Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

ORIGINAL ARTICLE

Optimal control of a linear system subject to partially specified input noise

Alexander Erreygers* | Jasper De Bock | Gert de Cooman | Arthur Van Camp

Foundations Lab for Imprecise Probabilities, Department of Electronics and Information Systems, Ghent University, Belgium

Correspondence
*Alexander Erreygers, Technologiepark-Zwijnaarde 125, B-9052 Ghent, Belgium. Email: alexander.erreygers@ugent.be

Abstract

One of the most basic problems in control theory is that of controlling a discrete-time linear system subject to uncertain noise with the objective of minimising the expectation of a quadratic cost. If one assumes the noise to be white, then solving this problem is relatively straightforward. However, white noise is arguably unrealistic: noise is not necessarily independent and one does not always precisely know its expectation. We first recall the optimal control policy without assuming independence, and show that in this case computing the optimal control inputs becomes infeasible. In a next step, we assume only knowledge of lower and upper bounds on the conditional expectation of the noise, and prove that this approach leads to tight lower and upper bounds on the optimal control inputs. The analytical expressions that determine these bounds are strikingly similar to the usual expressions for the case of white noise.

KEYWORDS:
Linear system, quadratic cost, optimal control, partially specified noise.

1 | INTRODUCTION

We consider the problem of controlling a scalar discrete-time linear system with perfect state information subject to stochastic input noise, with the objective of minimising the expectation of a quadratic cost. It is well known—see for instance Root[1]—that this problem, often referred to as the Linear-Quadratic Control (LQC) problem, is greatly simplified if the noise model is assumed to be (wide-sense) white. However, the underlying assumptions of the white noise model are not always satisfied. In particular, it is often difficult to provide a precise value for the expectation of the noise, and the values of the noise at different times are not necessarily independent.

For this reason, we introduce the notion of a partially specified noise model, which does away with the aforementioned issues of the white noise model. First and foremost, it is specified simply by providing lower and upper bounds on the conditional expectation of the input noise, rather than providing a precise value for it. Second, the model does not include an independence assumption in the classical sense; the only independence-like assumption that we make is that our knowledge about the conditional expectation (i.e., the lower and upper bounds) does not depend on the noise history. The resulting model can therefore be seen as an extension of the wide-sense white noise model.

Our main contribution consists in using the expectation bounds of the partially specified noise model to determine bounds on the optimal control input. More specifically, we show that this optimal control input combines state feedback with noise feedforward and provide a precise expression for the former and tight bounds on the latter. Quite remarkably, these bounds on the

Abbreviations: LQC, linear-quadratic control noise feedforward can be tractably computed using an interval-arithmetic version of the well-known backwards recursion that defines the noise feedforward in the case of (wide-sense) white noise. As an immediate consequence, we also find that dropping the independence assumption—as we do—does not result in wider bounds on the optimal control policy, but rather leads to the exact same bounds as would be obtained *with* independence.

Because we end up with tight bounds on the noise feedforward instead of an exact value for it, we need a secondary decision criterion to select a feedforward from the obtained interval. While at first sight this might seem disadvantageous, we argue that using a secondary decision criterion can actually be beneficial, as it allows one to take into account additional requirements—e.g., state or input constraints—that usually make solving the optimisation problem much harder. We also argue why dropping the independence assumption is essential to make this approach reasonable. Furthermore, our bounds on the feedforward can be used as a measure for the sensitivity of the optimal control policy with respect to misspecification of the (white) noise model. In particular, we can easily check the effect of small changes to the expectation of the noise—i.e., narrow bounds on the conditional expectations—on the (bounds on the) optimal control policy.

As for the connection with existing literature, we should note that we are not the first to take into account the sensitivity of the optimal control policy to imprecision or errors in the noise model. Several other authors have proposed alternative noise models that are more realistic, or less prone to modelling errors, than white noise. Examples of such alternative noise models are the Linear Gaussian Vacuous Mixture,[2] constrained uncertainty,[3–8] uncertainty theory,[9] and relative entropy constraints.[10] Our approach and the mentioned alternative approaches differ in multiple ways.

A first difference is the noise model that is adopted. We use the partially specified noise model, which is a (non-white) probabilistic noise model without the extra assumption that the precise value of the conditional expectation of the noise is known: it is only assumed to have (possibly conservative) lower and upper bounds. Several authors[3–5] assume that the noise is known to be within an easily parametrised set. Benavoli and Chisci[2] assume that the initial state, input noise and output noise are independent, and that their probability distribution is given by a convex combination of a known Gaussian distribution and an unknown arbitrary/vacuous distribution. Petersen[10] assumes that the joint distribution of the initial state, input noise and output noise is unknown, but he only considers joints that have some allowable distance to a product of Gaussian distributions. Chen and Zhu[9] model the additive input noise as "independent uncertain linear variables" with finite support.

A second difference is the optimality criterion. A popular choice is a type of minimax optimality criterion,[2–5,10] in the sense that one is interested in the control policy that minimises the (expectation of) the worst-case cost. This criterion usually leads to a single optimal control input that has to be computed numerically. We minimise the expected quadratic cost, but do so for every probabilistic noise model that satisfies the bounds on the expectation of the noise. In this way, we end up with an interval of optimal control inputs, whose lower and upper bounds have analytical expressions that are easy to evaluate. These bounds lead to a tight interval of conceivably optimal control inputs, which allows (and requires) the use of a secondary decision criterion to select a unique control input to apply from this interval.

A third difference is that the mentioned alternatives study a multi-dimensional system, with perfect[3–5,9] or imperfect[2,10] state information, while we restrict our attention to a one-dimensional system with perfect state information. Nevertheless, all of the material in Sections 2 to 4 can be readily generalised to multi-dimensional systems. However, generalising the material in Sections 5 and 6 to multi-dimensional systems, or also taking into account imperfect state information, seems less immediate, unfortunately.

Finally, while the conditional expectation is assumed to be bounded, we do not impose constraints on the actual value of the noise,[3–5] nor do we impose feasibility constraints on the states and control inputs.[2–5,9] These feasibility constraints are essential to *Model Predictive Control*; we refer to the review papers of Mayne,[6] Farina et al,[7] or Saltik et al[8]—and references therein—for a list of various ways to implement these constraints and a discussion of how they affect the solution of the control problem. Our reason for assuming perfect state information and imposing no feasibility constraints is that we want to focus on the new aspect of our approach—partially specified input noise—without complicating its treatment for extraneous reasons.

The remainder of the paper is structured as follows. We start in Section 2 by introducing some of the basic terminology and notation concerning the control of discrete-time linear systems. We then formalise the notion of a probabilistic noise model in Section 3, and use this to (re)define and solve the linear-quadratic control problem in Section 4. After introducing the partially specified noise model in Section 5, everything is finally set up to present our main contribution in Section 6. We end with a brief recap in Section 7. As most of the proofs are technical, and not an essential help in understanding the main ideas of the paper, we have moved them to the Appendix.

2 | A BRIEF INTRO TO LINEAR-QUADRATIC CONTROL

The optimal control of a discrete-time linear system with respect to a quadratic cost function has been studied since the late 1950s, and the formulation of this problem is well known. In this section, we briefly introduce this problem so as to familiarise the reader with the notation and terminology used throughout this contribution. Furthermore, we also mention explicitly all the assumptions that we require in the remainder.

We consider a *controller* that steers the state of a discrete-time scalar linear system with dynamics described by

$$X_{k+1} = a_k X_k + b_k u_k + W_k \quad \text{for all } k \in N := \{0, 1, \dots, n\}, \tag{1}$$

where n is a non-negative integer. † In this expression, X_{k+1} is the real-valued state, W_k is the real-valued (input) noise and u_k is the real-valued control input; a_k and b_k are system parameters with $b_k \neq 0$, as controlling the system is impossible otherwise. The controller's objective is to apply the control inputs $u_0, \dots, u_n$ that minimise the cost

$$J = \sum_{k=0}^{n} \left(r_k u_k^2 + q_{k+1} X_{k+1}^2 \right). \tag{2}$$

In this expression, r_k and q_{k+1} are real-valued parameters with $r_k > 0$ and $q_{k+1} \geq 0$. Hence, the cost J and its increments are always non-negative.

Prior to time 0 the controller is uncertain about the actual values of the noise $W_0, \dots, W_n$ in (1), and therefore also about the actual values of the states $X_1, \dots, X_{n+1}$ and the actual value of the cost J. Throughout this paper we follow the usual convention of denoting uncertain variables with upper case letters, while we use lower case letters to denote their actual values. We furthermore use the following notation for the noise, and similarly for the state and control input. Let k and ℓ be elements of N such that $k \leq \ell$. The tuple $(W_k, \dots, W_\ell)$ is denoted by $W_{k:\ell}$, and we also let W^ℓ denote $W_{0:\ell}$. For notational convenience, we let $W_{k:k-1}$ be the empty tuple. We also denote this empty tuple by $\diamond$. For the actual values of the uncertain noise, we use similar notations: $w_{k:\ell} := (w_k, \dots, w_\ell)$ and $w^\ell := w_{0:\ell}$. These tuples can take values in the sets $\mathscr{W}_{k:\ell} := \mathbb{R}^{\ell-k+1}$ and $\mathscr{W}^\ell := \mathscr{W}_{0:\ell}$, respectively. As before, we let $\mathscr{W}_{k:k-1} := \{\diamond\}$ and $w_{k:k-1} := \diamond$.

At time k and before applying the control input u_k, the controller observes x_k, the actual value of the state. Furthermore, we assume *perfect recall*, meaning that at every time $k \in N$ the controller knows the entire state history x^k.[11] It is then customary to determine the control input u_k as a function of x^k, which we will denote by ϕ_k. Such a function $\phi_k : \mathscr{X}^k \to \mathbb{R}$ is called a *feedback function*, and we call a tuple of feedback functions $\phi := (\phi_0, \phi_1, \dots, \phi_n)$ a *control policy*. We let Φ denote the set of all control policies.

Throughout this contribution we assume that (i) the control policy may depend implicitly on the controller's noise model, and that (ii) at time k the controller knows the previous feedback functions $\phi^{k-1} := (\phi_0, \dots, \phi_{k-1})$. This second assumption implies that given the state history x^k, the controller can determine the noise history w^{k-1} using (1). In the remainder, we will sometimes use the noise history w^{k-1} without mentioning explicitly that this noise history is computed from the state history x^k and the feedback functions ϕ^{k-1}. We do this in order not to obfuscate the notation too much, as the dependence on x^k and ϕ^{k-1} is usually clear from the context.

For all $k \in N$, all state histories $x^k \in \mathscr{X}^k$ and all control policies $\phi \in \Phi$, we define

$$J[\phi|x^k] := \sum_{\ell=k}^{n} \left(r_\ell \phi_\ell(x^k, X_{k+1:\ell})^2 + q_{\ell+1} X_{\ell+1}^2 \right). \tag{3}$$

As previously mentioned, it is clear from (1) and (3) that the cost $J[\phi|x^k]$ is implicitly dependent on the noise $W_{k:n}$ and therefore uncertain. Hence, in order to be completely correct, we ought to write $J[\phi|x^k, W_{k:n}]$ instead of $J[\phi|x^k]$; however, we will often write the latter and implicitly assume the dependence on the noise $W_{k:n}$.

Finally, it is customary to evaluate the performance of a control policy by means of its expected cost with respect to some probabilistic model. A control policy ϕ is then called (locally) *optimal* if it minimises the expected cost $E(J[\phi|x^k]|w^{k-1})$, and the LQC problem consists in finding such an optimal control policy. However, in order to formally define and solve this LQC problem, we first need to introduce conditional expectation operators.

†Note that the control horizon of length $n+2$ could start at any point in time, but that we have opted to start it at time 0 for notational convenience.

3 | PROBABILISTIC NOISE MODELS

In the remainder of this contribution we assume that the controller's uncertainty about the actual value of the noise W^k can be adequately modelled using a probabilistic model. The most popular probabilistic framework for doing so is that of measure-theoretic probability theory.[1,10–17] However, this measure-theoretic framework suffers from the fact that a lot of technicalities arise while using it; we refer to Bertsekas and Shreve[17] for a thorough discussion. Furthermore, the measure-theoretic framework also has the drawback that its elementary concepts are probability measures, and that the expectations we are actually interested in need to be derived from these measures.

In order to address these issues, an alternative—yet similar and mathematically equivalent—probabilistic framework was proposed by Whittle.[18] In contradistinction with the measure-theoretic approach, Whittle constructs his framework using an expectation operator as the elementary concept. He proposes five axioms that an expectation operator should satisfy. Combining these five axioms with a definition of independence and conditional expectation allows Whittle to derive the classical results of measure-theoretic probability theory.

In order to circumvent the technicalities that often arise when using the measure-theoretic approach and inspired by Whittle's approach to probability theory, we will model the controller's uncertainty about the noise W^n using a conditional expectation operator, the basic aspects of which are outlined below.

3.1 | Expectation-based probabilistic noise models

For all $k \in N$, the actual value w_k of the noise W_k is an element of the possibility space or sample space $\mathscr{W}_k = \mathbb{R}$. Any real-valued function on $\mathscr{W}^n$ is called an *uncertain variable*, and the set of all uncertain variables is denoted by $\mathscr{F}(\mathscr{W}^n)$. More generally, we will use $\mathscr{F}(S)$ to denote the set of all real-valued functions on a set S. Examples of uncertain variables are the uncertain state X_{k+1} and the remaining cost $J[\phi|x^k]$ induced by some control policy $\phi \in \Phi$ and some state history $x^k \in \mathscr{X}^k$.

As already mentioned in the introduction to this section, we use a conditional expectation operator $E(\cdot|\cdot)$ to model the controller's uncertainty about W^n. The expectation of the uncertain variable $f \in \mathscr{F}(\mathscr{W}^n)$, conditional on the knowledge that the actual value of W^{k-1} is w^{k-1}, is denoted by $E(f(W^n)|w^{k-1})$. Since this notation $E(f(W^n)|w^{k-1})$ is a bit lengthy, we will often shorten it to $E(f|w^{k-1})$. If $k = 0$, we write $E(\cdot)$ instead of $E(\cdot|\diamond)$.

A *conditional expectation operator* is an operator

$$E(\cdot|\cdot)\colon \mathscr{D} \times \cup_{k=0}^{n} \mathscr{W}^{k-1} \to \overline{\mathbb{R}} \colon (f, w^{k-1}) \mapsto E(f|w^{k-1}),$$

where the domain $\mathscr{D} \subseteq \mathscr{F}(\mathscr{W}^n)$ is some real linear space that includes all constant functions, and where $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ denotes the extended real number line. For any $k \in N$ and any $w^{k-1} \in \mathscr{W}^{k-1}$, $E(\cdot|w^{k-1})$ is taken to satisfy the following axioms.

(E1) $E(cf|w^{k-1}) = cE(f|w^{k-1})$ for all $f \in \mathscr{D}$ and all $c \in \mathbb{R}$,

(E2) $E(f + g|w^{k-1}) = E(f|w^{k-1}) + E(g|w^{k-1})$ for all $f, g \in \mathscr{D}$,

(E3) if $f \geq 0$, then $E(f|w^{k-1}) \geq 0$, for all $f \in \mathscr{D}$, [positivity]

(E4) $E(1|w^{k-1}) = 1$, [normalisation]

(E5) $E(f(W^n)|w^{k-1}) = E(f(w^{k-1}, W_{k:n})|w^{k-1})$ for all $f \in \mathscr{D}$.

Because the co-domain is the extended real number line, the equality in axiom (E2) is only imposed if the addition on the right hand side is well defined. Terminology-wise, we say that $E(f|w^{k-1})$ exists if f belongs to the domain $\mathscr{D}$.

Now fix some $f \in \mathscr{D}$. Then for any $k \in N$, we can consider the conditional expectation $E(f|w^{k-1})$ as a function on $\mathscr{W}^n$. We use $E(f|W^{k-1})$ to denote this function, which maps any $w^n \in \mathscr{W}^n$ to $E(f|w^{k-1})$. Note that this function $E(f|W^{k-1})$ is not necessarily real-valued nor necessarily an element of the domain $\mathscr{D}$; whenever it *is* real-valued and belongs to $\mathscr{D}$, we demand that

(E6) $E(E(f|W^{k-1})|w^{k-2}) = E(E(f|w^{k-2}, W_{k-1})|w^{k-2}) = E(f|w^{k-2})$ for all $w^{k-2} \in \mathscr{W}^{k-2}$.

Axiom (E6) is called the *law of iterated expectations*, and can be seen as the generalisation of the *law of total probability* to expectation operators. In classical measure-theoretic probability, where conditional expectation is not a primitive notion, (a simplified version of) this law is used to define conditional expectation. A conditional expectation operator E that satisfies (E1)–(E6) will be called a *probabilistic noise model*.

3.2 | White noise models

One thing that should be obvious from our definition of a probabilistic noise model is that specifying such a model is non-trivial. For any $k \in N$ and $f \in \mathscr{D}$, a different noise history w^{k-1} in $\mathscr{W}^{k-1} = \mathbb{R}^k$ can for instance lead to a different value of $E(f(W_k)|w^{k-1})$. In some situations, it might however be justified to assume that the value of $E(f(W_k)|w^{k-1})$ is the same for all values of w^{k-1}, which brings to mind a frequently made independence assumption.

This concept of independence is introduced as a product rule for probability measures in the measure-theoretic framework, and as a product rule for the expectation operator in Whittle's expectation framework.[18] We here use a slightly altered version of Whittle's product rule, which, for all $\ell \in N$, makes use of the notation $\mathscr{D}_\ell := \mathscr{D} \cap \mathscr{F}(\mathscr{W}_\ell)$ for those uncertain variables in the domain $\mathscr{D}$ that only depend on W_ℓ.

We say that a probabilistic noise model E is *independent* if for all $k \in N$, all $w^{k-1} \in \mathscr{W}^{k-1}$, and all $f_k, \dots, f_n$ in $\mathscr{D}_k, \dots, \mathscr{D}_n$ such that the product $E(f_k) \cdots E(f_n)$ is well-defined, it holds that the product $f_k \cdots f_n$ belongs to $\mathscr{D}$ and that

$$E(f_k \cdots f_n | w^{k-1}) = E(f_k) \cdots E(f_n).$$

As a consequence of this, it follows that for all $k, \ell \in N$ such that $k \leq \ell$, all $w^{k-1} \in \mathscr{W}^{k-1}$ and all $f_\ell \in \mathscr{D}_\ell$,

$$E(f_\ell | w^{k-1}) = E(f_\ell).$$

In particular, if the function $f_\ell(W_\ell) := W_\ell$ belongs to the domain, we find that

$$E(W_\ell | w^{k-1}) = E(W_\ell) \tag{4}$$

for all $k, \ell \in N$ such that $k \leq \ell$ and all $w^{k-1} \in \mathscr{W}^{k-1}$.

In this context of independent probabilistic noise models, to ensure that the optimal control problem is solvable, it is necessary to demand that some specific uncertain variables belong to the domain $\mathscr{D}$ and that their expectations are finite; see for instance Bertsekas.[19] In particular, it is necessary that:

(W1) $E(W_k)$ exists and is finite, for all k in N;

(W2) $E(W_k^2)$ exists and is finite, for all k in N.

If a probabilistic noise model E is independent and furthermore satisfies (W1) and (W2), then it is called *white*. Assumptions (W1) and (W2) ensure that at least one control policy has a finite expected cost, such that the optimal control problem studied in Section 4 is well-defined.

3.3 | Well-behaved probabilistic noise models

However, assuming that the noise is white—i.e., that it is independent—is not always justifiable. Therefore, we here propose a set of weaker assumptions on the probabilistic noise model that, as we will see, still ensure that the optimal control problem is solvable.

In particular, we call a probabilistic noise model E *well-behaved* if for all k, ℓ, i and j in N satisfying $k \leq \ell \leq i \leq j$ and all $w^{k-1} \in \mathscr{W}^{k-1}$

(B1) $E(W_\ell | w^{k-1})$ exists and is finite;

(B2) $E(E(W_i | W^\ell) E(W_j | W^\ell) | w^{k-1})$ exists and is finite;

(B3) $E(W_\ell | W^k) \in \mathscr{D}$;

(B4) $E(E(W_i | W^\ell) E(W_j | W^\ell) | W^k) \in \mathscr{D}$.

Assumptions (B1)–(B4) are (non-trivial) generalisations of assumptions (W1) and (W2) to more general, non-independent probabilistic noise models. In fact, a noise model is white if and only if it is well-behaved and independent. This can be immediately verified: for an independent well-behaved probabilistic noise model, (B1) and (B2) reduce to (W1) and (W2), and (B3) and (B4) are then redundant as they are an immediate consequence of (B1) and (B2).

The reason for assuming (B1) and (B2) is that they guarantee the existence of at least one control policy that has a finite expected (remaining) cost. (B3) and (B4) ensure that the law of iterated expectations can be used in some specific cases that appear in our proofs; see the Appendix.

3.4 | Wide-sense white noise models

It is clear that specifying a generic well-behaved probabilistic noise model is—in general—infeasible due to its complexity. One way to reduce this complexity is to assume independence, but this is often not justified. The main reason for assuming independence in our setting is that it simplifies the expression for the optimal control policy. However, to be able to execute this simplification we do not need to assume independence. As we will see, it actually suffices to assume that

$$E(W_k|w^{k-1}) = E(W_k) \text{ for all } k \in N \text{ and all } w^{k-1} \in \mathscr{W}^{k-1}. \tag{WSW}$$

A well-behaved probabilistic noise model E that satisfies (WSW) is called *wide-sense white*. Note that, as an immediate consequence of (E1), (E4) and (E6) and similarly to the independent case, every wide-sense white noise model E satisfies (4).

4 | LQC OF A SYSTEM SUBJECT TO PROBABILISTIC NOISE

Before we formulate the LQC problem using our well-behaved noise model, we first introduce some additional notation. For all $k \in N$ and $\phi \in \Phi$, we let $\Phi(\phi^{k-1}) := \{\psi \in \Phi : (\forall \ell \in \{0, \dots, k-1\})\ \psi_\ell = \phi_\ell\}$. Note that ϕ^{-1} is the empty tuple $\diamond$ and that $\Phi(\phi^{-1}) = \Phi$.

4.1 | Optimality

Let E be a well-behaved noise model. We say that a control policy $\hat{\phi} \in \Phi$ is *optimal* if it is an element of the set of optimal control policies

$$\operatorname{opt}(\Phi) := \left\{\phi \in \Phi : (\forall k \in N)(\forall x^k \in \mathscr{X}^k)(\forall \psi \in \Phi(\phi^{k-1}))\ E(J[\phi|x^k]|w^{k-1}) \leq E(J[\psi|x^k]|w^{k-1})\right\}, \tag{5}$$

where, as explained in Section 2, w^{k-1} is implicitly understood to be a function of x^k and ϕ^{k-1}. The rationale behind this definition is the following. The "$(\forall k \in N)(\forall x^k \in \mathscr{X}^k)$" part of (5) ensures that an optimal control policy is optimal for all state histories. The "$(\forall \psi \in \Phi(\phi^{k-1}))$" part of (5) ensures that locally, we only compare control policies that have applied the same feedback functions before the current time k.

It is important to emphasise that for a generic well-behaved probabilistic noise model, $E(J[\psi|x^k]|w^{k-1})$ is not guaranteed to exist for all $x^k \in \mathscr{X}^k$ and $\psi \in \Phi(\phi^{k-1})$. To ensure that this is the case, we will from here on limit Φ to the set of control policies ϕ for which:

(J1) $E(J[\phi|x^k]|w^{k-1})$ exists for all $k \in N$ and $x^k \in \mathscr{X}^k$;

(J2) $E(J[\phi|x^k, X_{k+1}]|w^{k-1}, W_k) \in \mathscr{D}$ for all $k \in N$ and $x^k \in \mathscr{X}^k$,

where w^{k-1} is derived from x^k and ϕ^{k-1}, and where X_{k+1} is a function of W_k and x^k determined by (1).

Assumption (J1) ensures that the optimality operator is well-defined, while assumption (J2) allows us to find the optimal control policy using dynamic programming. A more specific motivation for assuming (J2) can be found in the proof of Lemma 4 in the Appendix. For a well-behaved probabilistic noise model, one can verify that the trivial control policy θ, defined for all $k \in N$ and all $x^k \in \mathscr{X}^k$ as $\theta_k(x^k) := 0$, satisfies assumptions (J1) and (J2). Hence, for a well-behaved probabilistic noise model, the set Φ is non-empty. Furthermore, as we will show in Theorem 1, every well-behaved probabilistic noise model has a unique optimal control policy.

4.2 | Solution to the LQC problem with probabilistic noise

LQC problems were first studied in the late 1950s, and therefore it should not come as a surprise that LQC problems similar to ours have already been studied. In almost all cases, discrete-time LQC problems are studied in the context of multi-dimensional systems with imperfect state information and white noise. Two exceptions are Akashi and Nose [15] and Tse and Bar-Shalom [16], where the authors study an LQC problem with imperfect state information, without assuming white input and output noise. Tse and Bar-Shalom [16] assume noise with "known but arbitrary statistics", while Akashi and Nose [15] assume that "a priori probability distributions of all uncertain variables are known, and that each of these has a finite covariance matrix". The following result can be regarded as a formalised version of theirs, in terms of expectation operators, and in the special case of a one-dimensional system with perfect state information. We provide a full proof of this theorem in the Appendix.

Theorem 1. Let E be a well-behaved probabilistic noise model. Then the unique element of $\mathrm{opt}(\Phi)$ is $\hat{\phi}$, defined for all k in N and all x^k in $\mathscr{X}^k$ as

$$\hat{\phi}_k(x^k) := -\tilde{r}_k b_k \big(m_{k+1} a_k x_k + h_{k|w^{k-1}} \big). \tag{6}$$

The parameters $\tilde{r}_k$ and m_{k+1} in (6) are derived from the initial conditions $m_{n+1} := q_{n+1}$ and $\tilde{r}_{n+1} := 0$, and, for all $k \in N$, from the backwards recursive expressions

$$\tilde{r}_k := (r_k + b_k^2 m_{k+1})^{-1} \tag{7}$$

and

$$m_k := q_k + \tilde{r}_k a_k^2 r_k m_{k+1}. \tag{8}$$

For all $k \in N$ and all $w^{k-1} \in \mathscr{W}^{k-1}$, the feedforward $h_{k|w^{k-1}}$ is derived from the initial condition $h_{n+1|w^{k-1}} := 0$ and, for all $\ell \in N$ such that $\ell \geq k$, from the recursive expression

$$h_{\ell|w^{k-1}} := \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} h_{\ell+1|w^{k-1}} + m_{\ell+1} E(W_\ell | w^{k-1}), \tag{9}$$

where here and in the remainder, for notational convenience we let $a_{n+1} := 0$ and $r_{n+1} := 0$.

The optimal feedback function $\hat{\phi}_k$ defined in (6) should be a function of x^k, but at first sight it is a function of only x_k and w^{k-1}. Recall however that, as mentioned in Section 2, we implicitly assume the noise history w^{k-1} to be a function of the state history x^k and the feedback functions $\hat{\phi}^{k-1}$. This confirms that $\hat{\phi}_k$ is indeed a function of x^k, and therefore $\hat{\phi}$ is indeed a control policy.

The parameters that determine $\hat{\phi}_k$ need to be computed using backwards recursive expressions. This means that if we want to compute the parameters that determine the optimal feedback function at time 0, we need to compute the parameters that determine all the remaining future optimal feedback functions as well. This is a computational disadvantage, especially if we consider a long control horizon $n + 1$.

More problematically, the backwards recursive computations that are necessary to determine $h_{k|w^{k-1}}$ are—at least in general—intractable, as the following reasoning illustrates. Indeed, assume that the controller knows the noise history w^{k-1}. In order to compute the feedforward $h_{k|w^{k-1}}$, he then needs to know the conditional expectations $E(W_k|w^{k-1}), \dots, E(W_n|w^{k-1})$. While obtaining $E(W_k|w^{k-1})$ is still somewhat feasible, obtaining $E(W_{k+1}|w^{k-1}), \dots, E(W_n|w^{k-1})$ is usually not. For example, it follows from (E6) that

$$E(W_{k+1}|w^{k-1}) = E(E(W_{k+1}|w^{k-1}, W_k)|w^{k-1}).$$

Hence, in order to determine $E(W_{k+1}|w^{k-1})$, the controller first needs to know $E(W_{k+1}|w^k)$ for every $w_k \in \mathscr{W}_k$, after which he can use these values to compute $E(E(W_{k+1}|w^{k-1}, W_k)|w^{k-1})$. As $\mathscr{W}_k = \mathbb{R}$, this is typically infeasible. Iteratively determining $E(W_{k+2}|w^{k-1})$ is even harder, and so on. Fortunately, there are at least two specific cases where these computations do become tractable; see Corollary 1 and Theorem 2 further on.

Finally, we note that the expression for the optimal control policy $\hat{\phi}$ is strikingly similar to the well-known expression for the optimal control input of a system subject to deterministic noise. The similarity between these two is a property referred to as *certainty equivalence*. It was initially studied by Simon[13] and Theil[12] under the term first period certainty equivalence in the context of an economic planning problem, and later became a well-researched property in optimal control.[15,16]

4.3 | Solution to the classical stochastic LQC problem

In the classical approach to the LQC problem with probabilistic noise, the noise model is assumed to be white[1,14,19] or, more generally, wide-sense white. The popularity of this assumption in large part stems from the fact that (WSW) simplifies the calculation of the feedforward in the optimal solution (9) considerably. Executing this simplification yields—a stronger version of—the well-known solution to the classical LQC problem.[1,19] Alternatively, this result can be see as a special case of Theorem 2 further on.

Corollary 1. Let E be a wide-sense white noise model. Then the unique element of $\mathrm{opt}(\Phi)$ is $\hat{\phi}$, as defined in Theorem 1 by (6). For all $k \in N$ and all $w^{k-1} \in \mathscr{W}^{k-1}$, the feedforward $h_{k|w^{k-1}}$ is equal to h_k, which is derived from the initial condition $h_{n+1} := 0$ and, for all $k \in N$, from the backwards recursive relation

$$h_k := \tilde{r}_{k+1} a_{k+1} r_{k+1} h_{k+1} + m_{k+1} E(W_k). \tag{10}$$

Observe that in this case the only expectations that determine the feedforward—and hence the optimal control policy—are the marginal expectations $E(W_0), \dots, E(W_n)$, which simplifies specifying the noise model considerably. Therefore, and as evident from (10), computing the feedforward is no longer intractable.

There is also a special case in which, for wide-sense white noise models, the computational disadvantage of the backwards recursive computations disappears completely. This happens when the parameters of the system—a_k and b_k—as well as the parameters of the cost—r_k and q_{k+1}—are the same at all time instants $k \in N$, i.e., if $a_k = a$, $b_k = b$, $r_k = r$ and $q_{k+1} = q$. We then call the system *stationary*. If $E(W_k) = E(W)$ for all $k \in N$, then the wide-sense white noise model E is also called stationary. It is well known—see for example Bertsekas[19]—that in this case the parameters m_{k+1} and g_k that describe $\hat{\phi}_k$ converge to limit values in the limit for $n \to +\infty$. Moreover, these limit values can be calculated in a non-recursive manner, as stated by the following proposition.

Proposition 1. Assume that the linear system is stationary. In the limit for $n \to +\infty$, m_k—as defined in Theorem 1—converges to

$$m := \begin{cases} 0 & \text{if } q = 0, \\ \dfrac{(a^2-1)r + b^2 q + \sqrt{((a^2-1)r + b^2 q)^2 + 4b^2 qr}}{2b^2} & \text{otherwise.} \end{cases} \tag{11}$$

Even more, in the special case that $q = 0$, then $m_k = m = 0$ for all $k \in N$; if on the contrary $q > 0$ and $a = 0$, then $m_k = m = q$ for all $k \in N$. If moreover E is a stationary wide-sense white noise model, then h_k—as defined in Corollary 1—converges to

$$h := \begin{cases} 0 & \text{if } q = 0, \\ \dfrac{(r + b^2 m)m}{r + b^2 m - ar} E(W) & \text{otherwise.} \end{cases} \tag{12}$$

Again, in case $q = 0$, $h_k = h = 0$ for all $k \in N$; if on the contrary $q > 0$ but $a = 0$, then $h_k = h = qE(W)$ for all $k \in N$.

This result allows us to consider an infinitely long control horizon and use the a priori computed limit values m and h as the parameters of the feedback function at all times. Therefore, in this particular case, the disadvantage of the backwards recursive computations is indeed eliminated.

5 | PARTIALLY SPECIFIED NOISE MODELS

If we assume a generic well-behaved probabilistic noise model, as we did in Section 4.2, then the optimal control input is a unique function of this noise model, but determining the value of this function for a given noise model is—at least in general—intractable. Fortunately, if our knowledge about $E(W_k|w^{k-1})$ is independent of w^{k-1}—if E is wide-sense white—then, as we have seen in Section 4.3, determining the optimal control input does become tractable. However, while this assumption yields a nice result, one could argue that it is overly restrictive. Moreover, in order to specify a wide-sense white noise model, the controller needs to specify the precise value of the marginal expectations $E(W_0), \dots, E(W_n)$, which is not always possible.

Therefore, from here on, we will assume that the controller is not able to specify an exact or precise value for $E(W_k|w^{k-1})$. Instead, the controller can only assess a lower bound $\underline{E}(W_k) \in \mathbb{R}$ and upper bound $\overline{E}(W_k) \in \mathbb{R}$ for $E(W_k|w^{k-1})$, with $\underline{E}(W_k) \leq \overline{E}(W_k)$. In other words, we assume that the controller assesses that the noise can be adequately modelled using a well-behaved probabilistic noise model E, but that he only knows that for all $k \in N$ and all $w^{k-1} \in \mathscr{W}^{k-1}$,

$$\underline{E}(W_k) \leq E(W_k|w^{k-1}) \leq \overline{E}(W_k). \tag{PS}$$

We will call such a noise model E *partially specified*.

We would like to emphasise that the only independence assumption we make in a partially specified noise model E is that the controller's *knowledge*—which in this case is partial—about $E(W_k|w^{k-1})$ is independent of w^{k-1}, and *not* that $E(W_k|w^{k-1})$ is independent of w^{k-1}. Consequently, the partially specified noise model can be seen as an extension of the wide-sense white noise model that allows for partial or inexact specifications of the local conditional expectations. In fact, if the bounds of a partially specified noise model are degenerate, i.e., if $\underline{E}(W_k) = \overline{E}(W_k) = E(W_k)$ for all $k \in N$, then the partially specified noise actually degenerates to a wide-sense white noise model.

6 | LQC WITH PARTIALLY SPECIFIED NOISE

All prerequisites to present our main contribution are now in place. This section is entirely parallel to Section 4, be it that we now use the partially specified noise model instead of a well-behaved (or wide-sense white) probabilistic noise model.

6.1 | Possible optimality criteria

As the noise model is not precisely known, it should come as no surprise that we cannot precisely determine the optimal control policy either—unless the bounds of the partially specified noise model are degenerate. We therefore need an alternative optimality criterion that can handle the partially specified nature of our noise model. While there are plenty of alternatives—we refer to Troffaes[20] for an overview—we here use E-admissibility[20,21]. This simply means that we consider the set of all control policies that are optimal—in the sense of (5)—for at least one well-behaved noise model E that satisfies (PS). By Theorem 1, any such conceivably optimal control policy is a combination of the same state feedback and a possibly different noise feedforward.

The approach we will consider here is to determine tight lower and upper bounds on the feedforward terms of these conceivably optimal control policies. One way to interpret these bounds is then that they quantify the sensitivity of the optimal control policy to errors in the noise model. If the lower and upper bounds on the feedforward terms form a small interval, then the optimal control policy is not very sensitive to errors in the noise model; if on the contrary the bounds form a large interval, then the optimal control policy is rather more sensitive to modelling errors.

This approach should be contrasted with methods that try to derive a single control policy within this set, for example by minimising the worst or best case cost. On our approach, choosing among the conceivably optimal control policies is regarded as a second step, which we discuss in Section 6.4. Adopting a minimax[20]—minimising the worst case cost—strategy already at this stage in our current setting results in an optimisation problem that—in general—cannot be (efficiently) solved, as from Lemma 4 in the Appendix we know that the expression for the expected cost of a conceivably optimal control policy contains several second order terms, including for instance $E(W_k^2|w^{k-1})$ and $E(W_k E(W_{k+1}|W^k)|w^{k-1})$.

6.2 | Solution to the partially specified LQC problem

We consider the following theorem to be the main result of our contribution. It allows us to easily determine tight lower and upper bounds on the feedforward $h_{k|w^{k-1}}$—and hence on the optimal control input u_k—when the controller can only specify (noise history independent) lower and upper bounds on the conditional expectation of the noise.

Theorem 2. Let E be a partially specified noise model defined by the lower bounds $\underline{E}(W_0), \dots, \underline{E}(W_n)$ and the upper bounds $\overline{E}(W_0), \dots, \overline{E}(W_n)$. Then there is a unique optimal control policy $\hat{\phi}$, as defined in Theorem 1 by (6). For every $k \in N$ and every $w^{k-1} \in \mathscr{W}^{k-1}$, the exact value of the feedforward term $h_{k|w^{k-1}}$ in (6) cannot in general be determined exactly, but it holds that

$$\underline{h}_k \leq h_{k|w^{k-1}} \leq \overline{h}_k,$$

where the real-valued bounds $\underline{h}_k$ and $\overline{h}_k$ are derived from the initial condition

$$\underline{h}_{n+1} := 0 =: \overline{h}_{n+1}, \tag{13}$$

and, for all $\ell \in N$, from the recursive relations

$$\underline{h}_\ell := \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} \underline{h}_{\ell+1} + m_{\ell+1} \underline{E}(W_\ell), \tag{14}$$

$$\overline{h}_\ell := \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} \overline{h}_{\ell+1} + m_{\ell+1} \overline{E}(W_\ell), \tag{15}$$

if $a_{\ell+1} \geq 0$ or from the recursive relations

$$\underline{h}_\ell := \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} \overline{h}_{\ell+1} + m_{\ell+1} \underline{E}(W_\ell), \tag{16}$$

$$\overline{h}_\ell := \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} \underline{h}_{\ell+1} + m_{\ell+1} \overline{E}(W_\ell) \tag{17}$$

if $a_{\ell+1} < 0$. Moreover, the bounds $\underline{h}_k$ and $\overline{h}_k$ are tight, in the sense that if we fix some $k \in N$ and some $h_k \in [\underline{h}_k, \overline{h}_k]$, then there is a white probabilistic noise model that satisfies (PS) and for which the feedforward term—defined in (10)— at time k is equal to h_k.

In the special case that the bounds are degenerate—that is, if $\underline{E}(W_k) = \overline{E}(W_k)$ for all $k \in N$—this result reduces to Corollary 1. This should come as no surprise; we already know from Section 5 that a partially specified noise model degenerates to a wide-sense white noise model if it has degenerate bounds.

We would like to emphasise here that the partially specified noise model considers more than just all possible white noise models or all possible wide-sense white noise models that are compatible with the upper and lower bounds on the expectation of the noise. In fact, almost none of the well-behaved probabilistic noise models that satisfy the bounds of (PS) actually correspond to a (wide-sense) white noise model, since most of them do not satisfy (WSW)! We therefore find it rather remarkable that the bounds $\underline{h}_k$ and $\overline{h}_k$ are obtained as the bounds of an interval arithmetic version of (10), which is the recursive expression for the feedforward that corresponds to wide-sense white noise.

In fact, it follows from the last sentence of Theorem 2 that the obtained bounds on the feedforward $h_{k|w^{k-1}}$ are identical to those that we would have obtained if we had only considered white noise models—instead of all well-behaved noise models—that are compatible with the lower and upper bounds. Even stronger, one can verify that if all a_k's were non-negative, then considering only stationary white noise models still results in the same bounds. In other words, dropping the independence assumption—as we do—does not result in wider bounds on the feedforward! Of course, this begs the question why we put in all this effort in order to drop the independence assumption in the first place. A first argument for dropping the independence assumption is that it allows us to obtain the same result—i.e., the tight bounds on the feedforward—in a more general setting. So we can use simple formulas as if our noise models were independent, but our conclusions hold for a class of more complicated noise models, and we know which ones. A second, more involved, argument for dropping the independence assumption will be given in Section 6.4.

6.3 | Convergence due to stationarity

For wide-sense white noise models, as we know from Section 4.3, stationarity ensures that the parameters that characterise the optimal control policy converge to easily computable limit values. The following proposition shows that a similar result continues to hold for stationary partially specified noise models, that is, if $\underline{E}(W_k) = \underline{E}(W)$ and $\overline{E}(W_k) = \overline{E}(W)$ for all $k \in N$.

Proposition 2. Let E be a stationary partially specified noise model with stationary bounds $\underline{E}(W)$ and $\overline{E}(W)$, and assume that the linear system is stationary as well. In the limit for $n \to +\infty$ the bounds $\underline{h}_k$ and $\overline{h}_k$ on the feedforward term of the optimal control policy then converge to

$$\underline{h} := \begin{cases} 0 & \text{if } q = 0, \\ \dfrac{(r + b^2 m)m}{r + b^2 m - ar}\underline{E}(W) & \text{if } q > 0 \text{ and } a \geq 0, \\ (r + b^2 m)m\dfrac{(r + b^2 m)\underline{E}(W) + ar\overline{E}(W)}{(r + b^2 m)^2 - (ar)^2} & \text{if } q > 0 \text{ and } a < 0, \end{cases} \tag{18}$$

and

$$\overline{h} := \begin{cases} 0 & \text{if } q = 0, \\ \dfrac{(r + b^2 m)m}{r + b^2 m - ar}\overline{E}(W) & \text{if } q > 0 \text{ and } a \geq 0, \\ (r + b^2 m)m\dfrac{(r + b^2 m)\overline{E}(W) + ar\underline{E}(W)}{(r + b^2 m)^2 - (ar)^2} & \text{if } q > 0 \text{ and } a < 0. \end{cases} \tag{19}$$

Moreover, in the special case that $q = 0$, $\underline{h}_k = 0 = \overline{h}_k$ for all $k \in N$; if on the contrary $q > 0$ and $a = 0$, then $\underline{h}_k = q\underline{E}(W)$ and $\overline{h}_k = q\overline{E}(W)$ for all $k \in N$.

This result is an extension of Proposition 1 from wide-sense white noise models to the more general partially specified noise models. Here too, as in Section 4.3, this result eliminates the computational disadvantage of having to conduct backwards recursive computations.

6.4 | Selecting a control input

Consider now a practical situation where the controller's information about the noise can be adequately modelled by a partially specified noise model. Then as we know from Section 4, the optimal control input at time k is

$$u_k = \hat{\phi}_k(x^k) = -\tilde{r}_k b_k \big(m_{k+1} a_k x_k + h_{k|w^{k-1}}\big).$$

However, as we have argued in Section 6.1, due to the controllers limited information about the noise, he cannot determine the precise value of the feedforward term $h_{k|w^{k-1}}$ exactly. All he knows are the tight lower and upper bounds that we obtained for this feedforward in Theorem 2, which naturally induce lower and upper bounds on the control input: the optimal control input is guaranteed to be an element of a closed interval $[\underline{u}_k, \overline{u}_k]$. In particular, if $b_k > 0$, then

$$\underline{u}_k := -\tilde{r}_k b_k \big(m_{k+1} a_k x_k + \overline{h}_k\big) \qquad \text{and} \qquad \overline{u}_k := -\tilde{r}_k b_k \big(m_{k+1} a_k x_k + \underline{h}_k\big),$$

and if $b_k < 0$, then $\overline{h}_k$ and $\underline{h}_k$ switch places. Hence, based on his information about the noise—i.e., the partially specified noise model—the controller does not have a preference for any element of the interval $[\underline{u}_k, \overline{u}_k]$.

Of course, in a practical scenario the controller does need to choose a control input u_k. One possible—but naive—way to do this would be the following. Recall from our discussion in Section 6.2 that the bounds on the feedforward remain the same if we were to consider only white noise models. Hence, if the controller is comfortable with making an independence assumption, one way to select a control input is to simply apply the noise feedforward corresponding to a white noise model that is compatible with the bounds. This way, the problem is reduced to choosing precise values for the marginal expectation $E(W_\ell)$ in the interval $[\underline{E}(W_\ell), \overline{E}(W_\ell)]$ for all $\ell \in N$ a priori. However, this choice is arbitrary, since one of our two motivations for introducing the partially specified noise model was the assumption that the controller was not able to specify precise values for the marginal expectation in the first place!

Therefore, an arguably more sensible way to select a value u_k from the interval $[\underline{u}_k, \overline{u}_k]$ is to use a secondary decision criterion. Although one might regard the need for such an extra criterion as a drawback, we think that this is actually an important benefit of our approach, because it allows the controller to take into account additional requirements that would have made solving the original optimisation problem more difficult, if not impossible.

Indeed, the classic set up of the LQC problem—unconstrained and real-valued states and control inputs in combination with a linear-quadratic cost functional—is popular mainly because it leads to an optimisation problem that can be solved relatively easily, and not necessarily because it is a good model for reality. In fact, there are many possible extensions of the LQC problem that make it arguably more realistic or useful, such as extra requirements on the state and control input[3–5] or an exponential optimality criterion.[22] This extra realism comes at a price though, as solving the optimisation problem is usually (much) harder for the extended LQC problem than for the original LQC problem.

One practical example of an extension that results in a much harder LQC problem is the addition of a cost penalty for every non-zero control input. However, in our approach, dealing with this penalisation of non-zero control inputs using a secondary decision criterion is straightforward: the controller simply selects the element u_k of $[\underline{u}_k, \overline{u}_k]$ that has the smallest absolute value. The simple idea behind this choice is that if a non-zero control input has to be chosen, choosing the one with the smallest absolute value results in the lowest cost of the control at that time point. Given the available partial information about the probabilistic noise model E, one could argue that it is the most sensible policy that can be considered.

Another basic extension of the LQC problem is to impose that the states and/or control inputs are constrained.[3–5] Solving the resulting LQC problem is then more involved than solving the one without bounds, and an analytical solution is no longer possible. However, if we are working with a partially specified noise model, we can often satisfy these state and/or control input constraints using a secondary decision criterion, i.e., by appropriately selecting an element of $[\underline{u}_k, \overline{u}_k]$ that aims to satisfy these constraints locally.

A similar situation occurs if the control input can only take a finite number of values. If the number of possible values or the length of the control horizon is large, solving the LQC problem then becomes computationally intensive. Here too, the controller can use a secondary decision criterion to select, if possible, a suitable control input in $[\underline{u}_k, \overline{u}_k]$.

All the examples above have one thing in common: at every time point k and given the current state x_k, they select one element $u_k^\star$ from $[\underline{u}_k, \overline{u}_k]$. This is equivalent to choosing an element $h_k^\star$ from $[\underline{h}_k, \overline{h}_k]$ for every state x_k, and clearly defines a control policy. Ideally, such a control policy would be optimal with respect to at least one well-behaved noise model that satisfies (PS). Although we have reason to believe that this claim is—at least in a slightly different/weaker form—true, verifying or refuting it would require an excessively extensive argument, and we therefore leave this as a conjecture. We do argue however that in general, such a control policy *cannot* be optimal with respect to a (wide-sense) white noise model that satisfies (PS), which implies that the correctness of the claim—and therefore, the reasonability of the secondary decision criteria above—requires dropping the independence assumption, as allowed by our model.

The argument revolves around the following counterexample. Consider a control problem with $N = \{0, 1\}$, and assume that $a_k > 0$ and $q_{k+1} > 0$ for all $k \in N$. Furthermore, consider a partially specified noise model with $\underline{E}(W_0) < \overline{E}(W_0)$ and $\underline{E}(W_1) < \overline{E}(W_1)$. Assume that our choices are $h_0^\star = \underline{h}_0$ and $h_1^\star = \overline{h}_1$. We now claim that there is no (wide-sense) white noise model that satisfies (PS), $h_0 = h_0^\star$ and $h_1 = h_1^\star$. If this claim is true, then it follows from Corollary 1 that the control policy that corresponds to our choices cannot be optimal with respect to a (wide-sense) white noise model that satisfies (PS), hence completing our argument. We verify this claim as follows. First, we observe that from $h_1 = h_1^\star = \overline{h}_1$, (10) and (15) it follows that $E(W_1) = \overline{E}(W_1)$. Second, we apply (14) twice to yield

$$h_0^\star = \underline{h}_0 = \tilde{r}_1 a_1 r_1 q_2 \underline{E}(W_1) + m_1 \underline{E}(W_0).$$

Similarly, applying (10) twice yields

$$h_0 = \tilde{r}_1 a_1 r_1 q_2 E(W_1) + m_1 E(W_0).$$

We subtract the first equality from the second, to yield

$$h_0 - h_0^\star = \tilde{r}_1 a_1 r_1 q_2 (E(W_1) - \underline{E}(W_1) + m_1(E(W_0) - \underline{E}(W_0)).$$

Since $E(W_1) = \overline{E}(W_1) > \underline{E}(W_1)$ and $E(W_0) \geq \underline{E}(W_0)$—and because the relevant system parameters are positive—this implies that $h_0 - h_0^\star > 0$. Clearly, this contradicts with the requirement that $h_0 = h_0^\star$, which verifies our claim.

7 | CONCLUSION

Throughout this contribution, we were interested in determining the control policies that optimally control a linear system subject to input noise with respect to a quadratic cost. It is well known that if a controller's knowledge about the input noise can be modelled accurately by a white noise model, then there is a single optimal control policy, which can be easily determined from well-known backwards recursive expressions. The independence assumption made in the white noise model is however quite restrictive and often not justifiable. Fortunately, as we have seen—and as is essentially well-known—there is also a single optimal control policy if the controller's knowledge does not allow for assuming independence. In that case, however, we argued that determining the resulting optimal control inputs is, in general, computationally infeasible.

The special case where the local conditional expectation of the noise is independent of the noise history at all times deserves special attention. Indeed, we found that in this case the optimal control inputs can be determined using the same backwards recursive expressions from the case of white noise. While this is a nice result, this still requires a quite stringent assumption, and also requires that the controller is able to specify precise—exact—values for the (marginal) expectation of the noise at all times.

Therefore, we here considered the situation where the controller's knowledge about the noise is partial, in the sense that he can only specify lower and upper bounds on the conditional expectation of the noise, and where the noise is not required to be (wide-sense) white. Of course, as a consequence of the imprecision in the noise model, the optimal control input can no longer be uniquely determined. Nonetheless, we were able to show that there are tight—with respect to the controller's knowledge—lower and upper bounds on the optimal control inputs. Even more surprisingly, we found that these lower and upper bounds can be easily computed, using interval-arithmetic versions of the backwards recursive expressions from the case of (wide-sense) white noise. Additionally, we found that in the case of a stationary system, stationary noise model and long control horizon, similar to the (wide-sense) white noise case, the (lower and upper bounds for the) parameters that determine the optimal control input converge to limit values, which can be easily determined from closed-form expressions.

Inevitably, we have had to leave some questions unanswered. First and foremost, we have not thoroughly studied how to choose which value in the feedforward interval to actually apply. We have argued that this might be done using a secondary decision rule, which is a computationally tractable way of trying to satisfy some extra requirements on the state and/or control input, including requirements that would make solving the LQC problem hard when taken into account during optimisation. However, we have only presented a brief discussion on this use of secondary decision criteria, and a proper study using simulations and real-world applications is certainly necessary. We leave this for future work.

ACKNOWLEDGEMENTS

Jasper De Bock's and Gert de Cooman's research was partially funded by the European Commission's H2020 programme, through the UTOPIAE Marie Curie Innovative Training Network, H2020-MSCA-ITN-2016, Grant Agreement number 722734.

References

1. Root J. Optimum Control of Non-Gaussian Linear Stochastic Systems with Inaccessible State Variables. *SIAM Journal on Control* 1969; **7** (2): 317–323. DOI:10.1137/0307021
2. Benavoli A, Chisci L. Robust stochastic control based on imprecise probabilities. *IFAC Proceedings Volumes* 2011; **44** (1): 4606 – 4613. DOI: 10.3182/20110828-6-IT-1002.02081
3. Bemporad A, Borrell F, Morari M. Min-max control of constrained uncertain discrete-time linear systems. *IEEE Transactions on Automatic Control* 2003; **48** (9): 1600–1606. DOI: 10.1109/TAC.2003.816984
4. Mayne D, Seron M, Raković S. Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica* 2005; **41** (2): 219 – 224. DOI: 10.1016/j.automatica.2004.08.019
5. Bertsimas D, Brown D. Constrained stochastic LQC: a tractable approach. *IEEE Transactions on Automatic Control* 2007; **52** (10): 1826–1841. DOI: 10.1109/TAC.2007.906182
6. Mayne D. Model predictive control: Recent developments and future promise. *Automatica* 2014; **50** (12): 2967–2986. DOI: 10.1016/j.automatica.2014.10.128
7. Farina M, Giulioni L, Scattolini R. Stochastic linear Model Predictive Control with chance constraints – A review. *Journal of Process Control* 2016; **44**: 53 – 67. DOI: 10.1016/j.jprocont.2016.03.005
8. Saltık M, Özkan L, Ludlage J, Weiland S, Van den Hof P. An outlook on robust model predictive control algorithms: Reflections on performance and computational aspects. *Journal of Process Control* 2018; **61**: 77 -102. DOI: 10.1016/j.jprocont.2017.10.006
9. Chen Y, Zhu Y. Indefinite LQ optimal control with equality constraint for discrete-time uncertain systems. *Japan Journal of Industrial and Applied Mathematics* 2016; **33** (2): 361–378. DOI: 10.1007/s13160-016-0217-9
10. Petersen I. Minimax LQG Control. *International Journal of Applied Mathematics and Computer Science* 2006; **16** (3): 309–323.
11. Witsenhausen H. Separation of estimation and control for discrete time systems. *Proceedings of the IEEE* 1971; **59** (11): 1557–1566. DOI:10.1109/PROC.1971.8488
12. Theil H. A Note on Certainty Equivalence in Dynamic Planning. *Econometrica* 1957; **25** (2): 346–349. DOI:10.2307/1910260
13. Simon H. Dynamic Programming Under Uncertainty with a Quadratic Criterion Function. *Econometrica* 1956; **24** (1): 74-81. DOI:10.2307/1905261
14. Meier LI, Larson R, Tether A. Dynamic programming for stochastic control of discrete systems. *IEEE Transactions on Automatic Control* 1971; **16** (6): 767-775. DOI: 10.1109/TAC.1971.1099839
15. Akashi H, Nose K. On certainty equivalence of stochastic optimal control problem. *International Journal of Control* 1975; **21** (5): 857–863. DOI: 10.1080/00207177508922040
16. Tse E, Bar-Shalom Y. Generalized certainty equivalence and dual effect in stochastic control. *IEEE Transactions on Automatic Control* 1975; **20** (6): 817–819. DOI: 10.1109/TAC.1975.1101108
17. Bertsekas D, Shreve S. *Stochastic Optimal Control: The Discrete-Time Case*. Athena Scientific. 1996.

18. Whittle P. *Probability via Expectation*. Springer-Verlag New York. 2000.

19. Bertsekas D. *Dynamic Programming and Optimal Control*. Volume 1. Athena Scientific. 2005.

20. Troffaes MCM. Decision making under uncertainty using imprecise probabilities. *International Journal of Approximate Reasoning* 2007; **45**(1): 17–29. doi: 10.1016/j.ijar.2006.06.001

21. Levi I. *The enterprise of knowledge: An essay on knowledge, credal probability, and chance*. MIT press. 1983.

22. Whittle P. Risk-Sensitive Linear/Quadratic/Gaussian Control. *Advances in Applied Probability* 1981; **13** (4): 764–777. DOI:10.2307/1426972

How to cite this article: A. Erreygers, J. De Bock, G. de Cooman, and A. Van Camp (2019), Optimal control of a linear system subject to partially specified input noise, *International Journal of Robust and Nonlinear Control*. doi:10.1002/rnc.4583

APPENDIX

Lemma 1. The sequence $m_{n+1}, m_n, \dots, m_0$ that is defined by the initial condition $m_{n+1} := q_{n+1}$ and the recursive relation (8) is a monotonously increasing sequence of non-negative real numbers, which, provided the linear system is stationary, is bounded above by $q + ra^2/b^2$. Consequently, $\tilde{r}_n, \dots, \tilde{r}_0$, defined by (7), is a sequence of strictly positive real numbers.

Proof. Bertsekas makes this observation about the first sequence in the proof of Proposition 4.4.1 of Bertsekas.[19] The statement about the second sequence then follows directly from this and the strict positivity of r_k. □

Lemma 2. Let k be in N, w^{k-1} in $\mathscr{W}^{k-1}$ and let E be a conditional expectation operator with domain $\mathscr{D}$ that satisfies (E1)–(E5). Let f and g be two uncertain variables in $\mathscr{D}$ and assume that $E(f|w^{k-1})$ and $E(g|w^{k-1})$ are not both infinite. If $f \leq g$, then $E(f|w^{k-1}) \leq E(g|w^{k-1})$.

Proof. Since the domain $\mathscr{D}$ of E is a real linear space, $g-f$ belongs to $\mathscr{D}$; and by assumption $g-f \geq 0$. By (E3), $E(g-f|w^{k-1}) \geq 0$, from which the stated follows by (E2). □

Lemma 3. Let E be a well-behaved probabilistic noise model with domain $\mathscr{D}$ and fix some arbitrary $k \in N$ and some $w^{k-1} \in \mathscr{W}^{k-1}$. Let the real-valued function f on $\mathscr{W}^n$ be any linear combination of (i) constants; (ii) terms of the form W_ℓ or $E(W_\ell|W^k)$, with $\ell \in N$ such that $k \leq \ell$; and (iii) terms of the form $E(W_i|W^\ell)E(W_j|W^\ell)$—which includes terms of the form W_ℓ^2 and $W_\ell E(W_j|W^\ell)$ as special cases‡—or $E(E(W_i|W^\ell)E(W_j|W^\ell)|W^k)$—which includes terms of the form $E(W_\ell^2|W^k)$ and $E(W_\ell E(W_j|W^\ell)|W^k)$ as special cases§—with $\ell, i, j \in N$ such that $k \leq \ell \leq i \leq j$. Then f belongs to the domain $\mathscr{D}$, and $E(f|w^{k-1})$ is finite.

Proof. This proof is based on a trivial extension of (E1) and (E2). Let m be some strictly positive integer, $f_1, \dots, f_m$ some real-valued functions on $\mathscr{W}^n$ that belong to the domain $\mathscr{D}$ and $d_1, \dots, d_m$ some real numbers. It now follows from (E1) and (E2) that if $E(f_r|w^{k-1})$ is finite for all $r \in \{1, \dots, m\}$, then $\sum_{r=1}^m d_r f_r \in \mathscr{D}$ and

$$E\left(\sum_{r=1}^{m} d_r f_r \middle| w^{k-1}\right) = \sum_{r=1}^{m} d_r E(f_r|w^{k-1})$$

is finite.

Therefore, we can prove the statement as follows. First, we verify that every term f_i in the linear combination is contained in the domain $\mathscr{D}$. Second, we check that $E(f_i|w^{k-1})$ is finite. If this is the case, then the above reasoning implies that f—the function that is a linear combination of the terms in the statement—belongs to $\mathscr{D}$ and that $E(f|w^{k-1})$ is indeed finite.

‡Recall that by (E5), $E(W_\ell|W^\ell) = W_\ell$. Hence, terms of the form W_ℓ^2 are obtained by letting $\ell = i = j$, as then $E(W_\ell|W^\ell)E(W_\ell|W^\ell) = W_\ell^2$. Similarly, terms of the form $W_\ell E(W_j|W^\ell)$ are obtained by letting $\ell = i \leq j$, as then $E(W_\ell|W^\ell)E(W_j|W^\ell) = W_\ell E(W_j|W^\ell)$.

§As before, these two special cases follow by letting $\ell = i = j$ and $\ell = i \leq j$ and using (E5).

We first consider the constant terms. Let d be any real number. Observe that it follows from (E4) and (E1) that the constant function $g(W^n) := d$, which is contained in the domain $\mathscr{D}$, has expectation $E(g|w^{k-1}) = d$, which is clearly finite.

Second, we focus on the terms of the form W_ℓ or $E(W_\ell|W^k)$. By (B1), W_ℓ belongs to the domain $\mathscr{D}$ and $E(W_\ell|w^{k-1})$ is finite. By (B3), $E(W_\ell|W^k) \in \mathscr{D}$, such that we now only need to prove that $E(E(W_\ell|W^k)|w^{k-1})$ is finite. As $E(W_\ell|W^k) \in \mathscr{D}$ by (B3) and $W_\ell \in \mathscr{D}$ by (B1), it follows from (E6) that $E(E(W_\ell|W^k)|w^{k-1}) = E(W_\ell|w^{k-1})$. Recall that $E(W_\ell|w^{k-1})$ is finite by (B1), such that $E(E(W_\ell|W^k)|w^{k-1})$ is indeed finite.

Third, we focus on the terms of the form $E(W_i|W^\ell)E(W_j|W^\ell)$ and $E(E(W_i|W^\ell)E(W_j|W^\ell)|W^k)$. By (B2), $E(W_i|W^\ell)E(W_j|W^\ell)$ belongs to the domain $\mathscr{D}$ and $E(E(W_i|W^\ell)E(W_j|W^\ell)|w^{k-1})$ is finite. By (B4), $E(E(W_i|W^\ell)E(W_j|W^\ell)|W^k)$ belongs to the domain $\mathscr{D}$, such that what remains for us to prove is that these terms have finite conditional expectation. As $E(W_i|W^\ell)E(W_j|W^\ell) \in \mathscr{D}$ and $E(E(W_i|W^\ell)E(W_j|W^\ell)|W^k) \in \mathscr{D}$, it follows from (E6) that

$$E(E(E(W_i|W^\ell)E(W_j|W^\ell)|W^k)|w^{k-1}) = E(E(W_i|W^\ell)E(W_j|W^\ell)|w^{k-1}).$$

Recall that the right hand side of this equality is finite by (B2), such that $E(E(E(W_i|W^\ell)E(W_j|W^\ell)|W^k)|w^{k-1})$ is indeed finite. □

Lemma 4. Let E be a well-behaved probabilistic noise model. Fix some $\phi \in \Phi$, some $k \in N$, and some $x^k \in \mathscr{X}^k$, and let $w^{k-1} \in \mathscr{W}^{k-1}$ denote the noise history associated with ϕ and x^k according to (1). Then

$$\hat{c}(\phi^{k-1}, x^k) := (m_k - q_k)x_k^2 + 2\tilde{r}_k a_k r_k x_k h_{k|w^{k-1}} - \tilde{r}_k(b_k h_{k|w^{k-1}})^2 + 2\sum_{\ell=k}^{n-1} \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} E(W_\ell h_{\ell+1|W^\ell}|w^{k-1})$$
$$- \sum_{\ell=k}^{n-1} \tilde{r}_{\ell+1} b_{\ell+1}^2 E((h_{\ell+1|W^\ell})^2|w^{k-1}) + \sum_{\ell=k}^{n} m_{\ell+1} E(W_\ell^2|w^{k-1}) \tag{1}$$

is well defined—in the sense that all conditional expectation operators act on functions contained in their common domain $\mathscr{D}$—and finite. Furthermore, for any $x^k \in \mathscr{X}^k$ and any $\psi \in \Phi(\phi^{k-1})$,

$$\hat{c}(\phi^{k-1}, x^k) \leq E(J[\psi|x^k]|w^{k-1}). \tag{2}$$

Let $\hat{\phi}$ be defined as in Theorem 1, and let ψ be an arbitrary element of $\Phi(\phi^{k-1})$. If $\psi_k \neq \hat{\phi}_k$, then

$$(\exists x^k \in \mathscr{X}^k)\ \hat{c}(\phi^{k-1}, x^k) < E(J[\psi|x^k]|w^{k-1}). \tag{3}$$

Alternatively, if $\psi_\ell = \hat{\phi}_\ell$ for all $\ell \in N$ such that $\ell \geq k$, then

$$(\forall x^k \in \mathscr{X}^k)\ E(J[\psi|x^k]|w^{k-1}) = \hat{c}(\phi^{k-1}, x^k). \tag{4}$$

Proof. Throughout the proof, we let ϕ be an arbitrary element of Φ.

First, we verify that the expression for $\hat{c}(\phi^{k-1}, x^k)$ is indeed well defined and finite. To that end, we fix some $k \in N$ and some $x^k \in \mathscr{X}^k$. As mentioned in Section 2, we use w^{k-1} to denote the noise history associated with ϕ^{k-1} and x^k. It clearly suffices to show that $h_{k|w^{k-1}}$ is well-defined and finite, and that each of the conditional expectations in (1) is well-defined and finite.

For verifying that $h_{k|w^{k-1}}$ is well-defined and finite, we explicitly execute the recursion in (9), and observe that $h_{k|w^{k-1}}$ is a linear function of $E(W_k|w^{k-1}), \ldots, E(W_n|w^{k-1})$. As by (B1) these conditional expectations exist and are finite, $h_{k|w^{k-1}}$ is indeed well-defined and finite.

Similarly for the conditional expectations, we observe that $h_{\ell+1|W^\ell}$ is a linear combination of $E(W_{\ell+1}|W^\ell), \ldots, E(W_n|W^\ell)$. Consequently, the functions in the conditional expectations in (1) all satisfy the requirements of Lemma 3, from which it follows that these conditional expectations are well-defined and finite.

We now turn to proving the remainder of the statement—that is (2), (3) and (4)—using induction. In order to do that, for all $\psi \in \Phi$, all $k \in N$ and all $x^k \in \mathscr{X}^k$, we let

$$c(\psi, x^k) := E(J[\psi|x^k]|w^{k-1})$$

for notational convenience. Note that $c(\psi, x^k)$ is well-defined—in the sense that the conditional expectation exists—by (J1), implying that $J[\psi|x^k] \in \mathscr{D}$.

Now let $\psi \in \Phi(\phi^{n-1})$, and fix an arbitrary $x^n \in \mathscr{X}^n$. The set $\Phi(\phi^{n-1})$ is constructed in such a way that the noise history $w^{n-1} \in \mathscr{W}^{n-1}$ associated with any $\psi \in \Phi(\phi^{n-1})$ and the state history x^n is the same as that associated with ϕ and x^n. By (3),

$$c(\psi, x^n) = E(r_n \psi_n(x^n)^2 + q_{n+1} X_{n+1}^2|w^{n-1}).$$

Substituting X_{n+1} with its dynamics (1) yields

$$c(\psi, x^n) = E(q_{n+1}(a_n x_n + b_n \psi_n(x^n) + W_n)^2 + r_n \psi_n(x^n)^2 | w^{n-1}).$$

We expand the squares, use the linearity of the conditional expectation operator—which is allowed as the conditional expectation is finite by Lemma 3—and complete the squares containing $\psi_n(x^n)$, to yield

$$c(\psi, x^n) = \tilde{r}_n^{-1} s_n^2 + (m_n - q_n) x_n^2 + 2\tilde{r}_n a_n r_n x_n h_{n|w^{n-1}} - \tilde{r}_n (b_n h_{n|w^{n-1}})^2 + q_{n+1} E(W_n^2 | w^{n-1}), \tag{5}$$

where $h_{n|w^{n-1}}$, $\tilde{r}_n$ and m_n are as defined in Theorem 1 and

$$s_n := \psi_n(x^n) + \tilde{r}_n b_n (q_{n+1} a_n x_n + h_{n|w^{n-1}}). \tag{6}$$

The only term in our expression for $c(\psi, x^n)$ that is influenced by ψ_n—the component of the control policy ψ that we are free to choose—is $\tilde{r}_n^{-1} s_n^2$. By Lemma 1, this term is always non-negative, and zero if and only if $s_n = 0$. Setting $s_n = 0$ in (5) is the only way to minimise $c(\psi, x^n)$, which yields the minimum

$$c(\psi, x^n) \geq (m_n - q_n) x_n^2 + 2\tilde{r}_n a_n r_n x_n h_{n|w^{n-1}} - \tilde{r}_n (b_n h_{n|w^{n-1}})^2 + q_{n+1} E(W_n^2 | w^{n-1}) = \hat{c}(\phi^{n-1}, x^n). \tag{7}$$

Note that (7) agrees with (1) for $k = n$, such that (2) is true for $k = n$. Setting $s_n = 0$ in (6) yields

$$\psi_n(x^n) = -\tilde{r}_n b_n \left(q_{n+1} a_n x_n + h_{n|w^{n-1}} \right). \tag{8}$$

Hence, the expected remaining cost $c(\psi, x^n)$ is equal to its minimum $\hat{c}(\phi^{n-1}, x^n)$ for all $x^n \in \mathscr{X}^n$ if and only if $\psi_n = \hat{\phi}_n$, which agrees with (3) and (4) for $k = n$.

Next, we fix some $k \in N$ such that $k < n$, and assume that (2)–(4) hold for all $\ell \in N$ such that $\ell > k$. It follows from (3) that, for all any $x^k \in \mathscr{X}^k$ and all $\psi \in \Phi(\phi^{k-1})$,

$$J[\psi | x^k] = r_k \psi_k(x^k)^2 + q_{k+1} X_{k+1}^2 + J[\psi | x^k, X_{k+1}],$$

an equality between two (implicit) real-valued functions on $\mathscr{W}^n$. We now substitute X_{k+1} in the middle term of the right hand side with its dynamics (1), which, after expanding and regrouping the terms, yields

$$J[\psi | x^k] = f(W_k) + J[\psi | x^k, X_{k+1}], \tag{9}$$

where again both sides of the equality are (implicit) functions on $\mathscr{W}^n$ and where the real-valued function f on $\mathscr{W}_k$ is defined for all $w_k \in \mathscr{W}_k$ as

$$f(w_k) := (r_k + q_{k+1} b_k^2) \psi_k(x^k)^2 + q_{k+1}(a_k^2 x_k^2 + w_k^2) + 2q_{k+1} b_k \psi_k(x^k)(a_k x_k + w_k) + 2q_{k+1} a_k x_k w_k.$$

As $c(\psi, x^k) = E(J[\psi | x^k] | w^{k-1})$ is well-defined, it follows from (9) that $f(W_k) + J[\psi | x^k, X_{k+1}] \in \mathscr{D}$. We plan on applying (E6), so we now verify that also $E(f(W_k) + J[\psi | x^k, X_{k+1}] | w^{k-1}, W_k) \in \mathscr{D}$. To that end, first note that $f(W_k)$ satisfies Lemma 3, such that $f(W_k) \in \mathscr{D}$ and $E(f(W_k) | w^{k-1})$ is finite. Since $\mathscr{D}$ is a linear space, and because we already know that $J[\psi | x^k] \in \mathscr{D}$, (9) now implies that $J[\psi | x^k, X_{k+1}] \in \mathscr{D}$. Furthermore, it follows from (E5), (E1) and (E4) that $E(f(W_k) | W^k) = f(W_k)$. Combining our observations about f, we find that $E(E(f(W_k) | W^k) | w^{k-1})$ is finite because it is equal to $E(f(W_k) | w^{k-1})$, which is finite. As an immediate consequence of (E5), $E(E(f(W_k) | w^{k-1}, W_k) | w^{k-1})$ is equal to $E(E(f(W_k) | W^k) | w^{k-1})$ and hence finite (and therefore also well-defined), which implies that $E(f(W_k) | w^{k-1}, W_k) \in \mathscr{D}$. Also, recall that $E(J[\psi | x^k, X_{k+1}] | w^{k-1}, W_k) \in \mathscr{D}$ by (J2). Hence, since $\mathscr{D}$ is a linear space, $E(f(W_k) | w^{k-1}, W_k) + E(J[\psi | x^k, X_{k+1}] | w^{k-1}, W_k) \in \mathscr{D}$. However, since $f(W_k) \in \mathscr{D}$ and $J[\psi | x^k, X_{k+1}] \in \mathscr{D}$, it follows from applying (E2) that $E(f(W_k) + J[\psi | x^k, X_{k+1}] | w^{k-1}, W_k) = E(f(W_k) | w^{k-1}, W_k) + E(J[\psi | x^k, X_{k+1}] | w^{k-1}, W_k)$. Therefore, we find that $E(f(W_k) + J[\psi | x^k, X_{k+1}] | w^{k-1}, W_k) \in \mathscr{D}$.

We have now verified that both $f(W_k) + J[\psi | x^k, X_{k+1}] \in \mathscr{D}$ and $E(f(W_k) + J[\psi | x^k, X_{k+1}] | w^{k-1}, W_k) \in \mathscr{D}$. Hence,

$$\begin{aligned} c(\psi, x^k) &= E(J[\psi | x^k] | w^{k-1}) = E(f(W_k) + J[\psi | x^k, X_{k+1}] | w^{k-1}) \\ &= E(E(f(W_k) + J[\psi | x^k, X_{k+1}] | w^{k-1}, W_k) | w^{k-1}), \end{aligned}$$

where the first equality is the definition of $c(\psi, x^k)$, the second equality follows from (9) and the final equality follows from (E6). Using (E5), (E2), (E1) and (E4), we rewrite the argument—that is, the inner conditional expectation—of the outer conditional expectation, to yield

$$c(\psi, x^k) = E(f(W_k) + E(J[\psi | x^k, X_{k+1}] | w^{k-1}, W_k) | w^{k-1}). \tag{10}$$

For all $x_{k+1} \in \mathscr{X}_{k+1}$, we now define

$$c(\psi, x^k, x_{k+1}) := E(J[\psi | x^{k+1}] | w^{k-1}, v_k),$$

where $v_k \in \mathscr{W}_k$ is derived from a bijective linear function of x_{k+1} (and the "fixed" values $\psi_k(x^k)$ and x_k) given by (1). By construction, we may substitute $E(J[\psi|x^k, X_{k+1}]|w^{k-1}, W^k)$ with $c(\psi, x^k, X_{k+1})$—where X_{k+1} is a function of W_k—in (10), to yield

$$c(\psi, x^k) = E\Big((r_k + q_{k+1}b_k^2)\psi_k(x^k)^2 + q_{k+1}(a_k^2x_k^2 + W_k^2) + 2q_{k+1}b_k\psi_k(x^k)(a_kx_k + W_k) + 2q_{k+1}a_kx_kW_k + c(\psi, x^k, X_{k+1})\Big|w^{k-1}\Big),$$

where we have also re-substituted $f(W_k)$ with its full expression.

From the induction hypothesis, we know that—regardless of ψ_k—choosing $\psi_{k+1} = \hat{\phi}_{k+1}, ..., \psi_n = \hat{\phi}_n$ leads to the smallest possible point-wise value of $c(\psi, x^k, X_{k+1})$. Hence, because of Lemma 2, letting $\psi_{k+1} = \hat{\phi}_{k+1}, ..., \psi_n = \hat{\phi}_n$ and using the induction hypothesis yields

$$\begin{aligned} c(\psi, x^k) \geq E\Big(&(r_k + q_{k+1}b_k^2)\psi_k(x^k)^2 + q_{k+1}(a_k^2x_k^2 + W_k^2) + 2q_{k+1}b_k\psi_k(x^k)(a_kx_k + W_k) + 2q_{k+1}a_kx_kW_k \\ &+ (m_{k+1} - q_{k+1})X_{k+1}^2 + 2\tilde{r}_{k+1}a_{k+1}r_{k+1}X_{k+1}h_{k+1|W^k} - \tilde{r}_{k+1}(b_{k+1}h_{k+1|W^k})^2 \\ &+ 2\sum_{\ell=k+1}^{n-1}\tilde{r}_{\ell+1}a_{\ell+1}r_{\ell+1}E(W_\ell h_{\ell+1|W^\ell}|W^k) - \sum_{\ell=k+1}^{n-1}\tilde{r}_{\ell+1}b_{\ell+1}^2E((h_{\ell+1|W^\ell})^2|W^k) \\ &+ \sum_{\ell=k+1}^{n} m_{\ell+1}E(W_\ell^2|W^k)\Big|w^{k-1}\Big). \end{aligned} \tag{11}$$

In order to obtain the minimum for $c(\psi, x^k)$, we now simply need to correctly choose the value of $\psi_k(x^k)$. This is the main reason for assuming that (J2) holds, as this assumption allows us to use dynamic programming—optimisation in a backwards recursive manner—to find the optimal control input. We continue by again substituting X_{k+1} with its dynamics (1). After expanding the square and regrouping some terms, we find

$$\begin{aligned} c(\psi, x^k) \geq E\Big(&(r_k + m_{k+1}b_k^2)\psi_k(x^k)^2 + m_{k+1}(a_k^2x_k^2 + W_k^2) + 2b_k\psi_k(x^k)\big(m_{k+1}a_kx_k + m_{k+1}W_k + \tilde{r}_{k+1}a_{k+1}r_{k+1}h_{k+1|W^k}\big) \\ &+ 2m_{k+1}a_kx_kW_k + 2\tilde{r}_{k+1}a_{k+1}r_{k+1}a_kx_kh_{k+1|W^k} + 2\tilde{r}_{k+1}a_{k+1}r_{k+1}W_kh_{k+1|W^k} - \tilde{r}_{k+1}(b_{k+1}h_{k+1|W^k})^2 \\ &+ 2\sum_{\ell=k+1}^{n-1}\tilde{r}_{\ell+1}a_{\ell+1}r_{\ell+1}E(W_\ell h_{\ell+1|W^\ell}|W^k) - \sum_{\ell=k+1}^{n-1}\tilde{r}_{\ell+1}b_{\ell+1}^2E((h_{\ell+1|W^\ell})^2|W^k) \\ &+ \sum_{\ell=k+1}^{n} m_{\ell+1}E(W_\ell^2|W^k)\Big|w^{k-1}\Big). \end{aligned}$$

One can verify that all the terms in the above conditional expectation satisfy the requirements of Lemma 3. For most terms this is obvious, so we restrict our attention to the terms containing $h_{\ell|W^{\ell+1}}$. Note that, as mentioned before, it follows from (9) that $h_{\ell+1|W^\ell}$ is a linear combination of terms of the form $E(W_j|W^\ell)$ for $j > \ell$. Hence, from (B2) and (E2) it now follows that $E(W_\ell h_{\ell+1|W^\ell}|W^k)$ is equal to a linear combination of terms of the form $E(W_\ell E(W_j|W^\ell)|W^k)$ with $j > \ell$, which indeed satisfy the requirements of Lemma 3. Furthermore, it also follows from (B2) and (E2) that $E((h_{\ell+1|W^\ell})^2|W^k)$ is a linear combination of terms of the form $E(E(W_i|W^\ell)E(W_j|W^\ell)|W^k)$ with $\ell < i \leq j$, which also satisfy the requirements of Lemma 3. Hence, all terms in the above conditional expectation satisfy the requirements of Lemma 3, such that this conditional expectation is finite. We may therefore use the linearity of the expectation operator, to yield

$$\begin{aligned} c(\psi, x^k) \geq\ &(r_k + m_{k+1}b_k^2)\psi_k(x^k)^2 + m_{k+1}a_k^2x_k^2 + 2b_k\psi_k(x^k)(m_{k+1}a_kx_k + g_{k|w^{k-1}}) + 2a_kx_kg_{k|w^{k-1}} \\ &+ 2\tilde{r}_{k+1}a_{k+1}r_{k+1}E(W_kh_{k+1|W^k}|w^{k-1}) + 2\sum_{\ell=k+1}^{n-1}\tilde{r}_{\ell+1}a_{\ell+1}r_{\ell+1}E(E(W_\ell h_{\ell+1|W^\ell}|W^k)|w^{k-1}) \quad (12) \\ &- \tilde{r}_{k+1}b_{k+1}^2E((h_{k+1|W^k})^2|w^{k-1}) - \sum_{\ell=k+1}^{n-1}\tilde{r}_{\ell+1}b_{\ell+1}^2E(E((h_{\ell+1|W^\ell})^2|W^k)|w^{k-1}) \\ &+ m_{k+1}E(W_k^2|w^{k-1}) + \sum_{\ell=k+1}^{n} m_{\ell+1}E(E(W_\ell^2|W^k)|w^{k-1}), \quad (13) \end{aligned}$$

where

$$g_{k|w^{k-1}} := \tilde{r}_{k+1}a_{k+1}r_{k+1}E(h_{k+1|W^k}|w^{k-1}) + m_{k+1}E(W_k|w^{k-1}). \tag{14}$$

In order to continue, we first show that $g_{k|w^{k-1}}$ is equal to $h_{k|w^{k-1}}$ as defined in Theorem 1. To see this, recall from (9) that for all $w_k \in \mathscr{W}_k$, $h_{k+1|w^k}$ is derived from the initial condition $h_{n+1|w^k} = 0$ and, for all $\ell \in N$ such that $\ell \geq k+1$, from the recursive relation

$$h_{\ell|w^k} = \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} h_{\ell+1|w^k} + m_{\ell+1} E(W_\ell | w^k).$$

If we explicitly execute this recursion, we find that

$$h_{k+1|w^k} = \sum_{\ell=k+1}^{n} \gamma_\ell E(W_\ell | w^k), \tag{15}$$

where for all $\ell \in N$ such that $\ell \geq k+1$, γ_ℓ is a real number derived from $\tilde{r}_{\ell+1}, \dots, \tilde{r}_n$, $a_{\ell+1}, \dots, a_n$, $r_{\ell+1}, \dots, r_n$ and $m_{\ell+1}, \dots, m_{n+1}$. Similarly, we also find that $h_{k+1|W^k} = \sum_{\ell=k+1}^{n} \gamma_\ell E(W_\ell | W^k)$. Using (15) we can immediately verify that the conditions of Lemma 3 hold, such that $h_{k+1|W^k} \in \mathscr{D}$ and $E(h_{k+1|W^k} | w^{k-1})$ exists and is finite. Moreover, from the proof of Lemma 3 we know that then

$$E(h_{k+1|W^k} | w^{k-1}) = \sum_{\ell=k+1}^{n} \gamma_\ell E(E(W_\ell | W^k) | w^{k-1}).$$

As for all $\ell \in N$ such that $\ell \geq k+1$, $W_\ell \in \mathscr{D}$ by (B1) and $E(W_\ell | W^k) \in \mathscr{D}$ by (B3), it now follows from (E6) that

$$E(h_{k+1|W^k} | w^{k-1}) = \sum_{\ell=k+1}^{n} \gamma_\ell E(W_\ell | w^{k-1}).$$

It is now a matter of straightforward verification—and intuitively clear from (9)—that $E(h_{k+1|W^k} | w^{k-1}) = h_{k+1|w^{k-1}}$, where $h_{k+1|w^{k-1}}$ is as defined in Theorem 1. Consequently, we may substitute $E(h_{k+1|W^k} | w^{k-1})$ in (14) with $h_{k+1|w^{k-1}}$, to yield

$$g_{k|w^{k-1}} = \tilde{r}_{k+1} a_{k+1} r_{k+1} h_{k+1|w^{k-1}} + m_{k+1} E(W_k | w^{k-1}) = h_{k|w^{k-1}},$$

where the final equality follows from (9).

We continue simplifying (13), the expression for $c(\psi, x^k)$, by (i) substituting $g_{k|w^{k-1}}$ by $h_{k|w^{k-1}}$, (ii) applying the law of iterated expectations—which is allowed by Lemma 3—and (iii) incorporating some terms into the summations, to yield

$$\begin{aligned} c(\psi, x^k) \geq{} & (r_k + m_{k+1} b_k^2) \psi_k(x^k)^2 + m_{k+1} a_k^2 x_k^2 + 2 b_k \psi_k(x^k)(m_{k+1} a_k x_k + h_{k|w^{k-1}}) + 2 a_k x_{k+1} h_{k|w^{k-1}} \\ & + 2 \sum_{\ell=k}^{n-1} \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} E(W_\ell h_{\ell+1|W^\ell} | w^{k-1}) - \sum_{\ell=k}^{n-1} \tilde{r}_{\ell+1} b_{\ell+1}^2 E((h_{\ell+1|W^\ell})^2 | w^{k-1}) + \sum_{\ell=k}^{n} m_{\ell+1} E(W_\ell^2 | w^{k-1}). \end{aligned}$$

In order to simplify finding the minimising value of $\psi_k(x^k)$, we complete the squares containing $\psi_k(x^k)$, to yield

$$\begin{aligned} c(\psi, x^k) \geq{} & \tilde{r}_k^{-1} s_k^2 + (m_k - q_k) x_k^2 + 2 \tilde{r}_k a_k r_k x_k h_{k|w^{k-1}} - \tilde{r}_k b_k^2 h_{k|w^{k-1}}^2 + 2 \sum_{\ell=k}^{n-1} \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} E(W_\ell h_{\ell+1|W^\ell} | w^{k-1}) \\ & - \sum_{\ell=k}^{n-1} \tilde{r}_{\ell+1} b_{\ell+1}^2 E((h_{\ell+1|W^\ell})^2 | w^{k-1}) + \sum_{\ell=k}^{n} m_{\ell+1} E(W_\ell^2 | w^{k-1}), \end{aligned} \tag{16}$$

where m_k and $\tilde{r}_k$ are as defined in Theorem 1 and

$$s_k := \psi_k(x^k) + \tilde{r}_k b_k \left(m_{k+1} a_k x_k + h_{k|w^{k-1}} \right). \tag{17}$$

The only term in (16) that is influenced by our remaining choice of ψ_k is $\tilde{r}_k^{-1} s_k^2$. This term is minimised by demanding $\tilde{r}_k^{-1} s_k^2 = 0$, which is equal to demanding $s_k = 0$ by Lemma 1. By letting $\tilde{r}_k^{-1} s_k^2 = 0$ in (16) we obtain (2)—the expression for $\hat{c}(\phi^{k-1}, x^k)$—such that the first statement of this lemma is true.

If now $\psi_k \neq \hat{\phi}_k$, then there is at least one $x_\star^k \in \mathscr{X}^k$ such that $s_k \neq 0$. This implies that $c(\psi, x_\star^k) > \hat{c}(\phi^{k-1}, x_\star^k)$, such that the second statement of this lemma—(3)—is also true.

Finally, we confirm (4). Recall that the inequality in (11) is an equality if $\psi_{k+1} = \hat{\phi}_{k+1}, \dots, \psi_n = \hat{\phi}_n$. If this is the case, then $c(\psi, x^k) = \hat{c}(\phi^{k-1}, x^k)$ for all $x^k \in \mathscr{X}^k$ if we let $s_k = 0$ for all $x^k \in \mathscr{X}^k$ in (17). This can be achieved by choosing

$$\psi_k(x^k) = -\tilde{r}_k b_k \left(m_{k+1} a_k x_k + h_{k|w^{k-1}} \right) = \hat{\phi}_k(x^k), \tag{18}$$

which proves that (4) indeed holds. □

Proof of Theorem 1. Recall from (5) that the set of optimal control policies is defined as

$$\text{opt}(\Phi) := \left\{\phi \in \Phi : (\forall k \in N)(\forall x^k \in \mathscr{X}^k)(\forall \psi \in \Phi(\phi^{k-1}))\ E(J[\phi|x^k]|w^{k-1}) \leq E(J[\psi|x^k]|w^{k-1})\right\}.$$

We now use Lemma 4 to show that the control policy $\hat{\phi}$, as defined in Theorem 1, is indeed the only control policy in this set. Fix some $k \in N$, some $x^k \in \mathscr{X}^k$ and let $w^{k-1} \in \mathscr{W}^{k-1}$ be the noise history that is associated to x^k and $\hat{\phi}$ by (1). By combining (2) and (4), we find that

$$\hat{c}(\hat{\phi}^{k-1}, x^k) = E(J[\hat{\phi}|x^k]|w^{k-1}) \leq E(J[\psi|x^k]|w^{k-1})$$

for all $\psi \in \Phi(\hat{\phi}^{k-1})$. From the above inequality, we can infer that $\hat{\phi}$ is indeed an element of $\text{opt}(\Phi)$.

Next, consider any $\hat{\pi} \in \Phi$ such that $\hat{\pi} \neq \hat{\phi}$. Then there is at least one $k \in N$ such that $\hat{\pi}_k \neq \hat{\phi}_k$. Consider the smallest such k. Our choice of k obviously ensures that $\hat{\pi}^{k-1} = \hat{\phi}^{k-1}$, or equivalently that $\Phi(\hat{\pi}^{k-1}) = \Phi(\hat{\phi}^{k-1})$. From Lemma 4—more specifically from (4)—it follows that

$$(\forall x^k \in \mathscr{X}^k)\ E(J[\hat{\phi}|x^k]|w^{k-1}) = \hat{c}(\hat{\phi}^{k-1}, x^k) = \hat{c}(\hat{\pi}^{k-1}, x^k). \tag{19}$$

However, it also follows from Lemma 4—and more specifically from (3)—that there is an $x^k_\star \in \mathscr{X}^k$ such that

$$\hat{c}(\hat{\phi}^{k-1}, x^k_\star) = \hat{c}(\hat{\pi}^{k-1}, x^k_\star) < E(J[\hat{\pi}|x^k_\star]|w^{k-1}). \tag{20}$$

Combining (19) and (20) now yields that

$$E(J[\hat{\phi}|x^k_\star]|w^{k-1}) < E(J[\hat{\pi}|x^k_\star]|w^{k-1}).$$

As $\hat{\phi}^{k-1} \in \Phi(\hat{\pi}^{k-1})$, this inequality implies that $\hat{\pi}$ cannot be optimal. Hence, $\hat{\phi}$ is the unique element of $\text{opt}(\Phi)$. □

Proof of Proposition 1. This proposition is just an alternative statement of Proposition 4.4.1 of Bertsekas,[19] with the addition of a statement about the convergence of h_k. Let $N' := N \cup \{n+1\}$. The proof is more easily stated by considering the reversed sequences $\{m'_j\}_{j \in N'}$ and $\{h'_j\}_{j \in N'}$, defined as $m'_j := m_{n+1-j}$ and $h'_j := h_{n+1-j}$ for all $j \in N'$. Alternatively, these sequences can be derived from the initial conditions $m'_0 = q$ and $h'_0 = 0$ and, for all $j \in N$, from the recursive relation

$$m'_{j+1} = q + \tilde{r}'_{j+1} a^2 r m'_j, \tag{21}$$

$$h'_{j+1} = \tilde{r}'_j a r h'_j + m'_j E(W), \tag{22}$$

where

$$\tilde{r}'_j := \begin{cases} 0 & \text{if } j = 0, \\ (r + b^2 m'_{j-1})^{-1} & \text{otherwise.} \end{cases}$$

For $k \in N$ fixed, the limit of m_k for $n \to +\infty$ is equal to that of m'_j for $j \to +\infty$, and similarly for h_k and h'_j. Therefore, we can now focus on the limit behaviour of the infinite sequences $\{m'_j\}_{j \in \mathbb{N}}$ and $\{h'_j\}_{j \in \mathbb{N}}$. Throughout the remainder, we use $\mathbb{N}$ to denote the set of non-negative integers (including zero) and $\mathbb{N}_{>0}$ to denote the set of strictly positive integers.

From Lemma 1 it follows that m'_j converges for $j \to +\infty$. Let m be its limit value. In case $q = 0$, we can immediately verify from the initial condition $m'_0 = q = 0$ and the recursive relation (21) that $m'_j = 0$ for all $j \in \mathbb{N}$, which implies that $m = 0$. If on the contrary $q > 0$, then $\{m'_j\}_{j \in \mathbb{N}}$ is a monotonously increasing sequence of strictly positive real numbers. The limit value m of this sequence can be determined by setting m'_{j+1} and m'_j equal to m in (21)—i.e., assuming the convergence has occurred—and solving the resulting second order equation for m. Doing this yields two solutions:

$$m_+ = \frac{(a^2-1)r + b^2 q + \sqrt{((a^2-1)r + b^2 q)^2 + 4b^2 q r}}{2b^2}$$

and

$$m_- = \frac{(a^2-1)r + b^2 q - \sqrt{((a^2-1)r + b^2 q)^2 + 4b^2 q r}}{2b^2},$$

of which—since $b \neq 0$, $q > 0$ and $r > 0$—the first is strictly positive and the second is strictly negative. As m is the limit value of a strictly positive and monotonously increasing sequence, we may discard the second, strictly negative solution m_- and withhold only the first, strictly positive solution m_+. Note that if $q > 0$ and also $a = 0$, then it follows immediately from (21) that $m'_j = q$ for all $j \in \mathbb{N}$, which agrees with the obtained limit value m and the stated.

The convergence of $\{h'_j\}_{j \in \mathbb{N}}$ is proved a bit differently. In the special case that $q = 0$, it follows immediately from the initial value $h'_0 = 0$ and from setting $m'_j = 0$ in (22) that $h'_j = 0$ for all $j \in \mathbb{N}$. This agrees with the limit value h as defined in (12) of the statement. If on the contrary $q > 0$ but $a = 0$, then it also follows immediately from the initial value $h'_0 = 0$ and from setting $m'_j = q$ in (22) that $h'_j = qE(W)$ for all $j \in \mathbb{N}_0$, which also agrees with the limit value h as defined in (12) of the statement. The final trivial case for which we can immediately verify the limit value is $E(W) = 0$, as in this case it immediately follows from (22) that $h'_j = 0$ for all $j \in \mathbb{N}$.

In the remainder of the proof, without loss of generality, we assume that $q > 0$, $a \neq 0$ and $E(W) \neq 0$. Fix some arbitrary $j \in \mathbb{N}$, then we now posit that for all $k \in \mathbb{N}_{>0}$,

$$h'_{j+k} = \left(\prod_{\ell=0}^{k-1} \tilde{r}'_{j+\ell} ar\right) h'_j + \left(\sum_{\ell=0}^{k-1} \left(\prod_{i=\ell}^{k-2} \tilde{r}'_{j+i+1} ar\right) m'_{j+\ell}\right) E(W), \tag{23}$$

where the empty product is taken to equal 1 for notational convenience. We now prove the correctness of (23) using induction. For $k = 1$, (23) immediately reduces to (22). Next, we consider some $r \in \mathbb{N}_{>0}$, assume that (23) is valid for all $k \in \mathbb{N}_{>0}$ such that $k \leq r$, and prove that then (23) also holds for $k = r + 1$. By (22),

$$h'_{j+r+1} = \tilde{r}'_{j+r} ar h'_{j+r} + m'_{j+r} E(W).$$

If we substitute h'_{j+r} with its expression (23) and rewrite the resulting expression, we immediately find that (23) also holds for $k = r + 1$.

Before continuing with the proof, we observe that

$$\left|\frac{ar}{r + b^2 m}\right| < 1. \tag{24}$$

To verify that this indeed holds, we consider the cases $0 < |a| \leq 1$ and $|a| > 1$ separately. If $0 < |a| \leq 1$, (24) immediately follows from the strict positivity of r, b^2 and m:

$$|ar| \leq r < r + b^2 m = |r + b^2 m|.$$

If on the contrary $|a| > 1$, (24) follows from (11). Indeed, note that

$$\begin{aligned} |r + b^2 m| &= \left|r + \frac{1}{2}\left((a^2 - 1)r + b^2 q + \sqrt{((a^2 - 1)r + b^2 q)^2 + 4b^2 qr}\right)\right| \\ &> \left|r + (a^2 - 1)r + b^2 q\right| = \left|a^2 r + b^2 q\right| > |a^2 r| \geq |ar|. \end{aligned}$$

Consequently, we can fix some (arbitrarily small) strictly positive real number ϵ such that $\epsilon < m$—we previously proved that for $q > 0$, m is strictly positive—and

$$r + b^2 m > r + b^2(m - \epsilon) > |a| r.$$

If for all $\gamma \in [0, m)$ we let

$$c_\gamma := \frac{ar}{r + b^2(m - \gamma)},$$

then we have chosen ϵ such that $0 < |c_0| < |c_\epsilon| < 1$.

Next, observe that due to the convergence of $\{m'_j\}_{j\in\mathbb{N}}$ to m and because $\epsilon > 0$, there exists some $J \in \mathbb{N}_{>0}$ such that $m - \epsilon < m'_{J-1}$. As the sequence $\{m'_j\}_{j\in\mathbb{N}}$ moreover monotonously increases to m and because $\epsilon < m$, it follows that $0 < m - \epsilon < m'_{J+\ell-1} \leq m$ for all $\ell \in \mathbb{N}$. In order to proceed, we consider such a J and distinguish two cases based on the sign of a.

We first assume that $a > 0$, and recall that $b \neq 0$ and $r > 0$. It then follows that for all $k \in \mathbb{N}_0$,

$$\prod_{\ell=0}^{k-1} \tilde{r}'_{J+\ell} ar = \prod_{\ell=0}^{k-1} \frac{ar}{r + b^2 m'_{J+\ell-1}} \geq \prod_{\ell=0}^{k-1} \frac{ar}{r + b^2 m} = c_0^k \tag{25}$$

and

$$\prod_{\ell=0}^{k-1} \tilde{r}'_{J+\ell} ar = \prod_{\ell=0}^{k-1} \frac{ar}{r + b^2 m'_{J+\ell-1}} \leq \prod_{\ell=0}^{k-1} \frac{ar}{r + b^2(m - \epsilon)} = c_\epsilon^k. \tag{26}$$

Similarly, we also find that

$$\begin{aligned} \sum_{\ell=0}^{k-1} \left(\prod_{i=\ell}^{k-2} \tilde{r}'_{J+i+1} ar\right) m'_{J+\ell} &= \sum_{\ell=0}^{k-1} \left(\prod_{i=\ell}^{k-2} \frac{ar}{r + b^2 m'_{J+i}}\right) m'_{J+\ell} \geq (m - \epsilon) \sum_{\ell=0}^{k-1} \left(\prod_{i=\ell}^{k-2} \frac{ar}{r + b^2 m'_{J+i}}\right) \\ &\geq (m - \epsilon) \sum_{\ell=0}^{k-1} \left(\prod_{i=\ell}^{k-2} \frac{ar}{r + b^2 m}\right) \geq (m - \epsilon) \sum_{\ell=0}^{k-1} c_0^\ell = (m - \epsilon) \frac{1 - c_0^k}{1 - c_0} \end{aligned} \tag{27}$$

and

$$\sum_{\ell=0}^{k-1}\left(\prod_{i=\ell}^{k-2}\tilde{r}'_{J+i+1}ar\right)m'_{J+\ell}=\sum_{\ell=0}^{k-1}\left(\prod_{i=\ell}^{k-2}\frac{ar}{r+b^2m'_{J+i}}\right)m'_{J+\ell}\leq m\sum_{\ell=0}^{k-1}\left(\prod_{i=\ell}^{k-2}\frac{ar}{r+b^2(m-\epsilon)}\right)\leq m\frac{1-c_\epsilon^k}{1-c_\epsilon}. \tag{28}$$

Next, we need to take into account the sign of $E(W)$. From the recursive definition (22) and the signs of all involved parameters, it follows that h'_J is some non-negative real number if $E(W)>0$ and some non-positive real number if $E(W)<0$. We first consider the case $E(W)>0$. Combining (23) and (25)–(28) yields

$$c_0^k h'_J+(m-\epsilon)\frac{1-c_0^k}{1-c_0}E(W)\leq h'_{J+k}\leq c_\epsilon^k h'_J+m\frac{1-c_\epsilon^k}{1-c_\epsilon}E(W).$$

As $0<c_0<c_\epsilon<1$, taking the limit for $k\to+\infty$ for the lower and upper bound on h'_{J+k} yields

$$\limsup_{k\to+\infty} h'_{J+k}\leq\frac{1}{1-c_\epsilon}mE(W)=\frac{r+b^2(m-\epsilon)}{r+b^2(m-\epsilon)-ar}mE(W),$$
$$\liminf_{k\to+\infty} h'_{J+k}\geq\frac{1}{1-c_0}(m-\epsilon)E(W)=\frac{r+b^2m}{r+b^2m-ar}(m-\epsilon)E(W).$$

Next, we consider the case $E(W)<0$. Combining (23) and (25)–(28) now yields

$$c_\epsilon^k h'_J+m\frac{1-c_\epsilon^k}{1-c_\epsilon}E(W)\leq h'_{J+k}\leq c_0^k h'_J+(m-\epsilon)\frac{1-c_0^k}{1-c_0}E(W),$$

such that

$$\limsup_{k\to+\infty} h'_{J+k}\leq\frac{r+b^2m}{r+b^2m-ar}(m-\epsilon)E(W),$$
$$\liminf_{k\to+\infty} h'_{J+k}\geq\frac{r+b^2(m-\epsilon)}{r+b^2(m-\epsilon)-ar}mE(W).$$

As $\epsilon>0$ can be taken arbitrarily small, this proves the stated in the case that $a>0$.

The proof for $a<0$ is largely analogous, which is why we omit it. The only difference is that, due to the negativity of a, one needs to consider the bounds (25)–(28) for even and odd k separately. □

Lemma 5. Let E be a partially specified noise model defined by the lower bounds $\underline{E}(W_0),\dots,\underline{E}(W_n)$ and the upper bounds $\overline{E}(W_0),\dots,\overline{E}(W_n)$. For all $k\in N$, choose an arbitrary $\alpha_k\in[\underline{E}(W_k),\overline{E}(W_k)]$. There is at least one well-behaved probabilistic noise model E that agrees with the bounds, is white and has $E(W_k)=\alpha_k$ for all $k\in N$.

Proof. We can trivially construct such a white noise model E. For all $k\in N$, we let $E(W_k)=\alpha_k$ and $E(W_k^2)=1$. We can then trivially extend the domain and co-domain of E such that it satisfies (E1)–(E6) and is independent. As by construction the conditional expectation operator is independent and satisfies (W1) and (W2), it is indeed a white noise model. Also by construction it satisfies (PS), which proves the stated. □

Lemma 6. Let E be a partially specified noise model defined by the lower bounds $\underline{E}(W_0),\dots,\underline{E}(W_n)$ and the upper bounds $\overline{E}(W_0),\dots,\overline{E}(W_n)$. Then for all k,ℓ in N such that $k\leq\ell$ and all w^{k-1} in $\mathscr{W}^{k-1}$:

$$\underline{E}(W_\ell)\leq E(W_\ell|w^{k-1})\leq\overline{E}(W_\ell).$$

Proof. We prove this lemma using induction. Fix some arbitrary $\ell\in N$. By (PS), the stated holds for $k=\ell$. Consider now any $r\in N$ such that $r<\ell$ and assume that the stated holds for $k=r+1$. We will prove that this implies that the stated holds for $k=r$ as well. By (B3), $E(W_\ell|W^r)\in\mathscr{D}$ such that we may use the law of iterated expectations (E6):

$$E(W_\ell|w^{r-1})=E(E(W_\ell|W^r)|w^{r-1}).$$

From the induction hypothesis, we know that

$$\underline{E}(W_\ell)\leq E(W_\ell|w^r)\leq\overline{E}(W_\ell)$$

for all $w^r\in\mathscr{W}^r$. Combining Lemma 2 with the first inequality yields

$$E(E(W_\ell|W^r)|w^{r-1})\geq E(\underline{E}(W_\ell)|w^{r-1})=\underline{E}(W_\ell),$$

where the equality follows from (E1) and (E4). Using the second instead of the first inequality yields

$$E(E(W_\ell|W^r)|w^{r-1}) \leq E(\overline{E}(W_\ell)|w^{r-1}) = \overline{E}(W_\ell).$$

This way we have shown that indeed

$$\underline{E}(W_\ell) \leq E(W_\ell|w^{r-1}) \leq \overline{E}(W_\ell),$$

which finalises this proof. □

Proof of Theorem 2. Let E be the partially specified noise model defined by the lower bounds $\underline{E}(W_0), \dots, \underline{E}(W_n)$ and the upper bounds $\overline{E}(W_0), \dots, \overline{E}(W_n)$.

We first show that the feedforward term $h_{k|w^{k-1}}$ has a lower bound $\underline{h}_k$ and an upper bound $\overline{h}_k$ for all $w^{k-1} \in \mathscr{W}^{k-1}$. Let E be a well-behaved noise model that satisfies the bounds (PS), and fix some $k \in N$ and some $w^{k-1} \in \mathscr{W}^{k-1}$.

We will prove the bounds on $h_{k|w^{k-1}}$ by proving by induction that $h_{\ell|w^{k-1}} \in [\underline{h}_\ell, \overline{h}_\ell]$ for all $\ell \in N$ such that $\ell \geq k$. First, observe that if $\ell = n$, then (9) reduces to

$$h_{n|w^{k-1}} = m_{n+1} E(W_n|w^{k-1}).$$

By the non-negativity of $m_{n+1} = q_{n+1}$ and Lemma 6,

$$\underline{h}_n = m_{n+1}\underline{E}(W_n) \leq h_{n|w^{k-1}} \leq m_{n+1}\overline{E}(W_n) = \overline{h}_n.$$

Next, fix some $\ell \in N$ such that $k \leq \ell < n$ and assume that for all $j \in N$ such that $j > \ell$, $\underline{h}_j \leq h_{j|w^{k-1}} \leq \overline{h}_j$. By (9),

$$h_{\ell|w^{k-1}} = \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} h_{\ell+1|w^{k-1}} + m_{\ell+1} E(W_\ell|w^{k-1}).$$

By Lemmas 1 and 6, the second term in the expression for $h_{\ell|w^{k-1}}$ is bounded:

$$m_{\ell+1}\underline{E}(W_\ell) \leq m_{\ell+1} E(W_\ell|w^{k-1}) \leq m_{\ell+1}\overline{E}(W_\ell).$$

By the induction hypothesis and Lemma 1, the first term in the expression for $h_{\ell|w^{k-1}}$ is bounded too, but we need to distinguish between two cases. If $a_{\ell+1} \geq 0$, then

$$\tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} \underline{h}_{\ell+1} \leq \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} h_{\ell+1|w^{k-1}} \leq \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} \overline{h}_{\ell+1}.$$

If $a_{\ell+1} < 0$, then

$$\tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} \overline{h}_{\ell+1} \leq \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} h_{\ell+1|w^{k-1}} \leq \tilde{r}_{\ell+1} a_{\ell+1} r_{\ell+1} \underline{h}_{\ell+1}.$$

Combining the bounds on the first and the second term of the expression for $h_{\ell|w^{k-1}}$, we find that $\underline{h}_\ell \leq h_{\ell|w^{k-1}} \leq \overline{h}_\ell$. Hence, for all $\ell \in N$ such that $\ell \geq k$, we have found that $\underline{h}_\ell \leq h_{\ell|w^{k-1}} \leq \overline{h}_\ell$.

Now fix some $k \in N$ and some $h_k \in [\underline{h}_k, \overline{h}_k]$. Using Lemma 5, it is then possible to construct a white noise model that satisfies (PS) such that the resulting feedforward at time k is equal to h_k. This proves that the bounds $\underline{h}_k$ and $\overline{h}_k$ are tight. □

Proof of Proposition 2. The proof of this result is an extended version of the proof of Proposition 1. We will again consider the convergence of the "reversed" infinite sequences. The sequences $\{m'_j\}_{j\in\mathbb{N}}$ and $\{\tilde{r}'_j\}_{j\in\mathbb{N}}$ are defined as in the proof of Proposition 1. The bounds of the feedforward result in two sequences, $\{\underline{h}'_j\}_{j\in\mathbb{N}}$ and $\{\overline{h}'_j\}_{j\in\mathbb{N}}$, with initial conditions $\underline{h}'_0 = 0$ and $\overline{h}'_0 = 0$. If $a \geq 0$, then the sequences $\{\underline{h}'_j\}_{j\in\mathbb{N}}$ and $\{\overline{h}'_j\}_{j\in\mathbb{N}}$ are constructed recursively, for all $j \in \mathbb{N}$, by

$$\underline{h}'_{j+1} = \tilde{r}'_j a r \underline{h}'_j + m'_j \underline{E}(W) \qquad \text{and} \qquad \overline{h}'_{j+1} = \tilde{r}'_j a r \overline{h}'_j + m'_j \overline{E}(W).$$

These definitions result in sequences that are equivalent to the infinite sequence $\{h'_j\}_{j\in\mathbb{N}}$ of the proof of Proposition 1, but with $E(W) = \underline{E}(W)$ or $E(W) = \overline{E}(W)$. Hence, we can immediately conclude that the stated holds for $a \geq 0$.

Next, we consider the case $a < 0$. For all $j \in \mathbb{N}$, the recursive relations are now

$$\underline{h}'_{j+1} = \tilde{r}'_j a r \overline{h}'_j + m'_j \underline{E}(W) \qquad \text{and} \qquad \overline{h}'_{j+1} = \tilde{r}'_j a r \underline{h}'_j + m'_j \overline{E}(W).$$

In the case $q = 0$, we immediately infer from (the proof of) Proposition 1 and the above recursive expressions that $m'_k = 0 = m$, $\underline{h}'_k = 0$ and $\overline{h}'_k = 0$ for all $k \in \mathbb{N}$. Consequently, for any fixed $k \in N$,

$$\lim_{n\to+\infty} \underline{h}_k = \lim_{n\to+\infty} \underline{h}'_{n+1-k} = \lim_{j\to+\infty} \underline{h}'_j = 0$$

and

$$\lim_{n\to+\infty} \overline{h}_k = \lim_{n\to+\infty} \overline{h}'_{n+1-k} = \lim_{j\to+\infty} \overline{h}'_j = 0.$$

These limit values indeed correspond to those given in (18) and (19) for the case $q = 0$.

The stated therefore only remains unproven for $q > 0$ and $a < 0$. Depending on the sign of $\underline{E}(W)$ and $\overline{E}(W)$, we need to distinguish between three cases: $0 \leq \underline{E}(W) \leq \overline{E}(W)$, $\underline{E}(W) < 0 \leq \overline{E}(W)$ and $\underline{E}(W) \leq \overline{E}(W) < 0$. For each of these cases, we can prove the convergence of $\underline{h}'_{J+2k}$, $\overline{h}'_{J+2k}$, $\underline{h}'_{J+2k+1}$ and $\overline{h}'_{J+2k+1}$ using a similar strategy as outlined in the proof of Proposition 1, i.e., grouping the terms with odd or even powers of a. □