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Aequationes Mathematicae

On Morrison's definite integral

J. Arias de Reyna, M. L. Glasser and Y. Zhou

Abstract. As an application of Cauchy's Theorem we prove that

$$\int_0^1 \arctan\left(\frac{\operatorname{arctanh} x - \arctan x}{\pi + \operatorname{arctanh} x - \arctan x}\right)\frac{dx}{x} = \frac{\pi}{8}\log\frac{\pi^2}{8}$$

answering by this means a question posted in 1984 by J. A. Morrison in the Problem Section of the journal SIAM Review.

Mathematics Subject Classification. 30E20, 44A20.

1. Introduction

The Wiener–Hopf problem on the half-line with kernel depending on $e^{-|x|}$ occurs, for example, in the theory of the anomalous skin effect [5] (the subject of MLG's Ph.D. thesis [1]) and Gaussian–Markov Estimation theory [3]. In 1983 Murray Klamkin asked MLG to examine the definite integral

$$M = \int_0^1 \left\{\frac{2}{\pi}\arctan\left[\frac{2}{\pi}\arctan\frac{1}{x} + \frac{1}{\pi}\log\left(\frac{1+x}{1-x}\right)\right] - \frac{1}{2}\right\}\frac{dx}{x} \tag{1.1}$$

the evaluation of which arose in [3] and had been submitted as an unsolved problem to the SIAM Review by Morrison [2, p. 266]. In attempting to find its value the integral was transformed into several different forms; one of them

$$I_1 = \int_0^1 \arctan\left(\frac{\operatorname{arctanh} x - \arctan x}{\pi + \operatorname{arctanh} x - \arctan x}\right)\frac{dx}{x}, \tag{1.2}$$

is the subject of this note. The value $M = \frac{1}{2}\log(\pi/2\sqrt{2})$ was eventually conjectured by heuristic means (see Morrison [4] or also [2, p. 266]), and Morrison's integral was published as Problem 84-8 in the SIAM Review [4]. No solutions were submitted and the problem remained open for the subsequent 20 years until the Siam Review problem column was cancelled, and into the subsequent

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decade. In 2013, having discovered the question-and-answer internet site Mathematics Stack Exchange MLG decided to reopen the matter by submitting the modified integral (2) [6]. No solution has yet been recorded, but shortly thereafter a similar integral was posted by Vladimir Reshetnikov [7], which can be put in the form

$$I_2 = \int_0^1 \arctan\left(\frac{\operatorname{arctanh} x - \arcsin x}{2\pi + \operatorname{arctanh} x - \arcsin x}\right) \frac{dx}{x\sqrt{1-x^2}}. \tag{1.3}$$

In this note a complete evaluation of these integrals will be presented, thus bringing to an end an inquiry begun nearly a half-century ago.

We devote the remainder of this note to evaluating I_1 and then apply the result to Morrison's problem of finding the value of (1.1). In view of the elementary, but intriguing, nature of (1.2) and the novel application of a textbook technique required for its solution we feel that this note is of interest to the mathematical community.

2. Computation of I_1

2.1. Preparations

Proposition 2.1. *The integral in* (1.2) *is well defined.*

Proof. For $-1 < x < 1$ we have

$$\operatorname{arctanh} x = \int_0^x \frac{dt}{1-t^2}, \quad \arctan x = \int_0^x \frac{dt}{1+t^2}$$

so that

$$\operatorname{arctanh} x - \arctan x = \int_0^x \frac{2t^2}{1-t^4}\, dt. \tag{2.2}$$

Therefore,

$$f(x) := \frac{1}{\pi}(\operatorname{arctanh} x - \arctan x) \tag{2.3}$$

is a differentiable, strictly increasing and non negative function on $[0, 1)$. It follows that $\frac{f(x)}{1+f(x)}$ is continuous. Also $f(0) = 0$ so that

$$\arctan\left(\frac{\operatorname{arctanh} x - \arctan x}{\pi + \operatorname{arctanh} x - \arctan x}\right)\frac{1}{x} = \arctan\left(\frac{f(x)}{1+f(x)}\right)\frac{1}{x}$$

is continuous and bounded in $[0, 1)$. □

In this paper $\log z$ always denotes the main branch of the logarithmic function defined by $\log z = \log|z| + i\arg(z)$, with $|\arg(z)| < \pi$. This function is analytic in the complex plane with a cut along the negative real axis.

Let $\Omega \subset \mathbb{C}$ be the complex plane with four cuts, two along the real axis, one from 1 to $+\infty$, the other from -1 to $-\infty$, and two along the imaginary axis, one from i to $+i\infty$ the other from $-i$ to $-i\infty$. This is a star-shaped open set with center at 0.

Proposition 2.4. *The function $f(x)$ defined in (2.3) extends to an analytic function on the simply connected open set Ω and we have*

$$f(z) = \frac{1}{2\pi}\left(\log\frac{1+z}{1-z} + i\log\frac{1+iz}{1-iz}\right), \qquad z \in \Omega \tag{2.5}$$

and

$$f(z) = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{z^{4n+3}}{4n+3}, \qquad |z| < 1. \tag{2.6}$$

Proof. We may write

$$f(z) = \frac{2}{\pi}\int_0^z \frac{t^2\,dt}{1-t^4}, \qquad z \in \Omega. \tag{2.7}$$

It is clear that this defines an analytic function in Ω. We may integrate along the segment joining 0 to $z \in \Omega$, which by the star-shaped property of Ω is contained in Ω where the integrand $t^2(1-t^4)^{-1}$ is analytic.

When $|z| < 1$ we may integrate the Taylor expansion

$$\frac{t^2}{1-t^4} = \sum_{n=0}^{\infty} t^{4n+2}$$

which proves (2.6).

The expression $\frac{1+z}{1-z}$ is a negative real number only when z is real and $|z| > 1$. Therefore, $\log\frac{1+z}{1-z}$ is well defined and analytic in Ω. In the same way we show that $\log\frac{1+iz}{1-iz}$ is well defined and analytic in Ω. Therefore, the right hand side of (2.5) is an analytic function in Ω. Expanding in power series we see that the Taylor series of this right hand side function coincides with the power series in (2.6).

From this it is clear that we have equality in (2.5) for $|z| < 1$, and by analytic continuation we have equality for all $z \in \Omega$. □

Proposition 2.8. *We have*

$$\begin{aligned} I &:= \int_0^1 \arctan\left(\frac{\operatorname{arctanh} x - \arctan x}{\pi + \operatorname{arctanh} x - \arctan x}\right)\frac{dx}{x} \\ &= \operatorname{Im}\int_0^1 \log\left(1 + (1+i)f(z)\right)\frac{dz}{z}. \end{aligned} \tag{2.9}$$

Proof. For all positive real values x we have

$$\arctan\left(\frac{x}{1+x}\right) = \operatorname{Im}\log\left(1 + (1+i)x\right).$$

Therefore,

$$I = \int_0^1 \operatorname{Im} \log\left(1 + (1+i)f(x)\right) \frac{dx}{x}.$$

When $x = 0$ we have $f(x) = 0$, so that the above logarithm vanishes there. For x near 1 we have $|f(x)| \leq C \log(1-x)^{-1}$ by (2.5), so that the integrand is $\mathcal{O}(\log\log(1-x)^{-1})$. Therefore, the integral

$$\int_0^1 \log\left(1 + (1+i)f(x)\right) \frac{dx}{x}$$

is well defined, completing the proof. □

2.2. Application of Cauchy's theorem

Proposition 2.10. *The function*

$$G(z) := \frac{1}{z} \log\left(1 + (1+i)f(z)\right) \tag{2.11}$$

is an analytic function in the first quadrant.

Proof. We will show that $\operatorname{Re}\left(1 + (1+i)f(z)\right) > 0$ when z is in the first quadrant. Its log is well defined and by composition of analytic functions it will be analytic.

The bilinear function $w = \frac{1+z}{1-z}$ transforms the first quadrant into the points w with $\operatorname{Im}(w) > 0$ and $|w| > 1$. Then $\log \frac{1+z}{1-z} = a + i\varphi$ where $a > 0$ and $0 < \varphi < \pi$. The bilinear transform $w = \frac{1+iz}{1-iz}$ transforms the first quadrant in the points w with $\operatorname{Im}(w) > 0$ and $|w| < 1$. So $\log \frac{1+iz}{1-iz} = -b + i\theta$, where $b > 0$ and $0 < \theta < \pi$. Then, for z in the first quadrant, by (2.5)

$$f(z) = \frac{1}{2\pi}(a + i\varphi - \theta - ib),$$
$$\operatorname{Re}(1 + (1+i)f(z)) = 1 + \frac{a+b}{2\pi} - \frac{\varphi+\theta}{2\pi} > 0.$$

It is also clear that at $z = 0$ the function $G(z)$ is analytic, because $f(z)$ is analytic at $z = 0$ and has a zero there. □

The function $G(z)$ in Proposition 2.10 has branch points at $z = 1$ and $z = i$, but has well defined limits at all other points of the boundary (in fact it extends analytically at these points). This follows from the fact that for x real we have $f(x)$ real and $f(ix)$ purely imaginary, so that $1 + (1+i)f(z) \neq 0$ on the boundary of the first quadrant.

Proposition 2.12. *For $|z|>1$ in the first quadrant we have*

$$f(z)=\frac{i-1}{2}+\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(4n+1)z^{4n+1}}. \tag{2.13}$$

Proof. Take $z=(1+i)x$ with $x>1$ very large. Then by (2.5)

$$f(z)=\frac{1}{2\pi}\left(\log\frac{z^{-1}+1}{z^{-1}-1}+i\log\frac{(iz)^{-1}+1}{(iz)^{-1}-1}\right)$$

so that we have, for $x\to\infty$,

$$\frac{z^{-1}+1}{z^{-1}-1}=-1+\frac{i-1}{x}+\mathcal{O}(x^{-2}),\qquad \frac{(iz)^{-1}+1}{(iz)^{-1}-1}=-1+\frac{1+i}{x}+\mathcal{O}(x^{-2}).$$

It follows that for the main branch of log we have

$$\log\frac{z^{-1}+1}{z^{-1}-1}=\pi i+\log\frac{1+z^{-1}}{1-z^{-1}},\quad \log\frac{(iz)^{-1}+1}{(iz)^{-1}-1}=\pi i+\log\frac{1+(iz)^{-1}}{1-(iz)^{-1}}.$$

Therefore, for these values of z we have

$$f(z)=\frac{i-1}{2}+\frac{1}{2\pi}\left(\log\frac{1+z^{-1}}{1-z^{-1}}+i\log\frac{1+(iz)^{-1}}{1-(iz)^{-1}}\right)$$

so that we only need to use the known expansion in Taylor series to get the equality for $z=(1+i)x$ with $x>1$. Since $f(z)$ and the expansion are both analytic for $|z|>1$ on the first quadrant, we get the equality for z in the first quadrant with $|z|>1$. $\square$

Theorem 2.14. *We have*

$$\int_0^1 \arctan\left(\frac{\operatorname{arctanh}x-\arctan x}{\pi+\operatorname{arctanh}x-\arctan x}\right)\frac{dx}{x}=\frac{\pi}{8}\log\frac{\pi^2}{8}. \tag{2.15}$$

Proof. Let the path $C_{R,\varepsilon}$, consist of the positive real interval $[0,R]$ $(R>1)$ followed by the circular arc $Re^{i\theta}$, $0\le\theta\le\pi/2$, closed by the imaginary interval $[iR,0]$ and with semicircular indentations of radius $\varepsilon<1$ to avoid $z=1$, i. Then by Cauchy's Theorem to $G(z)$

$$\int_{C_{R,\varepsilon}} G(z)\,dz=0.$$

The integrals along the small semicircles tend to 0 when $\varepsilon\to0$, so that by Cauchy's Theorem

$$\begin{aligned}\int_0^1 G(x)\,dx+\int_1^R G(x)\,dx+iR\int_0^{\pi/2} G(Re^{ix})e^{ix}\,dx\\ -\int_0^i G(z)\,dz-i\int_1^R G(iy)\,dy=0.\end{aligned} \tag{2.16}$$

Two of the integrals combine to give our integral. In fact

$$\int_0^1 G(x)\,dx - \int_0^i G(z)\,dz = \int_0^1 G(x)\,dx - i\int_0^1 G(ix)\,dx,$$

and we have

$$G(x) - iG(x) = \frac{1}{x}\log\left(1+(1+i)f(x)\right) - \frac{i}{ix}\log\left(1+(1+i)f(ix)\right).$$

By (2.6) for $0<x<1$ we have $f(ix) = -if(x)$, so that

$$\begin{aligned} G(x) - iG(x) &= \frac{\log\left(1+(1+i)f(x)\right) - \log\left(1+(1-i)f(x)\right)}{x} \\ &= \frac{2i}{x}\operatorname{Im}\log\left(1+(1+i)f(x)\right). \end{aligned}$$

Then our two integrals are

$$\int_0^1 G(x)\,dx - \int_0^i G(z)\,dz = 2i\int_0^1 \operatorname{Im}\log\left(1+(1+i)f(x)\right)\frac{dx}{x}.$$

Applying (2.9) we get

$$\int_0^1 G(x)\,dx - \int_0^i G(z)\,dz = 2Ii. \tag{2.17}$$

Our Eq. (2.16), obtained by Cauchy's Theorem, may now be written as

$$2iI + \int_1^R G(x)\,dx + iR\int_0^{2\pi} G(Re^{ix})e^{ix}\,dx - i\int_1^R G(iy)\,dy = 0. \tag{2.18}$$

In a similar way we also find that

$$\int_1^R G(x)\,dx - i\int_1^R G(iy)\,dy = \int_1^R \left(G(x) - iG(ix)\right)dx$$

with

$$G(x) - iG(ix) = \frac{1}{x}\left\{\log\left(1+(1+i)f(x)\right) - \log\left(1+(1+i)f(ix)\right)\right\}.$$

Here $x>1$ and we substitute the values of $f(x)$ and $f(ix)$ given by (2.13)

$$\begin{aligned} 1+(1+i)f(x) &= \frac{2(1+i)}{\pi}\sum_{n=0}^\infty \frac{1}{(4n+1)x^{4n+1}}, \\ 1+(1+i)f(ix) &= \frac{2(1-i)}{\pi}\sum_{n=0}^\infty \frac{1}{(4n+1)x^{4n+1}}. \end{aligned}$$

We arrive at

$$\begin{aligned}\int_1^R G(x)\,dx - i\int_1^R G(iy)\,dy \\ = \int_1^R \Biggl\{\log\left(\frac{2(1+i)}{\pi}\sum_{n=0}^\infty \frac{1}{(4n+1)x^{4n+1}}\right) \\ - \log\left(\frac{2(1-i)}{\pi}\sum_{n=0}^\infty \frac{1}{(4n+1)x^{4n+1}}\right)\Biggr\}\frac{dx}{x}.\end{aligned}$$

Both logarithms have the same real part. Therefore, only the integrals of the imaginary parts remain

$$\int_1^R G(x)\,dx - i\int_1^R G(iy)\,dy = \int_1^R \frac{\pi i}{2}\frac{dx}{x} = \frac{\pi i}{2}\log R.$$

Substituting this in (2.18) we get

$$2iI + \frac{\pi i}{2}\log R + iR\int_0^{\pi/2} G(Re^{ix})e^{ix}\,dx = 0. \tag{2.19}$$

The last integral can be transformed in the following way

$$\begin{aligned}iR\int_0^{\pi/2} G(Re^{ix})e^{ix}\,dx &= iR\int_0^{\pi/2}\frac{1}{Re^{ix}}\log\bigl(1+(1+i)f(Re^{ix})\bigr)\,e^{ix}\,dx \\ &= i\int_0^{\pi/2}\log\bigl(1+(1+i)f(Re^{ix})\bigr)\,dx.\end{aligned}$$

Since $R>1$ and Re^{ix} is in the first quadrant, the function $f(z)$ can be computed by (2.13).

$$\begin{aligned}1+(1+i)f(Re^{ix}) &= \frac{2(1+i)}{\pi}\sum_{n=0}^\infty \frac{e^{-ix(4n+1)}}{(4n+1)R^{4n+1}} \\ &= \frac{2(1+i)}{\pi R}e^{-ix} + \frac{2(1+i)}{5\pi R^5}e^{-i5x} + \cdots\end{aligned}$$

Then for R large enough

$$\begin{aligned}\log\bigl(1+(1+i)f(Re^{ix})\bigr) &= \log\left(\frac{2(1+i)}{\pi R}e^{-ix}\right) \\ &\quad + \log\left(1+\frac{e^{-i4x}}{5R^4}+\frac{e^{-i8x}}{9R^8}+\cdots\right)\end{aligned}$$

and

$$
\begin{aligned}
i\int_0^{\pi/2} & \log\left(1+(1+i)f(Re^{ix})\right)\,dx \\
&= i\int_0^{\pi/2}\left\{\log\frac{2\sqrt{2}}{\pi R}+\left(\frac{\pi}{4}-x\right)i\right\}\,dx+\mathcal{O}(R^{-4})
\end{aligned}
$$

or[1]

$$i\int_0^{\pi/2}\log\left(1+(1+i)f(Re^{ix})\right)\,dx=-\frac{\pi i}{2}\log R+\frac{\pi i}{2}\log\frac{2\sqrt{2}}{\pi}+\mathcal{O}(R^{-4}).$$

Substituting this in (2.19) we get

$$2iI+\frac{\pi i}{2}\log\frac{2\sqrt{2}}{\pi}+\mathcal{O}(R^{-4})=0. \tag{2.20}$$

Taking limits for $R\to\infty$ we get

$$I=-\frac{\pi}{4}\log\frac{2\sqrt{2}}{\pi}=\frac{\pi}{8}\log\frac{\pi^2}{8}.$$

□

Corollary 2.21. *Reshetnikov's integral I_2 defined in (1.3) is equal to I_1.*

Proof. We begin by noting that by the substitution $x=\sin(2\theta)=2\tan\theta/(1+\tan^2\theta)$,

$$I_2=\int_0^{\pi/4}\arctan\left(\frac{\frac{1}{2}\operatorname{arctanh}\frac{2\tan\theta}{1+\tan^2\theta}-\theta}{\frac{1}{2}\operatorname{arctanh}\frac{2\tan\theta}{1+\tan^2\theta}+\pi-\theta}\right)\frac{\sec^2\theta}{\tan\theta}\,d\theta. \tag{2.22}$$

Next, by the substitution $u=\tan\theta$ and the elementary identity

$$\operatorname{arctanh}\left(\frac{2u}{1+u^2}\right)=2\operatorname{arctanh}u,\qquad -1<u<1,$$

we find that $I_2=I_1$. □

3. Solution of Morrison's problem

By noting the simple identities

$$
\begin{aligned}
\frac{2}{\pi}\arctan\frac{1}{x}&=1-\frac{2}{\pi}\arctan x, && x>0\\
\log\left(\frac{1+x}{1-x}\right)&=2\operatorname{arctanh}x, && -1<x<1 \qquad (3.1)\\
\arctan x-\arctan 1&=\arctan\frac{x-1}{x+1}, && x>-1,
\end{aligned}
$$

[1]The term $\mathcal{O}(R^{-4})$ is $=0$ but we do not need to prove this.

one has from (1.1)

$$M = \frac{2}{\pi}\int_0^1 \left\{\arctan\left(1-\frac{2}{\pi}(\arctan x - \operatorname{arctanh} x)\right) - \frac{\pi}{4}\right\}\frac{dx}{x}. \tag{3.2}$$

However, $\arctan 1 = \pi/4$, so by (3.1)

$$= \frac{2}{\pi}\int_0^\infty \arctan\left(\frac{\operatorname{arctanh} x - \arctan x}{\pi + \operatorname{arctanh} x - \arctan x}\right)\frac{dx}{x} = \frac{1}{2}\log\frac{\pi}{2\sqrt{2}} \tag{3.3}$$

completely solving Morrison's problem.

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J. Arias de Reyna
Facultad de Matemáticas
Univ. de Sevilla
Apdo. 1160, 41080 Sevilla, Spain
e-mail: arias@us.es

M. L. Glasser
Dpto. de Física Teórica, Facultad de Ciencias
Universidad de Valladolid
Paseo Belén 9, 47011 Valladolid, Spain
e-mail: laryg@clarkson.edu

Y. Zhou
Program in Applied and Computational Mathematics
Princeton University
Princeton, NJ 08544, USA
e-mail: yajunz@math.princeton.edu

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