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# DEVELOPING AND UTILIZING MULTIVARIATE STOCHASTIC WIRELESS CHANNEL MODELS 

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#### Abstract

Developing accurate channel models is paramount in designing efficient mobile communication systems. The focus of this dissertation is to understand the small-scale fading characteristics, develop mathematical tools that accurately capture these characteristics and utilize them in three different applications - diversity receivers, scheduling, packet duplication in dual connectivity scenarios.

This dissertation develops multivariate stochastic models for Rayleigh fading channels that incorporate factors such as the velocity of the users, angle of arrival distribution of signals, and carrier frequency. The developed models are more comprehensive than the existing ones. They capture the correlation characteristics of signals more accurately and are applicable to more practical scenarios. The developed models are the only ones that incorporate the spatial correlation structure suggested by 3GPP.

The models are used to derive analytical expressions for the output SNR of certain diversity receivers. Owing to our expressions, the output SNR performances of these receivers are now studied through their moments. The moments provide insight about the nature of these receivers' output SNR distribution, which is very useful in their reliability analysis.


Secondly, the models are used to capture the temporal evolution of the received

SNR. Temporal correlation characteristics of the SNR are exploited to decrease the number of variables in the downlink scheduling problem. This is achieved by making scheduling decisions less frequently for users with relatively higher coherence time. The results illustrate that the number of operations it takes to make scheduling decisions can be reduced by $33 \%$ with confidence probability of 0.7 and by $58 \%$ with confidence probability of 0.4 .

Finally, fade duration and non-fade duration characteristics of a Rayleigh fading channel are used to partially and randomly duplicate some packets when connected to multiple base stations. This is performed based on the small-scale fading statistics rather than the large-scale fading. Duplication based on large time scales can be wasteful and unnecessary, so it is shown using matrix exponential distributions how with low complexity to duplicate only when necessary. The results indicate that up to $50 \%$ of the resources at the duplicating base station can be liberated whilst meeting the target reliability measure.

## APPROVAL PAGE

The faculty listed below, appointed by the Dean of the School of Graduate Studies, have examined a dissertation titled "Developing and Utilizing Multivariate Stochastic Wireless Channel Models," presented by Mustafa Tekinay, candidate for the Doctor of Philosophy degree, and certify that in their opinion it is worthy of acceptance.

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## LIST OF ABBREVIATIONS

| 3GPP | Third Generation Partnership Project |
| :--- | :--- |
| 5G | fifth generation |
| AMC | adaptive modulation and coding |
| AWGN | additive white Gaussian noise |
| BS | base station |
| BW | bandwidth |
| CDF | channelative distribution function |
| CQI | equality indicator |
| EGC | mang Term Evolution exponential |
| LTE | master evolved node B |
| ME | multiple-input-multiple-output |
| MeNB | moment generating function |
| MIMO | maximal ratio combiner |
| MGF | packer azimuth spectrum |
| MRC | parrability density function ratio |
| PAS | PDF |

PPD partial packet duplication
RU resource unit
RV random variable
SC selection combiner
SD standard deviation
SeNB secondary evolved node B
SINR signal-to-interference-plus-noise ratio
SNR signal-to-noise ratio
UE user equipment
URLLC Ultra-Reliable and Low-Latency Communications

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## CHAPTER 1

## INTRODUCTION

Stochastic wireless channel modeling deals with the problem of developing mathematical tools that relate the behavior of the received signals to environmental factors and system parameters. There are many environmental factors. Some of them are geographic location (urban, rural, etc.), height, and velocity of the base station (BS) and the user equipment (UE). The system parameters include carrier frequency, transmit power and waveform of the signals, geometric structures of the multiple-input-multiple-output (MIMO) receiver and transmitter, bandwidth (BW) of the channel, length of the error-correction codes, and size of the allocated time and frequency resources.

Mathematical tools are then utilized to study the behavior of the signals at the transmitter and the receiver. With an objective in mind, the system parameter values are then chosen accordingly for a given environment. This is very useful because this process can be repeated for different scenarios on a computer instead of relying on trial and error methods using field tests. In short, we need these mathematical tools (channel models) for the design, simulation and planning of wireless systems [1].

The most notable work laying down the foundations of stochastic channel modeling and statistical communications analysis is the work of Stephen O. Rice of the Bell Laboratories in the mid 1940s [2]. The pioneering works of Rice can be found in [3-5] where he develops a theory of random noise. Ossana and Clarke apply this work to a wireless communications system where they incorporate mobility and
environmental factors [6, 7]. Gans develops a spectral density analysis for Clarke's model [8]. Jakes introduces the spatial diversity concept to avoid deep fades [9]. Bello extends the one dimensional models and explains the behavior of the signals using 2-D time-frequency statistical functions [10].

The aforementioned works describe the characteristics of the received signals; specifically the variation of the instantaneous received envelope in time-frequencyspace. They describe the instantaneous received envelope by the Rayleigh distribution. Molisch summarizes the reasons why Rayleigh distribution is widely used in wireless communications [1]:

- It provides an excellent approximation in a large number of practical scenarios, as confirmed by a multitude of measurements.
- It describes a worst case scenario where there is no line of light sight between the transmitter and the receiver. This is very useful in designing robust systems.
- It depends only on a single parameter, the mean received power. The entire signal statistics are known if this parameter is known.
- It is mathematically convenient.

In summary, the received signals are characterized as Rayleigh distributed stochastic processes. These models have been very successful in describing the radio waves in a statistical manner and have helped system designers to architect very efficient systems.

Having said that, there is still room for improvement in describing the received signals. There are two main arguments.

First, the inter-dependencies and intra-dependencies of time-frequency-space can be explained better by developing more sophisticated models $[1,11,12]$. This is the problem we address in this dissertation. Intra-dependencies focus on the dependencies across time (or frequency, or space) using 1-D correlation (or other statistical)
functions. Inter-dependencies focus on the dependencies between time and frequency and space which require multi-dimensional correlation functions.

Secondly, capturing the influence of the third dimension (elevation) is still lacking as the current models are only azimuth based $[13,14]$.

The essential reason to build new models is to move closer to the reality; more specifically, to the reality we can understand. This helps the researchers understand the existing systems more in depth and develop better ones for the future.

### 1.1 Goals

In this dissertation, we address the first argument that was mentioned above. Our goals are to:

1. Develop new channel models that capture the statistical variation in time-frequency-space more accurately.
2. Illustrate the significance of accurate wireless channel modeling through various applications.

The models we build rely on the Rayleigh distribution and stochastic processes. We define Rayleigh random variables (RVs) that are separated in time-frequencyspace. The dependencies between the RVs are quantified by correlation values. We build three different models that are more accurate than the existing ones and utilize them in three different applications.

### 1.2 Contributions

Here we highlight the dissertation contributions:

1. Derive the most comprehensive results regarding the quadrivariate Rayleigh distribution. Our model is the only one that can incorporate the Third Generation Partnership Project (3GPP) suggested spatial correlation structure.

Also, any four-branch receiver set-up regardless of the geometric arrangements (linear, circular...etc.) and the angle of arrival distribution can be studied.
2. Derive closed form expressions for the moments of the four-branch equal gain combiner's (EGC) and maximal ratio combiner's (MRC) output signal-to-noise ratio (SNR). The discrepancy between the independence assumption and our results goes up to $9 \%$ for the EGC receiver and $19 \%$ for the MRC receiver in terms of the mean output SNR. The discrepancy between the independence assumption and our results goes up to $16 \%$ for the EGC receiver and $57 \%$ for the MRC receiver in terms of the standard deviation of the output SNR. We show how inaccurate the independence assumption is up to first four moments. Our results enable one to carry out reliability analysis of these receivers.
3. Develop a novel bivariate SNR distribution that captures the temporal variation in the received SNR more accurately. This is utilized to decrease the number of computations a generic wireless scheduler requires. The results illustrate that the number of operations it takes to make scheduling decisions can be reduced by $33 \%$ with confidence probability of 0.7 and by $58 \%$ with confidence probability of 0.4 .
4. Develop a Markov chain model that describes the fade and non-fade durations of a Rayleigh fading channel. This is applied to the packet duplication (PD) feature in fifth generation (5G) cellular networks. We show that efficiency of the resource usage can be improved compared to existing works. The results indicate that up to $50 \%$ of the resources at the duplicating base station can be liberated whilst meeting the target reliability measure. This work is achieved by using matrix exponential (ME) distributions where residence times of Markov states as a group are studied rather than individual states.

In the rest of the dissertation we discuss the three applications each with their own motivation, problem, related work, results, conclusion and future work. Then we finalize with a more broad view of conclusions and future work.

## CHAPTER 2

# MOMENTS OF THE QUADRIVARIATE RAYLEIGH DISTRIBUTION WITH APPLICATIONS FOR DIVERSITY RECEIVERS 

### 2.1 Motivation

In wireless communications, the received signal amplitude can statistically be described by the Rayleigh distribution in urban and suburban environments where there are no line-of-sight paths [9]. Naturally, the signal samples are correlated in space, time, and frequency. Bandwidth of the signal, power azimuth spectrum (PAS), and relative motion of the UE are some key factors that influence the degree of the correlation. Therefore, the received signals at antennas of a four-branch diversity receiver can be modeled as Rayleigh random variables with some kind of spatial correlation model in between.

The correlation models in the literature make assumptions such as independence, constant correlation, exponential correlation or some other kind between received signals at each antenna. However, this limits the scenarios one can accurately model. For example, for a linear array of antennas, the constant correlation model would not be accurate as it assumes the correlation between all pairs of antennas are the same. This is not a very good assumption because the spatial correlation decreases as the distance between antennas increases. Therefore, a more accurate model should have less correlation between a farther pair of antennas compared to closer ones. One can easily provide other examples where the existing models would not be as realistic. As a result, an arbitrary correlation model without any assumptions is needed. In this way any receiver set-up can accurately be modeled regardless of the
geometric arrangements of them (linear, circular. . . etc.) or the PAS distribution. Li and Zhang [15] and the 3GPP [16] suggest a spatial correlation model for a diversity receiver at the UE. The suggested correlation model can be incorporated only into our expressions we derive here. The expressions in the literature would not be able to because of the assumptions they make.

It is often the case that a four-branch diversity receiver is studied through its outage probability and mean output SNR. The probability that the output SNR of the receiver is below a certain threshold is defined as outage probability. One needs to ideally obtain the joint cumulative distribution function (CDF) of the output SNR to analyze these properties ${ }^{1}$. A way of obtaining (or approximating) the CDF of the output SNR of a four-branch diversity receiver is by using the moments of the output SNR. This is referred to as the classical moment problem [17]. The higher the order of the moments, the more accurate the approximation to the CDF becomes. This also means qualitative properties of the output SNR such as skewness from the third moment and kurtosis, $\kappa$, from the fourth moment can be found. In a simplistic manner, these properties are measures of "asymmetry" and "tailedness" of the probability density function (PDF).

One motivation we have for finding higher order moments comes in the adaptation of Long Term Evolution (LTE) by public security entities. It is envisioned that the international market for this will surpass 2 billion Euros in 2019 [18]. Since dependable service is very important, the choice of a diversity receiver for public safety networks, we believe, will depend on not only the mean output SNR but also its variability and outage probability. How tailed the PDF is and which way the tail is ("left" or "right") are key properties that might help to study the reliability of a diversity receiver. Eggers et al. argue that tail statistics is very important in reliability analysis [19].

In addition to our specific interests and motivation for studying quadrivariate

[^0]Rayleigh RVs we mention above, there are other applications for them. An important one is a resource scheduler that exploits the time and frequency domain behaviors of the channel. Moments for four random variables capture the combined effects of frequency and time correlation (samples in a two dimensional plane).

### 2.2 Related Work

The PDF of the bivariate Rayleigh distribution is derived by Rice [3]. The corresponding cumulative distribution function is derived by Tan and Beaulieu [20]. A common method to derive the joint distribution for a higher number of Rayleigh RVs is to represent each random variable by an underlying complex Gaussian RV. The trivariate probability density function of Rayleigh RVs is derived by Miller [21] using this method. We also derived this distribution independently and reached the same result. Miller and Blumenson [22] provide an expression for the joint PDF of $N$ Rayleigh RVs, however, it is only valid when the inverse of the covariance matrix of the complex Gaussian RVs is tridiagonal. This type of correlation model is referred to as the exponential correlation model in the literature. Chen and Tellambura [23] extend the work of Miller and Blumenson for the quadrivariate case by relaxing the tridiagonal structure restriction on the inverse of the covariance matrix. Their method is limited to cases where the inverse of the aforementioned covariance matrix has zero values for the entries $(1,4)$ and $(4,1)$. Nadarajah and Kotz [24] provide simpler forms of the expressions derived by Chen and Tellambura. Le [25] shows that for the exponentially correlated case, the seven-nested infinite summation form derived by Chen and Tellambura can be reduced to a three-nested summation form.

The results in the discussed works so far are some form of an infinite series of either the modified Bessel functions, incomplete Gamma functions, or exponential functions.

There are also integral expressions in comparison to infinite series. These forms
of results are generally in terms of an integrand that is a function of magnitude (Rayleigh distributed) and phase random variables. Mallik [26] presents $N$-tuple integral expressions for the $N$-variate Rayleigh RVs with various correlation models. Beaulieu and Hemachandra [27] derive a single integral expression for the $N$-variate PDF with a limited correlation model. Beaulieu and Zhang [28], in a very recent work, derive a double integral form PDF of the quadrivariate Rayleigh distribution with arbitrary correlation model.

### 2.3 Contributions

To the best of our knowledge the results for the quadrivariate Rayleigh distribution in the literature either make assumptions such as independence, constant correlation, exponential correlation or some other kind between the RVs (signals at antennas), that results in a limited correlation model.

The only result with an arbitrary correlation model is a double integral form PDF given by Beaulieu and Zhang [28]. We derive the PDF in infinite series form. Owing to the infinite series form of our result, we obtain analytical expressions for the cumulative distribution function, joint moments, and moment generating function (MGF) with an arbitrary correlation model which we believe are not available in the literature.

Our results are novel and extend previous work by having no restrictions on the covariance matrix (as opposed to models in [22], [23] and [24]), being in infinite series forms (as opposed to integral forms in [26] and [28]) and providing expressions for the CDF, MGF and joint moments in addition to the PDF.

Furthermore, we derive all of the possible expressions for the PDF, joint moments and MGF where it can be reduced to simpler forms. We provide the corresponding correlation structure for each case. These cases represent different scenarios where the partial correlations between certain RVs are zero. We believe that only some of these reduced forms are available in the literature.

Using our results, we study performances of different diversity schemes. We utilize the CDF to compute the outage probability values for the four-branch selection combining receiver in different scenarios. We derive an analytical expression for the moments of the output SNR of the four-branch equal gain and maximal ratio combining receivers. We utilize them to compute the first, second, third and fourth moments of the output SNR in different scenarios. Our expressions consist of well known functions which make them readily available for computations. We believe that our findings regarding the aforementioned diversity receivers are new.

To sum up the contributions of the chapter, we develop a novel channel model that is the first to capture the 3GPP suggested spatial correlation structure. In other words, our model is the only one that can incorporate the 3GPP suggested spatial correlation values between RVs.

The rest of the chapter is organized as follows: List of notations regarding this chapter is given in Section 2.4. Representation of the correlated Rayleigh RVs is introduced in Section 2.5. In Section 2.6, the new results and analytical expressions are given. Section 2.7 presents applications where the results can be utilized. In Section 2.8, performance analysis of various diversity schemes in different scenarios are presented. Concluding remarks and future work are provided in Section 2.9. Derivations of the results are given in Appendices A-C.

### 2.4 List of Notations

$d_{\text {min }} \quad$ distance between closest pair of antennas
$D_{k}(\cdot) \quad$ parabolic cylinder function
$E[\cdot] \quad$ expectation operator
$f_{\boldsymbol{X}}(x) \quad$ probability density function of $X$
$F_{\boldsymbol{X}}(x) \quad$ cumulative distribution function of $X$
$I_{k}(\cdot) \quad k^{\text {th }}$ order modified Bessel function of the first kind

| $J_{k}(\cdot)$ | $k^{t h}$ order Bessel function of the first kind |
| :---: | :---: |
| $M_{\boldsymbol{X}}(s)$ | moment generating function of $X$ |
| $N_{o}$ | additive white Gaussian noise power spectral density at each branch |
| $P_{\text {out }}$ | outage probability |
| $r$ | Rayleigh random variable |
| $(\cdot)^{T}$ | transpose operator |
| $z$ | zero-mean Gaussian random variable |
| $\bar{Z}$ | row vector of Gaussian random variables |
| $\gamma_{k}$ | instantaneous SNR at the $k^{\text {th }}$ branch |
| $\bar{\gamma}_{k}, \bar{\gamma}$ | average SNR at the $k^{\text {th }}$ branch |
| $\gamma^{t h}$ | threshold SNR for outage probability |
| $\gamma(\cdot)$ | lower incomplete gamma function |
| $\Gamma(\cdot)$ | gamma function |
| $\zeta$ | variance of Gaussian random variables |
| $\kappa$ | kurtosis coefficient of output SNR |
| $\lambda$ | wavelength |
| $\mu$ | mean of power azimuth spectrum |
| $\rho_{\|i-j\|}$ | correlation coefficient between $i^{\text {th }}$ and $j^{\text {th }}$ Gaussian RV |
| $\sigma$ | standard deviation of output SNR |
| $\phi$ | conditional covariance between Gaussian RVs |
| $\Phi$ | inverse covariance matrix of Gaussian RVs |
| $\Psi$ | covariance matrix of Gaussian RVs |

### 2.5 Representation of Correlated Random Variables

We define random variables $z_{I_{1}}, z_{Q_{1}}, \ldots, z_{I_{4}}, z_{Q_{4}}$ as equal variance, $\zeta$, zero-mean Gaussian distributions (without loss of generality, i.e., $z_{I_{i}}, z_{Q_{i}}$ pair and $z_{I_{j}}, z_{Q_{j}}$ pair may have different variances in between for $i \neq j$ ). Their well-known joint PDF is given by [29] (7.18a):

$$
\begin{equation*}
f_{\boldsymbol{Z}}\left(z_{I_{1}}, z_{Q_{1}} \ldots, z_{I_{4}}, z_{Q_{4}}\right)=\frac{\exp \left(-\frac{\overline{\mathbf{Z}} \Psi^{-1} \overline{\mathbf{Z}}^{T}}{2}\right)}{(2 \pi)^{4}(\operatorname{det} \mathbf{\Psi})^{(1 / 2)}} \tag{2.1}
\end{equation*}
$$

where $(\cdot)^{T}$ denotes the transpose operator, $\overline{\boldsymbol{Z}}=\left[z_{I_{1}}, z_{Q_{1}}, \ldots, z_{I_{4}}, z_{Q_{4}}\right]$ and $\boldsymbol{\Psi}$ is a positive definite covariance matrix. We define Rayleigh random variables as $r_{i}=\left(z_{I_{i}}{ }^{2}+z_{Q_{i}}{ }^{2}\right)^{1 / 2}$ for $i \in\{1 \ldots 4\}$. Elements of the covariance matrix are defined as $\psi_{11}=E\left[z_{I_{1}} z_{I_{1}}\right], \psi_{12}=E\left[z_{I_{1}} z_{Q_{1}}\right], \psi_{13}=E\left[z_{I_{1}} z_{I_{2}}\right] \ldots \psi_{88}=E\left[z_{Q_{4}} z_{Q_{4}}\right]$ where $E[\cdot]$ denotes the expectation operator. Cross-covariance between the in-phase and quadrature components for all pairs are zero. In other words, $E\left[z_{I_{i}} z_{Q_{j}}\right]=0$ (hence, independent identically distributed) for $i, j \in\{1, \ldots, 4\}$.

The correlation coefficient between $i^{\text {th }}$ and $j^{\text {th }}$ RV is defined as

$$
\begin{equation*}
\rho_{|i-j|}=\frac{E\left[z_{I_{i}} z_{I_{j}}\right]}{\zeta}=\frac{E\left[z_{Q_{i}} z_{Q_{j}}\right]}{\zeta}, \quad i, j \in\{1, \ldots, 4\} \tag{2.2}
\end{equation*}
$$

Therefore, $\rho_{0}=1$. An illustration of this correlation structure for a linear array of antennas is shown in Fig. 2.1.


Figure 2.1: Correlation structure between antennas.

### 2.6 Statement of Results

### 2.6.1 Joint Density and Joint Cumulative Distribution Functions

Given a positive definite covariance matrix, $\boldsymbol{\Psi}$, (see previous section) the inverse covariance matrix, $\boldsymbol{\Phi}$ has the following form

$$
\boldsymbol{\Phi}=\boldsymbol{\Psi}^{-\mathbf{1}}=\frac{1}{\zeta}\left[\begin{array}{cccccccc}
\phi_{1} & 0 & \phi_{3} & 0 & \phi_{4} & 0 & \phi_{5} & 0  \tag{2.3}\\
0 & \phi_{1} & 0 & \phi_{3} & 0 & \phi_{4} & 0 & \phi_{5} \\
\phi_{3} & 0 & \phi_{2} & 0 & \phi_{6} & 0 & \phi_{4} & 0 \\
0 & \phi_{3} & 0 & \phi_{2} & 0 & \phi_{6} & 0 & \phi_{4} \\
\phi_{4} & 0 & \phi_{6} & 0 & \phi_{2} & 0 & \phi_{3} & 0 \\
0 & \phi_{4} & 0 & \phi_{6} & 0 & \phi_{2} & 0 & \phi_{3} \\
\phi_{5} & 0 & \phi_{4} & 0 & \phi_{3} & 0 & \phi_{1} & 0 \\
0 & \phi_{5} & 0 & \phi_{4} & 0 & \phi_{3} & 0 & \phi_{1}
\end{array}\right]
$$

where $\phi$ denotes the conditional covariance between Gaussian RVs. Specific entries of the inverse covariance matrix denote the conditional covariance between the corresponding RVs. For instance, $\phi_{3}$ denotes the conditional covariance between $z_{I_{1}}$ and $z_{I_{2}}, z_{Q_{1}}$ and $z_{Q_{2}}, z_{I_{3}}$ and $z_{I_{4}}, z_{Q_{3}}$ and $z_{Q_{4}}$ random variables. The conditional covariance between $z_{I_{1}}$ and $z_{I_{4}}, z_{Q_{1}}$ and $z_{Q_{4}}$ is denoted by $\phi_{5}$. The joint PDF and CDF of $N$ Rayleigh random variables are derived by Miller and Blumenson [22] given that the inverse covariance matrix of the underlying complex Gaussian RVs is tridiagonal. That is to say, $\phi_{4}=\phi_{5}=0$ corresponding to $\boldsymbol{\Phi}$. This form represents the well known exponential correlation model. The joint PDF and CDF of the quadrivariate Rayleigh distribution are derived by Chen and Tellambura [23] given that $\phi_{5}=0$.

Here we provide a more general result where there are no such restrictions on the inverse covariance matrix. In other words, our expression is valid when $\phi_{i} \neq 0$ for all $i$. This is significant because the spatial correlation matrix suggested by the

3GPP has non-zero for all $\phi_{i}$ values.
We derive the joint PDF of four Rayleigh random variables as (see Appendix A)

$$
\begin{align*}
& f_{\boldsymbol{R}}\left(r_{1}, r_{2}, r_{3}, r_{4}\right) \\
& \quad=(\operatorname{det} \boldsymbol{\Psi})^{-(1 / 2)} r_{1} r_{2} r_{3} r_{4} e^{-\left(\frac{\phi_{1}}{2 \zeta}\left(r_{1}^{2}+r_{4}^{2}\right)+\frac{\phi_{2}}{2 \zeta}\left(r_{2}^{2}+r_{3}^{2}\right)\right)} \\
& \quad \times \sum_{l=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} I_{l}\left(\frac{-\phi_{3}}{\zeta} r_{1} r_{2}\right) I_{j}\left(\frac{-\phi_{4}}{\zeta} r_{1} r_{3}\right)  \tag{2.4}\\
& \quad \times I_{l+j}\left(\frac{-\phi_{5}}{\zeta} r_{1} r_{4}\right) I_{l+m}\left(\frac{-\phi_{6}}{\zeta} r_{2} r_{3}\right) I_{l+j+m}\left(\frac{-\phi_{3}}{\zeta} r_{3} r_{4}\right) I_{m}\left(\frac{-\phi_{4}}{\zeta} r_{2} r_{4}\right)
\end{align*}
$$

where $I_{k}(\cdot)$ denotes the $k^{t h}$ order modified Bessel function of the first kind. We believe that this is a novel result and the most general result regarding the PDF of the quadrivariate Rayleigh distribution. Please note that the structure of this expression is valid for any given positive definite covariance matrix.

We derive the joint CDF of four Rayleigh random variables as (see Appendix B)

$$
\begin{align*}
& F_{\boldsymbol{R}}\left(r_{1}, r_{2}, r_{3}, r_{4}\right) \\
& \quad=\eta^{3} \sum_{l, m, j=-\infty}^{\infty} \sum_{b, q, b^{\prime}, h, f, h^{\prime}=0}^{\infty} \frac{\left(-2 \phi_{3} \eta\right)^{|l|+|l+j+m|+2 b^{\prime}+2 b}}{2^{\frac{\left(\nu_{1}+\nu_{2}+\nu_{3}+\nu_{4}\right)}{2}}} \\
& \quad \times \frac{\left(-2 \phi_{4} \eta\right)^{|j|+|m|+2 h^{\prime}+2 h}\left(-2 \phi_{5} \eta\right)^{|l+j|+2 f}\left(-2 \phi_{6} \eta\right)^{|l+m|+2 q}}{\left(\phi_{1} \eta\right)^{\frac{\nu_{1}+\nu_{4}+4}{2}}}\left(\phi_{2} \eta\right)^{\frac{\nu_{2}+\nu_{3}+4}{2}} b!f!h!(b+|l|)!(h+|j|)!  \tag{2.5}\\
& \quad \times \frac{\gamma\left(\frac{\nu_{1}+2}{2}, \frac{\phi_{1}}{2 \zeta} r_{1}^{2}\right)}{(f+|j+l|)!q!} \frac{\gamma\left(\frac{\nu_{2}+2}{2}, \frac{\phi_{2}}{2 \zeta} r_{2}^{2}\right)}{b^{\prime}!(q+|l+m|)!} \frac{\gamma\left(\frac{\nu_{3}+2}{2}, \frac{\phi_{2}}{2 \zeta} r_{3}^{2}\right)}{\left(b^{\prime}+|l+m+j|\right)!} \frac{\gamma\left(\frac{\nu_{4}+2}{2}, \frac{\phi_{1}}{2 \zeta} r_{4}^{2}\right)}{\left(h^{\prime}+|m|\right)!h^{\prime}!}
\end{align*}
$$

where $\eta=\frac{(\operatorname{det} \Psi)^{(1 / 2)}}{\zeta^{4}}, \zeta$ denotes the variance of the Gaussian RVs, $\gamma(\cdot)$ denotes the lower incomplete gamma function, $\nu_{1}=|l|+|j|+|j+l|+2 b+2 h+2 f, \nu_{2}=$ $|l|+|m|+|l+m|+2 b+2 q+2 h^{\prime}, \nu_{3}=|j|+|l+m|+|j+l+m|+2 h+2 q+2 b^{\prime}$, $\nu_{4}=|m|+|j+l|+|j+l+m|+2 f+2 b^{\prime}+2 h^{\prime}$. We believe that this is a new result and the most general result regarding the CDF of the quadrivariate Rayleigh distribution.

To study the performance of diversity receivers we derive moments and moment generating functions. Using (2.4), we derive the joint moments of four Rayleigh RVs as

$$
\begin{align*}
& E\left[r_{1}^{\beta_{1}} r_{2}^{\beta_{2}} r_{3}^{\beta_{3}} r_{4}^{\beta_{4}}\right] \\
& =\eta^{3}(\eta \zeta)^{\beta / 2} \sum_{l, m, j=-\infty}^{\infty} \sum_{b, q, b^{\prime}, h, f, h^{\prime}=0}^{\infty} \frac{\left(-2 \phi_{5} \eta\right)^{|l+j|+2 f}}{2^{\frac{\nu_{1}+\nu_{2}+\nu_{3}+\nu_{4}-\beta}{2}}} \\
& \times \frac{\left(-2 \phi_{4} \eta\right)^{|j|+|m|+2 h^{\prime}+2 h}\left(-2 \phi_{3} \eta\right)^{|l|+|l+j+m|+2 b^{\prime}+2 b}\left(-2 \phi_{6} \eta\right)^{|l+m|+2 q}}{\left(\phi_{1} \eta\right)^{\frac{\nu_{1}+\nu_{4}+4+\beta_{1}+\beta_{4}}{2}}\left(\phi_{2} \eta\right)^{\frac{\nu_{2}+\nu_{3}+4+\beta_{2}+\beta_{3}}{2}} b!h!h^{\prime}!(b+|l|)!(h+|j|)!f!}  \tag{2.6}\\
& \times \frac{\Gamma\left(\frac{\nu_{1}+\beta_{1}+2}{2}\right)}{q!b^{\prime}!(f+|j+l|)!} \frac{\Gamma\left(\frac{\nu_{2}+\beta_{2}+2}{2}\right)}{(q+|l+m|)!} \frac{\Gamma\left(\frac{\nu_{3}+\beta_{3}+2}{2}\right)}{\left(b^{\prime}+|l+m+j|\right)!} \frac{\Gamma\left(\frac{\nu_{4}+\beta_{4}+2}{2}\right)}{\left(h^{\prime}+|m|\right)!}, \\
& \beta_{1}, \beta_{2}, \beta_{3}, \beta_{4} \geq-1
\end{align*}
$$

where $\beta=\beta_{1}+\beta_{2}+\beta_{3}+\beta_{4}$ and $\Gamma(\cdot)$ denotes the gamma function. The derivation is very similar to what is presented in Appendix B. We utilize the integral representation of gamma function as given in [30] (6.1.1).

Using (2.4), we derive the joint moment generating function of the quadrivariate Rayleigh distribution as (see Appendix C)

$$
\begin{align*}
& M_{\boldsymbol{R}}\left(s_{1}, s_{2}, s_{3}, s_{4}\right)=E\left[e^{\left.s_{1} r_{1}+s_{2} r_{2}+s_{3} r_{3}+s_{4} r_{4}\right]}\right. \\
& \quad=\eta^{3} e^{\left(\frac{\zeta}{4 \phi_{1}}\left(s_{1}^{2}+s_{4}^{2}\right)+\frac{\zeta}{4 \phi_{2}}\left(s_{2}^{2}+s_{3}^{2}\right)\right)} \\
& \quad \times \sum_{l, m, j=-\infty}^{\infty} \sum_{b, q, b^{\prime}, h, f, h^{\prime}=0}^{\infty} \frac{\left(-2 \phi_{3} \eta\right)^{|l|+|l+j+m|+2 b^{\prime}+2 b}}{2^{\left(\nu_{1}+\nu_{2}+\nu_{3}+\nu_{4}\right)}\left(\phi_{1} \eta\right)^{\frac{\nu_{1}+\nu_{4}+4}{2}}} \\
& \quad \times \frac{\left(-2 \phi_{4} \eta\right)^{|j|+|m|+2 h^{\prime}+2 h}\left(-2 \phi_{5} \eta\right)^{|l+j|+2 f}\left(-2 \phi_{6} \eta\right)^{|l+m|+2 q}}{\left(\phi_{2} \eta\right)^{\frac{\nu_{2}+\nu_{3}+4}{2}} b!h!f!(b+|l|)!(h+|j|)!b^{\prime}!h^{\prime}!q!}  \tag{2.7}\\
& \quad \times \frac{\Gamma\left(\nu_{1}+2\right)}{(f+|j+l|)!} \frac{\Gamma\left(\nu_{2}+2\right)}{(q+|l+m|)!} \frac{\Gamma\left(\nu_{3}+2\right)}{\left(b^{\prime}+|l+m+j|\right)!} \frac{\Gamma\left(\nu_{4}+2\right)}{\left(h^{\prime}+|m|\right)!} \\
& \quad \times D_{-\left(\nu_{1}+2\right)}\left(\frac{-s_{1}}{\sqrt{\phi_{1} / \zeta}}\right) D_{-\left(\nu_{2}+2\right)}^{\left(\frac{-s_{2}}{\sqrt{\phi_{2} / \zeta}}\right) D_{-\left(\nu_{3}+2\right)}\left(\frac{-s_{3}}{\sqrt{\phi_{2} / \zeta}}\right)} \\
& \quad \times D_{-\left(\nu_{4}+2\right)}\left(\frac{-s_{4}}{\sqrt{\phi_{1} / \zeta}}\right)
\end{align*}
$$

where $D_{k}(\cdot)$ denotes the parabolic cylinder function [31] (9.240).

### 2.6.3 Reduced Forms

The results we provide for the joint PDF, CDF, moments and MGF in (2.4) - (2.7) represent the most general case. That is, $\phi_{i} \neq 0$ for all $i$. $\phi_{i}$ values are the entry values in the inverse covariance matrix (equation in (2.3)). This is the significance of our work.

However, if the underlying covariance matrix has a special structure, then the expressions will be reduced to simpler forms. Please note that the structure of the covariance matrix relies on the physical scenario one is interested in. As a result, if the correlation at the antennas can accurately be modeled by these special structures we provide in this section, the results will be simplified in terms of number of operations (number of infinite series terms).

There are a total of five cases where the expressions in (2.4) - (2.7) reduce to simpler forms. We present the joint PDF, moments, and MGF for each case. We also present the corresponding restriction on the correlation values $(\rho)$ between the underlying Gaussian RVs for each case. They are shown in (2.8) - (2.27).

To the best of our knowledge only two of these cases are previously analyzed. They can be found in [22] and [23]. Thus, we believe that the results in (2.8) - (2.27) are new except for $(2.20),(2.24)$ and (2.27).

Case When $\phi_{3}=0$
This case is applicable to a scenario where the first $\left(r_{1}\right)$ and second $\left(r_{2}\right)$ and the third $\left(r_{3}\right)$ and fourth $\left(r_{4}\right)$ Rayleigh RVs are conditionally independent (i.e., zero partial correlation in between). We derive the joint PDF of the quadrivariate

Rayleigh distribution for this case as

$$
\begin{align*}
& f_{\boldsymbol{R}}\left(r_{1}, r_{2}, r_{3}, r_{4}\right) \\
& \quad=(\operatorname{det} \boldsymbol{\Psi})^{-(1 / 2)} r_{1} r_{2} r_{3} r_{4} e^{-\left(\frac{\phi_{1}}{2 \zeta}\left(r_{1}^{2}+r_{4}^{2}\right)+\frac{\phi_{2}}{2 \zeta}\left(r_{2}^{2}+r_{3}^{2}\right)\right)}  \tag{2.8}\\
& \quad \times \sum_{l=-\infty}^{\infty} I_{l}\left(\frac{-\phi_{4}}{\zeta} r_{1} r_{3}\right) I_{l}\left(\frac{-\phi_{5}}{\zeta} r_{1} r_{4}\right) I_{l}\left(\frac{-\phi_{6}}{\zeta} r_{2} r_{3}\right) I_{l}\left(\frac{-\phi_{4}}{\zeta} r_{2} r_{4}\right)
\end{align*}
$$

We derive the joint moments of the quadrivariate Rayleigh distribution for this case as

$$
\begin{align*}
E & {\left[r_{1}^{\beta_{1}} r_{2}^{\beta_{2}} r_{3}^{\beta_{3}} r_{4}^{\beta_{4}}\right] } \\
& =\eta^{3}(\eta \zeta)^{\beta / 2} \sum_{l=-\infty}^{\infty} \sum_{h, f, h^{\prime}, q=0}^{\infty} \frac{\left(-2 \phi_{5} \eta\right)^{|l|+2 f}}{2^{\frac{\nu_{1}^{\prime}+\nu_{2}^{\prime}+\nu_{3}^{\prime}+\nu_{4}^{\prime}-\beta}{2}}} \\
& \times \frac{\left(-2 \phi_{4} \eta\right)^{|2 l|+2 h^{\prime}+2 h}\left(-2 \phi_{6} \eta\right)^{|l|+2 q}}{\left(\phi_{1} \eta\right)^{\frac{\nu_{1}^{\prime}+\nu_{4}^{\prime}+4+\beta_{1}+\beta_{4}}{2}}\left(\phi_{2} \eta\right)^{\frac{\nu_{2}^{\prime}+\nu_{3}^{\prime}+4+\beta_{2}+\beta_{3}}{2}} \frac{\Gamma\left(\frac{\nu_{1}^{\prime}+\beta_{1}+2}{2}\right)}{h!f!(h+|l|)!h^{\prime}!}}  \tag{2.9}\\
& \times \frac{\Gamma\left(\frac{\nu_{2}^{\prime}+\beta_{2}+2}{2}\right)}{(f+|l|)!q!} \frac{\Gamma\left(\frac{\nu_{3}^{\prime}+\beta_{3}+2}{2}\right)}{(q+|l|)!} \frac{\Gamma\left(\frac{\nu_{4}^{\prime}+\beta_{4}+2}{2}\right)}{\left(h^{\prime}+|l|\right)!}, \quad \quad \beta_{1}, \beta_{2}, \beta_{3}, \beta_{4} \geq-1
\end{align*}
$$

where $\beta=\beta_{1}+\beta_{2}+\beta_{3}+\beta_{4}$ and $\nu_{1}^{\prime}=|2 l|+2 h+2 f, \nu_{2}^{\prime}=|2 l|+2 q+2 h^{\prime}, \nu_{3}^{\prime}=$ $|2 l|+2 h+2 q, \nu_{4}^{\prime}=|2 l|+2 f+2 h^{\prime}$. We derive the joint moment generating function of the quadrivariate Rayleigh distribution for this case as

$$
\begin{align*}
& M_{\boldsymbol{R}}\left(s_{1}, s_{2}, s_{3}, s_{4}\right)=E\left[e^{\left.s_{1} r_{1}+s_{2} r_{2}+s_{3} r_{3}+s_{4} r_{4}\right]}\right. \\
& \quad=\eta^{3} e^{\left(\frac{\zeta}{4 \phi_{1}}\left(s_{1}^{2}+s_{4}^{2}\right)+\frac{\zeta}{4 \phi_{2}}\left(s_{2}^{2}+s_{3}^{2}\right)\right)} \\
& \quad \times \sum_{l=-\infty}^{\infty} \sum_{h, h^{\prime}, q, f=0}^{\infty} \frac{\left(-2 \phi_{5} \eta\right)^{|l|+2 f}}{} \frac{\left(-2 \phi_{4} \eta\right)^{|2 l|+2 h^{\prime}+2 h}\left(-2 \phi_{6} \eta\right)^{(l \mid+2 q}{ }^{\left(\nu_{2}^{\prime}+\nu_{2}^{\prime}+\nu_{3}^{\prime}+\nu_{4}^{\prime}\right)}\left(\phi_{1} \eta\right)^{\frac{\nu_{1}^{\prime}+\nu_{4}^{\prime}+4}{2}}}{\left(\nu_{2}\left(\nu_{1}^{\prime}+2\right)\right.} \\
& \quad \times \frac{\nu_{2}^{\prime}+\nu_{3}^{\prime}+4}{2} h!f!h!q!  \tag{2.10}\\
& \quad \times \frac{\Gamma\left(\nu_{2}^{\prime}+2\right)}{(f+|l|)!} \frac{\Gamma\left(\nu_{3}^{\prime}+2\right)!}{(h+|l|)!} \frac{\Gamma\left(\nu_{4}^{\prime}+2\right)}{\left(h^{\prime}+|l|\right)!} D_{-\left(\nu_{1}^{\prime}+2\right)}\left(\frac{-s_{1}}{\sqrt{\phi_{1} / \zeta}}\right) \\
& \quad \times D_{-\left(\nu_{2}^{\prime}+2\right)}\left(\frac{-s_{2}}{\sqrt{\phi_{2} / \zeta}}\right) D_{-\left(\nu_{3}^{\prime}+2\right)}\left(\frac{-s_{3}}{\sqrt{\phi_{2} / \zeta}}\right) D_{-\left(\nu_{4}^{\prime}+2\right)}\left(\frac{-s_{4}}{\sqrt{\phi_{1} / \zeta}}\right)
\end{align*}
$$

where $\nu_{1}^{\prime}=|2 l|+2 h+2 f, \nu_{2}^{\prime}=|2 l|+2 q+2 h^{\prime}, \nu_{3}^{\prime}=|2 l|+2 h+2 q, \nu_{4}^{\prime}=|2 l|+2 f+2 h^{\prime}$.

The joint PDF for this case is easily derived by substituting $\phi_{3}=0$ in (2.4) and algebraic manipulation. The joint moments and MGF, similarly, are obtained using (2.8). Please notice that the expressions presented here have less number of infinite series compared to the general case. This can be thought of as a tradeoff between the comprehensiveness (ability to capture more practical scenarios) of the model and the complexity. Similar argument can also be when results for the quadrivariate distribution are compared with the trivariate or bivariate distributions.

The expressions for this case are only valid when

$$
\begin{equation*}
\rho_{3}=\frac{\rho_{1}-\rho_{1}^{3}-\rho_{1} \rho_{2}+\rho_{1} \rho_{2}^{2}}{\rho_{2}-\rho_{1}^{2}} \tag{2.11}
\end{equation*}
$$

where $\rho_{|i-j|}=E\left[z_{I_{i}} z_{I_{j}}\right] / \zeta=E\left[z_{Q_{i}} z_{Q_{j}}\right] / \zeta$.

Case When $\phi_{4}=0$

This case is applicable to a scenario where the first $\left(r_{1}\right)$ and third $\left(r_{3}\right)$ and the second $\left(r_{2}\right)$ and fourth $\left(r_{4}\right)$ Rayleigh RVs are conditionally independent (i.e., zero partial correlation in between). We derive the joint PDF of the quadrivariate Rayleigh distribution for this case as

$$
\begin{align*}
& f_{\boldsymbol{R}}\left(r_{1}, r_{2}, r_{3}, r_{4}\right) \\
& \quad=(\operatorname{det} \boldsymbol{\Psi})^{-(1 / 2)} r_{1} r_{2} r_{3} r_{4} e^{-\left(\frac{\phi_{1}}{2 \zeta}\left(r_{1}^{2}+r_{4}^{2}\right)+\frac{\phi_{2}}{2 \zeta}\left(r_{2}^{2}+r_{3}^{2}\right)\right)}  \tag{2.12}\\
& \quad \times \sum_{l=-\infty}^{\infty} I_{l}\left(\frac{-\phi_{3}}{\zeta} r_{1} r_{2}\right) I_{l}\left(\frac{-\phi_{5}}{\zeta} r_{1} r_{4}\right) I_{l}\left(\frac{-\phi_{6}}{\zeta} r_{2} r_{3}\right) I_{l}\left(\frac{-\phi_{3}}{\zeta} r_{3} r_{4}\right)
\end{align*}
$$

We derive the joint moments of four Rayleigh RVs for this case as

$$
\begin{align*}
E & {\left[r_{1}^{\beta_{1}} r_{2}^{\beta_{2}} r_{3}^{\beta_{3}} r_{4}^{\beta_{4}}\right] } \\
& =\eta^{3}(\eta \zeta)^{\beta / 2} \sum_{l=-\infty}^{\infty} \sum_{b, b^{\prime}, q, f=0}^{\infty} \frac{\left(-2 \phi_{5} \eta\right)^{|l|+2 f}}{2^{\frac{\nu_{1}^{\prime}+\nu_{2}^{\prime}+\nu_{3}^{\prime}+\nu_{4}^{\prime}-\beta}{2}}} \\
& \times \frac{\left(-2 \phi_{3} \eta\right)^{|2 l|+2 b^{\prime}+2 b}\left(-2 \phi_{6} \eta\right)^{|l|+2 q}}{\left(\phi_{1} \eta\right)^{\frac{\nu_{1}^{\prime}+\nu_{4}^{\prime}+4+\beta_{1}+\beta_{4}}{2}}\left(\phi_{2} \eta\right)^{\frac{\nu_{2}^{\prime}+\nu_{3}^{\prime}+4+\beta_{2}+\beta_{3}}{2}} \frac{\Gamma\left(\frac{\nu_{1}^{\prime}+\beta_{1}+2}{2}\right)}{b!b^{\prime}!(b+|l|)!f!}}  \tag{2.13}\\
& \times \frac{\Gamma\left(\frac{\nu_{2}^{\prime}+\beta_{2}+2}{2}\right)}{(f+|l|)!q!} \frac{\Gamma\left(\frac{\nu_{3}^{\prime}+\beta_{3}+2}{2}\right)}{(q+|l|)!} \frac{\Gamma\left(\frac{\nu_{4}^{\prime}+\beta_{4}+2}{2}\right)}{\left(b^{\prime}+|l|\right)!}, \quad \quad \beta_{1}, \beta_{2}, \beta_{3}, \beta_{4} \geq-1
\end{align*}
$$

where $\beta=\beta_{1}+\beta_{2}+\beta_{3}+\beta_{4}$ and $\nu_{1}^{\prime}=|2 l|+2 b+2 f, \nu_{2}^{\prime}=|2 l|+2 q+2 b, \nu_{3}^{\prime}=$ $|2 l|+2 b^{\prime}+2 q, \nu_{4}^{\prime}=|2 l|+2 f+2 b^{\prime}$. We derive the joint moment generating function of the quadrivariate Rayleigh distribution for this case as

$$
\begin{align*}
& M_{\boldsymbol{R}}\left(s_{1}, s_{2}, s_{3}, s_{4}\right)=E\left[e^{\left.s_{1} r_{1}+s_{2} r_{2}+s_{3} r_{3}+s_{4} r_{4}\right]}\right. \\
& \quad=\eta^{3} e^{\left(\frac{\zeta}{4 \phi_{1}}\left(s_{1}^{2}+s_{4}^{2}\right)+\frac{\zeta}{4 \phi_{2}}\left(s_{2}^{2}+s_{3}^{2}\right)\right)} \\
& \quad \times \sum_{l=-\infty}^{\infty} \sum_{b, b^{\prime}, q, f=0}^{\infty} \frac{\left(-2 \phi_{5} \eta\right)^{|l|+2 f}}{2^{\left(\nu_{1}^{\prime}+\nu_{2}^{\prime}+\nu_{3}^{\prime}+\nu_{4}^{\prime}\right)}\left(\phi_{1} \eta\right)^{\frac{\nu_{1}^{\prime}+\nu_{4}^{\prime}+4}{2}}} \\
& \quad \times \frac{\left(-2 \phi_{3} \eta\right)^{|2 l|+2 b^{\prime}+2 b}\left(-2 \phi_{6} \eta\right)^{l l+2 q}}{\left(\phi_{2} \eta\right)^{\frac{\nu_{2}^{\prime}+\nu_{3}^{\prime}+4}{2}} b!f!b^{\prime}!q!} \frac{\Gamma\left(\nu_{1}^{\prime}+2\right)}{(f+|l|)!}  \tag{2.14}\\
& \quad \times \frac{\Gamma\left(\nu_{2}^{\prime}+2\right)}{(q+|l|)!} \frac{\Gamma\left(\nu_{3}^{\prime}+2\right)}{\left(b^{\prime}+|l|\right)!} \frac{\Gamma\left(\nu_{4}^{\prime}+2\right)}{(b+|l|)!} D_{-\left(\nu_{1}^{\prime}+2\right)}\left(\frac{-s_{1}}{\sqrt{\phi_{1} / \zeta}}\right) \\
& \quad \times D_{-\left(\nu_{2}^{\prime}+2\right)}\left(\frac{-s_{2}}{\sqrt{\phi_{2} / \zeta}}\right) D_{-\left(\nu_{3}^{\prime}+2\right)}\left(\frac{-s_{3}}{\sqrt{\phi_{2} / \zeta}}\right) D_{-\left(\nu_{4}^{\prime}+2\right)}\left(\frac{-s_{4}}{\sqrt{\phi_{1} / \zeta}}\right)
\end{align*}
$$

where $\nu_{1}^{\prime}=|2 l|+2 b+2 f, \nu_{2}^{\prime}=|2 l|+2 q+2 b, \nu_{3}^{\prime}=|2 l|+2 b^{\prime}+2 q, \nu_{4}^{\prime}=|2 l|+2 f+2 b^{\prime}$.

The joint PDF for this case is easily derived by substituting $\phi_{4}=0$ in (2.4) and algebraic manipulation. The joint moments and MGF, similarly, are obtained using (2.12). Please notice that the expressions presented here have less number of infinite series compared to the general case. This can be thought of as a tradeoff between the comprehensiveness (ability to capture more practical scenarios) of
the model and the complexity. Similar argument can also be when results for the quadrivariate distribution are compared with the trivariate or bivariate distributions.

The expressions for this case are only valid when

$$
\begin{equation*}
\rho_{3}=\frac{-\rho_{1}^{2}+\rho_{2}+\rho_{1}^{2} \rho_{2}-\rho_{2}^{3}}{\rho_{1}-\rho_{1} \rho_{2}} \tag{2.15}
\end{equation*}
$$

where $\rho_{|i-j|}=E\left[z_{I_{i}} z_{I_{j}}\right] / \zeta=E\left[z_{Q_{i}} z_{Q_{j}}\right] / \zeta$.

Case When $\phi_{6}=0$

This case is applicable to a scenario where the second $\left(r_{2}\right)$ and third $\left(r_{3}\right)$ Rayleigh RVs are conditionally independent (i.e., zero partial correlation in between). We derive the joint PDF of the quadrivariate Rayleigh distribution for this case as

$$
\begin{align*}
& f_{\boldsymbol{R}}\left(r_{1}, r_{2}, r_{3}, r_{4}\right) \\
& \quad=(\operatorname{det} \boldsymbol{\Psi})^{-(1 / 2)} r_{1} r_{2} r_{3} r_{4} e^{-\left(\frac{\phi_{1}}{2 \zeta}\left(r_{1}^{2}+r_{4}^{2}\right)+\frac{\phi_{2}}{2 \zeta}\left(r_{2}^{2}+r_{3}^{2}\right)\right)} \\
& \quad \times \sum_{l=0}^{\infty} \sum_{j=-\infty}^{\infty} \alpha_{l} I_{l}\left(\frac{-\phi_{3}}{\zeta} r_{1} r_{2}\right) I_{j}\left(\frac{-\phi_{4}}{\zeta} r_{1} r_{3}\right)  \tag{2.16}\\
& \quad \times I_{l+j}\left(\frac{-\phi_{5}}{\zeta} r_{1} r_{4}\right) I_{j}\left(\frac{-\phi_{3}}{\zeta} r_{3} r_{4}\right) I_{l}\left(\frac{-\phi_{4}}{\zeta} r_{2} r_{4}\right)
\end{align*}
$$

where $\alpha_{0}=1$ and $\alpha_{l}=2$ for $l \in \mathbb{Z}^{+}$. We derive the joint moments of four Rayleigh RVs for this case as

$$
\begin{align*}
E & {\left[r_{1}^{\beta_{1}} r_{2}^{\beta_{2}} r_{3}^{\beta_{3}} r_{4}^{\beta_{4}}\right] } \\
& =\eta^{3}(\eta \zeta)^{\beta / 2} \sum_{j=-\infty}^{\infty} \sum_{l, f, b, h, h^{\prime}, b^{\prime}=0}^{\infty} \frac{\alpha_{l}\left(-2 \phi_{5} \eta\right)^{|l+j|+2 f}}{2^{\frac{\nu_{1}^{\prime}+\nu_{2}^{\prime}+\nu_{3}^{\prime}+\nu_{4}^{\prime}-\beta}{2}}} \\
& \times \frac{\left(-2 \phi_{3} \eta\right)^{|l|+|j|+2 b^{\prime}+2 b}\left(-2 \phi_{4} \eta\right)^{|l|+|j|+2 h+2 h^{\prime}}}{\left(\phi_{1} \eta\right)^{\frac{\nu_{1}^{\prime}+\nu_{4}^{\prime}+4+\beta_{1}+\beta_{4}}{2}}\left(\phi_{2} \eta\right)^{\frac{\nu_{2}^{\prime}+\nu_{3}^{\prime}+4+\beta_{2}+\beta_{3}}{2}}} \frac{\Gamma\left(\frac{\nu_{1}^{\prime}+\beta_{1}+2}{2}\right)}{h!(b+|l|)!f!b!\left(h^{\prime}+|l|\right)!}  \tag{2.17}\\
& \times \frac{\Gamma\left(\frac{\nu_{2}^{\prime}+\beta_{2}+2}{2}\right)}{b^{\prime}!(h+|j|)!} \frac{\Gamma\left(\frac{\nu_{3}^{\prime}+\beta_{3}+2}{2}\right)}{(f+|l+j|)!} \frac{\Gamma\left(\frac{\nu_{4}^{\prime}+\beta_{4}+2}{2}\right)}{\left(b^{\prime}+|j|\right)!h^{\prime}!}, \quad \quad \beta_{1}, \beta_{2}, \beta_{3}, \beta_{4} \geq-1
\end{align*}
$$

where $\beta=\beta_{1}+\beta_{2}+\beta_{3}+\beta_{4}$ and $\nu_{1}^{\prime}=l+|j|+|j+l|+2 b+2 h+2 f, \nu_{2}^{\prime}=2 l+2 h^{\prime}+2 b$, $\nu_{3}^{\prime}=|2 j|+2 b^{\prime}+2 h, \nu_{4}^{\prime}=l+|j|+|l+j|+2 f+2 b^{\prime}+2 h^{\prime}$. We derive the joint moment generating function of the quadrivariate Rayleigh distribution for this case as

$$
\begin{align*}
& M_{\boldsymbol{R}}\left(s_{1}, s_{2}, s_{3}, s_{4}\right)=E\left[e^{\left.s_{1} r_{1}+s_{2} r_{2}+s_{3} r_{3}+s_{4} r_{4}\right]}\right. \\
& \quad=\eta^{3} e^{\left(\frac{\zeta}{4 \phi_{1}}\left(s_{1}^{2}+s_{4}^{2}\right)+\frac{\zeta}{4 \phi_{2}}\left(s_{2}^{2}+s_{3}^{2}\right)\right)} \\
& \quad \times \sum_{j=-\infty}^{\infty} \sum_{f, l, b, h, b^{\prime}, h^{\prime}=0}^{\infty} \frac{\alpha_{l}\left(-2 \phi_{5} \eta\right)^{|j+l|+2 f}}{2^{\left(\nu_{1}^{\prime}+\nu_{2}^{\prime}+\nu_{3}^{\prime}+\nu_{4}^{\prime}\right)}\left(\phi_{1} \eta\right)^{\frac{\nu_{1}^{\prime}+\nu_{4}^{\prime}+4}{2}}} \\
& \quad \times \frac{\left(-2 \phi_{3} \eta\right)^{|l|+|j|+2 b^{\prime}+2 b}\left(-2 \phi_{4} \eta\right)^{|l|+|j|+2 h+2 h^{\prime}}}{\left(\phi_{2} \eta\right)^{\frac{\nu_{2}^{\prime}+\nu_{3}^{\prime}+4}{2}} b!b^{\prime}!h^{\prime}!\left(h^{\prime}+|l|\right)!f!h!} \frac{\Gamma\left(\nu_{1}^{\prime}+2\right)}{(f+|l+j|)!}  \tag{2.18}\\
& \quad \times \frac{\Gamma\left(\nu_{2}^{\prime}+2\right)}{(b+|l|)!} \frac{\Gamma\left(\nu_{3}^{\prime}+2\right)}{\left(b^{\prime}+|j|\right)!} \frac{\Gamma\left(\nu_{4}^{\prime}+2\right)}{(h+|j|)!} D_{-\left(\nu_{1}^{\prime}+2\right)}\left(\frac{-s_{1}}{\sqrt{\phi_{1} / \zeta}}\right) \\
& \quad \times D_{-\left(\nu_{2}^{\prime}+2\right)}\left(\frac{-s_{2}}{\sqrt{\phi_{2} / \zeta}}\right) D_{-\left(\nu_{3}^{\prime}+2\right)}\left(\frac{-s_{3}}{\sqrt{\phi_{2} / \zeta}}\right) D_{-\left(\nu_{4}^{\prime}+2\right)}\left(\frac{-s_{4}}{\sqrt{\phi_{1} / \zeta}}\right)
\end{align*}
$$

where $\nu_{1}^{\prime}=l+|j|+|j+l|+2 b+2 h+2 f, \nu_{2}^{\prime}=2 l+2 h^{\prime}+2 b, \nu_{3}^{\prime}=|2 j|+2 b^{\prime}+2 h$, $\nu_{4}^{\prime}=l+|j|+|l+j|+2 f+2 b^{\prime}+2 h^{\prime}$.

The joint PDF for this case is easily derived by substituting $\phi_{6}=0$ in (2.4) and algebraic manipulation. The joint moments and MGF, similarly, are obtained using (2.16). Please notice that the expressions presented here have less number of infinite series compared to the general case. This can be thought of as a tradeoff between the comprehensiveness (ability to capture more practical scenarios) of the model and the complexity. Similar argument can also be when results for the quadrivariate distribution are compared with the trivariate or bivariate distributions.

The expressions for this case are only valid when

$$
\begin{equation*}
\rho_{3}=\frac{\rho_{1}^{2}+\rho_{2}^{2} \pm \sqrt{4 \rho_{1}^{2}+\rho_{1}^{4}-8 \rho_{1}^{2} \rho_{2}+2 \rho_{1}^{2} \rho_{2}^{2}+\rho_{2}^{4}}}{2 \rho_{1}} \tag{2.19}
\end{equation*}
$$

where $\rho_{|i-j|}=E\left[z_{I_{i}} z_{I_{j}}\right] / \zeta=E\left[z_{Q_{i}} z_{Q_{j}}\right] / \zeta$.

Case When $\phi_{5}=0$

This case is applicable to a scenario where the first $\left(r_{1}\right)$ and fourth $\left(r_{4}\right)$ Rayleigh RVs are conditionally independent (i.e., zero partial correlation in between). We derive the joint PDF of the quadrivariate Rayleigh distribution for this case as

$$
\begin{align*}
& f_{\boldsymbol{R}}\left(r_{1}, r_{2}, r_{3}, r_{4}\right) \\
& \quad=(\operatorname{det} \boldsymbol{\Psi})^{-(1 / 2)} r_{1} r_{2} r_{3} r_{4} e^{-\left(\frac{\phi_{1}}{2 \zeta}\left(r_{1}^{2}+r_{4}^{2}\right)+\frac{\phi_{2}}{2 \zeta}\left(r_{2}^{2}+r_{3}^{2}\right)\right)} \\
& \quad \times \sum_{l=0}^{\infty} \sum_{j=-\infty}^{\infty} \alpha_{l} I_{l}\left(\frac{-\phi_{3}}{\zeta} r_{1} r_{2}\right) I_{l}\left(\frac{-\phi_{4}}{\zeta} r_{1} r_{3}\right)  \tag{2.20}\\
& \quad \times I_{l+j}\left(\frac{-\phi_{6}}{\zeta} r_{2} r_{3}\right) I_{j}\left(\frac{-\phi_{3}}{\zeta} r_{3} r_{4}\right) I_{j}\left(\frac{-\phi_{4}}{\zeta} r_{2} r_{4}\right)
\end{align*}
$$

where $\alpha_{0}=1$ and $\alpha_{l}=2$ for $l \in \mathbb{Z}^{+}$. We derive the joint moments of four Rayleigh RVs for this case as

$$
\begin{align*}
E & {\left[r_{1}^{\beta_{1}} r_{2}^{\beta_{2}} r_{3}^{\beta_{3}} r_{4}^{\beta_{4}}\right] } \\
& =\eta^{3}(\eta \zeta)^{\beta / 2} \sum_{j=-\infty}^{\infty} \sum_{q, l, h^{\prime}, b^{\prime}, h, b=0}^{\infty} \frac{\alpha_{l}\left(-2 \phi_{6} \eta\right)^{|l+j|+2 q}}{2^{\frac{\nu_{1}^{\prime}+\nu_{2}^{\prime}+\nu_{3}^{\prime}+\nu_{4}^{\prime}-\beta}{2}}} \\
& \times \frac{\left(-2 \phi_{3} \eta\right)^{|l|+|j|+2 b^{\prime}+2 b}\left(-2 \phi_{4} \eta\right)^{|l|+|j|+2 h+2 h^{\prime}}}{\left(\phi_{1} \eta\right)^{\frac{\nu_{1}^{\prime}+\nu_{4}^{\prime}+4+\beta_{1}+\beta_{4}}{2}}\left(\phi_{2} \eta\right)^{\frac{\nu_{2}^{\prime}+\nu_{3}^{\prime}+4+\beta_{2}+\beta_{3}}{2}}} \frac{\Gamma\left(\frac{\nu_{1}^{\prime}+\beta_{1}+2}{2}\right)}{q!(b+|l|)!h!b!\left(h^{\prime}+|j|\right)!}  \tag{2.21}\\
& \times \frac{\Gamma\left(\frac{\nu_{2}^{\prime}+\beta_{2}+2}{2}\right)}{(h+|l|)!b^{\prime}!} \frac{\Gamma\left(\frac{\nu_{3}^{\prime}+\beta_{3}+2}{2}\right)}{(q+|l+j|)!} \frac{\Gamma\left(\frac{\nu_{4}^{\prime}+\beta_{4}+2}{2}\right)}{\left(b^{\prime}+|j|\right)!h^{\prime}!}, \quad \quad \beta_{1}, \beta_{2}, \beta_{3}, \beta_{4} \geq
\end{align*}
$$

where $\beta=\beta_{1}+\beta_{2}+\beta_{3}+\beta_{4}$ and $\nu_{1}^{\prime}=2 l+2 b+2 h, \nu_{2}^{\prime}=l+|j|+|l+j|+2 q+2 b+2 h^{\prime}$, $\nu_{3}^{\prime}=|l|+|j|+|l+j|+2 q+2 h+2 b^{\prime}, \nu_{4}^{\prime}=|2 j|+2 q+2 h^{\prime}$. We derive the joint moment
generating function of the quadrivariate Rayleigh distribution for this case as

$$
\begin{align*}
& M_{\boldsymbol{R}}\left(s_{1}, s_{2}, s_{3}, s_{4}\right)=E\left[e^{\left.s_{1} r_{1}+s_{2} r_{2}+s_{3} r_{3}+s_{4} r_{4}\right]}\right. \\
& \quad=\eta^{3} e^{\left(\frac{\zeta}{4 \phi_{1}}\left(s_{1}^{2}+s_{4}^{2}\right)+\frac{\zeta}{4 \phi_{2}}\left(s_{2}^{2}+s_{3}^{2}\right)\right)} \\
& \quad \times \sum_{j=-\infty}^{\infty} \sum_{q, l, b^{\prime}, h, b, h^{\prime}=0}^{\infty} \frac{\alpha_{l}\left(-2 \phi_{6} \eta\right)^{|l+j|+2 q}}{2^{\left(\nu_{1}^{\prime}+\nu_{2}^{\prime}+\nu_{3}^{\prime}+\nu_{4}^{\prime}\right)}\left(\phi_{1} \eta\right)^{\frac{\nu_{1}^{\prime}+\nu_{4}^{\prime}+4}{2}}} \\
& \quad \times \frac{\left(-2 \phi_{3} \eta\right)^{|l|+|j|+2 b^{\prime}+2 b}\left(-2 \phi_{4} \eta\right)^{|l|+|j|+2 h+2 h^{\prime}}}{\left(\phi_{2} \eta\right)^{\frac{\nu_{2}^{\prime}+\nu_{3}^{\prime}+4}{2}} b!b^{\prime}!h!!\left(h^{\prime}+|j|\right)!q!h^{\prime}} \frac{\Gamma\left(\nu_{1}^{\prime}+2\right)}{(q+|l+j|)!}  \tag{2.22}\\
& \quad \times \frac{\Gamma\left(\nu_{2}^{\prime}+2\right)}{(h+|l|)!} \frac{\Gamma\left(\nu_{3}^{\prime}+2\right)}{\left(b^{\prime}+|j|\right)!} \frac{\Gamma\left(\nu_{4}^{\prime}+2\right)}{(b+|l|)!} D_{-\left(\nu_{1}^{\prime}+2\right)}\left(\frac{-s_{1}}{\sqrt{\phi_{1} / \zeta}}\right) \\
& \quad \times D_{-\left(\nu_{2}^{\prime}+2\right)}\left(\frac{-s_{2}}{\sqrt{\phi_{2} / \zeta}}\right) D_{-\left(\nu_{3}^{\prime}+2\right)}\left(\frac{-s_{3}}{\sqrt{\phi_{2} / \zeta}}\right) D_{-\left(\nu_{4}^{\prime}+2\right)}\left(\frac{-s_{4}}{\sqrt{\phi_{1} / \zeta}}\right)
\end{align*}
$$

where $\nu_{1}^{\prime}=2 l+2 b+2 h, \nu_{2}^{\prime}=l+|j|+|l+j|+2 q+2 b+2 h^{\prime}, \nu_{3}^{\prime}=|l|+|j|+|l+j|+$ $2 q+2 h+2 b^{\prime}, \nu_{4}^{\prime}=|2 j|+2 q+2 h^{\prime}$.

The joint PDF for this case is easily derived by substituting $\phi_{5}=0$ in (2.4) and algebraic manipulation. The joint moments and MGF, similarly, are obtained using (2.20). Please notice that the expressions presented here have less number of infinite series compared to the general case. This can be thought of as a tradeoff between the comprehensiveness (ability to capture more practical scenarios) of the model and the complexity. Similar argument can also be when results for the quadrivariate distribution are compared with the trivariate or bivariate distributions.

The expressions for this case are only valid when

$$
\begin{equation*}
\rho_{3}=\frac{\rho_{1}^{3}-2 \rho_{1} \rho_{2}+\rho_{1} \rho_{2}^{2}}{\rho_{1}^{2}-1} \tag{2.23}
\end{equation*}
$$

where $\rho_{|i-j|}=E\left[z_{I_{i}} z_{I_{j}}\right] / \zeta=E\left[z_{Q_{i}} z_{Q_{j}}\right] / \zeta$.

Chen and Tellambura [23] derive the PDF for this correlation model. Our ex-
pression is in agreement with theirs.

Case When $\phi_{4}=\phi_{5}=0$
This case is applicable to a scenario where both the first $\left(r_{1}\right)$ and third $\left(r_{3}\right)$, the first $\left(r_{1}\right)$ and fourth $\left(r_{4}\right)$ and the second $\left(r_{2}\right)$ and fourth $\left(r_{4}\right)$ Rayleigh RVs are conditionally independent (i.e., zero partial correlation in between). We derive the joint PDF of the quadrivariate Rayleigh distribution for this case as

$$
\begin{align*}
& f_{\boldsymbol{R}}\left(r_{1}, r_{2}, r_{3}, r_{4}\right) \\
& \quad=(\operatorname{det} \boldsymbol{\Psi})^{-(1 / 2)} r_{1} r_{2} r_{3} r_{4} e^{-\left(\frac{\phi_{1}}{2 \zeta}\left(r_{1}^{2}+r_{4}^{2}\right)+\frac{\phi_{2}}{2 \zeta}\left(r_{2}^{2}+r_{3}^{2}\right)\right)}  \tag{2.24}\\
& \quad \times I_{0}\left(\frac{-\phi_{3}}{\zeta} r_{1} r_{2}\right) I_{0}\left(\frac{-\phi_{6}}{\zeta} r_{2} r_{3}\right) I_{0}\left(\frac{-\phi_{3}}{\zeta} r_{3} r_{4}\right)
\end{align*}
$$

We derive the joint moments of four Rayleigh RVs for this case as

$$
\begin{align*}
& E\left[r_{1}^{\beta_{1}} r_{2}^{\beta_{2}} r_{3}^{\beta_{3}} r_{4}^{\beta_{4}}\right] \\
& =\eta^{3}(\eta \zeta)^{\beta / 2} \sum_{b, q, b^{\prime}=0}^{\infty} \frac{\left(-2 \phi_{3} \eta\right)^{2 b+2 b^{\prime}}\left(-2 \phi_{6} \eta\right)^{2 q}}{2^{\frac{\nu_{1}^{\prime}+\nu_{2}^{\prime}+\nu_{3}^{\prime}+\nu_{4}^{\prime}-\beta}{2}}\left(\phi_{1} \eta\right)^{\frac{\nu_{1}^{\prime}+\nu_{4}^{\prime}+4+\beta_{1}+\beta_{4}}{2}}} \\
& \times \frac{\Gamma\left(\frac{\nu_{1}^{\prime}+\beta_{1}+2}{2}\right)}{\left(\phi_{2} \eta\right)^{\frac{\nu_{2}^{\prime}+\nu_{3}^{\prime}+4+\beta_{2}+\beta_{3}}{2}} \frac{\Gamma\left(\frac{\nu_{2}^{\prime}+\beta_{2}+2}{2}\right)}{b!b!} \frac{\Gamma\left(\frac{\nu_{3}^{\prime}+\beta_{3}+2}{2}\right)}{b^{\prime}!b^{\prime}!} \frac{\Gamma\left(\frac{\nu_{4}^{\prime}+\beta_{4}+2}{2}\right)}{q!q!},}  \tag{2.25}\\
& \quad \beta_{1}, \beta_{2}, \beta_{3}, \beta_{4} \geq-1
\end{align*}
$$

where $\beta=\beta_{1}+\beta_{2}+\beta_{3}+\beta_{4}$ and $\nu_{1}^{\prime}=2 b, \nu_{2}^{\prime}=2 b+2 q^{\prime}, \nu_{3}^{\prime}=2 q+2 b^{\prime}, \nu_{4}^{\prime}=$ $2 b^{\prime}$. We derive the joint moment generating function of the quadrivariate Rayleigh
distribution for this case as

$$
\begin{align*}
& M_{\boldsymbol{R}}\left(s_{1}, s_{2}, s_{3}, s_{4}\right)=E\left[e^{s_{1} r_{1}+s_{2} r_{2}+s_{3} r_{3}+s_{4} r_{4}}\right] \\
& \quad=\eta^{3} e^{\left(\frac{\zeta}{4 \phi_{1}}\left(s_{1}^{2}+s_{4}^{2}\right)+\frac{\zeta}{4 \phi_{2}}\left(s_{2}^{2}+s_{3}^{2}\right)\right)} \\
& \quad \times \sum_{b, q, b^{\prime}=0}^{\infty} \frac{\left(-2 \phi_{3} \eta\right)^{2 b+2 b^{\prime}}\left(-2 \phi_{6} \eta\right)^{2 q} \Gamma\left(\nu_{1}^{\prime}+2\right)}{2^{\left(\nu_{1}^{\prime}+\nu_{2}^{\prime}+\nu_{3}^{\prime}+\nu_{4}^{\prime}\right)}\left(\phi_{1} \eta\right)^{\frac{\nu_{1}^{\prime}+\nu_{4}^{\prime}+4}{2}}}  \tag{2.26}\\
& \quad \times \frac{\Gamma\left(\nu_{2}^{\prime}+2\right) \Gamma\left(\nu_{3}^{\prime}+2\right) \Gamma\left(\nu_{4}^{\prime}+2\right)}{\left(\phi_{2} \eta\right)^{\frac{\nu_{2}^{\prime}+\nu_{3}^{\prime}+4}{2}} b!b!b^{\prime}!b^{\prime}!q!q!} D_{-\left(\nu_{1}^{\prime}+2\right)}\left(\frac{-s_{1}}{\sqrt{\phi_{1} / \zeta}}\right) \\
& \quad \times D_{-\left(\nu_{2}^{\prime}+2\right)}\left(\frac{-s_{2}}{\sqrt{\phi_{2} / \zeta}}\right) D_{-\left(\nu_{3}^{\prime}+2\right)}\left(\frac{-s_{3}}{\sqrt{\phi_{2} / \zeta}}\right) D_{-\left(\nu_{4}^{\prime}+2\right)}\left(\frac{-s_{4}}{\sqrt{\phi_{1} / \zeta}}\right)
\end{align*}
$$

where $\nu_{1}^{\prime}=2 b, \nu_{2}^{\prime}=2 b+2 q^{\prime}, \nu_{3}^{\prime}=2 q+2 b^{\prime}, \nu_{4}^{\prime}=2 b^{\prime}$.

The joint PDF for this case is easily derived by substituting $\phi_{4}=\phi_{5}=0$ in (2.4) and algebraic manipulation. The joint moments and MGF, similarly, are obtained using (2.24). Please notice that the expressions presented here have less number of infinite series compared to the general case. This can be thought of as a tradeoff between the comprehensiveness (ability to capture more practical scenarios) of the model and the complexity. Similar argument can also be when results for the quadrivariate distribution are compared with the trivariate or bivariate distributions.

The expressions for this case are only valid when

$$
\begin{equation*}
\rho_{2}=\rho_{1}^{2}, \quad \rho_{3}=\rho_{1}^{3} \tag{2.27}
\end{equation*}
$$

where $\rho_{|i-j|}=E\left[z_{I_{i}} z_{I_{j}}\right] / \zeta=E\left[z_{Q_{i}} z_{Q_{j}}\right] / \zeta$.

This is the well known exponentially correlated case $\left(\phi_{3}=\phi_{6}\right)$. Our expression for the joint PDF is in agreement with the one shown in [22].

### 2.7 Applications

### 2.7.1 Outage Probability of the Four-Branch Selection Combiner

The selection combiner (SC) outputs the signal of the branch that has the highest SNR. Thus, the $K$-branch selection combiner's output SNR can be given as [32]

$$
\begin{equation*}
\gamma_{S C}=\max \left\{\gamma_{1}, \gamma_{2}, \ldots, \gamma_{K}\right\} \tag{2.28}
\end{equation*}
$$

where $\gamma_{k}$ denotes the $k^{\text {th }}$ branch's instantaneous SNR and is given as

$$
\begin{equation*}
\gamma_{k}=\frac{r_{k}^{2}}{N_{o}} \tag{2.29}
\end{equation*}
$$

where $N_{o}$ denotes the additive white Gaussian noise (AWGN) power spectral density at each branch. The average SNR at the $k^{\text {th }}$ branch can be given as

$$
\begin{equation*}
\bar{\gamma}_{k}=\bar{\gamma}=\frac{E\left[r_{k}^{2}\right]}{N_{o}}=\frac{2 \zeta}{N_{o}} \tag{2.30}
\end{equation*}
$$

Please note that $E\left[r_{k}^{2}\right]$ denotes the expected value of the received signal energy per symbol and is directly proportional to the transmitted symbol energy.

The probability that the output SNR is less than a threshold value, $\gamma^{\text {th }}$, is defined as the outage probability. Thus, for the four-branch SC it can be given as

$$
\begin{align*}
P_{\text {out }}\left(\gamma^{t h}\right) & =\operatorname{Pr}\left(0 \leq \gamma_{S C} \leq \gamma^{t h}\right) \\
& =F_{\boldsymbol{R}}\left(\sqrt{\frac{\gamma^{t h} 2 \zeta}{\bar{\gamma}_{1}}}, \sqrt{\frac{\gamma^{t h} 2 \zeta}{\bar{\gamma}_{2}}}, \sqrt{\frac{\gamma^{t h} 2 \zeta}{\bar{\gamma}_{3}}}, \sqrt{\frac{\gamma^{t h} 2 \zeta}{\bar{\gamma}_{4}}}\right) \tag{2.31}
\end{align*}
$$

where $F_{\boldsymbol{R}}$ is given by (2.5).

### 2.7.2 Moments of the Four-Branch Equal Gain Combiner's Output Signal-to-Noise Ratio

The equal gain combiner essentially adds up the signals at the branches with equal weights (after co-phasing) in order to produce the output signal. The $K$ branch equal gain combiner's output SNR is given as [32]

$$
\begin{equation*}
\gamma_{E G C}=\frac{1}{K N_{o}}\left(\sum_{k=1}^{K} r_{k}\right)^{2} \tag{2.32}
\end{equation*}
$$

Accordingly, the moments of the output SNR can be given as

$$
\begin{equation*}
E\left[\gamma_{E G C}^{n}\right]=\frac{1}{\left(K N_{o}\right)^{n}} E\left[\left(\sum_{k=1}^{K} r_{k}\right)^{2 n}\right] \tag{2.33}
\end{equation*}
$$

For the four-branch EGC, the moments of the output SNR can therefore be given as

$$
\begin{align*}
E\left[\gamma_{E G C}^{n}\right] & =\frac{1}{\left(4 N_{o}\right)^{n}} E\left[\left(r_{1}+r_{2}+r_{3}+r_{4}\right)^{2 n}\right] \\
& =\frac{1}{\left(4 N_{o}\right)^{n}} \sum_{\substack{\beta_{1}, \beta_{2}, \beta_{3}, \beta_{4}=0 \\
\beta_{1}+\beta_{2}+\beta_{3}+\beta_{4}=2 n}} \frac{(2 n)!}{\beta_{1}!\beta_{2}!\beta_{3}!\beta_{4}!} E\left[r_{1}^{\beta_{1}} r_{2}^{\beta_{2}} r_{3}^{\beta_{3}} r_{4}^{\beta_{4}}\right] \tag{2.34}
\end{align*}
$$

where $E\left[r_{1}^{\beta_{1}} r_{2}^{\beta_{2}} r_{3}^{\beta_{3}} r_{4}^{\beta_{4}}\right]$ is given by (2.6). Similar formulation for the three-branch EGC (using the trivariate Rayleigh distribution) is presented in [23]. Higher order moments provide useful information about the nature of the distribution of a RV. Therefore, (2.34) can be used to study the outage probability, bit error rate and variability of the output SNR of the four-branch EGC in a more rigorous manner.

# 2.7.3 Moments of the Four-Branch Maximal Ratio Combiner's Output Signal-to-Noise Ratio 

The maximal ratio combiner's output is a weighted (co-phased) sum of all branches. The $K$-branch MRC's output SNR is given as [32]

$$
\begin{equation*}
\gamma_{M R C}=\frac{1}{N_{o}} \frac{\left(\sum_{k=1}^{K} \omega_{k} r_{k}\right)^{2}}{\sum_{k=1}^{K} \omega_{k}^{2}} \tag{2.35}
\end{equation*}
$$

where $\omega_{k}$ denotes the weight of the $k^{\text {th }}$ branch. It can be shown that solving for the optimal weights to maximize the output SNR yields

$$
\begin{equation*}
\omega_{k}^{2}=\frac{r_{k}^{2}}{N_{o}} \tag{2.36}
\end{equation*}
$$

By substituting (2.36) into (2.35), it can be shown that the resulting output SNR becomes

$$
\begin{equation*}
\gamma_{M R C}=\frac{\sum_{k=1}^{K} r_{k}^{2}}{N_{o}}=\sum_{k=1}^{K} \gamma_{k} \tag{2.37}
\end{equation*}
$$

Thus, the moments of the output SNR can be given as

$$
\begin{equation*}
E\left[\gamma_{M R C}^{n}\right]=\frac{1}{N_{o}^{n}} E\left[\left(\sum_{k=1}^{K} r_{k}^{2}\right)^{n}\right] \tag{2.38}
\end{equation*}
$$

Accordingly, the moments of the four-branch MRC's output SNR can be given as

$$
\begin{align*}
E\left[\gamma_{M R C}^{n}\right] & =\frac{1}{N_{o}^{n}} E\left[\left(r_{1}^{2}+r_{2}^{2}+r_{3}^{2}+r_{4}^{2}\right)^{n}\right] \\
& =\frac{1}{N_{o}^{n}} \sum_{\substack{\beta_{1}, \beta_{2}, \beta_{3}, \beta_{4}=0 \\
\beta_{1}+\beta_{2}+\beta_{3}+\beta_{4}=n}} \frac{n!}{\beta_{1}!\beta_{2}!\beta_{3}!\beta_{4}!} E\left[r_{1}^{2 \beta_{1}} r_{2}^{2 \beta_{2}} r_{3}^{2 \beta_{3}} r_{4}^{2 \beta_{4}}\right] \tag{2.39}
\end{align*}
$$

where $E\left[r_{1}^{2 \beta_{1}} r_{2}^{2 \beta_{2}} r_{3}^{2 \beta_{3}} r_{4}^{2 \beta_{4}}\right]$ can be computed using (2.6). The result in (2.39) can be used to study the performance of the four-branch MRC in depth as it provides valuable information about the distribution of the output SNR.

### 2.8 Numerical Results

As illustrated in Fig. 2.2, we consider four equally spaced antennas in a linear arrangement where the closest pair is $d_{\text {min }}$ apart. The 3GPP suggests that the PAS at the UE can be modeled as a uniform distribution or a Laplacian distribution with standard deviation (SD) of $35^{\circ}$ [15], [16].


Figure 2.2: Distance and correlation structure between antennas.

We adopt four different scenarios, namely; independent, uniform PAS, Laplacian PAS with $\mu=60^{\circ}, S D=35^{\circ}$ and Laplacian PAS with $\mu=90^{\circ}, S D=35^{\circ}$. As the name suggests, the independent case represents a scenario where all the correlation coefficients are equal to zero.

In the case of the uniform PAS, the correlation coefficient between $i^{\text {th }}$ and $j^{\text {th }}$ antenna element is $\rho_{|i-j|}=E\left[z_{I_{i}} z_{I_{j}}\right] / \zeta=E\left[z_{Q_{i}} z_{Q_{j}}\right] / \zeta=J_{0}\left(2 \pi|i-j| d_{\text {min }} / \lambda\right)$ as given in [32] and [15] where $\lambda$ and $J_{0}(\cdot)$ denote the wavelength and the $0^{\text {th }}$ order Bessel function of the first kind.

The Laplacian PAS with $\mu$ and SD describes a scenario where the received PAS is distributed according to Laplacian distribution with mean $\mu$ and standard deviation, SD. The mean represents the angle relative to the plane of antenna elements. Hence, $\mu=0^{\circ}$ and $\mu=90^{\circ}$ represent the cases where the mean PAS is parallel and perpendicular to the direction of the plane of antenna elements respectively. For a given standard deviation, $\mu=90^{\circ}$ represents the case where the antenna elements are maximally correlated. The correlation coefficient values are obtained from [15] for the Laplacian PAS cases. Table 2.1 presents the correlation values for each of
the four scenarios
Table 2.1: Correlation Values for Different Scenarios

| Scenario | Correlation values |
| :---: | :---: |
| Independent | $\rho_{i}=0, \forall i$ |
| Uniform $P A S$ | $\rho_{\|i-j\|}=J_{0}\left(2 \pi\|i-j\| d_{\text {min }} / \lambda\right)$ |
| Laplacian $P A S, \mu=60^{\circ}$ | $\rho_{\|i-j\|}=\xi\left(\|i-j\| d_{\text {min }}, 60\right)$ |
| Laplacian $P A S, \mu=90^{\circ}$ | $\rho_{\|i-j\|}=\xi\left(\|i-j\| d_{\text {min }}, 90\right)$ |

where $\xi(\cdot)$ denotes equation (27) in [15].
We use our results given in (2.4) - (2.7) for all of these scenarios (except independent). Please note that these correlation structures can only be incorporated into our results because none of the RVs are conditionally independent of each other. In other words, the resulting inverse covariance matrix for all of these scenarios are such that all entries, $\phi_{i} \neq 0, \forall i$. All of the existing models assume that there is at least one $\phi_{i}$ that is equal to zero.

The outage probability values versus the normalized threshold SNR for the fourbranch SC are illustrated in Fig. 2.3. These values are obtained by using (2.31). The distance, $d_{\text {min }}$, between the closest pair of antennas is $0.6 \lambda$. As it can be seen,


Figure 2.3: Outage probability of the four-branch selection combiner, $d_{\text {min }}=0.6 \lambda$.
the outage probability values increase as the correlation between antenna elements
increase. This is a well-known result and is due to decrease in the individual variability of the random variables. There is an approximate 3 dB gap in SNR between the best (independent) and the worst case (Laplacian, $\mu=90^{\circ}$ ) at an outage probability of $10^{-4}$. The Laplacian PAS with $\mu=60^{\circ}$ causes more correlation between antenna elements than the uniform due to its mean and smaller spread.

Fig. 2.4 shows the four-branch EGC's and MRC's normalized mean output SNR, $\gamma / \bar{\gamma}$. These values are obtained by using (2.34) and (2.39) respectively. Please note that for the independent scenario, the values do not change when the distance between antennas change because the signals at antennas are assumed to be independent of each other. The values for the independent case are plotted as reference values. Higher correlation among antenna elements achieves higher mean SNR for the EGC. It is shown in [33] that any non-zero correlation coefficient, $E\left[z_{I_{i}} z_{I_{j}}\right], E\left[z_{Q_{i}} z_{Q_{j}}\right]$, between the Gaussian RVs yields a positive correlation coefficient between the resulting Rayleigh RVs. This is simply because the RV in interest is the magnitude of the two Gaussian RVs. Therefore, the Rayleigh RVs are positively correlated in all


Figure 2.4: Four-branch EGC and MRC normalized mean output SNR.
cases. As a result, $E\left[r_{i} r_{j}\right]>E\left[r_{i}\right] E\left[r_{j}\right]$ for any non-independent case. This explains why higher correlation causes higher mean SNR for the EGC (refer to (2.34)). The MRC's normalized mean output SNR is 4 (refer to (2.38)) as there are four branches.

The value does not change for the MRC because it adapts the weights to maximize the SNR. The upper bound of the EGC's mean output SNR is that of MRC's. This is because $E\left[r_{i} r_{j}\right]=E\left[r_{i}^{2}\right]$ as $\left|\rho_{i j}\right| \rightarrow 1$. As one can see the mean output SNR values start increasing more drastically as the distance between the closest pair of antennas becomes smaller than $2 \lambda$. The difference between the best and worst of EGC's mean output SNR can be as high as 0.23 depending on the PAS distribution (when $d_{\text {min }}=0.6 \lambda$ ). PAS is very closely related to angle of arrival distribution of the signals. Therefore, for the EGC diversity receiver, the output SNR is dependent on the angle of arrivals assuming there is no beam-forming. The discrepancy between the independence assumption and our results goes up to $9 \%$ for the EGC receiver and $19 \%$ for the MRC receiver. The MRC can achieve up to $18 \%$ higher mean output SNR than the EGC when $d_{\text {min }}=0.6 \lambda$.

Fig. 2.5 shows the standard deviation of the four-branch EGC ( $\sigma_{E G C}$ ) and MRC ( $\sigma_{M R C}$ ) output SNR. Again, please note that the values for the independent case are plotted as reference values. The signals at antennas are assumed to be independent regardless of the distance in between. Any non-independent case causes higher stan-


Figure 2.5: Standard deviation of the four-branch EGC and MRC output SNR.
dard deviation than the independent one because the inequality $E\left[r_{i}^{2} r_{j}^{2}\right]>E\left[r_{i}^{2}\right] E\left[r_{j}^{2}\right]$ holds when $\left|\rho_{i j}\right|>0$. The higher the correlation, the higher the increase is in SD. The output SNR for the MRC has larger spread about its mean than for the EGC.

The values for both receivers start changing more significantly when the distance between nearest pair of antennas becomes smaller than $2 \lambda$. The discrepancy between the independence assumption and our results goes up to $16 \%$ for the EGC receiver and $57 \%$ for the MRC receiver.

Fig. 2.6 shows the moment coefficients of skewness and kurtosis for the output SNR. They are respectively given by $E\left[(\gamma-E[\gamma])^{3}\right] / \sigma^{3}$ and $E\left[(\gamma-E[\gamma])^{4}\right] / \sigma^{4}$. The distance, $d_{\text {min }}$, between the closest pair of antennas is $3.6 \lambda$. Higher correlation causes higher skewness and higher kurtosis. This can be explained using a similar reasoning as the ones given above for the first and the second moments. Higher correlation expectedly increases the likelihood of extreme deviation (branches have either all high or all low SNR at any given time) events happening and this contributes to the higher skewness and kurtosis coefficients. The MRC's output SNR PDF has

(a) Skewness coefficient of the output (b) Kurtosis coefficient of the output SNR.
 SNR.

Figure 2.6: Skewness and kurtosis coefficients of the output SNR, $d_{\min }=3.6 \lambda$.
lower skewness and kurtosis coefficients than the EGC's. This can be explained by the ability of the MRC to output a higher SNR for a given probability compared to the EGC. An interesting point to mention is that the MRC's output SNR PDF (compared to the EGC's) has a higher SD but loosely speaking is less dispersed beyond the one standard deviation range from its mean (less widespread values for the output SNR in this range). All of the scenarios have higher kurtosis than a normal distribution $(\kappa=3)$.

To sum up, MRC's output SNR PDF has higher mean and spread but the contribution to the spread comes more from the values close to the mean compared to the EGC's. From a pictorial point of view the MRC's PDF looks more symmetrical and has a fatter main lobe around a higher mean than the EGC's.

### 2.9 Conclusion and Future Work

This chapter provides novel analytical expressions for the joint PDF, CDF, moments and MGF of the quadrivariate Rayleigh distribution with an arbitrary correlation model. The significance of our results is that one can use them to study any four-branch receiver set-up regardless of the geometric arrangements of them (linear, circular...etc.) or the PAS distribution. The existing models are not capable of this. For instance, our framework is the only one that can incorporate the spatial correlation model suggested by the 3GPP [16]. Moreover, we provide a comprehensive overview of the possible cases where the PDF, moments, and MGF can have simplified representations. Some of the works in the literature are found to be special cases of our results.

The performances of the EGC and MRC diversity receivers are studied through their output SNR moments. We show how inaccurate the independence assumption is up to first four moments. Our results show that higher the correlation between signals at the antennas, higher the moments for both receivers. The MRC can achieve up to $18 \%$ higher mean output SNR than the EGC. The discrepancy between the independence assumption and our results goes up to $9 \%$ for the EGC receiver and $19 \%$ for the MRC receiver in terms of the mean output SNR. The discrepancy between the independence assumption and our results goes up to $16 \%$ for the EGC receiver and $57 \%$ for the MRC receiver in terms of the standard deviation of the output SNR.

From a pictorial point of view the MRC's PDF looks more symmetrical and has a fatter main lobe around a higher mean than the EGC's. Higher correlation causes
higher spread and measure of "tailedness" for both receivers.
The performance analysis on the four-branch EGC and MRC diversity receivers we present in this chapter is the first analytical analysis in a 3GPP suggested correlation environment.

We hope that our findings on the higher moments will be useful in constructing approximate PDFs or CDFs of the output SNR distributions of EGC and MRC. This way, performances of these diversity receivers can be studied more in depth. Obtaining the PDF and CDF of the output SNR of these receivers will allow one to study the reliability of these receivers. This has not beed carried out yet in the literature.

The results provided here can also be utilized to study the time-frequency channel behavior more accurately because one does not have to assume conditional independence between the time and frequency samples of the received signal. Four RVs can be used to represent 4 RVs that are apart in time and frequency. Specifically, the first and second RV can be used to represent the time correlation, the first and third can be used to represent the frequency correlation, and finally the first and fourth can be used to represent the time and frequency correlation. Through this model, we will not lose serious amount of information anymore as was the case with 1-D correlation models [1].

## CHAPTER 3

## REDUCING COMPUTATION TIME OF A WIRELESS RESOURCE SCHEDULER BY EXPLOITING TEMPORAL CHANNEL CHARACTERISTICS

### 3.1 Motivation

The task of a wireless downlink scheduler is to allocate the limited amount of BW and power to UEs according to a predefined resource management scheme. The specific goal of a resource management scheme can vary. The most common ones are to maximize the total sum throughput of the system, maximize efficiency, maximize a fairness measure, etc. A scheduler also needs to consider the special requirements of the UEs, such as minimum throughput, maximum delay, etc. In LTE, the resources in the frequency domain are defined as 180 kHz wide. Frequency selectivity across these resources adds another dimension to the scheduling problem. Moreover, the dependency between the allocated power and achievable throughput makes the problem substantially harder to solve. Hence, providing an optimal solution to the LTE downlink resource allocation problem generally yields a high computation time. Many optimal proposed scheduling schemes are not implemented due to this and the fact that the decisions need to be made in real time [34]. A scheduler that wastes energy, money and does not satisfy the UEs is not a good option.

As a result, obtaining smart downlink scheduling schemes with low computation times is still a problem [35-40]. The 3GPP introduced mini-slots in 5G New Radio for the purpose of Ultra-Reliable and Low-Latency Communications (URLLC) further shrinking the time to make scheduling decisions from 1 ms (in LTE) to 0.125
ms [41]. This makes the problem of computation time even more relevant moving forward.

### 3.2 Related Work

In an LTE downlink scheduler, there are total of $N$ resource units (RUs) that are each defined as 180 kHz wide and 1 ms long in frequency and time domain respectively. The scheduling decisions are made according to a predefined resource management scheme to allocate these RUs to users with corresponding power amounts. This process is repeated every 1 ms . Many analytical approaches have been proposed to address the computation time problem of an LTE downlink scheduler. They mainly rely on computational complexity theory.

Perhaps the simplest way to reduce the computation time of the scheduler is to assign equal power to all RUs. This clearly reduces the number of variables. Assuming that the problem is formulated as an optimization problem, this also achieves a linear objective function in many schedulers which is very favorable. The drawback of this type of scheduler is the low efficiency. Ning et al. [42] propose to share the power equally to achieve polynomial time complexity at the cost of optimality of the solution. The same approach is taken by Zhang et al. [43] and it is shown that equal power allocation achieves almost as good as the water-filling algorithm especially when the number of users in the cell are high. Katoozian et al. [44] introduces individual power constraints for all RUs as opposed to total power constraint for the system. They achieve lower computation time because this essentially shrinks the state space of the problem.

Kwan et al. [45] assume that the channel is flat across all RUs. This considerably simplifies the problem since the frequency diversity is ignored. Their solution approaches the optimal one when the number of users in the cell grows.

Decoupling the RU and power allocation is another key method to significantly reduce the complexity [46-48]. The schedulers typically have two stages. The es-
timate of the number of RUs per user is first calculated considering that there are minimum throughput requirements. Then the RUs are allocated to users. This completes the first step. In the second step, optimal power allocations are carried out. However, these types of schedulers do not provide optimal solutions.

The integer variables of the RU allocation problem notoriously make the problem harder. Relaxation of these variables is a well-known way of simplifying the problem [49-51]. Schwarz et al. [52] utilize this method for their proportional fair scheduler. They show that their results suffer from $5 \%$ throughput reduction because of this relaxation. Wong et al. [53] achieve even lower time complexity for their proportional fair scheduler by relaxing the fairness measure. Their results indicate that their normalized fairness measure can be slightly different (less than 0.01) from the optimal value. Aggarwal et al. [54] investigate the effect of this relaxation in depth and find that the integer and continuous (relaxed) cases coincide under some scenarios.

Lagrangian relaxation methods are also adopted to tackle the computation time problem. Xiao et al. [55] achieve polynomial time complexity for their scheduler using this method. The key point in their work is the removal of coupling among RUs. Their solution approaches the optimal one when the number of RUs grow. A similar approach is taken to convert the optimization problem into a dual form where the optimal solution can be reached with less computation time [56].

Madan et al. [57] exploit the convexity of the problem and devise a faster computation algorithm than a conventional subgradient method. Zhang et al. [58] propose to schedule every user in a sequential manner and limit them to a single RU.

### 3.3 Contributions

The majority of the ideas discussed in the previous section arise from the computational complexity theory. Namely, the problem formulations are converted into more favorable forms at the cost of optimality. These solutions generally achieve
lower time complexity. Here, we do not seek to lower the time complexity but rather tackle the problem from a practical point of view and decrease the quantity of variables in the problem.

In this chapter, differently from the papers discussed in the previous section, we propose to take advantage of the physical characteristics of wireless channels. LTE performs scheduling at every 1 ms . To some UEs, we suggest to allocate resources that last for longer time durations than 1 ms . We derive and demonstrate that the duration of this time depends on the mean received SNR and Doppler shift experienced by the aforementioned UE. Higher mean received SNR and lower Doppler shift both lead to a higher probability that the instantaneous received SNR will stay relatively the same over time. In other words, the instantaneous received SNR of a UE is less likely to change over time when the mean received SNR is higher and the Doppler shift is lower. We assume that the scheduler has full knowledge of the estimated Doppler shift values for UEs. This can easily be achieved using reference signals designated for demodulation in LTE [59]. Our idea is illustrated in Fig. 3.1.


Figure 3.1: An example of resource allocations where decisions are made at $t_{0}$.

We call our approach time-windowed scheduling. It is clear that adopting this type of time-windowed scheduling reduces the number of RUs and users in the scheduling problem over a given time. Referring to Fig. 3.1, some UEs are allocated
at every 2 or 3 ms as opposed to 1 ms . This lowers the number of RUs and users in the problem at $t_{0}+1$ and $t_{0}+2 \mathrm{~ms}$ when compared to a conventional LTE scheduler. This results in a fewer number of operations reducing the time it takes to perform scheduling decisions. The key question we ask here is: What is the likelihood that a UE receives relatively the same instantaneous SNR after 1 ms has elapsed? We develop a stochastic framework to answer this question since there is certainly no deterministic way to predict the future in a practical macro cell environment.

We need to define what relatively the same SNR means in this context before we proceed. As far as an LTE scheduler is concerned, relatively the same SNR indicates that there is not a change large enough to select a different Adaptive Modulation and Coding (AMC) mode. The range of received SNR values corresponds to a channel quality indicator (CQI) value which in turn corresponds to an AMC mode. Therefore, we are interested in a stochastic framework that answers the question: What is the probability that a UE will stay in the same AMC mode after a given time?

The scheduler only should consider UEs with high probability of staying in the same AMC mode for time-windowed scheduling. If the scheduler does allocate resources longer than 1 ms and the UE's received SNR changes drastically during this time, then it might cause inefficient usage of resources. This is simply because this allocation might not be the optimal solution for this time duration. For instance, if a high mean SNR UE is allocated resources at time $t_{0}$ that last 5 ms , but its SNR lowers drastically at time $t_{0}+2 \mathrm{~ms}$, the already scheduled resources from time $t_{0}+2$ to $t_{0}+5 \mathrm{~ms}$ might be wasted or could have been utilized more efficiently. The scheduler takes this risk when it performs time-windowed scheduling. We define a parameter, called confidence probability, that is related to this risk. Therefore, our novel stochastic framework, namely, correlated bivariate SNR distribution, helps to assess our scheduling idea. Please note that the time-windowed approach of reducing computation time can be incorporated into any resource management scheme.

The rest of the chapter is organized as follows. List of notations regarding this
chapter is given in the next section. Section 3.5 discusses how we construct our novel stochastic framework. In Section 3.6, we use this framework to assess the performance of our proposed scheduling method in a sample scenario. Section 3.7 concludes the chapter.

### 3.4 List of Notations

| $A_{t}(\Delta t)$ | autocorrelation function |
| :---: | :---: |
| $b_{2}$ | joint PDF of two Rayleigh RVs for the noise |
| c | speed of light |
| $\tilde{c}_{m}(t)$ | minimum throughput requirement for UE $m$ in a sub-frame |
|  | duration at time $t$ |
| $c_{m}^{n}(t)$ | achievable data rate for UE $m$ on RU $n$ at time $t$ |
| $d$ | distance between the BS and the UE (km) |
| $d_{2}$ | joint PDF of desired signal and noise Rayleigh RVs |
| $E[\cdot]$ | expectation operator |
| $E_{0}$ | real amplitude of the local average E-field |
| $f_{c}$ | carrier frequency |
| $f_{D}$ | Doppler spread |
| $f_{g}$ | Doppler shift of the $g^{\text {th }}$ arriving wave |
| $\mathcal{F}^{-1}\{\cdot\}$ | inverse Fourier transform operator |
| $G$ | total number of arriving waves |
| $h_{b}$ | BS antenna height |
| $h_{g}$ | amplitude of the $g^{\text {th }}$ arriving wave |
| $h_{m}$ | UE antenna height |
| $I_{k}(\cdot)$ | $k^{\text {th }}$ order modified Bessel function of the first kind |


| $J_{k}(\cdot)$ | $k^{\text {th }}$ order Bessel function of the first kind |
| :---: | :---: |
| $k$ | correlation coefficient between Gaussian RVs |
| K | covariance matrix of Gaussian RVs |
| M | number of UEs in the cell |
| $n$ | Rayleigh RV, the envelope of the noise |
| $N$ | number of resource units in the scheduling problem |
| $P_{L}$ | path loss, average received desired signal |
| $P_{N}$ | average received noise |
| $r$ | Rayleigh RV, envelope of the desired signal |
| $\Re\{\cdot\}$ | real part of a complex number |
| $s_{2}$ | joint PDF of the ratio of the desired signal amplitude and |
|  | noise amplitude |
| $(\cdot)^{T}$ | transpose operator |
| $v$ | velocity of the UE |
| $w_{2}$ | joint PDF of two Rayleigh RVs for the desired signal |
| $x_{m}^{n}(t)$ | binary variable which is 1 if $\mathrm{UE} m$ is allocated $\mathrm{RU} n$ at time $t$ otherwise 0 |
| $z$ | received desired signal |
| $z_{I}, z_{Q}$ | zero-mean Gaussian random variable |
| Z | column vector of Gaussian RVs |
| $\alpha_{g}$ | angle of arrival of the $g^{\text {th }}$ wave |
| $\gamma$ | average received SNR |
| $\Gamma(\cdot)$ | gamma function |
| $\zeta, \zeta_{r}$ | variance of Gaussian RVs for the desired signal |
| $\zeta_{n}$ | variance of Gaussian RVs for the noise |

$\lambda$
$\xi_{2} \quad$ bivariate PDF of the received SNR
$\Xi_{2} \quad$ bivariate CDF of the received SNR
$\phi \quad$ phase of the $g^{t h}$ arriving wave
$\Psi_{t}(\Delta f) \quad$ Doppler power spectrum
$\omega_{g} \quad$ initial phase shift of the $g^{\text {th }}$ arriving wave

### 3.5 Stochastic Framework

We claim that not all of the UEs are likely to jump from one AMC mode to another every 1 ms due to their channel characteristics. Our goal here is to derive a probability distribution that allows us to associate probability values for temporal transitions of AMC modes depending on channel characteristics. To the best of our knowledge, such distribution does not exist in literature.

We start our derivation by defining the received signal distribution at a given instant in time. Then we provide the time correlation model we adopt. We finalize our derivation by developing a bivariate correlated SNR distribution. Note that we are deriving the distribution of the received SNR instead of just the received signal as the AMC mode selection depends on the received SNR.

### 3.5.1 Channel Model

The received signal at time $t$ for each tap (for an unmodulated carrier) is given by:

$$
\begin{equation*}
z(t)=\Re\left\{\left(\sum_{g=0}^{G} h_{g}(t) e^{-j \phi_{g}(t)}\right) e^{j 2 \pi f_{c} t}\right\} \tag{3.1}
\end{equation*}
$$

where $G$ represents total number of arriving waves (for one tap) and $\sum_{g=0}^{G} h_{g}(t)$ $e^{-j \phi_{g}(t)}$ represents the complex envelope of the received signal.

Based on the analysis by Clarke [6] and Rice [4], the E-field of the channel at a given instant in time, $t$, can be expressed as:

$$
\begin{equation*}
z(t)=E(t)=E_{0} \sum_{g=1}^{G} \cos \left(2 \pi f_{c} t-\phi_{g}\right) \tag{3.2}
\end{equation*}
$$

in the presence of scatterers and in the absence of a direct line of sight between the transmitter and the receiver where $E_{0}=\sum_{g} \mathbf{E}\left[h_{g}\right]$ represents the real amplitude of the local average E-field ${ }^{1}$, $f_{c}$ represents the carrier frequency and $\phi_{g}$ represents the random phase of the $g^{t h}$ arriving wave. The random phase $\phi_{g}=2 \pi f_{g} t+\omega_{g}$ where $f_{g}$ represents the Doppler shift. The Doppler shift $f_{g}=\frac{v \cos \alpha_{g}}{\lambda}$ where $v$ represents the velocity of the mobile, $\alpha_{g}$ represents the angle of arrival, and $\lambda$ represents the wavelength of the arriving wave. E-field can be expressed as:

$$
\begin{equation*}
z(t)=E(t)=z_{I}(t) \cos \left(2 \pi f_{c} t\right)+z_{Q}(t) \sin \left(2 \pi f_{c} t\right) \tag{3.3}
\end{equation*}
$$

where

$$
\begin{align*}
& z_{I}(t)=E_{0} \sum_{g=0}^{G} \cos \left(2 \pi f_{g} t+\omega_{g}\right)  \tag{3.4}\\
& z_{Q}(t)=E_{0} \sum_{g=0}^{G} \sin \left(2 \pi f_{g} t+\omega_{g}\right)
\end{align*}
$$

Assuming that $G$ is large enough and phases $\omega_{g}$ are independent of each other and angles of arrival $\alpha_{g}$ are uniform over the interval ${ }^{2}(0,2 \pi]$, Central Limit Theorem can be invoked to show that random variables $z_{I}$ and $z_{Q}$ can be approximated with a zeromean Gaussian distribution with equal variance $E_{0}^{2} / 2[60]$. Therefore, the envelope of the received E-field can be shown as: $|z(t)|=|E(t)|=\sqrt{z_{l}^{2}(t)+z_{Q}^{2}(t)}=r(t)$ and is Rayleigh distributed [61] (for a flat fading channel) as given in equation (3.5)

[^1]where $\zeta=E_{0}^{2} / 2$ represents average received power ${ }^{3}$.
\[

f_{r}(x)= $$
\begin{cases}\frac{x}{\zeta} e^{-x^{2} / 2 \zeta} & 0 \leq x  \tag{3.5}\\ 0 & x<0\end{cases}
$$
\]

## Time Correlation

The two dimensional autocorrelation function of the Wide Sense Stationary Uncorrelated Scattering channel is defined as [62]:

$$
\begin{align*}
A(\Delta \tau, \Delta t) & =E\left[h\left(\tau_{1}, t_{1}\right) h^{*}\left(\tau_{2}, t_{2}\right)\right]  \tag{3.6}\\
& =E\left[h(\tau, t) h^{*}(\tau+\Delta \tau, t+\Delta t)\right]
\end{align*}
$$

We inspect each dimension separately, by defining $A_{\tau}(\Delta \tau) \triangleq A(\Delta \tau, 0)$ and $A_{t}(\Delta t) \triangleq$ $A(0, \Delta t)$. The Doppler power spectrum of the channel is by definition given as: $\Psi_{t}(\Delta f)=\int_{-\infty}^{\infty} A_{t}(\Delta t) e^{-j \Delta f \cdot \Delta t} d \Delta t$ and is strictly non-zero for $\Delta f \in\left(-f_{D}, f_{D}\right)$ where $f_{D}$ is the Doppler spread. Doppler spread is $f_{D}=v f_{c} / c$ where $c$ represents the speed of light.

Gans developed a spectral analysis for Clarke's model providing a representation of channel frequency spectrum in terms of Doppler shift [8]. Assuming that angle of arrivals are uniform over the interval $(0,2 \pi]$, power spectral density of the received signal is given by:

$$
\Psi_{t}(\Delta f)= \begin{cases}\frac{E_{0}^{2}}{4 \pi f_{D} \sqrt{1-\left(\Delta f / f_{D}\right)^{2}}} & |\Delta f| \leq f_{D}  \tag{3.7}\\ 0 & \text { otherwise }\end{cases}
$$

Equation (3.7) provides a band-limited description of the correlation in time domain. Plot of an example Doppler power spectrum is shown in Fig. 3.2. The inverse Fourier transform of the Doppler power spectrum is the well known $0^{t h}$ order Bessel function

[^2]

Figure 3.2: Doppler power spectrum $\left(f_{c}=700 \mathrm{MHz}, v=20 \mathrm{~m} / \mathrm{s}, f_{D}=46.6 \mathrm{~Hz}\right)$ based on equation (3.7).
of the first kind $\left(J_{0}(\cdot)\right)[30]$ (9.1.18). Therefore, the autocorrelation function of the received signal is given by:

$$
\begin{align*}
\mathbf{E}[z(t) z(t+\Delta t)] & =\mathcal{F}^{-1}\left\{\Psi_{t}(\Delta f)\right\} \\
& =\frac{E_{0}^{2}}{2} J_{0}\left(2 \pi f_{D} \Delta t\right) \cos \left(2 \pi f_{c} \Delta t\right) \tag{3.8}
\end{align*}
$$

where

$$
\begin{equation*}
J_{0}(x)=\frac{1}{\pi} \int_{0}^{\pi} \cos (x \cos (\theta)) d \theta \tag{3.9}
\end{equation*}
$$

Fig. 3.3 depicts the autocorrelation function $\left(J_{0}\left(2 \pi f_{D} \Delta t\right)\right)$ and corresponding signal fading of two mobiles at different speeds with the same carrier frequency. One can easily notice that higher the speed, the faster the channel variations.

(a) Autocorrelation function (Clarke's (b) Received $r(t)$. Zoomed in section model [6]).
 represents 25 ms .

Figure 3.3: Temporal behavior of received signal at different speeds.

## Correlated Received SNR

We are interested in finding the probability of a UE staying in the same AMC mode over some time. Therefore, we need to derive the joint distribution of received

SNR random variables over time since AMC mode selection depends on received SNR. Specifically, we derive a bivariate correlated received SNR distribution where the random variables represent two samples of received SNR separated by some time.

We start by deriving the joint distribution of two received signals (envelope) separated by time. We define the random variables $z_{I_{1}}, z_{Q_{1}}, z_{I_{2}}$ and $z_{Q_{2}}$ as equal variance ( $\zeta$ ) zero-mean Gaussian distributions. They represent the in-phase and quadrature parts of the received signal sampled at different time or frequency. The joint probability density function of them is given by equation (3.10) [29]:

$$
\begin{equation*}
w_{2}\left(z_{I_{1}}, z_{Q_{1}}, z_{I_{2}}, z_{Q_{2}}\right)=\frac{\exp \left(-\frac{\mathbf{Z}^{T} \mathbf{K}^{-1} \mathbf{z}}{2}\right)}{(2 \pi)^{2}(\operatorname{det} \mathbf{K})^{(1 / 2)}} \tag{3.10}
\end{equation*}
$$

where (. $)^{T}$ is the transpose operator, $\mathbf{Z}$ and $\mathbf{K}$ are given as:

$$
\mathbf{Z}=\left[\begin{array}{l}
z_{I_{1}}  \tag{3.11}\\
z_{Q_{1}} \\
z_{I_{2}} \\
z_{Q_{2}}
\end{array}\right] \quad \mathbf{K}=\zeta\left[\begin{array}{cccc}
1 & 0 & k & 0 \\
0 & 1 & 0 & k \\
k & 0 & 1 & 0 \\
0 & k & 0 & 1
\end{array}\right]
$$

assuming that cross-correlation values between in-phase and quadrature parts are zero. This is accurate if the angle distribution of arriving signal components are uniform [32]. Correlation value $0 \leq|k| \leq 1$ comes from function depicted in Fig. 3.3a. Plugging equations shown in (3.11) back to (3.10) and changing coordinates to polar with $z_{I_{1}}=r_{1} \cos \left(\theta_{1}\right), z_{Q_{1}}=r_{1} \sin \left(\theta_{1}\right), z_{I_{2}}=r_{2} \cos \left(\theta_{2}\right), z_{Q_{2}}=r_{2} \sin \left(\theta_{2}\right)$ yields the distribution density in terms of magnitude and phase:

$$
\begin{gather*}
w_{2}\left(r_{1}, r_{2}, \theta_{1}, \theta_{2}\right)=\frac{r_{1} r_{2}}{\left(2 \pi \zeta_{r}\right)^{2}\left(1-k^{2}\right)} \exp \left(-\frac{r_{1}^{2}+r_{2}^{2}-2 k r_{1} r_{2} \cos \left(\theta_{2}-\theta_{1}\right)}{2 \zeta_{r}\left(1-k^{2}\right)}\right),  \tag{3.12}\\
0 \leq r_{1}, r_{2} \leq \infty, \quad 0 \leq \theta_{1}, \theta_{2} \leq 2 \pi
\end{gather*}
$$

One can show that

$$
\begin{align*}
w_{2}\left(r_{1}, r_{2}\right) & =\int_{0}^{2 \pi} \int_{0}^{2 \pi} w_{2}\left(r_{1}, r_{2}, \theta_{1}, \theta_{2}\right) d \theta_{1} d \theta_{2}=\frac{r_{1} r_{2} \exp \left(\frac{-\left(r_{1}^{2}+r_{2}^{2}\right)}{2 \zeta_{r(1}\left(1-k^{2}\right)}\right)}{\zeta_{r}^{2}\left(1-k^{2}\right)}  \tag{3.13}\\
& \times I_{0}\left(\frac{k r_{1} r_{2}}{\zeta_{r}\left(1-k^{2}\right)}\right), \quad 0 \leq r_{1}, r_{2} \leq \infty
\end{align*}
$$

where $I_{0}(\cdot)$ represents the $0^{t h}$ order modified Bessel function of the first kind and is given by [30] (9.6.16):

$$
\begin{equation*}
I_{0}(x)=\frac{1}{\pi} \int_{0}^{\pi} e^{ \pm x \cos (\theta)} d \theta \tag{3.14}
\end{equation*}
$$

Note that as $k \rightarrow 0, w_{2}\left(r_{1}, r_{2}\right)$ can be expressed as multiplication of two independent Rayleigh distribution densities (equation (3.5)).

The envelope of the received (band-limited) noise (assuming that it is measured successfully at the BS and white) is Rayleigh distributed since in-phase and quadrature components are assumed to have independent Gaussian distributions with equal variance, $\zeta_{n}$, [63]. The samples of noise are assumed to have zero correlation; therefore joint distribution density of two noise random variables can be given by:

$$
\begin{equation*}
b_{2}\left(n_{1}, n_{2}\right)=\frac{n_{1} n_{2}}{\zeta_{n}^{2}} e^{-\left(n_{1}^{2}+n_{2}^{2}\right) / 2 \zeta_{n}}, \quad 0 \leq n_{1}, n_{2} \leq \infty \tag{3.15}
\end{equation*}
$$

The joint distribution density of received signal and noise (no correlation in between) is given by:

$$
\begin{align*}
& d_{2}\left(r_{1}, r_{2}, n_{1}, n_{2}\right) \\
& \quad=w_{2}\left(r_{1}, r_{2}\right) b_{2}\left(n_{1}, n_{2}\right) \\
& =\frac{r_{1} r_{2} n_{1} n_{2} \exp \left(\frac{-\left(r_{1}^{2}+r_{2}^{2}\right)}{\left.2 \zeta_{r\left(1-k^{2}\right)}\right) \exp \left(\frac{-\left(n_{1}^{2}+n_{2}^{2}\right)}{2 \zeta_{n}}\right)} \zeta_{r}^{2} \zeta_{n}^{2}\left(1-k^{2}\right)\right.}{\zeta_{0}\left(\frac{k r_{1} r_{2}}{\zeta_{r}\left(1-k^{2}\right)}\right),}  \tag{3.16}\\
& 0 \leq r_{1}, r_{2}, n_{1}, n_{2} \leq \infty
\end{align*}
$$

Next we make a change of variables, $\beta_{1}=\frac{r_{1}}{n_{1}}, \beta_{2}=\frac{r_{2}}{n_{2}}, \mu_{1}=n_{1}$, and $\mu_{2}=n_{2}$. The

Jacobian is:

$$
|J|=\left|\begin{array}{cccc}
\mu_{1} & 0 & \beta_{1} & 0  \tag{3.17}\\
0 & \mu_{2} & 0 & \beta_{2} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right|=\mu_{1} \mu_{2}
$$

After the change of variables the bivariate received signal to noise amplitude distribution density can be given as $d_{2}\left(\beta_{1} \mu_{1}, \beta_{2} \mu_{2}, \mu_{1}, \mu_{2}\right) \times|J|=$

$$
\begin{align*}
& \frac{\beta_{1} \beta_{2} \mu_{1}^{3} \mu_{2}^{3} \exp \left(\frac{-\left(\left(\beta_{1} \mu_{1}\right)^{2}+\left(\beta_{2} \mu_{2}\right)^{2}\right)}{2 \zeta_{r\left(1-k^{2}\right)}}\right) \exp \left(\frac{-\left(\mu_{1}^{2}+\mu_{2}^{2}\right)}{2 \zeta_{n}}\right)}{\zeta_{r}^{2} \zeta_{n}^{2}\left(1-k^{2}\right)} I_{0}\left(\frac{k \beta_{1} \mu_{1} \beta_{2} \mu_{2}}{\zeta_{r}\left(1-k^{2}\right)}\right),  \tag{3.18}\\
& 0 \leq \beta_{1}, \beta_{2}, \mu_{1}, \mu_{2} \leq \infty
\end{align*}
$$

Next, we integrate over $\mu_{1}$. The modified Bessel function can be replaced by an infinite series [31] (8.447.1):

$$
\begin{equation*}
I_{0}(x)=\sum_{l=0}^{\infty}\left(\frac{x}{2}\right)^{2 l} \frac{1}{(l!)^{2}} \tag{3.19}
\end{equation*}
$$

The expression becomes:

$$
\begin{align*}
& \frac{\beta_{1} \beta_{2} \mu_{2}^{3} \exp \left(\frac{-\left(\left(\beta_{2} \mu_{2}\right)^{2}\right)}{2 \zeta_{r}\left(1-k^{2}\right)}\right) \exp \left(\frac{-\left(\mu_{2}^{2}\right)}{2 \zeta_{n}}\right) \sum_{l=0}^{\infty}\left(\frac{k \beta_{1} \beta_{2} \mu_{2}}{2 \zeta_{r}\left(1-k^{2}\right)}\right)^{2 l}}{\zeta_{r}^{2} \zeta_{n}^{2}\left(1-k^{2}\right)(l!)^{2}}  \tag{3.20}\\
& \times \int_{0}^{\infty} \mu_{1}^{2 l+3} \exp \left(\frac{-\mu_{1}^{2}\left(\beta_{1}^{2} \zeta_{n}+\zeta_{r}\left(1-k^{2}\right)\right)}{2 \zeta_{r} \zeta_{n}\left(1-k^{2}\right)}\right) d \mu_{1}
\end{align*}
$$

The integrand can be put in a form, where $\kappa$ represents the constant coefficient in the exponent:

$$
\begin{gather*}
\int_{0}^{\infty} \mu_{1}^{2 l+3} e^{-\kappa \mu_{1}^{2}} d \mu_{1} \\
\lambda=\kappa \mu_{1}^{2} \quad d \lambda=2 \kappa \mu_{1} d \mu_{1}  \tag{3.21}\\
\int_{0}^{\infty} \frac{(\sqrt{\lambda / \kappa})^{2 l+3} e^{-\lambda}}{2(\sqrt{\lambda / \kappa}) \kappa} d \lambda=\frac{1}{2 \kappa^{l+2}} \int_{0}^{\infty} \lambda^{l+1} e^{-\lambda} d \lambda=\frac{\Gamma(l+2)}{2 \kappa^{l+2}}
\end{gather*}
$$

where $\Gamma(\cdot)$ represents the Gamma function $[30]$ (6.1.1) and is given by:

$$
\begin{equation*}
\Gamma(x)=\int_{0}^{\infty} a^{x-1} e^{-a} d a, \quad \Re(x)>0 \tag{3.22}
\end{equation*}
$$

Next, we integrate over $\mu_{2}$ :

$$
\begin{equation*}
\frac{\beta_{1} \beta_{2} \sum_{l=0}^{\infty}\left(\frac{k \beta_{1} \beta_{2}}{2 \zeta_{r}\left(1-k^{2}\right)}\right)^{2 l} \Gamma(l+2)}{2 \zeta_{r}^{2} \zeta_{n}^{2}\left(1-k^{2}\right)(l!)^{2} \kappa^{l+2}} \int_{0}^{\infty} \mu_{2}^{2 l+3} \exp \left(\frac{-\mu_{2}^{2}\left(\beta_{2}^{2} \zeta_{n}+\zeta_{r}\left(1-k^{2}\right)\right)}{2 \zeta_{r} \zeta_{n}\left(1-k^{2}\right)}\right) d \mu_{2} \tag{3.23}
\end{equation*}
$$

We can do the same substitution we did in (3.21) since the integrand is of the same form. We now have the expression for signal amplitude to noise amplitude ratio distribution density, $s_{2}\left(\beta_{1}, \beta_{2}\right)=$

$$
\begin{equation*}
\frac{\beta_{1} \beta_{2} \sum_{l=0}^{\infty}\left(\frac{k \beta_{1} \beta_{2}}{2 \zeta_{r}\left(1-k^{2}\right)}\right)^{2 l} \Gamma^{2}(l+2)}{4 \zeta_{r}^{2} \zeta_{n}^{2}\left(1-k^{2}\right)(l!)^{2}\left(\frac{\beta_{1}^{2} \zeta_{n}+\zeta_{r}\left(1-k^{2}\right)}{2 \zeta_{r} \zeta_{n}\left(1-k^{2}\right)}\right)^{2+l}\left(\frac{\beta_{2}^{2} \zeta_{n}+\zeta_{r}\left(1-k^{2}\right)}{2 \zeta_{r} \zeta_{n}\left(1-k^{2}\right)}\right)^{2+l}} \tag{3.24}
\end{equation*}
$$

After simplification, the density $s_{2}\left(\beta_{1}, \beta_{2}\right)$ becomes:

$$
\begin{equation*}
\sum_{l=0}^{\infty} \frac{4 \beta_{1} \beta_{2} \Gamma^{2}(l+2)\left(k \beta_{1} \beta_{2}\right)^{2 l} \zeta_{r}^{2} \zeta_{n}^{2+2 l}\left(1-k^{2}\right)^{3}}{\left(\beta_{1}^{2} \zeta_{n}^{2} \beta_{2}^{2}+\zeta_{n} \beta_{1}^{2} \zeta_{r}\left(1-k^{2}\right)+\zeta_{r} \zeta_{n} \beta_{2}^{2}\left(1-k^{2}\right)+\zeta_{r}^{2}\left(1-k^{2}\right)^{2}\right)^{2+l}(l!)^{2}} \tag{3.25}
\end{equation*}
$$

Next, we make a change of variables, $\varphi_{1}=\beta_{1}^{2}$ and $\varphi_{2}=\beta_{2}^{2}$ to derive the signal to noise ratio density. The Jacobian of the transformation is:

$$
|J|=\left|\begin{array}{cc}
\frac{1}{2 \sqrt{\varphi_{1}}} & 0  \tag{3.26}\\
0 & \frac{1}{2 \sqrt{\varphi_{2}}}
\end{array}\right|=\frac{1}{4 \sqrt{\varphi_{1} \varphi_{2}}}
$$

The SNR distribution density, $\xi_{2}\left(\varphi_{1}, \varphi_{2}\right)$ can then be given by:

$$
\begin{gather*}
\sum_{l=0}^{\infty} \frac{\Gamma^{2}(l+2)\left(k \sqrt{\varphi_{1}} \sqrt{\varphi_{2}}\right)^{2 l} \zeta_{r}^{2} \zeta_{n}^{2+2 l}\left(1-k^{2}\right)^{3}}{\left(\varphi_{1} \varphi_{2} \zeta_{n}^{2}+\zeta_{n} \zeta_{r} \varphi_{1}\left(1-k^{2}\right)+\zeta_{n} \zeta_{r} \varphi_{2}\left(1-k^{2}\right)+\zeta_{r}^{2}\left(1-k^{2}\right)^{2}\right)^{2+l}(l!)^{2}}  \tag{3.27}\\
0 \leq \varphi_{1}, \varphi_{2} \leq \infty
\end{gather*}
$$

We provide a contour plot of this density for two UEs with different speeds in

Fig. 3.4. For the UE with higher correlation coefficient i.e., a slower moving user, the density is focused more towards the center of the plot illustrating that it is likely to have SNR values close to each other and not as likely to be toward another AMC mode. However, for the other UE, the density is spread out over all pairs of values illustrating that it is more likely for the SNR values to not be close to each other.

(a) $1.25 \mathrm{~m} / \mathrm{s}$ speed, correlation coeffi- (b) $27 \mathrm{~m} / \mathrm{s}$ speed, correlation coefficient, cient, $k=0.998$.

Figure 3.4: Bivariate received SNR density function (13 dB average), samples 10 ms apart.

We derived the received SNR distribution density (equations (3.16)-(3.27)) from received amplitude distribution densities. Specifically, we first transform joint amplitude density to signal to noise amplitude density, then transform that to SNR. In other words, the SNR density is obtained mathematically:

$$
\begin{equation*}
\xi_{2}\left(\varphi_{1}, \varphi_{2}\right)=\left(\left(r_{1} / n_{1}\right)^{2},\left(r_{2} / n_{2}\right)^{2}\right) \tag{3.28}
\end{equation*}
$$

We also provide the derivation of the SNR density by changing the order of the transformations. This yields the same result. The derivation can be found in Appendix D.

We now show that the expression in (3.27) is indeed a probability distribution
density. We integrate over $\varphi_{1}$ followed by $\varphi_{2}$ values:

$$
\begin{align*}
& \int_{0}^{\infty} \frac{\varphi_{1}^{l}}{\left(\varphi_{1} \varphi_{2} \zeta_{n}^{2}+\zeta_{n} \zeta_{r} \varphi_{1}\left(1-k^{2}\right)+\right.} d \varphi_{1} \\
& \left.\zeta_{n} \zeta_{r} \varphi_{2}\left(1-k^{2}\right)+\zeta_{r}^{2}\left(1-k^{2}\right)^{2}\right)^{2+l}  \tag{3.29}\\
& =\frac{\zeta_{n}^{-(l+1)}\left(\varphi_{2} \zeta_{n}+\zeta_{r}\left(1-k^{2}\right)\right)^{-(2+l)}}{\left(1-k^{2}\right)(l+1) \zeta_{r}}, \\
& \zeta_{r}>0, k<1 \\
& \int_{0}^{\infty} \frac{\varphi_{2}^{l}}{\left(\varphi_{2} \zeta_{n}+\zeta_{r}\left(1-k^{2}\right)\right)^{2+l}} d \varphi_{2}=\frac{\zeta_{n}^{-(l+1)}}{\left(1-k^{2}\right)(1+l) \zeta_{r}}, \quad \zeta_{r}>0, k<1 \tag{3.30}
\end{align*}
$$

After simplification, expression becomes:

$$
\begin{equation*}
\sum_{l=0}^{\infty} \frac{\Gamma^{2}(l+2) k^{2 l}\left(1-k^{2}\right)}{(l!)^{2}(l+1)^{2}} \tag{3.31}
\end{equation*}
$$

Since $\Gamma(x)=(x-1)$ ! [30] (6.1.6) for integer values,

$$
\begin{equation*}
\sum_{l=0}^{\infty} \frac{((l+1)!)^{2} k^{2 l}\left(1-k^{2}\right)}{(l!)^{2}(l+1)^{2}}=\sum_{l=0}^{\infty} k^{2 l}\left(1-k^{2}\right)=1 \tag{3.32}
\end{equation*}
$$

We continue by finding the cumulative distribution:

$$
\begin{align*}
& \int_{0}^{v_{1}} \frac{\varphi_{1}^{l}}{\left(\varphi_{1} \varphi_{2} \zeta_{n}^{2}+\zeta_{n} \zeta_{r} \varphi_{1}\left(1-k^{2}\right)+\right.} d \varphi_{1} \\
& \left.\zeta_{n} \zeta_{r} \varphi_{2}\left(1-k^{2}\right)+\zeta_{r}^{2}\left(1-k^{2}\right)^{2}\right)^{2+l}  \tag{3.33}\\
& =\frac{\left(\frac{v_{1}}{v_{1} \zeta_{n}+\zeta_{r}\left(1-k^{2}\right)}\right)^{l+1}\left(\zeta_{n} \varphi_{2}+\zeta_{r}\left(1-k^{2}\right)\right)^{-(2+l)}}{\left(1-k^{2}\right)(l+1) \zeta_{r}} \\
& \int_{0}^{v_{2}} \frac{\varphi_{2}^{l}}{\left(\varphi_{2} \zeta_{n}+\zeta_{r}\left(1-k^{2}\right)\right)^{2+l}} d \varphi_{2} \\
& =\frac{\left(\frac{v_{2}}{v_{2} \zeta_{n}+\zeta_{r}\left(1-k^{2}\right)}\right)^{l+1}}{\left(1-k^{2}\right)(l+1) \zeta_{r}}, \quad \zeta_{r}>0, k<1 \tag{3.34}
\end{align*}
$$

After simplification, cumulative distribution function of received SNR becomes:

$$
\begin{equation*}
\Xi_{2}\left(v_{1}, v_{2}\right)=\frac{\sum_{l=0}^{\infty} \Gamma^{2}(l+2) k^{2 l} \zeta_{n}^{2+2 l}\left(1-k^{2}\right)\left(v_{1} v_{2}\right)^{l+1}}{(l!)^{2}(l+1)^{2}\left(v_{1} \zeta_{n}+\zeta_{r}\left(1-k^{2}\right)\right)^{l+1}\left(v_{2} \zeta_{n}+\zeta_{r}\left(1-k^{2}\right)\right)^{l+1}} \tag{3.35}
\end{equation*}
$$

So far, we have derived probability density (equation (3.27)) and cumulative distribution (equation (3.35)) of bivariate received SNR where $\zeta_{r}, \zeta_{n}, k$ represent mean signal power, mean noise power, and normalized correlation coefficient respectively. Please note that this distribution can be utilized in spatial correlation or frequency correlation models as well.

Moreover, even though we state that this represents the bivariate received SNR, the same distribution can be thought of as bivariate received SIR since the noise is Rayleigh distributed in our model. This makes our model applicable to a multi-cell environment.

As a side product of our research, we have independently derived the distribution density of three Rayleigh random variables adopting the same methodology. Our result is consistent with the one given in [23]. We provide our derivation in Appendix E. This can be used to improve the accuracy of our model.

### 3.5.2 Probability of Staying in the Same Adaptive Modulation and Coding Mode

Now we are ready to answer the question we posed in the Contributions section. The 3GPP-LTE standard defines 15 AMC modes with different constellation sizes and coding rates [62]. The downlink scheduler adopts one of these modes according to CQI that the UE reports. The UE picks the CQI index that indicates the highest AMC mode that achieves $10 \%$ block error rate. There is no common way of picking the CQI value for UEs since it depends on user specific antenna configurations, codes, and methods of effective SNR calculation. In order to provide consistency among users, we refer to the work of Mehlführer et al. [64] for the LTE specific relationship between received SNR and CQI values for a single-input single-output
(SISO) additive white Gaussian noise (AWGN) channel. Table 3.1 presents the relationship between the received SNR and the AMC mode.

Table 3.1: Received SNR and AMC mode Relationship

| AMC mode | SNR (dB) |
| :---: | :---: |
| 0 | $x<-6.5$ |
| 1 | $-6.5 \leq x<-4.5$ |
| 2 | $-4.5 \leq x<-2.5$ |
| 3 | $-2.5 \leq x<-0.5$ |
| 4 | $-0.5 \leq x<1$ |
| 5 | $1 \leq x<2.5$ |
| 6 | $2.5 \leq x<5$ |
| 7 | $5 \leq x<7.5$ |
| 8 | $7.5 \leq x<9.5$ |
| 9 | $9.5 \leq x<10.5$ |
| 10 | $10.5 \leq x<12$ |
| 11 | $12 \leq x<14$ |
| 12 | $14 \leq x<16$ |
| 13 | $16 \leq x<18$ |
| 14 | $18 \leq x<20.5$ |
| 15 | $20.5 \leq x$ |

We define the probability of a UE staying in the same AMC mode as:

$$
\begin{equation*}
P\left(X<v_{2} \leq Y \mid X<v_{1} \leq Y\right)=\frac{\Xi_{2}(Y, Y)-\Xi_{2}(X, Y)-\Xi_{2}(Y, X)+\Xi_{2}(X, X)}{\Xi_{1}(Y)-\Xi_{1}(X)} \tag{3.36}
\end{equation*}
$$

where $\Xi_{2}$ is the bivariate cumulative received SNR distribution given in (3.35), $v_{1}$ and $v_{2}$ represent the received SNR at different times, $\Xi_{1}(v)$ represents the marginal SNR distribution, and $X, Y$ represent the cut-off values for AMC modes as given in Table 3.1.

Utilizing (3.36) we define the probability of staying in the same AMC mode regardless of the current AMC mode as:

$$
\begin{equation*}
\sum_{k=1}^{15} P\left(X_{k}<v_{2} \leq Y_{k+1} \mid X_{k}<v_{1} \leq Y_{k+1}\right) \times P\left(X_{k}<v_{1} \leq Y_{k+1}\right) \tag{3.37}
\end{equation*}
$$

Please note that $X_{k}$ and $Y_{k}$ represent the cut-off SNR values in Table 3.1.

### 3.6 Performance Evaluation

### 3.6.1 Sample Scheduling Problem

To illustrate the benefits of our ideas, we adopt a common formulation for the scheduling problem. It is given in (3.38). This is typical a binary integer programming problem.
$\underset{x}{\operatorname{maximize}} \sum_{n=1}^{N} \sum_{m=1}^{M} c_{m}^{n}\left(t_{0}\right) \cdot x_{m}^{n}\left(t_{0}\right)$
subject to

$$
\begin{align*}
& \sum_{n=1}^{N} c_{m}^{n}\left(t_{0}\right) \cdot x_{m}^{n}\left(t_{0}\right) \geq \tilde{c}_{m}\left(t_{0}\right), \quad \forall m  \tag{3.38b}\\
& \sum_{m=1}^{M} x_{m}^{n}\left(t_{0}\right)=1, \quad \forall n  \tag{3.38c}\\
& x_{m}^{n}\left(t_{0}\right) \in\{0,1\}, \quad \forall n, \forall m \tag{3.38d}
\end{align*}
$$

where $N$ denotes the number of RUs, $M$ denotes the number of UEs in the cell, $t_{0}$ denotes time, $c_{m}^{n}\left(t_{0}\right)$ denotes the achievable data rate for UE $m$ on RU $n$ at time $t_{0}, \tilde{c}_{m}\left(t_{0}\right)$ denotes the minimum throughput requirement for UE $m$ in a sub-frame duration at time $t_{0}$ and $x_{m}^{n}\left(t_{0}\right)$ denotes the binary variables which are 1 if $\mathrm{UE} m$ is allocated RU $n$ at time $t_{0}$, otherwise 0 . Constraint (4b) indicates that every UE has a minimum throughput requirement and constraint (4c) indicates that only one UE can be allocated on a RU at a given time. Minimum throughput constraints are assumed to be non-negative and constant for all $t$. In short, the problem maximizes the total downlink throughput with UEs having minimum throughput constraints. Please note that we are assuming equal power allocation. The reason for this is twofold. First, this is a very common problem formulation. Secondly, the work in [43] suggests that equal power allocation in an orthogonal frequency division multiple access setting achieves nearly as good as an optimal solution especially
when the total number of users in the system is high. The formulation of the problem is not very critical for this work because we are investigating the effect of the time-windowed scheduling effect on the computation time. Similar study can be done for other types of scheduling formulations.

The parameters for our sample scenario is presented in Table 3.2. They are obtained from [62].

Table 3.2: Sample Scenario Parameters

| BW | 1.4 MHz |
| :---: | :---: |
| Number of RUs | 6 |
| Total number of UEs | 6 |
| Speed of UEs | $1.25 \mathrm{~m} / \mathrm{s}$ |
| BS height | 60 m |
| UE height | 2 m |
| Transmit power | 1 W |
| Antenna gain | 0 dB |
| Carrier frequency, $f_{c}$ | 700 MHz |
| Average noise power | -115 dB |

For the large scale path loss, we adopt the Hata model for medium sized cities [62]:

$$
\begin{align*}
P_{L}(d B)= & \left(44.9-6.55 \log _{10}\left(h_{b}\right)\right) \log _{10}(d)+69.55+26.16 \log _{10}\left(f_{c}\right)  \tag{3.39}\\
& -13.82 \log _{10}\left(h_{b}\right)+0.8+\left(1.1 \log _{10}\left(f_{c}\right)-0.7\right) h_{m}-1.56 \log _{10}\left(f_{c}\right)
\end{align*}
$$

where $h_{b}$ is the BS antenna height (m), $d$ is the distance between the BS and the $\mathrm{UE}(\mathrm{km}), f_{c}$ is the carrier frequency $(\mathrm{MHz}), h_{m}$ is the UE antenna height (m). The average received SNR is then given by $\gamma(d B)=P_{L}(d B)-P_{N}(d B)$ where $P_{N}$ is the average noise power received.

The distances between the UEs and the BS are $\{6.5,5,3.5,3,2.2,1.5\} \mathrm{km}$ yielding $\{0.23,4,9.17,11.4,15.9,21.4\} \mathrm{dB}$ of mean SNR after plugging in equation (3.39). Therefore, User 1 has a mean SNR of 0.23 dB , User 2 has a mean SNR of 4 dB , etc. These distance values are picked so that we have a uniform distribution of UEs across AMC modes. We compute time correlation coefficients, $k$, from equation
(3.8) with 700 MHz carrier frequency for different $\Delta \mathrm{t}$ values assuming all have a constant $1.25 \mathrm{~m} / \mathrm{s}$ speed.

### 3.6.2 Numerical Results

Temporal Transitions of AMC Modes

We start by calculating the probability values for UEs to stay in the same AMC mode (refer to Section 3.4.2 for details on how we compute these probabilities.). They are presented in Fig. 3.5a for a time lag of 1 ms . As one can see, users with relatively lower mean SNR values (Users 1 and 2), have non-significantly different values for modes 1-9. They tend to have smaller probability values for the modes above 9 because it is not very likely that these users will stay in a good channel condition since they have low mean SNRs. In fact User 1 has the highest probability of staying in AMC mode 1 at time $t_{0}+1 \mathrm{~ms}$ given that it is at AMC mode 1 at time $t_{0}$.

Users with relatively higher mean SNRs (Users 5 and 6) have drastically different values for the AMC mode 15. This is due to the nature of the AMC modes. Any SNR value that is higher than the cut-off level for AMC mode 15 still results in AMC mode 15 selection. This phenomena is explained by the tail of the SNR distribution. They are very likely to stay in the mode 15 but not as much in the lower AMC mode because they are more likely to land on AMC mode 15 regardless of the current AMC mode.

(a) P (staying in the corresponding AMC mode).

(b) P (staying in the corresponding AMC mode or higher).

Figure 3.5: Comparison of maintaining the mode or achieving higher, time lag is 1 ms.


Figure 3.6: Comparison of maintaining the mode or achieving higher (avg values).

Next, we compute the probability of staying in the same AMC mode regardless of the current $\left(t_{0}\right)$ AMC mode (refer to Section 2.2 for details on how we compute these probabilities.). We are interested in the the behavior of the probability of staying in the same mode vs time lag. These values are depicted in Fig. 3.6a.

There are several deductions to make. The probability of staying in the same AMC mode diminishes for all UEs as the time lag increases. This is due to the decrease in correlation coefficient as the time lag increases. Secondly, UEs with higher mean SNRs have higher probability values overall. Due to the $1.25 \mathrm{~m} / \mathrm{s}$ mobility of users, the correlation coefficient values range from 0.95 to 0.85 . The slow linear decline in probability values advances all the way down to zero correlation in a similar fashion. Therefore, this brings us to our last deduction, which is that mean SNR has a higher impact than the correlation coefficient values. Even for a zero correlation coefficient (fast fading), the high mean SNR UEs are more likely to stay in the same channel condition than a low mean SNR UE with a very high correlation coefficient.

In our computations, we do not include the probabilities of staying in Mode 0 . This is because UEs are not allowed to transmit in this mode and we are only interested in UEs that can transmit for scheduling purposes. The probability values presented would be higher if we had not omitted them.

Next, we focus on the probabilities where UEs either stay in same or achieve higher AMC mode. Our reasoning to make this decision is two-fold. Due to the fact
that we adopt an environment where UEs have minimum throughput requirement, we are only interested if the channel changes to a worse AMC mode potentially not satisfying aforementioned UE. Secondly, a channel changing to a higher AMC mode does not harm the overall throughput of the system. We depict individual AMC mode probability values in Fig. 3.5b. The values are decreasing as the AMC modes are increasing because it is less likely to change to a higher AMC mode once already in a relatively high AMC mode.

Similarly, probability of staying in the same or changing to a higher AMC mode regardless of the current AMC mode for UEs are depicted in Fig. 3.6b. The overall picture looks very similar; however, the probability values are significantly higher for every user compared to Fig. 3.6a.

## Computation Time Analysis

Now, we are ready to evaluate the performance of time-windowed scheduling. Recall that we propose to allocate slow varying SNR UEs resources that last longer than 1 ms (Refer to Fig. 3.1). We assess our idea in terms of number of operations it takes for the scheduler to perform allocation decisions.

Eisenbrand [65] showed that for a fixed number of constraints, $x$, binary coded at most of length $y$, the computational complexity can be given by: $x+\log (x) y$. Relying on their work, the scheduling problem given in (3.38), can be solved with

$$
\begin{equation*}
(M+N)+\log (M+N) M N \tag{3.40}
\end{equation*}
$$

operations assuming that there are a maximum number of users, $M$ and $N$. For the scenario we adopt (refer to Table 3.2), the number of operations during 2 ms for the conventional program is

$$
\begin{equation*}
2 \times((12)+\log (12) 36)=203 \tag{3.41}
\end{equation*}
$$

because we have 6 UEs and 6 RUs. For the case where we schedule only UE 6 at a periodicity of 2 ms (time-window $=2 \mathrm{~ms}$ ), the proposed method's number of operations will be

$$
\begin{equation*}
12+\log (12) 36+10+\log (10) 25=169 \tag{3.42}
\end{equation*}
$$

for 2 ms period. This reduction is related to a probability value where UE 6 will stay in the same mode or higher. We define this probability value as confidence probability. Looking at Fig. 3.6b, we see that 0.73 is the probability it will stay constant or achieve better. Therefore, we can then say that with a probability of 0.73 , the number of operations can be reduced from 203 to 169 .

We carry out similar analysis for different UEs and for different durations of time windows. Fig. 3.7 illustrates the trade-off between the number of operations and probability of UEs staying either in the same or achieving a higher AMC mode. The confidence probability values 0.73 - 0.7 represent the cases where only UE 6 is considered for time-windowed scheduling. The values $0.44-0.4$ represent the cases where both UE 6 and UE 5 are considered. It is clear that there is a positive relationship between the confidence probability and number of operations. This can also be interpreted as higher is the risk of potential non-optimal usage of resources, higher is the improvement in computation time. The number of operations can be


Figure 3.7: Comparison between the conventional and the proposed scheme (avg. per 1 ms ).
decreased from 101 to 68 by considering UE 6 for 50 ms windowed scheduling with a confidence probability of 0.7 . It can further be reduced to 42 by considering both UE 6 and 5 for 50 ms scheduling. This reduces the confidence probability to 0.4 .

Interestingly, the case where UE $6 \& 5$ are considered for 2 ms scheduling is a worse option than the one where only UE 6 is considered for 35 ms . Thus, we conclude that it is a better idea to include small number of very high mean SNR UEs in the scheme for a relatively long time-window than to have large number of relatively high mean SNR UEs for a smaller time-window. This idea can be generalized to higher numbers of UEs and RUs. For a cell with 100 UEs where 10 of them have SNR around 30 dB and speed of $1.25 \mathrm{~m} / \mathrm{s}$, the number of operations can be reduced from 25200 to 17804 operations with confidence probability of 0.4.

### 3.7 Conclusion and Future Work

We argue that the computation time can be lowered by exploiting the temporal behavior of wireless channels. We show that this is achieved by scheduling slow varying SNR UEs (not likely to transition between AMC modes) less frequently for longer durations than others. This results in a lower number of variables for the scheduling problem over a given time. This in turn yields less number of operations the scheduler needs to perform for allocation decisions compared to a conventional LTE scheduler. The necessary input values are the mean SNR and Doppler shift estimations which suffice for the scheduler to perform time-windowed allocations.

Essentially, higher mean SNR and lower speeds of UEs yield larger reduction in the number of operations. The contribution of the mean SNR towards the improvement in the computation time is higher than the speed. Using more UEs for time-windowed scheduling, increases the likelihood of non-optimal usage of resources. The results illustrate that the number of operations it takes to make scheduling decisions can be reduced by $33 \%$ with confidence probability of 0.7 and by $58 \%$ with confidence probability of 0.4 .

Lowering the computation time of an LTE scheduler is important because many of the optimal allocation schemes are not suitable for implementation due to their high computation times. We believe that even more studies on wireless channel
characteristics should be carried out in attempt to lower the computation time of a downlink LTE scheduler. Perhaps, a similar study can be done exploiting frequency correlation among RUs. Various other aspects of a wireless system can be studied using the bivariate SNR distribution derived here. The time and frequency intervals of pilot signals can be investigated more in depth. The distribution can also be used in 2 by 2 MIMO system where the RVs can represent the signals at the antennas. The performances of diversity receivers can be studied.

## CHAPTER 4

## PARTIAL PACKET DUPLICATION IN 5G: CONTROL OF FADE AND NON-FADE DURATION OUTAGES USING MATRIX EXPONENTIAL DISTRIBUTIONS

### 4.1 Motivation, Related Work and Contributions

5G wireless communication systems are expected to host a wide range of new applications and technologies. The tactile Internet, vehicle-to-vehicle communication, professional audio, smart grid, and industrial automation are some examples [66-68]. In order to meet the strict requirements of these applications, the 3GPP has recently introduced the concept of URLLC [69]. Motivated by this, they are incorporating so called PD functionality into the existing networks [70].

A thorough overview of packet duplication implementation in 5G is presented in [71]. There are ongoing discussions on how the specific implementation should be. The main idea is to transmit redundant packets over two independent channels to decrease packet error ratio (PER). The UE has connections to two access points, namely; master evolved node B (MeNB) and secondary evolved node B (SeNB). The MeNB transmits at all times where the SeNB might not.

There are several works that study the performance of PD in LTE. Simulation based study in [72] finds that PD improves the latency by $80 \%$ for 0.1 PER. The work in [73] shows that the outage probability can be improved from $10^{-2}$ to $10^{-4}$ at -6 dB signal-to-interference-plus-noise ratio (SINR) [73]. [41]. Resource utilization analysis is carried out in [41]. The study focuses on the relationship between modulation and coding levels and average error ratio.

The mentioned papers study the performance of PD by observing how much improvement it brings in terms of reliability, latency and efficiency. The underlying assumption is that PD is either active or not at all times depending on the link SINRs.

Our aim is different. Instead of duplicating at all times, we try to find out when to activate and when to deactivate PD to satisfy a given target reliability. The motivation comes from the possibility that secondary link might be providing higher reliability than needed. Our approach achieves (probabilistic) optimal packet duplication preventing waste of resources at the SeNB. Our model incorporates tunable parameters for activation and deactivation. We call this operation partial packet duplication (PPD) because we propose to switch between duplication and no-duplication modes even with relatively stable link SINRs. This can be thought as semi-dual connectivity.

Secondly, existing works use the average PER as a reliability measure. Here, we build a model where we study time dynamics of the fading signal using ME distributions. This allows us to examine the distribution of the packet errors in addition to the average PER. The continuous duration of the erroneous time (fade duration) is more significant quality of experience measure than the average error ratio $[74,75]$. The fade duration outage is investigated for two selection combined links in [76]. However, this is different than the PPD case we introduce here because it assumes duplication at all times. The continuous duration of the error-free time (non-fade duration) is also of interest; especially in professional live audio use cases in 5G [75].

The contributions of this chapter are as follows. It provides solution to the question: How long should PD stay active and not active to meet a given target reliability measure? It extends previous works by considering fade and non-fade duration outages in the context of partial packet duplication.

The rest of the chapter is organized as follows. List of notations regarding this chapter is given in the next section. Section 4.3 describes the system model.

Performance analysis of the proposed method is discussed in Section 4.4. Section 4.5 concludes the chapter.

### 4.2 List of Notations

| B | bad state and no duplication |
| :---: | :---: |
| $\mathrm{B}^{\prime}$ | bad state and duplication |
| B | sub-generator matrix |
| $\boldsymbol{B}_{F F}$ | transition matrix for failure state |
| $\boldsymbol{B}_{S S}$ | transition matrix for success state |
| c | speed of light |
| $e^{\prime}$ | column vector of ones |
| $f_{c}$ | carrier frequency |
| $f_{m}$ | maximum Doppler frequency |
| $f(t)$ | probability density function |
| F | failure state |
| $F(t)$ | cumulative distribution function |
| G | good state and no duplication |
| $\mathrm{G}^{\prime}$ | good state and duplication |
| $\boldsymbol{L}_{F S}$ | transition matrix from failure state to success state |
| $\boldsymbol{L}_{S F}$ | transition matrix from success state to failure state |
| $N_{R}$ | level crossing rate |
| $p$ | initial state row vector |
| $(\cdot)^{\prime}$ | transpose operator |
| $Q$ | generator matrix |
| $R$ | threshold value |

success state
velocity of the UE
$\lambda_{1} \quad$ transition rate from good state to bad state for link 1
$\lambda_{2} \quad$ transition rate from good state to bad state for link 2
$\lambda_{D} \quad$ transition rate from no-duplication mode to duplication mode
$\lambda_{N D} \quad$ transition rate from duplication mode to no-duplication mode
$\mu_{1} \quad$ transition rate from bad state to good state for link 1
$\mu_{2}$
transition rate from bad state to good state for link 2
$\pi \quad$ steady state solution row vector
$\rho \quad$ normalized threshold value
$\tau$
tolerable continuous erroneous and error-free time
$\bar{\tau}_{f} \quad$ average fade duration
$\bar{\tau}_{n f} \quad$ average non-fade duration

### 4.3 System Model

The received signal amplitude can statistically be well described by the Rayleigh distribution where there are no line-of-sight paths [9]. The quality of a wireless link is typically characterized by the received SINR. It is shown in [74] that with a very large number of interferers and a relatively large fade margin ${ }^{1}$, the quality of the link can be characterized by the desired Rayleigh fading signal.

When the signal level is above some predefined threshold value, we assume that the information is received successfully. We call this state the good (G) state. Similarly, when the signal level is below this threshold value, we assume that an error occurs. We call this state the bad (B) state. We utilize a two-state Markov

[^3]chain to model the transitions between the G and B states. Öhmann and Fetweiss show that this is an accurate model for moderate fading margin values in a Rayleigh fading environment [76]. This relies on the result Rice derives where he shows that the fade duration distribution becomes exponential when the threshold value tends to infinity [5]. Please note that fade duration represents the time duration signal stays below this threshold. Similarly, non-fade duration represents the time duration signal stays above this threshold.

The Markov chain model for partial packet duplication is shown in Fig. 4.1 where $\lambda_{1}$ and $\lambda_{2}$ denote the transition rates from state G to B for link 1 and link 2 respectively. The transition rates from state $B$ to $G$ for link 1 and 2 are denoted by $\mu_{1}$ and $\mu_{2}$ respectively. Rates $\mu$ and $\lambda$ are inversely related to average fade and non-fade durations. The average fade duration, $\bar{\tau}_{f}$, and level crossing rate, $N_{R}$, for Rayleigh fading are given in [60]. Using these two expressions, one can find the average non-fade duration, $\bar{\tau}_{n f}$, by using the relationship $\bar{\tau}_{n f}=\left(1 / N_{R}\right)-\bar{\tau}_{f}$. The rates are expressed as

$$
\begin{align*}
& \lambda=\frac{1}{\bar{\tau}_{n f}}=\sqrt{2 \pi} f_{m} \rho  \tag{4.1}\\
& \mu=\frac{1}{\bar{\tau}_{f}}=\frac{\sqrt{2 \pi} f_{m} \rho}{e^{\rho^{2}}-1} \tag{4.2}
\end{align*}
$$

where $f_{m}$ denotes the maximum Doppler frequency and $\rho=R / R_{r m s}$ denotes the


Figure 4.1: Markov chain model for the partial packet duplication.
specified threshold value $R$, normalized to the local rms amplitude of the fading envelope. The maximum Doppler frequency can be expressed as $f_{m}=v f_{c} / c$ where $v$ denotes the velocity difference between the receiver and the transmitter, $f_{c}$ denotes the carrier frequency, and $c$ denotes the speed of light.

Rates $\lambda_{D}$ and $\lambda_{N D}$ denote the transition rates into duplication mode and noduplication mode respectively. Therefore, state G,B represents the case where link 1 is in a good state (above threshold), link 2 is in a bad state (below threshold) and link 2 is not duplicating. Similarly, state $\mathrm{G}^{\prime}, \mathrm{B}^{\prime}$ represents the case where link 1 is in a G state, link 2 is in a B state and link 2 is duplicating.

We are interested in the continuous erroneous time and error-free time for the UE. Hence, we group the states accordingly. The grey colored states represent the erroneous states. We define failure, F, and success, S, states as

$$
\begin{gather*}
F=\left\{B G, B B, \quad B^{\prime} B^{\prime}\right\}  \tag{4.3}\\
S=\left\{G G, G B, \quad G^{\prime} G^{\prime}, G^{\prime} B^{\prime}, \quad B^{\prime} G^{\prime}\right\} \tag{4.4}
\end{gather*}
$$

We are not showing the steady state solution, $\boldsymbol{\pi}$, due to space limitation. Boldface lowercase letters denote row vectors and boldface uppercase letters denote matrices throughout this correspondence. The steady state probabilities for failure states (4.3) can be added up to find the average error ratio.

By definition, fade duration outage occurs when the continuous erroneous time exceeds some value. Non-fade duration outage occurs when the continuous errorfree time is less than some value. Thus, in terms of our model, staying in the F state longer than some value causes a fade duration outage. Similar reasoning can be formed for a non-fade duration outage.

To study the fade duration and non-fade duration outages, we construct the residence time distributions for failure and success states. Please note that the residence times of the individual states within the state F and S do not concern us. We are only interested in the total time spent in state F and total time spent in
state S .
We employ matrix exponential distributions. A matrix exponential distribution is represented by the notation $\operatorname{ME}(\boldsymbol{p}, \boldsymbol{B})$. The probability density function, $f(t)$, and the cumulative distribution function, $F(t)$, for an ME are given as [77]

$$
\begin{gather*}
f(t)=\boldsymbol{p} \exp (\boldsymbol{B} t) \boldsymbol{b}^{\prime}  \tag{4.5}\\
F(t)=1-\boldsymbol{p} \exp (\boldsymbol{B} t) \boldsymbol{e}^{\prime} \tag{4.6}
\end{gather*}
$$

where $\boldsymbol{b}^{\prime}=-\boldsymbol{B} \boldsymbol{e}^{\prime}$ and $\boldsymbol{e}^{\prime}$ is a column vector of ones. Vector $\boldsymbol{p}$ is called the initial state vector. Matrix $\boldsymbol{B}$ is called the sub-generator matrix. We refer the reader to [77] and [78] for more information on ME distributions.

The Markov model in Fig. 4.1 can be described by the following generator matrix.

$$
\boldsymbol{Q}=\left[\begin{array}{ll}
\boldsymbol{B}_{S S} & \boldsymbol{L}_{S F}  \tag{4.7}\\
\boldsymbol{L}_{F S} & \boldsymbol{B}_{F F}
\end{array}\right]
$$

where $\boldsymbol{B}_{F F}$ describes the internal transition rates in F and $\boldsymbol{L}_{F S}$ describes the transition rates from F to S. Same reasoning applies to $\boldsymbol{B}_{S S}$ and $\boldsymbol{L}_{S F}$. The initial state vector $\boldsymbol{p}$ (as in (4.5)) for failure and success processes are as follows.

$$
\begin{equation*}
\boldsymbol{p}_{F}=\frac{\boldsymbol{\pi}_{S} \boldsymbol{L}_{S F}}{\boldsymbol{\pi}_{S} \boldsymbol{L}_{S F} \boldsymbol{e}_{F}^{\prime}}, \quad \boldsymbol{p}_{S}=\frac{\boldsymbol{\pi}_{F} \boldsymbol{L}_{F S}}{\boldsymbol{\pi}_{F} \boldsymbol{L}_{F S} \boldsymbol{e}_{S}^{\prime}}, \tag{4.8}
\end{equation*}
$$

where $\boldsymbol{\pi}=\left[\boldsymbol{\pi}_{S}, \boldsymbol{\pi}_{F}\right]$ and $\boldsymbol{e}^{\prime}$ is a column vector of ones. Please note that $\boldsymbol{\pi}$ denotes the steady state solution.

Consequently, the distribution of residence time in F and S states are given as

$$
\begin{gather*}
F_{F}(t)=1-\boldsymbol{p}_{F} \exp \left(\boldsymbol{B}_{F F} t\right) \boldsymbol{e}_{F}^{\prime}  \tag{4.9}\\
F_{S}(t)=1-\boldsymbol{p}_{S} \exp \left(\boldsymbol{B}_{S S} t\right) \boldsymbol{e}_{S}^{\prime} \tag{4.10}
\end{gather*}
$$

### 4.4 Control of Fade and Non-Fade Duration Outages

We define two separate reliability measures:

$$
\begin{gather*}
\mathbb{P}(\text { failure duration }>\tau)=1-F_{F}(\tau)  \tag{4.11}\\
\mathbb{P}(\text { success duration } \leq \tau)=F_{S}(\tau) \tag{4.12}
\end{gather*}
$$

We illustrate their behavior with respect to $\lambda_{D}$ and $\lambda_{N D}$ in Fig. 4.2 and Fig. 4.3. Please note that the plots do not depict the cumulative distribution function with respect to $\tau$. The tolerable duration of erroneous time ( $\tau$ in (4.11)) is fixed to 30 ms. The minimum needed duration of error-free time ( $\tau$ in (4.12)) is fixed to 100 ms . These values are obtained for the use case of professional live audio in 5G [75].


Figure 4.2: Failure duration outage probability. $\rho_{1}=0.13$ (fading margin $=18 \mathrm{~dB}$ ), $\rho_{2}=0.16$ (fading margin $=16 \mathrm{~dB}$ ). $\tau=0.03 \mathrm{sec}, v=1.45 \mathrm{~m} / \mathrm{s}, f_{c}=700 \mathrm{MHz}$.

The distribution of residence time in F , tends to an exponential distribution with parameter $\mu_{1}$ around point B in Fig. 4.2.

$$
\begin{equation*}
\lim _{\substack{\lambda_{D \rightarrow 0} \\ \lambda_{N D} \rightarrow \infty}} F_{F}(t)=1-e^{-\mu_{1} t} \tag{4.1.1}
\end{equation*}
$$

Similarly, the distribution tends to an exponential distribution with parameter $\mu_{1}+$
$\mu_{2}$ around point D .

$$
\begin{equation*}
\lim _{\substack{\lambda_{D} \rightarrow \infty \\ \lambda_{N D} \rightarrow 0}} F_{F}(t)=1-e^{-\left(\mu_{1}+\mu_{2}\right) t} \tag{4.14}
\end{equation*}
$$

The distribution around point A tends to hyperexponential which is mixture of distributions around point B and D .

$$
\begin{equation*}
\lim _{\substack{\lambda_{D} \rightarrow 0 \\ \lambda_{N D} \rightarrow 0}} F_{F}(t)=1-\alpha_{1} e^{-\mu_{1} t}-\alpha_{2} e^{-\left(\mu_{1}+\mu_{2}\right) t} \tag{4.15}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha_{1}=\frac{\pi_{G G} \lambda_{1}+\pi_{G B} \lambda_{1}}{\pi_{G G} \lambda_{1}+\pi_{G B} \lambda_{1}+\pi_{G^{\prime} B^{\prime}} \lambda_{1}+\pi_{B^{\prime} G^{\prime}} \lambda_{2}}, \quad \alpha_{2}=1-\alpha_{1} \tag{4.16}
\end{equation*}
$$

Through results in (4.13) - (4.15), it is evident that duplicating at all times (point D) provides lower failure duration outage probability than not duplication at all times (point B). The distribution tends to exponential with parameter $\lambda_{D}$ when both $\lambda_{D}$ and $\lambda_{N D}$ become relatively high.

$$
\begin{equation*}
\lim _{\lambda_{D}, \lambda_{N D} \gg \lambda_{1}, \lambda_{2}, \mu_{1}, \mu_{2}} F_{F}(t)=1-e^{-\lambda_{D} t} \tag{4.17}
\end{equation*}
$$

In this case, the transitions between $\mathrm{B}, \mathrm{G}$ and $\mathrm{B}^{\prime}, \mathrm{G}^{\prime}$ states dominate. This transi-


Figure 4.3: Success duration outage probability. $\rho_{1}=0.13$ (fading margin $=18 \mathrm{~dB}$ ), $\rho_{2}=0.16$ (fading margin $=16 \mathrm{~dB}$ ). $\tau=0.1 \mathrm{sec}, v=1.45 \mathrm{~m} / \mathrm{s}, f_{c}=700 \mathrm{MHz}$.
tion represents switching duplication on and off when the primary link is bad and secondary link is good ${ }^{2}$. This means that turning duplication on and off at high rates provides more reliability than duplicating at all times. It is interesting to note that the outage probability does not only depend on $\frac{\lambda_{D}}{\lambda_{N D}}$ but the actual values of the $\lambda_{D}$ and $\lambda_{N D}$.

We reach to a couple of approximation results regarding the residence time distribution in S :

$$
\begin{gather*}
\lim _{\substack{\lambda_{D} \rightarrow 0 \\
\lambda_{N D} \rightarrow \infty}} F_{S}(t)=1-e^{-\lambda_{1} t}  \tag{4.18}\\
\lim _{\lambda_{D}, \lambda_{N D} \gg \lambda_{1}, \lambda_{2}, \mu_{1}, \mu_{2}} F_{S}(t)=1-e^{-\lambda_{N D} t} \tag{4.19}
\end{gather*}
$$

These results represent the cases near point E and G in Fig. 4.3 respectively. As is the case with the failure duration outages, duplicating all times performs better than not duplicating all times. In contrast to failure duration outages, decreasing the average times spent in each mode (maintaining $\frac{\lambda_{D}}{\lambda_{N D}}=1$ ) increases the success duration outage probability.

Our goal is to find range of values for $\lambda_{D}$ and $\lambda_{N D}$ to satisfy target reliability criteria for both failure and success duration outages. We pick the target reliability criteria as 0.1 for both outage cases. Therefore, we search for values where both plots in Fig. 4.2 and Fig. 4.3 are below -1. This region is depicted in Fig. 4.4.

No duplication at all does not meet the target reliability of 0.1 for both outage cases. Duplicating at all times meets target reliability for both cases. However, it wastes resources at the SeNB. The reliability is also met where $\lambda_{D}=\lambda_{N D}=10^{1.12}=$ 13.18. This means that duplication is active for 76 ms on average before deactivation and vice versa. Compared to conventional packet duplication where the secondary link duplicates all times, this utilizes the resources at the SeNB only half of the time. Proposed PPD liberates $50 \%$ of the resources at the SeNB whilst meeting the reliability target. Fig. 4.5 depicts the reliable region for a different case where

[^4]

Figure 4.4: Ranges of $\lambda_{D}$ and $\lambda_{N D}$ to satisfy 0.1 outage probability. $\rho_{1}=0.13$ (fading margin $=18 \mathrm{~dB}$ ), $\rho_{2}=0.16$ (fading margin $=16 \mathrm{~dB}$ ). $\tau=0.03 \mathrm{sec}$ for failure outage, $\tau=0.1 \mathrm{sec}$ for success outage, $v=1.45 \mathrm{~m} / \mathrm{s}, f_{c}=700 \mathrm{MHz}$.
the fading margin for links are lower compared to the case in Fig. 4.4. This either implies that the threshold value is higher or the average received power is lower. In this case, duplicating at all times does not satisfy the reliability criterion for the


Figure 4.5: Ranges of $\lambda_{D}$ and $\lambda_{N D}$ to satisfy 0.1 outage probability. $\rho_{1}=0.16$ (fading margin $=16 \mathrm{~dB}$ ), $\rho_{2}=0.25$ (fading margin $=12 \mathrm{~dB}$ ). $\tau=0.03 \mathrm{sec}$ for failure outage, $\tau=0.1 \mathrm{sec}$ for success outage, $v=1.25 \mathrm{~m} / \mathrm{s}, f_{c}=700 \mathrm{MHz}$.
failure duration outages. This is because of the lower fading margins. PPD satisfies the target reliability of 0.1 when $\lambda_{D}=10^{1.54}=34.67$ and $\lambda_{N D}=10^{0.79}=6.17$. This
means that reliability target is satisfied with $15 \%$ of the time not duplicating.

### 4.5 Conclusion and Future Work

Packet duplication using dual connectivity has recently been introduced to increase reliability of wireless systems. We propose partial packet packet duplication for professional live audio use case in 5G. PPD satisfies the reliability target with less resource utilization than a common packet duplication method.

In the case of professional live audio, the continuous duration of packet errors and error-free time are of interest. We develop a framework where time distribution of continuous erroneous duration and continuous error-free duration are characterized with ME distributions. This allows us to investigate the temporal distribution of packet errors in addition to the PER. This framework is not limited to live audio case and is applicable to other use cases as well.

The results indicate that up to $50 \%$ of the resources at the duplicating base station can be liberated whilst meeting the target reliability measure.

There are many variables that influence the PPD's performance such as Doppler shift, average received SNR, threshold value, and the number of SeNBs. As a future work, a more in depth sensitivity analysis of PPD should be carried out. A more rigorous numerical analysis can reveal different performance improvements under different scenarios.

In this work, we model the fade and non-fade distributions of a Rayleigh faded signal with exponential distributions. However, our preliminary simulation based study shows that these distributions can be more accurately modeled with different distributions than exponential depending on the threshold value ${ }^{3}$. Rice derives the closed form expressions for the distributions when the threshold value approached zero and infinity [5]. We study how the distribution functions change with respect to the threshold value. We are working on deriving expressions for these distributions

[^5]such that they can easily be incorporated into our models. A promising method is to tabulate all of the moments of these distributions from simulation and try to fit them with ME moment matching theory [79]. This way, instead of having scalar values, $\lambda$ and $\mu$, to describe the fade and non-fade duration distributions, one would have matrices. This will capture the fade and non-fade behavior of signals more accurately without costing minimal mathematical inconvenience.

We also hope that this chapter's work exemplifies an application of ME distributions in the field wireless communications. We believe that ME distributions can be and should be used to describe the fading statistics of signals.

## CHAPTER 5

## CONCLUSION AND FUTURE WORK

Developing accurate wireless channels and understanding how the signals behave over time-frequency-space leads to smarter and more efficient systems. In the case of deriving the most comprehensive quadrivariate Rayleigh distribution, we show that the performances of certain diversity receivers can now be better understood. This is because this model is the only distribution that can model the 3GPP suggested spatial correlation structure. In the case of the SC receiver, our model shows that the existing models are optimistic and inaccurate in terms of reliability. The discrepancy between the independence assumption and our results goes up to $9 \%$ for the EGC receiver and $19 \%$ for the MRC receiver in terms of the mean output SNR. The discrepancy between the independence assumption and our results goes up to $16 \%$ for the EGC receiver and $57 \%$ for the MRC receiver in terms of the standard deviation of the output SNR. We show how inaccurate the independence assumption is up to first four moments. Our results enable one to carry out reliability analysis of these receivers. Our results can be used to better study the behavior of signals in time and frequency without loss of information. In other words, one does not have to shrink 2-D correlation functions into two 1-D functions.

Secondly, we show that understanding how the Doppler shift and average SNR influence the AMC mode selection can be very useful. This is achieved by deriving a novel bivariate SNR distribution. We find out that higher average SNR and lower Doppler shift values yield less frequent variations in the received SNR over time. A wireless scheduler can use this information to allocate larger resource blocks (i.e.,
longer in time duration) to mentioned users, hence reducing the computation time of the scheduling process. The results illustrate that the number of operations it takes to make scheduling decisions can be reduced by $33 \%$ with confidence probability of 0.7 and by $58 \%$ with confidence probability of 0.4 . A similar exploitation can be studied in the frequency domain. Various other investigations can be carried out using our bivariate SNR distribution. For example, analysis of time or frequency intervals between pilot signals can be carried out.

In the case of packet duplication in 5G, we have shown that efficiency of resource usage can be increased by up to $50 \%$ for a live audio application. This was achieved by building a Markov chain model to capture fade and non-fade statistics of a Rayleigh faded signal. Switching duplication on and off based on a small-scale fading statistics is carried out. As future work, these statistics can be better modeled by ME distributions rather than scalar based exponential distributions. There is a very good case for this because based on our simulation work, we observed that these distributions show different behavior than exponential distributions. Using the theory of ME moment matching, more accurate models can be developed. The usefulness of ME distributions is that they are very easy to incorporate into the existing Markov chain models. The framework developed also is applicable to different use cases and is not limited to live audio case.

To sum up, this dissertation illustrated the significance of understanding and developing mathematical tools for wireless signals through three different applications. We show that the performances of four-branch diversity receivers can be studied more accurately and more in depth, computation time of a scheduler can be decreased, and efficiency of resource usage in dual connectivity scenarios can be increased by using partial packet duplication.

## APPENDIX A

## DERIVATION OF THE PDF OF THE QUADRIVARIATE RAYLEIGH DISTRIBUTION

The joint PDF of zero-mean equal variance ( $\zeta$ ) Gaussian RVs $\boldsymbol{Z}=\left\{z_{I_{1}}, z_{Q_{1}}, \ldots\right.$, $\left.z_{I_{4}}, z_{Q_{4}}\right\}$, is given by $(2.1)$ where $(\cdot)^{T}$ denotes the transpose operator, $\overline{\boldsymbol{Z}}=\left[z_{I_{1}}, z_{Q_{1}}\right.$, $\left.\ldots, z_{I_{4}}, z_{Q_{4}}\right]^{T}$ and $\boldsymbol{\Psi}$ is a positive definite covariance matrix [29] (7.18a). We define $\rho_{|i-j|}=E\left[z_{I_{i}} z_{I_{j}}\right] / \zeta=E\left[z_{Q_{i}} z_{Q_{j}}\right] / \zeta$ for $i, j \in\{1, \ldots, 4\}$. Substituting $z_{I_{k}}=r_{k} \cos \left(\theta_{k}\right)$, $z_{Q_{k}}=r_{k} \sin \left(\theta_{k}\right)$ yields the joint PDF of $\mathbf{R}=\left\{r_{1}, r_{2}, r_{3}, r_{4}\right\}$ and $\boldsymbol{\Theta}=\left\{\theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}\right\}$ :

$$
\begin{align*}
& f_{\mathbf{R}, \boldsymbol{\Theta}}\left(r_{1}, \theta_{1}, \ldots, r_{4}, \theta_{4}\right) \\
& \quad=\frac{r_{1} r_{2} r_{3} r_{4}}{(2 \pi)^{4}(\operatorname{det} \Psi)^{(1 / 2)}} e^{-\left(\frac{\phi_{1}}{2 \zeta}\left(r_{1}^{2}+r_{4}^{2}\right)+\frac{\phi_{2}}{2 \zeta}\left(r_{2}^{2}+r_{3}^{2}\right)\right)} \\
& \quad \times e^{-\left(\frac{\phi_{3}}{\zeta} r_{1} r_{2} \cos \left(\theta_{1}-\theta_{2}\right)+\frac{\phi_{4}}{\zeta} r_{1} r_{3} \cos \left(\theta_{1}-\theta_{3}\right)\right)}  \tag{A.1}\\
& \quad \times e^{-\left(\frac{\phi_{5}}{\zeta} r_{1} r_{4} \cos \left(\theta_{1}-\theta_{4}\right)+\frac{\phi_{6}}{\zeta} r_{2} r_{3} \cos \left(\theta_{2}-\theta_{3}\right)\right)} \\
& \quad \times e^{-\left(\frac{\phi_{4}}{\zeta} r_{2} r_{4} \cos \left(\theta_{2}-\theta_{4}\right)+\frac{\phi_{3}}{\zeta} r_{3} r_{4} \cos \left(\theta_{3}-\theta_{4}\right)\right)}
\end{align*}
$$

We substitute $e^{A \cos (x)}=I_{0}(A)+2 \sum_{a=1}^{\infty} I_{a}(A) \cos (a x)$ as given in [30] (9.6.34) and integrate over phases, $\boldsymbol{\Theta}$. After algebraic manipulation we reach (2.4).

## APPENDIX B

## DERIVATION OF THE CDF OF THE QUADRIVARIATE RAYLEIGH DISTRIBUTION

We expand $I_{l}(x)=\left(\frac{x}{2}\right)^{l} \sum_{k=0}^{\infty} \frac{\left(x^{2} / 4\right)^{k}}{k!\Gamma(l+k+1)}$ as given in [30] (9.6.10) where $\Gamma(\cdot)$ denotes the gamma function and integrate (2.4). We start with $r_{1}$ :

$$
\begin{equation*}
\int_{0}^{r_{1}^{\prime}} e^{-\frac{\phi_{1} r_{1}^{2}}{2 \zeta}} r_{1}^{\nu_{1}+1} d r_{1} \tag{B.1}
\end{equation*}
$$

We make substitution $u=\frac{\phi_{1} r_{1}^{2}}{2 \zeta}$.

$$
\begin{align*}
& =\frac{1}{2}\left(\frac{2 \zeta}{\phi_{1}}\right)^{\frac{\nu_{1}+2}{2}} \int_{0}^{\frac{\phi_{1}\left(r_{1}^{\prime}\right)^{2}}{2 \zeta}} e^{-u} u^{\frac{\nu_{1}}{2}} d u  \tag{B.2}\\
& =\frac{1}{2}\left(\frac{2 \zeta}{\phi_{1}}\right)^{\frac{\nu_{1}+2}{2}} \gamma\left(\frac{\nu_{1}+2}{2}, \frac{\phi_{1}\left(r_{1}^{\prime}\right)^{2}}{2 \zeta}\right)
\end{align*}
$$

where $\nu_{1}=|l|+|j|+|j+l|+2 b+2 h+2 f$. We rely on the integral representation of lower incomplete gamma function as given in [30] (6.5.2) reaching the final expression in (B.2). We repeat the same integration process for $r_{2}, r_{3}, r_{4}$. After algebraic manipulation we reach (2.5).

## APPENDIX C

## DERIVATION OF THE MGF OF THE QUADRIVARIATE RAYLEIGH DISTRIBUTION

By definition,

$$
\begin{align*}
& M_{\boldsymbol{R}}\left(s_{1}, s_{2}, s_{3}, s_{4}\right)=E\left[e^{s_{1} r_{1}+s_{2} r_{2}+s_{3} r_{3}+s_{4} r_{4}}\right] \\
& \quad=\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{s_{1} r_{1}+s_{2} r_{2}+s_{3} r_{3}+s_{4} r_{4}} f_{\boldsymbol{R}}\left(r_{1}, r_{2}, r_{3}, r_{4}\right) d r_{1} d r_{2} d r_{3} d r_{4} \tag{C.1}
\end{align*}
$$

Using (2.4) and substituting modified Bessel function of the first kind with its infinite series representation, as given in [30] (9.6.10), we proceed:

$$
\begin{align*}
& \int_{0}^{\infty} e^{-\frac{\phi_{1} r_{1}^{2}}{2 \zeta}+s_{1} r_{1}} r_{1}^{\nu+1} d r_{1} \\
& \left.=\left(\frac{\phi_{1}}{\zeta}\right)^{-\frac{\nu_{1}+2}{2}} \Gamma\left(\nu_{1}+2\right) e^{\left(\frac{s_{1}^{2} \zeta}{4 \phi_{1}}\right.}\right) D_{-\left(\nu_{1}+2\right)}\left(\frac{-s_{1}}{\sqrt{\phi_{1} / \zeta}}\right) \tag{C.2}
\end{align*}
$$

where $\nu_{1}=|l|+|j|+|j+l|+2 b+2 h+2 f$ and $D_{k}(\cdot)$ denotes the parabolic cylinder function [31] (9.240). We rely on the equation given in [31] (3.462.1) reaching the final expression in (C.2). We repeat the same integration process for $r_{2}, r_{3}, r_{4}$. After algebraic manipulation we reach (2.7).

## APPENDIX D

## DERIVATION OF BIVARIATE SNR DENSITY

Here, we derive bivariate SNR density by first converting joint amplitude density distribution to joint power density distribution before concluding with ratio of signal to noise. We start by joint signal and noise amplitude distribution density, equation $(3.16), d_{2}\left(r_{1}, r_{2}, n_{1}, n_{2}\right)=$

$$
\begin{gather*}
\frac{r_{1} r_{2} n_{1} n_{2} \exp \left(\frac{-\left(r_{1}^{2}+r_{2}^{2}\right)}{2 \zeta_{r}\left(1-k^{2}\right)}\right) \exp \left(\frac{-\left(n_{1}^{2}+n_{2}^{2}\right)}{2 \zeta_{n}}\right)}{\zeta_{r}^{2} \zeta_{n}^{2}\left(1-k^{2}\right)} I_{0}\left(\frac{k r_{1} r_{2}}{\zeta_{r}\left(1-k^{2}\right)}\right),  \tag{D.1}\\
0 \leq r_{1}, r_{2}, n_{1}, n_{2} \leq \infty \\
\beta_{1}=r_{1}^{2}, \beta_{2}=r_{2}^{2}, \beta_{3}=n_{1}^{2}, \beta_{4}=n_{2}^{2} \\
|J|=1 /\left(16 \sqrt{\beta_{1} \beta_{2} \beta_{3} \beta_{4}}\right)  \tag{D.2}\\
I_{0}(x)=\sum_{l=0}^{\infty}\left(\frac{x}{2}\right)^{2 l} \frac{1}{(l!)^{2}}  \tag{D.3}\\
=\frac{\exp \left(\frac{-\left(\beta_{1}+\beta_{2}\right)}{2 \zeta_{r}\left(1-k^{2}\right)}\right) \exp \left(\frac{-\left(\beta_{3}+\beta_{4}\right)}{2 \zeta_{n}}\right) \sum_{l=0}^{\infty}\left(\frac{k \sqrt{\beta_{1}} \sqrt{\beta_{2}}}{2 \zeta_{r}\left(1-k^{2}\right)}\right)^{2 l}}{16 \zeta_{r}^{2} \zeta_{n}^{2}\left(1-k^{2}\right)(l!)^{2}}  \tag{D.4}\\
\varphi_{1}=\frac{\beta_{1}}{\beta_{3}}, \varphi_{2}=\frac{\beta_{2}}{\beta_{4}}, \varphi_{3}=\beta_{3}, \varphi_{4}=\beta_{4},  \tag{D.5}\\
|J|=\varphi_{3} \varphi_{4}  \tag{D.6}\\
=\frac{\varphi_{3} \varphi_{4} \exp \left(\frac{-\left(\varphi_{1} \varphi_{3}+\varphi_{2} \varphi_{4}\right)}{2 \zeta_{r}\left(1-k^{2}\right)}\right) \exp \left(\frac{-\left(\varphi_{3}+\varphi_{4}\right)}{2 \zeta_{n}}\right)}{16 \zeta_{r}^{2} \zeta_{n}^{2}\left(1-k^{2}\right)(l!)^{2}} \sum_{l=0}^{\infty}\left(\frac{k \sqrt{\varphi_{1} \varphi_{3}} \sqrt{\varphi_{2} \varphi_{4}}}{2 \zeta_{r}\left(1-k^{2}\right)}\right)^{2 l}
\end{gather*}
$$

Integrate over $\varphi_{3}$ :

$$
\begin{equation*}
\int_{0}^{\infty} \varphi_{3}^{l+1} \exp \left(\frac{-\varphi_{1} \varphi_{3}}{2 \zeta_{r}\left(1-k^{2}\right)}\right) \exp \left(\frac{-\varphi_{3}}{2 \zeta_{n}}\right) d \varphi_{3}=\frac{2^{2+l} \Gamma(2+l)}{\left(\frac{1}{\zeta_{n}}+\frac{\varphi_{1}}{\left(1-k^{2}\right) \zeta_{r}}\right)^{2+l}} \tag{D.7}
\end{equation*}
$$

Integration over $\varphi_{4}$ yields similar expression. The SNR distribution density, $\xi\left(\varphi_{1}, \varphi_{2}\right)$, becomes:

$$
\begin{align*}
& =\sum_{l=0}^{\infty} \frac{k^{2 l} \varphi_{1}^{l} \varphi_{2}^{l} 2^{2 l+4} \Gamma^{2}(2+l)}{2^{2 l} 16 \zeta_{r}^{2 l}\left(1-k^{2}\right)^{2 l} \zeta_{r}^{2} \zeta_{n}^{2}\left(1-k^{2}\right)(l!)^{2}\left(\frac{1}{\zeta_{n}}+\frac{\varphi_{1}}{\left(1-k^{2}\right) \zeta_{r}}\right)^{2+l}\left(\frac{1}{\zeta_{n}}+\frac{\varphi_{2}}{\left(1-k^{2}\right) \zeta_{r}}\right)^{2+l}}(\mathrm{D}  \tag{D.8}\\
& =\sum_{l=0}^{\infty} \frac{\Gamma^{2}(2+l)\left(k \sqrt{\varphi_{1}} \sqrt{\varphi_{2}}\right)^{2 l} \zeta_{r}^{2} \zeta_{n}^{2+2 l}\left(1-k^{2}\right)^{3}}{\left(\varphi_{1} \varphi_{2} \zeta_{n}^{2}+\zeta_{n} \zeta_{r} \varphi_{1}\left(1-k^{2}\right)+\zeta_{n} \zeta_{r} \varphi_{2}\left(1-k^{2}\right)+\zeta_{r}^{2}\left(1-k^{2}\right)^{2}\right)^{2+l}(l!)^{2}},  \tag{D.9}\\
& 0 \leq \varphi_{1}, \varphi_{2} \leq \infty
\end{align*}
$$

## APPENDIX E

## DERIVATION OF TRIVARIATE DISTRIBUTION DENSITY OF RAYLEIGH RANDOM VARIABLES

Joint density function, $w_{2}\left(z_{I_{1}}, z_{Q_{1}}, z_{I_{2}}, z_{Q_{2}}, z_{I_{3}}, z_{Q_{3}}\right)$ is given by

$$
\begin{equation*}
=\frac{\exp \left(-\frac{\mathbf{Z}^{T} \mathbf{K}^{-1} \mathbf{Z}}{2}\right)}{(2 \pi)^{3}(\operatorname{det} \mathbf{K})^{(1 / 2)}} \tag{E.1}
\end{equation*}
$$

where

$$
\mathbf{Z}=\left[\begin{array}{l}
z_{I_{1}}  \tag{E.2}\\
z_{Q_{1}} \\
z_{I_{2}} \\
z_{Q_{2}} \\
z_{I_{3}} \\
z_{Q_{3}}
\end{array}\right] \quad \mathbf{K}=\zeta\left[\begin{array}{cccccc}
1 & 0 & k_{1} & 0 & k_{2} & 0 \\
0 & 1 & 0 & k_{1} & 0 & k_{2} \\
k_{1} & 0 & 1 & 0 & k_{1} & 0 \\
0 & k_{1} & 0 & 1 & 0 & k_{1} \\
k_{2} & 0 & k_{1} & 0 & 1 & 0 \\
0 & k_{2} & 0 & k_{1} & 0 & 1
\end{array}\right]
$$

We change coordinates to polar, $z_{I_{1}}=r_{1} \cos \left(\theta_{1}\right), z_{Q_{1}}=r_{1} \sin \left(\theta_{1}\right), z_{I_{2}}=r_{2} \cos \left(\theta_{2}\right)$, $z_{Q_{2}}=r_{2} \sin \left(\theta_{2}\right), z_{I_{3}}=r_{3} \cos \left(\theta_{3}\right), z_{Q_{3}}=r_{3} \sin \left(\theta_{3}\right),|J|=r_{1} r_{2} r_{3}$.

Density, $w_{3}\left(r_{1}, r_{2}, r_{3}, \theta_{1}, \theta_{2}, \theta_{3}\right)=$

$$
\begin{equation*}
\frac{r_{1} r_{2} r_{3} \exp -\left(\frac{r_{1}^{2} \sigma_{1}+r_{2}^{2} \sigma_{4}+r_{3}^{2} \sigma_{1}+2 r_{1} r_{2} \sigma_{2} \cos \left(\theta_{1}-\theta_{2}\right)+2 r_{1} r_{3} \sigma_{3} \cos \left(\theta_{1}-\theta_{3}\right)+2 r_{2} r_{3} \sigma_{2} \cos \left(\theta_{2}-\theta_{3}\right)}{2 \zeta\left(k_{2}-1\right)\left(2 k_{1}^{2}-k_{2}-1\right)}\right)}{(2 \pi)^{3} \zeta^{3}\left(k_{2}-1\right)\left(2 k_{1}^{2}-k_{2}-1\right)} \tag{E.3}
\end{equation*}
$$

where $\sigma_{1}=1-k_{1}^{2}, \sigma_{2}=k_{1}\left(k_{2}-1\right), \sigma_{3}=k_{1}^{2}-k_{2}, \sigma_{4}=1-k_{2}^{2}$. Next, we make a change of variables $x_{1}=\theta_{1}-\theta_{2}, x_{2}=\theta_{2}-\theta_{3}$, and $x_{3}=\left(\theta_{1}+\theta_{2}+\theta_{3}\right) / 3 .|J|=1$.

Integration over $x_{i}$ 's would yield the density in terms of magnitude. Therefore,

$$
\begin{align*}
& \int_{0}^{2 \pi} \int_{0}^{2 \pi} \int_{0}^{2 \pi} e^{-\left(A \cos \left(x_{1}\right)+B \cos \left(x_{2}\right)+C \cos \left(x_{1}+x_{2}\right)\right)} d x_{1} d x_{2} d x_{3} \\
& =2 \pi \int_{0}^{2 \pi} \int_{0}^{2 \pi}\left(I_{0}(A)+2 \sum_{k=1}^{\infty} I_{k}(A) \cos \left(k x_{1}\right)\right)\left(I_{0}(B)+2 \sum_{l=1}^{\infty} I_{l}(B) \cos \left(l x_{2}\right)\right) \\
& \times\left(I_{0}(C)+2 \sum_{j=1}^{\infty} I_{j}(C) \cos \left(j\left(x_{1}+x_{2}\right)\right)\right) d x_{1} d x_{2} \tag{E.4}
\end{align*}
$$

where

$$
\begin{equation*}
e^{-\left(A \cos \left(x_{1}\right)\right)}=I_{0}(A)+2 \sum_{k=1}^{\infty} I_{k}(A) \cos \left(k x_{1}\right) \tag{E.5}
\end{equation*}
$$

as given in [30] (9.6.34) and $I_{k}(A)$ represents the modified Bessel function. We are not providing all of the steps due to space limitations here. Since the region of integration is periodic, all terms integrate to zero but of this form:

$$
\begin{equation*}
\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} I_{k}(A) I_{j}(C) \int_{0}^{2 \pi} \cos \left(k x_{1}\right) \cos \left(j x_{1}\right) d x_{1}=\pi \sum_{k=1}^{\infty} I_{k}(A) I_{k}(C) \tag{E.6}
\end{equation*}
$$

The solution to equation (E.4) and the density are given in equations. (E.7) and (E.8) respectively.

$$
\begin{align*}
&(2 \pi)^{3}\left(I_{0}(A) I_{0}(B) I_{0}(C)+2 \sum_{k=1}^{\infty} I_{k}(A) I_{k}(B) I_{k}(C)\right)  \tag{E.7}\\
& w_{3}\left(r_{1}, r_{2}, r_{3}\right)=\frac{r_{1} r_{2} r_{3} \exp -\left(\frac{r_{1}^{2} \sigma_{1}+r_{2}^{2} \sigma_{4}+r_{3}^{2} \sigma_{1}}{2 D \zeta}\right)}{\zeta^{3} D} \\
& \times\left(I_{0}(A) I_{0}(B) I_{0}(C)+2 \sum_{k=1}^{\infty} I_{k}(A) I_{k}(B) I_{k}(C)\right) \tag{E.8}
\end{align*}
$$

where $A=2 r_{1} r_{2} \sigma_{2} / 2 \zeta D, B=2 r_{2} r_{3} \sigma_{2} / 2 \zeta D, C=2 r_{1} r_{3} \sigma_{3} / 2 \zeta D, D=\left(k_{2}-1\right)\left(2 k_{1}^{2}-\right.$ $\left.k_{2}-1\right)$.

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## VITA

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[^0]:    ${ }^{1}$ It is not in the scope of this dissertation to attempt to approximate the CDF of the output SNR of the investigated diversity receivers.

[^1]:    ${ }^{1} E_{0}^{2} / 2$ is the average power received without small scale fading (based on path loss and shadowing alone).
    ${ }^{2}$ Angles of arrival do not have to be uniformly distributed as long as they are random for the zero-mean Gaussian approximation to hold.

[^2]:    ${ }^{3}$ The received signal power follows an exponential distribution.

[^3]:    ${ }^{1}$ Fade margin is defined as the difference between the average SINR and the level of SINR where signal quality is not acceptable (threshold).

[^4]:    ${ }^{2}$ Note that the duplication is turned on and off proactively; without knowing the instantaneous quality of the links. This is the main premise of the packet duplication to limit packet loss.

[^5]:    ${ }^{3}$ We are not showing these results in the dissertation because they are preliminary and we have not found a mathematical framework that fits them.

