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Modelling quantum black holes

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Novel bound states are obtained for manifolds with singular potentials. These singular potentials require proper boundary conditions across boundaries. The number of bound states matches nicely with what we would expect for black holes. Also they serve to model membrane mechanism for the black hole horizons in simpler contexts. The singular potentials can also mimic expanding boundaries elegantly, thereby obtaining appropriately tuned radiation rates.

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1. Introduction

W. Pauli[1] remarked the boundaries were the creation of the devil. Bekenstein's area law[2,3] for the entropy of black hole prescribes that the microscopic states live close to the horizon and the number of such states grow rapidly with area. One can proceed at least in the case of large black holes without the detailed requirements of quantum geometry to study quantum black holes. Such attempts have been made earlier by 't Hooft through the brick wall model,[4] and Bekenstein and Mukhanov.[5] The entropy is understood in some context by entanglement of those bound states inside the horizon which are inaccessible to asymptotic observer with the outside.[6,7]

In this paper, we elaborate an earlier proposal[8] by one of us that the existence of bound states in the black hole geometries follows from the study of self-adjoint extensions of the Laplacian near the horizon. Near horizon geometry of black holes present a singular potential to the particles and can be studied through quantum mechanics with special boundary conditions. Not only they lead to localised states on the boundary their number also scales with area. This in statistical mechanics or in quantum field theory (QFT) context translates as an area law for the entropy.[9] Similar is the case of von Neumann entropy when we trace over unobserved bound states for a distant observer. The quantum physics on manifolds with boundaries introduces novel features. They appear in varied situations like Casimir effect,[10–13] quantum Hall effect, topological insulators[14,15] or quantum gravity contexts like black hole, or de Sitter spacetime with cosmological horizon.[16,17] Many of the novel features stem from studying correct boundary conditions which are physically relevant as well mathematically correct to make the Hamiltonian a self-adjoint operator in properly extended domains in the Hilbert space of L^2 functions.

The garden variety boundary conditions are Dirichlet and Neumann for which either the function or the normal derivative vanishes. However, as shown for the Laplacian a more general class of boundary condition is possible, a particular example being Robin boundary condition. Here the function and the normal are related on the boundary $(\psi + \kappa\partial\psi)|_{\partial\mathcal{M}} = 0$. Dirichlet and Neumann are extreme limits of the Robin boundary condition. But more importantly it introduces a fundamental length κ into the theory.[18,19] Such a parameter will emerge from the coarse grained structure of the underlying spacetime or crystalline lattice. Typically it can be related to Planck length in a semiclassical gravity context.

But the boundaries are obtained in reality through singular potentials or point interactions in spacetime and such potentials are also subjected to self-adjointness conditions on unbounded operators.[20–22] The well-known potential of this kind in one dimension is the $\delta(x)$ which introduces discontinuity in the derivative of the wave functions. A more general potential of the same type is the $\delta'(x)$ potential which has the new feature of introducing discontinuities in the wave function itself. In spite of such discontinuities, the Hamiltonian remains self-adjoint and the quantum theory describes well-defined unitary evolution. This approach allows study of quantum fields over bounded regions in terms of interesting and meaningful questions that can be answered. One can sacrifice the self-adjointness with special boundary conditions like purely incoming waves leading to quasi-normal modes (QNMs) which are also linked to ringing modes of stellar objects including black holes.[23]

We will consider quantum theory with point interactions of the type which is a combination of δ and δ' potentials. Such a combination in addition to being more general, is also necessitated for several reasons. They arise naturally when we consider self-adjoint extensions of Dirac operator with singular δ potential.[24] But for us it brings new features like what we anticipate from membrane paradigm[16,17] for black holes through a new parameter. Our constructions can easily be extended to curved backgrounds too.

Introduction of singular distributions as potentials also help in introducing time-dependent boundaries and associated radiation. We will study in this communication in Sec. 2, a model Schrödinger equation in $\mathbb{R}^2$ with singular point interaction potentials $a\delta(r - R) + 2b\delta'(r - R)$. We consider in Sec. 2.1, the scattering states and in Sec. 2.2, the bound states. In Sec. 2.3, we explain the extension to three dimensions. We also remark about BTZ black hole in this context. In Sec. 3, we consider moving boundaries through singular potentials and explain how this program can be carried out.[25]

2. The Model

The general study of point interactions of the free Hamiltonian in one dimension $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$ is due to Kurasov[26,27] and uses von Neumann's theory of symmetric unbounded operators with identical deficiency indices.[20] The general analysis of self-adjointness of Laplacian in higher-dimensional manifolds with boundaries is more complex due to infinite deficiency indices. But it is possible to relate them directly to boundary conditions of functions and normal derivatives on the boundaries.[21,22] In cases like ours, the presence of isometries simplifies the problem considerably and provide exact solutions.[18]

Consider a Schrödinger Hamiltonian equation in $\mathbb{R}^2$ for stationary states with a singular potential along a circle of radius X:

$$\left[-\frac{\hbar^2}{2m}\Delta_{\boldsymbol{\rho}} + a\delta(\rho - X) + b\delta'(\rho - X)\right]\Psi(\rho,\theta) = E\Psi(\rho,\theta)\,. \tag{1}$$

In order to work with dimensionless quantities, let us introduce new variables and parameters.

$$\boldsymbol{\rho} = \frac{\hbar}{mc}\mathbf{r}\,, \qquad X = \frac{\hbar}{mc}R\,, \qquad w_0 = \frac{2a}{\hbar c}\,, \qquad w_1 = \frac{mb}{\hbar^2}\,, \qquad \lambda = \frac{2E}{mc^2}\,, \tag{2}$$

such that (1) becomes with $\varphi(\mathbf{r}) = \Psi(r,\theta)$

$$-\Delta_{\mathbf{r}}\varphi(\mathbf{r}) + w_0\delta(r-R)\varphi(\mathbf{r}) + 2w_1\delta'(r-R)\varphi(\mathbf{r}) = \lambda\varphi(\mathbf{r})\,. \tag{3}$$

This new parametrization corresponds to lengths being measured in the units of Compton wavelength of the particle. The origin of w_0 is related to the underlying background geometry and is independent of "m". On the other hand w_1 is related to the mass m.

The crucial question is to find the domain of wave functions $\varphi(\mathbf{r})$ that makes H_0 self-adjoint. As these functions and their derivatives have a discontinuity at $r = R$, we have to define the products of the form $\delta(r-R)\varphi(\mathbf{r})$ and $\delta'(r-R)\varphi(\mathbf{r})$ in (1). The form for these products are given as

$$\begin{aligned}
\delta(r-R)\varphi(\mathbf{r}) &= \frac{\varphi(R^+,\theta)+\varphi(R^-,\theta)}{2}\delta(r-R)\,,\\
\delta'(r-R)\varphi(\mathbf{r}) &= \frac{\varphi(R^+,\theta)+\varphi(R^-,\theta)}{2}\delta'(r-R)\\
&\quad - \frac{\varphi'(R^+,\theta)+\varphi'(R^-,\theta)}{2}\delta(r-R)\,,
\end{aligned} \tag{4}$$

where $f(R^+,\theta)$ and $f(R^-,\theta)$ are the right and left limits of the function $f(\mathbf{r})$ as $r \to R$, respectively. The problem is separable and can be reduced to a 1D radial problem with a central potential. In order to obtain a self-adjoint extension of the Hamiltonian, we have to find a domain on which this extension acts, namely given by a space of square integrable functions satisfying matching conditions at the point R. The radial functions in the domain of the Hamiltonian H are functions in the Sobolev space $W_2^2(\mathbb{R}/\{S^2(R)\})$ such that at $r = R$ satisfy the following matching conditions given by an $\mathrm{SL}(2,R)$ matrix:[28]

$$\begin{pmatrix} \varphi(R^+) \\ \varphi'(R^+) \end{pmatrix} = \begin{pmatrix} \dfrac{1+w_1}{1-w_1} & 0 \\ \dfrac{w_0}{1-w_1^2} & \dfrac{1-w_1}{1+w_1} \end{pmatrix} \begin{pmatrix} \varphi(R^-) \\ \varphi'(R^-) \end{pmatrix}. \tag{5}$$

Note that in the case of w_1 being zero it goes to known discontinuities in normal derivatives.[24]

2.1. *Scattering states*

For scattering theory, we solve Schrödinger equation with plane waves and positive energy $\lambda = k^2$. For each angular momentum n we obtain the following Schrödinger 1D problem for $\varphi(r,\theta) = \mathcal{R}(r)e^{in\theta}$:

$$\frac{d^2\mathcal{R}}{dr^2} + \frac{1}{r}\frac{d\mathcal{R}}{dr} + (w_0\delta(r-R) + 2w_1\delta'(r-R))\mathcal{R} - \left(\lambda + \frac{n^2}{r^2}\right)\mathcal{R} = 0\,, \tag{6}$$

with $\lambda = +k^2$ and suitable boundary conditions. The general scattering solution is given by

$$\mathcal{R}(r) = \begin{cases} J_n(kr)\,, & r < R\,, \\ A(k,n)J_n(kr) + B(k,n)Y_n(kr)\,, & r > R\,, \end{cases} \tag{7}$$

where J_n and Y_n are the Bessel functions and $A(k,n)$ and $B(k,n)$ constants to be determined through matching boundary conditions.

$$\begin{aligned} B(k,n) &= J_n(x)(4kw_1RJ_{n-1}(x) - J_n(x)(4w_1n + w_0R))\,, \\ A(k,n) &= J_n(x)(k(w_1+1)^2RY_{n-1}(x) - Y_n(x)(4w_1L + w_0R)) \\ &\quad - k(w_1-1)^2RJ_{n-1}(x)Y_n(x)\,, \end{aligned}$$

where we have defined $x = kR$. This complicated looking expression can be checked to coincide with expected results for hard sphere ($w_1 = -1$). Note that when $w_1 = 1$, the exterior side of the Disc D_2 is seen by the quantum particle as Robin boundary condition while the inside face is Dirichlet. On the other hand, for $w_1 = -1$ this is the other way round. The phase shifts are given by $\tan(\delta_n) = -B(k,n)/A(k,n)$[29] where A, B are given above.

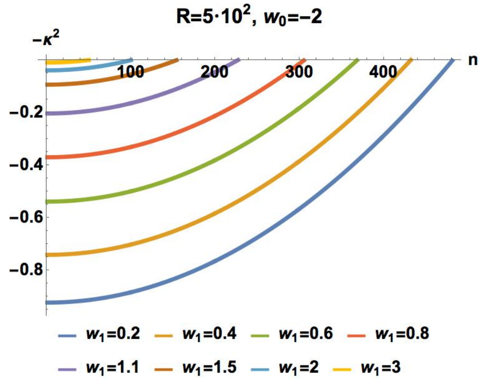

Fig. 1. Bound states $E(w_1)$.

2.2. *Bound states*

The Schrödinger equation for the bound states is Eq. (6) with $\lambda = -\kappa^2 < 0$. The solutions in the regions $r < R$ and $r > R$ are the modified Bessel functions of the first and second kinds: $\mathcal{R}(r) = cI_n(\kappa r)$ and $dK_n(\kappa r)$. We match the boundary conditions at $r = R$ using Eq. (5). We rewrite: $\alpha = \frac{1+\omega_1}{1-\omega_1}$, $\beta = \frac{2\omega_0}{1-w_1^2}$. We get (with $x = \sqrt{|\lambda|}R = \kappa R$)

$$x\left(\alpha\frac{K_n'(x)}{K_n(x)} - \alpha^{-1}\frac{I_n'(x)}{I_n(x)}\right) = \bar{\beta}\,, \tag{8}$$

where $\bar{\beta} = \beta R$. We can simplify the above equation using Bessel function identities to get

$$-x\left(\frac{\alpha K_{n-1}}{K_n} + \frac{\alpha^{-1}I_{n-1}}{I_n}\right) - n(\alpha - \alpha^{-1}) = \bar{\beta}\,. \tag{9}$$

Now we can look for a maximum value of $n = n_m$. It is easy to work out. This gives

$$n_m = \left\lfloor -\frac{\bar{\beta}}{\alpha + \alpha^{-1}} \right\rfloor, \tag{10}$$

where $\lfloor q \rfloor$ denotes the integer part of q. Hence the maximum number of bound states are still proportional to the radius of the circle, but with a renormalised constant $\frac{w_0}{2(1+w_1^2)}$. For the special case of $w_1 = 0$, we get maximum number n_m of bound states is the nearest integer lower than $\bar{w}_0$, which is same as our earlier result.[8,18] For the case when w_1 is small when we can drop w_1^2 term, we get the number of bound states is unaltered. This is to be expected since the singular potential can be written as shifted singular potential: $V(r) \approx w_0\delta(r - R + \frac{2w_1}{w_0})$.

Energy eigenvalues are obtained numerically for different values of R, w_0 and w_1 by solving Eq. (9). Similarly we can obtain expectation values $\langle r_n \rangle$ by using appropriate $\mathcal{R}(r)$ in the two regions. The graphs demonstrate the number of bound

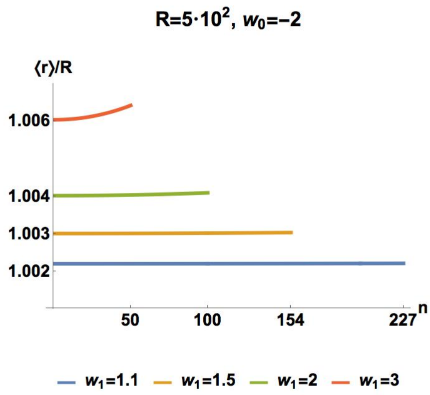

Fig. 2. $\langle r_n\rangle/R$ for large w_1.

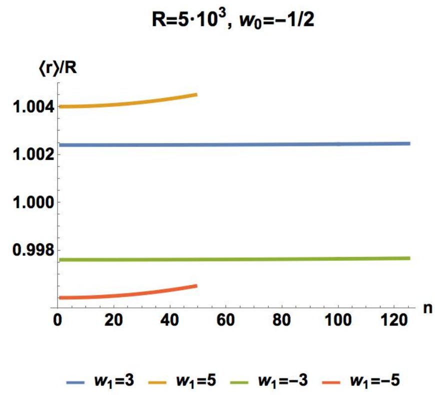

Fig. 3. Expectation $\langle r_n\rangle/R$ for large R.

states as well as energy eigenvalues (Fig. 1). They also explicitly show that they are localised close to the boundary and deviate externally or internally when we increase the coefficient of δ' potential (Figs. 2 and 3). Interestingly, the states are localised outside (inside) the boundary for positive (negative) w_1 respectively. Moreover, w_1 can be tuned to reduce the probability of finding the particle inside model black hole to be small. Also note that higher angular momenta states move closer to (away from) the boundary for negative (positive) w_1. We will remark about this in the conclusions.

2.3. *Three dimensions and BTZ black hole*

We present in this section the results for bound states in $d = 3$. For that we consider Schrödinger equation on $\mathbb{R}^3$ with a singular potential along a sphere: $V(r) = w_0\delta(r-R) + 2w_1\delta'(r-R)$. The required Schrödinger equation in spherical polar coordinates has solutions $\varphi(r,\theta,\phi) = \mathcal{R}(r)Y_{lm}(\theta,\phi)$ where Y_{lm} are spherical harmonics solving angular part of the equations. The radial part of the equation is

$$\frac{d^2\mathcal{R}}{dr^2} + \frac{2}{r}\frac{d\mathcal{R}}{dr} - \frac{l(l+1)\mathcal{R}}{r^2} = \lambda\mathcal{R}\,. \tag{11}$$

The solutions in the regions $r < R$ and $r > R$ are modified spherical Bessel functions: $\mathcal{R}(r) = c\frac{I_{l+\frac{1}{2}}(\sqrt{\lambda}r)}{\sqrt{r}}$ and $d\frac{K_{l+\frac{1}{2}}(\sqrt{\lambda}r)}{\sqrt{r}}$. Again matching the boundary conditions at $r = R$ and using Eq. (5), Eq. (9) gets modified to

$$-x\left(\frac{\alpha K_{l-\frac{1}{2}}}{K_{l+\frac{1}{2}}} + \frac{\alpha^{-1}I_{l-\frac{1}{2}}}{I_{l+\frac{1}{2}}}\right) - (l+1)(\alpha - \alpha^{-1}) = \bar{\beta}\,. \tag{12}$$

The maximum angular momentum allowed $l_{\max} = \frac{\bar{\beta}+\alpha}{\alpha+\alpha^{-1}}$. Each angular momentum l has degeneracy of $2l+1$ states. Hence the number of states up to $l_{\max} \propto 2l_{\max}^2$ we get the number of bound states $\propto R^2$.

We will briefly mention the toy model of black hole in 3D, BTZ black hole.[30,31] For simplicity we consider BTZ black hole metric with angular momentum $J = 0$. The metric is given by

$$ds^2 = -\left(\frac{r^2}{l^2} - M\right) dt^2 + \left(\frac{r^2}{l^2} - M\right)^{-1} dr^2 + r^2\, d\phi^2 \,. \tag{13}$$

Here, cosmological constant $\frac{1}{l^2}$ and M is the mass of the black hole and horizon is at $r = r_+ = \sqrt{M}l$. The solution of the scalar field in the presence of this metric along with singular potentials $\delta(r - r_+)$ and $\delta'(r - r_+)$ can be written as: $e^{-i\omega t + im\phi}\mathcal{R}(r)$. Defining $z = \frac{r^2 - r_+^2}{r^2}$, $\mathcal{F}(z) = z^{i\alpha}(1 - z)^{-\beta}\mathcal{R}(z)$, one gets the hypergeometric differential equation for $\mathcal{F}(z)$

$$z(1 - z)\frac{d^2\mathcal{F}}{dz^2} + (c - (1 + a + b))\frac{d\mathcal{F}}{dz} + ab\mathcal{F} = 0 \,, \tag{14}$$

where a, b, c, α and β are constants defined terms of r_+, l, m and ω. This hypergeometric equation has singularities at $z = 0, 1$. To obtain the bound states one should match the boundary conditions at $z = 0$ for the hypergeometric functions $\mathcal{F}(a, b, c, z)$.

3. Expanding Boundaries

In this brief section, we explore the singular potentials for introduction of time dependence in boundaries. The simplest case is in one dimension with $x > 0$: If the boundary is moving with uniform velocity $x = vt$ it can be studied as a quantum mechanical problem with delta function potential $\delta(x - vt)$. The solution is easy to get as $\psi(x, t) \propto e^{-|\kappa(x - vt)|}e^{-i(\kappa^2 - v^2/2)t - vx}$.

We can easily extend this analysis to a boundary with an acceleration "g" with a singular potential $\delta(x - \frac{gt^2}{2})$. This is unitarily equivalent to the static singular potential and an additional gravitational potential mgx. This can be seen by using the unitary transformations: $\phi(x, t) = UV\psi(x, t)$ where $V(x, t) = e^{-i\frac{g^2t^2p_x}{2}}$ and $U(x, t) = e^{igxt + i\frac{g^2t^3}{6}}$. The solutions for linear gravitational potential are given by Airy functions.

Similarly, we can consider $\mathbb{R}^2 - D$. If the disc is expanding it is better to convert the question to a delta function potential which is expanding.

Berry and Klein[32] showed the time-dependent

$$H(r, p, l(t)) = \frac{p^2}{2m} + \frac{1}{l^2}V(r/l) \,, \tag{15}$$

can be simplified if the time dependence is of the form $l(t) = \sqrt{at^2 + 2bt + c}$. It becomes in a comoving frame

$$H(\rho, \pi, k) = \frac{\pi^2}{2m} + V(\rho) + \frac{1}{2}k\rho^2 \,, \tag{16}$$

where $\rho = r/l$ and $k = m(ac - b^2)$ which is conserved in $\rho, \tau \equiv \int^t \frac{dt}{l^2(t)}$. The expanding disc in $\mathbb{R}^2$ and ball in $\mathbb{R}^3$ will come under this class of Hamiltonians. Consider the Hamiltonian[18] in $\mathbb{R}^2$

$$H = -\Delta + g\delta(r - e^f R(0))\,. \tag{17}$$

By rescaling r we get the potential as $e^f\delta(r - R(0))$. The time dependence is shifted to the strength of potential. This is analogous to changing the Hamiltonian to a time-dependent one by keeping the domain of the wave functions in the Hilbert space same for all times.

Applying Berry–Klein transformation[32] we can convert the problem in a co-moving frame to a time independent potential with a delta function along a ring. This will also correspond to generalised pantographic change of Anza *et al.*[33] This has important consequences for the rate of emission or in expanding statistical ensembles with new boundary conditions.

4. Conclusions

In this paper, we have approached the question of quantum black hole through straightforward analysis of quantum theory on manifolds with boundaries or equivalently singular potentials. While our study is in Euclidean space it can be applied to curved background also since point interactions are local. This can parallel the recent approach to understand black holes through conventional notions of particles and forces treating black holes just like atoms, molecules (see 't Hooft[34]). These require analysis through self-adjoint extensions of operator domains. Our analysis surprisingly brings out the importance of both δ and δ' potentials. There are a number bound states localised close to the boundary and is proportional to the area. As pointed out in the Introduction,[9] they relate to entropy in QFT. Hence the existence of correct behaviour of localised bound states on the boundary is a strong requirement for correct entropy. We also point out the role of δ' potential in extending the support of the bound states to enhanced length scales to allow for the possibility of quantum effects beyond Planck length.[35]

Following 't Hooft[4] one can consider scalar fields to vanish at a small distance away from the horizon. That is $\phi(R+h) = 0$. This is for small h equivalent to Robin boundary condition since by expanding around R we get $\phi(R) + h\phi'(R) \approx 0$. This boundary condition can also be obtained from δ function potential. Our potential is a generalisation of the potential which adds another parameter which allows the quantum effects to persist beyond the length parameter h. In Kruskal coordinates, one avoids the singularity of the metric at the horizon, but contain two copies of the spacetime. This is mimicked in our case of singular potentials connecting the two regions with suitable boundary conditions to maintain unitarity. Our generalised brick wall mechanism can be studied to obtain all the thermodynamic properties. Detailed analysis using these boundary conditions for the thermodynamic behaviour will be presented elsewhere with Rindler, BTZ and Schwarzschild background (under preparation).

These states are interestingly connected through spectrum generating algebra which is a sub-algebra of the Schrödinger group. By tuning the strength of the δ' potential, one can control the tunneling through the boundary. Lastly, if we scale the radius to ∞ keeping number of bound states fixed ($\frac{w_0}{w_1^2} \to 0$) the states become zero energy bound states and localised at the boundary and play significant role for asymptotic symmetries. The connections to QNM which arise from purely incoming modes is also intriguing. In addition, the singular potentials can be time-dependent to enable the analysis of expanding boundaries and associated radiation output. This study leads us to new avenues of exploration to situations where boundaries and boundary conditions are involved.[25]

Acknowledgments

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