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The anisotropic oscillator on curved spaces: A new exactly solvable model

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ABSTRACT

We present a new exactly solvable (classical and quantum) model that can be interpreted as the generalization to the two-dimensional sphere and to the hyperbolic space of the two-dimensional anisotropic oscillator with any pair of frequencies ω_x and ω_y. The new curved Hamiltonian H_κ depends on the curvature κ of the underlying space as a deformation/contraction parameter, and the Liouville integrability of H_κ relies on its separability in terms of geodesic parallel coordinates, which generalize the Cartesian coordinates of the plane. Moreover, the system is shown to be superintegrable for commensurate frequencies $\omega_x : \omega_y$, thus mimicking the behaviour of the flat Euclidean case, which is always recovered in the $\kappa \to 0$ limit. The additional constant of motion in the commensurate case is, as expected, of higher-order in the momenta and can be explicitly deduced by performing the classical factorization of the Hamiltonian. The known $1 : 1$ and $2 : 1$ anisotropic curved oscillators are recovered as particular cases of H_κ, meanwhile all the remaining $\omega_x : \omega_y$ curved oscillators define new superintegrable systems. Furthermore, the quantum Hamiltonian $\hat{H}_\kappa$ is fully constructed and studied by following a quantum factorization approach. In the case of commensurate frequencies, the Hamiltonian $\hat{H}_\kappa$ turns out to be quantum superintegrable and leads to a new exactly solvable quantum model. Its corresponding spectrum, that exhibits a maximal degeneracy, is explicitly given as

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an analytical deformation of the Euclidean eigenvalues in terms of both the curvature κ and the Planck constant $\hbar$. In fact, such spectrum is obtained as a composition of two one-dimensional (either trigonometric or hyperbolic) Pösch–Teller set of eigenvalues.

1. Introduction

The aim of this paper is to present a (classical and quantum) integrable generalization on the sphere $\mathbf{S}^2$ and on the hyperbolic plane $\mathbf{H}^2$ of a unit mass two-dimensional anisotropic oscillator Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(\omega_x^2 x^2 + \omega_y^2 y^2), \tag{1.1}$$

where $(x, y) \in \mathbb{R}^2$ are Cartesian coordinates, (p_x, p_y) their conjugate momenta and the frequencies (ω_x, ω_y) are arbitrary real numbers.

It is well-known that the Euclidean system (1.1) is always integrable (in the Liouville sense) due to its obvious separability in Cartesian coordinates. On the other hand, for commensurate frequencies $\omega_x : \omega_y$ the Hamiltonian (1.1) defines a superintegrable oscillator, since an "additional" integral of motion of higher-order in the momenta arises (see [1–3] and references therein). We recall that in the classical case the superintegrability property ensures that all the bounded trajectories are closed, thus leading to Lissajous curves, while in the quantum case superintegrability gives rise to maximal degeneracy of the spectrum.

To the best of our knowledge, the only two known generalizations on the sphere and the hyperbolic space of the superintegrable anisotropic oscillator (1.1) are the 1 : 1 and 2 : 1 cases. In the classical case, both systems arise within the classification of superintegrable systems on $\mathbf{S}^2$ and $\mathbf{H}^2$ that are endowed with constants of motion that are quadratic in the momenta (see [4–8]).

In this paper, we present the generalization of this result for arbitrary commensurate frequencies $\omega_x : \omega_y$, and the classical and quantum superintegrability of the proposed Hamiltonian H_κ, where κ stands for the curvature of the surface, will be proven by making use of a factorization approach. We remark that some preliminary results on the classical Hamiltonian H_κ have recently been anticipated in [9]. The key point of our approach is that, in the same way as Cartesian coordinates are the natural ones to write the anisotropic oscillator in the Euclidean plane, so are the geodesic parallel coordinates on constant curvature surfaces to express the curved anisotropic oscillators. Moreover, in order to show that our systems consist in a deformation of the Euclidean anisotropic oscillator we have adopted a notation depending explicitly on the curvature parameter κ. For $\kappa > 0$ we have the system defined on the sphere, for $\kappa < 0$ on the hyperboloid, while in the limit $\kappa \to 0$ we will recover all the well-known Euclidean results.

In Sections 2 and 3, we start by reviewing the integrability properties of classical and quantum anisotropic oscillators on the Euclidean plane $\mathbf{E}^2$ through the factorization approach introduced in [10–15] (see also [16–18] and references therein). In Section 4, we will revisit the well-known 1 : 1 and 2 : 1 curved oscillators [6,19,20], and we will show that both Hamiltonians can be written in a simple and unified way if we express them in terms of the so called geodesic parallel coordinates on $\mathbf{S}^2$ and $\mathbf{H}^2$ (see [6,21–23]). This fact turns out to be the keystone for the generalization of the system in the case of arbitrary frequencies, which is presented in Section 5 and is shown to be superintegrable through a factorizaton approach similar to that used in the Euclidean case.

The quantization of the previous result is presented in Section 6, where the ladder and shift operators for $\hat{H}_\kappa$ are explicitly constructed. From them, higher-order symmetries leading to the quantum superintegrability are straightforwardly obtained. Sections 7 and 8 are devoted to a detailed analysis of the spectral problem of the quantum commensurate oscillator on the sphere and the hyperbolic space, respectively. In particular, the maximal degeneracy of the energy levels will be explicitly shown. A final section including some remarks and open problems close the paper, and some technical tools or proofs that are needed along the paper have been included in the Appendices.

2. The classical factorization method

In this section we review the Hamiltonian (1.1) from the classical factorization viewpoint introduced in [15–18]. Although the results here presented are quite elementary, they are useful in order to present the approach that we will follow in the curved case. If we denote

$$\omega_x = \gamma\omega_y, \qquad \omega_y = \omega, \qquad \gamma \in \mathbb{R}^+/\{0\}, \tag{2.1}$$

the Hamiltonian (1.1) can be rewritten, in terms of the parameter γ and the frequency ω, as

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{\omega^2}{2}\left((\gamma x)^2 + y^2\right). \tag{2.2}$$

Now, we introduce the new canonical variables

$$\xi = \gamma x, \qquad p_\xi = p_x/\gamma, \qquad \xi \in \mathbb{R}, \tag{2.3}$$

yielding

$$H = \frac{1}{2}\,p_y^2 + \frac{\omega^2}{2}\,y^2 + \gamma^2\left(\frac{1}{2}\,p_\xi^2 + \frac{\omega^2}{2\gamma^2}\,\xi^2\right). \tag{2.4}$$

The two one-dimensional Hamiltonians H^ξ and H^y given by

$$H^\xi = \frac{1}{2}\,p_\xi^2 + \frac{\omega^2}{2\gamma^2}\xi^2, \qquad H^y = \frac{1}{2}\,p_y^2 + \frac{\omega^2}{2}y^2, \qquad H = H^y + \gamma^2 H^\xi, \tag{2.5}$$

are indeed two integrals of the motion for H, since $\{H, H^\xi\} = \{H, H^y\} = \{H^\xi, H^y\} = 0$.

The factorization approach is based on the definition of the so-called "ladder functions" $B^\pm$, which are obtained by requiring that $H^\xi = B^+B^-$, and take the expression

$$B^\pm = \mp\frac{i}{\sqrt{2}}\,p_\xi + \frac{1}{\sqrt{2}}\frac{\omega}{\gamma}\,\xi. \tag{2.6}$$

They lead to the Poisson algebra

$$\{H^\xi, B^\pm\} = \mp i\,\frac{\omega}{\gamma}\,B^\pm, \qquad \{B^-, B^+\} = -i\,\frac{\omega}{\gamma}.$$

Therefore, the functions $(H^\xi, B^\pm, 1)$ generate the harmonic oscillator Poisson–Lie algebra $\mathfrak{h}_4$.

On the other hand, the "shift functions" $A^\pm$ also arise by imposing that $H^y = A^+A^-$, thus yielding

$$A^\pm = \mp\frac{i}{\sqrt{2}}\,p_y - \frac{\omega}{\sqrt{2}}\,y. \tag{2.7}$$

The four functions $(H^\xi, A^\pm, 1)$ span again the Poisson–Lie algebra $\mathfrak{h}_4$, since

$$\{H^y, A^\pm\} = \pm i\omega A^\pm, \qquad \{A^-, A^+\} = i\omega.$$

We stress that in this flat model the terms "ladder" and "shift" are fully equivalent and could be interchanged. However, in the curved cases both sets of functions will no longer be equivalent.

Consequently, the Hamiltonian (2.4) can be rewritten in terms of the above ladder and shift functions as

$$H = A^+A^- + \gamma^2 B^+B^-, \qquad \{H, B^\pm\} = \mp i\gamma\omega B^\pm, \qquad \{H, A^\pm\} = \pm i\omega A^\pm.$$

The remarkable point now is that if we consider a rational value for γ,

$$\gamma = \frac{\omega_x}{\omega_y} = \frac{m}{n}, \quad m, n \in \mathbb{N}^*, \tag{2.8}$$

we obtain two complex constants of motion $X^{\pm}$ for H (2.4) such that

$$\{H, X^{\pm}\} = 0,$$

where we have defined

$$X^{\pm} = (B^{\pm})^n (A^{\pm})^m, \qquad \bar{X}^{+} = X^{-}, \tag{2.9}$$

being $\bar{X}^{+}$ the complex conjugate of X^{+}. The four constants of motion so obtained $(H^{\xi}, H^{y}, X^{\pm})$, are not functionally independent since from the factorization properties of $A^{\pm}$ and $B^{\pm}$ it can be proven that

$$X^{+}X^{-} = (H^{\xi})^n (H^{y})^m.$$

In fact, these four functions generate a polynomial Poisson algebra, namely,

$$\begin{aligned}
&\{H^{\xi}, X^{\pm}\} = \mp i\omega \frac{n^2}{m} X^{\pm}, \qquad \{H^{y}, X^{\pm}\} = \pm i\omega\, m X^{\pm}, \\
&\{X^{+}, X^{-}\} = -i\frac{\omega}{m} (H^{\xi})^{n-1} (H^{y})^{m-1} \left(m^3 H^{\xi} - n^3 H^{y}\right).
\end{aligned} \tag{2.10}$$

Notice that the integrals of motion (2.9) are of $(m+n)$th-order in the momenta and, since $X^{\pm}$ are complex functions we can get real constants of motion given by

$$X = \frac{1}{2}(X^{+} + X^{-}), \qquad Y = \frac{1}{2i}(X^{+} - X^{-}). \tag{2.11}$$

The degree in the momenta for one of them is $(m+n)$ and $(m+n-1)$ for the other one. From (2.10) we find the following polynomial algebra of real symmetries

$$\begin{aligned}
&\{H^{\xi}, X\} = \omega \frac{n^2}{m} Y, \qquad \{H^{\xi}, Y\} = -\omega \frac{n^2}{m} X, \\
&\{H^{y}, X\} = -\omega\, mY, \qquad \{H^{y}, Y\} = \omega\, mX, \\
&\{X, Y\} = \frac{\omega}{2m} (H^{\xi})^{n-1} (H^{y})^{m-1} \left(m^3 H^{\xi} - n^3 H^{y}\right),
\end{aligned} \tag{2.12}$$

which is generated by the four real integrals (H^{ξ}, H^{y}, X, Y).

In this way, we have recovered all the well-known results on the (super)integrability of anisotropic oscillators [1–3] which can be summarized as follows.

Theorem 1. (i) *The Hamiltonian H (2.4) is integrable for any value of the real parameter γ, since it is endowed with a quadratic constant of motion given by either H^{ξ} or H^{y} (2.5).*

(ii) *When $\gamma = m/n$ is a rational parameter (2.8), the Hamiltonian (2.4) defines a superintegrable anisotropic oscillator with commensurate frequencies $\omega_x : \omega_y$ and the additional constant of motion is given by either X or Y in (2.11). The sets (H, H^{ξ}, X) and (H, H^{ξ}, Y) are formed by three functionally independent functions.*

2.1. The 1 : 1 oscillator

If $\gamma = 1$, we can set $m = n = 1$ such that $\omega_x = \omega_y = \omega$ and (2.3) gives $\xi = x$ and $p_\xi = p_x$. Hence, we recover the isotropic oscillator

$$H^{1:1} = \frac{1}{2}(p_x^2 + p_y^2) + \frac{\omega^2}{2}(x^2 + y^2),$$

and the integrals (2.11) reduce to

$$X = -\tfrac{1}{2}(p_x p_y + \omega^2 xy), \qquad Y = -\tfrac{1}{2}\omega(xp_y - yp_x). \tag{2.13}$$

Therefore, the quadratic integral X is one of the components of the Demkov–Fradkin tensor [24,25], meanwhile Y is proportional to the angular momentum

$$J = xp_y - yp_x. \tag{2.14}$$

Since $m+n$ is even, the symmetry with highest degree is X and the lowest one is given by Y.

2.2. The 2 : 1 oscillator

For $\gamma = 2$, we take $m = 2$ and $n = 1$. Thus, $\omega_x = 2\omega_y = 2\omega$, $\xi = 2x$ and $p_\xi = p_x/2$. The Hamiltonian (2.4) and the integrals (2.11) read

$$\begin{aligned}
H^{2:1} &= \frac{1}{2}p_y^2 + \frac{\omega^2}{2}y^2 + 4\left(\frac{1}{2}p_\xi^2 + \frac{\omega^2}{8}\xi^2\right) = \frac{1}{2}(p_x^2+p_y^2) + \frac{\omega^2}{2}\left(4x^2+y^2\right),\\
X &= -\frac{\omega}{4\sqrt{2}}\left(p_y(\xi p_y - 4yp_\xi) - \omega^2\xi y^2\right) = -\frac{\omega}{2\sqrt{2}}\left(p_y J - \omega^2 xy^2\right),\\
Y &= \frac{1}{2\sqrt{2}}\left(p_\xi p_y^2 + \omega^2 y(\xi p_y - yp_\xi)\right) = \frac{1}{4\sqrt{2}}\left(p_x p_y^2 + \omega^2 y(4xp_y - yp_x)\right).
\end{aligned} \tag{2.15}$$

The quadratic symmetry X, which involves the angular momentum J (2.14), is the constant considered in the literature (see e.g. [6,26]), whilst Y is a cubic integral. In this sense, the 2 : 1 oscillator can be considered as a superintegrable system with *quadratic* constants of motion. In fact, the 1 : 1 and 2 : 1 oscillators are the only anisotropic Euclidean oscillators endowed with quadratic integrals (see the classifications [6,27]), and all the remaining $m : n$ oscillators have higher-order symmetries. In this case, since $m+n$ is odd, the highest $(m+n)$-degree symmetry is Y while X is of lowest order $(m+n-1)$.

2.3. The 1 : 3 oscillator

In this case, we have that $\gamma = 1/3$, $m = 1$, $n = 3$, $\omega_x = \omega_y/3 = \omega/3$, $\xi = x/3$ and $p_\xi = 3p_x$. The Hamiltonian (2.4) is now given by

$$H^{1:3} = \frac{1}{2}p_y^2 + \frac{\omega^2}{2}y^2 + \frac{1}{9}\left(\frac{1}{2}p_\xi^2 + \frac{9\omega^2}{2}\xi^2\right) = \frac{1}{2}(p_x^2+p_y^2) + \frac{\omega^2}{2}\left(\frac{1}{9}x^2 + y^2\right),$$

and the integrals (2.11) are

$$\begin{aligned}
X &= \frac{1}{4}\left(27p_x^3p_y - 9\omega^2 xp_x(xp_y - 3yp_x) - \omega^4x^3y\right),\\
Y &= \frac{\omega}{4}\left(27p_x^2J - \omega^2x^2(xp_y - 9yp_x)\right).
\end{aligned}$$

Since the symmetry X is quartic in the momenta but Y (that includes the angular momentum (2.14)) is cubic, H is a cubic superintegrable system.

Notice that, obviously, the 1 : 2 and 3 : 1 oscillators with $\gamma = 1/2$ and $\gamma = 3$ define equivalent systems to the previous oscillators via the interchange $x \leftrightarrow y$. And, clearly, any $m : n$ oscillator (with γ) is equivalent to the $n : m$ one (with $1/\gamma$). Surprisingly enough, this (trivial) fact from the Euclidean viewpoint will no longer hold when the curvature of the space is non-vanishing, as we will explicitly show in Section 5.

3. The quantum factorization method

In order to study the quantum analogue of the Hamiltonian (1.1), let us introduce the standard definitions for quantum position $(\hat{x}, \hat{y})$ and momentum $(\hat{p}_x, \hat{p}_y)$ operators

$$\hat{x}\Psi(x,y) = x\Psi(x,y), \qquad \hat{p}_x\Psi(x,y) = -i\hbar\,\frac{\partial\Psi(x,y)}{\partial x}, \qquad [\hat{x}, \hat{p}_x] = i\hbar,$$

and similarly for $(\hat{y}, \hat{p}_y)$. Hereafter, the hat will be suppressed for the $\hat{x}, \hat{y}$ position operators to simplify the presentation. Hence, as it is well-known, the quantum version of the Hamiltonian (1.1) reads

$$\hat{H} = \frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}(\omega_x^2 x^2 + \omega_y^2 y^2) = -\frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}(\omega_x^2 x^2 + \omega_y^2 y^2).$$

By introducing the frequency ω (2.1) and the new variable ξ (2.3) we find that

$$\hat{H} = -\frac{\hbar^2}{2}\,\frac{\partial^2}{\partial y^2} + \frac{\omega^2}{2}\,y^2 + \gamma^2\left(-\frac{\hbar^2}{2}\,\frac{\partial^2}{\partial \xi^2} + \frac{\omega^2}{2\gamma^2}\,\xi^2\right), \tag{3.1}$$

and the corresponding eigenvalue equation is given by

$$\hat{H}\,\Psi(\xi, y) = E\,\Psi(\xi, y).$$

From (3.1) we get the one-dimensional Hamiltonian operators

$$\hat{H}^\xi = -\frac{\hbar^2}{2}\,\frac{\partial^2}{\partial \xi^2} + \frac{\omega^2}{2\gamma^2}\,\xi^2, \qquad \hat{H}^y = -\frac{\hbar^2}{2}\,\frac{\partial^2}{\partial y^2} + \frac{\omega^2}{2}\,y^2, \qquad \hat{H} = \hat{H}^y + \gamma^2\hat{H}^\xi, \tag{3.2}$$

such that $[\hat{H}, \hat{H}^\xi] = [\hat{H}, \hat{H}^y] = [\hat{H}^\xi, \hat{H}^y] = 0$.

Now, we look for factorized solutions, $\Psi(\xi, y) = \Xi(\xi)\,Y(y)$, where the component functions $\Xi(\xi)$ and $Y(y)$ satisfy the following one-dimensional eigenvalue equations

$$\hat{H}^\xi\;\Xi(\xi) = E^\xi\;\Xi(\xi), \qquad \hat{H}^y\,Y(y) = E^y\,Y(y).$$

The factorizations of these systems are the standard ones

$$\hat{H}^\xi = \hat{B}^+\hat{B}^- + \lambda^B, \qquad \hat{H}^y = \hat{A}^+\hat{A}^- + \lambda^A, \tag{3.3}$$

and yield the following ladder

$$\hat{B}^\pm = \mp\frac{\hbar}{\sqrt{2}}\,\frac{\partial}{\partial\xi} + \frac{\omega}{\sqrt{2}\gamma}\,\xi, \qquad \lambda^B = \frac{\hbar\omega}{2\gamma},$$

and shift operators

$$\hat{A}^\pm = \mp\frac{\hbar}{\sqrt{2}}\,\frac{\partial}{\partial y} - \frac{\omega}{\sqrt{2}}\,y, \qquad \lambda^A = -\frac{\hbar\omega}{2}.$$

The commutation rules of the two sets of operators $(\hat{H}^\xi, \hat{B}^\pm)$ and $(\hat{H}^y, \hat{A}^\pm)$ read

$$[\hat{H}^\xi, \hat{B}^\pm] = \pm\frac{\hbar\omega}{\gamma}\hat{B}^\pm, \qquad [\hat{B}^-, \hat{B}^+] = \frac{\hbar\omega}{\gamma},$$
$$[\hat{H}^y, \hat{A}^\pm] = \mp\hbar\omega\hat{A}^\pm, \qquad [\hat{A}^-, \hat{A}^+] = -\hbar\omega,$$

and each set generates the harmonic oscillator Lie algebra $\mathfrak{h}_4$. Hence, we find that

$$\hat{H}^\xi = \hat{B}^+\hat{B}^- + \frac{\hbar\omega}{2\gamma} = \hat{B}^-\hat{B}^+ - \frac{\hbar\omega}{2\gamma}, \qquad \hat{H}^y = \hat{A}^+\hat{A}^- - \frac{\hbar\omega}{2} = \hat{A}^-\hat{A}^+ + \frac{\hbar\omega}{2}. \tag{3.4}$$

The eigenvalues of $\hat{H}^\xi$ and $\hat{H}^y$ corresponding to the respective eigenfunctions $\Xi^\mu(\xi)$ and $Y^\nu(y)$ turn out to be

$$E^{\xi,\mu} = \frac{\hbar\omega}{2\gamma} + \mu\,\frac{\hbar\omega}{\gamma}, \qquad E^{y,\nu} = \frac{\hbar\omega}{2} + \nu\,\hbar\omega, \quad \mu,\nu = 0,1,2,\ldots. \tag{3.5}$$

Next, according to (3.2) and (3.4) the Hamiltonian (3.1) can be written as

$$\hat{H} = \frac{1}{2}\big(\hat{A}^+\hat{A}^- + \hat{A}^-\hat{A}^+\big) + \frac{\gamma^2}{2}\big(\hat{B}^+\hat{B}^- + \hat{B}^-\hat{B}^+\big),$$

where

$$[\hat{H},\hat{B}^\pm] = \pm\gamma\hbar\omega\hat{B}^\pm, \qquad [\hat{H},\hat{A}^\pm] = \mp\hbar\omega\hat{A}^\pm.$$

Finally, the eigenvalue of the wave function $\Psi^{\mu,\nu}(\xi,y) = \Xi^\mu(\xi)Y^\nu(y)$, corresponding to (3.5) reads

$$E^{\mu,\nu} = E^{y,\nu} + \gamma^2 E^{\xi,\mu} = \hbar\omega\left(\tfrac{1}{2}(\gamma+1) + \gamma\mu + \nu\right), \quad \mu,\nu = 0,1,2,\ldots. \tag{3.6}$$

Furthermore, if $\gamma = m/n$, with $m, n \in \mathbb{N}^*$, as in the classical case, we obtain "additional" higher-order symmetries for $\hat{H}$ (3.1), beyond $\hat{H}^\xi$ and $\hat{H}^y$, since the operators

$$\hat{X}^\pm = (\hat{B}^\pm)^n\,(\hat{A}^\pm)^m \qquad \text{are such that} \qquad [\hat{H},\hat{X}^\pm] = 0. \tag{3.7}$$

From the factorization property (3.3) of $\hat{A}^\pm$ and $\hat{B}^\pm$ we find that

$$\hat{X}^+\hat{X}^- = \left(\hat{H}^\xi - \frac{\hbar\omega}{2\gamma}\right)^n \left(\hat{H}^y + \frac{\hbar\omega}{2}\right)^m.$$

Therefore, the four constants of motion $(\hat{H}^y, \hat{H}^\xi, \hat{X}^\pm)$ are algebraically dependent, as they should be. On the other hand, the sets $(\hat{H}^y, \hat{H}^\xi, \hat{X}^+)$ or $(\hat{H}^y, \hat{H}^\xi, \hat{X}^-)$ are algebraically independent. Notice that, obviously, we could also consider the quantum observables $\hat{X}$ and $\hat{Y}$ defined by following (2.11) and therefore the sets $(\hat{H}^y, \hat{H}^\xi, \hat{X}^\pm)$ or $(\hat{H}^y, \hat{H}^\xi, \hat{X}, \hat{Y})$ will close a polynomial symmetry algebra similar to (2.10) or (2.12).

As a straightforward consequence, if $\gamma = m/n$ is a rational parameter (2.8), the energy levels of $\hat{H}$ will be degenerate, since (3.6) can be written in the form

$$E^{\mu,\nu} = \hbar\omega\left(\frac{1}{2}\left(\frac{m}{n} + 1\right) + \frac{m\,\mu + n\,\nu}{n}\right), \quad \mu,\nu = 0,1,2,\ldots$$

and the energy will be the same for all pairs (μ,ν) of quantum numbers for which $m\,\mu + n\,\nu$ takes the same value. Moreover, the corresponding eigenstates are connected by means of the $\hat{X}^\pm$ operators.

Summarizing, the quantum counterpart of Theorem 1 can be stated as follows.

Theorem 2. (i) *The Hamiltonian $\hat{H}$ (3.1) commutes with the operators $\hat{H}^\xi$ and $\hat{H}^y$ (3.2) and defines an integrable quantum system for any value of the real parameter γ. The discrete spectrum of $\hat{H}$ depends on two quantum numbers and is given by $E^{\mu,\nu}$ (3.6).*

(ii) *Whenever $\gamma = m/n$ is a rational parameter, the Hamiltonian $\hat{H}$ commutes with the operators $\hat{X}^\pm$ (3.7). The sets $(\hat{H}, \hat{H}^\xi, \hat{X}^+)$ and $(\hat{H}, \hat{H}^\xi, \hat{X}^-)$ are formed by three algebraically independent operators. Therefore, the quantum anisotropic oscillator with commensurate frequencies $\omega_x : \omega_y$ is a superintegrable quantum model, and the spectrum $E^{\mu,\nu}$ is degenerate.*

4. The (known) 1 : 1 and 2 : 1 curved oscillators

The 1 : 1 and 2 : 1 cases are the only superintegrable Euclidean oscillators with integrals of motion that are quadratic in the momenta, and they are the only ones whose corresponding curved generalizations are well-known. In this section we will recall both of them, and we will show that by rewriting these two Hamiltonians in terms of geodesic parallel coordinates we will find the keystone in order to propose the curved analogue of the generic $m : n$ oscillator.

The isotropic 1 : 1 oscillator is usually expressed in geodesic polar coordinates since in this way the isotropic oscillator potential on $\mathbf{S}^2$ (characterized by the curvature $\kappa = 1$) and $\mathbf{H}^2$ (with $\kappa = -1$) is simply written in terms of the functions $\tan^2 r$ and $\tanh^2 r$, respectively, where the variable r is just the geodesic distance from the particle to the centre of force. In this way, the first-order expansion for both potentials around $r = 0$ gives the Euclidean potential function r^2. This system is also known as the Higgs oscillator [28,29] and has been widely studied (see [4–6,19,30–34] and references therein).

In order to be able to consider simultaneously the two curved spaces, and to take the Euclidean limit as the zero curvature case, throughout the paper we will make use of the κ-dependent cosine and sine functions defined by

$$
\mathrm{C}_\kappa(u) \equiv \sum_{l=0}^{\infty}(-\kappa)^l \frac{u^{2l}}{(2l)!} = \begin{cases} \cos\sqrt{\kappa}\,u & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh\sqrt{-\kappa}\,u & \kappa < 0, \end{cases}
$$

$$
\mathrm{S}_\kappa(u) \equiv \sum_{l=0}^{\infty}(-\kappa)^l \frac{u^{2l+1}}{(2l+1)!} = \begin{cases} \dfrac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}\,u & \kappa > 0 \\ u & \kappa = 0 \\ \dfrac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}\,u & \kappa < 0. \end{cases}
$$

Obviously, the κ-tangent is defined by

$$
\mathrm{T}_\kappa(u) \equiv \frac{\mathrm{S}_\kappa(u)}{\mathrm{C}_\kappa(u)}.
$$

Some relations involving these κ-functions can be found in [6,21] and, more extensively, in [35]. For instance:

$$
\begin{aligned}
&\mathrm{C}_\kappa^2(u) + \kappa\,\mathrm{S}_\kappa^2(u) = 1, \qquad \mathrm{C}_\kappa(2u) = \mathrm{C}_\kappa^2(u) - \kappa\,\mathrm{S}_\kappa^2(u), \qquad \mathrm{S}_\kappa(2u) = 2\mathrm{S}_\kappa(u)\mathrm{C}_\kappa(u), \\
&\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{C}_\kappa(u) = -\kappa\,\mathrm{S}_\kappa(u), \qquad \frac{\mathrm{d}}{\mathrm{d}u}\mathrm{S}_\kappa(u) = \mathrm{C}_\kappa(u), \qquad \frac{\mathrm{d}}{\mathrm{d}u}\mathrm{T}_\kappa(u) = \frac{1}{\mathrm{C}_\kappa^2(u)}.
\end{aligned} \tag{4.1}
$$

In terms of the curvature κ and the geodesic polar coordinates (r, ϕ) the complete Higgs Hamiltonian is given by (see, for instance, [19])

$$
H_\kappa^{1:1} = \mathcal{T}_\kappa + U_\kappa^{1:1} = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{\mathrm{S}_\kappa^2(r)}\right) + \frac{\omega^2}{2}\mathrm{T}_\kappa^2(r), \tag{4.2}
$$

and the specific expressions for $\mathbf{S}^2$ ($\kappa = 1$) and $\mathbf{H}^2$ ($\kappa = -1$) are straightforwardly obtained.

On the other hand, the superintegrable 2 : 1 curved oscillator, with Hamiltonian $H_\kappa^{2:1} = \mathcal{T}_\kappa + U_\kappa^{2:1}$, was firstly introduced in the classification carried out in [6] and it has been recently studied in detail in [19,20]. In geodesic polar variables the potential $U_\kappa^{2:1}$ adopts the (rather cumbersome) expression

$$
U_\kappa^{2:1} = \frac{\omega^2}{2}\left(\frac{4\mathrm{T}_\kappa^2(r)\cos^2\phi}{\left(1 - \kappa\,\mathrm{S}_\kappa^2(r)\sin^2\phi\right)\left(1 - \kappa\,\mathrm{T}_\kappa^2(r)\cos^2\phi\right)^2} + \frac{\mathrm{S}_\kappa^2(r)\sin^2\phi}{1 - \kappa\,\mathrm{S}_\kappa^2(r)\sin^2\phi}\right). \tag{4.3}
$$

Therefore, a glimpse on (4.2) and (4.3) makes evident that the generalization of these potentials for the arbitrary $m : n$ case is far from being obvious.

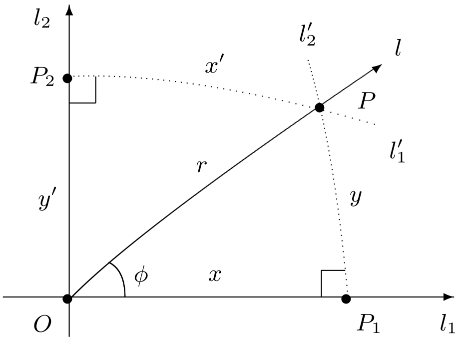

Fig. 1. Schematic representation of the geodesic coordinates (x, y), (x', y') and (r, ϕ) of a point P on a curved space.

However, things drastically change if we make use of the geodesic parallel coordinates for $\mathbf{S}^2$ and $\mathbf{H}^2$, that are described in detail in Appendix A. In order to define them, we take an origin O in the space, two base geodesics l_1, l_2 orthogonal at O and the geodesic l that joins a point P (the particle) and O (see Fig. 1). The geodesic polar coordinates (r, ϕ) are defined by the distance r between O and P measured along l and the angle ϕ of l relative to l_1. Let P_1 be the intersection point of l_1 with its orthogonal geodesic l_2' through P. Then, the geodesic parallel coordinates (x, y) are just defined by the distance x between O and P_1 measured along l_1 and the distance y between P_1 and P measured along l_2'. Notice that a similar set of coordinates (x', y') can also be formed by considering the intersection point P_2 of l_2 with its orthogonal geodesic l_1' through P and that generally $(x, y) \neq (x', y')$ if $\kappa \neq 0$ (see [6,21]). It is straightforward to realize that on $\mathbf{E}^2$ with $\kappa = 0$, the coordinates $(x, y) = (x', y')$ reduce to Cartesian coordinates and (r, ϕ) give the usual polar ones.

In fact, the geodesic parallel coordinates turn out to be the closest to the Cartesian ones on these two curved spaces, and they will be indeed the appropriate ones in order to write the curved anisotropic oscillator Hamiltonians. This statement can be made evident if we consider the known curved 1 : 1 and 2 : 1 cases. Namely, if we apply to (4.2) the relations (A.3) and the first identity given in (4.1), we obtain two equivalent forms for the potential $U_\kappa^{1:1}$ written in geodesic parallel coordinates (see [6]):

$$U_\kappa^{1:1} = \frac{\omega^2}{2}\left(\mathrm{T}_\kappa^2(x) + \frac{\mathrm{T}_\kappa^2(y)}{\mathrm{C}_\kappa^2(x)}\right) = \frac{\omega^2}{2}\left(\frac{\mathrm{T}_\kappa^2(x)}{\mathrm{C}_\kappa^2(y)} + \mathrm{T}_\kappa^2(y)\right). \tag{4.4}$$

By using the same transformation, the potential $U_\kappa^{2:1}$ (4.3) is shown to take the following expression in terms of geodesic parallel coordinates

$$U_\kappa^{2:1} = \frac{\omega^2}{2}\left(\frac{\mathrm{T}_\kappa^2(2x)}{\mathrm{C}_\kappa^2(y)} + \mathrm{T}_\kappa^2(y)\right). \tag{4.5}$$

As a consequence, from both expressions $U_\kappa^{1:1}$ (4.4) and $U_\kappa^{2:1}$ (4.5) it is natural to propose the following expression for the generic curved anisotropic oscillator potential:

$$U_\kappa^{\gamma} = \frac{\omega^2}{2}\left(\frac{\mathrm{T}_\kappa^2(\gamma x)}{\mathrm{C}_\kappa^2(y)} + \mathrm{T}_\kappa^2(y)\right), \quad \gamma \in \mathbb{R}^+/\{0\}. \tag{4.6}$$

In the next two sections, we will show that this Ansatz is correct, since (4.6) provides an integrable system both in the classical and quantum contexts, that can be exactly solved by making use of the factorization approach in terms of geodesic parallel coordinates. Moreover, in the commensurate case $\gamma = m/n$ the system turns out to be superintegrable due to the existence of an additional symmetry. We also remark that, generically, U_κ^{γ} and $U_\kappa^{1/\gamma}$ will define two different systems, in contradistinction with what happens in the Euclidean case.

5. The generic classical curved anisotropic oscillator

Let us consider the following Hamiltonian function of a particle with unit mass written in terms of geodesic parallel coordinates (x, y)

$$H_\kappa = \mathcal{T}_\kappa + U_\kappa^\gamma = \frac{1}{2}\left(\frac{p_x^2}{\mathrm{C}_\kappa^2(y)} + p_y^2\right) + \frac{\omega^2}{2}\left(\frac{\mathrm{T}_\kappa^2(\gamma x)}{\mathrm{C}_\kappa^2(y)} + \mathrm{T}_\kappa^2(y)\right), \tag{5.1}$$

where ω and γ are positive real constants, and the potential term is just (4.6). Obviously, this Hamiltonian for $\kappa = 0$ reproduces the Euclidean anisotropic oscillator (2.2), while for $\kappa > 0$ gives a system defined on $\mathbf{S}^2$ and for $\kappa < 0$ on $\mathbf{H}^2$. For $\kappa < 0$, the Hamiltonian is well defined for any real values of x and y (A.7). However, if $\kappa > 0$ in order to avoid a multivalued Hamiltonian, restrictions on the domain of the coordinates (A.6) arise from the term $\mathrm{T}_\kappa(\gamma x)$, namely:

$$\mathbf{S}^2\ (\kappa > 0):\quad -\frac{\pi}{2\sqrt{\kappa}} < \gamma x < \frac{\pi}{2\sqrt{\kappa}},\quad -\frac{\pi}{2\sqrt{\kappa}} < y < \frac{\pi}{2\sqrt{\kappa}},\quad \gamma \geq \frac{1}{2}. \tag{5.2}$$

$$\mathbf{H}^2\ (\kappa < 0):\ \ x, y \in \mathbb{R},\quad \gamma \in \mathbb{R}^+/\{0\}. \tag{5.3}$$

By assuming that $\kappa \neq 0$ and by using the relation

$$1 + \kappa \mathrm{T}_\kappa^2(u) = 1/\mathrm{C}_\kappa^2(u), \tag{5.4}$$

which can be derived from (4.1), the Hamiltonian H_κ can be rewritten as

$$H_\kappa = \frac{p_y^2}{2} + \frac{1}{\mathrm{C}_\kappa^2(y)}\left(\frac{p_x^2}{2} + \frac{\omega^2}{2\kappa\,\mathrm{C}_\kappa^2(\gamma x)}\right) - \frac{\omega^2}{2\kappa},\quad \kappa \neq 0. \tag{5.5}$$

After introducing the new variable $\xi = \gamma x$ (2.3) with domain given by (5.2) and (5.3), the Hamiltonian (5.5) takes the form

$$H_\kappa = \frac{p_y^2}{2} + \frac{\gamma^2}{\mathrm{C}_\kappa^2(y)}\left(\frac{p_\xi^2}{2} + \frac{\omega^2}{2\kappa\gamma^2\mathrm{C}_\kappa^2(\xi)}\right) - \frac{\omega^2}{2\kappa},\quad \kappa \neq 0.$$

In this way the total Hamiltonian can be rewritten as

$$H_\kappa = \frac{p_y^2}{2} + \frac{\gamma^2 H_\kappa^\xi}{\mathrm{C}_\kappa^2(y)} - \frac{\omega^2}{2\kappa}, \tag{5.6}$$

where the constant of the motion H_κ^ξ is given by

$$H_\kappa^\xi = \frac{p_\xi^2}{2} + \frac{\omega^2}{2\kappa\gamma^2\mathrm{C}_\kappa^2(\xi)},\qquad \{H_\kappa, H_\kappa^\xi\} = 0. \tag{5.7}$$

Therefore, H_κ defines an integrable system for any value of ω and γ. We remark that the integral H_κ^ξ is, in fact, a one-dimensional Higgs-type oscillator (4.2) with "frequency" ω/γ on the variable ξ, since it can be rewritten through (5.4) as

$$H_\kappa^\xi = \frac{p_\xi^2}{2} + \frac{\omega^2}{2\gamma^2}\mathrm{T}_\kappa^2(\xi) + \frac{\omega^2}{2\kappa\gamma^2}.$$

Note that its Euclidean limit H^ξ (2.5) is recovered in the form

$$\lim_{\kappa\to 0}\left(H_\kappa^\xi - \frac{\omega^2}{2\kappa\gamma^2}\right) = \frac{1}{2}p_\xi^2 + \frac{\omega^2}{2\gamma^2}\xi^2.$$

In what follows, we will factorize the classical Hamiltonians H_κ^ξ and H_κ for a generic value of γ, by taking into account that expressions (5.6) and (5.7) correspond to classical Pöschl–Teller Hamiltonians, whose factorization properties have been previously considered in [15]. Moreover, when γ is a rational number the factorization approach will lead us to the superintegrability of the complete system (5.6).

5.1. Ladder functions

Firstly, we search for some functions $B_\kappa^\pm(\xi)$ that generate a Poisson algebra of the type

$$\{H_\kappa^\xi, B_\kappa^\pm\} = \mp i f(H_\kappa^\xi) B_\kappa^\pm,$$

for some function f. They will be the so-called *ladder* functions for H_κ^ξ, and can be found to be [15]

$$B_\kappa^\pm = \mp \frac{i}{\sqrt{2}} \mathrm{C}_\kappa(\xi)\, p_\xi + \frac{\mathcal{E}_\kappa}{\sqrt{2}} \mathrm{S}_\kappa(\xi), \tag{5.8}$$

where

$$\mathcal{E}_\kappa(p_\xi, \xi) = \sqrt{2\kappa H_\kappa^\xi}. \tag{5.9}$$

In fact, it is straightforward to check that

$$H_\kappa^\xi = B_\kappa^+ B_\kappa^- + \frac{\omega^2}{2\kappa\gamma^2}, \tag{5.10}$$

and the following Poisson algebra is obtained

$$\{H_\kappa^\xi, B_\kappa^\pm\} = \mp i\, \mathcal{E}_\kappa\, B_\kappa^\pm, \qquad \{B_\kappa^-, B_\kappa^+\} = -i\, \mathcal{E}_\kappa. \tag{5.11}$$

As a consequence,

$$\{H_\kappa, B_\kappa^\pm\} = \mp i\, \frac{\gamma^2 \mathcal{E}_\kappa}{\mathrm{C}_\kappa^2(y)}\, B_\kappa^\pm, \tag{5.12}$$

where from (5.7) we know that the function $\mathcal{E}_\kappa$ (5.9) is a constant of the motion for H_κ.

5.2. Shift functions

Next, we look for *shift* functions $A_\kappa^\pm(y)$ that factorize H_κ. This implies the search for a Poisson algebra of the type

$$\{H_\kappa, A_\kappa^\pm\} = \pm i\, g(\mathcal{E}_\kappa, y) A_\kappa^\pm,$$

for a certain function g including the potential in (5.6). By taking into account that $\mathcal{E}_\kappa$ (5.9) is a constant of the motion for H_κ (5.6) and by imposing that the Hamiltonian can be factorized in the form

$$H_\kappa = A_\kappa^+ A_\kappa^- + \lambda_\kappa^A, \tag{5.13}$$

we obtain the shift functions

$$A_\kappa^\pm = \mp \frac{i}{\sqrt{2}} p_y - \frac{\gamma \mathcal{E}_\kappa}{\sqrt{2}} \mathrm{T}_\kappa(y), \qquad \lambda_\kappa^A = \frac{1}{2\kappa} \left(\gamma^2 \mathcal{E}_\kappa^2 - \omega^2\right), \tag{5.14}$$

and the Poisson algebra

$$\{H_\kappa, A_\kappa^\pm\} = \pm i\, \frac{\gamma \mathcal{E}_\kappa}{\mathrm{C}_\kappa^2(y)}\, A_\kappa^\pm, \qquad \{A_\kappa^-, A_\kappa^+\} = i\, \frac{\gamma \mathcal{E}_\kappa}{\mathrm{C}_\kappa^2(y)}. \tag{5.15}$$

5.3. Additional symmetries

The superintegrability of the Hamiltonian H_κ for rational values $\gamma = m/n$ is now easily deduced from the factorization approach. In fact, from (5.12) and (5.15), a straightforward computation shows that the ladder and shift functions provide two additional integrals of the motion for H_κ:

$$\{H_\kappa, X_\kappa^\pm\} = 0, \qquad \text{where} \qquad X_\kappa^\pm = (B_\kappa^\pm)^n (A_\kappa^\pm)^m. \tag{5.16}$$

From the factorization properties of $B^{\pm}$ and $A^{\pm}$ given in (5.10) and (5.13), we conclude that

$$X^{+}X^{-}=\left(H_{\kappa}^{\xi}-\frac{\omega^{2}}{2\kappa\gamma^{2}}\right)^{n}\left(H_{\kappa}-\gamma^{2}\left(H_{\kappa}^{\xi}-\frac{\omega^{2}}{2\kappa\gamma^{2}}\right)\right)^{m}.$$

Therefore, the four constants of motion $(H_\kappa, H_\kappa^\xi, X_\kappa^\pm)$ are functionally dependent. However, any of the sets $(H_\kappa, H_\kappa^\xi, X_\kappa^+)$ or $(H_\kappa, H_\kappa^\xi, X_\kappa^-)$ is formed by functionally independent integrals.

In principle, the constants of motion $X_\kappa^\pm$ are complex and they include powers of the square root $\mathcal{E}_\kappa$ (5.9). We have two situations giving rise to real constants of motion X_κ and Y_κ:

(i) If $m+n$ is even we have

$$X_\kappa^\pm = \pm i\,\mathcal{E}_\kappa Y_\kappa + X_\kappa . \tag{5.17}$$

(ii) If $m+n$ is odd we find

$$X_\kappa^\pm = \mathcal{E}_\kappa X_\kappa \pm i\,Y_\kappa . \tag{5.18}$$

The symmetries X_κ and Y_κ are polynomial in the momenta, whose degrees are $m+n$ and $m+n-1$ for case (i), and $m+n-1$ and $m+n$ in case (ii), respectively [36]. It can also be proven that the algebraic structure generated by the Poisson brackets of the sets of integrals of motion $(H_\kappa, H_\kappa^\xi, X_\kappa^\pm)$ or $(H_\kappa, H_\kappa^\xi, X_\kappa, Y_\kappa)$ also gives rise to a polynomial algebra, as in the Euclidean case. Therefore, the generalization of Theorem 1 to the sphere and the hyperbolic space can be stated as follows:

Theorem 3. (i) *For any value of the real anisotropy parameter γ, the Hamiltonian H_κ (5.1) defines an integrable anisotropic curved oscillator on $\mathbf{S}^2$ and $\mathbf{H}^2$, whose (quadratic) constant of motion is given by H_κ^ξ (5.7).*

(ii) *When $\gamma = m/n$ is a rational parameter, H_κ defines a superintegrable anisotropic curved oscillator and the additional constant of motion is given by either X_κ or Y_κ in (5.17) and (5.18). The sets $(H_\kappa, H_\kappa^\xi, X_\kappa)$ and $(H_\kappa, H_\kappa^\xi, Y_\kappa)$ are formed by three functionally independent functions.*

As far as the (flat) Euclidean limit $\kappa \to 0$ is concerned we remark that, despite the expressions (5.6) and (5.7) are only defined if $\kappa \neq 0$, all the remaining ones have a well defined Euclidean limit. The latter can be achieved by taking into a account the following limits of the integrals H_κ^ξ (5.7) and $\mathcal{E}_\kappa$ (5.9)

$$\lim_{\kappa\to 0}\kappa H_\kappa^\xi = \frac{\omega^2}{2\gamma^2}, \qquad \lim_{\kappa\to 0}\mathcal{E}_\kappa = \frac{\omega}{\gamma}. \tag{5.19}$$

Hence, when $\kappa = 0$, we find that the curved Hamiltonian H_κ (5.1) reduces to H (2.4), the curved ladder functions $B_\kappa^\pm$ (5.8) to $B^\pm$ (2.6), the curved shift functions $A_\kappa^\pm$ (5.14) to $A^\pm$ (2.7), λ_κ^A to $\gamma^2 H^\xi$ (2.5), and the curved integrals $X_\kappa^\pm$ (5.16) to $X^\pm$ (2.9).

5.4. Examples

We now illustrate the results described by Theorem 3 through the particular cases with $\gamma = \{1, 2, 1/2\}$. The first two ones generalize the Euclidean oscillators presented in Sections 2.1 and 2.2, while the third one allows us to show explicitly the non-equivalence among the curved potentials U_κ^γ and $U_\kappa^{1/\gamma}$, despite of the fact that when $\kappa = 0$ both of them lead to equivalent Euclidean potentials.

5.4.1. The $\gamma = 1$ curved (Higgs) oscillator

We set $\gamma = m = n = 1$ so that $\xi = x$ and $p_\xi = p_x$. The Hamiltonian H_κ (5.1) (with potential $U_\kappa^{\gamma=1} = U_\kappa^{1:1}$ (4.4)) and also in the form (5.6) reads

$$H_\kappa^{\gamma=1} = \frac{1}{2}\left(\frac{p_x^2}{\mathrm{C}_\kappa^2(y)} + p_y^2\right) + \frac{\omega^2}{2}\left(\frac{\mathrm{T}_\kappa^2(x)}{\mathrm{C}_\kappa^2(y)} + \mathrm{T}_\kappa^2(y)\right) = \frac{p_y^2}{2} + \frac{H_\kappa^x}{\mathrm{C}_\kappa^2(y)} - \frac{\omega^2}{2\kappa},$$

where the quadratic integral $H_\kappa^x \equiv H_\kappa^\xi$ is given by

$$H_\kappa^x = \frac{p_x^2}{2} + \frac{\omega^2}{2\kappa\,\mathrm{C}_\kappa^2(x)}.$$

The polynomial integrals (5.17) turn out to be

$$\begin{aligned} X_\kappa &= -\frac{1}{2}\left(\mathrm{C}_\kappa(x)p_x p_y + \mathcal{E}_\kappa^2 \mathrm{S}_\kappa(x)\mathrm{T}_\kappa(y)\right), \\ Y_\kappa &= -\frac{1}{2}\left(\mathrm{S}_\kappa(x)p_y - \mathrm{C}_\kappa(x)\mathrm{T}_\kappa(y)p_x\right). \end{aligned} \tag{5.20}$$

Notice that the integral Y_κ is proportional to the (curved) angular momentum J_κ which in geodesic parallel and polar variables can be shown to be given by [6,21]

$$J_\kappa = \mathrm{S}_\kappa(x)p_y - \mathrm{C}_\kappa(x)\mathrm{T}_\kappa(y)p_x = p_\phi. \tag{5.21}$$

Under the flat limit $\kappa \to 0$, we recover the results of the Euclidean isotropic oscillator given in Section 2.1. In particular, the integrals X_κ and $\mathcal{E}_\kappa Y_\kappa$ from (5.20) and the curved angular momentum (5.21) reduce to (2.13) and (2.14), since $\mathcal{E}_\kappa \to \omega$.

5.4.2. The $\gamma = 2$ curved oscillator

In this case, we choose $\gamma = m = 2$ and $n = 1$ so that $\xi = 2x$ and $p_\xi = p_x/2$. Thus, the Hamiltonian H_κ (5.1) with potential $U_\kappa^{\gamma=2} = U_\kappa^{2:1}$ (4.5) is given by

$$H_\kappa^{\gamma=2} = \frac{1}{2}\left(\frac{p_x^2}{\mathrm{C}_\kappa^2(y)} + p_y^2\right) + \frac{\omega^2}{2}\left(\frac{\mathrm{T}_\kappa^2(2x)}{\mathrm{C}_\kappa^2(y)} + \mathrm{T}_\kappa^2(y)\right) = \frac{p_y^2}{2} + \frac{4H_\kappa^\xi}{\mathrm{C}_\kappa^2(y)} - \frac{\omega^2}{2\kappa}, \tag{5.22}$$

where

$$H_\kappa^\xi = \frac{p_\xi^2}{2} + \frac{\omega^2}{8\kappa\,\mathrm{C}_\kappa^2(\xi)} = \frac{p_x^2}{8} + \frac{\omega^2}{8\kappa\,\mathrm{C}_\kappa^2(2x)}.$$

From the additional symmetries $X_\kappa^\pm$ (5.16) we find that the polynomial integrals (5.18) read

$$X_\kappa = -\frac{1}{2\sqrt{2}}\left(\left[\mathrm{S}_\kappa(2x)p_y - 2\mathrm{C}_\kappa(2x)\mathrm{T}_\kappa(y)p_x\right]p_y - 4\mathcal{E}_\kappa^2 \mathrm{S}_\kappa(2x)\mathrm{T}_\kappa^2(y)\right), \tag{5.23}$$

$$Y_\kappa = \frac{1}{4\sqrt{2}}\left(\mathrm{C}_\kappa(2x)p_x p_y^2 + 4\mathcal{E}_\kappa^2 \mathrm{T}_\kappa(y)\left[2\mathrm{S}_\kappa(2x)p_y - \mathrm{C}_\kappa(2x)\mathrm{T}_\kappa(y)p_x\right]\right). \tag{5.24}$$

If $\kappa \to 0$, the limit (5.19) yields $\mathcal{E}_\kappa \to \omega/2$ and the integrals $\mathcal{E}_\kappa X_\kappa$ and Y_κ reduce to (2.15), thus reproducing the results of Section 2.2.

5.4.3. The $\gamma = 1/2$ curved oscillator

Now we fix $\gamma = 1/2$, $m = 1$ and $n = 2$. Then $\xi = x/2$, $p_\xi = 2p_x$ and the corresponding Hamiltonian H_κ (5.1) with potential $U_\kappa^{\gamma=1/2}$ (4.6) is given by

$$H_\kappa^{\gamma=1/2} = \frac{1}{2}\left(\frac{p_x^2}{\mathrm{C}_\kappa^2(y)} + p_y^2\right) + \frac{\omega^2}{2}\left(\frac{\mathrm{T}_\kappa^2(\frac{x}{2})}{\mathrm{C}_\kappa^2(y)} + \mathrm{T}_\kappa^2(y)\right) = \frac{p_y^2}{2} + \frac{H_\kappa^\xi}{4\mathrm{C}_\kappa^2(y)} - \frac{\omega^2}{2\kappa}, \tag{5.25}$$

where

$$H_\kappa^\xi = \frac{p_\xi^2}{2} + \frac{2\omega^2}{\kappa\,\mathrm{C}_\kappa^2(\xi)} = 2p_x^2 + \frac{2\omega^2}{\kappa\,\mathrm{C}_\kappa^2(\frac{x}{2})}.$$

Notice that, due to the term $\mathrm{T}_\kappa^2(\frac{x}{2})$ in the potential, the Hamiltonian (5.25) defines a different/non-equivalent curved oscillator to the previous $\gamma = 2$ case (5.22) which involved the term $\mathrm{T}_\kappa^2(2x)$ in the potential.

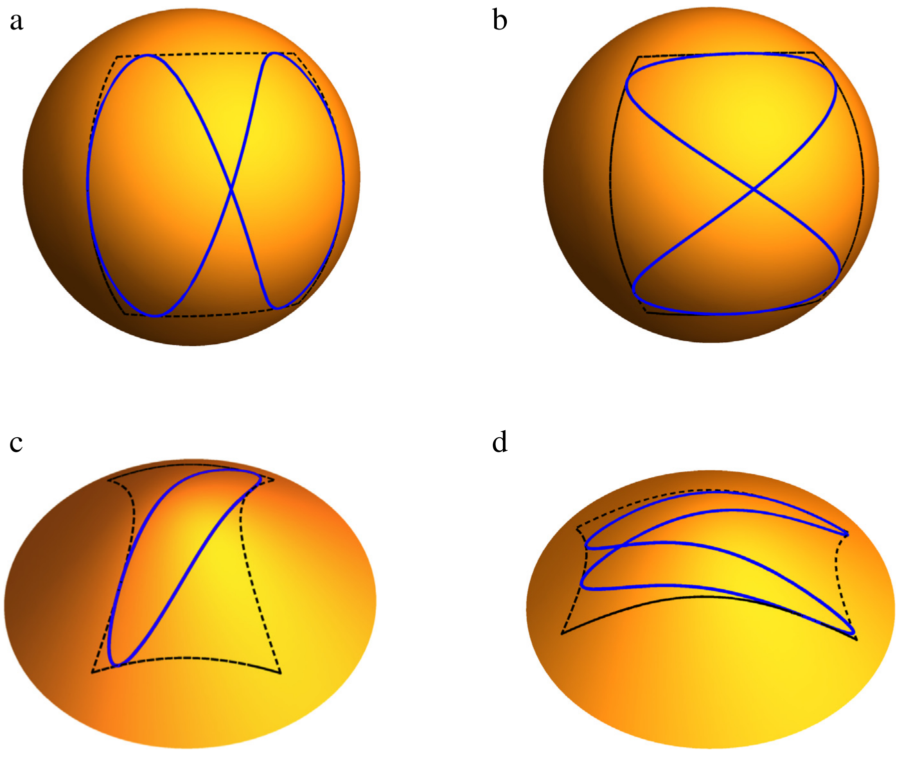

Fig. 2. Examples of trajectories for the Hamiltonian H_κ (5.1): (a) on the sphere for $\gamma = 1/2$, (b) on the sphere for $\gamma = 2$, (c) on the hyperboloid for $\gamma = 1$, and (d) on the hyperboloid for $\gamma = 2$.

The additional integrals (5.18) for $H_\kappa^{\gamma=1/2}$ read

$$X_\kappa = -\frac{1}{4\sqrt{2}} \left(4\left[\mathrm{S}_\kappa(x)p_y - \mathrm{C}_\kappa^2(\tfrac{x}{2})\mathrm{T}_\kappa(y)p_x\right]p_x + \varepsilon_\kappa^2 \mathrm{S}_\kappa^2(\tfrac{x}{2})\mathrm{T}_\kappa(y)\right), \tag{5.26}$$

$$Y_\kappa = \frac{1}{2\sqrt{2}} \left(4\mathrm{C}_\kappa^2(\tfrac{x}{2})p_x^2 p_y - \varepsilon_\kappa^2\left[\mathrm{S}_\kappa^2(\tfrac{x}{2})p_y - \mathrm{S}_\kappa(x)\mathrm{T}_\kappa(y)p_x\right]\right). \tag{5.27}$$

We finally remark that the above three anisotropic curved oscillators are the only ones within the family (5.1) which are quadratic (in the momenta) superintegrable systems. All the remaining ones with a rational γ, are also superintegrable Hamiltonians but the additional integral is always of higher-order in the momenta. Some trajectories on the sphere and on the hyperboloid for these three systems are plotted in Fig. 2.

6. Quantum anisotropic curved oscillators

In order to construct the quantum analogue of the Hamiltonian H_κ (5.1), let us consider the Laplace–Beltrami (LB) operator on a two-dimensional (curved) space

$$\Delta_{\mathrm{LB}} = \sum_{i,j=1}^{2} \frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j,$$

where g^{ij} is the inverse of the metric tensor g_{ij} and g is the determinant. In terms of the geodesic parallel and geodesic polar coordinates (see Eq. (A.4) in Appendix A) we obtain the following Laplacian operator on $\mathbf{S}^2$ and $\mathbf{H}^2$ with curvature parameter κ [21]:

$$\Delta_{\rm LB}=\frac{1}{\mathrm{C}_\kappa^2(y)}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\kappa\,\mathrm{T}_\kappa(y)\frac{\partial}{\partial y}=\frac{\partial^2}{\partial r^2}+\frac{1}{\mathrm{T}_\kappa(r)}\frac{\partial}{\partial r}+\frac{1}{\mathrm{S}_\kappa^2(r)}\frac{\partial^2}{\partial\phi^2}.$$

Next, if we assume the so-called LB quantization prescription on curved spaces (see [37–40] and references therein) the free quantum Hamiltonian of a unit mass particle will read

$$\hat{\mathcal{T}}_\kappa=-\frac{\hbar^2}{2}\Delta_{\rm LB}.$$

Now, we can define the quantum curved anisotropic oscillator Hamiltonian as $\hat{H}_\kappa=\hat{\mathcal{T}}_\kappa+\hat{U}_\kappa^\gamma$ with the potential (4.6), which in terms of geodesic parallel coordinates is given by

$$\hat{H}_\kappa=-\frac{\hbar^2}{2}\left(\frac{1}{\mathrm{C}_\kappa^2(y)}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\kappa\,\mathrm{T}_\kappa(y)\frac{\partial}{\partial y}\right)+\frac{\omega^2}{2}\left(\frac{\mathrm{T}_\kappa^2(\gamma x)}{\mathrm{C}_\kappa^2(y)}+\mathrm{T}_\kappa^2(y)\right), \tag{6.1}$$

where γ and ω are real positive parameters and the domains of x and y are shown in (5.3).

By applying the relation (5.4) to the terms $\mathrm{T}_\kappa^2(\gamma x)$ and $\mathrm{T}_\kappa^2(y)$ and after the change of variable $\xi=\gamma x$ with domain (5.2) and (5.3), the quantum Hamiltonian reads

$$\hat{H}_\kappa=-\frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2}+\frac{\hbar^2}{2}\kappa\,\mathrm{T}_\kappa(y)\frac{\partial}{\partial y}+\frac{\gamma^2}{\mathrm{C}_\kappa^2(y)}\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\xi^2}+\frac{\omega^2}{2\kappa\gamma^2\mathrm{C}_\kappa^2(\xi)}\right)-\frac{\omega^2}{2\kappa},\qquad\kappa\neq0. \tag{6.2}$$

Thus, we can write $\hat{H}_\kappa$ in terms of a one-dimensional symmetry operator $\hat{H}_\kappa^\xi$ such that $[\hat{H}_\kappa,\hat{H}_\kappa^\xi]=0$, namely

$$\hat{H}_\kappa=-\frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2}+\frac{\hbar^2}{2}\kappa\,\mathrm{T}_\kappa(y)\frac{\partial}{\partial y}+\frac{\gamma^2\hat{H}_\kappa^\xi}{\mathrm{C}_\kappa^2(y)}-\frac{\omega^2}{2\kappa},\qquad\kappa\neq0, \tag{6.3}$$

$$\hat{H}_\kappa^\xi=-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\xi^2}+\frac{\omega^2}{2\kappa\gamma^2\mathrm{C}_\kappa^2(\xi)}. \tag{6.4}$$

Remark that $\hat{H}_\kappa^\xi$, which is the quantization of (5.7), is just the quantum Pöschl–Teller Hamiltonian written simultaneously in its trigonometric ($\kappa>0$) and hyperbolic ($\kappa<0$) versions (see [41] and references therein). Note that the Euclidean oscillator in ξ is obtained under the limit

$$\lim_{\kappa\to0}\left(\hat{H}_\kappa^\xi-\frac{\omega^2}{2\kappa\gamma^2}\right)=-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\xi^2}+\frac{\omega^2}{2\gamma^2}\xi^2.$$

Now, the eigenvalue equation for $\hat{H}_\kappa$ that we want to solve is

$$\hat{H}_\kappa\Psi_\kappa(\xi,y)=E_\kappa\Psi_\kappa(\xi,y), \tag{6.5}$$

where we are looking for factorizable solutions in the form $\Psi_\kappa(\xi,y)=\Xi_\kappa^\epsilon(\xi)\,Y_\kappa^{\gamma\epsilon}(y)$. If the function $\Xi_\kappa^\epsilon(\xi)$ fulfils the eigenvalue equation

$$\hat{H}_\kappa^\xi\;\Xi_\kappa^\epsilon(\xi)=E_\kappa^\xi\;\Xi_\kappa^\epsilon(\xi),\qquad\text{with}\qquad\epsilon=\sqrt{2\kappa E_\kappa^\xi}, \tag{6.6}$$

then, the second component $Y_\kappa^{\gamma\epsilon}(y)$ of the solutions for (6.5) can be obtained as a one-dimensional eigenvalue problem

$$\hat{H}_\kappa Y_\kappa^{\gamma\epsilon}(y)=E_\kappa Y_\kappa^{\gamma\epsilon}(y) \tag{6.7}$$

for the Hamiltonian

$$\hat{H}_\kappa=-\frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2}+\frac{\hbar^2}{2}\kappa\,\mathrm{T}_\kappa(y)\frac{\partial}{\partial y}+\frac{(\gamma\epsilon)^2}{2\kappa\,\mathrm{C}_\kappa^2(y)}-\frac{\omega^2}{2\kappa},\qquad\kappa\neq0. \tag{6.8}$$

In particular, we shall deal separately with each of the two one-dimensional problems (6.6) and (6.7) by means of the factorization approach [11,15]. In this way we will find ladder operators $\hat{B}_\kappa^\pm$ for $\hat{H}_\kappa^\xi$ (6.4) and shift operators $\hat{A}_\kappa^\pm$ for $\hat{H}_\kappa$ (6.3) or (6.8). As in the classical case, we will be able to deduce "additional" quantum symmetries for $\hat{H}_\kappa$ (6.2) when γ be a rational number.

6.1. Ladder operators for the Hamiltonian $\hat{H}_\kappa^\xi$

In order to find the ladder operators $\hat{B}_\kappa^\pm$ for $\hat{H}_\kappa^\xi$ (6.4), we express the corresponding eigenvalue equation (6.6) as

$$\left(-\frac{\hbar^2}{2}\mathrm{C}_\kappa^2(\xi)\,\frac{\partial^2}{\partial\xi^2}-\mathrm{C}_\kappa^2(\xi)\,E_\kappa^\xi\right)\Xi_\kappa^\epsilon(\xi)=\left(-\frac{\omega^2}{2\kappa\gamma^2}\right)\Xi_\kappa^\epsilon(\xi).$$

Now, we define the diagonal operator $\hat{\mathcal{E}}_\kappa$ that acts on the space of eigenfunctions $\Xi_\kappa^\epsilon(\xi)$ in the form

$$\hat{\mathcal{E}}_\kappa\,\Xi_\kappa^\epsilon(\xi)=\epsilon\,\Xi_\kappa^\epsilon(\xi) \tag{6.9}$$

where ϵ was defined in (6.6). Next we define the operator

$$\hat{h}_\kappa=-\frac{\hbar^2}{2}\mathrm{C}_\kappa^2(\xi)\frac{\partial^2}{\partial\xi^2}-\mathrm{C}_\kappa^2(\xi)\,\frac{(\hat{\mathcal{E}}_\kappa)^2}{2\kappa},$$

that can be factorized in terms of two first-order $\hat{\mathcal{E}}_\kappa$-dependent differential operators plus another diagonal operator in the form

$$\hat{h}_\kappa=\hat{B}_\kappa^-\hat{B}_\kappa^++\hat{\lambda}_\kappa^B, \tag{6.10}$$

where

$$\begin{aligned}\hat{B}_\kappa^-&=\frac{\hbar}{\sqrt{2}}\mathrm{C}_\kappa(\xi)\frac{\partial}{\partial\xi}+\frac{1}{\sqrt{2}}\mathrm{S}_\kappa(\xi)\,\hat{\mathcal{E}}_\kappa,\\ \hat{B}_\kappa^+&=-\frac{\hbar}{\sqrt{2}}\mathrm{C}_\kappa(\xi)\frac{\partial}{\partial\xi}+\frac{1}{\sqrt{2}}\mathrm{S}_\kappa(\xi)\,\hat{\mathcal{E}}_\kappa,\\ \hat{\lambda}_\kappa^B&=-\frac{\hat{\mathcal{E}}_\kappa}{2\kappa}(\hat{\mathcal{E}}_\kappa+\hbar\kappa).\end{aligned}$$

These operators can be called *pure-ladder* ones in order to stress that they correspond to different eigenvalues for $\hat{H}_\kappa^\xi$ (see [42]) and their action on the eigenfunctions of $\hat{H}_\kappa^\xi$ is straightforwardly shown to be

$$\hat{B}_\kappa^+\Xi_\kappa^\epsilon\propto\Xi_\kappa^{\epsilon+\hbar\kappa},\qquad \hat{B}_\kappa^-\Xi_\kappa^{\epsilon+\hbar\kappa}\propto\Xi_\kappa^\epsilon.$$

In this way, by acting on the subspace of eigenfunctions Ξ_κ^ϵ we find that

$$\left[\hat{\mathcal{E}}_\kappa,\hat{B}_\kappa^\pm\right]=\pm\hbar\kappa\,\hat{B}_\kappa^\pm\quad\Longleftrightarrow\quad\hat{B}_\kappa^\pm\,\hat{\mathcal{E}}_\kappa=(\hat{\mathcal{E}}_\kappa\mp\hbar\kappa)\hat{B}_\kappa^\pm. \tag{6.11}$$

Therefore,

$$\left[(\hat{\mathcal{E}}_\kappa)^2,\hat{B}_\kappa^\pm\right]=\hbar\kappa\left(\pm2\hat{\mathcal{E}}_\kappa-\hbar\kappa\right)\hat{B}_\kappa^\pm,$$

that is,

$$\left[\hat{H}_\kappa^\xi,\hat{B}_\kappa^+\right]=\hbar\left(\hat{\mathcal{E}}_\kappa-\frac{1}{2}\hbar\kappa\right)\hat{B}_\kappa^+,\qquad\left[\hat{H}_\kappa^\xi,\hat{B}_\kappa^-\right]=-\hat{B}_\kappa^-\,\hbar\left(\hat{\mathcal{E}}_\kappa-\frac{1}{2}\hbar\kappa\right). \tag{6.12}$$

Likewise, we find that

$$\left[\hat{B}_\kappa^-,\hat{B}_\kappa^+\right]=\hbar\,\hat{\mathcal{E}}_\kappa. \tag{6.13}$$

Note that the Lie brackets (6.12) and (6.13) are just the quantum analogues of the Poisson algebra (5.11) and that a pure quantum-curvature term $\hbar\kappa/2$ arises, which is obviously negligible at both the classical curved and quantum flat (Euclidean) frameworks.

Finally, the commutation rules between the operators $\hat{B}_\kappa^\pm$ and the complete Hamiltonian $\hat{H}_\kappa$ (6.3) are shown to be

$$[\hat{H}_\kappa,\hat{B}_\kappa^\pm]=\frac{\hbar\gamma^2}{\mathrm{C}_\kappa^2(y)}\left(\pm\hat{\mathcal{E}}_\kappa-\frac{1}{2}\hbar\kappa\right)\hat{B}_\kappa^\pm,$$

which can be compared with the Poisson algebra (5.12).

6.2. Shift operators for the Hamiltonian $\hat{H}_\kappa$

By making use of the operator $\hat{\mathcal{E}}_\kappa$ (6.9), the Hamiltonian $\hat{H}_\kappa$ (6.3), acting on the eigenfunctions (6.7) can be rewritten as

$$\hat{H}_\kappa(\gamma\hat{\mathcal{E}}_\kappa) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2} + \frac{\hbar^2}{2}\kappa\,\mathrm{T}_\kappa(y)\frac{\partial}{\partial y} + \frac{(\gamma\hat{\mathcal{E}}_\kappa)^2}{2\kappa\,\mathrm{C}_\kappa^2(y)} - \frac{\omega^2}{2\kappa}, \tag{6.14}$$

where we have stressed its dependence on $\gamma\hat{\mathcal{E}}_\kappa$. Now, it can be proven that $\hat{H}_\kappa$ can be factorized in terms of two first-order differential operators plus a diagonal one, namely

$$\hat{H}_\kappa = \hat{A}_\kappa^+\hat{A}_\kappa^- + \hat{\lambda}_\kappa^A, \tag{6.15}$$

where

$$\hat{A}_\kappa^+ = -\frac{\hbar}{\sqrt{2}}\frac{\partial}{\partial y} - \frac{1}{\sqrt{2}}(\gamma\hat{\mathcal{E}}_\kappa - \hbar\kappa)\mathrm{T}_\kappa(y),$$
$$\hat{A}_\kappa^- = \frac{\hbar}{\sqrt{2}}\frac{\partial}{\partial y} - \frac{\gamma\hat{\mathcal{E}}_\kappa}{\sqrt{2}}\mathrm{T}_\kappa(y),$$
$$\hat{\lambda}_\kappa^A = \frac{\gamma\hat{\mathcal{E}}_\kappa}{2\kappa}(\gamma\hat{\mathcal{E}}_\kappa - \hbar\kappa) - \frac{\omega^2}{2\kappa}.$$

These expressions are worth to be compared with (5.13) and (5.14). Then, the action of the shift operators on the eigenfunctions $Y_\kappa^{\gamma\epsilon}$ can be shown to be of the type

$$\hat{A}_\kappa^+ Y_\kappa^{\gamma\epsilon-\hbar\kappa} \propto Y_\kappa^{\gamma\epsilon}, \qquad \hat{A}_\kappa^- Y_\kappa^{\gamma\epsilon} \propto Y_\kappa^{\gamma\epsilon-\hbar\kappa}.$$

Consequently, these operators change the parameter $\gamma\epsilon \to \gamma\epsilon \pm \hbar\kappa$ of the eigenfunctions, but keep the energy E_κ constant. In this sense, they are called *pure-shift* operators, and the following commutation relations can be derived

$$[\hat{H}_\kappa, \hat{A}_\kappa^+] = -\hbar\,\hat{A}_\kappa^+\left(\frac{2\gamma\hat{\mathcal{E}}_\kappa - \hbar\kappa}{2\mathrm{C}_\kappa^2(y)}\right), \qquad [\hat{H}_\kappa, \hat{A}_\kappa^-] = \hbar\left(\frac{2\gamma\hat{\mathcal{E}}_\kappa - \hbar\kappa}{2\mathrm{C}_\kappa^2(y)}\right)\hat{A}_\kappa^-,$$

where a quantum-curvature contribution $\hbar\kappa$ appears again. These commutation relations are straightforwardly proven to be equivalent to the following so-called "intertwining relations"

$$\hat{A}_\kappa^-\hat{H}_\kappa(\gamma\hat{\mathcal{E}}_\kappa) = \hat{H}_\kappa(\gamma\hat{\mathcal{E}}_\kappa - \hbar\kappa)\hat{A}_\kappa^-, \qquad \hat{A}_\kappa^+\hat{H}_\kappa(\gamma\hat{\mathcal{E}}_\kappa - \hbar\kappa) = \hat{H}_\kappa(\gamma\hat{\mathcal{E}}_\kappa)\hat{A}_\kappa^+. \tag{6.16}$$

6.3. Quantum symmetries

So far we have all the ingredients to construct the "additional" symmetry operators $\hat{X}_\kappa^\pm$ for the quantum Hamiltonian $\hat{H}_\kappa$ (6.2) in the rational $\gamma = m/n$ case, that can be defined as

$$\hat{X}_\kappa^\pm = (\hat{A}_\kappa^\pm)^m(\hat{B}_\kappa^\pm)^n, \quad m, n \in \mathbb{N}^*. \tag{6.17}$$

The proof that $[\hat{H}_\kappa, \hat{X}_\kappa^\pm] = 0$ when $\gamma = m/n$ can be obtained by direct computation through the action on the subspace of eigenfunctions for $\hat{H}_\kappa$ (see Appendix B for details). The set of symmetries $(\hat{H}_\kappa, \hat{H}_\kappa^\xi, \hat{X}_\kappa^\pm)$ is not algebraically independent; due to the factorization properties (6.10) and (6.15), the products $\hat{X}_\kappa^+\hat{X}_\kappa^-$ and $\hat{X}_\kappa^-\hat{X}_\kappa^+$ are functions of $\hat{H}_\kappa, \hat{H}_\kappa^\xi$. Similarly to the classical cases (5.17) and (5.18), we can define real polynomial quantum symmetries $\hat{X}_\kappa$ and $\hat{Y}_\kappa$ of orders $(m+n)$ or $(m+n-1)$ in the momentum operators. These sets of symmetries close a polynomial algebra (see [36] for more details).

All the above results are summarized as follows.

Theorem 4. (i) *The quantum Hamiltonian* $\hat{H}_\kappa$ (6.1) *defines an integrable quantum system for any value of the parameter* γ, *since it commutes with the operator* $\hat{H}_\kappa^\xi$ (6.4).

(ii) *When* γ *is a rational parameter,* $\hat{H}_\kappa$ *defines a superintegrable anisotropic quantum curved oscillator with additional symmetry operators given by* (6.17). *The sets* $(\hat{H}_\kappa, \hat{H}_\kappa^\xi, \hat{X}_\kappa^+)$ *and* $(\hat{H}_\kappa, \hat{H}_\kappa^\xi, \hat{X}_\kappa^-)$ *are formed by three algebraically independent operators.*

Let us illustrate this result through some particular systems.

6.4. Symmetries of the 1 : 1 case

The simplest case of the quantum anisotropic curved oscillator corresponds to set $\gamma = 1$ which, in fact, gives rise to the isotropic case. In this case we have the symmetry operators

$$\hat{X}_\kappa^\pm = (\hat{A}_\kappa^\pm)(\hat{B}_\kappa^\pm) = \pm \hat{Y}_\kappa \hat{\mathcal{E}}_\kappa + \hat{X}_\kappa,$$

where $\hat{X}_\kappa$ is a polynomial symmetry of degree two, while $\hat{Y}_\kappa$ has degree one in the momentum operators. These symmetries take the explicit form

$$\begin{aligned}
\hat{X}_\kappa &= \frac{\hbar^2}{2}\mathrm{C}_\kappa(\xi)\partial_\xi\partial_y - \frac{1}{2}\mathrm{S}_\kappa(\xi)\mathrm{T}_\kappa(y)(\hat{\mathcal{E}}_\kappa)^2,\\
\hat{Y}_\kappa &= -\frac{\hbar}{2}\big(\mathrm{S}_\kappa(\xi)\partial_y - \mathrm{C}_\kappa(\xi)\mathrm{T}_\kappa(y)\partial_\xi\big),
\end{aligned}$$

where $\xi = x$. Recall that, according to (6.6) and (6.9), we can replace $(\hat{\mathcal{E}}_\kappa)^2$ by $2\kappa\hat{H}_\kappa^\xi$.

If we take the classical limit $\hbar \to 0$ (so $-i\hbar\partial_\xi \to p_\xi$ and $\hat{\mathcal{E}}_\kappa \to \mathcal{E}_\kappa$), we recover the classical symmetries on the curved spaces given in (5.20): $\hat{X}_\kappa \to X_\kappa$ and $\hat{Y}_\kappa \to iY_\kappa$. Furthermore, in the limit $\kappa \to 0$, they become the classical Euclidean expressions shown in (2.13).

6.5. Symmetries of the 2 : 1 case

When $\gamma = 2$ the symmetries come from the operators

$$\hat{X}_\kappa^\pm = (\hat{A}_\kappa^\pm)^2(\hat{B}_\kappa^\pm) = \hat{X}_\kappa \hat{\mathcal{E}}_\kappa \pm \hat{Y}_\kappa,$$

where $\hat{X}_\kappa$ is a polynomial symmetry of degree two, while $\hat{Y}_\kappa$ has degree three in the momentum operators. Explicitly, we have

$$\begin{aligned}
\hat{X}_\kappa &= \frac{2}{\sqrt{2}}\mathrm{S}_\kappa(\xi)\mathrm{T}_\kappa^2(y)(\hat{\mathcal{E}}_\kappa)^2 + \frac{\hbar^2\kappa}{2\sqrt{2}}\mathrm{S}_\kappa(\xi)\mathrm{T}_\kappa(y)\partial_y + \frac{\hbar^2}{2\sqrt{2}}\mathrm{S}_\kappa(\xi)\partial_y^2\\
&\quad - \frac{\hbar^2}{\sqrt{2}}\mathrm{C}_\kappa(\xi)\left(\frac{1}{\mathrm{C}_\kappa^2(y)} + \kappa\mathrm{T}_\kappa^2(y)\right)\partial_\xi - \frac{2\,\hbar^2}{\sqrt{2}}\mathrm{C}_\kappa(\xi)\mathrm{T}_\kappa(y)\partial_\xi\partial_y,\\
\hat{Y}_\kappa &= \frac{2\hbar}{\sqrt{2}}\left(\mathrm{S}_\kappa(\xi)\left(\frac{1}{\mathrm{C}_\kappa^2(y)} - \frac{1}{2}\right) + \mathrm{S}_\kappa(\xi)\mathrm{T}_\kappa(y)\partial_y - \mathrm{C}_\kappa(\xi)\mathrm{T}_\kappa^2(y)\partial_\xi\right)(\hat{\mathcal{E}}_\kappa)^2\\
&\quad - \frac{\hbar^3\kappa}{2\sqrt{2}}\mathrm{C}_\kappa(\xi)\mathrm{T}_\kappa(y)\partial_\xi\partial_y - \frac{\hbar^3}{2\sqrt{2}}\mathrm{C}_\kappa(\xi)\partial_\xi\partial_y^2.
\end{aligned}$$

In the limit $\hbar \to 0$, these symmetries turn into the classical counterparts of (5.23) and (5.24): $\hat{X}_\kappa \to X_\kappa$ and $\hat{Y}_\kappa \to iY_\kappa$; recall that now $\xi = 2x$. If on this latter result we take the flat limit $\kappa \to 0$ we recover the Euclidean constants of motion of (2.15).

6.6. Symmetries of the 1 : 2 case

If we set $\gamma = 1/2$, the symmetries read

$$\hat{X}^{\pm}_{\kappa} = (\hat{A}^{\pm}_{\kappa})(\hat{B}^{\pm}_{\kappa})^2 = \hat{X}_{\kappa}\hat{\mathcal{E}}_{\kappa} \pm \hat{Y}_{\kappa},$$

where, as in the previous case, $\hat{X}_{\kappa}$ is a second-order symmetry while $\hat{Y}_{\kappa}$ is a third-order one. These operators take the following form

$$\begin{aligned}\hat{X}_{\kappa} &= -\frac{1}{4\sqrt{2}}\mathrm{S}^2_{\kappa}(\xi)\mathrm{T}_{\kappa}(y)(\hat{\mathcal{E}}_{\kappa})^2 - \frac{\hbar^2}{4\sqrt{2}}\mathrm{C}^2_{\kappa}(\xi)\mathrm{T}_{\kappa}(y)\partial^2_{\xi}\\&\quad+\frac{\hbar^2}{4\sqrt{2}}\big(2\mathrm{C}_{\kappa}(2\xi)\partial_y + \mathrm{S}_{\kappa}(2\xi)(\kappa\mathrm{T}_{\kappa}(y)\partial_{\xi} + 2\partial_{\xi}\partial_y)\big),\\\hat{Y}_{\kappa} &= \frac{\hbar}{4\sqrt{2}}\left(\mathrm{T}_{\kappa}(y)(\mathrm{C}_{\kappa}(2\xi) + \mathrm{S}_{\kappa}(2\xi)\partial_{\xi}) - 2\mathrm{S}^2_{\kappa}(\xi)\partial_y\right)(\hat{\mathcal{E}}_{\kappa})^2\\&\quad+\frac{\hbar^3}{2\sqrt{2}}\mathrm{C}_{\kappa}(\xi)\big(2\kappa\,\mathrm{S}_{\kappa}(\xi)\partial_{\xi}\partial_y - \mathrm{C}_{\kappa}(\xi)\partial^2_{\xi}\partial_y\big).\end{aligned}$$

Under the limit $\hbar \to 0$, these symmetries give rise to the curved classical functions (5.26) and (5.27) provided that $\xi = x/2$.

In the same manner, other $m : n$ quantum oscillators can straightforwardly be worked out, and, obviously, the expressions for their symmetries become rather cumbersome.

7. Spectrum of the anisotropic oscillator on the sphere

Due to the different properties of the spectra in the $\kappa > 0$ and the $\kappa < 0$ cases, both quantum systems are worth to be analysed separately. Let us firstly consider the anisotropic oscillator on the sphere $\mathbf{S}^2$ with arbitrary positive curvature $\kappa > 0$. Take the eigenfunctions $\Psi^E_{\kappa} = \Xi^{\epsilon}_{\kappa}\, Y^{\gamma\epsilon}_{\kappa}$, given in terms of the eigenfunctions of the one-dimensional Hamiltonians (6.6) and (6.7). The eigenvalue equation (6.6) $\hat{H}^{\xi}_{\kappa}\; \Xi^{\epsilon}_{\kappa}(\xi) = E^{\xi}_{\kappa}\; \Xi^{\epsilon}_{\kappa}(\xi)$, for Ξ^{ϵ}_{κ}, corresponds to the well-known trigonometric Pöschl–Teller Hamiltonian [42,43] whose eigenvalues are given by

$$\begin{aligned}E^{\xi}_{\kappa} \equiv E^{\mu}_{\kappa,\chi} &= \frac{1}{2\kappa}\big(\chi + (\mu+1)\hbar\kappa\big)^2\\&= \frac{\hbar}{4}(1+2\mu)\sqrt{\hbar^2\kappa^2 + \frac{4\omega^2}{\gamma^2}} + \frac{\hbar^2\kappa}{4}\left(1+2\mu+2\mu^2\right) + \frac{\omega^2}{2\kappa\gamma^2}, \quad \mu = 0,1,2,\ldots \qquad (7.1)\end{aligned}$$

where the positive parameter χ is given by

$$\chi(\chi+\hbar\kappa) = \frac{\omega^2}{\gamma^2}, \quad \chi = \frac{1}{2}\left(\sqrt{\hbar^2\kappa^2 + 4\omega^2/\gamma^2} - \hbar\kappa\right). \qquad (7.2)$$

Having in mind (6.6), we will also use the parameter ϵ_{μ} corresponding to such eigenvalues:

$$\epsilon_{\mu} = \sqrt{2\kappa E^{\xi}_{\kappa}} = \chi + (\mu+1)\hbar\kappa. \qquad (7.3)$$

On the other hand, the eigenvalue equation for the Hamiltonian (6.8), $\hat{H}_{\kappa} Y^{\gamma\epsilon}_{\kappa}(y) = E_{\kappa} Y^{\gamma\epsilon}_{\kappa}(y)$, satisfied by the second function $Y^{\gamma\epsilon}_{\kappa}$, also corresponds to a modified trigonometric Pöschl–Teller Hamiltonian with eigenvalues

$$E^{\nu}_{\kappa,\gamma\epsilon} = \frac{1}{2\kappa}\left(\gamma\epsilon + \nu\hbar\kappa\right)\left(\gamma\epsilon + (\nu+1)\hbar\kappa\right) - \frac{\omega^2}{2\kappa}, \quad \nu = 0,1,2,\ldots$$

Therefore, by replacing ϵ in this expression by the values ϵ_μ above obtained in (7.3), we get the energy eigenvalues of the whole two-dimensional Hamiltonian (6.2), namely

$$\begin{aligned} E_\kappa \equiv E_\kappa^{\mu,\nu} &= \frac{1}{2\kappa}\big(\gamma\chi + [\gamma(\mu+1)+\nu]\hbar\kappa\big)\big(\gamma\chi + [\gamma(\mu+1)+\nu+1]\hbar\kappa\big) - \frac{\omega^2}{2\kappa} \\ &= \gamma^2\left(E_{\kappa,\chi}^{\mu} - \frac{\omega^2}{2\kappa\gamma^2}\right) + \frac{\hbar\gamma}{2}\big(\chi + (\mu+1)\hbar\kappa\big)(2\nu+1) + \frac{\hbar^2\kappa}{2}\nu(\nu+1). \end{aligned} \tag{7.4}$$

It is worth stressing that in this expression the role of the curvature is essential, since it gives rise to a quadratic dependence in terms of the quantum numbers μ,ν instead of the linear Euclidean one. The corresponding eigenfunctions (6.5) take the form

$$\Psi_\kappa^{\mu,\nu}(\xi,y) = \Xi_\kappa^{\epsilon_\mu}(\xi)Y_{\kappa,\nu}^{\gamma\epsilon_\mu}(y).$$

According to (7.4), two of these eigenfunctions $\Psi_\kappa^{\mu,\nu}(\xi,y)$ and $\Psi_\kappa^{\mu',\nu'}(\xi,y)$ will have the same energy $E_\kappa^{\mu,\nu} = E_\kappa^{\mu',\nu'}$ if

$$\gamma(\mu'-\mu) + \nu' - \nu = 0, \tag{7.5}$$

which is only satisfied when $\gamma = m/n$ with $m,n \in \mathbb{N}^*$, and these states are connected by the symmetry operators (6.17). Therefore, we can conclude that:

Theorem 5. (i) *The spectrum of the quantum Hamiltonian $\hat{H}_\kappa$ (6.2) on the sphere with $\kappa > 0$ is given, for any value of the parameter γ, by (7.4) where the parameter χ is written in (7.2).*
(ii) *When $\gamma = m/n$ with $m,n \in \mathbb{N}^*$, the spectrum (7.4) is degenerate, and the degeneracy is the same as in the Euclidean case.*

Some comments on the Euclidean limit of the above results seem to be pertinent. Although most of the computations have been carried out for $\kappa \neq 0$, the flat limit $\kappa \to 0$ can adequately be performed on the final results. Explicitly, the limit $\kappa \to 0$ of the parameters χ (7.2) and ϵ_μ (7.3) as well as of the spectrum $E_\kappa^\xi \equiv E_{\kappa,\chi}^\mu$ (7.1) is achieved as

$$\lim_{\kappa\to 0}\chi = \lim_{\kappa\to 0}\epsilon_\mu = \frac{\omega}{\gamma}, \qquad \lim_{\kappa\to 0}\left(E_{\kappa,\chi}^\mu - \frac{\omega^2}{2\kappa\gamma^2}\right) = \frac{\hbar\omega}{2\gamma} + \mu\frac{\hbar\omega}{\gamma}, \tag{7.6}$$

such that the latter is just the spectrum $E^{\xi,\mu}$ (3.5) of the one-dimensional quantum Euclidean Hamiltonian $\hat{H}^\xi$ (3.2). And the complete spectrum $E_\kappa \equiv E_\kappa^{\mu,\nu}$ (7.4) directly reduces to $E^{\mu,\nu}$ (3.6) corresponding to the two-dimensional quantum Euclidean Hamiltonian $\hat{H}$ (3.1):

$$\lim_{\kappa\to 0} E_\kappa^{\mu,\nu} = \hbar\omega\left(\tfrac{1}{2}(\gamma+1) + \gamma\mu + \nu\right) \equiv E^{\mu,\nu}. \tag{7.7}$$

8. Spectrum of the anisotropic oscillator on the hyperboloid

Now, we consider the hyperbolic space $\mathbf{H}^2$ with arbitrary negative curvature $\kappa = -|\kappa|$ and we follow the same approach as in the sphere $\mathbf{S}^2$ with $\kappa > 0$. We anticipate that, although the same algebraic method holds in both spaces, the solution to the eigenvalue problem is quite different.

Let us consider the Hamiltonian $\hat{H}_\kappa \equiv \hat{H}_{-|\kappa|}$ (6.14), where the factorized solutions take the form $\Psi_{-|\kappa|}^E = \Xi_{-|\kappa|}^\epsilon Y_{-|\kappa|}^{\gamma\epsilon}$ in the same way as in the previous section. Now, the eigenvalue equation (6.6) $\hat{H}_{-|\kappa|}^\xi \Xi_{-|\kappa|}^\epsilon(\xi) = E_{-|\kappa|}^\xi \Xi_{-|\kappa|}^\epsilon(\xi)$, for $\Xi_{-|\kappa|}^\epsilon$, corresponds to the hyperbolic Pöschl–Teller Hamiltonian [42,43] whose eigenvalues are

$$\begin{aligned} E_{-|\kappa|}^\xi \equiv E_{\kappa,\chi}^\mu &= -\frac{1}{2|\kappa|}\left(\chi - (\mu+1)\hbar|\kappa|\right)^2 \\ &= \frac{\hbar}{4}(1+2\mu)\sqrt{\hbar^2|\kappa|^2 + \frac{4\omega^2}{\gamma^2}} - \frac{\hbar^2|\kappa|}{4}\left(1+2\mu+2\mu^2\right) - \frac{\omega^2}{2|\kappa|\gamma^2}, \quad \mu = 0,1,2,\ldots,\mu_{\max}, \end{aligned}$$

where the parameter $\chi > \hbar|\kappa|$ is given by

$$\chi(\chi - \hbar|\kappa|) = \frac{\omega^2}{\gamma^2}, \quad \chi = \frac{1}{2}\left(\sqrt{\hbar^2\,|\kappa|^2 + 4\omega^2/\gamma^2} + \hbar|\kappa|\right). \tag{8.1}$$

Therefore, there is only a finite number of bounded states. The value $\mu_{\max}$ is the maximum integer such that

$$\mu_{\max} < \frac{\chi}{\hbar|\kappa|} - 1. \tag{8.2}$$

Again, instead of $E^{\xi}_{-|\kappa|}$ we will use the parameter (6.6),

$$\epsilon_\mu = \sqrt{-2|\kappa| E^{\xi}_{-|\kappa|}} = \chi - (\mu + 1)\hbar|\kappa|. \tag{8.3}$$

The eigenvalue equation (6.7), $\hat{H}_{-|\kappa|} Y^{\gamma\epsilon}_{-|\kappa|}(y) = E_{-|\kappa|} Y^{\gamma\epsilon}_{-|\kappa|}(y)$, satisfied by the function $Y^{\gamma\epsilon}_{-|\kappa|}$, is the one of a modified hyperbolic Pöschl–Teller Hamiltonian with eigenvalues

$$E^{\nu}_{-|\kappa|,\gamma\epsilon} = -\frac{1}{2|\kappa|}\left(\gamma\epsilon - \nu\hbar|\kappa|\right)\left(\gamma\epsilon - (\nu + 1)\hbar|\kappa|\right) + \frac{\omega^2}{2|\kappa|}, \quad \nu = 0, 1, 2, \ldots, \nu_{\max}. \tag{8.4}$$

Such a maximum value $\nu_{\max}$ of the quantum number ν is the maximum integer such that

$$\nu_{\max} < \frac{\gamma\epsilon}{\hbar|\kappa|} - 1. \tag{8.5}$$

Next, replacing ϵ in (8.4) by the values ϵ_μ obtained in (8.3), we obtain the energy eigenvalues of the quantum Hamiltonian $\hat{H}_\kappa \equiv \hat{H}_{-|\kappa|}$ (6.2):

$$\begin{aligned} E_{-|\kappa|} \equiv E^{\mu,\nu}_{-|\kappa|} &= -\frac{1}{2|\kappa|}\big(\gamma\chi - [\gamma(\mu + 1) + \nu]\,\hbar|\kappa|\big)\big(\gamma\chi - [\gamma(\mu + 1) + \nu + 1]\,\hbar|\kappa|\big) + \frac{\omega^2}{2|\kappa|} \\ &= \gamma^2\left(E^{\mu}_{\kappa,\chi} + \frac{\omega^2}{2|\kappa|\gamma^2}\right) + \frac{\hbar\gamma}{2}\big(\chi - (\mu + 1)\hbar|\kappa|\big)(2\nu + 1) - \frac{\hbar^2\,|\kappa|}{2}\nu(\nu + 1). \end{aligned} \tag{8.6}$$

Clearly, the number of eigenvalues is finite because of the constraints on μ and ν, that is, we have a fixed value of χ (8.1), which determines $\mu_{\max}$ (8.2) and, then, for any allowed value of μ there is a maximum value for ν, $\nu_{\max}(\mu)$ determined by (8.5). Consequently, on $\mathbf{H}^2$ there is a finite number of bound states in contradistinction with the $\mathbf{S}^2$ case. The corresponding eigenfunctions are written in the form

$$\Psi^{\mu,\nu}_{-|\kappa|}(\xi, y) = \Xi^{\epsilon_\mu}_{-|\kappa|}(\xi)\, Y^{\gamma\epsilon_\mu}_{-|\kappa|,\nu}(y).$$

The degeneracy of the energy levels can be discussed in a similar way as on $\mathbf{S}^2$: $\Psi^{\mu,\nu}_{-|\kappa|}(\xi, y)$ and $\Psi^{\mu',\nu'}_{-|\kappa|}(\xi, y)$ will have the same energy $E^{\mu,\nu}_{-|\kappa|}$ whenever the constraint (7.5) is fulfilled. This means that the degeneration can take place only when $\gamma = m/n$ for $m, n \in \mathbb{N}^*$.

Summing up, we have shown that:

Theorem 6. (i) *The spectrum of the quantum Hamiltonian $\hat{H}_{-|\kappa|}$ (6.2) on the hyperbolic space with $\kappa = -|\kappa| < 0$ is given, for any value of the parameter γ, by a finite number of eigenvalues (8.6) where the parameter χ is written in (8.1).*

(ii) *When $\gamma = m/n$ with $m, n \in \mathbb{N}^*$, the spectrum (8.6) is degenerate, and the degeneracy is the same as in the Euclidean case.*

The Euclidean limit of the above results can be easily obtained in the same way as in (7.6) and (7.7). Finally, we should mention that for negative curvature $\kappa = -|\kappa|$, there exist unbounded states which may come from any of the one-dimensional hyperbolic Pöschl–Teller potentials which take part in the total Hamiltonian. This feature, of course, is not present for the anisotropic oscillator on $\mathbf{S}^2$.

9. Concluding remarks

In this work we have identified the form of the generic classical and quantum anisotropic oscillator system on the sphere and the hyperboloid, in such a way that this new Hamiltonian keeps the integrability properties of the known Euclidean anisotropic oscillator for any value of the frequencies. Moreover, when the frequencies are commensurate, superintegrability arises and, in particular, the previously known curved anisotropic oscillators which correspond to the ratio $1:1$ or $2:1$ are recovered.

There are two key points in the approach here presented. The first one consists in a formulation depending on the curvature parameter κ, in such a way that the algebraic treatment is simultaneous for both the sphere and the hyperboloid. At the same time, this viewpoint allows us to get the Euclidean expressions in the limit $\kappa \to 0$, thus showing explicitly that our systems are indeed curved integrable deformations of the Euclidean anisotropic oscillator. Of course, there are also many properties that depend on the sign of κ, and they have to be treated separately. For instance, the spectrum of the quantum anisotropic oscillator on $\mathbf{S}^2$ is purely discrete (and has infinite values), whilst a (finite) discrete spectrum plus a continuous one arises for the system on $\mathbf{H}^2$, as it has been explicitly discussed.

The second key point is the choice of geodesic parallel coordinates on the curved surfaces in order to get the simplest possible expression of the corresponding anisotropic oscillators. These coordinates turn out to be the appropriate curved analogue of the Euclidean Cartesian coordinates, which are used to write the planar anisotropic oscillator in a separable manner.

In order to get a unified approach to find the symmetries for both, the classical and quantum systems, we have applied a factorization approach [15,18,36,44]. This method turns out to be helpful in order to highlight the correspondence between the classical and quantum algebraic symmetries. In this respect, it is worth to stress that in the quantum context we have kept the quantum constant $\hbar$ in all the expressions. This is quite relevant when we compare quantum and classical results through the limit $\hbar \to 0$ (together with other considerations).

We also recall that other methods have also been designed to deal with the symmetries of this kind of systems; for instance we can mention an action–angle (or Hamilton–Jacobi) based procedure considered for the classical systems in [45] and the type of recurrence relation arguments used in the quantum counterparts [46]. Similar approaches for a family of superintegrable models were applied in [47], a coalgebra procedure has been developed in [48], while other different viewpoints/approaches can be found in [49–53].

There are several open problems related to anisotropic curved oscillators, which are worth to be investigated in the near future. For instance: (i) The construction of the anisotropic curved oscillators in three (and N) dimensions. (ii) The formulation of anisotropic oscillators on $(1+1)$ relativistic spacetimes: AdS, dS and Minkowski [54] (see also [55,56] for the oscillator problem on the $SO(2,2)$ hyperboloid). (iii) The generalization of the so-called caged anisotropic oscillator studied in [57] to curved spaces by adding centrifugal terms [19,22,23].

Finally, we remark that in order to keep the length of this paper under reasonable limits, we have not included the full study of the algebra generated by the symmetries in the generic case. However, this structure can straightforwardly be derived by following the procedure shown in [36].

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Appendix A

The three Riemannian spaces $\mathbf{S}^2$, $\mathbf{E}^2$ and $\mathbf{H}^2$ can be simultaneously studied by considering that their constant Gaussian curvature $\kappa = \pm 1/R^2$ plays the role of a deformation/contraction parameter

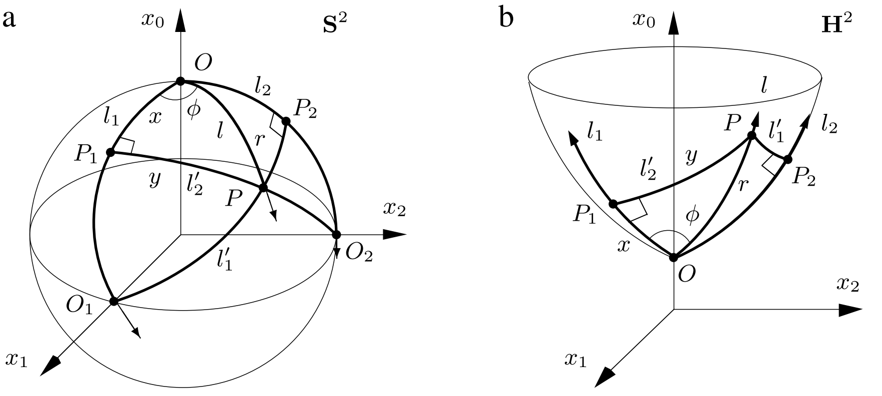

Fig. 3. Ambient (x_0, x_1, x_2), geodesic parallel (x, y) and geodesic polar (r, ϕ) coordinates. The origin is $O = (1, 0, 0)$. (*a*) On the sphere $\mathbf{S}^2$ with $\kappa = +1$; $O_1 = (0, 1, 0)$ and $O_2 = (0, 0, 1)$. (*b*) On the hyperboloid with $\kappa = -1$ and $x_0 \geq 1$.

(see [6,20–23] and references therein). These three spaces can be embedded in the linear space $\mathbb{R}^3$ with ambient or Weierstrass coordinates (x_0, x_1, x_2) subjected to the constraint

$$\Sigma_\kappa \equiv x_0^2 + \kappa \left(x_1^2 + x_2^2\right) = 1, \tag{A.1}$$

such that the origin corresponds to the point $O = (1, 0, 0) \in \mathbb{R}^3$. If $\kappa = 1/R^2 > 0$, we recover the sphere $\mathbf{S}^2$ and when $\kappa = -1/R^2 < 0$, we find the two-sheeted hyperboloid. The null curvature case can be understood as a flat contraction $\kappa = 0$ (i.e. the limit $R \to \infty$), giving rise to two Euclidean planes $x_0 = \pm 1$ with Cartesian coordinates (x_1, x_2). We shall identify the hyperbolic space $\mathbf{H}^2$ with the upper sheet of the hyperboloid with $x_0 \geq 1$ and the Euclidean space $\mathbf{E}^2$ with the plane $x_0 = +1$. Although $\kappa \neq 0$ can always be scaled to ± 1, the explicit presence of the curvature parameter will make evident all the deformation processes from the Euclidean systems to the curved ones and, conversely, the contraction from the latter to the former ones.

In this approach, the metric on the curved spaces comes from the usual metric in $\mathbb{R}^3$ divided by the curvature κ and restricted to Σ_κ (A.1):

$$\mathrm{d}s^2 = \frac{1}{\kappa}\left(\mathrm{d}x_0^2 + \kappa\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2\right)\right)\Big|_{\Sigma_\kappa} = \frac{\kappa\,(x_1\mathrm{d}x_1 + x_2\mathrm{d}x_2)^2}{1 - \kappa\left(x_1^2 + x_2^2\right)} + \mathrm{d}x_1^2 + \mathrm{d}x_2^2. \tag{A.2}$$

The ambient coordinates (x_0, x_1, x_2) can be parametrized in terms of two intrinsic variables in different ways (see, e.g., [20]). In particular, we can consider geodesic polar (r, ϕ) and geodesic parallel coordinates (x, y) (see Fig. 1). The parametrization of the ambient coordinates in the variables (x, y) and (r, ϕ), so fulfilling the constraint (A.1), reads [6,21–23]

$$\begin{aligned}
x_0 &= \mathrm{C}_\kappa(x)\mathrm{C}_\kappa(y) = \mathrm{C}_\kappa(r),\\
x_1 &= \mathrm{S}_\kappa(x)\mathrm{C}_\kappa(y) = \mathrm{S}_\kappa(r)\cos\phi,\\
x_2 &= \mathrm{S}_\kappa(y) = \mathrm{S}_\kappa(r)\sin\phi,
\end{aligned} \tag{A.3}$$

which on $\mathbf{E}^2$ with $\kappa = 0$ reduce to

$$x_0 = 1, \qquad x_1 = x = r\cos\phi, \qquad x_2 = y = r\sin\phi.$$

In the three spaces, the coordinates x, y and r have dimensions of length, while $\phi \in [0, 2\pi)$ is always an ordinary angle. However, on $\mathbf{S}^2$ with $\kappa = 1/R^2$, the dimensionless variable r/R is usually considered instead of r. All these coordinates are represented in Fig. 3.

By introducing (A.3) in the metric (A.2) we find the metrics

$$\mathrm{d}s^2 = \mathrm{C}_\kappa^2(y)\mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + \mathrm{S}_\kappa^2(r)\mathrm{d}\phi^2. \tag{A.4}$$

The kinetic energy Lagrangian for a free particle moving on these spaces can be straightforwardly derived from (A.4). Explicitly, let (p_x, p_y) and (p_r, p_ϕ) be, in this order, the conjugate momenta for (x, y) and (r, ϕ). Then, the free kinetic energy Hamiltonian $\mathcal{T}_\kappa$ that generates the geodesic dynamics in the curved space is given by

$$\mathcal{T}_\kappa = \frac{1}{2}\left(\frac{p_x^2}{\mathrm{C}_\kappa^2(y)} + p_y^2\right) = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{\mathrm{S}_\kappa^2(r)}\right). \tag{A.5}$$

Indeed, when $\kappa = 0$ we recover the Euclidean kinetic energy.

$$\mathcal{T}_0 = \frac{1}{2}\left(p_x^2 + p_y^2\right) = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right).$$

In order to guarantee that (A.5) is well defined, we find that the coordinates (x, y) and r have to be defined in the following intervals:

$$\mathbf{S}^2\ (\kappa > 0): \quad -\frac{\pi}{\sqrt{\kappa}} < x \leq \frac{\pi}{\sqrt{\kappa}}, \quad -\frac{\pi}{2\sqrt{\kappa}} < y < \frac{\pi}{2\sqrt{\kappa}}, \quad 0 < r < \frac{\pi}{\sqrt{\kappa}}. \tag{A.6}$$

$$\mathbf{H}^2\ (\kappa < 0): \quad -\infty < x < \infty, \quad -\infty < y < \infty, \quad 0 < r < \infty. \tag{A.7}$$

Appendix B

In order to check that $\hat{X}_\kappa^\pm$ are a pair of symmetries of the Hamiltonian $\hat{H}_\kappa$ (6.2) when the coefficient γ takes the rational value, let us use the following notation for the Hamiltonian (6.3)

$$\hat{H}_\kappa = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2} + \frac{\hbar^2}{2}\kappa\,\mathrm{T}_\kappa(y)\frac{\partial}{\partial y} + \frac{(\gamma\hat{\mathcal{E}}_\kappa)^2}{2\kappa\,\mathrm{C}_\kappa^2(y)} - \frac{\omega^2}{2\kappa} \equiv \hat{H}_\kappa\big(\gamma\hat{\mathcal{E}}_\kappa\big),$$

where we recall the definition (6.9) of the operator $\hat{\mathcal{E}}_\kappa$. Let us consider, for instance, the product $\hat{X}_\kappa^+\hat{H}_\kappa(\gamma\hat{\mathcal{E}}_\kappa)$ and let us move the operators $\hat{X}_\kappa^+$ to the r.h.s. of the Hamiltonian. First, by making use of (6.11) we move the $\hat{B}_\kappa^+$ operators:

$$\hat{X}_\kappa^+\hat{H}_\kappa(\gamma\hat{\mathcal{E}}_\kappa) = (\hat{A}_\kappa^+)^m(\hat{B}_\kappa^+)^n\hat{H}_\kappa(\gamma\hat{\mathcal{E}}_\kappa) = (\hat{A}_\kappa^+)^m\hat{H}_\kappa\big(\gamma(\hat{\mathcal{E}}_\kappa - n\hbar\kappa)\big)(\hat{B}_\kappa^+)^n.$$

Next, we translate the $\hat{A}_\kappa^+$ operators to the r.h.s. by means of the intertwining (6.16) and require that we should obtain $\hat{H}_\kappa(\gamma\hat{\mathcal{E}}_\kappa)\hat{X}_\kappa^+$, that is,

$$\hat{X}_\kappa^+\hat{H}_\kappa(\gamma\hat{\mathcal{E}}_\kappa) = \hat{H}_\kappa\big(\gamma(\hat{\mathcal{E}}_\kappa - n\hbar\kappa) + m\hbar\kappa\big)(\hat{A}_\kappa^+)^m(\hat{B}_\kappa^+)^n = \hat{H}_\kappa(\gamma\hat{\mathcal{E}}_\kappa)\hat{X}_\kappa^+.$$

Therefore, we get the condition

$$\gamma\hat{\mathcal{E}}_\kappa = \gamma(\hat{\mathcal{E}}_\kappa - n\hbar\kappa) + m\hbar\kappa \quad \Longleftrightarrow \quad \gamma = m/n$$

and in that case $\hat{X}_\kappa^+$ (and $\hat{X}_\kappa^-$ as well) will be a symmetry operator of the Hamiltonian. Notice that the operator products in (6.17) are written in terms of the operator $\hat{\mathcal{E}}_\kappa$. In order to write the computation explicitly, we will use the following notation:

$$\hat{B}_\kappa^+ \to \hat{B}_\kappa^+(\hat{\mathcal{E}}_\kappa), \qquad \hat{A}_\kappa^+ \to \hat{A}_\kappa^+(\gamma\hat{\mathcal{E}}_\kappa).$$

Then, the symmetry operators read

$$\begin{aligned}\hat{X}_\kappa^+ &= \hat{A}_\kappa^+\big(\gamma(\hat{\mathcal{E}}_\kappa - n\hbar\kappa) + m\hbar\kappa\big) \ldots \hat{A}_\kappa^+\big(\gamma(\hat{\mathcal{E}}_\kappa - n\hbar\kappa) + \hbar\kappa\big)\,\hat{B}_\kappa^+(\hat{\mathcal{E}}_\kappa) \ldots \hat{B}_\kappa^+(\hat{\mathcal{E}}_\kappa),\\ \hat{X}_\kappa^- &= \hat{A}_\kappa^-\big(\gamma(\hat{\mathcal{E}}_\kappa + n\hbar\kappa) - (m-1)\hbar\kappa\big) \ldots \hat{A}_\kappa^-\big(\gamma(\hat{\mathcal{E}}_\kappa + n\hbar\kappa)\big)\,\hat{B}_\kappa^-(\hat{\mathcal{E}}_\kappa) \ldots \hat{B}_\kappa^-(\hat{\mathcal{E}}_\kappa),\end{aligned}$$

where these operators are always assumed to act on the space of eigenfunctions $\Psi_\kappa(\xi, y) = \Xi_\kappa^\epsilon(\xi)\, Y_\kappa^{\gamma\epsilon}(y)$ of $\hat{H}_\kappa^\xi$ with eigenvalues given by (6.6) and (6.7).

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