



Kernel Approximation Approach to the L_1 Optimal Sampled-Data Controller Synthesis Problem

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Abstract: This paper is concerned with a new framework called the kernel approximation approach to the L_1 optimal controller synthesis problem of sampled-data systems. On the basis of the lifted representation of sampled-data systems, which contains an input operator and an output operator, this paper introduces a method for approximating the kernel function of the input operator and the hold function of the output operator by piecewise constant functions. Through such a method, the L_1 optimal sampled-data controller synthesis problem could be (almost) equivalently converted into the discrete-time l_1 optimal controller synthesis problem. This paper further establishes an important inequality that forms the theoretical validity of the kernel approximation approach for tackling the L_1 optimal sampled-data controller synthesis problem.

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1. INTRODUCTION

There have been a number of studies on sampled-data systems taking into account of their intersample behavior, and the disturbance rejection problem has been regarded as one of the most important issues in the studies of sampled-data systems. For instance, the H_2 problem of sampled-data systems is dealt with in Bamieh and Pearson (1992a); Chen and Francis (1991); Hagiwara and Araki (1995); Khargonekar and Sivashankar (1991); Mirkin et al. (1999a,b) for evaluating the effect of impulse disturbance inputs, while the H_∞ problem of sampled-data systems is considered in Bamieh and Pearson (1992b); Kabamba and Hara (1993); Mirkin et al. (1999a,b); Tadmor (1992); Toivonen (1992) for reducing the energy of the output for the worst disturbance inputs among those with unit energy. The main idea in these studies can be interpreted as providing discretization procedures of the continuous-time generalized plant by which the H_2 or H_∞ norm of the discrete-time system obtained by connecting the resulting discrete-time generalized plant and the discrete-time controller (approximately) coincides with that of the original sampled-data system.

On the other hand, these control objectives do not suitably match control applications such as collision avoidance of robot systems and protection of chemical systems from being overly pressured. In this sense, the L_1 problem of sampled-data systems, which aims at minimizing their

L_∞ -induced norm, has been brought to the attention of control community since its control objective effectively matches such applications. However, in contrast to the cases of the H_2 and H_∞ problems of sampled-data systems, developing a discretization method of the continuous-time generalized plant so that the discrete-time l_∞ -induced norm of the discrete-time system obtained by connecting the resulting discrete-time generalized plant and the discrete-time controller (approximately) coincides with the L_∞ -induced norm of the original sampled-data system is a non-trivial task. In this regard, the pioneering studies on the L_1 problem of sampled-data systems (Bamieh et al., 1993; Dullerud and Francis, 1992; Sivashankar et al., 1992) approximate a sampled-data system by a discrete-time system through the idea of the fast-sample/fast-hold (FSFH) approximation, a technique developed in another study on the digital redesign of discrete-time controllers (Keller and Anderson, 1992). Even though it is shown in these studies that the l_∞ -induced norm of the approximating discrete-time systems converges to the L_∞ -induced norm of the original sampled-data systems as the FSFH approximation parameter M tends to infinity, these studies do not give any explicit result for evaluating how close the l_∞ -induced norm for a given M is to the exact value of the L_∞ -induced norm.

To remedy this, the present authors introduced an interpretation of the FSFH approximation method through the idea of the fast-lifting technique (Hagiwara and Umeda, 2008), which further led to a more general approach called the input approximation approach (Kim and Hagiwara, 2014). Through this approach the present authors first

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derived in Kim and Hagiwara (2015a) readily computable upper and lower bounds of the L_∞ -induced norm of sampled-data systems. Furthermore, similar results were established in Kim and Hagiwara (2016b) through a different idea called the kernel approximation approach (Kim and Hagiwara, 2015b), and this new approach is also based on the fast-lifting technique which has the parameter M . Indeed, it was shown in Kim and Hagiwara (2016b) that even though the associated convergence rates in the kernel approximation approach are qualitatively the same as those in the input approximation approach, the approximation errors through the former approximation approach are smaller than those through the latter approximation approach under the same fast-lifting parameter M .

Stimulated by the above success of the kernel approximation approach in the L_1 analysis of sampled-data systems, this paper aims at establishing a parallel result in the L_1 synthesis of sampled-data systems. In connection with this, we first remark that the L_1 synthesis of sampled-data systems has been studied in Bamieh et al. (1993); Dullerud and Francis (1992) through the FSFH approximation, while the latest study (Kim and Hagiwara, 2016a) is most sophisticated in its use of the fast-lifting treatment together with the input approximation approach and thus it could have a much clearer link with the kernel approximation approach. Hence, this paper aims at extending the basic ideas in Kim and Hagiwara (2016a) with the input approximation approach to the kernel approximation approach and showing that the advantage of the kernel approximation approach is inherited to the synthesis phase. More specifically, we confine ourselves, as a preliminary study, to the piecewise constant approximation scheme applied to the synthesis via the kernel approximation approach, and show its advantage over the corresponding counterpart via the input approximation approach in Kim and Hagiwara (2016a).

To this end, two types of ‘constant approximations’ for functions on the interval $[0, h/M)$ are introduced with the fast-lifting technique (where h is the sampling width), one for the kernel functions of an input operator and the other for the hold function of an output operator. Through such treatment, the L_1 optimal sampled-data controller synthesis problem is approximately converted into the discrete-time l_1 optimal controller synthesis problem. It is further shown by establishing an inequality that the associated convergence rate about the approximation errors is in the order of $1/M$. Even though this rate itself remains the same as that of the input approximation approach, our arguments show that the former approach can lead to a smaller approximation error than the latter approach for the L_1 optimal controller synthesis problem of sampled-data systems under the same fast-lifting parameter M .

2. MATHEMATICAL PRELIMINARIES

In this section, we provide some mathematical preliminaries. The notations \mathbb{N} , \mathbb{R}_p^ν ($p = 1, \infty$) and $R(\cdot)$ are used to denote the the set of positive integers, the Banach spaces of ν -dimensional real vectors equipped with vector p -norm and the range of an operator, respectively.

We use the notations $\|\cdot\|_p$ ($p = 1, \infty$) to denote either the $L_p[0, h)$ ($p = 1, \infty$) norm of a vector function, i.e.,

$$\|f(\cdot)\|_1 := \sum_i \int_0^h |f_i(t)| dt$$

$$\|f(\cdot)\|_\infty := \max_i \operatorname{ess\,sup}_{0 \leq t < h} |f_i(t)|$$

(or those with h replaced by h/M or ∞), the $L_p[0, h)$ -induced norm (or those with h/M or ∞ instead of h) of an operator. On the other hand, the p -norm of a matrix or a vector is denoted by $|\cdot|_p$ ($p = 1, \infty$). The distinction about the same norm symbols for different types of objects will be clear from the context.

For a Banach space X , its dual space is denoted by X^* . If we let X and Y be Banach spaces and consider a linear operator $\mathbf{T} : X \rightarrow Y$, its adjoint is denoted by $\mathbf{T}^* : Y^* \rightarrow X^*$. For the given Banach spaces X and Y , suppose that there exists unique Banach spaces, denoted by X_* and Y_* , such that their dual spaces $(X_*)^*$ and $(Y_*)^*$ coincide with X and Y , respectively. Then, if there exists an operator $\mathbf{T}_* : Y_* \rightarrow X_*$ such that $(\mathbf{T}_*)^* = \mathbf{T}$, then \mathbf{T}_* is called the preadjoint of $\mathbf{T} : X \rightarrow Y$. Not every operator has a preadjoint, but those operators we deal with in this paper do; it suffices to note that for $X = (L_\infty[0, h])^\nu$ and $X = \mathbb{R}_\infty^n$, a unique X_* such that $(X_*)^* = X$ is $X_* = (L_1[0, h])^\nu$ and $X_* = \mathbb{R}_1^n$, respectively. See Brown and Tvrdý (1980, 1981); Lindner (2006) for more details.

For a Banach space X , the notation l_X is used to denote the space of all X -valued sequences. $\mathcal{F}(G, H)$ denotes the so-called lower linear-fractional-transformation (LFT) given by $G_{11} + G_{12}H(I - G_{22}H)^{-1}G_{21}$.

3. L_1 OPTIMAL CONTROL PROBLEM OF SAMPLED-DATA SYSTEMS

Consider the linear time-invariant (LTI) sampled-data system Σ_{SD} shown in Fig. 1, where P denotes the continuous-time LTI generalized plant, while Ψ , \mathcal{H} and \mathcal{S} denote the discrete-time LTI controller, the zero-order hold and the ideal sampler, respectively, operating with sampling period h in a synchronous fashion. Solid lines and dashed lines in Fig. 1 represent continuous-time signals and discrete-time signals, respectively. Suppose that P and Ψ are given respectively by

$$P: \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x \end{cases} \quad \Psi: \begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k \\ u_k = C_\Psi \psi_k + D_\Psi y_k \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}_\infty^n$, $w(t) \in \mathbb{R}_\infty^{n_w}$, $u(t) \in \mathbb{R}_\infty^{n_u}$, $z(t) \in \mathbb{R}_\infty^{n_z}$, $y(t) \in \mathbb{R}_\infty^{n_y}$, $\psi_k \in \mathbb{R}_\infty^{n_\Psi}$, $y_k = y(kh)$ and $u(t) = u_k$ ($kh \leq t < (k+1)h$).

To facilitate the treatment of the sampled-data system Σ_{SD} viewed as an h -periodic mapping from $w \in L_\infty^{n_w}$ to $z \in L_\infty^{n_z}$, we review the lifting technique as follows (Bamieh et al., 1991; Toivonen, 1992; Yamamoto, 1994). Given $f(t) \in L_\infty^\nu$, its lifting $\{\hat{f}_k\}_{k=0}^\infty \in l_{(L_\infty[0, h])^\nu}$

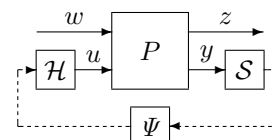


Fig. 1. Sampled-data system Σ_{SD} .

is defined by $\widehat{f}_k(\theta) = f(kh + \theta)$ ($0 \leq \theta < h$). For $\{\widehat{f}_k\}_{k=0}^\infty \in l_{(L_\infty[0,h])^\nu}$, we simply call

$$\|\{\widehat{f}_k\}_{k=0}^\infty\|_\infty := \sup_k \|\widehat{f}_k\|_\infty (= \|f\|_\infty) \quad (2)$$

the $l_{[0,h]}^\infty$ norm. Similar convention is also applied to that with h replaced by $h' := h/M$ for an $M \in \mathbb{N}$. If $\mathbf{F} : X \rightarrow Y$ and either X or Y is $l_{L_\infty[0,h]^\nu}$, we sometimes call the induced norm $\|\mathbf{F}\|_\infty := \sup_{x \in X \setminus \{0\}} \|\mathbf{F}x\|_Y / \|x\|_X$ the L_∞ -induced norm.

Applying lifting to w and z together with discretizing u and y leads to the (partially) lifted representation of the continuous-time generalized plant P described by

$$\widehat{P} : \begin{cases} x_{k+1} = A_d x_k + \mathbf{B}_1 \widehat{w}_k + B_{2d} u_k \\ \widehat{z}_k = \mathbf{C}_1 x_k + \mathbf{D}_{11} \widehat{w}_k + \mathbf{D}_{12} u_k \\ y_k = C_{2d} x_k \end{cases} \quad (3)$$

with $x_k := x(kh)$, $u_k = u(kh)$ and $y_k = y(kh)$, the matrices

$$A_d = \exp(Ah) : \mathbb{R}_\infty^n \rightarrow \mathbb{R}_\infty^n$$

$$B_{2d} = \int_0^h \exp(A\theta) B_2 d\theta : \mathbb{R}_\infty^{n_u} \rightarrow \mathbb{R}_\infty^n$$

$$C_{2d} = C_2 : \mathbb{R}_\infty^n \rightarrow \mathbb{R}_\infty^{n_y}$$

and the operators \mathbf{B}_1 , \mathbf{C}_1 , \mathbf{D}_{11} and \mathbf{D}_{12} defined as

$$\mathbf{B}_1 \widehat{w}_k = \int_0^h \exp(A(h-\theta)) B_1 \widehat{w}_k(\theta) d\theta : (L_\infty[0,h])^{n_w} \rightarrow \mathbb{R}_\infty^n \quad (4)$$

$$(\mathbf{C}_1 x_k)(\theta) = C_1 \exp(A\theta) x_k : \mathbb{R}_\infty^n \rightarrow (L_\infty[0,h])^{n_z} \quad (5)$$

$$\begin{aligned} (\mathbf{D}_{11} \widehat{w}_k)(\theta) &= \int_0^\theta C_1 \exp(A(\theta-\tau)) B_1 \widehat{w}_k(\tau) d\tau : \\ (L_\infty[0,h])^{n_w} &\rightarrow (L_\infty[0,h])^{n_z} \end{aligned} \quad (6)$$

$$\begin{aligned} (\mathbf{D}_{12} u_k)(\theta) &= \int_0^\theta C_1 \exp(A(\theta-\tau)) B_2 d\tau u_k + D_{12} u_k : \\ \mathbb{R}_\infty^{n_u} &\rightarrow (L_\infty[0,h])^{n_z} \end{aligned} \quad (7)$$

The mapping from $\{\widehat{w}_k\}_{k=0}^\infty$ to $\{\widehat{z}_k\}_{k=0}^\infty$ is derived by connecting Ψ to the above \widehat{P} , and we denote it by $\mathcal{F}(\widehat{P}, \Psi)$. Since the lifting technique is norm-preserving, we can see that the L_∞ -induced norm $\|\mathcal{F}(P, \mathcal{H}\Psi S)\|_\infty$ of Σ_{SD} equals $\|\mathcal{F}(\widehat{P}, \Psi)\|_\infty$. To derive alternative representation of $\mathcal{F}(\widehat{P}, \Psi)$, let us introduce $\mathbf{M}_1 := [\mathbf{C}_1 \ \mathbf{D}_{12}]$, which can also be defined by

$$\left(\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix} \right) (\theta) = M_1 \exp(A_2 \theta) \begin{bmatrix} x \\ u \end{bmatrix} \quad (8)$$

where $M_1 := [C_1 \ D_{12}]$ and $A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}$. Next, let us consider the (standard lifting-free) discrete-time plant

$$P_d : \begin{cases} x_{k+1} = A_d x_k + \eta_k + B_{2d} u_k \\ \zeta_k = \begin{bmatrix} I \\ 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ I \end{bmatrix} u_k \\ y_k = C_{2d} x_k \end{cases} \quad (9)$$

with $\eta_k \in \mathbb{R}_\infty^n$ and $\zeta_k \in \mathbb{R}_\infty^{n+n_u}$ and denote by $\mathcal{F}(P_d, \Psi)$ the mapping from η to ζ associated with the closed-loop system obtained by connecting Ψ to the above P_d . Then, $\mathcal{F}(\widehat{P}, \Psi)$ admits the representation

$$\mathcal{F}(\widehat{P}, \Psi) = \mathbf{M}_1 \mathcal{F}(P_d, \Psi) \mathbf{B}_1 + \mathbf{D}_{11} \quad (10)$$

Throughout the paper, let us assume that (A, B_1) is controllable and (M_1, A_2) is observable.

This paper aims at approximating the operators \mathbf{B}_1 , \mathbf{M}_1 and \mathbf{D}_{11} by using the idea of the kernel approximation approach (Kim and Hagiwara, 2015b) and deriving a discretization procedure of the continuous-time generalized plant P together with the associated convergence proof. It would be shown that the kernel approximation approach would be superior to the input approximation approach in the L_1 synthesis phase of sampled-data systems, as is the case with the L_1 analysis problem dealt with in Kim and Hagiwara (2016b).

4. KERNEL APPROXIMATION APPROACH TO SAMPLED-DATA SYSTEMS

This section tackles the L_1 optimal control problem of sampled-data systems by using the idea of the kernel approximation approach.

4.1 Review of the Fast-Lifting Treatment of Σ_{SD}

For the fast-lifting parameter $M \in \mathbb{N}$ and $h' := h/M$, fast-lifting (Hagiwara and Umeda, 2008) is defined as the mapping from $f \in (L_\infty[0,h])^\nu$ (or $f \in (L_1[0,h])^\nu$) to $\check{f} := [(f^{(1)})^T \ \dots \ (f^{(M)})^T]^T \in (L_\infty[0,h'])^{M\nu}$ (or $\check{f} \in (L_1[0,h'])^{M\nu}$), and is denoted by $\check{f} = \mathbf{L}_M f$, where

$$f^{(i)}(\theta') := f((i-1)h' + \theta') \quad (0 \leq \theta' < h') \quad (11)$$

Because \mathbf{L}_M is norm-preserving, it readily follows that

$$\|\mathcal{F}(\widehat{P}, \Psi)\|_\infty = \|\mathbf{L}_M \mathcal{F}(\widehat{P}, \Psi) \mathbf{L}_M^{-1}\|_\infty \quad (12)$$

where the right-hand side means the $l_{[0,h']^\infty}^\infty$ -induced norm (recall (2) for the definition of the $l_{[0,h']^\infty}^\infty$ norm).

Applying fast-lifting to \widehat{w}_k and \widehat{z}_k in the (partially) lifted generalized plant \widehat{P} leads to its fast-lifted counterpart

$$\widehat{P}_M = \text{diag}[\mathbf{L}_M, I] \widehat{P} \text{diag}[\mathbf{L}_M^{-1}, I] \quad (13)$$

Here, it readily follows that $\mathbf{L}_M \mathcal{F}(\widehat{P}, \Psi) \mathbf{L}_M^{-1} = \mathcal{F}(\widehat{P}_M, \Psi)$, and it admits the representation

$$\mathcal{F}(\widehat{P}_M, \Psi) = \mathbf{L}_M \mathbf{M}_1 \mathcal{F}(P_d, \Psi) \mathbf{B}_1 \mathbf{L}_M^{-1} + \mathbf{L}_M \mathbf{D}_{11} \mathbf{L}_M^{-1} \quad (14)$$

If we introduce \mathbf{M}'_1 , \mathbf{B}'_1 and \mathbf{D}'_{11} defined as \mathbf{M}_1 , \mathbf{B}_1 and \mathbf{D}_{11} , respectively, with the horizon $[0, h)$ replaced by $[0, h')$, together with the matrices

$$A'_d = \exp(Ah'), \quad A'_{2d} = \exp(A_2 h'), \quad J = \begin{bmatrix} I \\ 0 \end{bmatrix} : \mathbb{R}_\infty^{n+n_u} \rightarrow \mathbb{R}_\infty^n$$

then, as in the standard arguments employing fast-lifting (Hagiwara and Umeda, 2008), it readily follows that

$$\mathbf{L}_M \mathbf{D}_{11} \mathbf{L}_M^{-1} = \overline{\mathbf{M}'_1} \Delta^0_M \overline{\mathbf{B}'_1} + \overline{\mathbf{D}'_{11}} \quad (15)$$

$$\mathbf{L}_M \mathbf{M}_1 = \overline{\mathbf{M}'_1} A'_{2dM}, \quad \mathbf{B}_1 \mathbf{L}_M^{-1} = A'_{dM} \overline{\mathbf{B}'_1} \quad (16)$$

where

$$A'_{dM} := [(A'_d)^{M-1} \ \dots \ I], \quad A'_{2dM} := \begin{bmatrix} I \\ \vdots \\ (A'_{2d})^{M-1} \end{bmatrix} \quad (17)$$

$$\Delta^0_M := \begin{bmatrix} 0 & 0 & \dots & 0 \\ J & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (A'_{2d})^{M-2} J & \dots & J & 0 \end{bmatrix} \quad (18)$$

and $\overline{(\cdot)}$ denotes $\text{diag}\{(\cdot), \dots, (\cdot)\}$ consisting of M copies of (\cdot) .

The argument of the present paper is considered as a successive study for our preceding study (Kim and Hagiwara, 2016b); in the latter study, the kernel approximation approach (Kim and Hagiwara, 2015b) is applied to Σ_{SD} for tackling the L_1 analysis problem of sampled-data systems, but such a method is restricted to analysis and cannot be applied directly to synthesis. In other words, this paper deals the L_1 synthesis problem of sampled-data systems by applying the kernel approximation approach to Σ_{SD} under the piecewise constant approximation scheme. As a preliminary step to considering the treatment of Σ_{SD} via the kernel approximation approach, let us introduce the operators \mathbf{B}'_{k0} and \mathbf{M}'_{a0} defined respectively as

$$\mathbf{B}'_{k0}w = A'_d B_1 \int_0^{h'} w(\theta') d\theta' : (L_\infty[0, h'])^{n_w} \rightarrow \mathbb{R}^{n_z} \quad (19)$$

$$\left(\mathbf{M}'_{a0} \begin{bmatrix} x \\ u \end{bmatrix} \right) (\theta') = M_1 \begin{bmatrix} x \\ u \end{bmatrix} : \mathbb{R}^{n_x+n_u} \rightarrow (L_\infty[0, h'])^{n_z} \quad (20)$$

The above \mathbf{B}'_{k0} corresponds to the zero-order approximation of the kernel function $\exp(A(h' - \theta'))B_1 = A'_d \sum_{i=0}^{\infty} \frac{(-A\theta')^i}{i!} B_1$ of the operator \mathbf{B}'_1 ; the subscripts k and 0 stand for the kernel approximation and the zero-order approximation, respectively. Similarly, \mathbf{M}'_{a0} corresponds to the zero-order approximation of the hold function $M_1 \exp(A_2\theta') = M_1 \sum_{i=0}^{\infty} \frac{(A_2\theta')^i}{i!}$ of the operator \mathbf{M}'_1 (which was used also in the input approximation approach to the L_1 synthesis of sampled-data systems studied in Kim and Hagiwara (2016a)).

We consider replacing \mathbf{M}'_1 and \mathbf{B}'_1 with \mathbf{M}'_{a0} and \mathbf{B}'_{k0} , respectively, together with ignoring \mathbf{D}'_{11} in (14)–(16). Then, we have the following approximation of $\mathcal{F}(\widehat{P}_M, \Psi)$:

$$\mathcal{F}(\widehat{P}_{Mk0}, \Psi) := \overline{\mathbf{M}'_{a0} A'_{2dM} \mathcal{F}(P_d, \Psi) A'_{dM} \mathbf{B}'_{k0}} + \overline{\mathbf{M}'_{a0} \Delta_M^0 \mathbf{B}'_{k0}} \quad (21)$$

We call it piecewise constant kernel approximation of the sampled-data system Σ_{SD} , which alleviates the difficulty in the synthesis of the optimal controller Ψ that minimizes $\|\mathcal{F}(\widehat{P}, \Psi)\|_\infty = \|\mathcal{F}(\widehat{P}_M, \Psi)\|_\infty$.

4.2 Discretization of Continuous-Time Generalized Plant

This subsection provides a discretization procedure of the continuous-time generalized plant P developed through the piecewise constant kernel approximation treatment of Σ_{SD} . It converts the synthesis problem of an L_1 optimal controller Ψ for the sampled-data system Σ_{SD} into the discrete-time synthesis problem of an l_1 optimal controller relevant to the resulting generalized plant obtained by adequately modifying P_d in (9).

To derive such a discretized generalized plant, we consider replacing the operators \mathbf{B}'_{k0} and \mathbf{M}'_{a0} with appropriate finite-dimensional matrices. It readily follows from (20) and (21) that the output of $\mathcal{F}(\widehat{P}_{Mk0}, \Psi)$ is a constant function determined by the matrix M_1 . Furthermore, it readily follows from (19) that the following relation holds, where a constant function in $(L_\infty[0, h'])^{n_w}$ is denoted by w_d .

$$\{\mathbf{B}'_{k0}w \mid \|w\|_\infty \leq 1\} = \{\mathbf{B}'_{k0}w_d \mid \|w_d\|_\infty \leq 1\} \quad (22)$$

This clearly implies that the input of $\mathcal{F}(\widehat{P}_{Mk0}, \Psi)$ can always be assumed to be a constant function when we evaluate $\|\mathcal{F}(\widehat{P}_{Mk0}, \Psi)\|_\infty$ (and the action of \mathbf{B}'_{k0} can virtually be described by the matrix $A'_d B_1 h'$ as seen by (19)). Thus, $\|\mathcal{F}(\widehat{P}_{Mk0}, \Psi)\|_\infty$ coincides with the l_∞ -induced norm of the discrete-time system obtained by replacing the operators \mathbf{B}'_{k0} and \mathbf{M}'_{a0} in $\mathcal{F}(\widehat{P}_{Mk0}, \Psi)$ in (21) with $A'_d B_1 h'$ and M_1 , respectively. Combining the above arguments and interpreting the resulting discrete-time system (as the feedback connection of P_{Mk0d} given below and Ψ) leads to the following result.

Theorem 1. Let us consider the discrete-time generalized plant given by

$$P_{Mk0d} : \begin{cases} x_{k+1} = A_d x_k + B_{Mk0} w_k + B_{2d} u_k \\ z_k = C_{Mk0} x_k + D_{Mk01} w_k + D_{Mk02} u_k \\ y_k = C_{2d} x_k \end{cases} \quad (23)$$

where

$$B_{Mk0} := A'_{dM} \overline{A'_d B_1 h'} \quad (24)$$

$$D_{Mk01} := \overline{M_1 \Delta_M^0 A'_d B_1 h'} \quad (25)$$

$$[C_{Mk0} \ D_{Mk02}] := \overline{M_1 A'_{2dM}} \quad (C_{Mk0} \in \mathbb{R}^{M n_z \times n}, \ D_{Mk02} \in \mathbb{R}^{M n_z \times n_u}) \quad (26)$$

Let us denote by $\|\mathcal{F}(P_{Mk0d}, \Psi)\|_\infty$ the l_∞ -induced norm of the discrete-time system consisting of P_{Mk0d} and Ψ . Then, the $l_{[0, h']^\infty}$ -induced norm $\|\mathcal{F}(\widehat{P}_{Mk0}, \Psi)\|_\infty$ coincides with the l_∞ -induced norm $\|\mathcal{F}(P_{Mk0d}, \Psi)\|_\infty$, i.e., $\|\mathcal{F}(\widehat{P}_{Mk0}, \Psi)\|_\infty = \|\mathcal{F}(P_{Mk0d}, \Psi)\|_\infty$.

Theorem 1 obviously means that the synthesis problem about $\inf_{\Psi} \|\mathcal{F}(\widehat{P}_{Mk0}, \Psi)\|_\infty$ obtained by the piecewise constant kernel approximation of Σ_{SD} is *equivalently* converted into the discrete-time l_1 optimal controller synthesis problem about $\inf_{\Psi} \|\mathcal{F}(P_{Mk0d}, \Psi)\|_\infty$. Thus, the remaining task is to construct a theoretical basis of the piecewise constant kernel approximation for tackling the L_1 optimal control problem of sampled-data systems through dealing with $\mathcal{F}(\widehat{P}_{Mk0}, \Psi)$, which is merely an approximation of $\mathcal{F}(\widehat{P}_M, \Psi)$. To this end, we derive an error bound between $\|\mathcal{F}(\widehat{P}_{Mk0}, \Psi)\|_\infty (= \|\mathcal{F}(P_{Mk0d}, \Psi)\|_\infty)$ and $\|\mathcal{F}(\widehat{P}, \Psi)\|_\infty$, whose details and implications are discussed in the following subsections.

4.3 Error Analysis for Kernel Approximation Approach

This subsection is devoted to providing the error analysis in the piecewise constant kernel approximation and shows that the associated convergence rate is in the order of $1/M$ with the underlying fast-lifting parameter M . To evaluate the error in the approximation of $\|\mathcal{F}(\widehat{P}, \Psi)\|_\infty = \|\mathcal{F}(\widehat{P}_M, \Psi)\|_\infty$ by $\|\mathcal{F}(P_{Mk0d}, \Psi)\|_\infty$, we first introduce the operators $\mathbf{J}'_{k0} : (L_\infty[0, h'])^{n_w} \rightarrow (L_\infty[0, h'])^{n_w}$ and $\mathbf{H}'_{a0} : (L_\infty[0, h'])^{n_z} \rightarrow (L_\infty[0, h'])^{n_z}$ described respectively as

$$(\mathbf{J}'_{k0}w)(\theta') := B_1^T \exp(A^T(h' - \theta')) W_{h'}^{-1} \mathbf{B}'_{k0}w \quad (27)$$

$$(\mathbf{H}'_{a0}z)(\theta') = z(0) \quad (0 \leq \theta' < h') \quad (28)$$

where $W_{h'}$ is the controllability Grammian defined as

$$W_{h'} := \int_0^{h'} \exp(A(h' - \theta')) B_1 B_1^T \exp(A^T(h' - \theta')) d\theta' \quad (29)$$

and its inverse in (27) is ensured to exist by the controllability assumption of (A, B_1) . Then, we can see from (19) and (20) that

$$\mathbf{B}'_{k0} = \mathbf{B}'_1 \mathbf{J}'_{k0}, \quad \mathbf{M}'_{a0} = \mathbf{H}'_{a0} \mathbf{M}'_1 \quad (30)$$

Hence, by defining $\mathbf{J}_{Mk0} := \mathbf{L}_M^{-1} \overline{\mathbf{J}'_{k0}}$, $\mathbf{H}_{Ma0} := \overline{\mathbf{H}'_{a0}} \mathbf{L}_M$ and $\mathbf{D}_{Mk0} := \overline{\mathbf{M}'_{a0}} \Delta_M^0 \overline{\mathbf{B}'_{k0}}$, it follows from (15), (16) and (21) that $\mathcal{F}(\widehat{P}_{Mk0}, \Psi)$ can be also represented as

$$\mathcal{F}(\widehat{P}_{Mk0}, \Psi) = \mathbf{H}_{Ma0} \mathbf{M}_1 \mathcal{F}(P_d, \Psi) \mathbf{B}_1 \mathbf{J}_{Mk0} + \mathbf{D}_{Mk0} \quad (31)$$

We next introduce ‘finite-rank portions’ of $\mathcal{F}(\widehat{P}_M, \Psi)$ in (14) and $\mathcal{F}(\widehat{P}_{Mk0}, \Psi)$ in (31) described respectively by

$$\begin{aligned} \mathcal{F}^0(\widehat{P}_M, \Psi) &:= \mathbf{L}_M \mathbf{M}_1 \mathcal{F}(P_d, \Psi) \mathbf{B}_1 \mathbf{L}_M^{-1} \\ &= \mathcal{F}(\widehat{P}_M, \Psi) - \mathbf{L}_M \mathbf{D}_{11} \mathbf{L}_M^{-1} \end{aligned} \quad (32)$$

$$\begin{aligned} \mathcal{F}^0(\widehat{P}_{Mk0}, \Psi) &:= \mathbf{H}_{Ma0} \mathbf{M}_1 \mathcal{F}(P_d, \Psi) \mathbf{B}_1 \mathbf{J}_{Mk0} \\ &= \mathcal{F}(\widehat{P}_{Mk0}, \Psi) - \mathbf{D}_{Mk0} \end{aligned} \quad (33)$$

It is obvious from the comparison between the above two equations that evaluating $\mathbf{J}_{Mk0} - \mathbf{L}_M^{-1}$ and $\mathbf{H}_{Ma0} - \mathbf{L}_M$ is important in the error analysis (and this is why \mathbf{J}'_{k0} and \mathbf{H}'_{a0} were introduced in such a way that (30) is satisfied, so that (21) can be rewritten in the form of (31)). The following lemma is associated with such an evaluation.

Lemma 2. We have the following properties regarding the preadjoints \mathbf{B}_{1*} and \mathbf{J}_{Mk0*} and the operators \mathbf{M}_1 and \mathbf{H}_{Ma0} .

a) There exists a constant K_{Bk0} such that

$$\|(\mathbf{L}_M - \mathbf{J}_{Mk0*})|_{R(\mathbf{B}_{1*})}\|_1 \leq \frac{K_{Bk0}}{M} \quad (34)$$

where $R(\mathbf{B}_{1*})$ denotes the range of \mathbf{B}_{1*} (to which $\mathbf{L}_M - \mathbf{J}_{Mk0*}$ is restricted) and is viewed as a subset of $(L_1[0, h])^{n_w}$.

b) There exists a constant K_{Ck0} such that

$$\|(\mathbf{L}_M - \mathbf{H}_{Ma0})|_{R(\mathbf{M}_1)}\|_\infty \leq \frac{K_{Ck0}}{M} \quad (35)$$

where $R(\mathbf{M}_1)$ denotes the range of \mathbf{M}_1 (to which $\mathbf{L}_M - \mathbf{H}_{Ma0}$ is restricted) and is viewed as a subset of $(L_\infty[0, h])^{n_z}$.

From Lemma 2, we can obtain the following result.

Proposition 3. There exists a constant K_{k0}^0 independent of Ψ , such that

$$\|\mathcal{F}^0(\widehat{P}_{Mk0}, \Psi) - \mathcal{F}^0(\widehat{P}_M, \Psi)\|_\infty \leq \frac{K_{k0}^0}{M} \|\mathcal{F}^0(\widehat{P}_M, \Psi)\|_\infty \quad (36)$$

In comparison between (32) and (33), we see that evaluating $\mathbf{D}_{Mk0} - \mathbf{L}_M \mathbf{D}_{11} \mathbf{L}_M^{-1}$ is also very important in the error analysis, for which we refer to the following result (Kim and Hagiwara, 2016b, Lemma 1).

Lemma 4. There exists a constant K_{k0}^1 such that

$$\|\mathbf{D}_{Mk0} - \mathbf{L}_M \mathbf{D}_{11} \mathbf{L}_M^{-1}\|_\infty \leq \frac{K_{k0}^1}{M} \quad (37)$$

From Proposition 3 and Lemma 4, we provide the following main result on the error analysis of the piecewise constant kernel approximation.

Theorem 5. The following inequality holds:

$$\begin{aligned} \left(1 - \frac{K_{k0}^0}{M}\right) \|\mathcal{F}(\widehat{P}, \Psi)\|_\infty - \frac{K_{k0}^1}{M} &\leq \|\mathcal{F}(\widehat{P}_{Mk0}, \Psi)\|_\infty \\ &\leq \left(1 + \frac{K_{k0}^0}{M}\right) \|\mathcal{F}(\widehat{P}, \Psi)\|_\infty + \frac{K_{k0}^1}{M} \end{aligned} \quad (38)$$

Combining Theorems 1 and 5 leads to the following result.

Corollary 6. The inequality holds:

$$\begin{aligned} \left(1 - \frac{K_{k0}^0}{M}\right) \|\mathcal{F}(\widehat{P}, \Psi)\|_\infty - \frac{K_{k0}^1}{M} &\leq \|\mathcal{F}(P_{Mk0d}, \Psi)\|_\infty \\ &\leq \left(1 + \frac{K_{k0}^0}{M}\right) \|\mathcal{F}(\widehat{P}, \Psi)\|_\infty + \frac{K_{k0}^1}{M} \end{aligned} \quad (39)$$

Remark 1. Through the input approximation approach under the piecewise constant approximation scheme in Kim and Hagiwara (2014, 2015a), we could derive the inequality

$$\begin{aligned} \left(1 - \frac{K_{i0}^0}{M}\right) \|\mathcal{F}(\widehat{P}, \Psi)\|_\infty - \frac{K_{i0}^1}{M} &\leq \|\mathcal{F}(P_{Mi0d}, \Psi)\|_\infty \\ &\leq \left(1 + \frac{K_{i0}^0}{M}\right) \|\mathcal{F}(\widehat{P}, \Psi)\|_\infty + \frac{K_{i0}^1}{M} \end{aligned} \quad (40)$$

with appropriately defined discrete-time generalized plant P_{Mi0d} and constants K_{i0}^0 and K_{i0}^1 . With regards to the comparison of the inequalities (39) and (40), we can show that the constants K_{k0}^0 and K_{k0}^1 introduced in this paper can be shown to be smaller than K_{i0}^0 and K_{i0}^1 , respectively. Thus, it is expected that the kernel approximation approach can lead to a smaller approximation error than the input approximation approach under the same parameter M for the L_1 optimal controller synthesis problem of sampled-data systems.

4.4 Validity of Piecewise Constant Kernel Approximation

This subsection provides theoretical validity of the piecewise constant kernel approximation for tackling the L_1 optimal controller synthesis for Σ_{SD} by using the arguments in Corollary 6. Let $\gamma_{\text{opt}} := \inf_{\Psi} \|\mathcal{F}(\widehat{P}, \Psi)\|_\infty$ and take an M . Suppose that Ψ_{Mk0} is an ε -suboptimal controller with respect to $\gamma_{Mk0} := \inf_{\Psi} \|\mathcal{F}(P_{Mk0d}, \Psi)\|_\infty$, i.e., $\|\mathcal{F}(P_{Mk0d}, \Psi_{Mk0})\|_\infty \leq \gamma_{Mk0} + \varepsilon$ ($\varepsilon > 0$). Let $M_0 \in \mathbb{N}$ be the minimum such that $M_0 > K_{k0}$. Then, for $M \geq M_0$, the first inequality of (39) implies that

$$\begin{aligned} \gamma_{\text{opt}} &\leq \|\mathcal{F}(\widehat{P}, \Psi_{Mk0})\|_\infty \\ &\leq \left(1 + \frac{K_{k0}^0}{M - K_{k0}^0}\right) (\gamma_{Mk0} + \varepsilon) + \frac{K_{k0}^1}{M - K_{k0}^0} \end{aligned} \quad (41)$$

On the other hand, it readily follows from the second inequality of (39) that

$$\gamma_{Mk0} \leq \left(1 + \frac{K_{k0}^0}{M}\right) \gamma_{\text{opt}} + \frac{K_{k0}^1}{M} \quad (42)$$

Substituting this into (41) and taking a sufficiently large M such that $M \geq M_0$, we see that

$$\gamma_{\text{opt}} \leq \|\mathcal{F}(\widehat{P}, \Psi_{Mk0})\|_\infty \leq \gamma_{\text{opt}} + \varepsilon + \frac{X}{M} \quad (43)$$

where

$$X := \frac{2K_{k0}^0 \gamma_{\text{opt}}}{1 - K_{k0}^0/M_0} + \frac{2K_{k0}^1}{1 - K_{k0}^0/M_0} + \frac{K_{k0}^0 \varepsilon}{1 - K_{k0}^0/M_0} \quad (44)$$

This obviously means that the performance with a sub-optimal Ψ_{Mk_0} with respect to $\inf_{\Psi} \|\mathcal{F}(P_{Mk_0d}, \Psi)\|_{\infty}$ (sufficiently close to the infimal performance) converges to that of the L_1 optimal controller for Σ_{SD} in the order of $1/M$.

5. CONCLUSION

This is the *first paper* to deal with the use of the kernel approximation approach in controller synthesis and we specifically employed the piecewise constant approximation scheme to give a preliminary study of the L_1 optimal control problem of sampled-data systems. With these approximations, the continuous-time generalized plant P in the sampled-data system was eventually approximated by a discrete-time generalized plant. More precisely, we first established Theorem 1, whose implication is that the L_1 optimal control problem of sampled-data systems is approximately converted into an appropriately constructed discrete-time l_1 optimal control problem. Furthermore, to develop a mathematical basis for the kernel approximation approach in the L_1 optimal controller synthesis problem of sampled-data systems, we next established Theorem 5 or the inequality (38) through the arguments of preadjoint operators. This inequality showed that the convergence rate associated with the kernel approximation approach is $1/M$ with respect to the fast-lifting parameter M . Here, it should be stressed that even though this convergence rate is qualitatively the same as that in the existing input approximation with the piecewise constant approximation scheme, the approximation error through the new kernel approximation approach is smaller than that through the existing input approximation approach under the same fast-lifting parameter M .

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