Differential Forms, Linked Fields and the u-Invariant

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Abstract

We associate an Albert form to any pair of cyclic algebras of prime degree p over a field F with $\mathrm{char}(F) = p$ which coincides with the classical Albert form when $p = 2$. We prove that if every Albert form is isotropic then $H^4(F) = 0$. As a result, we obtain that if F is a linked field with $\mathrm{char}(F) = 2$ then its u-invariant is either $0, 2, 4$ or 8.

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1. Introduction

Given a field F, a quaternion algebra over F is a central simple F-algebra of degree 2. The maximal subfields of quaternion division algebras over F are quadratic field extensions of F. When $\mathrm{char}(F) \neq 2$, all quadratic field extensions are separable. When $\mathrm{char}(F) = 2$, there are two types of quadratic field extensions: the separable type which is of the form $F[x : x^2 + x = \alpha]$ for some $\alpha \in F \setminus \{\lambda^2 + \lambda : \lambda \in F\}$, and the inseparable type which is of the form $F[\sqrt{\alpha}]$ for some $\alpha \in F^\times \setminus (F^\times)^2$. In this case, any quaternion division algebra contains both types of field extensions, which can be seen by its symbol presentation

$$[\alpha, \beta)_{2,F} = F\langle x, y : x^2 + x = \alpha, y^2 = \beta, yxy^{-1} = x + 1\rangle .$$

If $\mathrm{char}(F) = p$ for some prime $p > 0$, we let $\wp(F)$ denote the additive subgroup $\{\lambda^p - \lambda : \lambda \in F\}$. Then we may consider cyclic division algebras over F of degree p. Any such algebra admits a symbol presentation

$$[\alpha, \beta)_{p,F} = F\langle x, y : x^p - x = \alpha, y^p = \beta, yxy^{-1} = x + 1\rangle$$

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where $\alpha \in F \setminus \wp(F)$ and $\beta \in F^\times \setminus (F^\times)^p$. In particular, these algebras contain both cyclic separable field extensions of F (e.g. $F[x]$) of degree p and purely inseparable field extensions of F of degree p (e.g. $F[y]$).

Two quaternion F-algebras are called *linked* if they share a common maximal subfield. When $\mathrm{char}(F) = 2$, the notion of linkage can be refined to *separable linkage* and *inseparable linkage* depending on the type of quadratic field extension of the center they share. Inseparable linkage implies separable linkage, but the converse does not hold in general (see [Lam02]). This observation was extended to Hurwitz algebras in [EV05] and to quadratic Pfister forms in [Fai06]. We similarly call cyclic p-algebras of prime degree p over a field F *separably linked* (resp. *inseparably linked*) if they share a common maximal subfield that is a cyclic separable (resp. purely inseparable) extension of F of degree p. The above linkage result for quaternion algebras was generalized to this setting in [Cha15].

A field F is called *linked* if every two quaternion F-algebras are linked. When $\mathrm{char}(F) = 2$, a field F is called inseparably linked if every two quaternion F-algebras are inseparably linked. Note that any inseparably linked field is clearly linked.

The u-invariant of a field F, denoted by $u(F)$, is defined to be the maximal dimension of an anisotropic nonsingular quadratic form over F of finite order in W_qF. Note that when -1 can be written as a sum of squares in F, and in particular when $\mathrm{char}(F) = 2$, every form in I_qF is of finite order. It was proven in [EL73, Main Theorem] that if F is a linked field with $\mathrm{char}(F) \neq 2$ then the possible values $u(F)$ can take are $0, 1, 2, 4$ and 8. For fields F of characteristic 2, it was shown in [Bae82, Theorem 3.1] that F is inseparably linked if and only if $u(F) \leqslant 4$. In particular, this means that a linked field F with $u(F) = 8$ is not inseparably linked. For example, the field of iterated Laurent series in two variables $\mathbb{F}_2((\alpha))((\beta))$ over $\mathbb{F}_2$ is linked by [AJ95, Corollary 3.5], but not inseparably linked, because its u-invariant is 8. There are also many examples of inseparably linked fields, such as local fields, global fields and Laurent series over perfect fields (see [CDL16, Section 6]). In [Fai06, Theorem 3.3.10] it was shown that if F is a linked field and $I_q^4F = 0$ (see Section 2) then $u(F)$ is either $0, 2, 4$ or 8. We are interested in removing the assumption that $I_q^4F = 0$ from this result.

We approach this problem from the more general setting of differential forms over fields of characteristic p (see Section 3). We associate an Albert form to any pair of cyclic algebras of degree p over a field F with $\mathrm{char}(F) = p$ which coincides with the classical Albert form when $p = 2$. We prove that if every Albert form is isotropic then $H^4(F) = 0$. When $p = 2$, this means that if F is linked then $I_q^4F = 0$. Together with [Fai06, Theorem 3.3.10], this gives that the possible values of $u(F)$ are $0, 2, 4$ and 8.

2. Bilinear and Quadratic Pfister Forms

We recall certain results and terminology we use from quadratic form theory. We refer to [EKM08, Chapters 1 and 2] for standard notation, basic results and as a general reference on quadratic forms.

Let F be a field of characteristic 2. A symmetric bilinear form over F is a map $B : V \times V \to F$ satisfying $B(v, w) = B(w, v)$, $B(cv, w) = cB(v, w)$ and $B(v + w, t) = B(v, t) + B(w, t)$ for all $v, w, t \in V$ and $c \in F$ where V is an n-dimensional F-vector space. A symmetric bilinear form B is degenerate if there exists a vector $v \in V \setminus \{0\}$ such that $B(v, w) = 0$ for all $w \in V$. If such a vector does not exist, we say that B is nondegenerate. Two symmetric bilinear forms $B : V \times V \to F$ and $B' : W \times W \to F$ are isometric if there exists an isomorphism $M : V \to W$ such that $B(v, v') = B'(Mv, Mv')$ for all $v, v' \in V$.

A quadratic form over F is a map $\varphi : V \to F$ such that $\varphi(av) = a^2\varphi(v)$ for all $a \in F$ and $v \in V$ and the map defined by $B_\varphi(v, w) = \varphi(v + w) - \varphi(v) - \varphi(w)$ for all $v, w \in V$ is a bilinear form. The bilinear form B_φ is called the polar form of φ and is clearly symmetric. Two quadratic forms $\varphi : V \to F$ and $\psi : W \to F$ are isometric if there exists an isomorphism $M : V \to W$ such that $\varphi(v) = \psi(Mv)$ for all $v \in V$. We are interested in the isometry classes of quadratic forms, so when we write $\varphi = \psi$ we actually mean that they are isometric.

We say that φ is singular if B_φ is degenerate, and that φ is nonsingular if B_φ is nondegenerate. Every nonsingular form φ is even dimensional and can be written as

$$\varphi = [\alpha_1, \beta_1] \perp \cdots \perp [\alpha_n, \beta_n]$$

for some $\alpha_1, \ldots, \beta_n \in F$, where $[\alpha, \beta]$ denotes the two-dimensional quadratic form $\psi(x, y) = \alpha x^2 + xy + \beta y^2$ and $\perp$ denotes the orthogonal sum of quadratic forms.

We say that a quadratic form $\varphi : V \to F$ is isotropic if there exists a vector $v \in V \setminus \{0\}$ such that $\varphi(v) = 0$. If such a vector does not exist, we say that φ is anisotropic. The unique nonsingular two-dimensional isotropic quadratic form is $\mathbb{H} = [0, 0]$, which we call the hyperbolic plane. A hyperbolic form is an orthogonal sum of hyperbolic planes. We say that two nonsingular quadratic forms are Witt equivalent if their orthogonal sum is a hyperbolic form.

We denote by $\langle \alpha_1, \ldots, \alpha_n \rangle$ the diagonal bilinear form given by $(x, y) \mapsto \sum_{i=1}^n \alpha_i x_i y_i$. Given two symmetric bilinear forms $B_1 : V \times V \to F$ and $B_2 : W \times W \to F$, the tensor product of B_1 and B_2 denoted $B_1 \otimes B_2$ is the unique F-bilinear map $B_1 \otimes B_2 : (V \otimes_F W) \times (V \otimes_F W) \to F$ such that

$$(B_1 \otimes B_2)\left((v_1 \otimes w_1), (v_2 \otimes w_2)\right) = B_1(v_1, v_2) \cdot B_2(w_1, w_2)$$

for all $w_1, w_2 \in W, v_1, v_2 \in V$. A bilinear n-fold Pfister form over F is a symmetric bilinear form isometric to $\langle 1, \alpha_1 \rangle \otimes \cdots \otimes \langle 1, \alpha_n \rangle$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in F^\times$. We denote such a form by $\langle\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle\rangle$. By convention, the bilinear 0-fold Pfister form is $\langle 1 \rangle$.

Let $B : V \times V \to F$ be a symmetric bilinear form over F and $\varphi : W \to F$ be a quadratic form over F. We may define a quadratic form $B \otimes \varphi : V \otimes_F W \to F$ determined by the rule that $(B \otimes \varphi)(v \otimes w) = B(v, v) \cdot \varphi(w)$ for all $w \in W, v \in V$. We call this quadratic form the tensor product of B and φ. A quadratic n-fold Pfister form over F is a tensor product of a bilinear $(n-1)$-fold Pfister form $\langle\langle \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \rangle\rangle$ and a two-dimensional quadratic form $[1, \beta]$ for some $\beta \in F$. We denote such a form by $\langle\langle \alpha_1, \ldots, \alpha_{n-1}, \beta]]$. Quadratic n-fold Pfister forms are isotropic if and only if they are hyperbolic (see [EKM08, (9.10)]).

The Witt equivalence classes of nonsingular quadratic forms over F form an abelian group, called the Witt group of F, with $\perp$ as the binary group operation and $\mathbb{H}$ as the zero element. We denote this group by $I_q F$ or $I_q^1 F$. This group is generated by scalar multiples of quadratic 1-fold Pfister forms. Let $I_q^n F$ denote the subgroup generated by scalar multiples of quadratic n-fold Pfister forms over F.

Let $\varphi = [\alpha_1, \beta_1] \perp \cdots \perp [\alpha_n, \beta_n]$ be a nonsingular quadratic form. The Arf invariant of φ, denoted $\triangle(\varphi)$, is the class of $\alpha_1\beta_1 + \cdots + \alpha_n\beta_n$ in the additive group $F/\wp(F)$ (see [EKM08, §13]). The Arf invariant only depends on the class of the form φ in $I_q F$. An Albert form over a field of characteristic 2 is a 6-dimensional nonsingular quadratic form with trivial Arf invariant. To any central simple algebra isomorphic to the tensor product of two quaternion algebras over F, we may associate the Witt class of the orthogonal sum of the two norm forms of the quaternion algebras (these norm forms are 2-fold quadratic Pfister forms). This uniquely determines a similarity class of Albert forms. Conversely, every similarity class of Albert forms determines such a central simple algebra over F (see [MS89] for more details).

3. Differential Forms

Let F be a field of characteristic $p > 0$. For $a \in F$, we denote the extension of F isomorphic to $F[T]/(T^p - T - a)$ by F_a. If $a \notin \wp(F)$, then this is a cyclic field extension of degree p and we denote the norm map by $N_{F_a/F} : F_a \to F$. Otherwise F_a is an étale extension isomorphic to $F \times \ldots \times F$ (p times), and one defines a norm map by taking the determinant of the F-linear map given by multiplying by an element of F_a. We again denote this map by $N_{F_a/F}$. It is easily seen that $N_{F_a/F}$ has a non-trivial zero if and only if F_a is not a field if and only if $a \notin \wp(F)$.

The space $\Omega^1(F)$ of absolute differential 1-forms over F is defined to be the F-vector space generated by symbols da, $a \in F$, subject to the relations given by additivity, $d(a + b) = da + db$, and the product rule, $d(ab) = adb + bda$. In particular, one has $d(F^p) = 0$ for $F^p = \{a^p \mid a \in F\}$, and $d : F \to \Omega^1(F)$ is an F^p-derivation.

The space of n-differentials $\Omega^n(F)$ ($n \geqslant 1$) is then defined by the n-fold exterior power, $\Omega^n(F) := \bigwedge^n(\Omega^1(F))$, which is therefore an F-vector space generated by symbols $da_1 \wedge \ldots \wedge da_n$, $a_i \in F$. The derivation d extends to an operator $d : \Omega^n(F) \to \Omega^{n+1}(F)$ by $d(a_0 da_1 \wedge \ldots \wedge da_n) = da_0 \wedge da_1 \wedge \ldots \wedge da_n$. We put $\Omega^0(F) = F$, $\Omega^n(F) = 0$ for $n < 0$, and $\Omega(F) = \bigoplus_{n \geqslant 0} \Omega^n(F)$, the algebra of differential forms over F with multiplication naturally defined by

$$(a_0 da_1 \wedge \ldots \wedge da_n)(b_0 db_1 \wedge \ldots \wedge db_m) = a_0 b_0 da_1 \wedge \ldots \wedge da_n \wedge db_1 \wedge \ldots \wedge db_m \,.$$

Note that the wedge product is anti-commutative. That is $da \wedge db = -db \wedge da$.

There exists a well-defined group homomorphism $\Omega^n(F) \to \Omega^n(F)/d\Omega^{n-1}(F)$, the Artin-Schreier map $\wp$, which acts on logarithmic differentials as follows:

$$b\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n} \longmapsto (b^p - b)\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n}$$

We define $H^{n+1}(F) := \operatorname{coker}(\wp)$. The connection between the groups $H^{n+1}(F)$ and quadratic forms was shown by Kato [Kato82]:

Theorem 3.1. *Let F be a field of characteristic 2. Then there is an isomorphism $\alpha_{n,F} : H^{n+1}(F) \xrightarrow{\sim} I_q^{n+1}(F)/I_q^{n+2}(F)$ defined on generators as follows:*

$$b\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n} \longmapsto \langle\langle a_1, \ldots, a_n, b]] \mod I_q^{n+2}(F)\,.$$

The p-torsion part of the Brauer group of F is known to be isomorphic to $H^2(F)$ (see [GS06, Section 9.2]). The isomorphism is given by

$$[\alpha, \beta)_{p,F} \mapsto \alpha\frac{d\beta}{\beta}\,.$$

The following lemma records certain equalities for later use.

Lemma 3.2. *Take $a_1 \ldots, a_n \in F^\times$ and $b \in F \setminus \wp(F)$. Let $0 \neq \beta = N_{F_b/F}(u)$ for some $u \in F_b$.*

(*a*) *For all $i = 1, \ldots, n$ we have*

$$b\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n} = (b + a_i)\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n} \mod d\Omega^{n-1}(F)\,.$$

(*b*) *For all $i = 1, \ldots, n$ we have in $H^{n+1}(F)$*

$$b\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n} = b\frac{da_1}{a_1} \wedge \ldots \wedge \frac{d(a_i\beta)}{a_i\beta} \wedge \ldots \wedge \frac{da_n}{a_n}\,.$$

(*c*) *For all $i = 1, \ldots, n$ we have in $H^{n+1}(F)$*

$$b\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n} = (b + a_i\beta)\frac{da_1}{a_1} \wedge \ldots \wedge \frac{d(a_i\beta)}{a_i\beta} \wedge \ldots \wedge \frac{da_n}{a_n}\,.$$

Proof. In all the statements, it suffices to consider the case $i = 1$. Note that as

$$d\left(b^{-1}\right) \wedge db = d\left(\frac{b^{p-1}}{b^p}\right) \wedge db = -\frac{b^{p-2}}{b^p} db \wedge db = 0$$

for all $b \in F^\times$ we have

$$d\left(b\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n}\right) = db \wedge \frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n} \in d\Omega^{n-1}(F)\,.$$

We first show (*a*). We have that

$$a_1\frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n} = da_1 \wedge \frac{da_2}{a_2} \wedge \ldots \wedge \frac{da_n}{a_n} = d\left(a_1\frac{da_2}{a_2} \wedge \ldots \wedge \frac{da_n}{a_n}\right) \in d\Omega^{n-1}(F)\,.$$

Hence the result follows from the additivity of d.

For (*b*), it suffices to consider the case $n = 1$. In this case, the result follows from [BO13, VII.1.9, (2)] via identifying cyclic p-algebras and symbols in $H^2(F)$. Statement (*c*) then follows immediately from (*a*) and (*b*). □

4. Albert p-forms

Let F be a field of characteristic $p > 0$. For $\alpha, \beta \in F$ and $\gamma, \delta \in F^\times$ the map $A(\alpha, \beta, \gamma, \delta) : F_{\alpha+\beta} \oplus F_\alpha \oplus F_\beta \to F$ given by

$$(x, y, z) \mapsto N_{F_{\alpha+\beta}/F}(x) + \gamma N_{F_\alpha/F}(y) + \delta N_{F_\beta/F}(z)$$

is called an Albert p-form. By the pure part of the Albert p-form $A(\alpha, \beta, \gamma, \delta)$ we mean the restriction of $A(\alpha, \beta, \gamma, \delta)$ to $F \oplus F_\alpha \oplus F_\beta \to F$.

Remark 4.1. Note that for $p = 2$ an Albert p-form is an Albert form as defined in Section 2. We also note the following:

1. If the Albert p-form above has a non-trivial zero, then the cyclic algebras $[\alpha, \gamma)_{p,F}$ and $[\beta, \delta)_{p,F}$ are separably linked. If $p = 2$, then the converse also holds.
2. If the pure part of the Albert p-form above has a nontrivial zero, then the cyclic algebras $[\alpha, \gamma)_{p,F}$ and $[\beta, \delta)_{p,F}$ are inseparably linked. If $p = 2$, then the converse also holds.

Proof. The 'if' statements follow immediately from [Cha17, Lemma 2.2]. The converse statements for $p = 2$ can be found in [MS89]. $\square$

Lemma 4.2. *Take $\alpha \in F$ and $\beta \in F^\times$. Then there exist $\alpha_1, \alpha_2 \in F$ and $u \in F^\times$ such that $\alpha = \alpha_1 + \alpha_2$ and*

$$\alpha \frac{d\beta}{\beta} = \alpha_1 \frac{d\beta}{\beta} = \alpha_2 \frac{d\beta u}{\beta u} \in H^2(F)\,.$$

Proof. If $\alpha \in \wp(F)$ then the result is trivial. Otherwise let $t = \frac{\alpha\beta - \alpha}{\beta}$, $\alpha_1 = \alpha + \beta t^p$ and $\alpha_2 = \alpha - \beta(t^p - t + \alpha)$. Then $\alpha = \alpha_1 + \alpha_2$. If $t = 0$ then $\beta = 1$ and again the result is trivial. If $t \neq 0$ then both t^p and $u = -(t^p - t + \alpha)$ are norms of elements in the field F_a (using $-1 = (-1)^p$). Hence applying Lemma 3.2, (c) gives the result. $\square$

Theorem 4.3. *Let F be a field of characteristic p. If every Albert p-form over F has a non-trivial zero then $H^4(F) = 0$.*

Proof. Let $\alpha \in F \setminus \wp(F), \beta, \gamma, \delta \in F^\times$ and $\omega = \alpha \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma} \wedge \frac{d\delta}{\delta} \in H^4(F)$. By Lemma 4.2 there exist $\alpha_1, \alpha_2 \in F$ and $u \in F^\times$ such that $\alpha = \alpha_1 + \alpha_2$ and

$$\alpha \frac{d\beta}{\beta} = \alpha_1 \frac{d\beta}{\beta} = \alpha_2 \frac{d\beta u}{\beta u} \in H^2(F)\,. \tag{1}$$

If α_2 or $\alpha_1 \in \wp(F)$, we have that $\omega = 0 \in H^4(F)$. Therefore we may assume that F_{α_2}/F and F_{α_1}/F are non-trivial extensions. In particular the respective norm forms have no non-trivial zeros.

By the hypothesis, the Albert p-form $A(\alpha_1, \alpha_2, \gamma, \delta)$ has a non-trivial zero. That is, there exist $x \in F_\alpha$, $y \in F_{\alpha_1}$ and $z \in F_{\alpha_2}$ not all zero such that

$$N_{F_\alpha/F}(x) + \gamma N_{F_{\alpha_1}/F}(y) + \delta N_{F_{\alpha_2}/F}(z) = 0\,. \tag{2}$$

If $x = 0$, which holds if and only if $N_{F_a/F}(x) = 0$, then

$$\gamma N_{F_{\alpha_1}/F}(y) = -\delta N_{F_{\alpha_2}/F}(z) = \delta N_{F_{\alpha_2}/F}(-z) \neq 0\,.$$

Fix $r = N_{F_{\alpha_1}/F}(y)$ and $s = N_{F_{\alpha_2}/F}(-z)$. Then by (1) and Lemma 3.2, (c) we have

$$\begin{aligned}\omega &= \alpha_1\frac{d\beta}{\beta}\wedge\frac{d\gamma}{\gamma}\wedge\frac{d\delta}{\delta} = \alpha_1\frac{d\beta}{\beta}\wedge\frac{d\gamma r}{\gamma r}\wedge\frac{d\delta}{\delta}\\ &= \alpha_2\frac{d\beta u}{\beta u}\wedge\frac{d\gamma r}{\gamma r}\wedge\frac{d\delta}{\delta} = \alpha_2\frac{d\beta u}{\beta u}\wedge\frac{d\gamma r}{\gamma r}\wedge\frac{d\delta s}{\delta s} = 0\,,\end{aligned}$$

where the last equality follows from $\gamma r = \delta s$.

Assume now that $x \neq 0$, and hence $N_{F_a/F}(x) \neq 0$. Let $\eta \in F_a$ be such that $N_{F_a/F}(\eta) = \alpha$. Fix $q = N_{F_a/F}(\eta x^{-1})$. Multiplying (2) by q and using the multiplicativity of the norm form gives $\alpha + q\gamma N_{F_{\alpha_1}/F}(y) + q\delta N_{F_{\alpha_2}/F}(z) = 0$. As q is also a norm of an element in F_α, Lemma 3.2, (c) gives

$$\omega = \alpha\frac{d\beta}{\beta}\wedge\frac{dq\gamma}{q\gamma}\wedge\frac{dq\delta}{q\delta}\,.$$

Hence we may assume that $\omega = \alpha\frac{d\beta}{\beta}\wedge\frac{d\gamma}{\gamma}\wedge\frac{d\delta}{\delta}$ and that $\alpha+\gamma N_{F_{\alpha_1}/F}(y)+\delta N_{F_{\alpha_2}/F}(z) = 0$. Fix

$$r = \begin{cases} N_{F_{\alpha_1}}(y) & \text{if } y \neq 0\\ 1 & \text{otherwise}\end{cases} \quad\text{and}\quad s = \begin{cases} N_{F_{\alpha_2}}(z) & \text{if } z \neq 0\\ 1 & \text{otherwise}\end{cases}\,.$$

Then by (1) and Lemma 3.2, (c) we have for some $u \in F^\times$

$$\begin{aligned}\omega &= \alpha_1\frac{d\beta}{\beta}\wedge\frac{d\gamma}{\gamma}\wedge\frac{d\delta}{\delta} = \alpha_1\frac{d\beta}{\beta}\wedge\frac{d\gamma r}{\gamma r}\wedge\frac{d\delta}{\delta} = \alpha_2\frac{d\beta u}{\beta u}\wedge\frac{d\gamma r}{\gamma r}\wedge\frac{d\delta}{\delta}\\ &= \alpha_2\frac{d\beta u}{\beta u}\wedge\frac{d\gamma r}{\gamma r}\wedge\frac{d\delta s}{\delta s} = \alpha\frac{d\beta}{\beta}\wedge\frac{d\gamma r}{\gamma r}\wedge\frac{d\delta s}{\delta s}\\ &= (\alpha + \gamma N_{F_{\alpha_1}/F}(y) + \delta N_{F_{\alpha_2}/F}(z))\frac{d\beta}{\beta}\wedge\frac{d\gamma r}{\gamma r}\wedge\frac{d\delta s}{\delta s} = 0\,.\end{aligned}$$

□

Theorem 4.4. *Let F be a field of characteristic p. Suppose that for all $\alpha \in F$ and $\gamma, \delta \in F^\times$, the pure part of the Albert p-form $A(\alpha, \alpha, \gamma, \delta)$ has a non-trivial zero. Then $H^3(F) = 0$.*

Proof. Let $\alpha, \gamma, \delta \in F^\times$ and $\omega = \alpha\frac{d\gamma}{\gamma}\wedge\frac{d\delta}{\delta} \in H^3(F)$. We may assume that $\alpha \notin \wp(F)$ and hence that the field extension F_α/F is non-trivial. Consider the Albert p-form $A(\alpha, \alpha, \gamma, \delta)$. By the hypothesis, there exist $x \in F$ and $y, z \in F_\alpha$ not all zero such that

$$x^p + \gamma N_{F_\alpha/F}(y) + \delta N_{F_\alpha/F}(z) = 0\,.$$

Clearly at least one of y or z must be non-zero. Fix

$$r = \begin{cases} N_{F_\alpha/F}(y) & \text{if } y \neq 0\\ 1 & \text{otherwise}\end{cases} \quad\text{and}\quad s = \begin{cases} N_{F_\alpha/F}(-z) & \text{if } z \neq 0\\ 1 & \text{otherwise}\end{cases}\,.$$

Then by Lemma 3.2, (*b*) we have

$$\omega = \alpha \frac{d\gamma r}{\gamma r} \wedge \frac{d\delta s}{\delta s} .$$

As γr and δs differ by an element in F^p we have $\frac{d\gamma r}{\gamma r} \wedge \frac{d\delta s}{\delta s} = 0$ and hence $\omega = 0$. □

Remark 4.5. Over fields of characteristic 2, the pure parts of Albert 2-forms of the type $A(\alpha, \alpha, \gamma, \delta)$ correspond (up to scaling) to 5-dimensional Pfister neighbours (see [EKM08, (23.10)]) and hence it is clear that if all these forms are isotropic, then $I_q^3 F$ is trivial.

By Remark 4.1 and Theorem 4.3 have that $H^4(F) = 0$ for a linked field F of characteristic 2.

Question 4.6. *Let F be a field of characteristic $p > 2$. If every two cyclic algebras of degree p over F are separably linked, is $H^4(F) = 0$? If every two such algebras are inseparably linked, is $H^3(F) = 0$?*

5. Linked Fields of characteristic 2

The following result was shown in [Fai06].

Theorem 5.1 ([Fai06, Theorem 3.3.10])**.** *If F is a linked field with* $\mathrm{char}(F) = 2$ *and $I_q^4 F = 0$ then the possible values its u-invariant can take are* $0, 2, 4$ *and* 8.

Since a field of characteristic 2 being linked is equivalent to every Albert form over F being isotropic by Remark 4.1, combining Theorem 5.1, Theorem 4.3 and Theorem 3.1 gives the following result:

Corollary 5.2. *If F is a linked field with* $\mathrm{char}(F) = 2$ *then the possible values its u-invariant can take are* $0, 2, 4$ *and* 8.

Corollary 5.2 follows more directly from Theorem 5.1 if one can show that every 4-fold Pfister form contains an Albert form as a subform, as then clearly $I_q^4(F) = 0$ if the field is linked. As this result is also of independent interest, we give a proof below. The computations are similar to those used in Theorem 4.3. We use the following well-known isometry. This can be derived from, for example, [DQ17, (2.4) and (2.6)]. It can also be directly derived from Lemma 3.2 and Theorem 3.1.

Lemma 5.3. *Assume* $\mathrm{char}(F) = 2$. *Let $b \in F^\times$, $a \in F$, $x, y \in F$ not both zero and $\beta = x^2 + xy + ay^2$. Then we have $\langle\langle b, a]] \simeq \langle\langle b\beta, a + b\beta]]$.*

Lemma 5.4. *Assume* $\mathrm{char}(F) = 2$. *Let π be a 2-fold Pfister form over F and $\lambda_i \in F^\times$ for $i = 1, 2, 3$. The the form $\rho = \langle \lambda_1, \lambda_2, \lambda_3 \rangle \otimes \pi$ contains an Albert subform.*

Proof. Let $\alpha \in F$ and $\beta \in F^\times$ such that $\pi = \langle\langle \beta, \alpha]]$. Fix $t = \frac{\alpha + \beta\alpha}{\beta}$, $\alpha_1 = \alpha + \beta t^2$ and $\alpha_2 = \alpha + \beta(t^2 + t + \alpha)$. Then $\alpha = \alpha_1 + \alpha_2$. Using Lemma 5.3, we see that the forms $[1, \alpha]$, $[1, \alpha_1]$ and $[1, \alpha_2]$ are all subforms of π. Consequently, the form

$$\psi = \lambda_1[1, \alpha] \perp \lambda_2[1, \alpha_1] \perp \lambda_3[1, \alpha_2]$$

is a subform of ρ. The Arf invariant of this form is $\alpha + \alpha_1 + \alpha_2 = 0$. Therefore ψ is an Albert form. □

Corollary 5.5. *Given a field F with* $\mathrm{char}(F) = 2$*, every* 4*-fold Pfister form contains an Albert subform.*

Proof. Let $\varphi = \langle\langle \delta, \gamma, \beta, \alpha]]$ be a 4-fold Pfister form over F and let $\pi = \langle\langle \beta, \alpha]]$. Then $\rho = \pi \perp \delta\pi \perp \gamma\pi$ is a subform of φ. The form ρ, and hence φ, contains an Albert subform by Lemma 5.4. □

Theorem 5.6. *If F is a linked field with* $\mathrm{char}(F) = 2$ *then* $I_q^4 F = 0$*.*

Proof. Let φ be a 4-fold Pfister form over F. By Corollary 5.5 it contains an Albert subform ρ. Since F is linked, ρ must be isotropic. Therefore φ is hyperbolic. □

Remark 5.7. A result analogous to Lemma 5.4 holds for those fields F with $\mathrm{char}(F) \neq 2$ containing a square root of -1. Let $\pi = \langle\langle \alpha, \beta \rangle\rangle$ be a 2-fold Pfister form over F and let $\lambda_1, \lambda_2, \lambda_3$ be three arbitrary elements in $F^\times$. Then $\rho = \langle \lambda_1, \lambda_2, \lambda_3 \rangle \otimes \pi$ contains the subform

$$\lambda_1 \langle \alpha, \beta \rangle \perp \lambda_2 \langle \alpha, \alpha\beta \rangle \perp \lambda_3 \langle \alpha\beta, \beta \rangle .$$

This has trivial discriminant, and so it is an Albert form. Hence, every 4-fold Pfister form over F contains an Albert form.

This is not the case in general for fields that do not contain a square root of -1. For example, the unique anisotropic 4-fold Pfister form $\langle\langle -1, -1, -1, -1 \rangle\rangle$ over $\mathbb{R}$ clearly has no Albert subform, as all Albert forms over $\mathbb{R}$ are isotropic.

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Bibliography

References

[AJ95] R. Aravire and B. Jacob, *p-algebras over maximally complete fields*, Proc. Sympos. Pure Math. **58** (1995), 27–49.

[Bae82] R. Baeza, *Comparing u-invariants of fields of characteristic* 2, Bol. Soc. Brasil. Mat. **13** (1982), no. 1, 105–114.

[BO13] G. Berhuy and F. Oggier, *An introduction to central simple algebras and their applications to wireless communications*, Mathematical surveys and monographs, vol. 191, 2013.

[Cha15] A. Chapman, *Common subfields of p-algebras of prime degree*, Bull. Belg. Math. Soc. Simon Stevin **22** (2015), no. 4, 683–686.

[Cha17] A. Chapman, *Symbol length of p-algebras of prime exponent*, J. Algebra Appl. **16** (2017), no. 5, 1750136, 9.

[CDL16] A. Chapman, A. Dolphin and A. Laghribi, *Total linkage of quaternion algebras and Pfister forms in characteristic two*, J. Pure and Appl. Algebra **220**(2016), no. 11, 3676–3691.

[DQ17] A. Dolphin and A. Quéguiner-Mathieu, *Symplectic involutions, quadratic pairs and function fields of conics*, **221**(2017), no. 8, 1966–1980.

[EKM08] R. Elman, N. Karpenko and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications, vol. 56, Amer. Math. Soc., Providence, RI, 2008.

[EL73] R. Elman and T. Y. Lam, *Quadratic forms and the u-invariant II*, Invent. Math. **21** (1973), 125–137.

[EV05] A. Elduque and O. Villa, *A note on the linkage of Hurwitz algebras*, Manuscripta Math. **117** (2005), no. 1, 105–110.

[Fai06] F. Faivre, *Liaison des formes de Pfister et corps de fonctions de quadriques en caractéristique* 2, 2006, Thesis (Ph.D.)–Université de Franche-Comté.

[GS06] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, Cambridge, 2006.

[Kato82] K. Kato, *Symmetric bilinear forms, quadratic forms and Milnor K-theory in characteristic two*. Invent. Math. **66** (1982), no. 3, 493–510.

[Lam02] T. Y. Lam, *On the linkage of quaternion algebras*, Bull. Belg. Math. Soc. Simon Stevin **9** (2002), no. 3, 415–418.

[MS89] P. Mammone and D.B. Shapiro, *The Albert quadratic form for an algebra of degree four*, Proc. Amer. Math. Soc. **105** (1989), no. 3, 525–530.