega, \mathbf{x}) \mid \omega \in \Omega\,,\; \mathbf{x} \in P(\omega)\} \subseteq \Omega \times \mathbb{R}^n$$

is a closed τ_A-invariant subbundle of $\Omega\times\mathbb{R}^n$ of dimension d_P. In particular, this is the case if Ω is minimal.

Proof. (i) This proof is completely analogous to that of Theorem 3.6.

(ii) Theorem 4.5(i) ensures that $P(\omega)$ is a linear space for all $\omega\in\Omega$, and by hypothesis its dimension is always d_P. It follows from here and Theorem 4.5(iii) that $U_A(t,\omega)\cdot P(\omega) = P(\omega{\cdot}t)$ for all $t\in\mathbb{R}$ and $\omega\in\Omega$. And finally, the analogous property of Theorem 3.6(x) for $P(\omega)$ proves the closed character of P. □

The next result shows that, under the assumption of constant dimension for the subspaces $\operatorname{Im} L_1^-(\omega)$ and $E(\omega)$ (which in particular holds if Ω is minimal), there exist two closed invariant subbundles L_0^- and L_E^- of L^- such that the sections $l_0^-(\omega)$ of the first one intersect the vertical Lagrange plane $l_v \equiv \left[\begin{smallmatrix} 0_n \\ I_n \end{smallmatrix}\right]$, and such that the first components of the vectors of the sections $l_E^-(\omega)$ of the second one belong to $E(\omega)$ for all $\omega\in\Omega$. Recall that the flows τ_A and τ_H are defined by (2.3) and (2.6).

Theorem 4.8. *Suppose that* Hypotheses 4.3 *hold. Let $l^\pm(\omega) \equiv \left[\begin{smallmatrix} L_1^\pm(\omega) \\ L_2^\pm(\omega) \end{smallmatrix}\right] \in \mathcal{L}_\mathbb{R}$ be the Lagrange planes and let $L^\pm = \{(\omega,\mathbf{z}) \mid \mathbf{z}\in l^\pm(\omega)\}$ be the τ_H-invariant subbundles provided by the exponential dichotomy. Define $l_0^-(\omega) = \big\{\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{smallmatrix}\right] \in l^-(\omega)\big\}$, $l_E^-(\omega) = \{\left[\begin{smallmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{smallmatrix}\right] \in l^-(\omega) \mid \mathbf{z}_1 \in E(\omega)\}$, and the subsets of L^- given by*

$$L_0^- = \{(\omega,\mathbf{z}) \mid \omega\in\Omega\,,\; \mathbf{z}\in l_0^-(\omega)\} \quad \textit{and} \quad L_E^- = \{(\omega,\mathbf{z}) \mid \omega\in\Omega\,,\; \mathbf{z}\in l_E^-(\omega)\}\,.$$

(i) *If $d_P(\omega) = d_P$ is constant on Ω, then L_0^- is a closed τ_H-invariant subbundle of L^- of dimension $n - d_P$.*

(ii) *If $d_E(\omega) = d_E$ and $d_P(\omega) = d_P$ are constant on Ω, then L_E^- is a closed τ_H-invariant subbundle of L^- of dimension $n - d_P + d_E$.*

In particular, the assertions in (i) *and* (ii) *hold if Ω is minimal.*

Proof. (i) It is clear that $l_0^-(\omega)$ is a vector space for each $\omega\in\Omega$. Note also that the vector $\left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{smallmatrix}\right]$ belongs to $l^-(\omega)$ if and only if there exists a vector $\mathbf{x}_0\in\mathbb{R}^n$ such that $\left[\begin{smallmatrix} L_1^-(\omega) \\ L_2^-(\omega) \end{smallmatrix}\right] \mathbf{x}_0 = \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{z}_2 \end{smallmatrix}\right]$, i.e., $\mathbf{x}_0 \in \operatorname{Ker} L_1^-(\omega)$, which has constant dimension given by $n - d_P(\omega) = n - d_P$. This together with the fact that the rank of $\left[\begin{smallmatrix} L_1^-(\omega) \\ L_2^-(\omega) \end{smallmatrix}\right]$ is n shows that $\dim l_0^-(\omega) = n - d_P$ for all $\omega\in\Omega$, as stated.

Fix $\omega\in\Omega$ and represent $U(t,\omega)\left[\begin{smallmatrix}L_1^-(\omega)\\ L_2^-(\omega)\end{smallmatrix}\right]=\left[\begin{smallmatrix}L_1^-(t,\omega)\\ L_2^-(t,\omega)\end{smallmatrix}\right]$. Theorem 4.5(v) ensures that the map $t\mapsto \operatorname{Ker} L_1^-(t,\omega)$ is constant, and hence the equalities $U(t,\omega)\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2\end{smallmatrix}\right]=U(t,\omega)\left[\begin{smallmatrix}L_1^-(\omega)\\ L_2^-(\omega)\end{smallmatrix}\right]\mathbf{c}=\left[\begin{smallmatrix}L_1^-(t,\omega)\\ L_2^-(t,\omega)\end{smallmatrix}\right]\mathbf{c}=\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2(t)\end{smallmatrix}\right]$ prove the invariance of L_0^-.

Finally, we must check that $\lim_{k\to\infty} l_0^-(\omega_k)=l_0^-(\omega)$ in the Grassmannian manifold $\mathcal{G}_{n-d_P}(\mathbb{R}^{2n})$ in the case that $\lim_{k\to\infty}\omega_k=\omega$. The compactness of $\mathcal{G}_{n-d_P}(\mathbb{R}^{2n})$ provides a subsequence (ω_{k_j}) such that $\lim_{j\to\infty} l_0^-(\omega_{k_j})=g\in\mathcal{G}_{n-d_P}(\mathbb{R}^{2n})$. Moreover, for each j we can choose a representant of $l_0^-(\omega_{k_j})$ of the form $\left[\begin{smallmatrix}0\\ G_{k_j}\end{smallmatrix}\right]$ for some $n\times(n-d_p)$ matrix G_{k_j} of rank $n-d_P$; here 0 denotes the $n\times(n-d_P)$ null matrix. Thus, we deduce from Proposition 1.24 of [10] that g has a representant of the same form, and hence $g=l_0^-(\omega)$, which shows that the subbundle is closed.

(ii) It is also clear that $l_E^-(\omega)$ is a vector subspace for each $\omega\in\Omega$. Moreover, $\left[\begin{smallmatrix}\mathbf{z}_1\\ \mathbf{z}_2\end{smallmatrix}\right]\in l_E^-(\omega)$ if and only if there exists a vector $\mathbf{c}\in\mathbb{R}^n$ such that $\left[\begin{smallmatrix}L_1^-(\omega)\\ L_2^-(\omega)\end{smallmatrix}\right]\mathbf{c}=\left[\begin{smallmatrix}\mathbf{z}_1\\ \mathbf{z}_2\end{smallmatrix}\right]$ and $\mathbf{z}_1\in E(\omega)\subseteq P(\omega)=\operatorname{Im} L_1^-(\omega)$. Let $L_1^-(\omega)\,\mathbf{c}_j$, $j=1,\dots d_E$ be a basis of $E(\omega)$, and consider the vectors $\left[\begin{smallmatrix}L_1^-(\omega)\\ L_2^-(\omega)\end{smallmatrix}\right]\mathbf{c}_j$ for $j=1,\dots,d_E$, which are linearly independent. We also take a basis of $l_0^-(\omega)$ given by $\left[\begin{smallmatrix}L_1^-(\omega)\\ L_2^-(\omega)\end{smallmatrix}\right]\mathbf{c}_j=\left[\begin{smallmatrix}\mathbf{0}\\ L_2^-(\omega)\,\mathbf{c}_j\end{smallmatrix}\right]$ for $j=d_E+1,\dots,d_E+n-d_P$. This means that $\{\mathbf{c}_{d_E+1},\dots,\mathbf{c}_{d_E+n-d_P}\}$ is a basis of $\operatorname{Ker} L_1^-(\omega)$. Then, it is easy to deduce that the vectors $\left[\begin{smallmatrix}L_1^-(\omega)\\ L_2^-(\omega)\end{smallmatrix}\right]\mathbf{c}_j$ for $j=1,\dots,d_E+n-d_P$ form a basis of $l_E^-(\omega)$, so that $\dim l_E^-(\omega)=n-d_P+d_E$, as stated. The closed character of L_E^- is a consequence of Theorem 3.6(x). Finally we check the invariance. Let $\left[\begin{smallmatrix}L_1^-(\omega)\\ L_2^-(\omega)\end{smallmatrix}\right]\mathbf{c}\in l_E^-(\omega)$, so that $L_1^-(\omega)\,\mathbf{c}\in E(\omega)$. Then $U(t,\omega)\left[\begin{smallmatrix}L_1^-(\omega)\\ L_2^-(\omega)\end{smallmatrix}\right]\mathbf{c}=\left[\begin{smallmatrix}L_1^-(t,\omega)\\ L_2^-(t,\omega)\end{smallmatrix}\right]\mathbf{c}$ and consequently,

$$\begin{aligned}\frac{d}{dt}\,L_1^-(t,\omega)\,\mathbf{c}&=H_1(\omega{\cdot}t)\,L_1^-(t,\omega)\,\mathbf{c}+H_3(\omega{\cdot}t)\,L_2^-(t,\omega)\,\mathbf{c}\\&=A(\omega{\cdot}t)\,L_1^-(t,\omega)\,\mathbf{c}+B(\omega{\cdot}t)\,\mathbf{u}(t)\end{aligned}$$

for $\mathbf{u}(t)=R^{-1}(\omega{\cdot}t)\left[-g^T(\omega{\cdot}t)+B^T(\omega{\cdot}t)\,\mathbf{c}\right]$. Thus, $L_1^-(t,\omega)\,\mathbf{c}=\mathbf{x}(t,\omega,L_1^-(\omega)\,\mathbf{c},\mathbf{u})$ and since $L_1^-(\omega)\,\mathbf{c}\in E(\omega)$, we conclude from Proposition 3.4(iv) that $L_1^-(t,\omega)\,\mathbf{c}\in E(\omega{\cdot}t)$, which proves the invariance.

The last assertion of the theorem follows immediately from (i), (ii), and Theorems 3.6(v) and 4.7(v). □

4.1. **Null controllable sets for the initial family.** The results seen so far in this section relate the properties of the null controllable sets $E(\omega)$ for the time-reversed control systems (3.3) to the properties of the subbundle L^- provided by the exponential dichotomy. But they can be easily translated to the null controllable sets $\widetilde{E}(\omega)$ for the initial control systems (3.1) and the subbundle L^+, as we will explain.

Note that we can associate the family of time-reversed quadratic functionals

$$\mathcal{Q}_\omega^-(t,\mathbf{x},\mathbf{u})=\frac{1}{2}\left(\langle\mathbf{x},G(\omega{\cdot}(-t))\,\mathbf{x}\rangle+2\,\langle\mathbf{x},g(\omega{\cdot}(-t))\,\mathbf{u}\rangle+\langle\mathbf{u},R(\omega{\cdot}(-t))\,\mathbf{u}\rangle\right)\qquad(4.6)$$

to the time-reversed control systems (3.3), with the same functions G, g and R as before. Hence, we can consider the family of linear Hamiltonian systems

$$\mathbf{z}' = H^-(\omega{\cdot}(-t))\,\mathbf{z}\,, \quad \omega\in\Omega\,, \tag{4.7}$$

where

$$H^-(\omega) = \left[\begin{array}{cc} -A(\omega)+B(\omega)R^{-1}(\omega)g^T(\omega) & B(\omega)R^{-1}(\omega)B^T(\omega) \\ G(\omega)-g(\omega)R^{-1}(\omega)g^T(\omega) & A^T(\omega)-g(\omega)R^{-1}(\omega)B^T(\omega) \end{array}\right].$$

It is easy to check that $\left[\begin{smallmatrix}\mathbf{x}(t)\\ \mathbf{y}(t)\end{smallmatrix}\right]$ solves (4.3) if and only if $\left[\begin{smallmatrix}\mathbf{x}(-t)\\ -\mathbf{y}(-t)\end{smallmatrix}\right]$ solves (4.7). As a mater of fact,

$$U^-(t,\omega) = \left[\begin{array}{cc} U_1(-t,\omega) & -U_3(-t,\omega) \\ -U_2(-t,\omega) & U_4(-t,\omega) \end{array}\right] \tag{4.8}$$

is the fundamental matrix solution of (4.7) with $U^-(0,\omega)=I_{2n}$, since that of (4.3) is $U(t,\omega) = \left[\begin{smallmatrix} U_1(t,\omega) & U3(t,\omega)\\ U_2(t,\omega) & U_4(t,\omega)\end{smallmatrix}\right]$. Therefore, the family (4.3) admits an exponential dichotomy over Ω, with Lagrange planes $l^\pm(\omega)\in\mathcal{L}_\mathbb{R}$, if and only if the family (4.7) admits an exponential dichotomy over Ω (for the time reversed flow $(t,\omega)\mapsto\omega{\cdot}(-t)$), with Lagrange planes

$$\widetilde{l}^\pm(\omega) = \{\left[\begin{smallmatrix}\mathbf{w}_1\\ \mathbf{w}_2\end{smallmatrix}\right] \mid \left[\begin{smallmatrix}\mathbf{w}_1\\ -\mathbf{w}_2\end{smallmatrix}\right]\in l^\mp(\omega)\}\in\mathcal{L}_\mathbb{R}\,. \tag{4.9}$$

It is also easy to check that the σ-ergodic measures on Ω agree with the σ^--ergodic ones. In addition, the rotation number for the family (4.7) with respect to m_0 agrees with that of (4.3): this assertion can be proved using the definition

$$\alpha(m_0) = \lim_{t\to\pm\infty}\frac{1}{t}\,\arg\det(U_1(t,\omega)-iU_2(t,\omega))\,,$$

which is valid for m_0-a.e. $\omega\in\Omega$ (see [16] and Chapter 2 of [10]), and the equality (4.8). Altogether, these properties imply that Hypotheses 4.3 hold for (4.3) if and only if they hold for (4.7).

Assume hence that Hypotheses 4.3 hold, and use the notation and results of Theorem 4.4 in order to define

$$\widetilde{P}(\omega) = \left\{\mathbf{x}_0\in\mathbb{R}^n \mid \lim_{\varepsilon\to0^+} -\mathbf{x}_0^T\,M_\varepsilon^+(\omega)\,\mathbf{x}_0<\infty\right\}$$

for each $\omega\in\Omega$. Note that $-M_\varepsilon^+(\omega)=\widetilde{M}_\varepsilon^-(\omega)$, where $\widetilde{M}_\varepsilon^\pm$ represent the Weyl functions for the ε-perturbed systems which are defined from (4.7) as (4.4) is defined from (4.3). That is, $\widetilde{P}(\omega)$ is defined for the time-reversed problem in the same way as $P(\omega)$ for the initial one. From here, it is not difficult to translate the results of Theorems 4.5, 4.7 and 4.8. We prove some of the most important properties.

Theorem 4.9. *Suppose that* Hypotheses 4.3 *hold. Let* $l^\pm(\omega)\equiv\left[\begin{smallmatrix}L_1^\pm(\omega)\\ L_2^\pm(\omega)\end{smallmatrix}\right]\in\mathcal{L}_\mathbb{R}$ *be the Lagrange planes and let* $L^\pm=\{(\omega,\mathbf{z})\mid\mathbf{z}\in l^\pm(\omega)\}$ *be the* τ_H*-invariant subbundles provided by the exponential dichotomy. Then,*

(i) $\widetilde{P}(\omega)$ *is a vector subspace of* $\mathbb{R}^n$ *satisfying* $\widetilde{P}(\omega)=\operatorname{Im}L_1^+(\omega)$*, and* $\widetilde{E}(\omega)\subseteq\widetilde{P}(\omega)$ *for each* $\omega\in\Omega$*.*

(ii) *If* $\dim\widetilde{P}(\omega)=\widetilde{d}_P$ *is constant on* Ω*, then* $\widetilde{P}=\{(\omega,\mathbf{x})\mid\omega\in\Omega,\,\mathbf{x}\in\widetilde{P}(\omega)\}$ *is a closed* τ_A*-invariant subbundle of* $\Omega\times\mathbb{R}^n$ *of dimension* $\widetilde{d}_P$*.*

(iii) *Define* $l_0^+(\omega)=\left\{\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{z}_2\end{smallmatrix}\right]\in l^+(\omega)\right\}$ *and* $L_0^+=\{(\omega,\mathbf{z})\mid\omega\in\Omega,\,\mathbf{z}\in l_0^+(\omega)\}$*. If* $\dim\widetilde{P}(\omega)=\widetilde{d}_P$ *is constant on* Ω*, then* L_0^+ *is a closed* τ_H*-invariant subbundle of* L^+ *of dimension* $n-\widetilde{d}_P$*.*

(iv) *Define* $l^+_{\widetilde E}(\omega)=\{\left[\begin{smallmatrix}\mathbf z_1\\ \mathbf z_2\end{smallmatrix}\right]\in l^+(\omega)\mid \mathbf z_1\in\widetilde E(\omega)\}$ *and* $L^+_{\widetilde E}=\{(\omega,\mathbf z)\mid\omega\in\Omega,\ \mathbf z\in l^+_{\widetilde E}(\omega)\}$*. If* $\dim\widetilde E(\omega)=\widetilde d_E$ *and* $\dim\widetilde P(\omega)=\widetilde d_P$ *are constant on* Ω*, then* $L^+_{\widetilde E}$ *is a closed* τ_H*-invariant subbundle of* L^+ *of dimension* $n-\widetilde d_P+\widetilde d_E$*.*

In particular, the assertions in (ii)*,* (iii) *and* (iv) *hold if* Ω *is minimal.*

Proof. (i) From (4.9), if $l^+(\omega)\equiv\left[\begin{smallmatrix}L_1^+(\omega)\\ L_2^+(\omega)\end{smallmatrix}\right]$, then $\widetilde l^-(\omega)\equiv\left[\begin{smallmatrix}L_1^+(\omega)\\ -L_2^+(\omega)\end{smallmatrix}\right]$, and the result is a direct consequence of Theorem 4.5 applied to the time-reversed setting.

(ii) It follows from Theorem 4.7(ii) that $\widetilde P$ is a closed subbundle of dimension $\widetilde d_P$. It also proves that $\widetilde P$ is invariant for the skew-product flow induced on $\Omega\times\mathbb R^n$ by $\mathbf x'=-A(\omega{\cdot}(-t))\,\mathbf x$, with base (Ω,σ^-). In other words, $\widetilde P(\omega{\cdot}(-t))=U_A(-t,\omega){\cdot}\widetilde P(\omega)$ for all $t\in\mathbb R$ and $\omega\in\Omega$, so that $\widetilde P(\omega)=U_A(-t,\omega{\cdot}t){\cdot}\widetilde P(\omega{\cdot}t)$ and hence $\widetilde P(\omega{\cdot}t)=U_A(t,\omega){\cdot}\widetilde P(\omega)$. This proves the τ_A-invariance of $\widetilde P$.

(iii) Note that from (4.9) we deduce that $l_0^+(\omega)=\widetilde l_0^-(\omega)$. Hence, Theorem 4.8(i) applied to the time-reversed problem proves that L_0^+ is a τ_{H^-}-invariant closed subbundle of L^+ of dimension $n-\widetilde d_P$. The τ_H-invariant character follows from (4.8).

(iv) Note that $\dim E(\omega)$ is also constant on Ω, and $E(\omega)=\widetilde E(\omega)$ for all $\omega\in\Omega$: see Proposition 3.10(i) and Theorem 3.9(i). Applying now Theorem 4.8(ii) to the time-reversed setting shows that $\widetilde L^-_{\widetilde E}=\big\{(\omega,\left[\begin{smallmatrix}\mathbf w_1\\ \mathbf w_2\end{smallmatrix}\right])\mid\left[\begin{smallmatrix}\mathbf w_1\\ \mathbf w_2\end{smallmatrix}\right]\in\widetilde l^-_{\widetilde E}(\omega)\big\}$ is a τ_{H^-}-invariant closed subbundle of $\widetilde L^-$. Therefore, from this fact, (4.9) and (4.8) we conclude that $L^+_{\widetilde E}=\{\big(\omega,\left[\begin{smallmatrix}\mathbf z_1\\ \mathbf z_2\end{smallmatrix}\right]\big)\mid(\omega,\left[\begin{smallmatrix}\mathbf z_1\\ -\mathbf z_2\end{smallmatrix}\right])\in\widetilde L^-_{\widetilde E}\}$ is a τ_H-invariant closed subbundle of L^+ of dimension $n-\widetilde d_P+\widetilde d_E$, as stated. □

Corollary 4.10. *Suppose that* Hypotheses 4.3 *hold. If* $\dim\widetilde E(\omega)=\widetilde d_E$ *is constant on* Ω*, then*

$$E(\omega)=\widetilde E(\omega)\subseteq P(\omega)\cap\widetilde P(\omega)=\operatorname{Im}L_1^-(\omega)\cap\operatorname{Im}L_1^+(\omega)$$

for all $\omega\in\Omega$*. In particular, this happens if* Ω *is minimal.*

Proof. Under these conditions, $\dim E(\omega)$ is also constant on Ω, and $E(\omega)=\widetilde E(\omega)$ for all $\omega\in\Omega$: see Proposition 3.10(i) and Theorem 3.9(i). Therefore, the result follows from Theorems 4.5(i) and 4.9(i). □

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