



# Properties of the Discrete Hilbert Transform

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## Abstract

The asymptotic behavior of the distribution function of the Hilbert transform of sequences from the class  $l_1$  is studied. The concept of  $Q$ -summability of series is introduced; using this notion, it is shown that the Hilbert transform of a sequence from the class  $l_1$  is  $Q$ -summable and is  $Q$ -sum is zero.

**Keywords** Discrete Hilbert transform · Asymptotic behavior of the distribution function ·  $Q$ -integral ·  $Q$ -summability

**Mathematics Subject Classification** 44A15 · 26A39 · 40G99

## 1 Introduction

Let  $\{b_n\}_{n \in \mathbb{Z}} \in l_1$ . The sequence

$$\tilde{b}_n = \sum_{m \neq n} \frac{b_m}{n - m}, \quad n \in \mathbb{Z},$$

is called the Hilbert transform of the sequence  $\{b_n\}_{n \in \mathbb{Z}}$ .

M. Riesz (see [18, see also 10, 15]) proved that if  $\{b_n\}_{n \in \mathbb{Z}} \in l_p$ ,  $p > 1$ , then  $\{\tilde{b}_n\}_{n \in \mathbb{Z}} \in l_p$  and the inequality

$$\|\tilde{b}_n\|_p \leq C_p \|b_n\|_p \quad (1)$$

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holds. Weighted analogues of (1) are investigated in the works [7–9, 13, 14, 16, 17, 21].

If  $\{b_n\}_{n \in \mathbb{Z}} \in l_1$ , then the sequence  $\{\tilde{b}_n\}_{n \in \mathbb{Z}}$  belong to the class  $\bigcap_{p>1} l_p$ , but it does not belong to the class  $l_1$ . In this case, R. Hunt, B. Muckenhoupt and R. Wheeden (see [14]) proved that the distribution function  $\tilde{b}(\lambda) = \sum_{\{n \in \mathbb{Z}: |\tilde{b}_n| > \lambda\}} 1$  of the Hilbert transform of the sequence  $\{b_n\}_{n \in \mathbb{Z}}$  satisfies the condition

$$\forall \lambda > 0: \left| \tilde{b}(\lambda) \right| \leq \frac{C_0}{\lambda} \sum_{n \in \mathbb{Z}} |b_n|, \quad (2)$$

where  $C_0$  is an absolute constant. Note that for the sequence  $\{b_n\}_{n \in \mathbb{Z}} \in l_1$  the series  $\sum_{n \in \mathbb{Z}} \tilde{b}_n$  does not converge even in the sense of the principal value, i.e., in the sense

$$\sum_{n \in \mathbb{Z}} \tilde{b}_n = \lim_{N \rightarrow \infty} \sum_{|n| \leq N} \tilde{b}_n.$$

In the present paper, we study the asymptotic behavior of the distribution function  $\tilde{b}(\lambda)$  of the Hilbert transform of a sequence  $\{b_n\}_{n \in \mathbb{Z}} \in l_1$  as  $\lambda \rightarrow 0$  (Theorem 1). We introduce the concept of  $Q$ -summability of series and, using this notion, prove that the Hilbert transform of a sequence  $\{b_n\}_{n \in \mathbb{Z}} \in l_1$  is  $Q$ -summable and its  $Q$ -sum is zero (Theorem 2).

## 2 Asymptotic Behavior of the Distribution Function of the Discrete Hilbert Transform

**Theorem 1** *Let  $\{b_n\}_{n \in \mathbb{Z}} \in l_1$ . Then*

$$\lim_{\lambda \rightarrow 0+} \lambda \cdot \tilde{b}(\lambda) = 2 \left| \sum_{n \in \mathbb{Z}} b_n \right|, \quad (3)$$

where  $\tilde{b}(\lambda) = \sum_{\{n \in \mathbb{Z}: |\tilde{b}_n| > \lambda\}} 1$  is the distribution function of the Hilbert transform of  $\{b_n\}_{n \in \mathbb{Z}}$ .

We first prove the auxiliary lemma.

**Lemma 1** *Let  $\{b_n\}_{n \in \mathbb{Z}} \in l_1$  and  $\sum_{n \in \mathbb{Z}} b_n = 0$ . Then*

$$\tilde{b}(\lambda) = o\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow 0+. \quad (4)$$

**Proof of Lemma 1** We first assume that a sequence  $\{b_n\}_{n \in \mathbb{Z}} \in l_1$  is concentrated on some finite interval  $[-m, m]$ , that is,  $b_n = 0$  for  $|n| > m$ . For every  $|k| < m$  and  $|n| > 2m$ , we have

$$|n - k| \geq |n| - |k| > |n| - m > |n| - \frac{|n|}{2} = \frac{|n|}{2}, \quad \left|n - \frac{1}{2}\right| \geq |n| - \frac{1}{2} \geq \frac{|n|}{2}.$$

Therefore, in this case, for  $|n| > 2m$  from the equality

$$\begin{aligned} \tilde{b}_n &= \sum_{k \neq n} \frac{b_k}{n - k} = \sum_{|k| \leq m} \frac{b_k}{n - k} = \sum_{|k| \leq m} \frac{b_k}{n - k} - \frac{1}{n - 1/2} \sum_{k \in Z} b_k \\ &= \sum_{|k| \leq m} \frac{b_k}{n - k} - \frac{1}{n - 1/2} \sum_{|k| \leq m} b_k = \sum_{|k| \leq m} \frac{k - 1/2}{(n - k)(n - 1/2)} b_k \end{aligned}$$

we obtain

$$\left| \tilde{b}_n \right| \leq \sum_{|k| \leq m} \frac{|k - 1/2|}{|n - k||n - 1/2|} b_k \leq \frac{4}{n^2} \sum_{|k| \leq m} |k - 1/2| |b_k|. \quad (5)$$

Denote

$$M_0 = \sum_{|k| \leq m} |k - 1/2| |b_k|.$$

Then it follows from (5) that

$$\begin{aligned} \left\{ n \in Z : \left| \tilde{b}_n \right| > \lambda \right\} &\subset \{n \in Z : |n| \leq 2m\} \cup \left\{ n \in Z \setminus [-2m, 2m] : \frac{4}{n^2} M_0 > \lambda \right\} \\ &= \{n \in Z : |n| \leq 2m\} \cup \left\{ n \in Z \setminus [-2m, 2m] : |n| < 2\sqrt{M_0/\lambda} \right\}. \end{aligned}$$

Hence we have

$$\tilde{b}(\lambda) = \sum_{\{n \in Z : |\tilde{b}_n| > \lambda\}} 1 \leq \sum_{\{n \in Z : |n| \leq 2m\}} 1 + \sum_{\{n \in Z \setminus [-2m, 2m] : |n| < 2\sqrt{M_0/\lambda}\}} 1 \leq 4m + 4\sqrt{M_0/\lambda} + 2,$$

whence the asymptotic Eq. (4) follows.

Now let us consider the general case when a sequence  $\{b_n\}_{n \in Z} \in l_1$  is concentrated on  $Z$ . It follows from the condition  $\{b_n\}_{n \in Z} \in l_1$  that, for any  $\varepsilon > 0$ , there exist a number  $m_\varepsilon \in N$  satisfying

$$\sum_{|n| > m_\varepsilon} |b_n| < \frac{\varepsilon}{8C_0},$$

where  $C_0$  is the constant from (2). Setting

$$b'_n = \begin{cases} 0, & \text{for } |n| > m_\varepsilon \\ b_n - \frac{1}{2m_\varepsilon + 1} \sum_{|k| \leq m_\varepsilon} b_k, & \text{for } |n| \leq m_\varepsilon \end{cases}, \quad b''_n = \begin{cases} b_n, & \text{for } |n| > m_\varepsilon \\ \frac{1}{2m_\varepsilon + 1} \sum_{|k| \leq m_\varepsilon} b_k, & \text{for } |n| \leq m_\varepsilon \end{cases}$$

we have

$$b_n = b'_n + b''_n,$$

the sequence  $\{b'_n\}_{n \in \mathbb{Z}} \in l_1$  is concentrated on the finite interval  $[-m_\varepsilon, m_\varepsilon]$ ,

$$\begin{aligned} \sum_{n \in \mathbb{Z}} b'_n &= \sum_{|n| \leq m_\varepsilon} b'_n = \sum_{|n| \leq m_\varepsilon} \left[ b_n - \frac{1}{2m_\varepsilon + 1} \sum_{|k| \leq m_\varepsilon} b_k \right] \\ &= \sum_{|n| \leq m_\varepsilon} b_n - \frac{1}{2m_\varepsilon + 1} (2m_\varepsilon + 1) \sum_{|k| \leq m_\varepsilon} b_k = 0; \end{aligned}$$

and the sequence  $\{b''_n\}_{n \in \mathbb{Z}} \in l_1$  satisfies the inequality

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |b''_n| &= \sum_{|n| > m_\varepsilon} |b''_n| + \sum_{|n| \leq m_\varepsilon} |b''_n| = \sum_{|n| > m_\varepsilon} |b_n| + \sum_{|n| \leq m_\varepsilon} \left| \frac{1}{2m_\varepsilon + 1} \sum_{|k| \leq m_\varepsilon} b_k \right| \\ &= \sum_{|n| > m_\varepsilon} |b_n| + \left| \sum_{|k| \leq m_\varepsilon} b_k \right| = \sum_{|n| > m_\varepsilon} |b_n| + \left| \sum_{k \in \mathbb{Z}} b_k - \sum_{|k| > m_\varepsilon} b_k \right| \\ &= \sum_{|n| > m_\varepsilon} |b_n| + \left| \sum_{|k| > m_\varepsilon} b_k \right| \leq 2 \sum_{|n| > m_\varepsilon} |b_n| < \frac{\varepsilon}{4C_0}. \end{aligned} \tag{6}$$

Since the sequence  $\{b'_n\}_{n \in \mathbb{Z}} \in l_1$  is concentrated on  $[-m_\varepsilon, m_\varepsilon]$  and  $\sum_{n \in \mathbb{Z}} b'_n = 0$ , then for the sequence  $\{b'_n\}_{n \in \mathbb{Z}} \in l_1$  Eq. (4) is satisfied, and therefore, there exists  $\lambda(\varepsilon) > 0$  such that, for  $0 < \lambda < \lambda(\varepsilon)$ ,

$$\lambda \tilde{b}'\left(\frac{\lambda}{2}\right) < \frac{\varepsilon}{2}, \tag{7}$$

where  $\tilde{b}'(\lambda) = \sum_{\{n \in \mathbb{Z}: |\tilde{b}'_n| > \lambda\}} 1$ . On the other hand, from (2) and (6) it follows that

$$\lambda \tilde{b}''\left(\frac{\lambda}{2}\right) \leq 2C_0 \sum_{n \in \mathbb{Z}} |b''_n| < \frac{\varepsilon}{2} \tag{8}$$

for any  $\lambda > 0$ , where  $\tilde{b}''(\lambda) = \sum_{\{n \in \mathbb{Z}: |\tilde{b}''_n| > \lambda\}} 1$ .

Since  $\tilde{b}_n = \tilde{b}'_n + \tilde{b}''_n$  for every  $n \in \mathbb{Z}$ , we have, for every  $\lambda > 0$ ,

$$\{n \in \mathbb{Z} : |\tilde{b}_n| > \lambda\} \subset \{n \in \mathbb{Z} : |\tilde{b}'_n| > \lambda/2\} \cup \{n \in \mathbb{Z} : |\tilde{b}''_n| > \lambda/2\}. \tag{9}$$

For any  $0 < \lambda < \lambda(\varepsilon)$  from inequalities (7), (8) and inclusion (9) we get

$$\tilde{b}(\lambda) = \sum_{\{n \in \mathbb{Z}: |\tilde{b}_n| > \lambda\}} 1 \leq \sum_{\{n \in \mathbb{Z}: |\tilde{b}'_n| > \lambda/2\}} 1 + \sum_{\{n \in \mathbb{Z}: |\tilde{b}''_n| > \lambda/2\}} 1 = \tilde{b}'(\lambda/2) + \tilde{b}''(\lambda/2) < \frac{\varepsilon}{\lambda}.$$

This shows that equality (4) is valid for all  $\{b_n\}_{n \in \mathbb{Z}} \in l_1$  satisfying the condition  $\sum_{n \in \mathbb{Z}} b_n = 0$ . This completes the proof of Lemma 1.

**Proof of Theorem 1** In the case  $\sum_{n \in \mathbb{Z}} b_n = 0$  the assertion of the theorem follows from Lemma 1. Let us consider the case  $\sum_{n \in \mathbb{Z}} b_n = \alpha \neq 0$ . Puttig

$$b'_n = \begin{cases} b_n, & \text{for } n \neq 0 \\ b_0 - \alpha, & \text{for } n = 0 \end{cases}, \quad b''_n = \begin{cases} 0, & \text{for } n \neq 0 \\ \alpha, & \text{for } n = 0 \end{cases}$$

we have  $b_n = b'_n + b''_n$  and  $\sum_{n \in \mathbb{Z}} b'_n = 0$ . It follows from Lemma 1 that

$$\tilde{b}'(\lambda) = o\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow 0+. \quad (10)$$

Since  $\tilde{b}''_n = \frac{\alpha}{n}$  for  $n \neq 0$ ,  $\tilde{b}''_0 = 0$ , we have

$$\tilde{b}''(\lambda) = \sum_{\{n \in \mathbb{Z}: |\tilde{b}''_n| > \lambda\}} 1 = \sum_{\{n \in \mathbb{Z} \setminus \{0\}: |\alpha/n| > \lambda\}} 1 = \sum_{\{n \in \mathbb{Z} \setminus \{0\}: |n| < \alpha/\lambda\}} 1 = 2[|\alpha/\lambda|], \quad (11)$$

where  $[|\alpha/\lambda|]$  is the integer part of the number  $|\alpha/\lambda|$ .

Since  $b_n = \tilde{b}'_n + \tilde{b}''_n$  for every  $n \in \mathbb{Z}$ , we have, for any  $0 < \varepsilon < 1$ ,

$$\begin{aligned} \{n \in \mathbb{Z}: |\tilde{b}_n| > \lambda\} &\subset \{n \in \mathbb{Z}: |\tilde{b}'_n| > \varepsilon\lambda\} \cup \{n \in \mathbb{Z}: |\tilde{b}''_n| > (1 - \varepsilon)\lambda\} \\ \{n \in \mathbb{Z}: |\tilde{b}_n| > \lambda\} &\supset \{n \in \mathbb{Z}: |\tilde{b}''_n| > (1 + \varepsilon)\lambda\} \setminus \{n \in \mathbb{Z}: |\tilde{b}'_n| > \varepsilon\lambda\}. \end{aligned}$$

Hence

$$\begin{aligned} \tilde{b}(\lambda) &= \sum_{\{n \in \mathbb{Z}: |\tilde{b}_n| > \lambda\}} 1 \leq \sum_{\{n \in \mathbb{Z}: |\tilde{b}'_n| > \varepsilon\lambda\}} 1 + \sum_{\{n \in \mathbb{Z}: |\tilde{b}''_n| > (1 - \varepsilon)\lambda\}} 1 \\ &= \tilde{b}'(\varepsilon\lambda) + \tilde{b}''((1 - \varepsilon)\lambda), \\ \tilde{b}(\lambda) &= \sum_{\{n \in \mathbb{Z}: |\tilde{b}_n| > \lambda\}} 1 \geq \sum_{\{n \in \mathbb{Z}: |\tilde{b}''_n| > (1 + \varepsilon)\lambda\}} 1 - \sum_{\{n \in \mathbb{Z}: |\tilde{b}'_n| > \varepsilon\lambda\}} 1 \\ &= \tilde{b}''((1 + \varepsilon)\lambda) - \tilde{b}'(\varepsilon\lambda). \end{aligned}$$

Now, using (10), (11),

$$\begin{aligned} \limsup_{\lambda \rightarrow 0^+} \lambda \cdot \tilde{b}(\lambda) &\leq \limsup_{\lambda \rightarrow 0^+} \lambda \cdot \left[ \tilde{b}'(\varepsilon\lambda) + \tilde{b}''((1-\varepsilon)\lambda) \right] \\ &= \limsup_{\lambda \rightarrow 0^+} \lambda \cdot 2 \left[ \left| \frac{\alpha}{(1-\varepsilon)\lambda} \right| \right] = \frac{2|\alpha|}{1-\varepsilon}, \end{aligned} \tag{12}$$

$$\begin{aligned} \liminf_{\lambda \rightarrow 0^+} \lambda \cdot \tilde{b}(\lambda) &\geq \liminf_{\lambda \rightarrow 0^+} \lambda \cdot \left[ \tilde{b}''((1+\varepsilon)\lambda) - \tilde{b}'(\varepsilon\lambda) \right] \\ &= \liminf_{\lambda \rightarrow 0^+} \lambda \cdot 2 \left[ \left| \frac{\alpha}{(1+\varepsilon)\lambda} \right| \right] = \frac{2|\alpha|}{1+\varepsilon}. \end{aligned} \tag{13}$$

Since  $\varepsilon$  is arbitrary, it follows from (12), (13) that

$$\liminf_{\lambda \rightarrow 0^+} \lambda \cdot \tilde{b}(\lambda) = \limsup_{\lambda \rightarrow 0^+} \lambda \cdot \tilde{b}(\lambda) = 2|\alpha|.$$

Hence (3) holds. This completes the proof of Theorem 1.

### 3 Q-Summability of Series and the Hilbert Transform

For a measurable complex function  $f$  on an interval  $[a, b] \subset \mathbb{R}$ , we set

$$\begin{aligned} [f(x)]_n &= [f(x)]^n = f(x) \quad \text{for } |f(x)| \leq n, \\ [f(x)]_n &= n \cdot \operatorname{sgn} f(x), \quad [f(x)]^n = 0 \quad \text{for } |f(x)| > n, \quad n \in \mathbb{N}, \end{aligned}$$

where  $\operatorname{sgnz} = z/|z|$  for  $z \neq 0$  and  $\operatorname{sgn}0 = 0$ .

In 1929, Titchmarsh [22] introduced the notions of  $Q$ - and  $Q'$ -integrals of a function measurable on  $[a, b]$ .

**Definition 1** If the finite limit  $\lim_{n \rightarrow \infty} \int_a^b [f(x)]_n dx$  ( $\lim_{n \rightarrow \infty} \int_a^b [f(x)]^n dx$ , respectively) exists, then  $f$  is said to be  $Q$ -integrable ( $Q'$ -integrable, respectively) on  $[a, b]$ ; that is,  $f \in Q[a, b]$  ( $f \in Q'[a, b]$ ). The value of this limit is referred to as the  $Q$ -integral ( $Q'$ -integral) of this function and is denoted by

$$({Q}) \int_a^b f(x) dx \quad \left( (Q') \int_a^b f(x) dx \right).$$

As in Definition 1, one defines the  $Q$ - and  $Q'$ -integrals for a function measurable on the real axis  $\mathbb{R}$ .

Given a measurable complex function  $f$  on the real axis  $\mathbb{R}$ , we set

$$[f(x)]_{\delta, \lambda} = [f(x)]^{\delta, \lambda} = f(x) \quad \text{for } \delta \leq |f(x)| \leq \lambda,$$

$$[f(x)]_{\delta, \lambda} = [f(x)]^{\delta, \lambda} = 0 \text{ for } |f(x)| < \delta,$$

$$[f(x)]_{\delta, \lambda} = \lambda \operatorname{sgn} f(x), \quad [f(x)]^{\delta, \lambda} = 0 \text{ for } |f(x)| > \lambda, \quad 0 < \delta < \lambda.$$

**Definition 2** If the finite limit  $\lim_{\substack{\delta \rightarrow 0+ \\ \lambda \rightarrow +\infty}} \int_R [f(x)]_{\delta, \lambda} dx$

( $\lim_{\delta \rightarrow 0+} \int_R [f(x)]^{\delta, \lambda} dx$ , respectively) exists, then  $f$  is said to be  $Q$ -integrable ( $Q'$ -integrable, respectively) on  $R$ ; that is,  $f \in Q(R)$  ( $f \in Q'(R)$ ). The value of this limit is referred to as the  $Q$ -integral ( $Q'$ -integral) of this function and is denoted by

$$({Q}) \int_R f(x) dx \quad \left( (Q') \int_R f(x) dx \right).$$

A very uncomfortable fact impeding the application of  $Q$ -integrals and  $Q'$ -integrals when dealing with diverse problems of function theory is the absence of the additivity property; that is, the  $Q$ -integrability ( $Q'$ -integrability) of two functions does not imply the  $Q$ -integrability ( $Q'$ -integrability) of their sum. If one adds the conditions

$$\delta m\{x \in R : |f(x)| > \delta\} = o(1), \quad \delta \rightarrow 0+, \tag{14}$$

$$\lambda m\{x \in R : |f(x)| > \lambda\} = o(1), \quad \lambda \rightarrow +\infty, \tag{15}$$

to the definition of  $Q$ -integrability ( $Q'$ -integrability) of a function  $f$ , then the  $Q$ -integral and  $Q'$ -integral coincide ( $Q(R) = Q'(R)$ ) and these integrals become additive.

**Definition 3** If  $f \in Q'(R)$  (or  $f \in Q(R)$ ) and conditions (14) and (15) holds, then  $f$  is said to be  $A$ -integrable on  $R$ ; that is,  $f \in A(R)$ . In this case, the limit  $\lim_{\substack{\delta \rightarrow 0+ \\ \lambda \rightarrow +\infty}} \int_R [f(x)]^{\delta, \lambda} dx$  (or the limit  $\lim_{\substack{\delta \rightarrow 0+ \\ \lambda \rightarrow +\infty}} \int_R [f(x)]_{\delta, \lambda} dx$ ) is denoted by

$$(A) \int_R f(x) dx.$$

Properties of  $Q$ - and  $Q'$ -integrals were investigated in [2, 3, 6, 11, 12, 22]; for the applications of  $A$ -,  $Q$ - and  $Q'$ -integrals in the theory of functions of real and complex variables we refer the reader to [1–6, 19, 20, 22, 23].

We need the following theorem proved in [3] and [4].

**Theorem A [4, Theorem 4]** Let  $\nu$  be a finite complex measure on the real axis  $R$ . Then

$$(Q') \int_R (H\nu)(x) dx = 0,$$

where  $(H\nu)(x) = \frac{1}{\pi} \int_R \frac{d\nu(t)}{x-t}$  is the Hilbert transform of the measure  $\nu$ .

**Definition 4** We denote by  $M(R; C)$  the class of measurable complex-valued functions  $f$  on  $R$  for which the finite limits  $\lim_{\lambda \rightarrow +\infty} \lambda m\{z \in R : |f(z)| > \lambda\}$  and  $\lim_{\delta \rightarrow 0+} \delta m\{z \in R : |f(z)| > \delta\}$  exist.

**Remark 1** Note that the Hilbert transform of a finite complex measure belong to the class of functions  $M(R; C)$  (see [4]).

**Theorem B [3, Theorem 2.3]** If a function  $f \in M(R; C)$  is  $Q'$ -integrable on  $R$  and a function  $g$  is  $A$ -integrable on  $R$ , then their sum  $f + g \in M(R; C)$  is  $Q'$ -integrable on  $R$ , and

$$(Q') \int_R [f(x) + g(x)] dx = (Q') \int_R f(x) dx + (A) \int_R g(x) dx.$$

Similar to the definition of the  $Q$ -integral, we define the  $Q$ -sum of series. Let  $\{a_n\}_{n \in Z}$  be a sequence of complex numbers.

**Definition 5** If the finite limit  $\lim_{\lambda \rightarrow 0+} \sum_{\{n \in Z: |a_n| \geq \lambda\}} a_n$  exists, then the series  $\sum_{n \in Z} a_n$  is said to be  $Q$ -summable, and the value of this limit is referred to as the  $Q$ -sum of this series and is denoted by

$$(Q) \sum_{n \in Z} a_n.$$

$Q$ -summable series does not enjoy the additivity property; that is, the  $Q$ -summability of two series does not imply the  $Q$ -summability of their sum. If one adds the condition

$$\sum_{\{n \in Z: |a_n| > \lambda\}} 1 = o\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow 0+ \tag{16}$$

to the definition of  $Q$ -summability of a series  $\sum_{n \in Z} a_n$ , then the  $Q$ -sum become additive.

**Definition 6** If a series  $\sum_{n \in Z} a_n$  is  $Q$ -summable and condition (16) holds, then the series  $\sum_{n \in Z} a_n$  is said to be  $A$ -summable, and the limit  $\lim_{\lambda \rightarrow 0+} \sum_{\{n \in Z: |a_n| \geq \lambda\}} a_n$  is denoted in this case by  $(A) \sum_{n \in Z} a_n$ .

**Theorem 2** Let  $\{b_n\}_{n \in Z} \in l_1$ . Then the series  $\sum_{n \in Z} \tilde{b}_n$  is  $Q$ -summable and the equation

$$(Q) \sum_{n \in Z} \tilde{b}_n = 0 \tag{17}$$

holds.

**Proof of Theorem 2** Define the function  $f(x)$  to be  $2\pi b_n$  for  $x \in [n - 1/4, n + 1/4]$ ,  $n \in Z$  and 0 elsewhere, the function  $F(x)$  to be  $\tilde{b}_n$  for  $x \in [n - 1/2, n + 1/2]$ ,  $n \in Z$  and



$$G(x) = (Hf)(x) - F(x).$$

We first show that  $G_1(x) \in L_1(\mathbb{R})$ .

For every  $x \in [n - 1/2, n + 1/2]$ ,  $x \neq n \pm 1/4$  we have

$$\begin{aligned} G(x) &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{x-t} dt - \tilde{b}_n = \frac{1}{\pi} \sum_{m \in \mathbb{Z}_{m-1/4}} \int_{m-1/4}^{m+1/4} \frac{2\pi b_m}{x-t} dt - \sum_{m \neq n} \frac{b_m}{n-m} \\ &= \left( \sum_{m \neq n} 2b_m \int_{m-1/4}^{m+1/4} \frac{dt}{x-t} + 2b_n \int_{n-1/4}^{n+1/4} \frac{dt}{x-t} \right) - \sum_{m \neq n} \frac{b_m}{n-m} \\ &= \sum_{m \neq n} 2b_m \left( \int_{m-1/4}^{m+1/4} \left( \frac{1}{x-t} - \frac{1}{n-m} \right) dt \right) + 2b_n \int_{n-1/4}^{n+1/4} \frac{dt}{x-t} = G_1(x) + G_2(x). \end{aligned} \tag{18}$$

Let  $m \neq n$ . Then, for every  $x \in [n - 1/2, n + 1/2]$  and  $t \in [m - 1/4, m + 1/4]$ , since

$$|x - n| \leq 1/2, \quad |m - t| \leq 1/4, \quad |x - t| \geq |n - m| - |x - n| - |m - t| \geq |n - m| - 3/4$$

we get

$$\left| \frac{1}{x-t} - \frac{1}{n-m} \right| = \frac{|n-x+t-m|}{|x-t| \cdot |n-m|} \leq \frac{1/2+1/4}{|n-m| \cdot (|n-m| - 3/4)} = \frac{3}{|n-m| \cdot (4|n-m| - 3)}. \tag{19}$$

Therefore, for every  $x \in [n - 1/2, n + 1/2]$ ,

$$|G_1(x)| \leq \sum_{m \neq n} 2|b_m| \cdot \int_{m-1/4}^{m+1/4} \left| \frac{1}{x-t} - \frac{1}{n-m} \right| dt \leq \sum_{m \neq n} 3|b_m| \cdot \frac{1}{|n-m| \cdot (4|n-m| - 3)}. \tag{20}$$

From inequality (20) it follows that

$$\int_{\mathbb{R}} |G_1(x)| dx = \sum_{n \in \mathbb{Z}_{n-1/2}} \int_{n-1/2}^{n+1/2} |G_1(x)| dx \leq \sum_{n \in \mathbb{Z}} \sum_{m \neq n} 3|b_m| \cdot \frac{1}{|n-m| \cdot (4|n-m| - 3)}. \tag{21}$$

Since for every  $m \in \mathbb{Z}$  the series  $\sum_{n \neq m} \frac{1}{|n-m| \cdot (4|n-m| - 3)} = \sum_{k \neq 0} \frac{1}{|k| \cdot (4|k| - 3)}$  is convergent, we have from (21)

$$\begin{aligned} \int_R |G_1(x)| dx &\leq \sum_{m \in \mathbb{Z}} 3|b_m| \sum_{n \neq m} \frac{1}{|n-m| \cdot (4|n-m|-3)} \\ &= 3 \sum_{m \in \mathbb{Z}} |b_m| \cdot \sum_{k \neq 0} \frac{1}{|k| \cdot (4|k|-3)} = 3 \sum_{k \neq 0} \frac{1}{|k| \cdot (4|k|-3)} \cdot \sum_{m \in \mathbb{Z}} |b_m| \end{aligned}$$

and, therefore,  $G_1(x) \in L_1(R)$ .

Let us show that  $G_2(x) \in L_1(R)$ .

For every  $n \in \mathbb{Z}$  we subdivide the set  $[n-1/2, n+1/2] \setminus \{n-1/4, n+1/4\}$  into four parts:  $[n-1/2, n-1/4)$ ,  $(n-1/4, n]$ ,  $(n, n+1/4)$ ,  $(n+1/4, n+1/2)$ .

If  $x \in [n-1/2, n-1/4)$ , then

$$G_2(x) = 2b_n \int_{n-1/4}^{n+1/4} \frac{dt}{x-t} = 2b_n [-\ln(n+1/4-x) + \ln(n-1/4-x)];$$

For every  $x \in R$  and  $\delta > 0$ , the equality

$$\text{v.p.} \int_{x-\delta}^{x+\delta} \frac{dt}{x-t} = \text{v.p.} \int_{-\delta}^{\delta} \frac{du}{-u} = 0$$

holds. Therefore, if  $x \in (n-1/4, n]$ , then

$$\begin{aligned} G_2(x) &= 2b_n \text{v.p.} \int_{n-1/4}^{n+1/4} \frac{dt}{x-t} = 2b_n \left( \text{v.p.} \int_{x-(x-n+1/4)}^{x+(x-n+1/4)} \frac{dt}{x-t} + \int_{x+(x-n+1/4)}^{n+1/4} \frac{dt}{x-t} \right) \\ &= 2b_n \int_{2x-n+1/4}^{n+1/4} \frac{dt}{x-t} = 2b_n [-\ln(n+1/4-x) + \ln(x-n+1/4)] \end{aligned}$$

if  $x \in (n, n+1/4)$ , then

$$\begin{aligned} G_2(x) &= 2b_n \text{v.p.} \int_{n-1/4}^{n+1/4} \frac{dt}{x-t} = 2b_n \left( \text{v.p.} \int_{n-1/4}^{x-(n+1/4-x)} \frac{dt}{x-t} + \int_{x-(n+1/4-x)}^{x+(n+1/4-x)} \frac{dt}{x-t} \right) \\ &= 2b_n \int_{n-1/4}^{2x-n-1/4} \frac{dt}{x-t} = 2b_n [-\ln(n+1/4-x) + \ln(x-n+1/4)]; \end{aligned}$$

if  $x \in (n + 1/4, n + 1/2)$ , then

$$G_2(x) = 2b_n \int_{n-1/4}^{n+1/4} \frac{dt}{x-t} = 2b_n [-\ln(x-n-1/4) + \ln(x-n+1/4)].$$

This shows that for every  $x \in [n - 1/2, n + 1/2)$ ,  $x \neq n \pm 1/4$  we have

$$G_2(x) = 2b_n [-\ln|x-n-1/4| + \ln|x-n+1/4|] \quad (22)$$

and, therefore,

$$|G_2(x)| \leq |2b_n| \left[ \ln \frac{1}{|x-n-1/4|} + \ln \frac{1}{|x-n+1/4|} \right].$$

Now, for every  $n \in \mathbb{Z}$ ,

$$\int_{n-1/2}^{n+1/2} |G_2(x)| dx \leq |2b_n| \cdot M_1,$$

where  $M_1 = \int_{n-1/2}^{n+1/2} \left[ \ln \frac{1}{|x-n-1/4|} + \ln \frac{1}{|x-n+1/4|} \right] dx = \int_{-1/2}^{1/2} \left[ \ln \frac{1}{|u-1/4|} + \ln \frac{1}{|u+1/4|} \right] du$ . Therefore,

$$\int_{\mathbb{R}} |G_2(x)| dx = \sum_{n \in \mathbb{Z}} \int_{n-1/2}^{n+1/2} |G_2(x)| dx \leq 2M_1 \sum_{n \in \mathbb{Z}} |b_n|.$$

It follows that  $G_2(x) \in L_1(\mathbb{R})$ , and hence  $G(x) \in L_1(\mathbb{R})$ .

Now we prove that the series  $\sum_{n \in \mathbb{Z}} \tilde{b}_n$  is  $Q$ -summable and Eq. (17) holds.

Since  $F(x) = (Hf)(x) - G(x)$ ,  $Hf \in M(\mathbb{R}; \mathbb{C})$  (see: Remark 1) and  $G(x) \in L_1(\mathbb{R})$ , it follows from Theorems A and B that the function  $F(x)$  is  $Q'$ -integrable on  $\mathbb{R}$ , and moreover,

$$(Q') \int_{\mathbb{R}} F(x) dx = (Q') \int_{\mathbb{R}} (Hf)(x) dx - \int_{\mathbb{R}} G(x) dx = - \int_{\mathbb{R}} G(x) dx \quad (23)$$

The function  $F(x)$  is bounded and by definition for every  $\lambda > 0$

$$\{x \in \mathbb{R} : |F(x)| > \lambda\} = \bigcup_{n \in \mathbb{Z}} \{x \in [n - 1/2, n + 1/2) : |\tilde{b}_n| > \lambda\} = \bigcup_{\{n: |\tilde{b}_n| > \lambda\}} [n - 1/2, n + 1/2),$$

hence

$$\begin{aligned}
 (Q') \int_R F(x)dx &= \lim_{\substack{\lambda \rightarrow 0+ \\ \delta \rightarrow +\infty}} \int_{\{x \in R: \lambda < |F(x)| < \delta\}} F(x)dx = \lim_{\lambda \rightarrow 0+} \int_{\{x \in R: |F(x)| > \lambda\}} F(x)dx \\
 &= \lim_{\lambda \rightarrow 0+} \sum_{\{n: |\tilde{b}_n| > \lambda\}} \int_{n-1/2}^{n+1/2} \tilde{b}_n dx = \lim_{\lambda \rightarrow 0+} \sum_{\{n: |\tilde{b}_n| > \lambda\}} \tilde{b}_n = (Q) \sum_{n \in Z} \tilde{b}_n. \tag{24}
 \end{aligned}$$

It follows from (23) and (24) that the series  $\sum_{n \in Z} \tilde{b}_n$  is  $Q$ -summable and the equation

$$(Q) \sum_{n \in Z} \tilde{b}_n = - \int_R G(x)dx. \tag{25}$$

holds. For every  $n \in Z$  it follows from (22) that

$$\begin{aligned}
 \int_{n-1/2}^{n+1/2} G_2(x)dx &= 2b_n \left[ - \int_{n-1/2}^{n+1/2} \ln|x - n - 1/4|dx + \int_{n-1/2}^{n+1/2} \ln|x - n + 1/4|dx \right] \\
 &= 2b_n \left[ - \int_{-1/2}^{1/2} \ln|u - 1/4|du + \int_{-1/2}^{1/2} \ln|u + 1/4|du \right] \\
 &= 2b_n \left[ \int_{-1/2}^{1/2} \ln|u - 1/4|d(-u) + \int_{-1/2}^{1/2} \ln|u + 1/4|du \right] \\
 &= 2b_n \left[ - \int_{-1/2}^{1/2} \ln|-z - 1/4|dz + \int_{-1/2}^{1/2} \ln|u + 1/4|du \right] = 0.
 \end{aligned}$$

Therefore,

$$\int_R G_2(x)dx = \sum_{n \in Z} \int_{n-1/2}^{n+1/2} G_2(x)dx = 0. \tag{26}$$

By (19) for every  $m \in Z$  and  $t \in [m - 1/4, m + 1/4]$  the series  $\sum_{n \neq m} \int_{n-1/2}^{n+1/2} \left| \frac{1}{x-t} - \frac{1}{n-m} \right| dx$  is convergent. Hence

$$\begin{aligned}
 & \sum_{n \neq m} \int_{n-1/2}^{n+1/2} \left( \frac{1}{x-t} - \frac{1}{n-m} \right) dx \\
 &= \lim_{p \rightarrow \infty} \left( \sum_{n=m-p}^{m-1} \int_{n-1/2}^{n+1/2} \left( \frac{1}{x-t} - \frac{1}{n-m} \right) dx + \sum_{n=m+1}^{m+p} \int_{n-1/2}^{n+1/2} \left( \frac{1}{x-t} - \frac{1}{n-m} \right) dx \right). \quad (27)
 \end{aligned}$$

Since for every  $m \in Z$  and  $p \in N$

$$\begin{aligned}
 \sum_{n=m-p}^{m-1} \frac{1}{n-m} + \sum_{n=m+1}^{m+p} \frac{1}{n-m} &= \left( -\frac{1}{p} - \frac{1}{p-1} - \dots - \frac{1}{2} - 1 \right) \\
 &+ \left( 1 + \frac{1}{2} + \dots + \frac{1}{p-1} + \frac{1}{p} \right) = 0,
 \end{aligned}$$

it follows from (27) that

$$\begin{aligned}
 & \sum_{n \neq m} \int_{n-1/2}^{n+1/2} \left( \frac{1}{x-t} - \frac{1}{n-m} \right) dx \\
 &= \lim_{p \rightarrow \infty} \left( \sum_{n=m-p}^{m-1} \int_{n-1/2}^{n+1/2} \frac{1}{x-t} dx + \sum_{n=m+1}^{m+p} \int_{n-1/2}^{n+1/2} \frac{1}{x-t} dx \right) = \lim_{p \rightarrow \infty} \left( \int_{m-p-1/2}^{m-1/2} \frac{1}{x-t} dx + \int_{m+1/2}^{m+p+1/2} \frac{1}{x-t} dx \right) \\
 &= \lim_{p \rightarrow \infty} [\ln(t-m+1/2) - \ln(t-m+p+1/2) + \ln(m+p+1/2-t) - \ln(m+1/2-t)] \\
 &= \ln(t-m+1/2) - \ln(m+1/2-t).
 \end{aligned}$$

Therefore, for every  $m \in Z$ ,

$$\begin{aligned}
 & \int_{m-1/4}^{m+1/4} \left[ \sum_{n \neq m} \int_{n-1/2}^{n+1/2} \left( \frac{1}{x-t} - \frac{1}{n-m} \right) dx \right] dt = \int_{m-1/4}^{m+1/4} [\ln(t-m+1/2) - \ln(m+1/2-t)] dt \\
 &= \int_{-1/4}^{1/4} [\ln(u+1/2) - \ln(1/2-u)] du = \int_{-1/4}^{1/4} \ln(u+1/2) du + \int_{-1/4}^{1/4} \ln(1/2-u) d(-u) \\
 &= \int_{-1/4}^{1/4} \ln(u+1/2) du - \int_{-1/4}^{1/4} \ln(1/2+z) dz = 0.
 \end{aligned}$$

Now it follows from Fubini's theorem that

$$\begin{aligned}
 \int_{\mathbb{R}} G_1(x) dx &= \sum_{n \in Z} \int_{n-1/2}^{n+1/2} G_1(x) dx = \sum_{n \in Z} \int_{n-1/2}^{n+1/2} \left[ \sum_{m \neq n} 2b_m \int_{m-1/4}^{m+1/4} \left( \frac{1}{x-t} - \frac{1}{n-m} \right) dt \right] dx \\
 &= \sum_{m \in Z} 2b_m \left[ \int_{m-1/4}^{m+1/4} \left[ \sum_{n \neq m} \int_{n-1/2}^{n+1/2} \left( \frac{1}{x-t} - \frac{1}{n-m} \right) dx \right] dt \right] = 0. \quad (28)
 \end{aligned}$$

Now from Eqs. (18), (25), (26) and (28) we finally obtain (17). This completes the proof of theorem 2.

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