An Asymptotic Algebraic Solution to a Canonic Romer Endogenous Growth Model

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Abstract The purpose of this paper is to provide an algebraic solution to a fully specified Romer endogenous growth model. The proposed model has three main virtues. First, taking Romer's [1986] model as the starting point, we build a completely and explictly micro-founded competitive general equilibrium model. Second, this version consistently incorporates all the suggestions in Romer [1986] concerning externalities and complementarity between knowledge and physical capital. Lastly, our model has an asymptotic algebraic solution that allows the dynamics of the variables to be completely described, and does not require a characterization through a phase plane geometric analysis. The result is then a canonic Romer model of endogenous growth, fully specified, tractable and coherent.

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1. Introduction

Today, economic growth is central to the study of macroeconomics. Within the economic growth literature, much of the modern theoretical and empirical work focuses on increasing returns to scale in production. The reference paper to study the role played by this assumption on economic growth is Romer's [1986] model of endogenous growth. This author proposed an economy in which knowledge is an input in production that has increasing marginal productivity due to its external effects and that implies a production function displaying increasing returns to scale, an idea afterwards considered by Lucas [1988] and Murphy, Shleifer and Vishny [1989] among others.

Romer's [1986] model is very appealing given its intuitive richness and is one of the key-stones in the analysis of endogenous growth. However, unlike the other reference models in economic growth, for the present it has not been possible to algebraically solve an endogenous growth model *á la* Romer or to formulate a canonic version incorporating Romer's [1986] suggestions. In the literature on dynamic macroeconomics, it is very usual to formulate canonic versions of the models with the objective of counting on an algebraic solution illustrating the model behavior. That is the case for the Ramsey-Cass-Koopmans model when the depreciation rate is the unity, of the Lucas [1988], Rebelo [1991] and Barro [1990] models of endogenous growth, of the Brock [1982] and Abel [1988] asset pricing models, of the Uzawa-Lucas model of endogenous growth in Bethmann [2007], of the AK model, physical and human capital model, learning-by-doing model and public-goods endogenous growth model in Barro and Sala-i-Martin [1995], etc. The interested reader can also consult Stokey and Lucas with Prescott [1989] and Manuelli and Sargent [1987], where numerous models are analyzed and exactly solved. As Bethmann [2007] explains, the advantages of algebraic solutions are obvious, since they allow the model's implications to be easily studied, provide useful examples for sharpening economic intuition, are very interesting for teaching purposes, and constitute a benchmark to evaluate computational approximated solutions. For Romer's [1986] model, the existence of an algebraic solution is particularly interesting, since until now it has not been possible to algebraically solve an endogenous growth model based on increasing returns nor to count on a canonic reference model to study the dynamics.

This is in part due to the formal complexity underlying Romer's [1986] model, a two sector general equilibrium model of endogenous growth based on the increasing returns of knowledge, very difficult to formulate and to solve. Indeed, although Romer's [1986] model is a closed general equilibrium model, its microfoundation is simplified to the maximum for the sake of tractability. Firstly, the economy general equilibrium is reduced to a simple maximization problem of a representative consumer subject to a single constraint given by the accumulation law of knowledge. Secondly, although Romer's [1986] model is a two sector general equilibrium model, all inputs other than knowledge are in fixed supply and thus can not be accumulated, and the production function of knowledge does not provide the level of knowledge but its growth rate. Additionally, since increasing returns in production are the consequence of externalities associated to knowledge, the equivalence between the social planner's problem and the competitive equilibrium, a very useful result in growth theory in Romer's words, is only guaranteed under government intervention. In this respect, although Romer studies a reduced formulation of the social planner's problem, he does not discuss the characterization of the competitive equilibrium with government intervention as the solution of the social planner's problem, he only points out this possibility. Finally, given the complexity of the model and the impossibility to obtain an exact algebraic solution or a set of steady state equations, the dynamics of the variables must be analyzed using the phase plane. The phase plane analysis, however, exclusively provides qualitative conclusions on the variable dynamics, is complicated to apply in this kind of models, and is also sometimes insufficient to study interesting aspects of the variable behaviors.

These inconveniences would disappear if a canonic version of Romer's [1986] model with an algebraic solution could be found. More precisely, the primary motivation of this paper is to build a fully specified and explicitly micro-founded Romer's [1986] endogenous growth model, incorporating all his suggestions, formulating the competitive general equilibrium model as a social planner's problem, and providing an asymptotic algebraic solution for a particular case allowing the dynamics of the model to be exactly described. To do so, we will follow Romer's [1986] indications. Indeed, the author himself suggested that, in order to explicitly state and formalize the model from the microeconomics point of view, knowledge and physical capital could be used in fixed proportions in production, increasing returns might be the consequence of externalities associated to knowledge, and the government could implement lump sum taxation and subsidies to firms to support the social planner's problem as a competitive general equilibrium.

The paper is organized as follows. After this introduction, section 2 presents the model, defines the competitive general equilibrium and obtains the equivalent social planner's problem. The asymptotic algebraic solution of the proposed canonic Romer [1986] model is calculated and analyzed in section 3, where the dynamics of the model variables are also discussed. Finally, section 4 concludes and summarizes the paper's main contributions.

2. The Model

As explained above, our intention is to faithfully follow the model and suggestions in Romer [1986] and to build a competitive general equilibrium model displaying long-run growth on the basis of externalities and increasing returns associated to knowledge.

In particular, we will consider that in the economy there are three types of agents, namely households, firms and government. Each household maximizes her/his discounted utility subject to her/his budget constraint, given by

$$C_t + [K_{t+1} - K_t(1-\delta_k)] + [h_{t+1} - h_t(1-\delta_h)] + T_t \leq w_t l_t + r_t K_t + m_t h_t,$$

where C, K, h, T, l, $\bar{l}$, w, r, m, δ_k, δ_h and t are, respectively, the consumption of good, the participation in physical capital, the household level of knowledge, the taxes paid to government, the labor supply, the time endowment, the labor input price, the physical capital input price, the knowledge input price, the depreciation rate of physical capital, the depreciation rate of knowledge, and the period of time. The budget constraint simply says that, each period, the household's wealth, given by the remuneration to the labor, physical capital and knowledge used in production, is used to acquire consumption good, to increase her/his participation in the physical capital input, to accumulate knowledge input, and to pay taxes. It is then assumed, as in Romer [1986], that there is a trade-off between consumption today and knowledge that can be used to increase consumption tomorrow.

Firms are the second type of agent. As in Romer [1986], it will be assumed that the production function of each firm incorporates knowledge as an input, that knowledge used by a firm has a positive external effect on the production possibilities of other firms, and that the production functions display constant returns to scale in production factors other than knowledge. Following Romer's [1986] suggestions, it will also be assumed that physical capital and knowledge are used in fixed proportions in production. A convenient way to capture these features is through the production function

$$y^i = F(K_t^i, l_t^i, h_t^1, h_t^2, \ldots, h_t^J) = \min\{aK_t^i, h_t^i\}^{\alpha}(l_t^i)^{1-\alpha}(\prod_{j\neq i} h_t^j)^{\rho},$$

$$i = 1, 2, \ldots, J,$$

where a is the number of units of knowledge that combine with one unit of physical capital to produce output, α, $(1-\alpha)$ and ρ are the input elasticities, $i = 1, 2, \ldots, J$ is the superscript denoting the firm, and J denotes the total number of firms. The term $(\prod_{j\neq i} h_t^j)^\rho$ captures the external effects of the knowledge used by the other firms on firm i's production.

Government, the third agent in our economy, collects lump-sum taxes from the agents and internalizes the externalities in production through optimal pigouvian taxes and subsidies to firms. As is well known, under this kind of government intervention, the competitive general equilibrium -defined through the usual conditions of utility maximizing consumers, profit maximizing firms, and market clearing-, can be formulated as the solution of a social planner's problem, an idea pointed out by Romer in his reference paper. In particular, normalizing the time endowment to unity, the equivalent social planner's problem can be written

$$\max_{C_t,h_t} \sum_{t=0}^{\infty} \beta^t U(C_t)$$

$$s.t. \quad C_t + bh_{t+1} \leq h_t^\gamma + h_t[1-\delta)], \tag{1}$$

$$C_t, h_{t+1} \geq 0,$$

$$t = 0, 1, \ldots, \infty,$$

$$h_0 \quad \text{historically given,}$$

where β is the discount factor, U is the instantaneous utility function, $b = 1 + \frac{1}{a}$, $\gamma = \alpha + \rho$ and $\delta = \delta_h - \frac{1-\delta_K}{a}$. The interested reader can find the definition of the competitive equilibrium of this economy and the proof of its equivalence with the social planner's problem in the Appendix A. Applying the recursive theory[1],

[1] See for instance Stokey and Lucas with Prescott [1989].

this social planner's problem (1) can be written

$$v(h_t) = \max_{h_{t+1} \in \Gamma[h_t]} \{U(f(h_t) - bh_{t+1}) + \beta v(h_{t+1}\}, \tag{2}$$

where

$$\Gamma[h_t] = [0, \frac{f(h_t)}{b}],$$

$$f(h_t) = h_t^{\gamma} + h_t(1 - \delta).$$

Here v is called the value function, and the function describing the household level of knowledge, h, is called the policy function. According to Theorem 4.15 in Stokey and Lucas with Prescott [1989], if a solution exists for the recursive problem (2) and it verifies the Euler equation and the transversality condition, then it is also the solution of the social planner's problem (1). Note that proof of Theorem 4.15 in Stokey and Lucas with Prescott [1989] does not require the boundedness hypothesis for the instantaneous utility function. Then, since the existence of a unique solution is ensured by Theorem 4.6 in Stokey and Lucas with Prescott [1989] when the instantaneous utility function is bounded, we only need to ensure the existence of a solution for the recursive problem (2) in the unbounded case. In this respect, theorem 6 in Rincón-Zapatero and Rodríguez-Palmero [2003] ensure the existence of a unique solution for all $\beta \in (0, 1)$ whenever $\gamma \in (0, 1]$, and the existence of a solution for all $\beta \in (0, 1/\gamma)$ whenever $\gamma > 1$. This is the case for problem (2), which will be algebraically solved for a particular case in the next section.

3. A Canonic Endogenous Growth Model Based on Increasing Returns

Unlike Romer's [1986] general equilibrium model, formulated by considering the maximization problem of an individual agent who takes as given the path of knowledge accumulation, ours is a standard and explicitly micro-founded general equilibrium model. Indeed, the role of externalities in generating increasing returns is made explicit, the accumulation law for knowledge has not been exogenously imposed, and the economy is completely described through a social planner's problem which directly arises from utility maximizing consumers, profit maximizing firms, and market clearing conditions.

Additionally, the proposed model can be algebraically solved for the particular case in which $\gamma = 2$ and $U(C_t) = \ln(C_t)$, something that allows the behavior of the variables to be completely described. This is an obvious virtue of our model. As is well known, in Romer's [1986] endogenous growth model the dynamics of the variables must be characterized through a phase plane geometric analysis. This is due to the impossibility of obtaining an algebraic solution even for the simple examples considered by the author. This difficulty has compelled to the use of the phase plane analysis in order to examine the behavior of the variables. However and as explained in the introductory section, this method, that exclusively provides qualitative conclusions on the variable dynamics, is not only complicated to apply in this kind of models, but is also sometimes insufficient to study interesting aspects of the variable behaviors[2]. These inconveniences disappear once an algebraic solution is calculated, since then it is possible to completely describe the dynamics of the model and to stress the models's implications.

In our case, when $\gamma = 2$ (therefore, $f(h_t) = h_t^2 + h_t(1-\delta)$) and $U(C_t) = U(h_t, h_{t+1}) = \ln(f(h_t) - bh_{t+1}) = \ln(C_t)$, the asymptotic law of motion for knowledge is given by

$$h_{t+1} = \frac{\sqrt{4\beta^2 f(h_t)^2 + b^2(1-\delta)^2} + 2\beta f(h_t) - b(1-\delta)}{2b}. \tag{3}$$

It is worth noting that the convexity of the correspondence Γ is not required, since, as explained above, we ensure that equation (3) is the solution of problem

[2] For instance, in Romer's [1986] first example, it is not possible to describe the asymptotic rates of growth

(1) by verifying the Euler equation and the transversality condition, and not by applying the convergence of policy functions result in Theorem 4.9, Stokey and Lucas with Prescott [1989]. Indeed, we obtained the solution (3) for the policy function using the "guess and verify" method and testing the Euler equation and the transversality condition, but not applying the Contraction Mapping Theorem. It is also worth noting that, whenever $\delta = 1$ and $b = 1$ (therefore, $f(h_t) = h_t^2$), the expression for h_{t+1} above defined becomes $h_{t+1} = 2\beta h_t^2$, which is the well-known policy function solving problem (2) when the parameters take the aforementioned values $\delta = 1$ and $b = 1$.

In fact, given the initial condition $h = h_0$, it is possible to show that for h_0 large enough, equation (3) is the asymptotic policy function of the recursive formulation of the general equilibrium, since the Euler equation

$$\beta \frac{\partial U}{\partial x}(h_{t+1}, h_{t+2}) + \frac{\partial U}{\partial y}(h_t, h_{t+1}) = \tag{4}$$

$$\beta \frac{2h_{t+1} + (1-\delta)}{f(h_{t+1}) - bh_{t+2}} - \frac{b}{f(h_t) - bh_{t+1}} = 0$$

is verified with the desired accuracy degree, and the transversality condition

$$\lim_{t\to\infty} \beta^t h_t \frac{\partial U}{\partial x}(h_t, h_{t+1}) = \tag{5}$$

$$\lim_{t\to\infty} \beta^t h_t \frac{2h_t + 1 - \delta}{f(h_t) - bh_{t+1}} = 0$$

holds. Furthermore, since $b\,h_{t+1} \leq 2\beta f(h_t)$, it follows that h_{t+1} belongs to the interior of $\Gamma[h_t]$ whenever $\beta < 1/\gamma = 1/2$, and then is feasible.

The initial condition has a double justification. Firstly, as Romer [1986] explains, when $\gamma > 1$, the production function $y = F(h_t, l_t) = h_t^\gamma l_t^{1-\alpha}$ is reasonable only for large values of h_0, since for small values of h_0 the marginal productivity of knowledge is also implausibly small. Second, given that the parameter a represents the number of units of knowledge/human capital that combines with one unit of physical capital, it is logical to think that in the economy there exists at least one unit of physical capital and therefore a presumably large quantity of units a of knowledge, given the intellectual effort embodied in one unit of physical capital. Thus, it is also reasonable to assume that, since a is large, so is h_0, the initial household level of knowledge. As we have commented on, for h_0 large enough, the policy function h_{t+1} verifies the transversality condition and asymptotically satisfies the Euler equation, and then it is the asymptotic policy function for problem (1). Specifically, the Euler equation tends to zero as t goes to infinity, which means that for t large enough (or, equivalently, for h_0 large enough), the Euler equation can be numerically considered as zero. For instance, for the particular values $b = 1.0001, \delta = 0.02$, and $\beta = 0.4$, the absolute value of the Euler equation is less than 10^{-10} for $h_0 \geq 592$, and less than 10^{-15} whenever $h_0 \geq 10521$. On the other hand, to derive that the transversality condition holds, it is enough to consider that the following inequality is verified

$$\begin{aligned}\left|\beta^t h_t \frac{2h_t + 1 - \delta}{f(h_t) - bh_{t+1}}\right| &= \beta^t \left|\frac{f(h_t) + h_t^2}{f(h_t) - bh_{t+1}}\right| \\ &\leq \beta^t \left|\frac{f(h_t)}{f(h_t)(1-2\beta)}\right| + \beta^t \left|\frac{h_t^2}{f(h_t)(1-2\beta)}\right| \\ &= \beta^t \frac{1}{1-2\beta} + \beta^t \frac{1}{(1-2\beta)} \frac{h_t^2}{h_t^2 + (1-\delta)h_t},\end{aligned}$$

and the fact that $\lim_{t\to\infty} \beta^t \frac{1}{1-2\beta} + \beta^t \frac{1}{(1-2\beta)} \frac{h_t^2}{h_t^2+(1-\delta)h_t} = 0$ for all $\beta < 1/2$, since $\lim_{t\to\infty} h_t = \infty$ due to the properties of h_{t+1}.

It is worth noting again that, once the appropriate values for the parameters are introduced, equation (3) also provides the algebraic solution for the social planner's problems associated to the Ramsey-Cass-Koopmans and AK models, and can then be considered as the generic policy function for a growth model.

From equation (3), it is straightforward to obtain that the growth rate of knowledge, γ_h, is given by

$$\gamma_h = \frac{h_{t+1} - h_t}{h_t} =$$

$$\frac{\sqrt{4\beta^2 h_t^2 + b^2(1-\delta)^2}}{2bh_t} + \tag{6}$$
$$\frac{2\beta f(h_t) - 2bh_t - b(1-\delta)}{2bh_t}.$$

This growth rate of knowledge is positive when h is greater than its steady state value $\overline{h}$, and negative below this value. The economy then does not converge to the steady state, given by[3]

$$\overline{h} = \frac{b - \beta(1-\delta)}{2\beta}.$$

From expressions (3) and (6) it is immediate to obtain that

$$\lim_{h_t \to \infty} \gamma_h = \infty, \quad \frac{d\gamma_h}{dh_t} > 0, \quad \lim_{h_t \to \infty} \frac{d\gamma_h}{dh_t} = \frac{2\beta}{b},$$

$$\frac{dh_{t+1}}{dh_t} > 0 \quad \frac{d^2 h_{t+1}}{dh_t^2} > 0,$$

and then we can conclude that knowledge grows without bound in an exponential form, at a positive and increasing rate, which, for its part, increases asymptotically approaching to a constant. Since $\frac{d\gamma_h}{dh_t} > 0$, initial levels of knowledge are positively correlated with later growth rates, and then economies with different initial values of knowledge will display divergence over time.

Given that physical capital and knowledge are used in a fixed proportion, the growth rate of physical capital, γ_k, coincides with γ_h. Regarding the growth rate of output, γ_y, since $y_t = h_t^2$, it is immediate that

$$\gamma_y = \gamma_h^2 + 2\gamma_h, \quad \frac{d\gamma_y}{dh_t} = 2\gamma_h \frac{d\gamma_h}{dh_t} + 2\frac{d\gamma_h}{dh_t}.$$

[3] In the general case, for any $\gamma > 1$, the expression for the steady state is given by $\overline{h} = (\frac{b-\beta(1-\delta)}{\gamma\beta})^{\frac{1}{\gamma-1}}$.

From equation (6), it can be concluded that, assuming $h > h_0$,

$$\gamma_y > 0, \quad \frac{d\gamma_y}{dh_t} > 0, \quad \lim_{h_t \to \infty} \frac{d\gamma_y}{dh_t} = \infty.$$

Therefore, output grows without bound at a positive and increasing rate, which asymptotically tends to infinity. This is also the behavior of consumption, given each period by the expression

$$C_t = h_t^2 + (1-\delta)h_t - bh_{t+1} =$$

$$\frac{2(1-\beta)f(h_t) + b(1-\delta) - \sqrt{4\beta^2 f(h_t)^2 + b^2(1-\delta)^2}}{2} \tag{7}$$

which is always positive, since $bh_{t+1} \leq 2\beta f(h_t)$ whenever $\beta < 1/2$. After some algebra, it can be concluded that the growth rate of consumption, γ_c, verifies

$$\gamma_c > 0, \quad \frac{d\gamma_c}{dh_t} > 0, \quad \lim_{h_t \to \infty} \frac{d\gamma_c}{dh_t} = \infty,$$

and consumption grows without bound at a positive and increasing rate, which asymptotically tends to infinity.

Since $\frac{dy_t}{dh_t} > 0$ and $\frac{dy_t}{dh_t} > 0$, given that $\frac{d\gamma_h}{dh_t} > 0$, economies with different initial values of knowledge will display divergent output and consumption trajectories. Our model therefore allows all the implications of Romer's [1986] model of endogenous growth to be obtained. In particular, the canonic formulation that we have considered originates growth rates increasing over time, the amplification of small disturbances by the action of private agents, and large countries growing faster than small countries.

4. Conclusions

Taking the model and suggestions in Romer [1986] as the starting point, we have built a completely and explicitly micro-founded competitive general equilibrium model displaying endogenous growth based on increasing returns, and with an asymptotic algebraic solution. The microeconomic suggestions in Romer's [1986] model have been incorporated and explicited, and the result is a canonic Romer's model of endogenous growth, fully specified, tractable and coherent. One of the main virtues of the model is the existence of an asymptotic algebraic solution that allows the dynamics of the variables to be completely described, not requiring a characterization through a phase plane geometric analysis. All the implications in Romer's [1986] model are straightway obtained, in particular growth rates increasing over time and divergence across economies with different initial values of knowledge. These features can be easily analyzed through the algebraic expression of the policy function, that describes the accumulation law for knowledge and then the dynamics of all variables. It is worth mentioning that equation (3) can be considered as the generic policy function for a growth model, since it also provides the exact algebraic solution for the social planner's problems associated to the Ramsey-Cass-Koopmans and AK models once the appropriate values for the parameters are introduced.

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Appendix: The General Equilibrium and the Social Planner's Problem

As explained above, in the economy there are three types of agents, namely households, firms and government. The household's problem is

$$\max_{C_t,K_t,h_t,l_t} \sum_{t=0}^{\infty} \beta^t U(C_t)$$

$$s.t. \quad C_t + [K_{t+1} - K_t(1-\delta_K)] + [h_{t+1} - h_t(1-\delta_h)] + T_t \leq w_t l_t + r_t K_t + m_t h_t, \quad \text{(I)}$$

$$C_t \geq 0,$$

$$0 \leq l_t \leq \bar{l},$$

$$t = 0, \ldots, \infty,$$

$$K_0, h_0 \quad \text{historically given,}$$

where β, U, C, K, h, T, l, $\bar{l}$, w, r, m, δ_K, δ_h and t are, respectively, the discount factor, the instantaneous utility function, the good consumption, the participation in physical capital, the household level of knowledge, the taxes paid to government, the labor supply, the time endowment, the labor input price, the physical capital input price, the knowledge input price, the depreciation rate of physical capital, the depreciation rate of knowledge, and the period of time.

Note that unlike Romer's [1986) model, where the households maximize their utility subject to the constraint given by the accumulation law of knowledge, exogenously imposed, our household's problem is the standard in a competitive general equilibrium model.

Firms, the second type of agent, operate according to the production function

$$y^i = F(K_t^i, l_t^i, h_t^1, h_t^2, \ldots, h_t^J) = \min\{aK_t^i, h_t^i\}^{\alpha}(l_t^i)^{1-\alpha} \prod_{j\neq i} (h_t^j)^{\rho},$$

$$i = 1, 2, \ldots, J,$$

where a is the number of units of knowledge that combine with one unit of physical capital to produce output, α, $(1-\alpha)$ and ρ are the input elasticities, $i = 1, 2, \ldots, J$ is the superscript denoting the firm, and J denotes the total number of firms. The term $\prod_{j \neq i}(h_t^j)^\rho$ captures the externalities associated to knowledge. Without any loss of generality, it will be assumed that the number of firms is $J = 2$.

Government, the third agent in our economy, collects lump-sum taxes from the agents and internalizes the externalities in production through optimal pigouvian taxes and subsidies to firms. Under this kind of government intervention, the competitive general equilibrium can be formulated as the solution of a social planner's problem, an idea pointed out by Romer in his reference paper. Formally, assuming internalization of the external economies associated to knowledge through optimal taxes and subsidies, each firm's problem is

$$\max_{K_t^i, l_t^i, h_t^i, h_t^j} \Pi^i = \min\{aK_t^i, h_t^i\}^\alpha (l_t^i)^{1-\alpha}(h_t^j)^\rho - w_t l_t^i - r_t K_t^i - m_t h_t^i + S_t h_t^i - S_t h_t^j + P_t,$$

$$i, j = 1, 2, \quad i \neq j,$$

where S_t is the subsidy/tax to knowledge input and P_t is a lump-sum subsidy to profits. Since firms are profit-maximizing, given the perfect complementarity between physical capital and knowledge, $aK_t^i = h_t^i$, and then the firm's problem becomes

$$\max_{K_t^i, l_t^i, h_t^i, h_t^j} \Pi^i = (h_t^i)^\alpha (l_t^i)^{1-\alpha}(h_t^j)^\rho - w_t l_t^i - h_t^i(m_t + \frac{r_t}{a} - S_t) - S_t h_t^j + P_t,$$

$$i, j = 1, 2, \quad i \neq j.$$

The first order necessary and sufficient conditions are

$$\alpha(h_t^i)^{\alpha-1}(h_t^j)^\rho (l_t^i)^{1-\alpha} = \frac{r_t}{a} + m_t - S_t,$$

$$\rho(h_t^i)^\alpha (h_t^j)^{\rho-1}(l_t^i)^{1-\alpha} = S_t,$$

$$(1-\alpha)(h_t^i)^{\alpha}(h_t^j)^{\rho}(l_t^i)^{-\alpha} = w_t,$$

$$i, j = 1, 2, \quad i \neq j.$$

It is clear that $l_t^1 = l_t^2 = l_t$ and $h_t^1 = h_t^2 = h_t$, and then the former equations can be written

$$(\alpha+\rho)(h_t)^{\alpha+\rho-1}(l_t)^{1-\alpha} = \frac{r_t}{a} + m_t,$$

$$(1-\alpha)(h_t)^{\alpha+\rho}(l_t)^{-\alpha} = w_t,$$

which are the first order necessary and sufficient conditions for the problem

$$\max_{h_t, l_t} \Pi = h_t^{\alpha+\rho}(l_t)^{1-\alpha} - w_t l_t - h_t(\frac{r_t}{a} + m_t) + P_t. \qquad \text{(II)}$$

Therefore, each firm's profits are

$$\Pi = -\rho h_t^{\alpha+\rho}(l_t)^{1-\alpha} + P_t,$$

which depend on the government lump-sum subsidy to profits. When $P_t = \rho h_t^{\alpha+\rho}(l_t)^{1-\alpha}$, then $\Pi = 0$, and the production sector is in a long-run equilibrium.

Applying the usual reasonings, when population is constant -as in Romer [1986]-, the long-run competitive general equilibrium of this economy can be formulated in per-capita terms as follows[4]:

Definition 1 (Long-Run Competitive General Equilibrium) *Sequences $\{C_t\}$, $\{h_t\}$, $\{K_t\}$, $\{l_t\}$, $\{w_t\}$, $\{r_t\}$, $\{m_t\}$, $\{T_t\}$ and $\{P_t\}$ such that:*

- *Given the sequences $\{w_t\}$, $\{r_t\}$, $\{m_t\}$ and $\{T_t\}$, the sequences $\{C_t\}$, $\{h_t\}$, $\{K_t\}$ and $\{l_t\}$ solve the representative consumer's problem (I).*
- *Given the sequences $\{w_t\}$, $\{r_t\}$, $\{m_t\}$ and $\{P_t\}$, the sequences $\{h_t\}$, $\{K_t\}$ and $\{l_t\}$ solve the representative firm's problem (II) and verify $aK_t = h_t$.*

[4] See for instance Rubinstein [1974) and Cooley and Prescott [1994). For an exhaustive proof, consult Gutiérrez [2002].

– *The sequences* $\{C_t\}$, $\{h_t\}$, $\{K_t\}$ *and* $\{l_t\}$ *verify the market clearing condition*

$$C_t + K_{t+1} + h_{t+1} \leq h_t^{\alpha+\rho}(l_t)^{1-\alpha} + K_t(1-\delta_K) + h_t(1-\delta_h).$$

– *Government sequences of lump-sum taxes* $\{S_t\}$ *and subsidies* $\{P_t\}$ *verify*

$$T_t = P_t = \rho h_t^{\alpha+\rho}(l_t)^{1-\alpha}.$$

This formulation of Romer's [1986) model as a long-run competitive general equilibrium is equivalent to a very simple social planner's problem. From the first order necessary conditions in the firm's problem, the perfect complementarity between physical capital and knowledge, and the equality $T_t = P_t = \rho h_t^{\alpha+\rho}(l_t)^{1-\alpha}$, the household's budget constraint becomes

$$C_t + h_{t+1}(1+\frac{1}{a}) \leq h_t^{\alpha+\rho}(l_t)^{1-\alpha} + h_t[1-\delta_h + \frac{1}{a}(1-\delta_K)].$$

The long-run competitive general equilibrium is therefore equivalent to the following social planner's problem:

$$\max_{C_t,h_t,l_t} \sum_{t=0}^{\infty} \beta^t U(C_t)$$

$$s.t. \quad C_t + h_{t+1}(1+\frac{1}{a}) \leq h_t^{\alpha+\rho}(l_t)^{1-\alpha} + h_t[1-\delta_h + \frac{1}{a}(1-\delta_K)],$$

$$C_t \geq 0,$$

$$0 \leq l_t \leq \bar{l},$$

$$t = 0, 1, \ldots, \infty,$$

$$h_0 \quad \text{historically given.}$$

It is straightforward that defining $b = 1 + \frac{1}{a}$, $\gamma = \alpha + \rho$ and $\delta = \delta_h - \frac{1-\delta_K}{a}$ and normalizing the time endowment to unity, the former problem becomes problem (I).

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