Characterization of Quasi L_∞/L_2 Hankel Norms of Sampled-Data Systems

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Abstract: This paper is concerned with the Hankel operator of sampled-data systems. The Hankel operator is usually defined as a mapping from the past input to the future output and its norm plays an important role in evaluating the performance of systems. Since the continuous-time mapping between the input and output is periodically time-varying (h-periodic, where h denotes the sampling period) in sampled-data systems, it matters when to take the time instant separating the past and the future when we define the Hankel operator for sampled-data systems. This paper takes an arbitrary $\Theta \in [0, h)$ as such an instant, and considers the quasi L_∞/L_2 Hankel operator defined as the mapping from the past input in $L_2(-\infty, \Theta)$ to the future output in $L_\infty[\Theta, \infty)$. The norm of this operator, which we call the quasi L_∞/L_2 Hankel norm at Θ, is then characterized in such a way that its numerical computation becomes possible. Then, regarding the computation of the L_∞/L_2 Hankel norm defined as the supremum of the quasi L_∞/L_2 Hankel norms over $\Theta \in [0, h)$, some relationship is discussed between the arguments through such characterization and an alternative method developed in an earlier paper that is free from the computations of quasi L_∞/L_2 Hankel norms. A numerical example is studied to confirm the validity of the arguments in this paper.

Keywords: Linear multivariable systems, Time-varying systems, Disturbance rejection, Sampled-data systems, Hankel operator, H_2 norm.

1. INTRODUCTION

Many systems in engineering and science belong to the class of dynamical systems, which are characterized as such systems whose output depends not only on the present input but also on the past input. The mapping from the past input to the future output is generally called the Hankel operator, and its norm is called the Hankel norm. The study on the Hankel operator/norm is important in model reduction and related studies, and many significant results have been obtained in continuous-time linear time-invariant (LTI) systems, e.g., Lin and Kung (1982); Glover (1982); Zhou (1995); Hung and Glover (1986).

The Hankel operator is usually defined as a mapping from the past input in $L_2(-\infty, 0)$ to the future output in $L_2[0, \infty)$, and the Hankel norm is defined accordingly. On the other hand, there is also a study that is interested in the Hankel operator defined as a mapping from $L_2(-\infty, 0)$ to $L_\infty[0, \infty)$ (Wilson, 1989), which we call the L_∞/L_2 Hankel operator for simplicity. Unlike in the former case dealing with the usual Hankel operator (which we call the L_2/L_2 Hankel operator), it has been clarified for continuous-time LTI systems that the associated Hankel norm (which we call the L_∞/L_2 Hankel norm) equals the corresponding induced norm from $L_2[0, \infty)$ to $L_\infty[0, \infty)$. In continuous-time LTI systems, this induced norm is further known to have a very close relationship with the H_2 norm (Wilson, 1989; Rotea, 1993; Chellabonia and Haddad, 2000).

This paper is concerned with the L_∞/L_2 Hankel operator/norm of sampled-data systems, in which the generalized plant has the continuous-time disturbance w and the continuous-time controlled output z, and the effect of the past w on the future z is studied. To make this statement more rigorous, however, we first need to note a very important and essential feature of sampled-data systems: due to the periodic action of the 'sampled-data controller' consisting of the discrete-time controller together with the hold and sampling devices, the continuous-time mapping between w and z is h-periodic, where h denotes the sampling period. Hence, it deeply matters when to take the time instant that separates the 'past' about the input w and the 'future' about the output z in defining the Hankel operator. In Chongsrid and Hara (1995)[1], this issue was completely neglected and the past and the future were simply separated at time 0 *under the treatment in which the time* 0 *is also an instant at which the sampler takes*

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[1] Note that the study in Chongsrid and Hara (1995) corresponds to the L_2/L_2 Hankel operator (more precisely, a quasi L_2/L_2 Hankel operator in the term of the present paper) while the present paper deals with the L_∞/L_2 Hankel operator.

its action. This inappropriate treatment was amended in our recent study (Hagiwara et al., 2016) by first taking an arbitrary $\Theta \in [0, h)$ as the instant separating the past and the future and then considering the mapping from $w \in L_2(-\infty, \Theta)$ to $z \in L_\infty[\Theta, \infty)$, which we call the quasi L_∞/L_2 Hankel operator at Θ. The associated norm (i.e., the quasi L_∞/L_2 Hankel norm at Θ) was then considered and, roughly speaking, the worst value of this norm over $\Theta \in [0, h)$ was defined as *the L_∞/L_2 Hankel norm* while the quasi L_∞/L_2 Hankel operator at such Θ that corresponds to the worst value was defined as *the L_∞/L_2 Hankel operator.*

What has been shown in our recent study (Hagiwara et al., 2016) on sampled-data systems is that

(i) as in the continuous-time LTI case, the induced norm from $L_2[0, \infty)$ to $L_\infty[0, \infty)$ (Zhu and Skelton, 1995; Kim and Hagiwara, 2015)coincides with *the L_∞/L_2 Hankel norm*;

(ii) the L_∞/L_2 Hankel operator/norm can actually be characterized in such an alternative approach that completely avoids the reference to the quasi L_∞/L_2 Hankel operators/norms.

However, characterizing the quasi L_∞/L_2 Hankel norm for a given $\Theta \in [0, h)$ in such a way that the norm can be computed readily is still an open problem. What the present paper tackles is exactly this open problem.

The notation in this paper is as follows. $\mathbb{N}$ and $\mathbb{R}^\nu$ denote the set of positive integers and the set of ν-dimensional real vectors, respectively. We further use the notation $\mathbb{N}_0$ to imply $\mathbb{N} \cup \{0\}$. The ∞-norm of vectors is denoted by $|x|_\infty$ $(= \max_{i=1,\dots,\nu} |x_i|)$. Furthermore, we use the notation $d_{\max}(\cdot)$ to denote the maximum diagonal entry of a real symmetric matrix.

2. LIFTING TREATMENT OF SAMPLED-DATA SYSTEMS AND QUASI L_∞/L_2 HANKEL NORM AT Θ

2.1 Lifting Treatment of Sampled-Data Systems

We consider the stable LTI sampled-data system Σ_{SD} shown in Fig. 1, where P denotes the continuous-time LTI generalized plant while Ψ, $\mathcal{H}$ and $\mathcal{S}$ denote the discrete-time LTI controller, the zero-order hold and the ideal sampler, respectively, operating with sampling period h in a synchronous fashion. Solid lines and dashed lines are used to represent continuous-time and discrete-time signals, respectively, in this figure. Let P and Ψ be described by

$$P : \begin{cases} \dfrac{dx}{dt} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x \end{cases} \tag{1}$$

$$\Psi : \begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k \\ u_k = C_\Psi \psi_k + D_\Psi y_k \end{cases} \tag{2}$$

respectively, where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$, $u(t) \in \mathbb{R}^{n_u}$, $z(t) \in \mathbb{R}^{n_z}$, $y(t) \in \mathbb{R}^{n_y}$, $\psi_k \in \mathbb{R}^{n_\Psi}$, $y_k = y(kh)$ and $u(t) = u_k$ $(kh \le t < (k+1)h)$.

The sampled-data system Σ_{SD} viewed as a continuous-time mapping between w and z is periodically time-
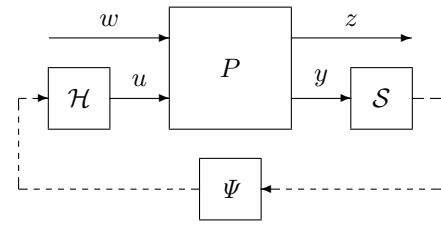

Fig. 1. Sampled-data system Σ_{SD}.

varying. To deal with Σ_{SD} as if it were a time-invariant system, we apply the lifting technique (Bamieh and Pearson, 1992a; Toivonen, 1992; Yamamoto, 1994), which converts the continuous-time function $f(\cdot)$ to the sequence $\{\widehat{f}_k(\theta)\}$ of functions on $[0, h)$ given by

$$\widehat{f}_k(\theta) = f(kh + \theta) \quad (0 \le \theta < h) \tag{3}$$

In accordance with the above equation for lifting, we assume that the sampling instants are given by the integer multiples of h (and thus time 0 is a sampling instant)[2]. Hence, we have the lifted representation of sampled-data systems Σ_{SD} given by

$$\begin{cases} \xi_{k+1} = \mathcal{A}\xi_k + \mathcal{B}\widehat{w}_k \\ \widehat{z}_k = \mathcal{C}\xi_k + \mathcal{D}\widehat{w}_k \end{cases} \tag{4}$$

with $\xi_k := [x_k^T \ \psi_k^T]^T$ $(x_k := x(kh))$, the matrix

$$\mathcal{A} = \begin{bmatrix} A_d + B_{2d} D_\Psi C_{2d} & B_{2d} C_\Psi \\ B_\Psi C_{2d} & A_\Psi \end{bmatrix} : \mathbb{R}^{n+n_\Psi} \to \mathbb{R}^{n+n_\Psi} \tag{5}$$

and the operators

$$\mathcal{B} = J_\Sigma \mathbf{B}_1 : L_2[0, h) \to \mathbb{R}^{n+n_\Psi} \tag{6}$$

$$\mathcal{C} = \mathbf{M}_1 C_\Sigma : \mathbb{R}^{n+n_\Psi} \to L_\infty[0, h) \tag{7}$$

$$\mathcal{D} = \mathbf{D}_{11} : L_2[0, h) \to L_\infty[0, h) \tag{8}$$

where

$$A_d := \exp(Ah), \ B_{2d} := \int_0^h \exp(A\theta) B_2 d\theta, \ C_{2d} := C_2 \tag{9}$$

$$J_\Sigma := \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+n_\Psi) \times n}, \quad C_\Sigma := \begin{bmatrix} I & 0 \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix} \tag{10}$$

$$\mathbf{B}_1 w = \int_0^h \exp(A(h-\theta)) B_1 w(\theta) d\theta \tag{11}$$

$$\left(\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix} \right)(\theta) = M_1 \exp(A_2 \theta) \begin{bmatrix} x \\ u \end{bmatrix} \tag{12}$$

$$A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \quad M_1 := [C_1 \ D_{12}] \tag{13}$$

$$(\mathbf{D}_{11} w)(\theta) = \int_0^\theta C_1 \exp(A(\theta - \tau)) B_1 w(\tau) d\tau \tag{14}$$

Note that the matrix $\mathcal{A}$ is stable by the stability assumption of Σ_{SD}.

2.2 Quasi L_∞/L_2 Hankel Norm at Θ

In this section, we review the definition of the quasi L_∞/L_2 Hankel norm of sampled-data systems (Hagiwara et al., 2016) when we consider separating the past and the future at $\Theta \in [0, h)$. We also introduce some relevant notations.

Let $\Theta \in [0, h)$ be the time instant separating the past about the input w and the future about the output z.

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[2] We do not regard time 0 as the instant that separates the past and the future; such an instant will be denoted by $\Theta \in [0, h)$.

For each $\Theta \in [0, h)$, we consider the input w defined on $(-\infty, \Theta)$ such that its $L_2(-\infty, \Theta)$ norm defined as

$$||w(\cdot)||_{2-}^{[\Theta]} = \left(\int_{-\infty}^{\Theta} w(t)^T w(t) dt \right)^{1/2} \tag{15}$$

is well-defined. The function space of such w is denoted by $L_2(-\infty, \Theta)$. On the other hand, the output z is handled only on $[\Theta, \infty)$ (assuming that $w(t) = 0$ for $t \geq \Theta$) and regarded as an element of the function space $L_\infty[\Theta, \infty)$ defined as the set of z such that its $L_\infty[\Theta, \infty)$ norm defined as

$$||z(\cdot)||_{\infty}^{[\Theta]} = \operatorname*{ess\,sup}_{\Theta \leq t < \infty} |z(t)|_{\infty} \tag{16}$$

is well-defined. The mapping from $L_2(-\infty, \Theta)$ to $L_\infty[\Theta, \infty)$ is called the quasi L_∞/L_2 Hankel operator of the sampled-data system Σ_{SD} at Θ, which we denote by $\mathbf{H}^{[\Theta]}$. Its norm defined as

$$||\mathbf{H}^{[\Theta]}|| := \sup_{w \in L_2(-\infty, \Theta)} \frac{||z(\cdot)||_{\infty}^{[\Theta]}}{||w(\cdot)||_{2-}^{[\Theta]}} \tag{17}$$

is called the quasi L_∞/L_2 Hankel norm at Θ.

We further remark that *the L_∞/L_2 Hankel norm of the sampled-data systems Σ_{SD}*, which we denote by $||\Sigma_{\mathrm{SD}}||_{\mathrm{H}}$, is defined in Hagiwara et al. (2016) as

$$||\Sigma_{\mathrm{SD}}||_{\mathrm{H}} := \sup_{\Theta \in [0, h)} ||\mathbf{H}^{[\Theta]}|| \tag{18}$$

When the right hand side of the above equation is attained (at $\Theta = \Theta^\star$), $\mathbf{H}^{[\Theta^\star]}$ is defined in Hagiwara et al. (2016) as *the L_∞/L_2 Hankel operator of Σ_{SD}*. It is known that $\Theta^\star$ is well-defined when $D_{12} = 0$ (but not necessarily so otherwise).

What has been clarified in our preceding study (Hagiwara et al., 2016) is that (i) *the L_∞/L_2 Hankel norm can actually be computed without computing the quasi L_∞/L_2 Hankel norms at all*, and (ii) the L_∞/L_2 Hankel operator can also be characterized (when it exists) without dealing with the quasi L_∞/L_2 Hankel operators at all. Because of these facts, the preceding study did not even refer to an interesting problem of characterizing the quasi L_∞/L_2 Hankel norms in such a way that their numerical computation becomes possible. What the present paper is interested in, on the other hand, is precisely such characterization.

Remark 1. The notation $|| \cdot ||_2$ is also used for functions defined on the interval $[0, h)$, in which case (15) is modified accordingly.

3. CHARACTERIZING QUASI L_∞/L_2 HANKEL NORMS

3.1 Past-Input/Future-Output Relation of Σ_{SD} through the Lifting Treatment

An important preliminary step for our characterizing the quasi L_∞/L_2 Hankel norms is to represent the input/output relation of Σ_{SD} through the lifting treatment. Under the assumption that $x(-\infty) = 0$, $\psi_{-\infty} = 0$ and $w(t) = 0$, $t \geq \Theta$, the relationship between lifted representations $\{\widehat{w}_k\}_{k=-\infty}^{0}$ of the past input and $\{\widehat{z}_k\}_{k=0}^{\infty}$ of the future output can be described by the formal relation

$$\begin{bmatrix} \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \\ \widehat{z}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathcal{D} & \mathcal{CB} & \mathcal{CAB} & \mathcal{CA}^2\mathcal{B} & \cdots \\ \mathcal{CB} & \mathcal{CAB} & \mathcal{CA}^2\mathcal{B} & \cdots & \\ \mathcal{CAB} & \mathcal{CA}^2\mathcal{B} & & & \\ \mathcal{CA}^2\mathcal{B} & \vdots & & & \\ \vdots & & & & \end{bmatrix} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_{-1} \\ \widehat{w}_{-2} \\ \widehat{w}_{-3} \\ \vdots \end{bmatrix} \tag{19}$$

Let us take an arbitrary $\tau \in [\Theta, \Theta + h)$. Then, the output $z(kh + \tau)$ for each input $w \in L_2(-\infty, \Theta)$ is equal to the output $z(\tau)$ for another input obtained by shifting the original input w to the left by kh. Here, note that if $\tau \in [\Theta, \Theta + h)$ actually belongs to $\tau \in [0, h)$, then $z(\tau)$ is relevant to $\widehat{z}_0$, while if $\tau \in [h, 2h)$, then $z(\tau)$ is relevant to $\widehat{z}_1$. Similarly, $z(kh + \tau)$ is relevant to $\widehat{z}_k$ or $\widehat{z}_{k+1}$, depending on whether $\tau \in [0, h)$ or $\tau \in [h, 2h)$. Keeping these facts in mind, we see that the above observation about shifting w implies that only the first two block rows in the operator matrix on the right hand side of (19) matters when we are to characterize the quasi L_∞/L_2 Hankel norms. More precisely, by introducing

$$\mathbf{F}_1 = [\mathcal{D} \ \mathcal{CB} \ \mathcal{CAB} \ \mathcal{CA}^2\mathcal{B} \ \cdots] \tag{20}$$

$$\mathbf{F}_2 = [\mathcal{CB} \ \mathcal{CAB} \ \mathcal{CA}^2\mathcal{B} \ \mathcal{CA}^3\mathcal{B} \ \cdots] \tag{21}$$

and defining $\widehat{w} := [\widehat{w}_0^T, \widehat{w}_{-1}^T, \cdots]^T$ together with $||\widehat{w}||_{2-}^{[\Theta]} := (\sum_{k=-\infty}^{0} ||\widehat{w}_k||_2^2)^{1/2}$ under the assumption that $\widehat{w}_0(\theta) = 0$ $(\theta \geq \Theta)$ (which implies $||w||_{2-}^{[\Theta]} = ||\widehat{w}||_{2-}^{[\Theta]}$), we readily see the following relation about the quasi L_∞/L_2 Hankel norm at Θ:

$$||\mathbf{H}^{[\Theta]}|| = \sup_{||w||_{2-}^{[\Theta]} \leq 1} \ \sup_{\Theta \leq \tau < \Theta + h} |z(\tau)|_{\infty}$$

$$= \max \left\{ \sup_{||\widehat{w}||_{2-}^{[\Theta]} \leq 1} \ \sup_{\Theta \leq \theta < h} |(\mathbf{F}_1 \widehat{w})(\theta)|_{\infty}, \right.$$

$$\left. \sup_{||\widehat{w}||_{2-}^{[\Theta]} \leq 1} \ \sup_{0 \leq \theta < \Theta} |(\mathbf{F}_2 \widehat{w})(\theta)|_{\infty} \right\} \tag{22}$$

The following subsections are devoted to characterizing the quasi L_∞/L_2 Hankel norms $||\mathbf{H}^{[\Theta]}||$ through the above representation.

3.2 Characterization of $||\mathbf{H}^{[\Theta]}||$

We first note that $\widehat{w}_0(\tau) = 0$ for $\tau \in [\Theta, h)$. Then, it follows from (6)–(8) together with (11), (12) and (14) that, for $\theta \in [\Theta, h)$,

$$(\mathbf{F}_1 \widehat{w})(\theta) = (\mathcal{D}\widehat{w}_0)(\theta) + (\mathcal{CB}\widehat{w}_{-1})(\theta) + (\mathcal{CAB}\widehat{w}_{-2})(\theta) + \cdots$$

$$= \int_0^{\theta} D_\theta(\tau)\widehat{w}_0(\tau)d\tau + \sum_{k=0}^{\infty} \int_0^{h} C_\theta \mathcal{A}^k B_h(\tau)\widehat{w}_{-(k+1)}(\tau)d\tau$$

$$= \int_0^{\Theta} D_\theta(\tau)\widehat{w}_0(\tau)d\tau + \sum_{k=0}^{\infty} \int_0^{h} C_\theta \mathcal{A}^k B_h(\tau)\widehat{w}_{-(k+1)}(\tau)d\tau \tag{23}$$

and, for $\theta \in [0, \Theta)$,

$$(\mathbf{F}_2 \widehat{w})(\theta) = (\mathcal{C}\mathcal{B}\widehat{w}_0)(\theta) + (\mathcal{C}\mathcal{A}\mathcal{B}\widehat{w}_{-1})(\theta) + \cdots$$

$$= \int_0^h C_\theta B_h(\tau)\widehat{w}_0(\tau)d\tau + \sum_{k=1}^\infty \int_0^h C_\theta \mathcal{A}^k B_h(\tau)\widehat{w}_{-k}(\tau)d\tau$$

$$= \int_0^\Theta C_\theta B_h(\tau)\widehat{w}_0(\tau)d\tau + \sum_{k=1}^\infty \int_0^h C_\theta \mathcal{A}^k B_h(\tau)\widehat{w}_{-k}(\tau)d\tau \tag{24}$$

where

$$B_h(\tau) := J_\Sigma \exp(A(h-\tau))B_1 \tag{25}$$

$$D_\theta(\tau) := C_1 \exp(A(\theta-\tau))B_1 \mathbf{1}(\theta-\tau) \tag{26}$$

$$C_\theta := M_1 \exp(A_2\theta)C_\Sigma \tag{27}$$

($\mathbf{1}(t)$ denotes the step function.) Then, we can show that considering the ith entry and applying the continuous-time and discrete-time Cauchy-Schwarz inequalities to (23) and (24) as well as the triangle inequality leads to

$$\sup_{||\widehat{w}||_{2-}^{[\Theta]} \le 1} |(\mathbf{F}_1 \widehat{w})_i(\theta)| = (F_1^{[\Theta]}(\theta))_{ii}^{1/2} \tag{28}$$

$$\sup_{||\widehat{w}||_{2-}^{[\Theta]} \le 1} |(\mathbf{F}_2 \widehat{w})_i(\theta)| = (F_2^{[\Theta]}(\theta))_{ii}^{1/2} \tag{29}$$

where $(\mathbf{F}_1 \widehat{w})_i(\theta)$ and $(\mathbf{F}_2 \widehat{w})_i(\theta)$ denote the ith row of $(\mathbf{F}_1 \widehat{w})(\theta)$ and $(\mathbf{F}_2 \widehat{w})(\theta)$, respectively, and $(F_j^{[\Theta]}(\theta))_{ii}$ ($j = 1, 2$) denote the ith diagonal entry of the following symmetric matrices:

$$F_1^{[\Theta]}(\theta) := \int_0^\Theta D_\theta(\tau)D_\theta^T(\tau)d\tau$$
$$+ \sum_{k=0}^\infty \int_0^h C_\theta \mathcal{A}^k B_h(\tau)B_h^T(\tau)(\mathcal{A}^k)^T C_\theta^T d\tau \tag{30}$$

$$F_2^{[\Theta]}(\theta) := \int_0^\Theta C_\theta B_h(\tau)B_h^T(\tau)C_\theta^T d\tau$$
$$+ \sum_{k=1}^\infty \int_0^h C_\theta \mathcal{A}^k B_h(\tau)B_h^T(\tau)(\mathcal{A}^k)^T C_\theta^T d\tau \tag{31}$$

Hence, we are led from (22) to the following result, which is the main result of this paper.

Theorem 1. $||\mathbf{H}^{[\Theta]}||$ is given by

$$||\mathbf{H}^{[\Theta]}|| = \max \left\{ \sup_{\Theta \le \theta < h} d_{\max}^{1/2}\left(F_1^{[\Theta]}(\theta)\right), \right.$$
$$\left. \sup_{0 \le \theta < \Theta} d_{\max}^{1/2}\left(F_2^{[\Theta]}(\theta)\right) \right\} \tag{32}$$

4. RELATIONSHIP WITH PRECEDING STUDY ON THE L_∞/L_2 HANKEL NORM OF SAMPLED-DATA SYSTEMS

In this section, we discuss some relationship between the arguments in the preceding section and those in our preceding study on *the L_∞/L_2 Hankel norm of the sampled-data system Σ_{SD}*.

In our preceding study (Hagiwara et al., 2016), *the L_∞/L_2 Hankel norm* $||\Sigma_{\mathrm{SD}}||_{\mathrm{H}} = \sup_{\Theta \in [0,h)} ||\mathbf{H}^{[\Theta]}||$ was characterized with alternative arguments *that actually involve no reference to the computation of* $||\mathbf{H}^{[\Theta]}||$ ($\Theta \in [0, h)$). In fact, it was shown that

$$||\Sigma_{\mathrm{SD}}||_{\mathrm{H}} = \sup_{0 \le \theta < h} d_{\max}^{1/2}(F(\theta)) \tag{33}$$

with a readily computable matrix function $F(\theta)$. This $F(\theta)$ ($\theta \in [0, h)$) turns out to have a very close relationship to $F_1^{[\Theta]}(\theta)$ introduced in the preceding section. More specifically, we see that $F(\theta)$ equals $F_1^{[\Theta]}(\theta)$ with Θ set to θ. Since the arguments in Hagiwara et al. (2016) are based on the fact that $d_{\max}^{1/2}(F(\theta))$ ($\theta \in [0, h)$) gives the worst value of $|z(\theta)|_\infty$ for $w \in L_2(-\infty, \theta)$ such that $||w||_{2-}^{[\theta]} \le 1$, it readily follows (after replacing θ with Θ) that

$$d_{\max}^{1/2}(F(\Theta)) \le ||\mathbf{H}^{[\Theta]}|| \quad (\forall \Theta \in [0, h)) \tag{34}$$

To obtain some further insight into the relationship between the results in the preceding study (Hagiwara et al., 2016) and the arguments in the present study, we first claim that (34) can actually be replaced by the following stronger result:

$$d_{\max}^{1/2}(F(\Theta)) \le ||\mathbf{H}^{[\Theta]}|| \le \sup_{0 \le \theta < h} d_{\max}^{1/2}(F(\theta)), \ \forall \Theta \in [0, h) \tag{35}$$

Once we obtain this inequality, it is obvious that we are led to the following result:

$$\sup_{0 \le \Theta < h} ||\mathbf{H}^{[\Theta]}|| = \sup_{0 \le \theta < h} d_{\max}^{1/2}(F(\theta)) \tag{36}$$

Here, the right hand side implies *the L_∞/L_2 Hankel norm* $||\Sigma_{\mathrm{SD}}||_{\mathrm{H}}$ computed *with an alternative method (without referring to the quasi L_∞/L_2 Hankel norms)* through the established assertion (33) of the preceding study. On the other hand, the left hand side of (36) is nothing but the same L_∞/L_2 Hankel norm obtained through the quasi L_∞/L_2 Hankel norm computations over the interval $\Theta \in [0, h)$, as suggested by the definition in (18).

Regarding the derivation of (the second inequality in) (35), we only remark that the positive definiteness of $F_1^{[\theta]}(\theta) - F_1^{[\Theta]}(\theta) = F(\theta) - F_1^{[\Theta]}(\theta)$ for each $\theta \in [\Theta, h)$ and that of $F_1^{[\theta]}(\theta) - F_2^{[\Theta]}(\theta) = F(\theta) - F_2^{[\Theta]}(\theta)$ for each $\theta \in [0, \Theta]$ play a key role.

5. COMPUTATION METHOD OF $F_1^{[\Theta]}(\theta)$ AND $F_2^{[\Theta]}(\theta)$

To facilitate the numerical computation of $||\mathbf{H}^{[\Theta]}||$, we consider how to compute $F_j^{[\Theta]}(\theta)$ ($j = 1, 2$) in (30) and (31). We first introduce

$$W_\theta^{[\Theta]} := \int_0^\Theta \exp(A(\theta-\tau))B_1 B_1^T \exp(A^T(\theta-\tau))d\tau \tag{37}$$

whose numerical computation method is well known. It is easy to see that

$$\int_0^\Theta D_\theta(\tau)D_\theta^T(\tau)d\tau = C_1 W_\theta^{[\Theta]} C_1^T \tag{38}$$

$$\int_\Theta^h C_\theta B_h(\tau)B_h^T(\tau)C_\theta^T d\tau = C_\theta \begin{bmatrix} W_{h-\Theta}^{[h-\Theta]} & 0 \\ 0 & 0 \end{bmatrix} C_\theta^T$$

$$\int_0^h C_\theta \mathcal{A}^k B_h(\tau)B_h^T(\tau)(\mathcal{A}^k)^T C_\theta^T d\tau$$

$$= C_\theta \mathcal{A}^k \begin{bmatrix} W_h^{[h]} & 0 \\ 0 & 0 \end{bmatrix} (\mathcal{A}^k)^T C_\theta^T \quad (k \in \mathbb{N}_0) \tag{39}$$

Hence

$$F_1^{[\Theta]}(\theta) = C_1 W_\theta^{[\Theta]} C_1^T + C_\theta \sum_{k=0}^{\infty} \mathcal{A}^k \begin{bmatrix} W_h^{[h]} & 0 \\ 0 & 0 \end{bmatrix} (\mathcal{A}^k)^T C_\theta^T \tag{40}$$

$$F_2^{[\Theta]}(\theta) = C_\theta \sum_{k=0}^{\infty} \mathcal{A}^k \begin{bmatrix} W_h^{[h]} & 0 \\ 0 & 0 \end{bmatrix} (\mathcal{A}^k)^T C_\theta^T$$
$$- C_\theta \begin{bmatrix} W_{h-\Theta}^{[h-\Theta]} & 0 \\ 0 & 0 \end{bmatrix} C_\theta^T \tag{41}$$

where the infinite series on the right hand side can be computed easily by solving a discrete-time Lyapunov equation.

6. NUMERICAL EXAMPLE

In this section, we confirm the validity of the arguments about $||\mathbf{H}^{[\Theta]}||$ developed in the preceding section through a numerical example.

Let us consider the sampled-data systems Σ_{SD} associate with

$$A = \begin{bmatrix} -3 & 2 \\ -4 & 2 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$C_1 = [1 \ 0], \ C_2 = [1 \ 0], \ D_{12} = 0$$
$$A_\Psi = \begin{bmatrix} -2.1856 & 2.3760 \\ -1.1133 & 1.2103 \end{bmatrix}, \ B_\Psi = \begin{bmatrix} 0.0176 \\ 0.0090 \end{bmatrix}$$
$$C_\Psi = [-0.7610 \ 0.8273], \ D_\Psi = -0.2367 \tag{42}$$

and $h = 2$. We compute the quasi L_∞/L_2 Hankel norms $||\mathbf{H}^{[\Theta]}||$ at $\Theta \in [0, h)$.

The computation results are shown in Fig. 2 with the solid line together with $d_{\max}^{1/2}(F(\theta)) \ (= (F_1^{[\theta]}(\theta))^{1/2})$ shown with the dashed line. Fig. 3 is relevant to the computation of $||\mathbf{H}^{[\Theta]}||$ through (32); the solid line shows how $\theta^\star$ depends on $\Theta \in [0, h)$ (while the dashed lines correspond to $\theta^\star = \Theta$ and $\theta^\star = \Theta + h$), where $\theta^\star = \theta^\star(\Theta)$ is defined as such an instant that satisfies $||\mathbf{H}^{[\Theta]}|| = |z(\theta^\star)|_\infty$ for the worst $w \in L_2(-\infty, \Theta)$ of unit magnitude:

$$\theta^\star(\Theta) = \begin{cases} \arg \max_{\theta \in [\Theta, h]} d_{\max}^{1/2}(F_1^{[\Theta]}(\theta)) \\ \qquad (\text{if } ||\mathbf{H}^{[\Theta]}|| = \max_{\theta \in [\Theta, h]} d_{\max}^{1/2}(F_1^{[\Theta]}(\theta))) \\ h + \arg \max_{\theta \in [0, \Theta]} d_{\max}^{1/2}(F_2^{[\Theta]}(\theta)) \\ \qquad (\text{if } ||\mathbf{H}^{[\Theta]}|| = \max_{\theta \in [0, \Theta]} d_{\max}^{1/2}(F_2^{[\Theta]}(\theta))) \end{cases} \tag{43}$$

First, we can confirm the relations (35) and (36) from Fig. 2, where the latter implies that the gap about the second inequality in the former relation vanishes as Θ is swept over $[0, h)$ and the associated supremum is taken.

Furthermore, we can see from Figs. 2 and 3 that $||\mathbf{H}^{[\Theta]}|| = d_{\max}^{1/2}(F(\Theta))$ when Θ satisfies $\theta^\star(\Theta) = \Theta$ while $||\mathbf{H}^{[\Theta]}|| > d_{\max}^{1/2}(F(\Theta))$ when Θ satisfies $\theta^\star(\Theta) > \Theta$. This can be seen as a very natural consequence (and thus supports the validity of our computation results) if we recall (see the arguments below (33)) that $d_{\max}^{1/2}(F(\Theta)) \ (\Theta \in [0, h))$ gives the worst value of $|z(\Theta)|_\infty$ for $w \in L_2(-\infty, \Theta)$ such that $||w||_{2-}^{[\Theta]} \le 1$.
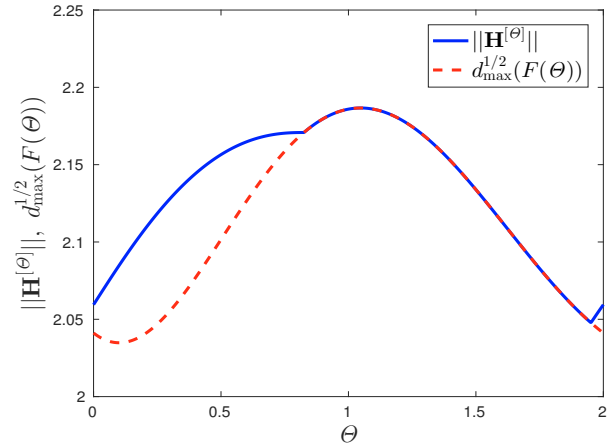

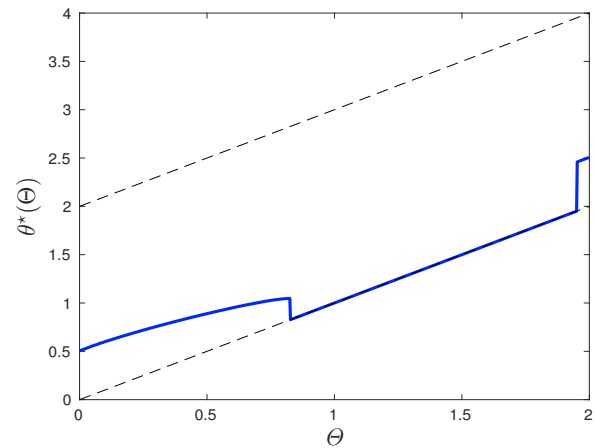Fig. 2. Quasi L_∞/L_2 Hankel norm $||\mathbf{H}^{[\Theta]}||$ at Θ

Fig. 3. Θ-Dependence of $\theta^\star$ defined by $||\mathbf{H}^{[\Theta]}|| = |z(\theta^\star)|_\infty$ for the worst $w \in L_2(-\infty, \Theta)$ of unit magnitude.

7. CONCLUSION

In this paper, we tackled the problem of characterizing the quasi L_∞/L_2 Hankel norm $||\mathbf{H}^{[\Theta]}||$ at $\Theta \in [0, h)$. We remark that the development in this paper allows us to compute the root mean square (RMS) of the quasi L_∞/L_2 Hankel norm $||\mathbf{H}^{[\Theta]}||$ over Θ in $[0, h)$. This value, as well as the L_∞/L_2 Hankel norm $||\Sigma_{\mathrm{SD}}||_H = \sup_{\Theta \in [0,h)} ||\mathbf{H}^{[\Theta]}||$ itself, could be used as new definitions of the (generalized) H_2 norm of the sampled-data system Σ_{SD}. This is because $||\mathbf{H}^{[\Theta]}||$ is independent of Θ and coincides with the H_2 norm for every $\Theta \in [0, h)$ in the special case when Σ_{SD} is actually a single-input single-output (SISO) linear time-invariant (LTI) continuous-time system (and thus both of the above two values also coincide with its H_2 norm). These definitions are believed to be different from the standard definition (Bamieh and Pearson, 1992b), and thus it would be important to study the mutual relationship among these definitions of the H_2 norm of sampled-data systems.

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