Vol. 50 (2019) *Acta Physica Polonica B* No 5

GLOBALLY MAXIMAL TIMELIKE GEODESICS IN STATIC SPHERICALLY SYMMETRIC SPACETIMES: RADIAL GEODESICS IN STATIC SPACETIMES AND ARBITRARY GEODESIC CURVES IN ULTRASTATIC SPACETIMES

Leszek M. Sokołowski, Zdzisław A. Golda

Astronomical Observatory, Jagiellonian University
Orla 171, 30-244 Kraków, Poland
and
Copernicus Center for Interdisciplinary Studies
Sławkowska 17, 31-016 Kraków, Poland

(Received January 23, 2019; accepted March 19, 2019)

This work deals with intersection points — conjugate points and cut points — of timelike geodesics emanating from a common initial point in special spacetimes. The paper contains three results. First, it is shown that radial timelike geodesics in static spherically symmetric spacetimes are globally maximal (have no cut points) in adequate domains. Second, in two ultrastatic spherically symmetric spacetimes, Morris–Thorne wormhole and global Barriola–Vilenkin monopole, it is found which geodesics have cut points (and these must coincide with conjugate points) and which ones are globally maximal on their entire segments. This result, concerning all timelike geodesics of both the spacetimes, is the core of the work. The third outcome deals with the astonishing feature of all ultrastatic spacetimes: they provide a coordinate system which faithfully imitates the dynamical properties of the inertial reference frame. We precisely formulate these similarities.

DOI:10.5506/APhysPolB.50.885

1. Introduction

This paper is a subsequent work in a sequence of papers on geodesic structure of spacetimes with high symmetries [1–6]. This research program has developed from the "twin paradox" in curved spacetime which has purely geometric nature. It comprises search for longest segments of timelike geodesics and consists in determining intersection points of some geodesics emanating from a common initial point. This, in turn, comprises two problems: the local one of finding out conjugate points where close geodesics may intersect, and the global problem of determining cut points being the end points of globally longest timelike curves connecting a given pair of points. The global problem is clearly more difficult. In both the problems, one needs an exact analytic description of timelike geodesics in terms of known functions, and numerical calculations play only a secondary and auxiliary role. This restricts the possible search to spacetimes with a large group of isometries and within them to specific classes of geodesic curves. An exception are ultrastatic spherically symmetric manifolds where all timelike geodesics can be given in explicit form in terms of integrals of metric functions. The static spherically symmetric spacetimes are most investigated in the search for conjugate and cut points on their distinguished geodesics, radial and circular.

In this studying particular spacetimes a question arises: do the same isometries of distinct spacetimes imply that their geodesics have the same (or at least similar) structure of conjugate and cut points? On the contrary, the example of de Sitter and anti-de Sitter spacetimes shows that this is not the case [2]. The spacetimes that have already been analyzed, indicate that no such general rule exists and a set of geodesic intersection points may be identified only after making a detailed investigation of some classes of geodesics in the given spacetime. Yet, it does not mean that one finds a kind of chaos in these spacetimes and two examples of common geodesic structure are known in static spherically symmetric manifolds: the radial timelike geodesics are globally maximal and the geodesic deviation equation on circular geodesics (if they exist) is the same in each of these spacetimes [4].

The paper is organized as follows. In Section 2, we give a complete and detailed proof of the proposition that the radial timelike geodesics in static spherically symmetric spacetimes are globally maximal on their segments lying in the domain of the chart explicitly exhibiting these isometries. The idea of the proof previously appeared in the printed version of [4], whereas the `arXiv` version of that work contains an incomplete proof. Section 3 discusses cut points and conjugate points of any timelike geodesic in a particular ultrastatic spherically symmetric spacetime: the Morris–Thorne wormhole. Using the same method in Section 4, we find possible cut points for geodesics in the global Barriola–Vilenkin monopole spacetime (conjugate points in this manifold were determined in [3]). These two sections contain main results of this work. Cut points in both the spacetimes, if exist, have similar features, yet there are also some differences. The latter arise due to the fact that the geometric structure of the wormhole spacetime is parameter-free, while that of B–V monopole depends on a metric parameter h and it essentially affects properties of geodesic curves. It has been known that all ultrastatic spacetimes (also without spherical symmetry) expressed in comoving coordinates may imitate the inertial reference frame. These similarities have usually been presented in an incomplete and imprecise way [7]. For this reason, we give in Appendix a proposition stating to what extent the comoving system in these spacetimes imitates (*i.e.* has the properties of) the inertial frame. Throughout the paper, we are dealing exclusively with timelike geodesics and this is not always marked.

2. Globally maximal segments of radial timelike geodesics in static spherically symmetric spacetimes

In this section, we consider static spherically symmetric (SSS) spacetimes; in these manifolds, radial timelike geodesics are distinguished by their geometric simplicity and physical relevance. Among all coordinate systems which are adapted to spherical symmetry, we single out the standard ones which make the metric explicitly time-independent and diagonal, what implies that the timelike Killing vector field is orthogonal to constant time spaces. Radial geodesics in these coordinates are those with their tangent vector spanned on vectors tangent to coordinate lines of time and radial variable. The metric of any SSS spacetime in the standard coordinates is

$$\mathrm{d}s^2 = e^{\nu(r)}\mathrm{d}t^2 - e^{\lambda(r)}\mathrm{d}r^2 - F^2(r)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right) , \tag{1}$$

where $t \in (-\infty, +\infty)$, arbitrary functions ν and λ are real for $r \in (r_m, r_M)$ and t and r have dimension of length. $F(r) = r$ for a generic SSS metric and $F(r) = \text{const}$ for some specific spacetimes (*e.g.* the Bertotti–Robinson metric). To avoid inessential complications, we assume that the spacetime is asymptotically flat at spatial infinity, *i.e.* ν and λ tend to zero for $r \to r_M \leq \infty$. Then the timelike Killing vector is normalized at spatial infinity and is $K^\alpha = \delta^\alpha_0$. Let a timelike geodesic be the worldline of a particle of mass m, then the conserved energy is $E = c\,K^\alpha p_\alpha = mc^2\,K^\alpha \dot{x}_\alpha$, and the energy per unit rest mass is $k \equiv \frac{E}{mc^2} > 0$ and is dimensionless. For metric (1), one finds

$$\dot{t} \equiv \frac{\mathrm{d}t}{\mathrm{d}s} = ke^{-\nu} . \tag{2}$$

In the rest of this section, we shall only consider radial timelike geodesics, $t = t(s)$, $r = r(s)$, $\theta = \text{const}$, $\varphi = \text{const}$, and we assume that they can be extended to infinity going outwards from any point $r_0 > r_m$. The universal integral of motion, $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 1$, allows one to replace the radial component of the geodesic equation by a first order expression

$$\dot{r} \equiv \frac{\mathrm{d}r}{\mathrm{d}s} = \left[e^{-(\nu+\lambda)}\left(k^2 - e^\nu\right)\right]^{\frac{1}{2}} . \tag{3}$$

Since $\nu \to 0$ at spatial infinity, one gets $k \geq 1$ and the geodesic may be extended inwards for all r such that $k^2 \geq e^\nu$. For simplicity, we assume that e^ν is bounded in the chart domain and $k^2 \geq e^\nu$ everywhere in it.

To show that radial geodesics are globally maximal, one considers a three-dimensional congruence of radial geodesics with the same energy k filling the entire chart domain. The congruence is described by the geodesic velocity field

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = \left(\dot{t}, \dot{r}, 0, 0\right) = \left[ke^{-\nu}, e^{-\frac{1}{2}(\nu+\lambda)}\left(k^2 - e^\nu\right)^{\frac{1}{2}}, 0, 0\right]. \tag{4}$$

We shall use this vector field to construct the comoving (Gauss normal geodesic, GNG) coordinate system adapted to this congruence. First, the field is rotation-free, $\nabla_{[\alpha}u_{\beta]} = 0$, hence it is a gradient field, $u_\alpha = \partial_\alpha \tau$ where $\tau = \tau(t, r)$ and the congruence is orthogonal to hypersurfaces $\tau = \text{const}$. One easily finds τ from equations

$$\frac{\partial \tau}{\partial t} = u_0 = k \qquad \text{and} \qquad \frac{\partial \tau}{\partial r} = u_1 = -e^{\frac{1}{2}(\lambda-\nu)}\left(k^2 - e^\nu\right)^{\frac{1}{2}}, \tag{5}$$

then

$$\tau = kt - \int \left[e^{\lambda-\nu}\left(k^2 - e^\nu\right)\right]^{\frac{1}{2}} \mathrm{d}r\,. \tag{6}$$

τ will be used as a new time coordinate. A new coordinate system is completed by introducing a new radial variable R whose coordinate lines must be orthogonal to τ lines. One postulates $R(t, r) = at + P(r)$, where $a = \text{const}$ and $P(r)$ will be determined by the orthogonality condition. The relationship for the differentials $\mathrm{d}\tau = k\mathrm{d}t + u_1\mathrm{d}r$ and $\mathrm{d}R = a\mathrm{d}t + P'\mathrm{d}r$, $P' = \frac{\mathrm{d}P}{\mathrm{d}r}$, is inverted to

$$\mathrm{d}r = \frac{1}{H}\left(k\mathrm{d}R - a\mathrm{d}\tau\right), \qquad \mathrm{d}t = \frac{1}{H}\left(P'\mathrm{d}\tau - u_1\mathrm{d}R\right), \tag{7}$$

where $H(r) \equiv kP' - au_1$. Inserting $\mathrm{d}t$ and $\mathrm{d}r$ into the line element (1), one finds that the coefficient at $2\mathrm{d}\tau\mathrm{d}R$ is $g'_{01} = -e^\nu P' u_1 + ake^\lambda$ and requiring $g'_{01} = 0$, one gets $P' = -ak\left[e^{\lambda-\nu}(k^2 - e^\nu)^{-1}\right]^{\frac{1}{2}}$. It turns out that the factor a is redundant and one puts $a = 1$, then the coordinate transformation is (6) and

$$R = t - k\int \left[e^{\lambda-\nu}\left(k^2 - e^\nu\right)^{-1}\right]^{\frac{1}{2}} \mathrm{d}r = t + P(r)\,. \tag{8}$$

One finds the inverse transformation first by determining $r = r(\tau, R)$. The latter arises from the difference

$$\tau - kR = \int \left[e^{\lambda-\nu}\left(k^2 - e^\nu\right)^{-1}\right]^{\frac{1}{2}} \mathrm{d}r \equiv W(r, k)\,. \tag{9}$$

Function W is positive and monotonically growing since $\frac{\mathrm{d}W}{\mathrm{d}r} > 0$. Hence, W is invertible in the whole range of $r_m < r < r_M$. One sees that $-\infty < \tau < +\infty$ and $-\infty < R < +\infty$, however, the chart domain does not cover the whole (τ, R) plane. Actually, τ and R vary in the strip

$$W(r_m, k) < \tau - kR < W(r_M, k)\,. \tag{10}$$

Next, $r = W^{-1}(\tau - kR)$ and inserting it into (8), one finds $t = R - P[W^{-1}(\tau - kR)]$. As an example, we take the Reissner–Nordström black hole, $M^2 > Q^2$. For it

$$e^{\nu} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = e^{-\lambda}\,, \tag{11}$$

here, $r \in (r_+, \infty)$ with $r_+ = M + \sqrt{M^2 - Q^2}$. We do not consider the maximally extended spacetime and assume existence of only one exterior asymptotically flat region. Outside the outer event horizon, there is $e^{\nu} < 1$ and one sets $k = 1$ for simplicity, then

$$W(r, 1) = \frac{2}{3}(2M)^{-\frac{1}{2}}\sqrt{r - \frac{Q^2}{2M}}\left(r + \frac{Q^2}{M}\right)\,. \tag{12}$$

W grows from $W(r_+) = \frac{r_+}{3M}\left(r_+ + \frac{Q^2}{M}\right)$ to infinity for $r \to \infty$, yet it cannot be effectively inverted since it requires solving a cubic equation. Only for $Q = 0$, one gets $r = (2M)^{\frac{1}{3}}\left[\frac{2}{3}(\tau - R)\right]^{\frac{2}{3}}$ and an explicit expression for $t = t(\tau, R)$ may be found (Eddington–Lemaître coordinates). For the Kottler (Schwarzschild–de Sitter) black hole for $0 < \Lambda < \frac{1}{9M^2}$, the corresponding formulae are quite complicated.

One sees from the construction that for arbitrary ν and λ, the domains for (t, r) and (τ, R) charts form the same region of the spacetime.

Transformations (6) and (8) provide a specific comoving system and the SSS metric (1) reads in it

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 - \left[k^2 - e^{\nu}\right]\mathrm{d}R^2 - F^2\left[r(\tau - R)\right]\mathrm{d}\Omega^2\,, \tag{13}$$

where as usual, $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$. The metric does not explicitly depend on $\lambda(r)$ since this function has been absorbed in the transformation, however, the metric depends implicitly on λ and explicitly on time τ via the inverse transformation $r = W^{-1}(\tau - R)$. For special spacetimes (Schwarzschild, Reissner–Nordström), metric (13) may be, as is well-known, extended to a domain larger than that in (t, r) chart.

Next, to show that the GNG system of (τ, R) coordinates is actually a comoving one in the sense that the congruence of radial timelike geodesics generating it is a family of coordinate lines for time τ (the particles are at rest), one transforms the velocity field u^α given in (4) to this system. One gets $\frac{d\tau}{ds} = 1$ and $\frac{dR}{ds} = 0$. Each radial geodesic is described by $\tau = \tau_0 + s$, $R = R_0$, $\theta = \theta_0$ and $\varphi = \varphi_0$.

Now, the proof that each radial geodesic extended to the whole range from r_m to r_M is globally maximal between any pair of its points, is immediate. Let $P_0(\tau_0, R_0, \theta_0, \varphi_0)$ and $P_1(\tau_1, R_0, \theta_0, \varphi_0)$ lie in strip (10), then they are connected by a unique radial geodesic γ. Its length is $s(\gamma) = \tau_1 - \tau_0$. For any timelike curve σ connecting P_0 and P_1 and parametrized by τ, $x^\mu = x^\mu(\tau)$, its length is

$$s(\sigma)$$
$$= \int_{\tau_0}^{\tau_1} \left[1 - \left(k^2 - e^\nu\right)\left(\frac{dR}{d\tau}\right)^2 - F^2\left(\frac{d\theta}{d\tau}\right)^2 - F^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2\right]^{\frac{1}{2}} d\tau < \tau_1 - \tau_0 \,,$$

hence γ is globally the longest curve.

3. Future cut points on arbitrary timelike geodesics and globally maximal curves in the Morris–Thorne wormhole

In most SSS spacetimes, the geodesic equation for timelike curves cannot be effectively integrated (to give a parametric description) besides special cases such as radial and circular lines, hence finding the longest curve joining two arbitrary (chronologically related) points is a hopeless task. Yet explicit formulae for arbitrary timelike geodesics in terms of integrals have been found for ultrastatic spherically symmetric (USSS) spacetimes. Generic ultrastatic spacetimes (without spherical symmetry) are described in [7] and arbitrary timelike geodesics in any USSS spacetime are given in [3]. Even in the USSS case, these integral formulae for parametric description of geodesic lines do not allow one to determine whether a given curve is globally maximal on its sufficiently long segment. One must instead separately study particular USSS spacetimes in which any timelike geodesic is explicitly expressed in terms of known functions. In this section, we investigate the geodesic structure of the Morris–Thorne wormhole. Its properties are discussed in [8] and for our purpose, we only need its metric expressed in a chart covering the entire manifold. In [8], it was defined as a special case of USSS spacetimes with the metric

$$ds^2 = dt^2 - \frac{r^2}{r^2 - a^2}dr^2 - r^2 d\Omega^2 \,, \tag{14}$$

where $a = \text{const} > 0$, *i.e.* it is metric (1) with $\nu = 0$ and $F(r) = r$. However, this chart has a boundary $r = a$ which is a coordinate singularity and the spacetime can be extended beyond it. To this end, one computes the length of the radial line $t = \text{const}$ from the boundary $r = a$ to a point $r = r_0$, $l(a, r_0) = \int_a^{r_0} (-\mathrm{d}s^2)^{\frac{1}{2}} = \sqrt{r_0^2 - a^2}$.

One then introduces a new radial coordinate $l = \sqrt{r^2 - a^2}$ and the metric is [8]

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}l^2 - \left(l^2 + a^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right) , \tag{15}$$

and the singularity disappears. Here, $-\infty < t < \infty$ and θ and φ cover (in the usual sense) S^2. The original domain $a < r < \infty$ or $0 < l < \infty$ is now extended to entire real line, $-\infty < l < \infty$. The entire manifold has two flat spatial infinities, $l \to \pm\infty$. The former singularity is actually a regular hypersurface $l = 0$, the "wormhole throat". Metric (15) may be written as $\mathrm{d}s^2 = a^2 \mathrm{d}\bar{s}^2$ and the rescaled metric $\mathrm{d}\bar{s}^2$ with dimensionless coordinates t/a and l/a has the same geodesic structure as $\mathrm{d}s^2$. The rescaled metric is parameter-free, hence properties of geodesic curves on the wormhole manifold are independent of the value a.

For a timelike geodesic motion, the integral of energy is from (2) $\dot{t} = k$ and may be integrated to $t(s) = t_0 + ks$. The motion is "flat" in the sense that each geodesic lies in a 2-plane which in adapted to it angular coordinates is given by $\theta = \frac{\pi}{2}$ and the conserved angular momentum generated by the Killing field $K^\alpha_\varphi = \delta^\alpha_3$ is $g_{\alpha\beta} K^\alpha_\varphi p^\beta \equiv -mL$; L is the angular momentum per unit mass and

$$\dot{\varphi} \equiv \frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{L}{l^2 + a^2} , \tag{16}$$

L has dimensions of length. The geodesic equation for l is replaced by a modified version of (3), which reads

$$\dot{l}^2 = k^2 - 1 - \frac{L^2}{l^2 + a^2} . \tag{17}$$

For $k = 1$, one gets $L^2 = \dot{l}^2 = \dot{\varphi} = 0$, the motion is reduced to a rest (see Appendix) and the geodesic is a time coordinate line which is globally maximal. We are interested in all other geodesic curves, hence $k > 1$. In most formulae below, we shall use a dimensionless radial coordinate $x = \frac{l}{a}$.

Before searching for cut points on arbitrary nonradial geodesics, we deal with the following problem: is it possible to connect two points on a radial geodesic (points with the same angular coordinates) by another timelike geodesic?

3.1. A radial geodesic intersected twice by a nonradial geodesic

Let C be a radial geodesic with energy k_C, $\theta = \frac{\pi}{2}$, $\varphi = \varphi_0$. From (17), one gets $\dot{l} = \pm\sqrt{k_C^2 - 1}$ and one sees that inversion in time, $t \to -t$, maps a radial geodesic with $\dot{l} = +\sqrt{k_C^2 - 1}$ onto the geodesic with $\dot{l} = -\sqrt{k_C^2 - 1}$. Hence, it is sufficient to consider the segment of C with $\dot{l} > 0$ and $l > 0$. Choose a point $P_0(t_0, l_0)$ for $l_0 > 0$ on C, then $t(s) = t_0 + k_C s$ and $l(s) = \sqrt{k_C^2 - 1} s + l_0$.

Consider now a nonradial geodesic G which emanates from P_0 and goes to $l \to \infty$. We restrict it by requiring l_0 be the lowest value of $l(s)$ on G. If l attains minimum for $l = l_0$, then $\dot{l}(l_0) = 0$ and (17) implies $k_G^2 - 1 - \frac{L^2}{l_0 + a^2} = 0$.

In general, a nonradial geodesic depends on two parameters, k_G and L. The above restriction shows that L^2 is determined by k_G and l_0,

$$L^2 = \left(k_G^2 - 1\right)\left(l_0^2 + a^2\right), \tag{18}$$

and one has a one-parameter family of geodesics $G(k_G)$ emanating from P_0. (One may check that the acceleration $\ddot{l} > 0$ on G at P_0.) To simplify expressions below, one introduces parameter

$$p^2 \equiv \frac{k_G^2 - 1}{L^2} = \frac{1}{l_0^2 + a^2}, \qquad \text{hence} \qquad p^2 a^2 = \frac{a^2}{l_0^2 + a^2} < 1\,. \tag{19}$$

In terms of $x = \frac{l}{a}$ and $x_0 = \frac{l_0}{a}$, Eq. (17) reads for G

$$\dot{l}^2 = \left(k_G^2 - 1\right)\frac{x^2 - x_0^2}{x^2 + 1} \tag{20}$$

and its length from P_0 is

$$s(l) = a\left(k_G^2 - 1\right)^{-\frac{1}{2}} \int\limits_{x_0}^{x} \left[\frac{x^2 + 1}{x^2 - x_0^2}\right]^{\frac{1}{2}} \mathrm{d}x \equiv a\left(k_G^2 - 1\right)^{-\frac{1}{2}} H_s(x_0, x)\,, \tag{21}$$

where

$$\begin{aligned} H_s(x_0, x) \;=\; & x^{-1}\sqrt{\left(x^2 + 1\right)\left(x^2 - x_0^2\right)} + \sqrt{x_0^2 + 1} \\ & \times \left[\mathrm{F}\left[\arccos\left[\frac{x_0}{x}\right], \frac{1}{x_0^2 + 1}\right] - \mathrm{E}\left[\arccos\left[\frac{x_0}{x}\right], \frac{1}{x_0^2 + 1}\right]\right] \end{aligned}$$

and F is the incomplete elliptic integral of the first kind, and E is incomplete elliptic integral of the second kind, see [9, 10]. Also the angle φ on G is parametrized by l. From (16) and (20), one gets assuming $L > 0$,

$$\varphi(l) = \varphi_0 + \sqrt{x_0^2 + 1} \int\limits_{x_0}^{x} \left[\left(x^2 + 1\right)\left(x^2 - x_0^2\right)\right]^{-\frac{1}{2}} \mathrm{d}x \equiv \varphi_0 + \sqrt{x_0^2 + 1} H_\varphi(x_0, x)\,, \tag{22}$$

where

$$H_\varphi(x_0, x) = \mathrm{F}\left[\arccos\left[\frac{x_0}{x}\right], \frac{1}{x_0^2+1}\right].$$

The increase of the angle φ on $G(k_G)$ is independent of energy k_G and is the same for all geodesics of this family.

The necessary condition for G to intersect the radial C at some point $P_1(t_1, l_1)$ is that the increment of φ on G is 2π

$$\Delta\varphi \equiv \varphi(l_1) - \varphi_0 = \sqrt{x_0^2+1}H_\varphi(x_0, x_1) = 2\pi\,. \tag{23}$$

This is a transcendental equation for $x_1 = \frac{l_1}{a} < \infty$ and has an analytic solution in the form of $x_1 = x_0 \,\mathrm{nc}\left(2\pi, \frac{1}{1+x_0}\right)$, where nc is one of Jacobi elliptic functions [9, 10]. Rather surprisingly, it turns out that solutions do not exist for arbitrary values of x_0 and they exist in a narrow interval $0 < x_0 < x_{0M} = 0{,}0074705\ldots$ For $x_0 \to x_{0M}$, one finds $x_1 \to \infty$. The lower limit $x_0 = 0$ is unattainable since it is seen from (22) that $\frac{\mathrm{d}\varphi}{\mathrm{d}l}$ is divergent there as $\frac{1}{x}$ and $\varphi(l)$ behaves as $-\ln x$ (actually for $x_0 = 0$, integral (22) is an elementary function which is logarithmically divergent in the lower limit $x_0 \to 0$), hence, G winds up infinitely many times around $l = 0$ for infinitesimal l. For example, one takes two extreme cases:

- for $(1 + x_0)^{-1} = 0{,}99999$ corresponding to $x_0 \cong 0{,}003162$, there is $x_1 \cong 1{,}0315483397\ldots$,
- for $x_0 \cong 0{,}00747$, one gets $x_1 = 14\,098{,}2$.

The existence of finite solutions x_1 to Eq. (23) raises the question of whether it is possible for G to intersect C more than once. If n intersection points exist, $x_1 < x_2 < \cdots < x_n$, then each of them is a solution to equation analogous to (23),

$$\Delta\varphi = \sqrt{x_0^2+1}H_\varphi(x_0, x_n) = 2\pi n\,. \tag{24}$$

This equation is analytically solved for any natural n by $x_n = x_0 \,\mathrm{nc}$ $(2\pi n, \frac{1}{1+x_0})$. The solutions x_n exist for decreasing ranges of initial values of x_0. The value $x_{0M} \equiv x_{0M_1} = 0{,}0074705\ldots$ was found numerically, yet there are analytic arguments that it may be well-approximated by $x_{0M_1} = 2\,\mathrm{sech}(2\pi)$. In the limit $n \to \infty$, one gets an exact expression for the upper limit of the interval $0 < x_n < x_{0M_n}$,

$$x_{0M_n} = 2e^{-2\pi(n-1)}\,\mathrm{sech}(2\pi)\,, \tag{25}$$

and the interval length exponentially diminishes.

If the necessary condition holds, the sufficient condition for G to intersect C at P_1 is that the time coordinates of both the curves are equal at this point, $t_1 = t_0 + k_C s_C = t_0 + k_G s_G$, where their lengths are, respectively,

$$s_C = a(k_C^2 - 1)^{-\frac{1}{2}}(x_1 - x_0) \tag{26}$$

and $s_G = a(k_G^2 - 1)^{-\frac{1}{2}} H_s(x_0, x_1)$ from (21). Hence, the sufficient condition takes the form of

$$k_C \left(k_C^2 - 1\right)^{-\frac{1}{2}} (x_1 - x_0) = k_G \left(k_G^2 - 1\right)^{-\frac{1}{2}} H_s(x_0, x_1)\,. \tag{27}$$

For a given energy k_C on C and known solution $x_1(x_0)$ of Eq. (23), this is an equation for energy k_G on G and this means that C may be intersected at P_1 by only one geodesic out of the family $G(k_G)$. The solution of (27) is

$$k_G^2 = \frac{k_C^2 (x_1 - x_0)^2}{k_C^2 (x_1 - x_0)^2 - (k_C^2 - 1) H_s^2(x_0, x_1)}\,. \tag{28}$$

This formula makes sense if its denominator is positive, then $k_G^2 > 1$. The requirement that the denominator be positive is, in turn, a restriction imposed on k_C. Since $x_1 = x_1(x_0)$, one introduces a function

$$B(x_0) = \frac{H_s^2(x_0, x_1)}{(x_1 - x_0)^2} > 0 \tag{29}$$

and the requirement takes on the form of an inequality

$$k_C^2 (B(x_0) - 1) < B(x_0)\,. \tag{30}$$

A numerical computation applying Mathematica shows that in the allowed interval $0 < x_0 < 0,0074705\ldots$, function B is diminishing and everywhere $B(x_0) > 1$. Then, for given x_0, the energy k_C is restricted to the interval

$$1 < k_C^2 < \frac{B(x_0)}{B(x_0) - 1}\,. \tag{31}$$

For an admissible value of k_C, the length of G which emanates from P_0 and intersects C at P_1 is, from (21) and (28)

$$s_G = a \frac{\sqrt{B - k_C^2 (B - 1)}}{\sqrt{\left(k_C^2 - 1\right) B}} H_s(x_0, x_1)\,. \tag{32}$$

The ratio of the geodesic lengths is

$$\left[\frac{s_G}{s_C}\right]^2 = B - (B-1)k_C^2\,, \tag{33}$$

and it is clear that $s_G < s_C$ since $B(x_0) > 1$ and k_C^2 is in the allowed range. We note that this is not another proof of geodesic C being globally maximal because G is not the most generic nonradial geodesic intersecting C at P_0 and P_1: G cannot be extended for $l < l_0$ since its angular momentum is restricted by (18).

3.2. Future cut points on nonradial timelike geodesics

Now, we seek for globally maximal (globally longest) segments of nonradial geodesics in the wormhole spacetime. To this end, we briefly remind the necessary notions of Lorentzian geometry. The *Lorentzian distance function* $d(p,q)$ of two chronologically related points p and q (q is in the chronological future of p, $p \prec\!\prec q$) is the length of the longest timelike curve joining p and q. The curve G from p to q is said to be *globally maximal* if it is the longest one between these points, *i.e.* if $s(G) = d(p,q)$. The globally maximal curve (usually nonunique) is always a timelike geodesic (Theorem 4.13 of [11]).

We consider complete timelike geodesics: they are defined for all values of the canonical length parameter, $-\infty < s < +\infty$; in the Morris–Thorne wormhole, they extend to $l \to \pm\infty$. Usually, they are not globally maximal beyond some segment, in our notation: from P_0 to P_1. This gives rise to the notion of the cut point on a geodesic. Let G be a future directed timelike geodesic parametrized by its length s and let $P_0 = G(0)$ be a chosen point on it. Set

$$s_0 \equiv \sup\left\{s > 0\colon d\left(G(0), G(s)\right) = s\right\}\,.$$

If $0 < s_0 < \infty$, then $G(s_0)$ is said to be the *future timelike cut point* of $G(0)$ along G. For all $0 < s < s_0$, the geodesic G is the unique globally maximal curve from $G(0)$ to $G(s)$ and is globally maximal (not necessarily unique) on the segment from $G(0)$ to $G(s_0)$. For $s_1 > s_0$, there exists a future directed timelike curve K from $G(0)$ to $G(s_1)$ which is longer, $s(K) > s(G)$. In other terms: s_0 is the length of the longest globally maximal segment of G from $P_0 = G(0)$.

We, therefore, consider the full set of complete timelike geodesics $G(k,L)$ intersecting at some point P_0 and seek for another point P_1 where some of them intersect again. Due to spherical symmetry, each geodesic lies in space in its own "plane" $\theta = \frac{\pi}{2}$. If two geodesics lying in different planes intersect twice, the difference in the azimuthal angle φ between intersection points (measured in one of the planes) is $\Delta\varphi = \pi$. This geometric argument will be analytically shown below. We, therefore, focus our attention on double intersections of geodesics belonging to the same 2-plane. Then the intersection points are $P_0(t_0, l_0, \frac{\pi}{2}, \varphi)$ and $P_1(t_1, l_1, \frac{\pi}{2}, \bar{\varphi})$. For each complete geodesic, Eq. (17) holds for all values of l and the minimal value of $\dot{l}^2$ is attained for $l = 0$ and, at this point, there must be $\dot{l}^2 = k^2 - 1 - \frac{L^2}{a^2} > 0$, what implies $\left(k^2 - 1\right) \frac{a^2}{L^2} > 1$.

In fact, if $\dot{l}^2(l = 0) = 0$, then $\left(k^2 - 1\right) \frac{a^2}{L^2} = 1$ and one gets from the geodesic equation that $\ddot{l} = 0$ at this point and the unique solution of this equation is $l(s) = 0$ while $\dot{\varphi} = \frac{L}{a^2}$. This is a circular geodesic at the wormhole throat. As a side remark, we note that timelike circular geodesics with different L exist only at the throat, $l = 0$. As in (19), we denote $p^2 = (k^2 - 1)L^{-2}$, hence $p^2 a^2 > 1$. We assume that $\mathrm{d}l > 0$ for $\mathrm{d}s > 0$ and $-\infty < L < \infty$. In terms of p^2 and $x = \frac{l}{a}$, one finds from (17)

$$\frac{\mathrm{d}l}{\mathrm{d}s} = \frac{|L|}{a} \left(x^2 + 1\right)^{-\frac{1}{2}} \left[p^2 a^2 (x^2 + 1) - 1\right]^{\frac{1}{2}} . \tag{34}$$

This expression may be given a more convenient form if one introduces a parameter

$$b^2 \equiv 1 - \frac{1}{p^2 a^2} = 1 - \frac{L^2}{\left(k^2 - 1\right) a^2}, \qquad 0 < b^2 < 1 . \tag{35}$$

Integrating the inverse of formula (34), one gets the length of a generic nonradial geodesic $G(k, L)$ from l_0 to l

$$s(k, L, l) = \frac{a}{\sqrt{k^2 - 1}} \int\limits_{x_0}^{x} \left[\frac{x^2 + 1}{x^2 + b^2}\right]^{\frac{1}{2}} \mathrm{d}x \equiv \frac{a}{\sqrt{k^2 - 1}} J(b, x_0, x) . \tag{36}$$

Analogously to derivation of (22), one has in the general case from (16) and (34) that the angular coordinate along G is

$$\begin{aligned} \varphi(l_0, l) &= \varphi_0 + \frac{L}{|L|} \sqrt{1 - b^2} \int\limits_{x_0}^{x} \left[\left(x^2 + 1\right)\left(x^2 + b^2\right)\right]^{-\frac{1}{2}} \mathrm{d}x \\ &\equiv \varphi_0 + \frac{L}{|L|} \sqrt{1 - b^2} N(b, x_0, x) . \end{aligned} \tag{37}$$

Assume that two geodesics of the set, $G_1(k_1, L_1)$ and $G_2(k_2, L_2)$, intersect at P_1. First, one finds that G_1 and G_2 emanating from P_0 cannot intersect again if their angular momenta are of the same sign, $L_1 L_2 > 0$. In fact, let

$L_1 > 0$ and $L_2 > 0$, then φ_1 and φ_2 monotonically grow along the curves and their growth depends only on the corresponding values of b_1 and b_2. The necessary condition for the intersection at P_1 is that there exists $l_1 > l_0$ such that $\bar{\varphi} = \varphi_1(l_1) = \varphi_2(l_1)$. It is clear from (37) that difference $\varphi_1 - \varphi_2$ never vanishes if $b_1 \neq b_2$.

If $b_1 = b_2 \equiv b$, then one has $\varphi_1(l) = \varphi_2(l)$ for all l and the necessary condition holds trivially. The sufficient condition of intersection is that for some l_1, the time coordinates of G_1 and G_2 are the same, $t_1 = t_0 + k_1 s_1 = t_0 + k_2 s_2$. Then (36) implies

$$\frac{ak_1}{\sqrt{k_1^2 - 1}} J(b, x_0, x_1) = \frac{ak_2}{\sqrt{k_2^2 - 1}} J(b, x_0, x_1)\,, \tag{38}$$

or $k_1(k_1 - 1)^{-\frac{1}{2}} = k_2(k_2 - 1)^{-\frac{1}{2}}$ and the condition is satisfied by $k_1 = k_2 = k$. Finally, $b_1^2 = b_2^2$ implies $L_1^2 = L_2^2$ and $L_1 = L_2$ — the geodesics G_1 and G_2 are identical.

One infers that G_1 and G_2 can intersect only if $\varphi_1(l)$ grows monotonically ($L_1 > 0$) and $\varphi_2(l)$ decreases ($L_2 < 0$). At the intersection point, the difference between $\varphi_1(l_1) > 0$ and $\varphi_2(l_1) < 0$ is 2π. Applying (37), one gets the necessary condition

$$\sqrt{1 - b_1^2} N(b_1, x_0, x_1) + \sqrt{1 - b_2^2} N(b_2, x_0, x_1) = 2\pi\,. \tag{39}$$

This is an algebraic equation for $x_1 = \frac{l_1}{a}$ at known values of x_0, b_1 and b_2. If a solution exists, then the sufficient condition, equality of the time coordinates of G_1 and G_2, $k_1 s_1 = k_2 s_2$, requires the following equality to hold, being a direct generalization of (38)

$$\frac{k_1}{\sqrt{k_1^2 - 1}} J(b_1, x_0, x_1) = \frac{k_2}{\sqrt{k_2^2 - 1}} J(b_2, x_0, x_1)\,. \tag{40}$$

For $x_1 = x_1(x_0, b_1, b_2)$, this is a restriction on the geodesic parameters. Solving both (39) and (40) is a hard task, fortunately, for our purposes, it is sufficient to study a special case of these equations.

The Morris–Thorne wormhole is a globally hyperbolic spacetime and one may apply Theorem 9.12 in [11].

Theorem 3.1. *If $q = G(s_0)$ is the future cut point of $p = G(0)$ along the timelike geodesic G from p to q, then either one or possibly both of the following hold:*

- *(i) the point q is the first future conjugate point to p;*
- *(ii) there exist at least two future directed globally maximal geodesic segments from p to q.*

By definition, at a conjugate point, a non-zero Jacobi vector field along the geodesic vanishes. Jacobi fields are solutions to the approximate (linear) geodesic deviation equation hence any Jacobi field connects two infinitesimally close geodesic lines. If $G(k, L)$ is a fiducial geodesic surrounded by bundle of close geodesics determined by Jacobi fields on G, one should distinguish between geodesics lying in the 2-plane of G and those directed off the plane. The latter geodesics require a separate treatment given in Sec. 3.3 below and here we comment on close geodesics with $\theta(s) = \pi/2$. From the above formulae and discussion, one sees that in the wormhole spacetime, two infinitesimally close geodesics have their angular momenta of the same sign and their parameters b_1 and b_2 must be close, $b_2 = b_1 + \varepsilon$, $|\varepsilon| \ll 1$, what implies that they will never intersect.

It is then clear that in the case of "coplanar" curves, only distant (besides the end points) geodesics can intersect twice. Suppose that P_1 is the first future cut point to P_0 along G_1 and let G_2 be another globally maximal geodesic from P_0 to P_1 according to the theorem. Their lengths are equal, $s_1 = s_2$. At P_1, their time coordinates are the same, $t_1 = t_0 + k_1 s_1 = t_0 + k_2 s_2$, hence their energies are equal, $k_1 = k_2 = k$. Their parameters p_1 and p_2 satisfy $p_1^2 L_1^2 = k^2 - 1 = p_2^2 L_2^2$ and from (36) it follows

$$s_1 - s_2 = 0 = \frac{a}{\sqrt{k^2-1}} \int\limits_{x_0}^{x_1} \sqrt{x^2+1} \left[\frac{1}{\sqrt{x^2+b_1}} - \frac{1}{\sqrt{x^2+b_2}} \right] \mathrm{d}x \, . \tag{41}$$

For $b_1 \neq b_2$, the integrand is always either positive or negative and the integral cannot vanish. Hence, $b_1 = b_2$ and this implies $L_1^2 = L_2^2$, and then it follows that $L_2 = -L_1$.

One concludes that only two coplanar geodesics, $G_1(k, L)$ and $G_2(k, -L)$, may be both globally maximal between points P_0 and P_1. (Other geodesics emanating from P_0 lie in other 2-planes and these planes arise due to rotations of the $\theta = \frac{\pi}{2}$ "plane" of G_1 and G_2; in this way, the number of globally maximal pairs grows to infinity.) We remark that P_1 is the first cut point of P_0 in the sense that $x_1 = \frac{l_1}{a}$ is the closest to x_0 root of Eq. (39). Furthermore, it is clear that G_1 and G_2 are globally maximal between P_0 and P_1, or that their length $s = d(P_0, P_1)$. In fact, suppose on the contrary, that there exists a timelike geodesic $G_3(k_3, L_3)$ from P_0 to P_1 which is longer, $s(G_3) > s$. This implies $b_3 \neq b$. Let $L_3 L > 0$. By assumption, G_3 intersects G_1 at P_1 and this, as was shown above, is impossible. G_3 does not exist. We emphasize that G_3 is not excluded on the assumption that it is longer than G_1 and G_2, also G_3 shorter than these two cannot exist. Only two geodesics moving in the opposite directions in coordinate φ may intersect (modulo rotations of the "plane").

In conclusion, the first future cut point to P_0 on $G(k,L)$ is its first intersection point with $G(k,-L)$. In this case, the necessary condition (39) is simplified to

$$\sqrt{1-b^2}N(b,x_0,x_1)=\pi\,, \tag{42}$$

showing that the azimuthal angle increases by $\Delta\varphi=\pi$. Then the sufficient condition (40) trivially holds.

The indefinite integral N in (37) is the incomplete elliptic integral of the first kind denoted as $\mathrm{F}(x,k^2)$, hence Eq. (42) reads

$$\mathrm{F}\left[\arctan\left(\frac{x_1}{b}\right),1-b^2\right]-\mathrm{F}\left[\arctan\left(\frac{x_0}{b}\right),1-b^2\right]=\frac{\pi}{\sqrt{1-b^2}}\,. \tag{43}$$

This equation for $x_1=x_1(b,x_0)$ has an exact analytic solution in terms of the Jacobi elliptic function $\mathrm{sc}(x,k^2)$,

$$x_1=b\,\mathrm{sc}\left[\frac{\pi}{\sqrt{1-b^2}}+\mathrm{F}\left(\arctan\left(\frac{x_0}{b}\right),1-b^2\right)\right]\,. \tag{44}$$

There are two cases.

1. First, we consider the special case of timelike geodesics emanating from P_0 at the wormhole throat, $x_0=0$. In the limit $b\to 0$, the definite integral in (37) is an elementary function which is logarithmically divergent in the lower limit $x_0\to 0$. For $x_0=\varepsilon$, $|\varepsilon|\ll 1$, one finds $x_1\approx|\varepsilon|e^{\pi}$. For $x_0=0$ and finite b, one finds from (44) that x_1 rapidly grows to infinity for increasing b: finite solutions $x_1(b)$ exist only in the narrow interval $0<b<b_M=0{,}16780\ldots$ For $b>b_M$, timelike geodesics crossing the throat have no cut points to $P_0(x_0=0)$, hence are globally maximal up to $l\to+\infty$. The limit $b<b_M$ corresponds to $L^2>0{,}971843(k^2-1)a^2$.

   Since the geodesics $G_1(k,L)$ and $G_2(k,-L)$ which intersect at x_1 (if this point exists) are complete, one may also consider their behaviour for $l\to-\infty$. If the upper limit in integral (37) is $x<0$, then $\varphi_1(l)$ monotonically diminishes and $\varphi_2(l)$ monotonically grows for $l\to-\infty$. For $l<0$, geodesics G_1 and G_2 will intersect in the first past cut point $P_2(l_2)$, $l_2<0$, if $\varphi_1(l_2)-\varphi_2(l_2)=-2\pi$. The integrand in (37) is a symmetric function since it depends on x^2, hence the incomplete elliptic integral of the first kind is antysymmetric, $\mathrm{F}(-x,k^2)=-\mathrm{F}(x,k^2)$, therefore, if $x_1(b)$ is the lowest positive solution to (42), then $x_2=-x_1$ is the largest negative solution of the corresponding equation $\sqrt{1-b^2}\,N(b,0,x_2)=-\pi$.

One summarizes the case $x_0 = 0$ by stating that if $b(k, L)$ is in the interval $0 < b < 0,16780\ldots$, then the geodesics $G_1(k, L)$ and $G_2(k, -L)$ intersect at $l = l_1 > 0$ and at $l = l_2 = -l_1$. If $0,16780 < b < 1$, then geodesics G_1 and G_2 are globally maximal from $l = 0$ to $l \to \pm\infty$. The explicit form of G_1 (and correspondingly of G_2) is $t = t_0 + ks$, $s(k, L, l) = a(k^2 - 1)^{-\frac{1}{2}} J(b, x_0 = 0, \frac{l}{a})$ (Eq. (36)), $\theta = \frac{\pi}{2}$ and $\varphi(l)$ as in (37). G_1 is uniquely determined by its tangent vector $\dot{x}^\alpha$ at the initial crossing point $P_0(x_0 = 0)$

$$\dot{x}^\alpha = \left[k, \left(k^2 - 1 - \frac{L^2}{a^2}\right)^{\frac{1}{2}}, 0, \frac{L}{a^2}\right]. \tag{45}$$

At P_0, the ratio of the components of the tangent vector is

$$\frac{\dot{l}}{\dot{\varphi}} = \frac{a\,b}{\sqrt{1 - b^2}}. \tag{46}$$

It follows that if $b < b_M$, then $\frac{\dot{l}}{\dot{\varphi}} < a\,b_M(1 - b_M^2)^{-\frac{1}{2}} = 0,17021\ldots$ and the geodesics G_1 and G_2 have cut points to P_0 at $l = l_1$ and at $l = -l_1$; for $\frac{\dot{l}}{\dot{\varphi}} > 0,17021\ldots$, there are no cut points to $l_0 = 0$ and G_1 and G_2 are globally maximal up to $l \to \pm\infty$.

2. In the general case, geodesics $G_1(k, L)$ and $G_2(k, -L)$ emanate from P_0 for $x_0 > 0$. Again, the solution $x_1(b, x_0)$ given in (44) does not exist for all $x_0 > 0$ and $b^2 < 1$. On the plane (b, x_0), a finite solution exists for points in the domain bounded by the coordinate axes and the limiting curve corresponding to $x_1 = \infty$. The limiting curve is given by exact equation

$$-\,\mathrm{F}\left[\arctan\left(\frac{x_0}{b}\right), 1 - b^2\right] + \mathrm{K}\left(1 - b^2\right) = \frac{\pi}{\sqrt{1 - b^2}}, \tag{47}$$

where $\mathrm{K}(k^2)$ is the complete elliptic integral of the first kind, see figure 1. For small b, the root x_1 rapidly blows up to infinity for small x_0: one finds numerically from (47) that for $b = 10^{-7}$, the maximal value of x_0 is $x_0 = 0,08662\ldots$ Hence, in the strip $0 < b < 1$, $0 < x_0 < \infty$ most of the region corresponds to globally maximal geodesics.

3.3. Conjugate points on non-radial timelike geodesics

Let $G(k, L)$ be a general non-radial timelike geodesic lying in the 2-plane $\theta = \frac{\pi}{2}$. Its tangent vector is

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = \left[k, \frac{pL}{\sqrt{x^2 + 1}}\left(x^2 + b^2\right)^{\frac{1}{2}}, 0, \frac{L^2}{a^2(x^2 + 1)}\right]. \tag{48}$$

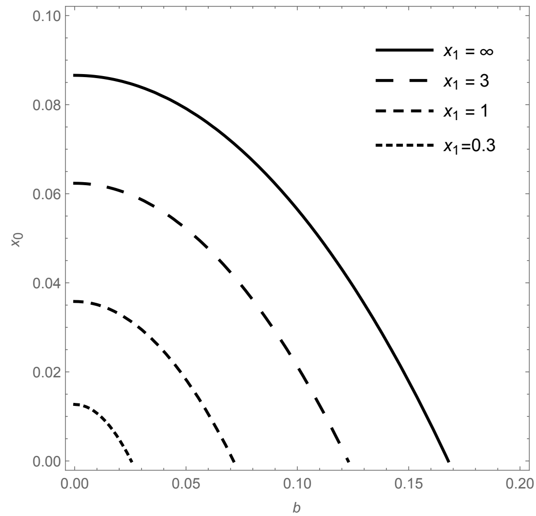

Fig. 1. Four curves representing values of b and x_0 corresponding to three finite values of x_1 and to $x_1 = \infty$. For $b > b_M = 0,16780\ldots$, there are no solutions.

G is viewed as a fiducial geodesic for a bundle of infinitesimally close to it geodesics connected to G by various Jacobi fields. These vector fields are solutions of the geodesic deviation equation on G, and the general formalism for finding these fields and conjugate points determined by them is presented in [4]. According to it, one introduces a spacelike basis triad along G, $e_a^\mu(s)$, $a = 1, 2, 3$ and any Jacobi vector field is spanned in this basis, $Z^\mu = \sum_a Z_a e_a^\mu$, where $Z_a(s)$ are called Jacobi scalars. The basis triad for a generic USSS spacetime is given [3] and after adopting it to (t, l, θ, φ) coordinates, it reads

$$\begin{aligned} e_1^\mu &= \left[0, \frac{1}{pa\sqrt{x^2+1}}, 0, -\frac{(x^2+b)^{\frac{1}{2}}}{a(x^2+1)}\right], \qquad e_2{}^\mu = \frac{1}{a\sqrt{x^2+1}}\delta_2^\mu, \\ e_3^\mu &= \left[\sqrt{k^2-1}, \frac{k}{\sqrt{x^2+1}}\left(x^2+b^2\right)^{\frac{1}{2}}, 0, \frac{k}{pa^2(x^2+1)}\right]. \end{aligned} \tag{49}$$

Also generic solutions for Jacobi scalars are given in [3]. The scalar Z_3 is a linear function $Z_3 = C_{31}s + C_{32}$ (C_{31} and C_{32} are arbitrary constants) in every USSS spacetime and it is clear that the deviation vector $Z^\mu = Z_3 e_3^\mu$ does not generate conjugate points on G. The scalar $Z_1(s)$ is expressed in terms of integrals of the metric functions and for the metric (15), one gets

$$\begin{aligned} Z_1(l(s)) &= C_1\left(x^2+b^2\right)^{\frac{1}{2}}\left[\int \frac{\left(x^2+1\right)^{\frac{1}{2}}}{\left(x^2+b^2\right)^{\frac{3}{2}}}\mathrm{d}x - C_0\right] \\ &\equiv C_1\left(x^2+b^2\right)^{\frac{1}{2}}\left[Y(b,x)-C_0\right] . \end{aligned} \tag{50}$$

By definition, the Jacobi vector field $Z^\mu = Z_1 e_1^\mu$ must vanish at the initial point P_0, $x_0 = \frac{l_0}{a}$, and this determines the constant C_0 as $C_0 = Y(b, x_0)$. Then there exists a conjugate point P_1 to P_0 for $x_1 \neq x_0$ if $Z^\mu(x_1) = 0$ or $Z_1(x_1) = 0$. However, $Y(b,x)$ is a monotonically increasing function and $Y(b,x_1) = C_0$ implies $x_1 = x_0$. Thus, one sees that also the geodesic generated from G by means of the field $Z_1 e_1^\mu$ (actually the factor C_1 in (50) shows that there is a one-dimensional family of geodesics determined by $Z_1 e_1^\mu$) may intersect G only once at P_0. The non-existence of intersection points of G with the geodesics generated by Jacobi vectors $Z_1 e_1^\mu$ and $Z_3 e_3^\mu$ is in full agreement with the result shown in Sec. 3.2. In fact, the basis vectors e_1^μ and e_3^μ have their θ-components equal zero, what means that the geodesics generated by the corresponding Jacobi vector fields also lie in $\theta = \frac{\pi}{2}$ plane, while we have shown above that among coplanar geodesics only $G(k,L)$ and $G(k,-L)$ may intersect twice. Therefore, the most interesting Jacobi field is that $Z_2 e_2^\mu$ which goes off the $\theta = \frac{\pi}{2}$ plane. It reads

$$Z_2 e_2^\mu = \left(C_{21}\cos\varphi(l) + C_{22}\sin\varphi(l)\right)\delta_2^\mu . \tag{51}$$

Assuming that the initial point P_0 has $\varphi_0 = 0$, one considers the field $Z_\mu = C_2 \delta_2^\mu \sin\varphi(l)$ and possible conjugate points to P_0 are $\varphi_n(l) = n\pi$, $n = 1, 2, \ldots$ The angle φ on G is given in (37) and the first conjugate point is for $\varphi(l_1) = \pm\pi$ or $\sqrt{1-b^2}N(b, x_0, x_1) = \pi$. This is exactly the point P_1 of intersection of geodesics $G(k,L)$ and $G(k,-L)$ determined by Eq. (42). It follows from the previous considerations that this is the first and a single cut points to P_0 on the geodesic $G(k,L)$. (Each conjugate point is a cut point, but, in general, cut points are not conjugate ones.)

4. Globally maximal timelike geodesics and future cut points in the global Barriola–Vilenkin monopole spacetime

This is another USSS spacetime being an approximate solution of Einstein's field equations with a source in the form of a triplet of scalar fields. This solutions describes the spacetime outside a monopole of negligibly small mass [12, 13] and reads

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{1}{h^2}\mathrm{d}r^2 - r^2\mathrm{d}\Omega\,, \tag{52}$$

here, h is a dimensionless constant, $0 < h < 1$ and $r \in (0, \infty)$. All timelike geodesics are "flat", $\theta = \frac{\pi}{2}$, and are explicitly given by [3]

$$t = t_0 + k\, s(r)\,, \tag{53}$$

$$s(r) = \frac{1}{h\, p^2 |L|} \left(\sqrt{p^2 r^2 - 1} - \sqrt{p^2 r_0^2 - 1} \right), \tag{54}$$

$$\varphi(r) = \frac{\mathrm{sgn} L}{h} \left[\arccos\left(\frac{1}{p\, r}\right) - \arccos\left(\frac{1}{p\, r_0}\right) \right]. \tag{55}$$

As in the wormhole spacetime, each geodesic is parametrized by conserved energy per unit mass, $k > 1$, and conserved angular momentum, $L \neq 0$. Actually, the above formulae do not comprise radial geodesics, $L = 0$, which are globally maximal and we deal here exclusively with non-radial geodesic curves. As previously, $p^2 = (k^2 - 1)L^{-2} > 0$. The starting point of any geodesic $G(k, L)$ is $P_0(t_0, r_0, \theta = \frac{\pi}{2}, \varphi_0 = 0)$ and from (54), one sees that G is an outgoing curve, $\frac{\mathrm{d}s}{\mathrm{d}r} > 0$, and $p^2 r_0^2 \geq 1$ implying that a non-radial geodesic cannot emanate from $r_0 = 0$ and $r_0 > 0$. One is interested in future cut points to P_0 along $G(k, L)$ and these may occur for $r_1 > r_0$. One then has $p\, r_0 \geq 1$ and $p\, r > 1$. One takes two geodesics, $G_1(k_1, L_1)$ and $G_2(k_2, L_2)$ emanating from P_0 and lying in the same 2-plane $\theta = \frac{\pi}{2}$ and seeks for an intersection point $P_1(t_1, r_1, \frac{\pi}{2}, \bar{\varphi})$. Since the azimuthal variable $\varphi(r)$ depends only on parameter p, one easily shows, as in the Morris–Thorne wormhole spacetime, that for $L_1 L_2 > 0$, the geodesics will never intersect. The Barriola–Vilenkin monopole spacetime is globally hyperbolic and one again applies Theorem 9.12 in [11]. It follows that if P_0 has a future cut point P_1 on $G(k, L)$, then it is an intersection point of this curve with $G(k, -L)$. Both the geodesics are determined by the same parameter p^2 and for each value of r, their lengths and the time coordinates are equal. Their azimuthal coordinates are $\varphi_1(r) > 0$ and $\varphi_2(r) = -\varphi_1 < 0$, hence if they intersect at P_1, then $\bar{\varphi} = \varphi_1(r_1)$ and $\varphi_1(r_1) - \varphi_2(r_1) = 2\pi$, what implies $\bar{\varphi} = \varphi_1(r_1) = \pi$. From (55), this equation reads

$$\arccos\left(\frac{1}{p\, r_1}\right) - \arccos\left(\frac{1}{p\, r_0}\right) = \pi h\,. \tag{56}$$

Its solution is

$$\frac{1}{r_1} = \frac{1}{r_0} \left[\cos \pi\, h - \sqrt{p^2 r_0^2 - 1} \sin \pi h \right]. \tag{57}$$

This solution makes sense if $r_1 > r_0$ and this actually comprises two conditions: $r_1 > 0$ and $\frac{r_0}{r_1} < 1$. The first condition is equivalent to $\cos \pi\, h - \sqrt{p^2 r_0^2 - 1} \sin \pi\, h > 0$. For $h < 1$, there is $\sin \pi\, h > 0$ and $\cos \pi\, h >$

$\sqrt{p^2r_0^2-1}\sin\pi h > 0$ implies $\cos\pi h > 0$ or $0 < h < \frac{1}{2}$, while $h \geq \frac{1}{2}$ is excluded. Then the inequality is solved by

$$p\,r_0 < \frac{1}{\sin\pi h}\,. \tag{58}$$

If $h < \frac{1}{2}$ and (58) holds, then one easily sees from (57) that $\frac{r_0}{r_1} < 1$ and the intersection point P_1 does exist. P_1 is the first future cut point to P_0 on both $G(k,L)$ and $G(k,-L)$ and since $\bar{\varphi} = \varphi_1(r_1) = \pi$, it follows from considerations in Sec. 5 in [3] that P_1 coincides with the first future conjugate point Q_1 to P_0 on $G(k,L)$. This conjugate point is generated by a Jacobi vector field going off the plane $\theta = \frac{\pi}{2}$ (this is Jacobi field $Z^\mu = Z_2 e_2^\mu$ in [3]) since close geodesics lying in this plane cannot intersect $G(k,L)$ again.

For $h < \frac{1}{2}$ and $p\,r_0 < \frac{1}{\sin\pi h}$, the conjugate point to P_0 on $G(k,L)$ is identical with the cut point proper and in the plane $\theta = \frac{\pi}{2}$ there are only two globally maximal geodesics intersecting there. If the plane is rotated, one finds infinite number of intersecting pairs, all with the same energy k and opposite angular momenta, L and $-L$. For $p\,r_0 \to \frac{1}{\sin\pi h}$, the intersection point goes to infinity, $r_1 \to \infty$ and for $p\,r_0 > \frac{1}{\sin\pi h}$, there are no cut (and conjugate) points and each timelike geodesic is globally maximal on its entire segment. For $h \geq \frac{1}{2}$, each timelike geodesic is globally maximal.

Next, one may ask if there are further future cut points on $G(k,L)$ beyond P_1. Let there are cut points $P_n(t_n, r_n, \frac{\pi}{2}, \bar{\varphi}_n)$, $n = 1, 2, \ldots$, to P_0. It is clear that these are successive intersection points of geodesics $G(k,L)$ and $G(k,-L)$. These points satisfy $\bar{\varphi}_n = \varphi_1(r_n) = n\pi$ and Eq. (56) is replaced by

$$\arccos\left(\frac{1}{p\,r_n}\right) - \arccos\left(\frac{1}{p\,r_0}\right) = \pi n h\,. \tag{59}$$

The same equation arises if one seeks for future conjugate points to P_0 on $G(k,L)$. The solution is a direct generalization of (57)

$$\frac{1}{r_n} = \frac{1}{r_0}\left[\cos(\pi n h) - \sqrt{p^2r_0^2-1}\,\sin(\pi n h)\right]. \tag{60}$$

The condition for $r_n > 0$ is $h < \frac{1}{2n}$. If the parameter h is sufficiently small, the condition for $r_n > r_0$ is

$$p\,r_0 < \frac{1}{\sin(n\pi h)}\,. \tag{61}$$

One sees that whether the given geodesic $G(k,L)$ is globally maximal on its inextendible segment crucially depends on the metric parameter h. Cut points may exist only for $h < \frac{1}{2}$ and whether they actually do exist depends on the product $p\,r_0$ determined by the conserved quantities k and L and the radial coordinate of the initial point.

Finally, one considers the problem of whether the division of Barriola–Vilenkin spacetimes into those containing solely globally maximal timelike geodesics, $h \geq \frac{1}{2}$ and these wherein some geodesics may have cut points, is in some way marked in the spacetime curvature. In other terms, whether the value $h = \frac{1}{2}$ is distinguished not only by the solutions of the geodesic deviation equation, but also by the curvature. It turns out that it is not. In fact, the Einstein tensor has two non-zero components, $G_{00} = (1 - h^2)r^{-2}$ and $G_{11} = (1 - h^{-2})r^{-2}$ and is generated by a material source with the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{8\pi G} G_{\mu\nu} = \mathrm{diag}\left[\rho, p_1, 0, 0\right]. \tag{62}$$

Here, $\rho > 0$ is energy density measured by an observer with proper velocity $u^\alpha = \delta^\alpha_0$ and p_1 is interpreted as a radial pressure. This tensor is of type I [17] and the weak energy condition is fulfilled for any $0 < h < 1$. Hence, the critical value $h = \frac{1}{2}$ is deeply encoded only in the geodesic structure of the monopole spacetime.

5. Conclusions

Our search for the cut points on timelike geodesic curves shows a rather complicated structure even in a very narrow class of ultrastatic spherically symmetric spacetimes. By comparison of two such spacetimes, the Morris–Thorne wormhole and the global Barriola–Vilenkin monopole, one finds both deep similarities and some differences. In M–T wormhole with metric (15), a generic timelike geodesic is described by elliptic integrals $\mathrm{F}(x, k^2)$ and $\mathrm{K}(k^2)$ and its future (and past) cut points are determined by these functions. The entire geometric structure of this spacetime is determined in terms of dimensionless coordinates (t, x, θ, φ) and the metric in these variables contains no free parameters. Cut points in the wormhole are intersection points of coplanar $(\theta = \pi/2)$ geodesics of equal energy and equal and oppositely directed angular momenta. These points are also conjugate points determined by infinitesimally close geodesics lying in different 2-planes which are close to each other and intersecting. If a fiducial geodesic has a cut point to a given initial point on it, it is intersected there by infinite number of other geodesics emanating from the initial point and lying in rotated 2-planes. The cut points exist only for energies and angular momenta belonging to very narrow intervals; beyond these intervals, timelike geodesics are globally maximal on their complete segments. This feature may be recognized only by analyzing the elliptic integrals determining the curves.

The geodesic structure of the Barriola–Vilenkin monopole is similar to that in the wormhole spacetime in that if a (non-radial) geodesic has a cut point proper, then it arises as an intersection point of two geodesics with equal energies and opposite angular momenta. In both the spacetimes the cut points, if exist, are also conjugate points determined by the Jacobi vector field directed off the $\theta = \pi/2$ plane in which the fiducial geodesic lies (the Jacobi field is proportional to the vector e_2^μ of the spacelike basis parallelly transported along the geodesic); this means that coplanar close geodesics cannot intersect twice. In the wormhole spacetime, the existence of a cut point to an initial P_0 on the given geodesic depends only on energy and spin of this geodesic and on location of P_0 (close or not to the "throat"). Yet in the B–V monopole spacetime, all timelike geodesics are globally maximal between any pair of their points if the metric parameter h is in the range of $1/2 \leq h < 1$ and cut points to some initial points on some geodesics appear if $0 < h < 1/2$. The critical value $h = 1/2$ is not distinguished by the spacetime curvature. A geodesic may have n cut points to the initial P_0, $n = 1, 2, \ldots$, if h is sufficiently small, $h < (2n)^{-1}$.

Clearly, all the USSS spacetimes inherit the feature that radial timelike geodesics are globally maximal.

The work of both the authors was supported by the John Templeton Foundation Grant "Conceptual Problems in Unification Theories" No. 60671.

Appendix

Ultrastatic spacetimes and inertial reference frames

In most textbooks on classical mechanics (see *e.g.* [14, 15]) and also in many ones on special relativity (*e.g.* [16]) it is claimed that the inertial reference frame may be uniquely defined in purely dynamical terms: as that frame in which a freely moving body (*i.e.* one which is not acted upon by external forces) moves uniformly with constant velocity or remains at rest. Discovery of ultrastatic spacetimes has shown that this definition is not unique since any coordinate system in these spacetimes which explicitly exhibits that they are ultrastatic, dynamically imitates the inertial frame. It has been found that in these spacetimes, there are no "true" gravitational forces, only "inertial" ones. More precisely, ultrastatic spacetimes are defined as those which admit a timelike Killing vector field which is covariantly constant $\nabla_\alpha K_\beta = 0$. This implies that the field does not accelerate (expand), rotate or deform and the reference frame determined by this vector field mimicks the notion of the inertial frame in Minkowski spacetime (see [7] and numerous references on inertial forces therein). In ultrastatic spherically symmetric spacetimes, it was shown that a free particle has a constant velocity relative to the comoving frame (its motion is uniform) [3].

Below, we prove a generic theorem showing to what extent a comoving coordinate system in a generic ultrastatic spacetime imitates the true inertial frame. From its definition, the timelike Killing vector characterizing any ultrastatic spacetime is a gradient, $K_\alpha = \partial_\alpha t$, and has a constant norm. Then it may be chosen as $K_\alpha = \delta_0^\alpha$ and the corresponding system is comoving (Gauss normal geodesic) with the metric [7]

$$\mathrm{d}s^2 = \mathrm{d}t^2 + g_{ik}(x^j)\mathrm{d}x^i\mathrm{d}x^k\,, \qquad i,j,k = 1,2,3\,, \tag{A.1}$$

where $\det(g_{ik}) < 0$. Distinct ultrastatic spacetimes differ in the three-metric g_{ik}, which is time-independent.

Proposition A.1. *A timelike geodesic motion in any ultrastatic spacetime may be described as a free motion in the constant time 3-space subject to covariant nonrelativistic Newtonian equations of motion with vanishing force. The free motion has following properties:*

(i) the 3-velocity has a constant norm, hence the motion is uniform;

(ii) its trajectory is a geodesic of the 3-space.

Proof. The timelike Killing vector K^α generates along any timelike geodesic the integral of energy per unit mass $k = \frac{E}{m\,c^2}$ and $\frac{\mathrm{d}t}{\mathrm{d}s} = k > 0$ from (2). The $t = \mathrm{const}$ space has the metric

$$\mathrm{d}\sigma^2 = \gamma_{ij}\left(x^k\right)\mathrm{d}x^i\mathrm{d}x^j\,, \tag{A.2}$$

where $\gamma_{ij} = -g_{ij}$ (the signature is $+---$). The metric γ_{ij} determines the connection $\bar{\Gamma}^i_{jk}(\gamma)$. The spacetime connection is $\Gamma^i_{jk}(g) = \bar{\Gamma}^i_{jk}(\gamma)$, other components of $\Gamma^\alpha_{\mu\nu}(g)$ vanish. The timelike geodesic equation has three independent components

$$\frac{\mathrm{D}}{\mathrm{d}s}\frac{\mathrm{d}x^i}{\mathrm{d}s} = \frac{\mathrm{d}^2x^i}{\mathrm{d}s^2} + \Gamma^i_{jk}(g)\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} = 0\,, \tag{A.3}$$

whereas $\frac{\mathrm{D}}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}k \equiv 0$. The connection $\bar{\Gamma}^i_{jk}(\gamma)$ in the space generates two absolute derivatives with respect to the scalars σ and t, $\frac{\bar{\mathrm{D}}}{\mathrm{d}s}$ and $\frac{\bar{\mathrm{D}}}{\mathrm{d}t}$. One defines the particle's 3-velocity $v^i \equiv \frac{\mathrm{d}x^i}{\mathrm{d}t}$ along the geodesic, which is a 3-vector with respect to purely spatial coordinate transformations. From $\frac{\mathrm{d}t}{\mathrm{d}s} = k$, one gets $v^i = \frac{1}{k}\frac{\mathrm{d}x^i}{\mathrm{d}s}$.

Next, one postulates, as in classical mechanics, that any particle motion in the space is subject to nonrelativistic Newtonian equations of motion,

$$\frac{\bar{\mathrm{D}}}{\mathrm{d}t}\left(m\,v^i\right) \equiv m\left[\frac{\mathrm{d}v^i}{\mathrm{d}t} + \bar{\Gamma}^i_{jk}v^j v^k\right] = F^i\left(t, x^j\right)\,, \tag{A.4}$$

where F^i is some external force. In the case of a free particle (no other interactions besides gravitation), one has $\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}x^i}{\mathrm{d}s} = k^2\frac{\mathrm{d}v^i}{\mathrm{d}t}$ and the three geodesic equations (A.3) take on the form

$$\frac{\mathrm{D}}{\mathrm{d}s}\frac{\mathrm{d}x^i}{\mathrm{d}s} \equiv k^2\left[\frac{\mathrm{d}v^i}{\mathrm{d}t} + \bar{\Gamma}^i_{jk}v^j v^k\right] = 0 \tag{A.5}$$

implying that the force in the Newtonian equations (A.4) vanish, $F^i = 0$. Hence, the free (geodesic) motion in the ultrastatic spacetime also manifests itself as a free motion in the space (if parametrized by time t as an external parameter).

The length V of the 3-velocity is $V^2 = \gamma_{ij}v^i v^j > 0$ and (A.4) shows that it is constant

$$\frac{\mathrm{d}}{\mathrm{d}t}V^2 = \frac{\bar{\mathrm{D}}}{\mathrm{d}t}\left(\gamma_{ij}v^i v^j\right) = 0\,,$$

or the motion in the space is uniform. Clearly, this motion is not rectilinear since straight lines, in general, do not exist in curved spaces, yet the trajectory $x^i = x^i(t)$ is a geodesic of the space. In fact, along the trajectory in the space, one has $\mathrm{d}x^i = v^i\mathrm{d}t$ and $\mathrm{d}\sigma^2 = V^2\mathrm{d}t^2$, hence $\mathrm{d}\sigma = V\mathrm{d}t$ and the relationship is linear, $\sigma = V\,t + \mathrm{const}$. The trajectory is parameterized by its length, $x^i = x^i(t(\sigma))$. Its tangent vector $\frac{\mathrm{d}x^i}{\mathrm{d}\sigma} = \frac{v^i}{V}$ satisfies

$$\frac{\bar{\mathrm{D}}}{\mathrm{d}\sigma}\frac{\mathrm{d}x^i}{\mathrm{d}\sigma} = \frac{\mathrm{d}t}{\mathrm{d}\sigma}\frac{\bar{\mathrm{D}}}{\mathrm{d}t}\left(\frac{v^i}{V}\right) = \frac{1}{V^2}\frac{\bar{\mathrm{D}}}{\mathrm{d}t}v^i = 0\,;$$

the trajectory in the space generated by the timelike geodesic in the spacetime is a geodesic of this space.

If an ultrastatic spacetime is also spherically symmetric (USSS), then radial timelike geodesics perfectly imitate the inertial motion in Minkowski space since they are straight lines on Minkowski 2-plane. In fact, in the adapted coordinates, the metric reads

$$\mathrm{d}s^2 = \mathrm{d}t^2 - e^{\lambda(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{A.6}$$

where $r \in (r_m, r_M)$. One introduces a new radial coordinate ρ by $\mathrm{d}r = e^{-\lambda/2}\mathrm{d}\rho$, then

$$\rho = \int e^{\lambda/2}\mathrm{d}r \equiv W(r)\,. \tag{A.7}$$

Function $W(r)$ is invertible in the whole range of r and $r = W^{-1}(\rho)$. The metric is now

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\rho^2 - r^2(\rho)\mathrm{d}\Omega^2\,. \tag{A.8}$$

A radial geodesic C is $t = t(s)$, $r = r(s)$, $\theta = \theta_0$ and $\varphi = \varphi_0$. In the coordinates $(t, \rho, \theta, \varphi)$, its tangent vector is $\dot{x}^\alpha = (\dot{t}, \dot{\rho}, 0, 0)$, where $\dot{t} = k$. Then the normalization $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 1$ yields $\dot{\rho} = +\sqrt{k^2-1}$ for outgoing C and, finally, $t = t_0 + ks$ and $\rho(s) = \sqrt{k^2-1}s + \rho_0$. C is a straight line on Minkowski plane (t, ρ). By a hyperbolic rotation on the plane, each radial C may be identified with a time coordinate line on this plane. (However, this hyperbolic rotation is not an ultrastatic transformation according to the definition given in [7].)

The genuine inertial frame exists only in Minkowski spacetime and should be defined in purely geometric terms.

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