A new measure of consensus with reciprocal preference relations: The correlation consensus degree

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Abstract

The achievement of a 'consensual' solution in a group decision making problem depends on experts' ideas, principles, knowledge, experience, etc. The measurement of consensus has been widely studied from the point of view of different research areas, and consequently different consensus measures have been formulated, although a common characteristic of most of them is that they are driven by the implementation of either distance or similarity functions. In the present work though, and within the framework of experts' opinions modelled via reciprocal preference relations, a different approach to the measurement of consensus based on the Pearson correlation coefficient is studied. The new correlation consensus degree measures the concordance between the intensities of preference for pairs of alternatives as expressed by the experts. Although a detailed study of the formal properties of the new correlation consensus degree shows that it verifies important properties that are common either to distance or to similarity functions between intensities of preferences, it is also proved that it is different to traditional consensus measures. In order to emphasise novelty, two applications of the proposed methodology are also included. The first one is used to illustrate the computation process and discussion of the results, while the second one covers a real life application that makes use of data from Clinical Decision-Making.

Keywords: Reciprocal preference relations, Consensus measure, Pearson correlation coefficient, Concordance opinions measure, Correlation consensus degree.

1. Introduction

Consensus reaching is an important component in decision making processes, and indeed it plays a key role in the resolution process of group decision making problems. One of the most significant current discussion in consensus research concerns the measurement and achievement of consensus from both a theoretical and applied points of view. On the one hand, establishing and characterising different methodologies to measure consensus have been addressed from a Social Choice perspective [1, 3, 13]. On the other hand, within the Decision Making Theory framework, modelling group decision making problems in order to reach a higher level of cohesiveness has been managed successfully [15, 32, 34, 38, 39, 65]. Outside of these main areas, it is possible to find other methodologies that use the idea of consensus in different ways to the aforementioned ones, with [41, 46] being representative examples of these methodologies.

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Despite the productive research on this area, consensus measurement is still an open-ended research question because the methodology to use in each case is an essential component of the problem. Up to now most studies on consensus measurement have focused on the use of distance/similarity function based measures and association measures, respectively. Among the distance functions used, and worth highlighting, are the Kemeny, Mahalanobis, Mannhattan, Jacard, Dice and Cosine distance functions [1, 4, 6, 17, 19, 29, 31]. Association measures are less widely used than distance functions but it is also possible to find the use of some of them such as the Kendall's coefficient, the Goodman-Kruskal's index and the Spearman's coefficient [18, 24, 35, 44, 58].

In this paper we focus on establishing a new consensus measure following the tradition of association measures. Our proposal is based on the original statistical correlation concept, the *Pearson correlation coefficient*. Therefore, this new measure is an alternative to the use of the aforementioned approaches. The Pearson correlation coefficient plays an important role in Statistics and Data Analysis and it is extensively used as a measure of the degree of linear dependence between two variables. It is easy to interpret as well as invariant to certain changes in the variables [52, 55, 57]. Specifically, in this paper the notion of dependence among elements from correlation coefficient as a measure of the cohesiveness between opinions is adopted. This seems natural because the measurement of consensus resembles the notion of a "measure of statistical correlation", in the sense that the maximum value 1 captures the notion of unanimity as a perfect relationship among agents' preferences (experts' preferences follow the same direction), while the minimum value -1 captures the notion of total disagreement (experts' preferences present a negative relationship). Furthermore, the higher the cohesiveness between experts' preferences, the more positive correlated the preferences are. Similarly, the lower the cohesiveness between experts' preferences, the more negative correlated the preferences are.

This new consensus measure will be developed within assumptions of experts' opinions or preferences being expressed by means of reciprocal preference relations, a framework that is currently of interest to the research community in decision theory under uncertainty [7, 27, 28, 45]. Under reciprocal preference relations, on the one hand and as it was mentioned above, the new proposed approach inherits advantages of previous approaches based on traditional distance/similarity and association measures. On the other hand, maximum consensus traditionally represents the case when experts provide the same preference intensities for each possible pair of alternatives. This, though, is not the only possible scenario of maximum consensus. Indeed, the proposal here put forward addresses this issue satisfactorily because maximum possible cohesiveness or consensus between experts' opinions does not necessary imply that all reciprocal preference relations have to coincide, and therefore all experts do not necessary need to have the same preference intensities in all possible pairs of alternatives. It is sufficient, though, that experts rank alternatives in the same way. To support all these claims, a set of properties verified by the new proposed measure of consensus, the *correlation consensus degree*, are proved. These properties ensure the suitability of the correlation consensus degree. Furthermore, in order to emphasise novelty, two applications of the proposed methodology are also included. The first one is used to illustrate the computation process and discussion of the results, while the second one covers a real life application that makes use of data from Clinical Decision-Making.

The rest of the paper is organised as follows. Section 2 contains a brief overview of the different approaches in literature to measure group cohesiveness. The basic notation and preliminaries are presented in Section 3. Section 4 provides the new approach to consensus measurement based on the Pearson correlation coefficient. In Section 5, properties of the new correlation consensus degree are studied. Section 6 presents two practical applications of the proposed methodology. Finally, some concluding remarks and future research are presented in Section 7.

2. Consensus measurement in the literature

A considerable amount of literature has been published on measuring and reaching consensus in group decision making problems. Consensus measurement is a prominent and active research subject in several areas such as Social Choice Theory and Decision Making Theory. A brief overview of how this issue has been addressed in recent literature from the aforementioned research areas is provided.

From the Social Choice Theory, the first serious discussions and analysis of consensus measurement from an Arrovian perspective emerged with Bosch's PhD Thesis [13], where both absolute and intrinsic measures of consensus were proposed, analysed and axiomatically characterised. From the point of view of considering consensus among a family of voters, McMorris and Powers [48] characterised consensus rules defined on hierarchies, while García-Lapresta and Pérez-Román [29] focused on how to measure consensus using complete preorders on alternatives and introduced a class of consensus measures based on seven well-known distances. Subsequently, Alcalde-Unzu and Vorstatz in [1] characterised a family of linear and additive consensus measures, whereas in [2] new ways to measure the similarity of preferences in a group of individuals were suggested. Alcantud, de Andrés Calle and Cascón [3] studied and characterised a class of consensus measure, called *referenced consensus measure*, that permits to produce a numerical social evaluation from purely ordinal individual information. This measure has to be specified by means of a voting mechanism and a measure of agreement between profiles of orderings and individual orderings. Moreover, Alcantud, de Andrés Calle and Cascón in [5] contributed to the formal and computational analysis of the aforementioned referenced consensus measure by focusing on two relevant and specific cases: the Borda and the Copeland rules under a Kemeny-type measure. There are, however, situations where each member of a population classifies a list of options as either acceptable or non-acceptable; either agree or disagree, etc., and therefore generating a dichotomous preference structure. Under this assumption, Alcantud, de Andrés Calle and Cascón [4] proposed the concept of *approval consensus measure* and gave axiomatic characterisations of two generic classes of such approval consensus measures. Alcantud, de Andrés Calle and González-Arteaga [6] introduced the use of the Mahalanobis distance for the analysis of the cohesiveness of a group of complete preorders and proved that arbitrary codifications of the preferences are incompatible with their formulation although affine transformations permit to compare profiles on the basis of such proposal. Finally, it is worth mentioning a distance-based approach to measure the degree of consensus considering approval information about alternatives as well as the rankings of them suggested by Erdamar et al. in [25].

From the Decision Making Theory, a considerable amount of contributions have been made since the 1980's. As such, it is worth mentioning the first preliminary work on reaching consensus and its measurements carried out by Kacprzyk and Fedrizzi [42], in which the concept of "degree of consensus" in the sense of expressing the degree to which "most of" the individuals in a group agree to "almost all of" the options. The point of departure of this paper being that the experts' opinions are expressed by fuzzy preference relations. Within this framework of preference representation, different consensus measurement based on similarity measures have been put forward by Herrera-Viedma, et al. [37] and Wu and Chiclana [63] for both complete and incomplete information environments. The case when experts' opinions are expressed by means of linguistic assessments has been extensively studied and it is worth mentioning the works of Ben-Arieh and Chen [12], Cabrerizo, Alonso and Herrera-Viedma [14], García-Lapresta, Pérez-Román [30], Herrera, Herrera-Viedma and Verdegay [36], Herrera-Viedma, et al. [40], Pérez-Asurmendi and Chiclana [53] and Wu, Chiclana and Herrera-Viedma [65]. Finally, models to reach consensus where experts assess their preferences using different preference representation structures (preference orderings, utility functions, multiplicative preference relations and fuzzy preference relations) have also been studied and proposed by Dong and Zhang

[23], Fedrizzi et al. [26] and Herrera-Viedma, Herrera and Chiclana [39]. The problem of measuring and reaching consensus with intuitionistic fuzzy preference relations and triangular fuzzy complementary preference relations have also been covered in detail by Wu and Chiclana in [62, 64].

To conclude, Table 1 summarises and classifies the approaches that have been reviewed in this Section.

Consensus measures in Social Choice Theory

Author(s)/Year	Framework	Measurement methodology
Bosch [13], 2005	Ordinal Inf.	
McMorris and Powers [48], 2009	Ordinal Inf.	
García-Lapresta and Pérez-Román [29], 2011	Ordinal Inf.	
Alcalde and Vorsatz [1], 2013	Ordinal Inf.	
Alcantud, de Andrés Calle and Cascón [3] [5], 2013	Ordinal Inf.	
Alcantud, de Andrés Calle and Cascón [4], 2013	Dichotomous Inf.	Based on different distances
Alcantud, de Andrés Calle and González-Arteaga [6], 2013	Ordinal Inf.	
Erdamar, et al. [25], 2014	Ordinal Inf.	
Alcalde and Vorsatz [2], 2015	Ordinal Inf.	

Consensus measures in Decision Making Theory

Author(s)/Year	Framework	Measurement methodology
Kacprzyk and Fedrizzi [42], 1988	Fuzzy Inf.	
Fedrizzi et al. [26], 2010	Fuzzy Inf.	
Herrera-Viedma et al. [37], 2007	Incomplete Fuz. Inf.	
Herrera, Herrera-Viedma and Verdegay [36], 1996	Linguistic Inf.	
Herrera-Viedma et al. [40], 2005	Linguistic Inf.	Based on collective solution
Cabrerizo, Alonso and Herrera-Viedma [14], 2009	Linguistic Inf.	
Wu and Chiclana [62–64], 2014	Incomplete Fuz. and Ling. Inf.	
García-Lapresta, Pérez-Román and Falcó [30], 2015	Linguistic Inf.	
Wu, Chiclana and Herrera-Viedma [65], 2015	Incomplete Linguistic Inf.	
Herrera-Viedma, Herrera and Chiclana [39], 2002	Different Inf.	Based on individual solution
Ben-Arieh and Chen [12], 2006	Linguistic Inf.	
Dong and Zhang [23], 2014	Different Inf.	

Table 1: Summary table of studies related to consensus measures

3. Preliminaries

This Section briefly presents the main concepts needed to make the paper self-contained, and as such a short review of the terminology and the concept of fuzzy binary relation are presented. The interested reader is advice to consult the following [7–9, 27, 28, 45, 50, 60].

Definition 1. *Let X be a non empty set. A* fuzzy binary relation $\mathcal{P}$ *on X is a fuzzy subset of the Cartesian product $X \times X$ characterised by its membership function $\mu_{\mathcal{P}} \colon X \times X \longrightarrow [0,1]$, where $\mu_{\mathcal{P}}(x_1, x_2) = p_{ij}$ represents the strength of the relation between x_1 and x_2.*

Henceforth, X is a finite set $X = \{x_1 \ldots, x_n\}$ $(n > 2)$, whose elements will be referred to as alternatives. Abusing notation, on occasions alternative x_i will be represented simply as i for convenience.

Definition 2. *A* reciprocal preference relation *on X is a fuzzy binary relation $\mathcal{P}$ where $\mu_{\mathcal{P}}(x_i, x_j) = p_{ij} \in [0,1]$ represents the partial preference intensity of element i over j and that verifies the following property: $p_{ij} + p_{ji} = 1 \ \forall x_i, x_j \in X$.*

In order to realise the meaning of a reciprocal preference relation, we suppose the following common situation: an expert compares two alternatives x_i and x_j. In this specific context, the

expert not only establishes that the alternative x_i is preferred to the alternative x_j, but also shows her/his intensity of preference between them by means of the value p_{ij}. So, the higher p_{ij}, the higher the preference intensity of alternative x_i over alternative x_j. Thus, $0 < p_{ij} < 0.5$ would indicate that x_j is preferred to x_i. If $p_{ij} = 0.5$ then alternatives x_i and x_j are equally preferred. When $0.5 < p_{ij} < 1$, x_i is preferred to x_j. Moreover, $p_{ij} = 0$ (resp. $p_{ij} = 1$) indicates that x_j (resp. x_i) is absolutely preferred to x_i (resp. x_j).

Let P be an $n \times n$ matrix that contains all the partial intensity degrees of a reciprocal preference relation on the set X:

$$
P = \begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1n} \\
p_{21} & p_{22} & \cdots & p_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n1} & p_{n2} & \cdots & p_{nn}
\end{pmatrix},
$$

verifying $0 \leq p_{ij} \leq 1$; $p_{ij} + p_{ji} = 1$ for $i, j \in \{1, \ldots, n\}$. The set of all these matrices $n \times n$ is denoted by $\mathbb{P}_{n \times n}$. Here it is also noticed that a reciprocal preference relation can also be mathematically represented by means of a vector, namely the *essential vector of preference intensities*.

Definition 3. *The* essential vector of preference intensities, V_P, *of a reciprocal preference relation* $P = (p_{ij})_{n \times n} \in \mathbb{P}_{n \times n}$ *is the vector made up with the* $\dfrac{n(n-1)}{2}$ *elements above its main diagonal:*

$$
\begin{aligned}
V_P &= \left(p_{12}, p_{13}, \ldots, p_{1n}, p_{23}, \ldots, p_{2n}, \ldots, p_{(n-1)\,n} \right) = \\
&= \left(v_1, \ldots, v_r, \ldots, v_{n(n-1)/2} \right).
\end{aligned}
$$

The reciprocity property of reciprocal preference relations allows the alternative definition of the essential vector of preference intensities of a reciprocal preference relation as the vector composed of the preference values below the main diagonal, $V_{P^t} = (p_{21}, p_{31}, \ldots, p_{n1}, p_{32}, \ldots, p_{n2}, \ldots, p_{n\,(n-1)})$.

4. A novel measurement of consensus based on the Pearson correlation coefficient

Based on the concept of correlation, specifically the Pearson correlation coefficient, this section introduces a new consensus measure for group decision making problems under reciprocal preference relations. First, we recall such a correlation coefficient and its properties as necessary to define the new correlation consensus degree and associated properties.

4.1. Pearson correlation coefficient

The measurement of the relationship strength among variables is an important issue in Statistical Analysis, and the *Pearson correlation coefficient* is a traditional tool used for that purpose [52, 55].

Definition 4. *Given a sample of n pairs of real values $\{(x_1, y_1), \ldots, (x_n, y_n)\}$, the* Pearson correlation coefficient *of the two n-dimensional vectors* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$, $cor(\mathbf{x}, \mathbf{y})$, *is computed as*

$$
cor(\mathbf{x}, \mathbf{y}) = \frac{\displaystyle\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\displaystyle\sum_{i=1}^{n}(x_i - \overline{x})^2}\sqrt{\displaystyle\sum_{i=1}^{n}(y_i - \overline{y})^2}}
$$

5

where $\overline{x} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i$ and $\overline{y} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} y_i$ are the arithmetic means of $\mathbf{x}$ and $\mathbf{y}$, respectively.

The standard interpretation of the Pearson correlation coefficient states that positive coefficient values point out a positive tendency relationship between $\mathbf{x}$ and $\mathbf{y}$ i.e., $\mathbf{x}$ and $\mathbf{y}$ increase (decrease) in the same direction. Negative correlation coefficient values point out towards a reverse direction between $\mathbf{x}$ and $\mathbf{y}$. In addition, the nearer the absolute correlation coefficient value is to 1, the stronger and more linear the tendency is. The Pearson correlation coefficient verifies the following well-known properties [57]:

1. $cor(\mathbf{x}, \mathbf{y}) \in [-1, 1]\ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

2. $cor(\mathbf{x}, \mathbf{y}) = cor(\mathbf{y}, \mathbf{x})\ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

3. $cor(\mathbf{x}, \mathbf{x}) = 1\ \forall \mathbf{x} \in \mathbb{R}^n$.

4. If $cor(\mathbf{x}, \mathbf{y}) = 1$ then there exists a perfect positive linear correlation between $\mathbf{x}$ and $\mathbf{y}$, i.e. $\exists a \in \mathbb{R}, b \in \mathbb{R}^+ : \mathbf{y} = a \cdot \mathbf{1} + b \cdot \mathbf{x}$ where $\mathbf{1} = (1, \dots, 1)$ is a vector of n ones. Respectively, if $cor(\mathbf{x}, \mathbf{y}) = -1$ there exists a perfect negative linear correlation between $\mathbf{x}$ and $\mathbf{y}$.

5. Let $\mathbf{x}' = a \cdot \mathbf{1} + b \cdot \mathbf{x}$ and $\mathbf{y}' = c \cdot \mathbf{1} + d \cdot \mathbf{y}$ be two vectors with $a, b, c, d \in \mathbb{R}$, b and d non zero and of equal sign (both positive or both negative). Then, $cor(\mathbf{x}', \mathbf{y}') = cor(\mathbf{x}, \mathbf{y})$.

4.2. A new consensus measure: Correlation consensus degree

From the Social Choice Theory perspective, the measurement of the degree of agreement in a group is associated the range $[0, 1]$, with 0 representing total lack of agreement and 1 unanimous agreement [1, 4, 13]. Also, as aforementioned in Section 1, the measurement of the degree of cohesiveness in a group has been based on the notion of distance or similarity between opinions or preferences of the members of such group. In this paper, a new way to measure the degree of consensus in a group based on the Pearson correlation coefficient, the *correlation consensus degree* ($\mathcal{CCD}$), is proposed within the framework of opinions on a set of elements, alternatives or options being represented by reciprocal preference relations.

A set of agents or experts will be represented by a finite subset $\mathbf{E} = \{1, 2, ..., m\}$ of natural numbers, $m \geq 2$. Assume that the m experts provide their pairwise preferences on a finite set of n alternatives, $n \geq 3$, $X = \{x_1, ..., x_n\}$ using fuzzy preference relations $\{P^{(1)}, \dots, P^{(m)}\}$. As per Definition 3, the essential vector of preference intensities associated to $P^{(k)}$ will be denoted by $V_{P^{(k)}}$.

Definition 5. The *correlation consensus degree*, $\mathcal{CCD}$, for reciprocal preference relations is a mapping $\mathcal{CCD} : \mathbb{P}_{n \times n} \times \mathbb{P}_{n \times n} \to [0, 1]$ that associates a pair of reciprocal preference relations $(P^{(1)}, P^{(2)})$ the following $[0,1]$-value:

$$\mathcal{CCD}(P^{(1)}, P^{(2)}) = \frac{1}{2}\left(1 + cor(V_{P^{(1)}}, V_{P^{(2)}})\right). \tag{1}$$

Given $P^{(1)}, P^{(2)} \in \mathbb{P}_{n \times n}$, the elaborated expression of $\mathcal{CCD}(P^{(1)}, P^{(2)})$ is

$$\mathcal{CCD}(P^{(1)}, P^{(2)}) = \frac{1}{2}\left(1 + \frac{\displaystyle\sum_{r=1}^{n(n-1)/2}\left(v_r^{(1)} - \overline{V}_{P^{(1)}}\right)\left(v_r^{(2)} - \overline{V}_{P^{(2)}}\right)}{\sqrt{\displaystyle\sum_{r=1}^{n(n-1)/2}\left(v_r^{(1)} - \overline{V}_{P^{(1)}}\right)^2}\sqrt{\displaystyle\sum_{r=1}^{n(n-1)/2}\left(v_r^{(2)} - \overline{V}_{P^{(2)}}\right)^2}}\right)$$

where $\overline{V}_{P^{(1)}} = \dfrac{1}{n(n-1)/2} \displaystyle\sum_{r=1}^{n(n-1)/2} v_r^{(1)}$ and $\overline{V}_{P^{(2)}} = \dfrac{1}{n(n-1)/2} \displaystyle\sum_{r=1}^{n(n-1)/2} v_r^{(2)}.$ [1]

Notice that the higher the value of $\mathcal{CCD}(P^{(1)}, P^{(2)})$, the more positive correlated the reciprocal preferences of $P^{(1)}$ and $P^{(2)}$ are. The maximum possible value $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$ implies that $cor(V_{P^{(1)}}, V_{P^{(2)}}) = 1$ which, contrary to previous consensus measures based on distance/similarity functions, does not necessarily implies that both reciprocal preference relations coincide. Consequently, $\mathcal{CCD}$ could be 1 even in cases when experts provide different preferences, although positive linearly correlated. On the other hand, the lower the value of $\mathcal{CCD}(P^{(1)}, P^{(2)})$, the more negative correlated the reciprocal preference intensities are, with $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 0$ representing the case when preferences are negative linearly correlated. The following proposition reflects these limit cases:

Proposition 1. *Let $P^{(1)}$, $P^{(2)} \in \mathbb{P}_{n \times n}$ be two reciprocal preference relation matrices. Then $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$ (resp. $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 0$) if and only if $\exists a \in \mathbb{R}$, $b > 0$ (resp. $b < 0$) such that: $p_{ij}^{(2)} = a + b \cdot p_{ij}^{(1)}$ $\forall i < j$; $p_{ij}^{(2)} = (1 - a - b) + b \cdot p_{ij}^{(1)}$ $\forall i > j$.*

Proof. Using Equation (1), we have that $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$ if and only if $cor(V_{P^{(1)}}, V_{P^{(2)}}) = 1$. Property 4 of the Pearson correlation coefficient (Section 4.1) implies that $\exists a \in \mathbb{R}, b \in \mathbb{R}^+$ such that $V_{P^{(2)}} = a \cdot \mathbf{1} + b \cdot V_{P^{(1)}}$, being $\mathbf{1} = (1, \ldots, 1)$ a vector of ones with suitable dimension, in this cases $n(n-1)/2$, i.e.:

$$p_{ij}^{(2)} = a + b \cdot p_{ij}^{(1)} \ \forall i < j.$$

When $j < i$, reciprocity of preferences means that

$$p_{ij}^{(2)} = 1 - p_{ji}^{(2)} = 1 - (a + b \cdot p_{ji}^{(1)}) = 1 - (a + b \cdot (1 - p_{ij}^{(1)})) = (1 - a - b) + b \cdot p_{ij}^{(1)}.$$

The proof for the case when $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 0$ is obtained accordingly. $\qquad\square$

Notice that if the set of alternatives is small, the experts can easily rank the alternatives and the possibility that the experts do it in a similar way (or opposite way) is high. Then, in this case the absolute value of the correlation coefficient tend to be close to 1. Meanwhile, when the set of alternatives is large, the experts may find it difficult to rank them (see [49]) and the possibility that the experts rank the alternatives in a similar way (or opposite way) is low. Then, in this case it is easy that the absolute value of the correlation coefficient becomes small.

The following proposition provides the sufficient condition for the correlation consensus degree to coincide for different pairs of reciprocal preference relations.

Proposition 2. *Let $P^{(1)}$, $P^{(2)} \in \mathbb{P}_{n \times n}$ be reciprocal preference relation matrices such that $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$, then*

$$\mathcal{CCD}(P, P^{(1)}) = \mathcal{CCD}(P, P^{(2)}) \quad \forall P \in \mathbb{P}_{n \times n}.$$

Proof. By Proposition 1, $\exists a \in \mathbb{R}$ and $b > 0$ such that $V_{P^{(2)}} = a \cdot \mathbf{1} + b \cdot V_{P^{(1)}}$. Applying Property 5 of the Pearson correlation coefficient (see Subsection 4.1) we have that $cor(V_P, V_{P^{(1)}}) = cor(V_P, V_{P^{(2)}}) \ \forall P \in \mathbb{P}_{n \times n}$ and by Definition 5 it is equivalent to $\mathcal{CCD}(P, P^{(1)}) = \mathcal{CCD}(P, P^{(2)}) \ \forall P \in \mathbb{P}_{n \times n}$. $\qquad\square$

The measurement of the degree of agreement among the preferences expressed by two or more experts can be captured by using a summary measure like the mean of all possible correlation consensus degrees between all different pairs of experts' reciprocal preference relations.

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[1]$\overline{V}_{P^{(i)}}$ summarizes the general level of uncertainty of the expert i on the set of alternatives.

The use of aggregation functions to merge inputs into a single output has been extensively analysed in literature [11, 28, 33, 43]. In the Decision Making context, the use of aggregation functions to derive the degree of agreement among a group of experts has been justified (see for example [11, 29, 31, 43, 47]). Recall that the main aim of considering aggregation functions is to produce an overall output that can be considered representative of the aggregated values by incorporating desirable properties. The *arithmetic mean* has been widely investigated and it is considered the most common central tendency aggregation function. All these considerations are used to motivate the definition of the new correlation group consensus measure, $\mathcal{CD}$, within a reciprocal preference relation framework.

Definition 6. Let $\mathbf{E}$ be a group of m experts with associated fuzzy preference relation relations $P^{(1)}, \ldots, P^{(m)} \in \mathbb{P}_{n \times n}$ on a set of alternatives X. The *group consensus degree* among the set of experts is

$$\mathcal{CD}(\mathbf{E}) = \frac{2}{m(m-1)} \sum_{k=1}^{m-1} \sum_{l=k+1}^{m} \mathcal{CCD}(P^{(k)}, P^{(l)}).$$

4.3. Consistency under maximum correlation consensus degree

Given a reciprocal preference relation on a set of alternatives, the concept of *non-dominance degree* introduced by Orlovsky [51] has been extensively used to rank the alternatives [10, 23, 38, 61, 63, 65, 66]. In the following, and in order to improve the understanding of the proposed correlation consensus degree, the consistency of the correlation consensus degree with Orlovsky's non-dominance degree is proved. Specifically, it is proved that when two reciprocal preference relations have a CCD equal to 1 then their Orlovsky's non-dominance degree orderings of the set of alternatives coincide. First, the concept of non-dominance degree is provided.

Given a reciprocal preference relation on a finite set of alternatives X, $P = (p_{ij})_{n \times n} \in \mathbb{P}_{n \times n}$, when $p_{ji} - p_{ij} > 0$ then alternative x_i is dominated by alternative x_j. Formally, it can be stated that alternative x_i is dominated by alternative x_j at degree $d(x_i, x_j) = \max\{p_{ji} - p_{ij}, 0\}$. Thus, the value $1 - d(x_i, x_j) = 1 - \max\{p_{ji} - p_{ij}, 0\}$ represents the degree of non-dominance of alternative x_i by alternative x_j. The degree up to which x_i is not dominated by any of the elements of X is known as the non-dominance degree of alternative x_i. This is summarised in the following definition.

Definition 7. *Let $P = (p_{ij})_{n \times n} \in \mathbb{P}_{n \times n}$ be a reciprocal preference relation on X. The* non-dominance degree *is a mapping $\mu_{ND} \colon X \longrightarrow [0, 1]$ such that*

$$\mu_{ND}(x_i) = \min_{j : j \neq i} \{1 - d(x_i, x_j)\},$$

where $d(x_i, x_j) = \max\{p_{ji} - p_{ij}, 0\}$.

The aforementioned non-dominance degree can be used to provide a total ordering of alternatives by means of the following rule:

$$x_i \succeq x_j \Leftrightarrow \mu_{ND}(x_i) \geq \mu_{ND}(x_j).$$

Notice that $p_{ij} - p_{ji} = -(p_{ji} - p_{ij})$, and therefore to compute $d(x_j, x_i) = \max\{p_{ji} - p_{ij}, 0\}$ when $j > i$, we use $d(x_j, x_i) = \max\{-(p_{ji} - p_{ij}), 0\}$. Now we are in disposition of introduce the following result.

Proposition 3. *Let $P^{(1)}, P^{(2)} \in \mathbb{P}_{n \times n}$ be two reciprocal preference relation matrices such that $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$ and $2a + b = 1$. The non-dominance based orderings of the set of alternatives derived from both reciprocal preference relation matrices are identical.*

Proof. Let $P^{(1)}, P^{(2)} \in \mathbb{P}_{n \times n}$ such that $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$. By Proposition 1, $\exists a \in \mathbb{R}$ and $b > 0$ such that $p_{ij}^{(2)} = a + b \cdot p_{ij}^{(1)} \ \forall i < j$ and $p_{ij}^{(2)} = 1 - p_{ji}^{(2)} = 1 - (a + b \cdot p_{ji}^{(1)}) = 1 - (a + b \cdot (1 - p_{ij}^{(1)})) = (1 - a - b) + b \cdot p_{ij}^{(1)} \ \forall i < j$.

1. Notice that:

   (a) If $i < j$ then

$$
\begin{aligned}
p_{ji}^{(2)} - p_{ij}^{(2)} &= [(1 - a - b) + b \cdot p_{ji}^{(1)}] - [a + b \cdot p_{ij}^{(1)}] = (1 - 2a - b) + b \cdot (p_{ji}^{(1)} - p_{ij}^{(1)}) = \\
&= b \cdot (p_{ji}^{(1)} - p_{ij}^{(1)}).
\end{aligned}
$$

   (b) If $i > j$ then

$$
\begin{aligned}
p_{ji}^{(2)} - p_{ij}^{(2)} &= [a + b \cdot p_{ji}^{(1)}] - [(1 - a - b) + b \cdot p_{ij}^{(1)}] = -(1 - 2a - b) + b \cdot (p_{ji}^{(1)} - p_{ij}^{(1)}) = \\
&= b \cdot (p_{ij}^{(1)} - p_{ji}^{(1)}).
\end{aligned}
$$

   Thus:
$$
\forall i, j : \ p_{ji}^{(2)} - p_{ij}^{(2)} = b \cdot (p_{ij}^{(1)} - p_{ji}^{(1)}).
$$

2. Let us denote by $\mu_{ND^{(1)}}(x_i)$ and $\mu_{ND^{(2)}}(x_i)$ the non-dominance choice degree associated to alternative x_i obtained from $P^{(1)}$ and $P^{(2)}$, respectively. It is:

$$
\mu_{ND^{(2)}}(x_i) = \min_{x_j \in X} \left\{ 1 - \max\{p_{ji}^{(2)} - p_{ij}^{(2)}, 0\} \right\}.
$$

   Because $b > 0$ we have that $p_{ji}^{(2)} - p_{ij}^{(2)}$ and $p_{ij}^{(1)} - p_{ji}^{(1)}$ are both negative, both positive or both equal to zero. Therefore, it is:

$$
\max\{p_{ji}^{(2)} - p_{ij}^{(2)}, 0\} = \max\{b \cdot (p_{ij}^{(1)} - p_{ji}^{(1)}), 0\} = b \cdot \max\{p_{ji}^{(1)} - p_{ij}^{(1)}, 0\}. \tag{2}
$$

   Let l be such that

$$
\mu_{ND^{(1)}}(x_i) = \min_{j : j \neq i} \left\{ 1 - \max\{p_{ji}^{(1)} - p_{ij}^{(1)}, 0\} \right\} = 1 - \max\{p_{li}^{(1)} - p_{il}^{(1)}, 0\}.
$$

   The following inequalities yield:

$$
1 - \max\{p_{li}^{(1)} - p_{il}^{(1)}, 0\} \leq 1 - \max\{p_{ji}^{(1)} - p_{ij}^{(1)}, 0\} \quad \text{for } j = 1, \dots, n.
$$

   They can be re-written equivalently as

$$
\max\{p_{li}^{(1)} - p_{il}^{(1)}, 0\} \geq \max\{p_{ji}^{(1)} - p_{ij}^{(1)}, 0\} \quad \text{for } j = 1, \dots, n.
$$

   Consequently,

$$
1 - b \cdot \max\{p_{li}^{(1)} - p_{il}^{(1)}, 0\} \leq 1 - b \cdot \max\{p_{ji}^{(1)} - p_{ij}^{(1)}, 0\} \quad \text{for } j = 1, \dots, n.
$$

   Relation (2) implies that

$$
\mu_{ND^{(2)}}(x_i) = \min_{j : j \neq i} \left\{ 1 - \max\{p_{ji}^{(2)} - p_{ij}^{(2)}, 0\} \right\} = 1 - \max\{p_{li}^{(2)} - p_{il}^{(2)}, 0\}. \tag{3}
$$

3. Finally, let us assume now that

$$\mu_{ND^{(1)}}(x_i) \leq \mu_{ND^{(1)}}(x_k).$$

Then there exist l and s such that

$$1 - \max\{p_{li}^{(1)} - p_{il}^{(1)}, 0\} = \mu_{ND^{(1)}}(x_i) \leq \mu_{ND^{(1)}}(x_k) = 1 - \max\{p_{sk}^{(1)} - p_{sk}^{(1)}, 0\}$$

The following inequality derives from it:

$$1 - b \cdot \max\{p_{li}^{(1)} - p_{il}^{(1)}, 0\} \leq 1 - b \cdot \max\{p_{sk}^{(1)} - p_{sk}^{(1)}, 0\}.$$

Applying again relation (2) and also expression (3), it can be concluded that

$$\mu_{ND^{(1)}}(x_i) \leq \mu_{ND^{(1)}}(x_k).$$

$\square$

5. Formal properties of the new consensus measure

As shown in Subsection 4.2, given a set of m experts $\mathbf{E}$, the following correlation consensus degree matrix can be computed

$$\mathcal{CCD} = (\mathcal{CCD}_{ij})$$

with $\mathcal{CCD}_{ij} = \mathcal{CCD}(P^{(i)}, P^{(j)})$. The following important properties are verified:

Reflexivity: $\mathcal{CCD}_{ii} = 1 \ \forall i$.

The proof is immediate from the properties of the Pearson correlation coefficient. This property rules out that the correlation consensus degree is a distance function, which will be pointed out at the end of this section.

Selfconsensus: $\mathcal{CCD}_{ij} \leq \mathcal{CCD}_{ii} \ \forall i, j$.

In other words, the correlation consensus degree between one expert and herself/himself is not lower than the correlation consensus degree with another expert. This is obvious from Definition 5 and the reflexibity property above.

Reciprocity: It was mentioned in Definition 3 that the essential vector of preference intensities of a reciprocal preference relation may also be defined as the vector with elements the preference values below the main diagonal of the reciprocal preference relation. Denoting by $\mathcal{CCD}_{ij}^t$ the correlation consensus degree between the reciprocal preference relations $P^{(i)}$ and $P^{(j)}$ using essential vector of preference intensities below their main diagonal, respectively, we have that:

$$\mathcal{CCD}_{ij}^t = \mathcal{CCD}_{ij} \quad \text{for } i, j = 1, \ldots, m.$$

Indeed, because $P^{(i)}$ and $P^{(j)}$ are reciprocal then we have that $V_{P^{(i)}t} = \mathbf{1} - V_{P^{(i)}}$ and $V_{P^{(j)}} = \mathbf{1} - V_{P^{(j)}t}$, respectively. Applying Property 4 of the Pearson correlation coefficient (Subsection 4.1), it is true that $cor(V_{P^{(i)}t}, V_{P^{(j)}t}) = cor(V_{P^{(i)}}, V_{P^{(j)}})$, and consequently it is $\mathcal{CCD}_{ij}^t = \mathcal{CCD}_{ij}$.

Symmetry: $\mathcal{CCD}_{ij} = \mathcal{CCD}_{ji}$ for $i, j = 1, \ldots, n$. The proof is straightforward from the symmetry property of the Pearson correlation coefficient.

Transitivity under the maximum: If $\mathcal{CCD}_{ij} = 1$ and $\mathcal{CCD}_{jk} = 1$ then $\mathcal{CCD}_{ik} = 1$.

In other words, when an expert has maximum correlation consensus degree with two different experts, then these two experts have also maximum correlation consensus degree. Indeed, from Proposition 1 we have that $V_{P(j)} = a \cdot \mathbf{1} + b \cdot V_{P(i)}$ for some $a \in \mathbb{R}$ and $b > 0$ and $V_{P(k)} = a' \cdot \mathbf{1} + b' \cdot V_{P(j)}$ for some $a' \in \mathbb{R}$ and $b' > 0$. Consequently, it is: $V_{P(k)} = a' \cdot \mathbf{1} + b' \cdot (a \cdot \mathbf{1} + b \cdot V_{P(i)}) = a' \cdot \mathbf{1} + b'a \cdot \mathbf{1} + b'b \cdot V_{P(i)}$, that is, $V_{P(k)} = a'' \cdot \mathbf{1} + b'' \cdot V_{P(i)}$ and because $b'' > 0$ it is $\mathcal{CCD}_{ik} = 1$.

Reversibility: The complementary reciprocal preference relation of a given reciprocal preference relation P, $\overline{P}$, is defined as follows: $\overline{P} = (1)_{n \times n} - P$. It is:

$$\mathcal{CCD}(P, \overline{P}) = 0.$$

It is obvious that $V_{\overline{P}} = \mathbf{1} - V_P$ and therefore applying Proposition 1 it is $\mathcal{CCD}(P, \overline{P}) = 0$.

The correlation consensus degree, $\mathcal{CCD}$, is neither a distance function, d, nor a similarity function, s. Firstly, $\mathcal{CCD}$ does not verify the property returning a zero value when an element is compared against itself, i.e. it does not verify $d(x, x) = 0$ [22]. Indeed, reflexivity property implies that $\mathcal{CCD}(P, P) = 1$ rather than $\mathcal{CCD}(P, P) = 0$. Secondly, a requirement for a similarity function [16, 22] is that the similarity between two objects takes value 1 if and only if the two objects are equal, i.e. $s(x, y) = 1$ iff $x = y$. This is not the case for $\mathcal{CCD}$ as two reciprocal preference relations do not necessarily need to coincide to have maximum correlation consensus degree, as the illustrative example 6.1 shows next.

6. Practical applications and discussion

In this Section we show the flexibility and applicability of our proposal. After discussing the basis of the measure we exemplify its use by means of two examples. The first one is an illustrative example that shows the various steps in our procedure and the interpretation of the results. The second one is a real example based on patients' health preferences.

6.1. An illustrative example

In this illustrative example we establish the following problem. We consider a set X of four alternatives $X = \{x_1, x_2, x_3, x_4\}$ and a set of four agents or experts $\mathbf{E} = \{1, 2, 3, 4\}$, who provide the following reciprocal preference relations on X:

$$P^{(1)} = \begin{pmatrix} 0.50 & 0.10 & 0.20 & 0.30 \\ 0.90 & 0.50 & 0.35 & 0.40 \\ 0.80 & 0.65 & 0.50 & 0.45 \\ 0.70 & 0.60 & 0.65 & 0.50 \end{pmatrix} \quad P^{(2)} = \begin{pmatrix} 0.50 & 0.15 & 0.25 & 0.35 \\ 0.85 & 0.50 & 0.40 & 0.45 \\ 0.75 & 0.60 & 0.50 & 0.50 \\ 0.65 & 0.55 & 0.50 & 0.50 \end{pmatrix}$$

$$P^{(3)} = \begin{pmatrix} 0.50 & 0.75 & 0.55 & 0.35 \\ 0.25 & 0.50 & 0.25 & 0.15 \\ 0.45 & 0.75 & 0.50 & 0.05 \\ 0.65 & 0.80 & 0.95 & 0.50 \end{pmatrix} \quad P^{(4)} = \begin{pmatrix} 0.50 & 0.40 & 0.20 & 0.60 \\ 0.60 & 0.50 & 0.40 & 0.70 \\ 0.80 & 0.60 & 0.50 & 0.80 \\ 0.40 & 0.30 & 0.10 & 0.50 \end{pmatrix}$$

Once experts' preference matrices have been described we proceed to the computations.

Selection of essential vectors of intensities of preferences. For $P^{(1)}$ the elements above of the main diagonal are:

$$P^{(1)} = \begin{pmatrix} 0.50 & \boxed{0.10} & \boxed{0.20} & \boxed{0.30} \\ 0.90 & 0.50 & \boxed{0.35} & \boxed{0.40} \\ 0.80 & 0.65 & 0.50 & \boxed{0.45} \\ 0.70 & 0.60 & 0.65 & 0.50 \end{pmatrix}$$

Thus, it is
$$V_{P^{(1)}} = (0.10, \ 0.20, 0.30, 0.35, 0.40, 0.45).$$

Similarly, the following essential vectors of intensities of preferences obtained:

$$V_{P^{(2)}} = (0.15, 0.25, 0.35, 0.40, 0.45, 0.50),$$
$$V_{P^{(3)}} = (0.75, 0.55, 0.35, 0.25, 0.15, 0.05),$$
$$V_{P^{(4)}} = (0.40, 0.20, 0.60, 0.40, 0.70, 0.80).$$

Computation of the correlation consensus degree matrix. The correlation consensus degree of all different pairs of essential vectors are computed For example, for correlation coefficient between $V_{P^{(1)}}$ and $V_{P^{(2)}}$ is:

$$cor(V_{P^{(1)}}, V_{P^{(2)}}) = \frac{0.085}{\sqrt{0.085} \cdot \sqrt{0.085}} = 1.$$

Using Equation (1), the correlation consensus degree between $P^{(1)}$ and $P^{(2)}$ would be $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$.

The correlation consensus degree matrix in this case is:

$$\mathcal{CCD} = \begin{pmatrix} 1 & 1 & 0 & 0.879 \\ 1 & 1 & 0 & 0.897 \\ 0 & 0 & 1 & 0.121 \\ 0.879 & 0.897 & 0.121 & 1 \end{pmatrix}$$

Computation of the group consensus degree. Finally, the average of all correlation consensus degrees is computed to derive the group consensus degree:

$$\mathcal{CD}(\mathbf{E}) = \frac{2}{12} \cdot (1 + 0 + 0.879 + 0 + 0.879 + 0.121) = \frac{2}{12} \cdot 2.879 = 0.480$$

On discussion, it is worth pointing out the following interesting issues arising from the given example:

- There is one case when the correlation consensus degree between two experts is maximum, i.e is equal to 1, which happens for the pair of experts e_1 and e_2 ($\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$). As previously mentioned and this example illustrates, this does not necessarily imply that both experts have the same preferences on all the possible pairs of alternatives, but that their preferences are positive linearly correlated as the top left scatter plot of the essential vectors $V_{P^{(2)}}$ versus $V_{P^{(1)}}$ in Figure 1 shows. Indeed, the higher the value of an element in $V_{P^{(1)}}$, the higher the corresponding element value of $V_{P^{(2)}}$. So, when one of the expert e_1 or e_2 increases her/his preference valuations, the other expert does the same and in a perfect linear way. Hence, there exists a maximum concordance between these two experts' reciprocal preference relations.

- Regarding experts e_1 and e_3, it is noted that $cor(V_{P^{(1)}}, V_{P^{(3)}}) = -1$ and consequently $\mathcal{CCD}(P^{(1)}, P^{(3)}) = 0$. Thus, the disagreement is maximum. Indeed, when one expert increases his/her preferences the other expert does the opposite and in a perfect linear way. This is reflected in the top right scatter plot of the essential vectors $V_{P^{(3)}}$ versus $V_{P^{(1)}}$ in Figure 1.
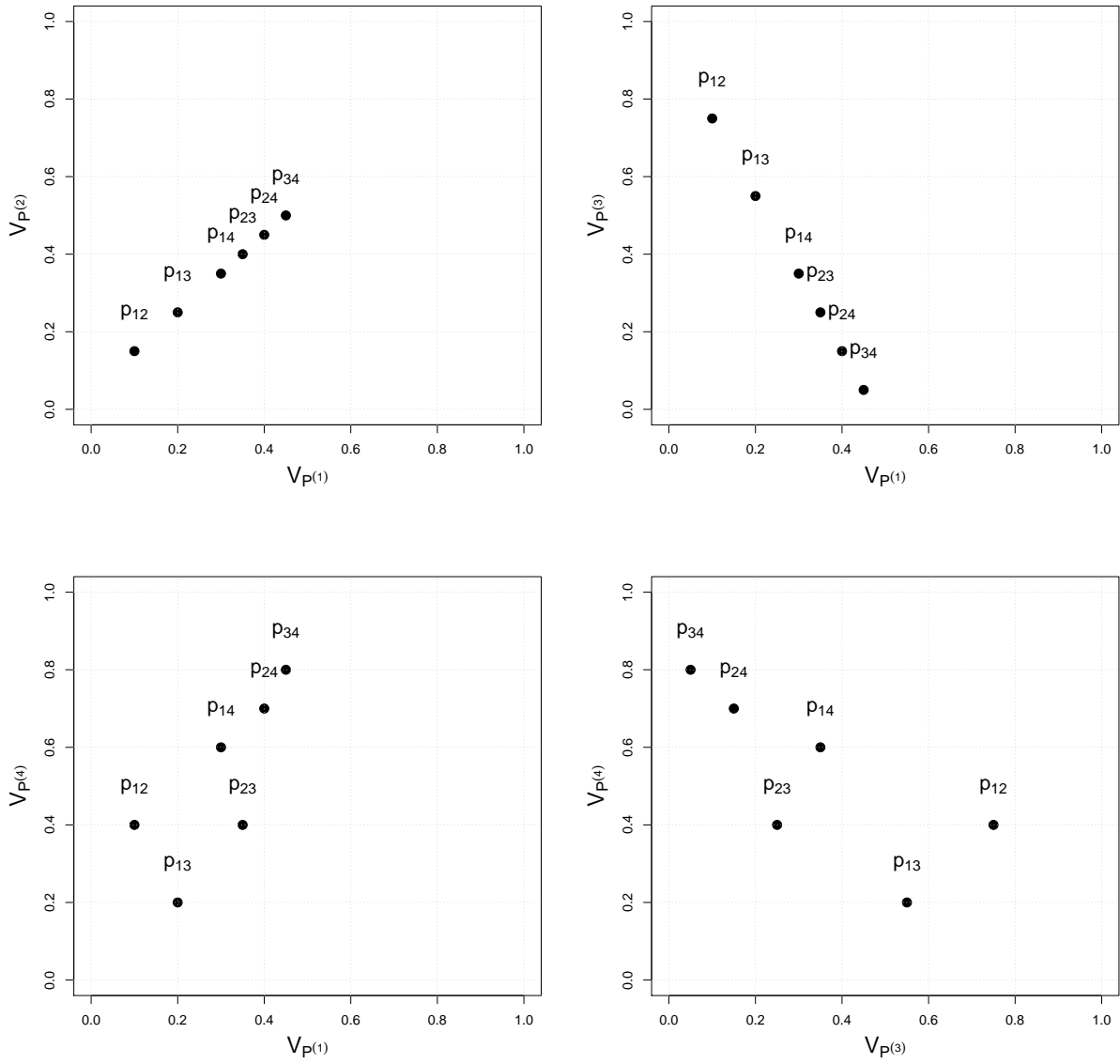
Figure 1: Plots of essential vectors of intensities of preferences (Subsection 6.1). Top left: case where $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$. Top right: case where $\mathcal{CCD}(P^{(1)}, P^{(3)}) = 0$. On the bottom plots, cases where $\mathcal{CCD}$ takes other non-extreme values.

- This example also shows a particular instance of Proposition 1 and Proposition 2 where $\mathcal{CCD}(P^{(1)}, P^{(2)}) = 1$. Indeed, Proposition 1 states that it is $V_{P^{(2)}} = a \cdot \mathbf{1} + b \cdot V_{P^{(1)}}$, which in this case results in $a = 0.05$, $b = 1$. The effect is that every essential pairwise intensity of preference is shifted to a new value using a constant amount. The preference relationship between one alternative and the rest of alternatives is essentially the same both experts, and consequently there is no real difference in the degree of agreement between for both experts when considering the set of alternatives as a whole. Indeed, it is worth remarking that the difference of preferences for both experts: $p_{13}^{(1)} - p_{12}^{(1)} = 0.20 - 0.10 = 0.10$ and $p_{13}^{(2)} - p_{12}^{(2)} = 0.25 - 0.15 = 0.10$; $p_{34}^{(1)} - p_{23}^{(1)} = 0.45 - 0.40 = 0.05$ and $p_{34}^{(2)} - p_{23}^{(2)} = 0.50 - 0.45 = 0.05$, etc. are the same for all pairs of alternatives compared. Thus, although the fuzzy relations $P^{(1)}$ and $P^{(2)}$ are not coincident, they are in the same tendency vein and they would lead to the same total ordering of the alternatives when the non-dominace degree is applied. As for Proposition 2, it is also true that the correlation consensus degrees between expert e_1 and experts e_3 and e_4 are the same as the correlation consensus degrees between expert e_2 and experts e_3 and e_4, respectively.

6.2. A real application: Concordance among patients' preferences

Recent developments in *Clinical Decision-Making* have led to a new interest on patient autonomy and their active involvement in decision making. Based on empirical evidences it has been tested that patients' choices related to take responsibility about treatment decisions differ among patients. Among others, age, sex, and type of clinical problem have been described as factors that can influence patients' choice. Due to these fats, it could be interesting to understand better patients' preferences in Clinical decision-making and the factors that could influence them (see e.g., De las Cuevas, Peñate and de Rivera [20], Robison and Thomson [54], Rodriguez et al. [56] and Tang et al. [59] among others). Most studies about patients' decision making preferences have been carried out by means of the use of the *Control Preference Scale* (CPS) introduced by Degner [21]. The CPS scale has been validated like an instrument clinically relevant to measure patients' preference roles in health care decision making. This scale gathers the level of control that patients prefer to have in their own medical decisions by selecting one of five possible alternatives, given in Table 2, when questioned "What is the statement that best describe your preferred role in decision making?".

Alternatives	Description
x_1	I prefer to make the final selection about which treatment I receive
x_2	I prefer to make the final selection of my treatment after seriously considering my doctor's opinion
x_3	I prefer that my doctor and I share responsibility for deciding which treatment is best for me
x_4	I prefer that my doctor makes the final decision about which treatment will be used, but seriously considers my opinion
x_5	I prefer to leave all decisions regarding my treatment to my doctor

Table 2: Control Preference Scale (CPS) [21]

In order to put in practice our proposal for measuring the cohesiveness among a group of agents or experts, the field experiment carried out by De las Cuevas, Peñate and Rivera in [20] was considered. In this study, the authors examined the concordance among psychiatric patients' preferences by means of a statistical approach based on a sample of 507 patients from the Community Mental Health Services on Tenerife Island, Spain. Patients were diagnosed by

the psychiatrists using the International Classification of Diseases and the CPS scale was used to gather patients' preferences. For our study, and to facilitate the process and the calculations, 12 patients were considered with 4 of them were diagnosed with *schizophrenia*, another 4 with *bipolar disorder* and other 4 with *obsessive compulsive disorder* (OCD). Each patient filled out a questionnaire based on the CPS scale adapted to our proposal (see Figure 2). Patients had to mark their degree of preference between pairs of options described in the CPS scale (Table 2 above), which are considered as the alternatives in our preference framework.
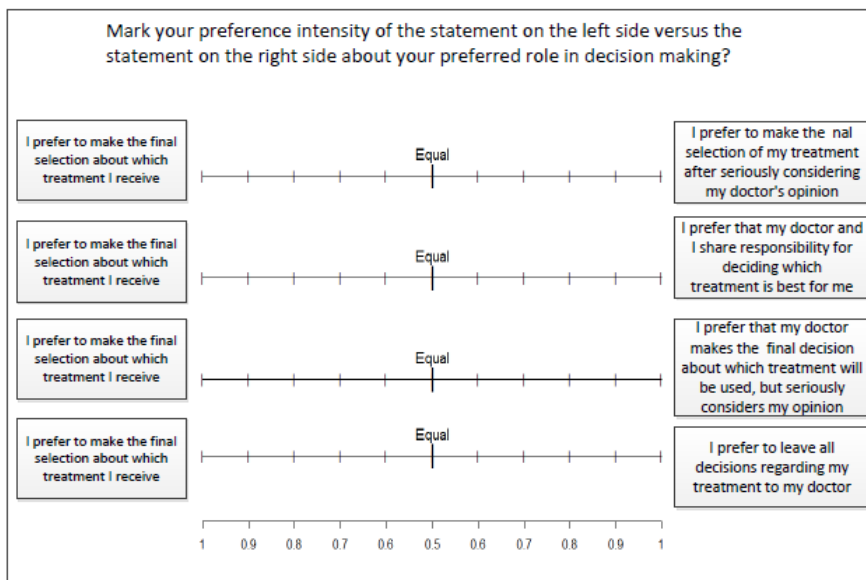
Figure 2: Questionnaire based on CPS.

Once patients' preferences were gathered (Table 3) we proceed to the apply computation process described in the previous illustrative example 6.1. Table 4 shows the correlation consensus degree between all pairs of patients (only the values $i \leq j$ are shown). Finally, the global consensus degree among all studied patients, $\mathcal{CD}$(patients), which measures the coherence among patients' preferences was:

$$\mathcal{CD}(\text{patients}) = 0.518$$

Taking into account the meaning of this measure as previously discussed, we can deduce that the low degree of coherence among all patients' preferences indicates heterogeneity among them. This fact could well respond to the combination of all patients' preferences without considering their diagnosed disorder. Indeed, when the consensus degree is computed within each collective of patients, i.e. by distinguishing patients according to their disorder, the following values are obtained:

- For patients suffering from schizophrenia: $\mathcal{CD}(\text{schizophrenia}) = 0.963$

- For patients suffering from bipolar disorder: $\mathcal{CD}(\text{bipolar}) = 0.961$

- For patients suffering from obsessive compulsive disorder: $\mathcal{CD}(\text{OCD}) = 0.985$

Figure 3 to Fig. 5 highlight the coherence among preferences inside the same collective of patients, while Fig. 6 to Fig. 8 highlight the disagreement among patients' preferences diagnosed with different disorders. As it was suspected, the coherence among the patients' preferences for each disorder separately is very high. This fact could add to the consideration of the type of disorder as a factor to be taken into account in Clinical Decision-Making.

Diagnoses	Patient	Intensities of preferences									
		p_{12}	p_{13}	p_{14}	p_{15}	p_{23}	p_{24}	p_{25}	p_{34}	p_{35}	p_{45}
Schizophrenia	1	0.3	0.2	0.1	0.1	0.4	0.3	0.1	0.5	0.1	0.1
	2	0.4	0.3	0.2	0.1	0.5	0.4	0.1	0.5	0.1	0.1
	3	0.2	0.2	0.2	0.1	0.3	0.2	0.1	0.3	0.1	0.1
	4	0.6	1.0	0.8	0.7	0.9	0.6	0.7	0.3	0.4	0.6
Bipolar disorder	5	0.6	1.0	0.8	0.7	0.9	0.6	0.7	0.3	0.4	0.6
	6	0.8	1.0	0.9	0.8	1.0	0.8	0.8	0.2	0.3	0.5
	7	0.1	0.1	0.1	0.5	0.5	0.5	0.9	0.5	0.9	0.9
	8	0.6	1.0	0.9	0.7	0.9	0.6	0.8	0.3	0.2	0.6
	9	0.1	0.1	0.1	0.5	0.5	0.5	0.9	0.5	0.9	0.9
OCD	10	0.2	0.1	0.3	0.5	0.5	0.5	0.8	0.5	0.7	0.9
	11	0.1	0.2	0.2	0.5	0.6	0.4	0.9	0.5	0.8	0.9
	12	0.1	0.2	0.2	0.5	0.5	0.4	0.8	0.5	0.9	0.9

Table 3: Patients' essential vector of preference intensities. OCD: Obsessive compulsive disorder. p_{ij} is the intensity of preference of alternative i versus alternative j.

	$P^{(1)}$	$P^{(2)}$	$P^{(3)}$	$P^{(4)}$	$P^{(5)}$	$P^{(6)}$	$P^{(7)}$	$P^{(8)}$	$P^{(9)}$	$P^{(10)}$	$P^{(11)}$	$P^{(12)}$
$P^{(1)}$	1.00	0.98	0.95	0.97	0.39	0.30	0.43	0.41	0.36	0.35	0.36	0.33
$P^{(2)}$		1.00	0.97	0.99	0.50	0.39	0.55	0.51	0.26	0.26	0.26	0.23
$P^{(3)}$			1.00	0.93	0.54	0.42	0.56	0.56	0.24	0.25	0.26	0.23
$P^{(4)}$				1.00	0.47	0.39	0.53	0.48	0.32	0.30	0.30	0.28
$P^{(5)}$					1.00	0.95	0.96	0.98	0.28	0.28	0.34	0.29
$P^{(6)}$						1.00	0.96	0.95	0.33	0.35	0.38	0.33
$P^{(7)}$							1.00	0.96	0.23	0.25	0.28	0.22
$P^{(8)}$								1.00	0.26	0.30	0.33	0.27
$P^{(9)}$									1.00	0.98	0.99	0.99
$P^{(10)}$										1.00	0.98	0.97
$P^{(11)}$											1.00	0.99
$P^{(12)}$												1.00

Table 4: Correlation consensus degree ($\mathcal{CCD}$) between pairs of patients.

7. Concluding remarks and future research

Research in the area of consensus measurement has advanced mainly in Social Choice Theory and Theory of Decision Making. In this work, a new consensus measure for reciprocal preference relations based on the classical definition of the Pearson correlation coefficient is studied. This new measure, the correlation consensus degree, pursues the measurement of the concordance between the intensities of pairwise preference values given by experts, decision makers or agents. This work open a new avenue to measure consensus. The correlation consensus degree between two reciprocal preference relations is neither a distance function nor a similarity function unlike the traditional consensus measures studied before. Nevertheless, the given correlation consensus degree verifies important properties that are common either to distances and/or similarities measures as well as additional properties that have been described in this work and that are different to traditional consensus measures properties. The novelty of the proposed correlation consensus measure as well as its application is shown with two examples. The first of the examples is used to illustrate the computation process and discussion of the results, while the second example covers a real life Clinical Decision-Making application. Both examples show the versatility and the applicability of the proposed measurement of consensus to a variety of real situations.

A future line of enquiry is the investigation of flexible consensus reaching processes based on the new correlation consensus degree. These processes would allow to produce a consensus solution by an iterative feedback mechanism accommodated to the this specific consensus measurement. We expect to conduct further investigations of these issues and report our findings in the future.

Acknowledgment

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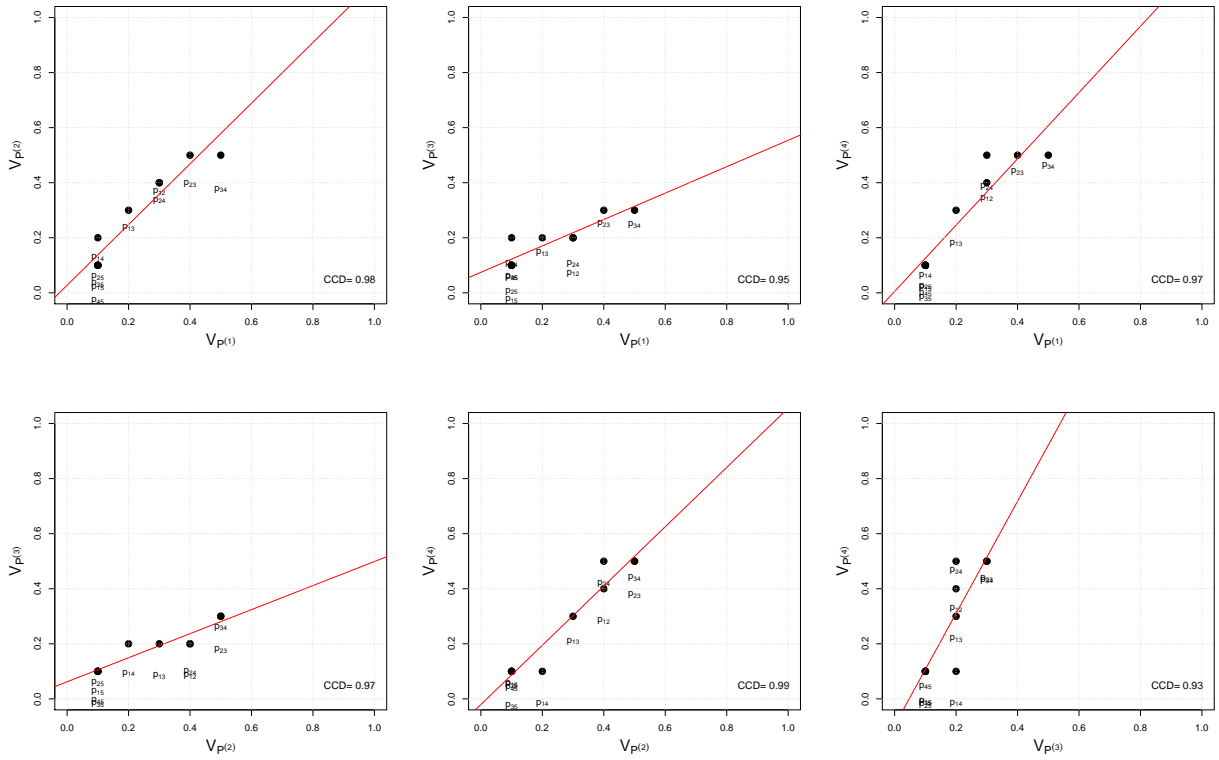
Figure 3: Plots of essential vectors of intensities of preferences corresponding to patients diagnosed Schizophrenia (Subsection 6.2) and the best adjusted line.
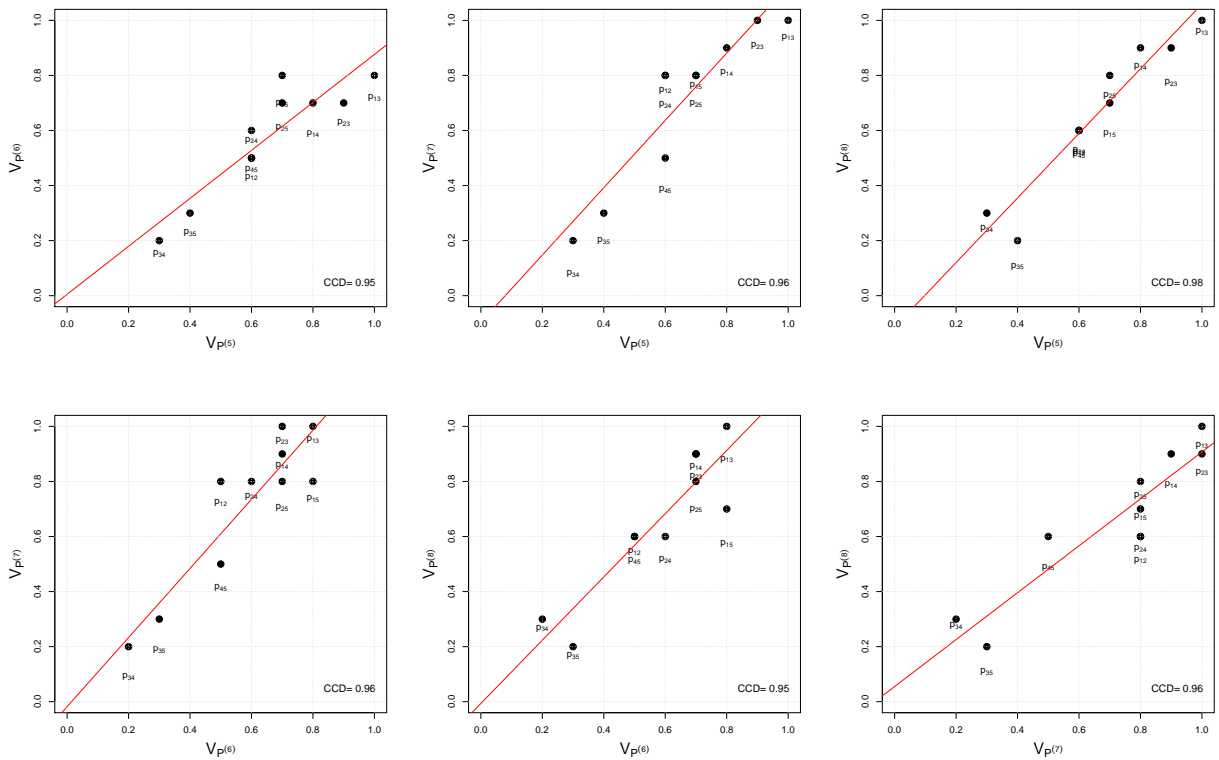

Figure 4: Scatterplots of essential vectors of intensities of preferences corresponding to patients diagnosed Bipolar disorder (subsection 6.2) and the best adjusted line.
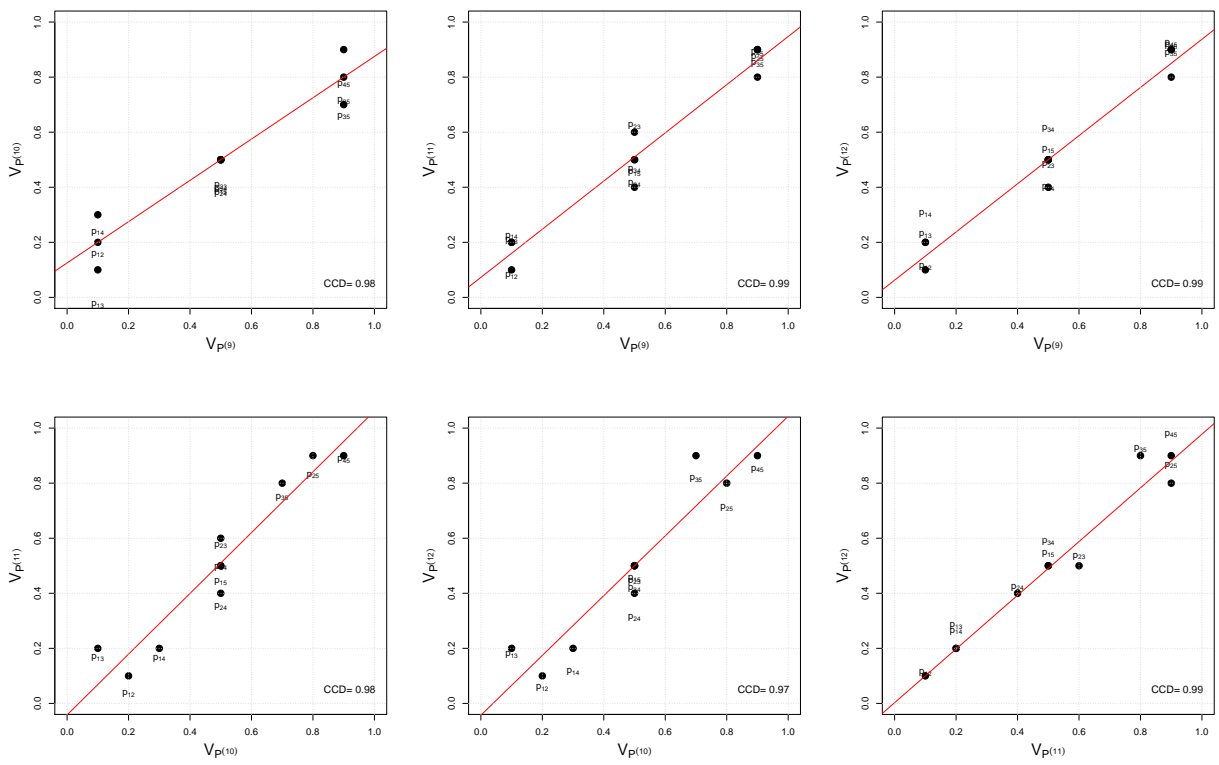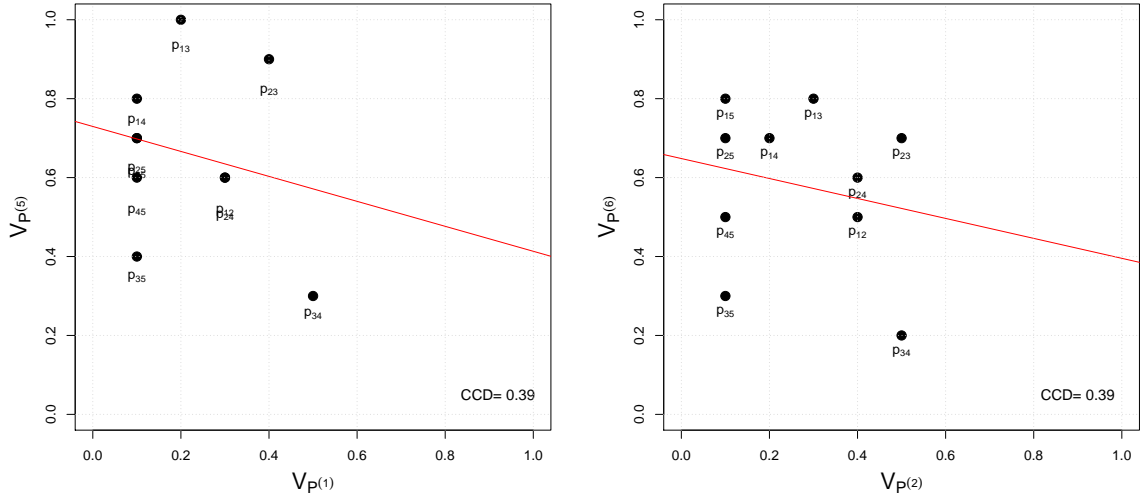
Figure 5: Plots of essential vectors of intensities of preferences corresponding to patients diagnosed OCD (Subsection 6.2) and the best adjusted line.

Figure 6: Plots of essential vectors of intensities of preferences corresponding to patients diagnosed Schizophrenia versus Bipolar disorder (Subsection 6.2) and the best adjusted line.
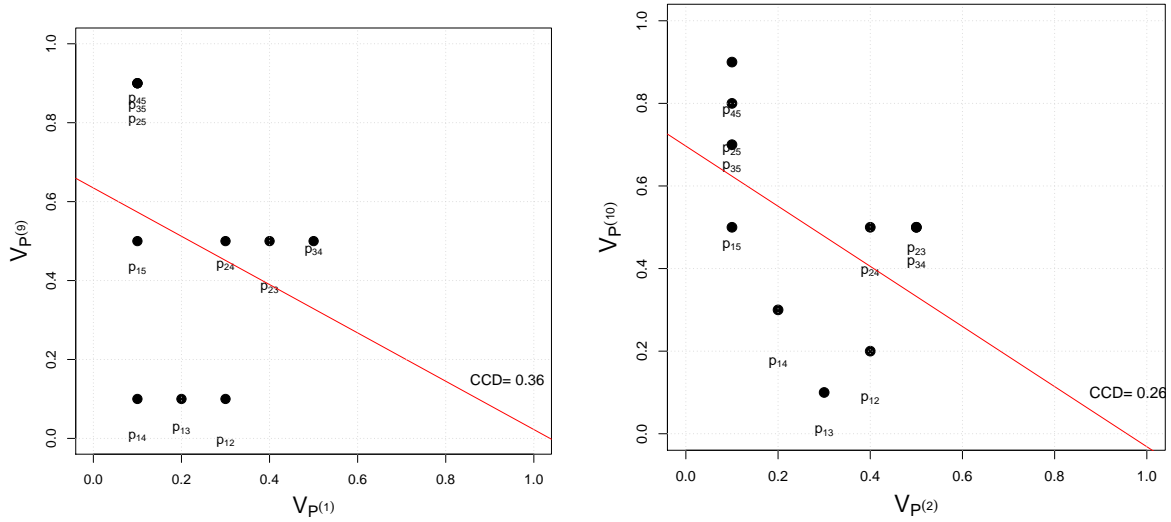

Figure 7: Plots of essential vectors of intensities of preferences corresponding to patients diagnosed Schizophrenia versus OCD (Subsection 6.2) and the best adjusted line.
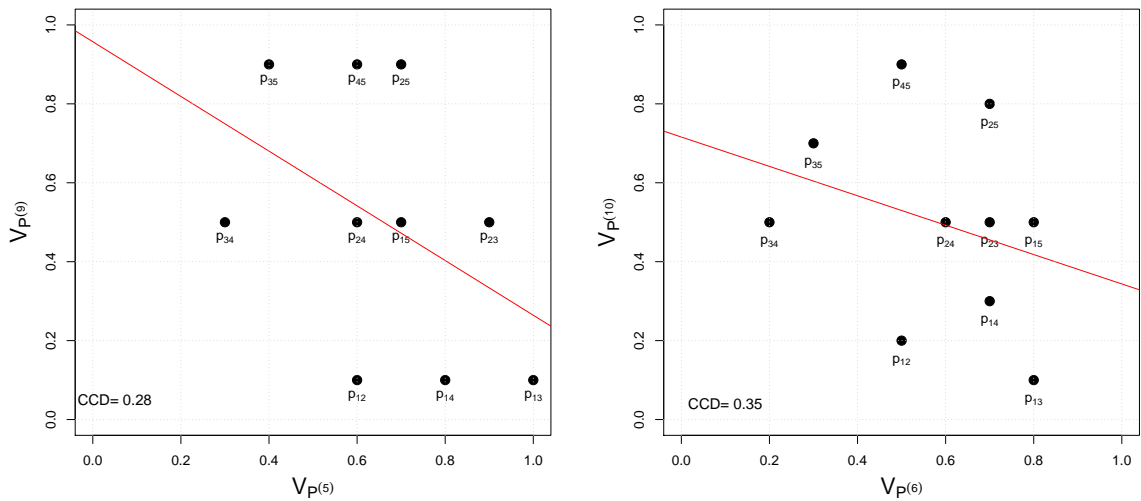

Figure 8: Plots of essential vectors of intensities of preferences corresponding to patients diagnosed Bipolar disorder versus OCD (Subsection 6.2) and the best adjusted line.