Approximate solutions to the quantum problem of two opposite charges in a constant magnetic field

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A B S T R A C T

We consider two particles of equal mass and opposite charge in a plane subject to a perpendicular constant magnetic field. This system is integrable but not superintegrable. From the quantum point of view, the solution is given by two fourth degree Hill differential equations which involve the energy as well as a second constant of motion. There are two solvable approximations in relation to the value of a parameter. Starting from each of these approximations, a consistent perturbation theory can be applied to get approximate values of the energy levels and of the second constant of motion.

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1. Introduction

In a previous paper [1] we have considered a kind of Landau system with two charged particles in a plane. The charges have the same absolute value and opposite sign and the same mass. These particles are subject to a constant perpendicular magnetic field. The same situation, under a different point of view has been discussed in some recent papers [2,3].

This model has considerable interest in Physics. For instance, it can be interpreted as a positronium system, or as a Frenkel or Mott–Wannier exciton [4]. Other applications have been studied in [5,6]. The model is also closely related to the system of a particle under two fixed gravity centers, a classical subject [7].

We have shown in [1] that this system may be studied from the point of view of either classical or quantum mechanics. The transition from the former to the latter is achieved through canonical quantization [8]. This system has four independent commuting constants of motion, or symmetries. Classically, the commutation is defined in terms of Poisson brackets. This system is integrable although not superintegrable.

In the classical analysis presented in [1], we have used two of these constants, written in compact form as the components of the two dimensional vector $\boldsymbol{\mu}$, in order to reduce by two the number of degrees of freedom, so that we have an effective two dimensional system.

The resulting Hamiltonian is a sum of a kinetic term plus an effective potential, which is given by the sum of a Coulomb potential plus a shifted harmonic oscillator. Along with this effective Hamiltonian, we have an additional constant of motion, denoted by T. This fact allows to separate the system in elliptic coordinates.

In the present Letter, we focus our interest on the quantum version of this model. Within this quantum context, the separation in elliptic coordinates of the effective system leads to a pair of equations. One is a fourth degree periodic Hill equation, while the second one is a similar modified Hill equation with hyperbolic functions [9,10]. Up to our knowledge, analytic solutions for these equations are not known.

Along this presentation, we shall discuss the possibility of obtaining approximate solutions of these equations by means of a procedure based on perturbation theory. In our calculations, we shall use $\mu := |\boldsymbol{\mu}|$ as natural perturbative parameter. This $\boldsymbol{\mu}$, to be defined in the next section (right after (6)), is a constant of motion and gives the position of the center of the displaced harmonic oscillator.

The zero order of perturbation will approximately describe the system either for $\mu \ll 1$ or for $\mu \gg 1$. In the first case, the Coulomb term will be dominant with respect to the oscillatory term, now used as a perturbation. The situation is reversed in the

second case, where the oscillatory term is dominant and the perturbation is given by the Coulomb part.

We shall see that in the zero order approximation valid for $\mu << 1$, or Coulomb approximation, both trigonometric and hyperbolic Hill equations become a pair of equations of the type discussed by Razavy in [11,12], which are solvable. These equations were known by some authors as the hyperbolic Wittaker–Hill equations. Nevertheless, since we are using as the reference the work by Razavy and following the use of some recent authors, we prefer to use the terminology *Razavy type* equations or simply, *Razavy* equations. Their solutions coincide with the bound solutions of the Coulomb problem and take significant values close to the origin of the potential. This means that the relative position between both particles keeps very small.

When $\mu >> 1$, we can again approximate both Hill equations by Razavy type equations. Their solutions describe the solutions for the harmonic oscillator in elliptic coordinates. These solutions are the bound states of the harmonic oscillator and correspond to much larger values of μ.

Using these zero order approximations as the unperturbed systems, we propose perturbations of first order. In this approach, we assume that the representation in terms of elliptic coordinates is valid for any order of perturbation.

This Letter is organized as follows: In Section 2, in order to orient the reader and for the sake of completeness, we summarize the results obtained in [1] with some additional information. We give in Section 3 the exact resolution of the zero order Razavy equations, where we pay an special attention to the correct choice of the boundary conditions and its consequences. In Section 4, we discuss the first order perturbative approach to solutions. Explicit expressions are left to Supplementary Material. We close our discussion with some Concluding Remarks.

2. Presentation of the problem

In this section, we briefly review the treatment given in [1]. Let us begin with the classical description of the model. The Hamiltonian describing two charged particles, with charges e and $-e$, of equal mass m, interacting among themselves by the Coulomb potential and subject to an external constant magnetic field perpendicular to the plane in which the particles move is given by ($c = 1$):

$$H = \frac{1}{2m}[(\mathbf{p}^{(1)} - eA(\mathbf{x}^{(1)}))^2 + (\mathbf{p}^{(2)} + eA(\mathbf{x}^{(2)}))^2] - \frac{e^2}{|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|} .$$
(1)

Here by $\mathbf{x}^{(k)}$, $\mathbf{p}^{(k)}$, $k = 1, 2$, we denote positions and linear momenta of both particles. The vector potential $\mathbf{A}(\mathbf{x})$ is taken in the symmetric gauge,

$$A_i(\mathbf{x}) = h \, \varepsilon_{ij} x_j, \qquad \mathbf{A}(\mathbf{x}) = (h x_2, -h x_1) ,$$
(2)

where ε_{ij} is the totally antisymmetric tensor in two dimensions. We are using the convention of summation over repeated indices. The magnetic field is parallel to the z axis with intensity $B = -2h$. For each particle k ($k = 1, 2$), we define a kinematic momentum $\boldsymbol{\pi}^{(k)}$ with components,

$$\pi_i^{(1)} = p_i^{(1)} - eA_i^{(1)} = p_i^{(1)} - e\varepsilon_{ij}x_j^{(1)}h ,$$
$$\pi_i^{(2)} = p_i^{(2)} + eA_i^{(2)} = p_i^{(2)} + e\varepsilon_{ij}x_j^{(2)}h .$$
(3)

Next, define the components, Π_i of the *total momentum* and the center of mass (c.o.m.) coordinates Q_i, $i = 1, 2$, by

$$\Pi_i := \pi_i^{(1)} + 2eh\varepsilon_{ij}x_j^{(1)} + \pi_i^{(2)} - 2eh\varepsilon_{ij}x_j^{(2)} ,$$
$$Q_i := \frac{1}{2}(x_i^{(1)} + x_i^{(2)})$$
(4)

and the relative momentum and coordinates as

$$\pi_i := \frac{1}{2}(\pi_i^{(1)} - \pi_i^{(2)}), \qquad q_i := x_i^{(1)} - x_i^{(2)}, \qquad i = 1, 2 .$$
(5)

Due to the fact that the total charge of the system vanishes, the functions $\{\mathbf{q}, \boldsymbol{\pi}, \mathbf{Q}, \boldsymbol{\Pi}\}$ constitute a canonical coordinate set. We could have equally discussed the case of two particles with different mass m_1, m_2; then, a similar canonical set would have been obtained.

In terms of these new coordinates, the initial Hamiltonian (1) has the following form:

$$H = \frac{1}{4m}\boldsymbol{\Pi}^2 - \frac{eh}{m}\,\varepsilon_{ij}\Pi_i q_j + \frac{1}{m}\,\boldsymbol{\pi}^2 + \frac{e^2h^2}{m}\mathbf{q}^2 - \frac{e^2}{q}$$
$$= \frac{1}{m}\,\boldsymbol{\pi}^2 + \frac{e^2h^2}{m}\,(\mathbf{q} + \frac{\boldsymbol{\mu}}{2})^2 - \frac{e^2}{q} ,$$
(6)

with $q := |\mathbf{q}|$, $\mu_j := -\varepsilon_{ij}\Pi_i/eh$ and $\mu := |\boldsymbol{\mu}|$. As the coordinates $\mathbf{Q}$ are cyclic, the components of the "total momentum" $\boldsymbol{\Pi}$ are constants of motion given in terms of $\boldsymbol{\mu}$. From (6), we conclude that the effective system consists of a particle, with a reduced mass $m/2$ and charge e, in the plane under the influence of a Coulomb potential set at the origin with charge $-e$, plus a shifted harmonic oscillator potential with angular frequency $\omega = 2eh/m = e|B|/m$, which is the cyclotron frequency, being $|B| = 2h$ the magnetic field intensity.

As is clear from (4), the constant of motion $\boldsymbol{\Pi}$ is the sum of the generators of magnetic translations for each particle, just as they are defined for the Landau system of a single particle in a constant magnetic field. In the Landau system, the values of $\boldsymbol{\Pi}$ give the center of the circular trajectories; in this case, the values of $\boldsymbol{\Pi}$ determine the relative position of the two centers of the Coulomb and oscillator effective potentials.

As the first term, $\frac{1}{4m}\boldsymbol{\Pi}^2 = \frac{(eh\mu)^2}{4m}$, in (6) is a constant, it will be hereafter dropped to simplify the expressions. Nevertheless, it will be recovered later in order to interpret the approximation $\mu >> 1$.

As shown in [1], this system has two independent constants of motion:

$$H = \frac{\boldsymbol{\pi}^2}{m} + U(\mathbf{q}) ; \qquad T := \pi_i \, g_{ij}(\mathbf{q})\pi_j + \Phi(\mathbf{q}) .$$
(7)

Here, H is the effective Hamiltonian given in (6), without the above mentioned constant term. The second constant of motion T includes a "kinetic term" given by[1]:

$$\pi_i g_{ij}(\mathbf{q})\pi_j = L^2 + (\mu_1\pi_2 - \mu_2\pi_1)L = \frac{1}{2}(L \cdot L' + L' \cdot L) ,$$
$$L := q_1\pi_2 - q_2\pi_1 ,$$
(8)

where L and L' are the angular momenta with respect to the origin and to the point $-\boldsymbol{\mu}$, respectively. By the way, this term has been already obtained by Erikson–Hill in [14] for the two center problem. The kinetic tensor can also be expressed as

$$g_{ij} = q_i^* q_j^* + \frac{1}{2}(\mu_i^* q_j^* + q_i^* \mu_j^*) ,$$
(9)

where, $q_i^* = \varepsilon_{ik}q_k$, $\mu_i^* = \varepsilon_{ik}\mu_k$. The "potential term" Φ is given by

$$\Phi(\mathbf{q}) = \frac{2me^2}{\mu^2}\frac{\mathbf{q} \cdot \boldsymbol{\mu}}{q} + \frac{e^2h^2}{4}(q^2\mu^2 - (\mathbf{q} \cdot \boldsymbol{\mu})^2) .$$
(10)

Note that T is a constant of motion in the sense that $\{H, T\} = 0$, where $\{\cdot, \cdot\}$ stands for Poisson bracket. Then with the help of T, we can separate the system using the confocal elliptic coordinates

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[1] Note that in the quantum case one has to use a symmetrized expression.

(α, β), where the foci are set at the origin and at $-\boldsymbol{\mu}$. These are defined by [1]:

$$\mu \cosh \alpha = q + |\mathbf{q} + \boldsymbol{\mu}|, \quad -\mu \cos \beta = q - |\mathbf{q} + \boldsymbol{\mu}|. \tag{11}$$

For instance, in the particular case $\boldsymbol{\mu} = (\mu, 0)$, we have the following expression of the Cartesian coordinates in terms of the elliptic coordinates:

$$x + \frac{\mu}{2} = \frac{\mu}{2} \cosh \alpha \cos \beta, \quad y = \frac{\mu}{2} \sinh \alpha \sin \beta. \tag{12}$$

According to (12), taking either the ranges $\alpha \in [0, +\infty)$, $\beta \in [-\pi, \pi)$ or $\alpha \in (-\infty, +\infty)$, $\beta \in [0, \pi)$, we cover the whole real plane.

Once we have introduced the model classically, we can proceed with its quantum version. In this context, we replace the functions H and T in (8) by the operators:

$$H := -\frac{1}{m} \partial_i \partial_i + U(\mathbf{q}); \qquad T := -\partial_i g_{ij}(\mathbf{q}) \partial_j + \Phi(\mathbf{q}),$$

$$\partial_i \equiv \frac{\partial}{\partial q_i}, \tag{13}$$

respectively ($\hbar = 1$) following the canonical quantization procedure [8]. In order to avoid unnecessary notational complications, we have use H and T for both classical functions and quantum operators. Then, the eigenfunction equations for these operators read, respectively:

$$(H - E)\Psi(\mathbf{x}) = 0, \qquad \left(\frac{1}{m} T - \mathcal{J}\right) \Psi(\mathbf{x}) = 0, \tag{14}$$

where the parameters E and $\mathcal{J}$ give the corresponding eigenvalues of the Hamiltonian H and the symmetry operator T. These equations can be separated in terms of the elliptic coordinates (α, β) with the usual Ansatz that the solution $\Psi(\alpha, \beta)$ be factorizable in terms of these coordinates [1]: $\Psi(\alpha, \beta) = \psi(\alpha)\phi(\beta)$. Thus, equations (15) that were coupled in Cartesian coordinates, give rise to a system of two ordinary second order differential equations in separated variables, for which their explicit form is given by

$$\frac{d^2\psi(\alpha)}{d\alpha^2} = m \left[A \sinh^4 \alpha - B \cosh \alpha - C \sinh^2 \alpha + \mathcal{J} \right] \psi(\alpha), \tag{15}$$

$$-\frac{d^2\phi(\beta)}{d\beta^2} = m \left[A \sin^4 \beta + B \cos \beta + C \sin^2 \beta + \mathcal{J} \right] \phi(\beta), \tag{16}$$

where

$$A = \frac{e^2 h^2}{m} \left(\frac{\mu}{2}\right)^4, \quad B = e^2 \frac{\mu}{2}, \quad C = E \frac{\mu^2}{4}. \tag{17}$$

First of all, note that although equations (15) and (16) are separated in the variables α and β, they are not separated in the variables E (through C) and $\mathcal{J}$, which are the constants of motion that we wish to obtain.

Next, these equations should satisfy some boundary conditions and, hence, we are facing to a sort of Sturm–Liouville problem. Since β is an angle, the solutions $\phi(\beta)$ of (16), which we are looking for, must be periodic with period 2π. Equation (15) can be considered as a Schrödinger equation for the function $\psi(\alpha)$ on the whole real line $-\infty < \alpha < \infty$. Thus, the solutions $\psi(\alpha)$ of (15) describing bound states should satisfy that $\psi(\alpha) \to 0$, sufficiently fast as $\alpha \to \pm\infty$. In addition, as the functions between brackets in (15) and (16) are even, then their respective solutions $\psi(\alpha)$ and $\phi(\beta)$ can be chosen to have a well defined parity. As we may conclude after the comment following (11), the range given by $\alpha \in (-\infty, \infty)$ and $\beta \in (-\pi, \pi)$ is overcomplete. Thus, in order that the solution $\Psi = \psi(\alpha)\phi(\beta)$ be a single valued function on $\mathbb{R}^2$, we must select the solutions $\psi(\alpha)$ and $\phi(\beta)$ having the same parity.

Note that equations (15) and (16) are indeed quite similar, which will help us to obtain solutions for this system. It is clear that (15) can be obtained from (16) by the replacement $\alpha \to i(\beta + \pi)$. Now, assume that we have found a solution $\psi(\alpha)$ of (15) with well defined parity, analytic on α and satisfying the proper boundary conditions. Then, the replacement $\alpha \to i(\beta + \pi)$ will give us a solution of (17) with the same parity and, hence, a single valued solution $\Psi(\alpha, \beta)$. Consequently, being given an analytic solution of (15) with the proper boundary conditions, this provides us with a complete solution to the problem.

Finally, we need to analyze the form of equations (15) and (16). To this end, let us consider the following type of second order differential equations:

$$\frac{d^2\psi(x)}{dx^2} + \left(A_0 + 2 \sum_{n=1}^{\infty} A_n \cos(2nx) \right) \psi(x) = 0, \tag{18}$$

where $A_0, A_1, A_2, \ldots$ are constants. This is a n-th degree Hill equation if $A_n \neq 0$ and $A_{n+1} = A_{n+2} = \cdots = 0$. The first degree Hill equation is called the Mathieu equation. The second degree Hill equation has been named the Whittaker–Hill equation. Up to our knowledge, no systematic studies of Hill equations of higher degree has been done. In particular (16) is a fourth degree Hill equation for which the solutions are not known.

In the present paper, we attempt to obtain approximate solutions for E and $\mathcal{J}$. This will be done in the next two sections and in the Supplementary Material.

3. Two solvable approximations

In this section, we shall discuss two approximations to the problem under discussion that have exact solution. These approximations are appropriate for small and high values of the parameter μ as defined in the previous section. Both approximations reduce the fourth degree Hill equations into the solvable Razavy equation. The first approximation is valid for $\mu << 1$ and is called the Coulomb limit as gives an energy spectrum of Coulomb type. The other is valid for $\mu >> 1$ and is named the harmonic oscillator limit as its energy spectrum is the corresponding to an off centered harmonic oscillator.

Observe that the Coulomb limit is just the special case $h = 0$, which is equivalent to $A = 0$. On the other hand, the harmonic oscillator limit, $B = 0$, can be taken $e \to 0$ with $h \to \infty$ if the product eh remains constant. In addition, since $A = O(\mu^4)$, $B = O(\mu)$ and $C = O(\mu^2)$, the large and small μ limits correspond to the harmonic oscillator and the Coulomb limit, respectively.

3.1. The Coulomb approximation $\mu << 1$

If we consider μ small, the term with coefficient A in (15) and (16) is negligible, due to the form of the coefficients given in (17). In our zeroth order approximation, $A = 0$. Then, equations (15) and (16) take the following form:

$$\frac{d^2\psi(\alpha)}{d\alpha^2} = -m \left[B \cosh \alpha + C \sinh^2 \alpha - \mathcal{J} \right] \psi(\alpha), \tag{19}$$

$$\frac{d^2\phi(\beta)}{d\beta^2} = -m \left[B \cos \beta + C \sin^2 \beta + \mathcal{J} \right] \phi(\beta). \tag{20}$$

The Schrödinger equation (19) has bound solutions with right boundary conditions in $\alpha \to \pm\infty$, provided that $C < 0$. According to (17), this means that the energy is negative, i.e., $E < 0$. This equation has exact solutions as it can be transformed into a Razavy equation, also called the hyperbolic Wittaker–Hill equation, which has the form [11,13,15,16]

$$-\frac{d^2\psi(x)}{dx^2} + (\zeta \cosh 2x - M)^2 \psi(x) = \lambda \psi(x). \tag{21}$$

Table 1
Values of the constants of motion and the eigenfunctions for $n = 0, 1, 2$ and $\mu << 1$.

$E_0 = -me^4$	$\lambda_{01} = 1 + \mu^2 m^2 e^4 \quad \mathfrak{J}_{01} = 0$
$\zeta_0 = \mu m e^2$	$\psi_{01}(\alpha) = \exp(-\frac{\zeta_0}{2}\cosh\alpha)$
$E_1 = -\frac{me^4}{4}$	$\mathfrak{J}_{11} = \frac{1+\mu m e^2}{4m}$
$\zeta_1 = \frac{\mu m e^2}{2}$	$\psi_{11}(\alpha) = \cosh(\alpha/2)\exp(-\frac{\zeta_1}{2}\cosh\alpha)$
	$\mathfrak{J}_{12} = \frac{1-\mu m e^2}{4m}$
	$\psi_{11}(\alpha) = \sinh(\alpha/2)\exp(-\frac{\zeta_1}{2}\cosh\alpha)$
$E_2 = -\frac{me^4}{9}$	$\mathfrak{J}_{21} = \frac{1}{m}$
$\zeta_2 = \frac{\mu m e^2}{3}$	$\psi_{21}(\alpha) = \sinh\alpha\,\exp(-\frac{\zeta_2}{2}\cosh\alpha)$
	$\mathfrak{J}_{22} = \frac{1+\sqrt{1+\frac{4}{9}\mu^2 m^2 e^4}}{2m}$
	$\psi_{22}(\alpha) = (\cosh\alpha - \frac{1-\sqrt{1+4\zeta_2^2}}{\zeta_2})\exp(-\frac{\zeta_2}{2}\cosh\alpha)$
	$\mathfrak{J}_{23} = \frac{1-\sqrt{1+\frac{4}{9}\mu^2 m^2 e^4}}{2m}$
	$\psi_{23}(\alpha) = (\cosh\alpha - \frac{1+\sqrt{1+4\zeta_2^2}}{\zeta_2})\exp(-\frac{\zeta_2}{2}\cosh\alpha)$

This transformation goes as follows: first perform that change of variables $\alpha = 2x$ and, then, some straightforward manipulations to obtain from (19) the following relation:

$$-\psi''(x) + \left(\sqrt{-4mC}\cosh 2x - \frac{2mB}{\sqrt{-4mC}}\right)^2 \psi(x)$$
$$= -\left(\frac{mB^2}{C} + 4mC + 4m\mathfrak{J}\right)\psi(x). \tag{22}$$

Equation (22) has already the form (21) with the following identifications:

$$\zeta = \sqrt{-4mC} = \mu\sqrt{-mE}, \tag{23}$$

$$M = \frac{2mB}{\sqrt{-4mC}} = e^2\sqrt{-\frac{m}{E}}, \tag{24}$$

$$-\lambda = \frac{mB^2}{C} + 4mC + 4m\mathfrak{J}. \tag{25}$$

The Razavy equation (21) is solvable and its solutions $\psi(x)$ are square integrable, provided that M be a positive integer, i.e., $M = n + 1$, with $n = 0, 1, 2, \ldots$.

Let us denote the energy levels by $E_n^{(C)}$, $n = 0, 1, 2 \ldots$, where the superscript (C) stand for Coulomb. Then, if $M^2 = (n+1)^2$, we have

$$E_n^{(C)} = -\frac{me^4}{(n+1)^2}. \tag{26}$$

Needless to say that the similarity of the energy levels in (26) with the energy levels for the Coulomb problem is obvious. We recall that our Hamiltonian (6) has a potential term of harmonic oscillator type and other of Coulomb type. We conclude that when $\mu << 1$, the Coulomb term predominates over the harmonic oscillator. However, not all energy levels (23) may be physically admissible, as we have not taken into account the boundary conditions yet.

Being fixed $E_C^{(n)}$, the value of ζ is obtained with (23). We denote it by ζ_n. Then, we replace $M = n + 1$ and ζ_n in the Razavy equation (21), so as to obtain the $n + 1$ known solutions for λ. We denote these solutions as $\lambda_{n,1}, \ldots, \lambda_{n,n+1}$. Each $\lambda_{n,r}$ determines a value $\mathfrak{J}_{n,r}$ through (25).

In summary, each energy level $E_C^{(n)}$ has a degeneracy of order $n + 1$, which is characterized by the $n + 1$ values of $\mathfrak{J}_{n,r}$. We denote their corresponding solutions as $\psi_{n,r}(x)$.

In Table 1, we give the values of E_n, ζ_n, $\mathfrak{J}_{n,r}$ and $\psi_{n,r}(\alpha)$, the latter in terms of the original variable $\alpha = 2x$, for $n = 0, 1, 2$. For

higher values of n, these polynomial solutions are of higher degree and, therefore, they became more and more complicated.

For equation (20), we just need to replace α by $i(\beta + \pi)$. Then, (20) becomes (19) which is already solved. The eigenfunctions are of the form $\phi_{n,r}(\beta)$ with the same values of the energy and $\mathfrak{J}_{n,r}$.

However, this is not the whole story because solutions of (20) must fulfill periodic boundary conditions of period 2π. In addition, they should be of the same parity as the solutions of (19). From Table 1, and after the transformation $\alpha \to i(\beta + \pi)$, it is rather trivial to prove that these conditions are satisfied for even values of n only. This shows that the correct solutions correspond to energy levels of the form

$$E_j^{(C)} = -\frac{me^4}{(2j+1)^2}, \qquad j = 0, 1, 2, \ldots \tag{27}$$

which precisely coincide with the energy levels of the Coulomb potential on the plane. The ground state solution $\Psi_{0,1}(\alpha, \beta)$ is given by

$$\Psi_{01} = \psi_{01}(\alpha)\phi_{01}(\beta) = N e^{-\frac{\zeta_0}{2}(\cosh\alpha - \cos\beta)} = N e^{-me^2 q}. \tag{28}$$

As a matter of fact, one can realize that the system (19)–(20) gives the two dimensional Coulomb problem in elliptic coordinates, see [17]. We have shown that the exact solutions to this problem can be obtained from the properties of the Razavy equation. See Figs. 1 and 2.

3.2. The harmonic oscillator approximation $\mu >> 1$

The second approximation has also exact solutions and corresponds to the case $\mu >> 1$, where the term containing B is dropped out after (17). The consequence is that the system behaves now as an out of center harmonic oscillator. For completeness, we write the equations (16) and (16) without the term on B:

$$\frac{d^2\psi(\alpha)}{d\alpha^2} = m\left[A\sinh^4\alpha - C\sinh^2\alpha + \mathfrak{J}\right]\psi(\alpha), \tag{29}$$

$$\frac{d^2\phi(\beta)}{d\beta^2} = -m\left[A\sin^4\beta + C\sin^2\beta + \mathfrak{J}\right]\phi(\beta). \tag{30}$$

After some straightforward transformations, (29) comes into

$$-\frac{d^2\psi(\alpha)}{d\alpha^2} + \left[\left(\sqrt{\frac{mA}{4}}\cosh(2\alpha) - \frac{1}{2}\sqrt{\frac{m}{A}}(A+C)\right)^2 \right.$$
$$\left. - \frac{m}{4A}(A+C)^2\right]\psi(\alpha)$$
$$= m\left(\frac{A}{4} + \frac{C}{2} + \mathfrak{J}\right)\psi(\alpha). \tag{31}$$

The resulting equation (31) is again a Razavy equation of solvable type with parameters given by

$$\zeta = \sqrt{\frac{mA}{4}}, \quad M = \frac{1}{2}\sqrt{\frac{m}{A}}(A+C) = n+1,$$

$$\lambda = \frac{m}{4A}(A+C)^2 + m\left(\frac{A}{4} + \frac{C}{2} + \mathfrak{J}^{(HO)}\right). \tag{32}$$

The superscript (HO) in (32) means harmonic oscillator. We know that (31) is solvable with square integrable solutions provided that $M = n + 1$ with $n = 0, 1, 2, \ldots$. Then, we recover A and C from (17) and use it in the second equation of the first row in (32) to conclude that

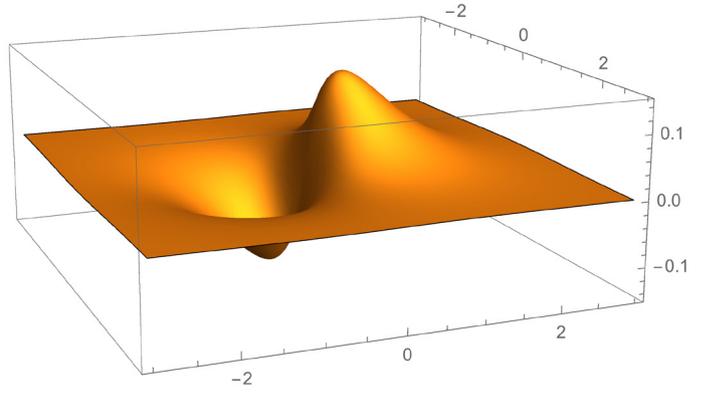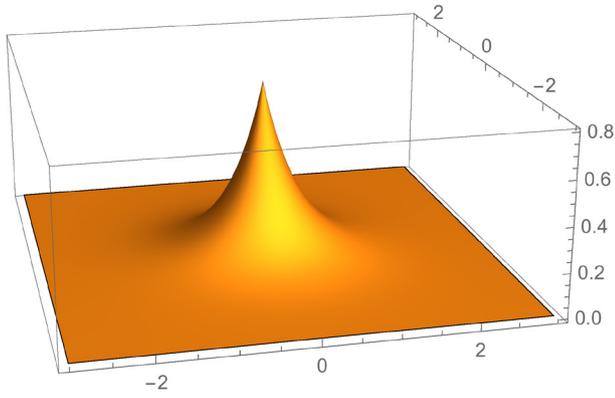$$E_n^{(HO)} = \frac{2eh}{m}(n+1) - \frac{(eh\mu)^2}{4m}, \tag{33}$$

Fig. 1. Graphics of the ground eigenfunction ψ_{01} (left) and the first excited eigenfunction ψ_{21} (right) in Table 1.
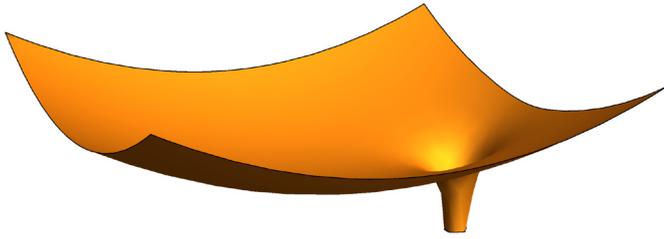

Fig. 2. Form of the potential.

Table 2
Values of the constants of motion and the eigenfunctions, up to normalization, for $n = 0, 1, 2$ and $\mu >> 1$ (the harmonic oscillator limit). The wavefunctions depend on the constant $\breve{e} = e\mu^2$ through ζ.

$E_0^* = \frac{2eh}{m}$	$\mathfrak{J}_{01} = \frac{-64\zeta + eh\mu^2(-16+eh\mu^2)}{64m}$
$\zeta = \frac{eh\mu^2}{8}$	$\psi_{01}(\alpha) = \exp(-\frac{\zeta_0}{2}\cosh 2\alpha)$
$E_1^* = \frac{4eh}{m}$	$\mathfrak{J}_{11} = \frac{-64 + 64(-2+\zeta) + eh\mu^2(-32+eh\mu^2)}{64m}$
$\zeta = \frac{eh\mu^2}{8}$	$\psi_{11}(\alpha) = \cosh(\alpha)\exp(-\frac{\zeta}{2}\cosh 2\alpha)$
	$\mathfrak{J}_{12} = \frac{-64+64(2+\zeta)+eh\mu^2(-32+eh\mu^2)}{64m}$
	$\psi_{11}(\alpha) = \sinh(\alpha)\exp(-\frac{\zeta}{2}\cosh 2\alpha)$
$E_2^* = \frac{6eh}{m}$	$\mathfrak{J}_{21} = \frac{-2304 + e\mu^2(64e^3m^2 + 9h(-48+eh\mu^2))}{576m}$
$\zeta = \frac{eh\mu^2}{8}$	$\psi_{21}(\alpha) = \sinh 2\alpha\exp(-\frac{\zeta}{2}\cosh 2\alpha)$
	$\mathfrak{J}_{22} = \frac{e\mu^2(64e^3m^2+9h(-48+eh\mu^2))-384(3+\sqrt{9+4e^4m^2\mu^2})}{576m}$
	$\psi_{22}(\alpha) = (\cosh 2\alpha - \frac{1-\sqrt{1+4\zeta^2}}{\zeta})\exp(-\frac{\zeta}{2}\cosh 2\alpha)$
	$\mathfrak{J}_{23} = \frac{e\mu^2(64e^3m^2+9h(-48+eh\mu^2))-384(-3+\sqrt{9+4e^4m^2\mu^2})}{576m}$
	$\psi_{23}(\alpha) = (\cosh 2\alpha - \frac{1+\sqrt{1+4\zeta^2}}{\zeta})\exp(-\frac{\zeta}{2}\cosh 2\alpha)$

which corresponds to the energy levels of a displaced harmonic oscillator. Note that the shift of energy in (33) coincides with the dropped term in the Hamiltonian (6). In our comments after equation (6), we have announced that that term should reappear. The values for the second constant of motion $\mathfrak{J}^{(HO)}$ and the wave functions are obtained as usual [11,13,15,16,22]. In Table 2, we list our results for the three first energy levels.

From the solutions of (29), we obtain the solutions of (30) just with the replacement $\alpha \rightarrow i(\beta + \pi)$, as in the Coulomb case. However, there is an important difference between both situations. We see in Table 1 that the functions $\psi_{nr}(\alpha)$ for Coulomb and n odd depend on $\alpha/2$ which implies that the functions $\phi_{nr}(\beta)$, for n odd, do not have periodicity with period 2π and, therefore, we had to discard these solutions. This is not the case now with the harmonic oscillator approximation, where all eigenfunctions $\psi_{nr}(\alpha)$ depend on α or 2α. Then, the functions $\phi_{nr}(\beta)$ obtained by this method are periodic with period 2π even for n odd. In consequence, all

energy values should be considered, even those with n odd. The ground state is now

$$\Psi(\alpha, \beta) = \exp(-\frac{\zeta_0}{2}(\cosh 2\alpha + \cos 2\beta))$$
$$\propto \exp(-\frac{eh}{2}(\mathbf{q} + \boldsymbol{\mu}/2)^2). \tag{34}$$

Note that this is the ground state of a harmonic oscillator centered at $-\boldsymbol{\mu}/2$. In fact, we have obtained the solutions for the harmonic oscillator separated in elliptic coordinates [18,19].

So far, we have proposed two exactly solvable approximations. What happens if $\mu = |\boldsymbol{\mu}|$ is neither too small nor too big? Then, we propose a perturbative treatment taking the previous approximations as the unperturbed solution. As the perturbed potential, we use the dropped term for each case. This will be the objective of the next section. Results are given in the Supplementary Material.

4. A perturbative approximation to the complete solution

In the sequel, we are dealing with the quantum case only. Therefore, expressions like H, H_0, H_1, T, etc., will henceforth represent operators. We follow this convention in order to use a notation as simple as possible.

In this section, we are looking for approximate solutions for equations (15) and (16). Let us begin with the unperturbed equations:

$$(H_0 - E_0)\Psi_0(\mathbf{x}) = 0, \qquad \left(\frac{1}{m}T_0 - \mathfrak{J}_0\right)\Psi_0(\mathbf{x}) = 0, \tag{35}$$

where H_0 and T_0 represent unperturbed versions of H and T in (13). They are assumed to be valid for the approximations $\mu << 1$ and $\mu >> 1$. Therefore, equations (35) lead to equations (19) and (20) for $\mu << 1$ and (29) and (30) for $\mu >> 1$. In the first case, the term containing A in equations (15) and (16) is omitted, while the term containing B drops out in the second.

We intend to give a perturbative first order approximation, for which we just write

$$H = H_0 + \epsilon H_1, \qquad T = T_0 + \epsilon T_1, \tag{36}$$

$$E = E_0 + \epsilon E_1, \qquad \mathfrak{J} = \mathfrak{J}_0 + \epsilon \mathfrak{J}_1, \tag{37}$$

$$\Psi = \Psi_0 + \epsilon \Psi_1. \tag{38}$$

Note that H_0 and T_0 commute and also does H and T. This is a crucial point here, as implies that the function Ψ showed in (38) is a simultaneous eigenfunction of H and T. Thus, replacing (36)–(37) in (14), we obtain

$$(H_0 + \epsilon\, H_1)(\Psi_0 + \epsilon\, \Psi_1) = (E_0 + \epsilon\, E_1)(\Psi_0 + \epsilon\, \Psi_1)\,, \tag{39}$$

$$\frac{1}{m}\,(T_0 + \epsilon\, T_1)\,(\Psi_0 + \epsilon\, \Psi_1) = (\mathfrak{J}_0 + \epsilon\, \mathfrak{J}_1)(\Psi_0 + \epsilon\, \Psi_1)\,. \tag{40}$$

Equations (39) and (40) yield the desired result using the standard procedure in perturbation theory [20,21]:

$$E_1 = \frac{\langle \Psi_0 | H_1 \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle}\,, \qquad \mathfrak{J}_1 = \frac{\langle \Psi_0 | T_1 \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle}\,. \tag{41}$$

To find H_1 and T_1, we first write $U(\mathbf{q})$ and $\Phi(\mathbf{q})$ in (7) in terms of elliptic coordinates. The explicit forms of $U(\mathbf{q})$ and $\Phi(\mathbf{q})$ are taken from [1] and their expression in terms of elliptic coordinates is the result of a straightforward calculation:

$$U(\mathbf{q}) = -\frac{e^2}{q} + \frac{e^2 h^2}{m}\,(q^2 + \mathbf{q}\cdot\boldsymbol{\mu})$$
$$= -\frac{2e^2}{\mu}\,\frac{1}{\cosh\alpha - \cos\beta} + \frac{e^2 h^2 \mu^2}{4m}\,(\sinh^2\alpha - \sin^2\beta)\,, \tag{42}$$

$$\Phi(\mathbf{q}) = \frac{2m\, e^2}{\mu^2}\,\frac{\mathbf{q}\cdot\boldsymbol{\mu}}{q} + \frac{e^2 h^2}{4}\,(q^2\mu^2 - (\mathbf{q}\cdot\boldsymbol{\mu})^2)$$
$$= -\frac{m e^2 \mu}{2}\,\frac{\cosh\alpha\cos\beta - 1}{\cosh\alpha - \cos\beta} + \frac{e^2 h^2 \mu^4}{16}\,\sinh^2\alpha\sin^2\beta\,. \tag{43}$$

For completeness, we here supply the expression of the "kinetic terms" of both constants of motion:

$$\frac{1}{m}\boldsymbol{\pi}^2 = \frac{4}{m\,\mu^2}\,\frac{1}{\sinh^2\alpha + \sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right) \tag{44}$$

$$\pi_i g_{ij}\pi_j = \frac{1}{\sinh^2\alpha + \sin^2\beta}\left(-\sin^2\beta\,\frac{\partial^2}{\partial\alpha^2} + \sinh^2\alpha\,\frac{\partial^2}{\partial\beta^2}\right)\,. \tag{45}$$

In order to assign an explicit form to the terms in the split (36)–(37), we go back to equations (7)–(8) to write them as

$$H = \frac{\boldsymbol{\pi}^2}{m} + U(\mathbf{q}) = \frac{\boldsymbol{\pi}^2}{m} + U_0(\mathbf{q}) + U_1(\mathbf{q}) = H_0 + U_1(\mathbf{q}) \tag{46}$$

and

$$T = \pi_i\, g_{ij}(\mathbf{q})\pi_j + \Phi(\mathbf{q}) = \pi_i\, g_{ij}(\mathbf{q})\pi_j + \Phi_0(\mathbf{q}) + \Phi_1(\mathbf{q})$$
$$= T_0 + \Phi_1(\mathbf{q})\,, \tag{47}$$

where $U_1(\mathbf{q}) = \epsilon\, H_1$ and $\Phi_1(\mathbf{q}) = \epsilon\, T_1$.

We observe from (44) that the kinetic term in H depends on μ^{-2}, so that it may be convenient redefining H by multiplying (46) by μ^2. The final result is

$$\widetilde{H} = \mu^2\,\frac{\boldsymbol{\pi}^2}{m} + \mu^2 U(\mathbf{q}) = \mu^2\,\frac{\boldsymbol{\pi}^2}{m} + \mu^2 U_0(\mathbf{q}) + \mu^2 U_1(\mathbf{q})$$
$$= \mu^2\,\frac{\boldsymbol{\pi}^2}{m} + \widetilde{U}_0(\mathbf{q}) + \widetilde{U}_1(\mathbf{q}) = \widetilde{H}_0 + \widetilde{U}_1(\mathbf{q})\,, \tag{48}$$

where the form of $\widetilde{H}_0$ is obvious. In the Supplementary Material, we give the explicit first order corrections in both cases.

5. Concluding remarks

Integrability and solvability are quite different concepts. This is illustrated in the present letter, where we have discussed an apparently very simple model: the Landau problem resulting of the consideration of two charged particles of opposite sign on a plane subject to a perpendicular constant magnetic field.

As shown in a previous paper, this system is equivalent to one in which a particle is subject to a Coulomb potential plus a shifted harmonic oscillator. In the search for solutions here labelled (15) and (16), we have used separation of variables in terms of elliptic

coordinates, a procedure which yields to two separated equations. Although these equations are not solvable, we have analyzed two solvable approximations depending on the values of a characteristic parameter μ.

For small values of the parameter μ, i.e., $\mu \ll 1$, we omit the highest order term on μ in the equations. This term contains the coefficient A, which depends on μ. The resulting situation is a Coulomb system expressed in elliptic coordinates described by two equations separated on the coordinates. These equations are exactly solvable of Razavy type and their solutions give the bound states for the Coulomb system. We justify the approximation by noting, making use of the approximate solutions, that the dropped term, i.e., $(A\sinh^4\alpha)\,\psi(\alpha)$, is very small for any real value of α.

For large values of μ, $\mu \gg 1$, the fundamental equations (15)–(16) can be simplified if we drop the term with the coefficient B. The resulting approximation is now equivalent to displaced harmonic oscillator in elliptic coordinates. After this simplification, equations (15)–(16) are again solvable Razavy equations. In Table 2, we give the wave functions corresponding to the lowest energy levels. These solutions spread out a center of potential located at $-\mu/2$. Again, an use of these solutions show that the dropped term, in this case $(B\cosh\alpha)\,\psi(\alpha)$ is very small for all real values of α.

Equations (15)–(16) as well as those resulting of the mentioned approximations can be obtained from each other just be using the correspondence $\alpha \to i(\beta + \pi)$. Consequently, once we have found a solution for the equation on α, we have a correspondent solution for the equation on β. However, not all these solutions are physical, only those fulfilling the correct boundary conditions for both variables.

Taking these solvable approximations as a point of departure, we have studied first order perturbative corrections. These perturbations show a break of the degeneracy of the initial energy levels. It is noteworthy that both zero order systems (Coulomb and oscillator) and the total system including the perturbation terms are separated with respect to the same elliptic coordinates.

The approximations $\mu \ll 1$ and $\mu \gg 1$ should be compared with the behavior of the system on the limits $\mu \to 0$ and $\mu \to \infty$, respectively. In the limit $\mu \to 0$, both potentials, Coulomb and harmonic oscillator remain. Both centers are located at the origin. In this limit, $T \sim L^2$ and elliptic coordinates become polar coordinates [3]. This is consistent with the rotational geometric symmetry around the z-axis of the system at this limit. The other situation comes in the limit $\mu \to \infty$. Here, after a translation of coordinates, the whole system becomes a harmonic oscillator centered at the origin, with Hamiltonian

$$H = \frac{\boldsymbol{\pi}^2}{m} + \frac{e^2 h^2}{m}\,\mathbf{q^2}\,.$$

Then, taking $\boldsymbol{\mu} := (\mu, 0)$, the second constant of motion T takes the form:

$$T \sim \frac{\pi_y^2}{m} + \frac{e^2 h^2}{m}\,q_y^2\,,$$

so that T appears in terms of Cartesian coordinates viewed as a limit of elliptic coordinates.

This two particle Landau system can be compared to the classical problem of a particle in the field produced by two Coulomb centers [7,22]. This is also separable in elliptic coordinates and it also has two solvable limits in terms of the distance between both centers. They are found when the distance between both centers becomes either zero or infinite. Then, the coordinates on the plane become polar or parabolic coordinates, respectively.

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.physleta.2016.03.038.

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1. Explicit expressions for E and $\mathcal{J}$ at first order: Coulomb case

We have to specify the form of the perturbations $U_1(\mathbf{q})$ and $\Phi_1(\mathbf{q})$. For the Coulomb approximation $\mu << 1$, the natural split of $U(\mathbf{q})$ in (42), main text, into $U_0(\mathbf{q})$ plus the perturbation $U_1(\mathbf{q})$ is

$$U_0(\mathbf{q}) = -\frac{2e^2}{\mu}\frac{1}{\cosh\alpha - \cos\beta}, \quad U_1(\mathbf{q}) = \frac{e^2 h^2 \mu^2}{4m}\left(\sinh^2\alpha - \sin^2\beta\right). \quad (1)$$

Clearly if $\mu << 1$, $U_1(\mathbf{q})$ is much smaller than $U_0(\mathbf{q})$. The split for $\Phi(\mathbf{q})$ in (43), main text, is also rather obvious:

$$\Phi_0(\mathbf{q}) = -\frac{me^2\mu}{2}\frac{\cosh\alpha\cos\beta - 1}{\cosh\alpha - \cos\beta}, \quad \Phi_1(\mathbf{q}) = \frac{e^2 h^2 \mu^4}{16}\sinh^2\alpha\sin^2\beta. \quad (2)$$

In this case, the perturbation parameter is given by $\epsilon = \mu^3$. If we redefine $\widetilde{e}^2 := e^2\mu$, where e is the charge, we have:

$$H_1 = \frac{\widetilde{e}^2 h^2}{4m}\left(\sinh^2\alpha - \sin^2\beta\right), \quad T_1 = \frac{\widetilde{e}^2 h^2}{16}\sinh^2\alpha\sin^2\beta. \quad (3)$$

In above equations we have used directly elliptic coordinates. Therefore, all ingredients in the integrals resulting from (41), main text, have to be written in terms of these coordinates. The differential $dx\,dy$ should also be changed according to the Jacobi theorem as follows:

$$dx\,dy \longmapsto \frac{\mu^2}{4}(\sinh^2\alpha + \sin^2\beta)d\alpha\,d\beta. \quad (4)$$

In (41), main text, the expression for Ψ_0 is the given for $\psi_{n,r}$ in Tables 1 and 2. For higher values of n, we should use the procedure in the standard bibliography. See References [11,12,15,16], main text.

Using the above equations and the wave function ψ_{01} of table 1, it can be shown that

$$\langle\psi_{01}|\psi_{01}\rangle = \frac{\mu^2}{4}[R_{(1,0)}S_{(0,0)} + R_{(0,0)}S_{(1,0)}] = \frac{\pi\mu^2}{2\zeta_0^2}, \quad (5)$$

where $\zeta_0 = m\widetilde{e}^2$. In turn

$$\langle\psi_{01}|H_1|\psi_{01}\rangle = \frac{\mu^2}{4}\frac{\widetilde{e}^2 h^2}{2m}\left[R_{(2,0)}(\zeta_0)S_{(0,0)}(\zeta_0) - R_{(0,0)}(\zeta_0)S_{(2,0)}(\zeta_0)\right]. \quad (6)$$

1

Then, using (41) main text, we have

$$E_1^{(0)} = \frac{\langle \psi_{01}|H_1|\psi_{01}\rangle}{\langle \psi_{01}^C|\psi_{01}^C\rangle} = \frac{\widetilde{e}^2 h^2 \zeta_0^2}{4\pi m}\left[R_{(2,0)}(\zeta_0)S_{(0,0)}(\zeta_0) - R_{(0,0)}(\zeta_0)S_{(2,0)}(\zeta_0)\right] .$$

(7)

For the second constant of motion, we obtain

$$\langle \psi_{01}|T_1|\psi_{01}\rangle = \frac{\widetilde{e}^2 h^2 \mu^2}{64}\left[R_{(2,0)}(\zeta_0)S_{(1,0)}(\zeta_0) + R_{(1,0)}(\zeta_0)S_{(2,0)}(\zeta_0)\right] .$$

(8)

Therefore,

$$\mathcal{J}_1^{(0)} = \frac{\widetilde{e}^2 h^2 \zeta_0^2}{32\pi}\left[R_{(2,0)}(\zeta_0)S_{(1,0)}(\zeta_0) + R_{(1,0)}(\zeta_0)S_{(2,0)}(\zeta_0)\right] .$$

(9)

For $n = 1$, the solutions are unphysical as it was shown in last section, and consequently we have to discard them. Thus, consider $n = 2$ and

$$\langle \psi_{21}|\psi_{21}\rangle = -\frac{\mu^2}{4}\left[R_{(2,0)}(\zeta_2)S_{(1,0)}(\zeta_2) + R_{(1,0)}(\zeta_2)S_{(2,0)}(\zeta_2)\right] ,$$

(10)

expression which cannot be reduced. Then

$$\begin{aligned}
\langle \psi_{21}|H_1|\psi_{21}\rangle &= -\frac{\mu^2}{4}\frac{\widetilde{e}^2 h^2}{2m}\left[R_{(3,0)}(\zeta_2)S_{(1,0)}(\zeta_2) - R_{(1,0)}(\zeta_2)S_{(3,0)}(\zeta_2)\right] \\
&= -\frac{15\pi\widetilde{e}^2 h^2 \mu^2}{m\zeta_2^6} .
\end{aligned}$$

(11)

Then,

$$E_1^{(2,1)} = \frac{\langle \psi_{21}|H_1|\psi_{21}\rangle}{\langle \psi_{21}|\psi_{21}\rangle} = \frac{60\pi\widetilde{e}^2 h^2}{m\zeta_2^6}\frac{1}{\left[R_{(2,0)}(\zeta_2)S_{(1,0)}(\zeta_2) + R_{(1,0)}(\zeta_2)S_{(2,0)}(\zeta_2)\right]} .$$

(12)

The second constant of motion takes the form

$$\mathcal{J}_1^{(2,1)} = \frac{\langle \psi_{21}|T_1|\psi_{21}\rangle}{\langle \psi_{21}|\psi_{21}\rangle} = \frac{\widetilde{e}^2 h^2}{16}\frac{R_{(3,0)}(\zeta_2)S_{(2,0)}(\zeta_2) + R_{(2,0)}(\zeta_2)S_{(3,0)}(\zeta_2)}{R_{(2,0)}(\zeta_2)S_{(1,0)}(\zeta_2) + R_{(1,0)}(\zeta_2)S_{(2,0)}(\zeta_2)} .$$

(13)

2

For the wave functions ψ_{22} and ψ_{23} we have to use the relations in Part 3 at the end of the present Supplementary Material, so as to obtain:

$$E_1^{(2,2)} = \frac{\langle \psi_{22}|H_1|\psi_{22}\rangle}{\langle \psi_{22}|\psi_{22}\rangle} = \frac{\widetilde{e}^2 h^2}{2m} \frac{R_{(2,2)}(a_-,\zeta_2)S_{(0,2)}(a_-,\zeta_2) - R_{(0,2)}(a_-,\zeta_2)S_{(2,2)}(a_-,\zeta_2)}{R_{(1,2)}(a_-,\zeta_2)S_{(0,2)}(a_-,\zeta_2) + R_{(0,2)}(a_-,\zeta_2)S_{(1,2)}(a_-,\zeta_2)}$$
(14)

and

$$\mathcal{J}_1^{(2,2)} = \frac{\langle \psi_{22}|T_1|\psi_{22}\rangle}{\langle \psi_{22}|\psi_{22}\rangle} = \frac{\widetilde{e}^2 h^2}{16} \frac{R_{(2,2)}(a_-,\zeta_2)S_{(1,2)}(a_-,\zeta_2) + R_{(1,2)}(a_-,\zeta_2)S_{(2,2)}(a_-,\zeta_2)}{R_{(1,2)}(a_-,\zeta_2)S_{(0,2)}(a_-,\zeta_2) + R_{(0,2)}(a_-,\zeta_2)S_{(1,2)}(a_-,\zeta_2)}$$
(15)

Finally, for ψ_{23} the result is identical to the last two equations, after the replacement of a_- by a_+:

$$E_1^{(2,3)} = \frac{\langle \psi_{23}|H_1|\psi_{23}\rangle}{\langle \psi_{23}|\psi_{23}\rangle} = \frac{\widetilde{e}^2 h^2}{2m} \frac{R_{(2,2)}(a_+,\zeta_2)S_{(0,2)}(a_+,\zeta_2) - R_{(0,2)}(a_+,\zeta_2)S_{(2,2)}(a_+,\zeta_2)}{R_{(1,2)}(a_+,\zeta_2)S_{(0,2)}(a_+,\zeta_2) + R_{(0,2)}(a_+,\zeta_2)S_{(1,2)}(a_+,\zeta_2)}$$
(16)

and

$$\mathcal{J}_1^{(2,3)} = \frac{\langle \psi_{23}|T_1|\psi_{23}\rangle}{\langle \psi_{23}|\psi_{23}\rangle} = \frac{\widetilde{e}^2 h^2}{16} \frac{R_{(2,2)}(a_+,\zeta_2)S_{(1,2)}(a_+,\zeta_2) + R_{(1,2)}(a_+,\zeta_2)S_{(2,2)}(a_+,\zeta_2)}{R_{(1,2)}(a_+,\zeta_2)S_{(0,2)}(a_+,\zeta_2) + R_{(0,2)}(a_+,\zeta_2)S_{(1,2)}(a_+,\zeta_2)}$$
(17)

From (12), (14) and (15), we observe that the degeneracy of the first excited level is broken, since $E_1^{(2,1)} \neq E_1^{(2,2)} \neq E_1^{(2,3)}$ which gives rise to three single energy levels. The same becomes true with all other energy levels.

2. Explicit expressions for E and $\mathcal{J}$ at first order: the harmonic oscillator

In the harmonic oscillator approximation, $\mu >> 1$, the roles of the functions in (1) and (2) are obviously interchanged. Let us use $\breve{e} = e\mu^2$. This gives

$$U_0(\mathbf{q}) = \frac{\breve{e}^2 h^2}{4m} \left(\sinh^2 \alpha - \sin^2 \beta\right), \qquad U_1(\mathbf{q}) = -\frac{2\breve{e}^2}{\mu^3} \frac{1}{\cosh \alpha - \cos \beta}, \quad (18)$$

$$\Phi_0(\mathbf{q}) = \frac{\breve{e}^2 h^2}{16} \sinh^2 \alpha \sin^2 \beta, \qquad \Phi_1(\mathbf{q}) = -\frac{m\breve{e}^2}{2\mu^3} \frac{\cosh \alpha \cos \beta - 1}{\cosh \alpha - \cos \beta}. \quad (19)$$

Now, the perturbation parameter is $\epsilon = \mu^{-3}$. Then, H_1 and T_1 in (36), main text, have the following explicit form:

$$H_1 = -2\breve{e}^2 \frac{1}{\cosh \alpha - \cos \beta}, \qquad T_1 = -\frac{m\breve{e}^2}{2} \frac{\cosh \alpha \cos \beta - 1}{\cosh \alpha - \cos \beta}. \quad (20)$$

In order to perform the integrals which correspond to the first order of approximation (41), main text, we write the measure (4) in the following form:

$$dx\,dy \longmapsto \frac{\mu^2}{4}(\cosh^2 \alpha - \cos^2 \beta)\,d\alpha\,d\beta \quad (21)$$

$$= \frac{\mu^2}{4}(\cosh \alpha - \cos \beta)(\cosh \alpha + \cos \beta)\,d\alpha\,d\beta.$$

Then, some singular denominators cancel out and the integrals should be quite similar as in the first case. With the help of the integrals V_j and W_j, we obtain the following results (here the functions $\psi_{n,r}$ are those corresponding to the harmonic oscillator approximation):

$$\langle \psi_{01} | \psi_{01} \rangle = \frac{\mu^2}{4} \left[V_2 W_0 + V_0 W_2 \right], \quad (22)$$

so that

$$\langle \psi_{01} | H_1 | \psi_{01} \rangle = -\frac{\mu e^2}{2} \left[V_1 W_0 + V_0 W_1 \right] \quad (23)$$

and

$$\langle \psi_{01} | \Phi_1 | \psi_{01} \rangle = -\frac{m e^2 \mu^3}{8} \left[(V_2 - V_0) W_1 + (W_2 - W_0) V_1 \right] \quad (24)$$

4

Using (22-24), we obtain for the first correction of the energy levels:

$$E_1^{(0)} = \frac{\langle\psi_{01}|H_1|\psi_{01}\rangle}{\langle\psi_{01}|\psi_{01}\rangle} = -2\breve{e}^2 \frac{V_1 W_0 + V_0 W_1}{V_2 W_0 + V_0 W_2} \tag{25}$$

and for the second constant of motion,

$$\eth_1^{(0)} = \frac{\langle\psi_{01}|\Phi_1|\psi_{01}\rangle}{\langle\psi_{01}|\psi_{01}\rangle} = -\frac{m\breve{e}^2}{2} \frac{(V_2 - V_0)W_1 + (W_2 - W_0)V_1}{V_2 W_0 + V_0 W_2} \tag{26}$$

For $n = 1$, we have

$$\langle\psi_{11}|\psi_{11}\rangle = \frac{\mu^2}{4}\left[V_4 W_0 + V_2 W_2\right], \tag{27}$$

$$E_1^{(1,1)} = \frac{\langle\psi_{11}|H_1|\psi_{11}\rangle}{\langle\psi_{11}|\psi_{11}\rangle} = -2\breve{e}^2 \frac{V_3 W_0 + V_2 W_1}{V_4 W_0 + V_2 W_2} \tag{28}$$

and

$$\eth_1^{(1,1)} = \frac{\langle\psi_{11}|\Phi_1|\psi_{11}\rangle}{\langle\psi_{11}|\psi_{11}\rangle} = -\frac{m\breve{e}^2}{2} \frac{(V_4 - V_2)W_1 + (W_2 - W_1)V_3}{V_4 W_0 + V_2 W_2} \tag{29}$$

For the second solution with $n = 1$, we obtain

$$\langle\psi_{12}|\psi_{12}\rangle = \frac{\mu^2}{4}\left[(V_4 - V_2)W_0 + (V_2 - V_0)W_2\right]. \tag{30}$$

The correction to the energy reads

$$E_2^{(1,2)} = \frac{\langle\psi_{12}|H_1|\psi_{12}\rangle}{\langle\psi_{12}|\psi_{12}\rangle} = -2\breve{e}^2 \frac{(V_3 - V_1)W_0 + (V_2 - V_0)W_1}{(V_4 - V_2)W_0 + (V_2 - V_0)W_2}, \tag{31}$$

and for the second constant of motion

$$\eth_2^{(1,2)} = \frac{\langle\psi_{12}|T_1|\psi_{1,2}\rangle}{\langle\psi_{12}|\psi_{12}\rangle} = -\frac{m\breve{e}^2}{2} \frac{(V_4 - 2V_2 + V_0)W_1 + (V_3 - V_1)(W_2 - W_0)}{(V_3 - V_1)W_0 + (V_2 - V_0)W_1}. \tag{32}$$

For $n = 2$, we obtain

$$\langle\psi_{21}|\psi_{21}\rangle = 4\mu^2 \left[V_6(W_2 - W_4) + V_4(W_6 - W_2) + V_2(W_4 - W_6)\right]. \tag{33}$$

The correction to the energy is

$$E_1^{(2,1)} = \frac{\langle \psi_{21}|H_1|\psi_{21}\rangle}{\langle \psi_{21}|\psi_{21}\rangle} = -2\breve{e}^2 \frac{(V_5 - V_3)(W_2 - W_4) + (V_4 - V_2)(W_3 - W_5)}{V_6(W_2 - W_4) + V_4(W_6 - W_2) + V_2(W_4 - W_6)},$$
$$(34)$$

while for the second constant of motion, we have

$$\mathfrak{J}_1^{(2,1)} = \frac{\langle \psi_{21}|\Phi_1|\psi_{21}\rangle}{\langle \psi_{21}|\psi_{21}\rangle}$$

$$= -\frac{m\breve{e}^2}{2} \frac{(V_6 - 2V_4 + V_2)(W_3 - W_5) + (V_5 - V_3)(2W_4 - W_6 - W_2)}{V_6(W_2 - W_4) + V_4(W_6 - W_2) + V_2(W_4 - W_6)}. \quad (35)$$

There are two other two solutions for $n = 2$, which are (the sign minus corresponds to the subindex 2 and the sign plus to 3)

$$\langle \psi_{2\mp}|\psi_{2\mp}\rangle = \frac{\mu^2}{4}[(2V_4 - V_2(1 + a_\mp))(2W_2 - 1 - a_\mp)$$

$$- (2V_2 - 1 - a_\mp)(2W_4 - W_2(1 + a_\mp))], \quad (36)$$

so that

$$E_2^{(2,\mp)} = \frac{\langle \psi_{2\mp}|H_1|\psi_{2\mp}\rangle}{\langle \psi_{2\mp}|\psi_{2\mp}\rangle} =$$

$$-2\breve{e}^2 \frac{(2V_3 - V_1(1 + a_\mp))(2W_2 - 1 - a_\mp) + (2V_2 - 1 - a_\mp)(2W_3 - W_1(1 + a_\mp))}{(2V_4 - V_2(1 + a_\mp))(2W_2 - 1 - a_\mp) - (2V_2 - 1 - a_\mp)(2W_4 - W_2(1 + a_\mp))}. (37)$$

We also have

$$\mathfrak{J}_2^{(2,\mp)} = \frac{\langle \psi_{2\mp}|\Phi_1|\psi_{2\mp}\rangle}{\langle \psi_{2\mp}|\psi_{2\mp}\rangle} =$$

$$-\frac{m\breve{e}^2\mu}{2} \frac{(2V_4 + (1 + a_\mp)(1 - V_2) - 2V_2)(2W_3 - W_1(1 + a_\mp))}{(2V_4 - V_2(1 + a_\mp))(2W_2 - 1 - a_\mp) - (2V_2 - 1 - a_\mp)(2W_4 - W_2(1 + a_\mp))} -$$

$$\frac{m\breve{e}^2}{2} \frac{(2V_3 - V_1(1 + a_\mp))(2W_4 + (1 + a_\mp)(1 - W_2) - 2W_2)}{(2V_4 - V_2(1 + a_\mp))(2W_2 - 1 - a_\mp) - (2V_2 - 1 - a_\mp)(2W_4 - W_2(1 + a_\mp))}. (38)$$

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We may consider higher values of n in order to obtain the corrections for the first and second constant of motion in terms of the elementary integrals given in Part 3, next in the present Supplementary Material.

3. Some integrals

We begin with the assumption $\mu << 1$ for the unperturbed Hamiltonian. In this case, as it was said before, the wave functions for β come after the change $\alpha \rightarrow i(\beta + \pi)$ which implies that $\cosh(i(\beta + \pi)) = -\cos \beta$ and $\sinh(i(\beta + \pi)) = -i \sin \beta$. The following four integrals will be used [1]

$$R_{(j,0)}(\zeta) = \int_0^\infty d\alpha\, e^{-\zeta \cosh \alpha} \sinh^{2j} \alpha = \int_1^\infty dy\, e^{-\zeta y}(y^2 - 1)^{j-1/2}$$

$$= \frac{2^j}{\sqrt{\pi}}\zeta^{-j}\Gamma(j + \frac{1}{2})K_j(\zeta). \tag{39}$$

$$R_{(j,2)}(a_\pm, \zeta) = \int_0^\infty d\alpha\, (\cosh \alpha - a_\pm)^2\, e^{-\zeta \cosh \alpha} \sinh^{2j} \alpha$$

$$= \int_1^\infty dy\, (y - a_\pm)^2\, e^{-\zeta y}(y^2 - 1)^{j-1/2} \tag{40}$$

$$= \frac{2^j}{\sqrt{\pi}}\zeta^{-j-1}\Gamma(j + \frac{1}{2})\left[(a_\pm^2 + 1)\zeta K_j(\zeta) + (2j + 1 - 2a_\pm\zeta)K_{j+1}(\zeta)\right].$$

$$S_{(j,0)}(\zeta) = \int_0^{2\pi} d\beta e^{\zeta \cos \beta} \sin^{2j} \beta = 2\sqrt{\pi}\Gamma(j + \frac{1}{2})\,_0\widetilde{F}_1(j + 1; \frac{\zeta^2}{4}), \tag{41}$$

$$S_{(j,2)}(a_\pm, \zeta) = \int_0^{2\pi} d\beta\, e^{\zeta \cos \beta} \sin^{2j} \beta(\cos \beta + a_\pm)^2$$

$$= \frac{2^{j+1}}{\zeta}\sqrt{\pi}(-\zeta)^j(-\zeta^2)^{-j}\left[(1 + a_\pm^2)\zeta I_j(\zeta) + (2a_\pm\zeta - 2j - 1)I_{j+1}(\zeta)\right], \tag{42}$$

where in the two first integrals we have introduced the transformation $\cosh \alpha = y$, which implies that $(y^2 - 1)^{-1/2}dy = d\alpha$. $K_j(\zeta)$ is the modified Bessel function of the second kind, $I_j(\zeta)$ is the modified Bessel function of the first kind and $_0\widetilde{F}_1$ is the regularized confluent hypergeometric function [2] and

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$$a_{\pm} = \frac{1 \pm \sqrt{1 + 4\zeta^2}}{\zeta} . \tag{43}$$

For the harmonic oscillator solutions, we must take care of the following integrals

$$W_j = \int_0^{2\pi} d\beta \; e^{-\zeta_0 \cos(2\beta)} \cos^j \beta$$

$$= \frac{1}{2}\sqrt{\pi}(1 + (-1)^j)^2 \, e^{\zeta} \, \Gamma\left(\frac{j+1}{2}\right) \, {}_1\widetilde{F}_1\left(\frac{j+1}{2}; \frac{j+2}{2}; -2\zeta\right) , \tag{44}$$

where ${}_1\widetilde{F}_1(a; b; z)$ is the regularized Hypergeometric function and $\Gamma(\frac{j+1}{2})$ is the Gamma function. The other integral to be computed reads

$$V_j = \int_0^\infty d\alpha \cosh^j \alpha e^{-\zeta_0 \cosh 2\alpha} = \int_1^\infty dy \frac{e^{-\zeta_0 y}}{\sqrt{y^2 - 1}} \left(\frac{y+1}{2}\right)^j , \tag{45}$$

which cannot be solved for general j, but for specific values, exact results can be found. In this manuscript these results read

$$V_0 = \frac{K_0(\zeta_0)}{2} , \qquad\qquad V_1 = \frac{1}{2}\sqrt{\frac{\pi}{2\zeta_0}} e^{-\zeta_0} , \tag{46}$$

$$V_2 = \frac{1}{4}\left[K_0(\zeta_0) + K_1(\zeta_0)\right] , \qquad\qquad V_3 = \sqrt{\frac{\pi}{2}} \frac{(1 + 4\zeta_0)}{8\zeta_0^{3/2}} e^{-\zeta_0} , \tag{47}$$

$$V_4 = \frac{1}{8}\left[2K_0(\zeta_0) + \left(\frac{1 + 2\zeta_0}{\zeta_0}\right)K_1(\zeta_0)\right] , \qquad V_5 = \sqrt{\frac{\pi}{2}} \frac{(3 + 8\zeta_0(1 + 2\zeta_0))}{32\zeta_0^{5/2}} e^{-\zeta_0} , \tag{48}$$

$$V_6 = \frac{1}{16}\left[\left(\frac{1 + 4\zeta_0}{\zeta_0}\right) K_0(\zeta_0) + \left(\frac{2 + \zeta_0(3 + 4\zeta_0)}{\zeta_0^2}\right)K_1(\zeta_0)\right] . \tag{49}$$

References

[1] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals Series and Products* (Academic, London, 1965).

[2] M. Abramovic, I.A. Stegun, *Handbook of Mathematical Functions* (US National Bureau of Standards, Applied Mathematical Series 55, Washington D.C., 1964).