

# A 2-Approximation Algorithm for the Complementary Maximal Strip Recovery Problem

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## Abstract

The Maximal Strip Recovery problem (MSR) and its complementary (CMSR) are well-studied NP-hard problems in computational genomics. The input of these dual problems are two signed permutations. The goal is to delete some gene markers from both permutations, such that, in the remaining permutations, each gene marker has at least one common neighbor. Equivalently, the resulting permutations could be partitioned into common strips of length at least two. Then MSR is to maximize the number of remaining genes, while the objective of CMSR is to delete the minimum number of gene markers. In this paper, we present a new approximation algorithm for the Complementary Maximal Strip Recovery (CMSR) problem. Our approximation factor is 2, improving the currently best  $7/3$ -approximation algorithm. Although the improvement on the factor is not huge, the analysis is greatly simplified by a compensating method, commonly referred to as the non-oblivious local search technique. In such a method a substitution may not always increase the value of the current solution (it sometimes may even decrease the solution value), though it always improves the value of another function seemingly unrelated to the objective function.

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## 1 Introduction

Maximal Strip Recovery (MSR) is a problem originally proposed to eliminate noise and ambiguities in genomic maps [4, 15]. In comparative genomics, a genetic map (interchangeably, a signed permutation) is represented by a sequence of  $n$  distinct gene markers (interchangeably, letters). A gene marker can appear in two different genomic maps, either positively or negatively. A *strip* (or, syntenic block) is a sequence of distinct markers that appears as subsequences in two maps, either directly or in a reversed and negated form.

Given two genetic maps  $G_1$  and  $G_2$  of length  $n$ , the problem *Maximal Strip Recovery* (MSR) [4, 15] is to find two subsequences of  $d$  strips (each of length at least two), denoted as  $G_i^*$ , for  $i = 1, 2$ , and find two signed permutations  $\pi_i$  of  $\langle 1, \dots, d \rangle$ , such that each sequence



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## 5:2 Approximation Algorithm for the CMSR Problem

$G_i^* = S_{\pi_i(1)} \dots S_{\pi_i(d)}$  (here  $S_{-j}$  denotes the reversed and negated sequence of  $S_j$ ) is a subsequence of  $G_i$ , and the total length of these  $S_j$ 's is maximized. Intuitively, those gene markers not included in  $G_1^*$  and  $G_2^*$  are noisy and ambiguous markers. The complementary problem of deleting the minimum number of noisy and ambiguous markers to have a feasible solution (i.e., every remaining marker must be in some strip) is called the *Complementary Maximal Strip Recovery*, which will be abbreviated as CMSR. We illustrate an example in Fig. 1. In this example, each integer in  $G_1$  and  $G_2$  represents a gene marker.

$$\begin{aligned}
G_1 &= \langle 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 \rangle \\
G_2 &= \langle -9, -4, -8, -7, -6, 1, 2, 3, -13, -12, -10, -5, -11 \rangle \\
S_1 &= \langle 1, 2, 3 \rangle \\
S_2 &= \langle 6, 7, 8, 9 \rangle \\
S_3 &= \langle 11, 12, 13 \rangle \\
\pi_1 &= \langle 1, 2, 3 \rangle \\
\pi_2 &= \langle -2, 1, -3 \rangle \\
G_1^* &= \langle 1, 2, 3, 6, 7, 8, 9, 11, 12, 13 \rangle \\
G_2^* &= \langle -9, -8, -7, -6, 1, 2, 3, -13, -12, -11 \rangle
\end{aligned}$$

■ **Figure 1** An example for the problem MSR and CMSR. MSR has a solution size of ten (with  $d = 3$  strips in  $G_1^*$  and  $G_2^*$ ; i.e.,  $(1,2,3), (6,7,8,9)$  and  $(11,12,13)$ ). CMSR has a solution size of three: the deleted markers are 4,5 and 10.

It was shown in [14] that both (the decision versions of) MSR and CMSR are NP-complete. (Readers are referred to [5] for the basic concepts in algorithms and NP-completeness.) A bit later, MSR was shown to be APX-hard [2, 9] and CMSR was also shown to be APX-hard [10]. For the positive results, in [4, 15], some heuristic approaches based on Max Independent Set and Max Clique were proposed and shown to be effective. In [3], a factor-4 polynomial-time approximation algorithm was proposed for MSR. In [7], a factor-3 polynomial-time approximation algorithm was proposed for CMSR and an  $O^*(3^k)$  FPT algorithm, where  $k$  is the parameter representing the minimum number of deleted genes, was also presented for CMSR. Currently, the best approximation factor for CMSR is 2.33 [13] and the best FPT algorithmic bound is  $O(2.36^k n^2)$  [1]. In 2014, Jiang and Zhu showed that CMSR admits a kernel of size  $78k$  [8]. This kernel was improved to  $58k$  more recently by Hu *et al.* [6] and then to  $42k$  by Li *et al.* [12]. Combined with these kernel bounds, CMSR can be solved in  $O(n^2 + 2.36^k k^2)$  time.

In this paper, we devise a new 2-approximation algorithm for the CMSR problem by a non-oblivious local search technique, initially proposed by Khanna *et al.* [11]. During the non-oblivious local search process, a substitution (or local update) may not always increase the value of the current solution, it sometimes makes the value unchanged or even decreased. (But it always improves the value of a function which is seemingly unrelated to the objective function – that is probably how the algorithm is named “non-oblivious”.) In other words, some strips with larger priority are preferred even though they cannot lead to a local minimum solution. While the idea is simple, the analysis is very involved. We hope to see more applications of this technique for problems in computational biology.

## 2 Preliminaries

We first present some formal definitions. Let  $G = \pi_1\pi_2\cdots\pi_n$  be a (signed) permutation and let  $\bar{G}$  be the reversed version of  $G$ , with all elements negated. A *substring* of  $G$  is a consecutive segment in  $G$ ,  $\pi_i\pi_{i+1}\cdots\pi_j$ , such that  $1 \leq i < j \leq n$  and its length is denoted as  $|\pi_i\pi_{i+1}\cdots\pi_j| = j - i + 1$ . A *subsequence* of  $G$  is a sequence of letters  $\pi_{l_1}\pi_{l_2}\cdots\pi_{l_k}$  such that  $1 \leq l_1 < l_2 < \cdots < l_k \leq n$ .

Let  $G_1$  and  $G_2$  be two input (signed) permutations over the same set of alphabet (letters). (We assume that  $G_1$  is the identity permutation  $I_n$  over  $[1..n]$  throughout this paper.) A (common) *strip*,  $\pi_{l_1}\pi_{l_2}\cdots\pi_{l_j}$  ( $2 \leq j \leq n$ ), is a common subsequence between  $G_1$  and  $G_2$  (or  $\bar{G}_2$ ), with at least two letters, and its length is  $j$ . An *existing* common strip is a common substring between  $G_1$  and  $G_2$  (or  $\bar{G}_2$ ), again with length at least two. Sometimes we also use  $S = [a, b]$  to denote a common strip, with  $a$  and  $b$  being its ending letters, and  $|S| = |[a, b]|$  to denote its length. An *adjacency* is a strip of length two. Hence, a pair of common strips of length  $l$  has  $l - 1$  adjacencies. In Figure 1, the strip  $(1, 2, 3)$  appears positively in  $G_2$  and the strip  $(6, 7, 8, 9)$  appears in reversed and negated form in  $G_2$  (or, appears positively in  $\bar{G}_2$ ); moreover,  $(1, 2, 3) = [1, 3]$  is an existing common strip with two adjacencies.

The **Complementary Maximal Strip Recovery (CMSR)** problem is formally defined as follows: Given two signed permutations  $G_1$  and  $G_2$  over the same set of alphabet of size  $n$ , delete the minimum number of letters from  $G_1$  and  $G_2$  to obtain  $G_1^*$  and  $G_2^*$  such that the remaining letters are all in some common strips; moreover, each common strip is a substring in  $G_1^*$  and  $G_2^*$ .

Two letters,  $a$  and  $b$ , form a *candidate adjacency*  $(a, b)$ , if  $ab$  or  $-b - a$  is a subsequence of  $G_1$  and  $G_2$ . Let  $IN_1(a, b)$  and  $IN_2(a, b)$  be the set of (interior) letters that appears in between  $a$  and  $b$  in  $G_1$  and  $G_2$  respectively. If  $|IN_1(a, b)| + |IN_2(a, b)| = i$ , then  $(a, b)$  is called an *i-candidate* adjacency. Obviously, a 0-candidate adjacency is an adjacency from a pair of existing common strips of  $G_1$  and  $G_2$ . An *i-candidate* adjacency, say  $(a, b)$ , becomes an adjacency whenever the letters in  $IN_1(a, b)$  and  $IN_2(a, b)$  are deleted. In Figure 1,  $(11, 12)$  is initially a 2-candidate adjacency and after the two markers 5 and 10 are deleted it becomes a strip (or a valid adjacency). The main idea of our algorithm is to identify some candidate adjacencies and delete the letters in between them.

## 3 Algorithm Description

In this section, we show the details of our algorithms. Since candidate adjacencies eventually form the final common strips after deleting the interior letters, the main idea of our algorithm is to identify some candidate adjacencies.

### 3.1 Preprocessing

Firstly, the algorithm preprocesses the two input permutations to identify some special subsequences, which could be put into the common strip components directly.

► **Definition 1** (*r-candidate subsequence*). A subsequence  $a_1a_2\cdots a_{r-1}a_r$  of  $G_1$  and  $G_2$  (or  $\bar{G}_2$ ),  $r \geq 3$ , is an *r-candidate subsequence*, if  $(a_i, a_{i+1})$  is a 1-candidate adjacency,  $1 \leq i \leq r - 1$ .

A letter  $c$  can *left-attach* (resp. *right-attach*) to a common strip  $[a, b]$ , if  $(c, a)$  (resp.  $(b, c)$ ) is a candidate adjacency, then a new common strip  $[c, b]$  (resp.  $[a, c]$ ) is generated, while deleting the letters in  $IN_1(c, a) \cup IN_2(c, a)$  (resp.  $IN_1(b, c) \cup IN_2(b, c)$ ). Throughout Algorithm 1, let  $CS_0$  be the current set of common strips. Initially,  $CS_0$  is empty.

**Algorithm 1** Preprocessing.

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- 1: Put all the existing common strips into  $CS_0$ .
  - 2: Compute a maximal set of non-overlapping  $r$ -candidate subsequences for  $r \geq 3$  and put them in  $CS_0$ .
  - 3: **for** each common strip  $[a, b] \in CS_0$  **do**
  - 4:   **while** there exists a letter  $c$ , such that  $(c, a)$  or  $(b, c)$  is a 1-candidate adjacency **do**
  - 5:     Delete the letter in  $In_1(c, a) \cup In_2(c, a)$  (or  $In_1(b, c) \cup In_2(b, c)$ ) from  $G_1$  and  $G_2$ , delete  $[a, b]$  from  $CS_0$ , and add  $[c, b]$  (or  $[a, c]$ ) to  $CS_0$ .
  - 6:   **end while**
  - 7: **end for**
- 

Note that keeping existing common strips immediately implies that the factor 2 is the best we could have. For example,  $G_1 = uabv \cdot xy$  and  $G_2 = -v - u \cdot -yab - x$ , keeping  $ab$  can only have an approximation factor 2 (as the optimal solution for CMSR is to delete  $a$  and  $b$ ). The reason why we cannot keep 2-candidate subsequences is that they could force the approximation factor to be beyond 2. For instance,  $G_1 = uv \cdot xa_1ya_2z$  and  $G_2 = -va_1a_2 - u \cdot -z - y - x$ , keeping  $a_1a_2$  would give us an approximation factor of 2.5 (as we need to delete 5 letters, while the optimal solution is to delete  $a_1$  and  $a_2$ ). We now use  $CS_0$  to denote the set of common strips in  $CS_0$  found by the preprocessing Algorithm 1 and will not alter  $CS_0$  henceforth. We then assign  $CS \leftarrow CS_0$  and try to improve  $CS$ . Next, we show how to obtain more common strips.

### 3.2 How to generate a common strip

A letter is *matched* if it belongs to a common strip in  $CS$ , otherwise it is *unmatched*. A matched letter could have either one or two adjacent letters in a common strip. For a matched letter  $b$ , which has exactly one adjacent letter, say  $a$ , such that  $(a, b)$  is a  $p$ -candidate adjacency, then  $b$  is of *type- $p$* . For a matched letter  $b$ , which has exactly two adjacent letters, say  $a$  and  $c$ , such that  $(a, b)$  is a  $p$ -candidate adjacency and  $(b, c)$  is a  $q$ -candidate adjacency, then  $b$  is of *type- $\min\{p, q\}$* .

The letters in  $In_1(a, b) \cup In_2(a, b)$  of a candidate adjacency  $(a, b)$  are *caught* in both  $G_1$  and  $G_2$ , if  $a$  and  $b$  do form an adjacency; otherwise they are *released*. There are three ways to generate a common strip in our algorithm.

- (I) **Join**: generate a common strip  $[a, b]$  from a  $p$ -candidate adjacency  $(a, b)$ , where  $p \leq 3$ .
- (II) **Attach**: generate a common strip  $[a, c]$  of length 3 by right-attaching a letter  $c$  to a common strip  $[a, b]$  of length 2, such that  $(b, c)$  is a  $p$ -candidate adjacency, where  $p \leq 2$ . Moreover, if  $(a, b)$  is a 3-candidate adjacency, then delete  $a$  from  $[a, c]$  to obtain the strip  $[b, c]$ , to release the 3 letters in  $IN_1(a, b) \cup IN_2(a, b)$ . The case that  $(c, a)$  is a  $p$ -candidate adjacency, where  $p \leq 3$ , is similar.
- (III) **Split**: generate two common strips  $[a, b]$  and  $[c, d]$ , both of length 2, by right-attaching a letter  $d$  to a common strip  $[a, b, c]$  of length 3, such that  $(c, d)$  is a  $p$ -candidate adjacency, where  $p \leq 3$ . The case that  $(d, a)$  is a  $p$ -candidate adjacency, where  $p \leq 3$ , is similar.

► **Property 1.** *Each common strip generated by Join, Attach, and Split operations is of length 2 or 3. Moreover, if  $[a, b]$  is a common strip of length 2, then  $(a, b)$  is a  $p$ -candidate adjacency with  $p \leq 3$ ; if  $[a, b, c]$  is a common strip of length 3, then  $(a, b)$  is a  $p$ -candidate adjacency and  $(b, c)$  is a  $q$ -candidate adjacency with  $p, q \leq 2$ .*

### 3.3 The Non-oblivious Local Search Algorithm

The main idea of our algorithm is a non-oblivious local search, which is outlined as Algorithm 2. The algorithm improves  $CS$  iteratively according to the objective function  $F$ . Let  $b_p$  be the number of type- $p$  letters in the current solution, where  $p \in \{1, 2, 3\}$ . Define

$$F = 15b_1 + 5b_2 + b_3.$$

Generally, the algorithm adopts local substitutions, which means substituting exactly one current strip, say  $S$ , in  $CS$  with some other strips, provided that the value of  $F$  could be increased.

Let  $CS = \{S_1, S_2, \dots, S_m\}$  be the current set of common strips, and  $G'_1$  and  $G'_2$  be the current common strip components. For a common strip, say  $S_j$ , let  $S_i$  and  $S_k$  be its proceeding and following common strips in  $G'_1$ , and  $S_{i'}$  and  $S_{k'}$  be its proceeding and following common strips in  $G'_2$  respectively. Define four letters to be  $L_{S_j}^1, L_{S_j}^2, R_{S_j}^1, R_{S_j}^2$  as follows. If the number of letters appear in between  $S_i$  and  $S_j$  is less than 4 in  $G_1$ , then  $L_{S_j}^1$  is the letter to the immediate right of  $S_i$ ; otherwise,  $L_{S_j}^1$  is the fourth letter to the left of  $S_j$ . The other three letters can be defined similarly.

$CS$  could be improved by performing the following three operations iteratively.

1. *0-substitution*: Generate new common strips according to (I), (II) and (III), provided that the value of  $F$  can be increased.
2. *1-substitution*: Substitute a common strip  $S$  by some other strip, which can be searched from  $L_S^1$  to  $R_S^1$  in  $G_1$  and from  $L_S^2$  to  $R_S^2$  in  $G_2$  and according to (I), (II) and (III), provided that the value of  $F$  can be increased.
3. *2-substitution*: Substitute two consecutive common strips  $S_u$  and  $S_v$  in  $G'_1$  (resp.  $G'_2$ ) by two other strips, which can be searched from  $L_{S_u}^1$  to  $R_{S_v}^1$  in  $G_1$  and from  $L_{S_u}^2$  to  $R_{S_v}^2$  as well as  $L_{S_v}^2$  to  $R_{S_u}^2$  in  $G_2$  according to (I), (II) and (III), provided that the value of  $F$  can be increased.

The pseudo-code of our algorithm is shown in **Algorithm 2**.

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**Algorithm 2** CMSR by Non-oblivious-Local-Search.

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- 1: Call Algorithm 1: Preprocessing.
  - 2: **while** (0-substitution, 1-substitution, or 2-substitution can be applied) **do**
  - 3:   Apply a 0-substitution, 1-substitution, or 2-substitution to improve  $CS$ .
  - 4: **end while**
- 

► **Theorem 2.** *The algorithm CMSR by Non-oblivious-Local-Search runs in  $O(n^2)$  time.*

**Proof.** First of all, it is easy to see that Algorithm 1 runs in  $O(n^2)$  time. It takes at most  $O(n)$  time to perform a 0-substitution. A 1-substitution on a common strip  $S$  tries to identify all possible 1, 2, 3-candidate adjacencies formed by letters from  $L_S^1$  to  $R_S^1$  in  $G_1$  and from  $L_S^2$  to  $R_S^2$  in  $G_2$ . There are at most 21 distinct letters in this range if  $S$  is of length 2, and at most 23 distinct letters if  $S$  is of length 3. Hence the number of possible 1, 2, 3-candidate adjacencies is bounded by a constant, and the number of its combination is also bounded by a constant. In this case, a candidate adjacency can be found in constant time. Similar argument holds for a 2-substitution.

The number of common strips in  $CS$  is obviously bounded by  $n$ . It remains to count the time spent on the While-Loop. Note that the While-Loop runs while  $CS$  is updated/refreshed, in other words, each round of the While-Loop implies a feasible substitution, and  $F$  must be

increased. Since the value of  $F$  increases by at least one each time, and the maximum value of  $F$  is bounded by  $15n$ , the While-Loop runs at most  $15n$  times.

Consequently, it takes  $O(n)$  time to find/perform a feasible substitution. And this procedure loops  $O(n)$  times. Therefore the time complexity of **Algorithm 2** is  $O(n^2)$ . ◀

#### 4 Performance Analysis

In this section, we show that **Algorithm 2** returns a 2-approximation for the complementary maximum strip recovery problem. To analyze the performance of **Algorithm 2**, we should compare it with the optimal solution. Here, we adopt the amortized analysis method.

Let  $G_1 = [\pi_1\pi_2 \cdots \pi_n] = I_n$  and  $G_2 = [\tilde{\pi}_1\tilde{\pi}_2 \cdots \tilde{\pi}_n]$  be two (signed) permutations over  $[1..n]$ . Let  $G_1^*$  and  $G_2^*$  be the optimal common strip components, each of which is composed of the set  $CS^*$  of common strips. Let  $OPT$  be the set of letters which are deleted from both  $G_1$  and  $G_2$  to obtain  $G_1^*$  and  $G_2^*$  respectively. As aforementioned,  $CS$  is the set of common strips computed by **Algorithm 2**, which constitute two feasible common strip components  $G_1'$  and  $G_2'$ . Let  $ALG$  be the set of letters which are deleted to obtain  $G_1'$  and  $G_2'$ . Without causing confusion,  $CS$  and  $CS^*$  can also be viewed as the sets of adjacencies, as well as the sets of preserved letters. Then we have,  $|OPT| + |CS^*| = |ALG| + |CS| = n$ .

Next, we review a property of the optimal solution presented in [7].

► **Lemma 3.** *There exists an optimal solution  $OPT$ , such that, for every existing common strip  $S$  of length 2 or greater, (1)  $S$  is either totally contained in  $OPT$ , or totally disjoint with  $OPT$ ; (2) if  $S$  is of length 4 or greater, then it is totally disjoint with  $OPT$ .*

This lemma and its proof also appear in [7], so we omit the proof here. The idea, following the example right after Algorithm 1, is that while existing length-2 or length-3 common strips could be deleted in any optimal solution, when the existing common strips are of length at least 4 then there is some optimal solution which keeps them. For example,  $G_1 = xabcdy \cdot uv$  and  $G_2 = -y - x \cdot -vabcd - u$ , then the optimal solution for MSR could either be  $abcd$  (by keeping the existing length-4 common strip  $abcd$ , the corresponding CMSR solution is to delete  $u, v, x$  and  $y$ ), or with two strips  $xy$  and  $uv$  (the corresponding CMSR solution is to delete  $a, b, c$  and  $d$ ).

Now, consider the imaginary permutations whenever we reinsert the letters of  $OPT \cap CS$  back into  $G_1^*$  and  $G_2^*$ . Some common strips could be broken into either shorter blocks or *isolates* (e.g., a single letter in either  $G_1$  or  $G_2$  which does not form adjacencies with its neighbors). For instance, the letters 9 and 11 in Fig. 1 are both isolates.

Without loss of generality, we only focus on the case when the adjacency  $(a, b)$  (and  $(x, y)$ ) appears positively in  $G_1$  and  $G_2$ . The other case when it appears in reversed and negated form in  $G_2$  is similar. But we omit the details.

► **Definition 4.** *Let  $(a, b)$  be an adjacency of a common strip in  $CS$ , where  $a = \pi_{i_1} = \tilde{\pi}_{i_2}$  and  $b = \pi_{j_1} = \tilde{\pi}_{j_2}$ . Let  $(x, y)$  be an adjacency of a common strip in  $CS^*$ , where  $x = \pi_{i_1^*} = \tilde{\pi}_{i_2^*}$  and  $y = \pi_{j_1^*} = \tilde{\pi}_{j_2^*}$ .*

*When  $(a, b) = (x, y)$ , we say that  $(a, b)$  **fully occupies**  $(x, y)$ .*

*Let  $\cap$  be the operation for computing the intersection of two intervals, which could be closed (i.e., in the form  $[i, j]$ ) or open (i.e., in the form  $(i, j)$ ). If  $[i_1, j_1] \cap [i_1^*, j_1^*] = [i_1^*, i_1^*] = [j_1, j_1]$  or  $[i_2, j_2] \cap [i_2^*, j_2^*] = [i_2^*, i_2^*] = [j_2, j_2]$ , we say that  $(a, b)$  **half-occupies**  $(x, y)$  via the letter  $b$ , then  $y$  belongs to the **occupying set** of  $(a, b)$ , denoted as  $O[(a, b)]$ . (The case  $(a, b)$  **half-occupies**  $(x, y)$  via  $a$  can be defined symmetrically, in which  $x$  belongs to  $O[(a, b)]$ .)*

Otherwise, if  $(i_1, j_1) \cap (i_1^*, j_1^*) \neq \emptyset$  or  $(i_2, j_2) \cap (i_2^*, j_2^*) \neq \emptyset$ , we say that  $(a, b)$  **breaks**  $(x, y)$ , then both  $x$  and  $y$  belong to the **breaking set** of  $(a, b)$ , denoted as  $B[(a, b)]$ .

► **Lemma 5.** Let  $(x, y)$  be an adjacency in  $CS^*$ , but not in  $CS$ . Then  $(x, y)$  is either broken or half-occupied by some adjacencies in  $CS$ , or  $(x, y)$  is a  $p$ -candidate adjacency with  $p \geq 4$ .

**Proof.** Since otherwise, **Algorithm 2** would keep running as the current solution can be improved. ◀

The **occupying set** of a common strip  $S = [a, \dots, b]$  in  $CS$ , denoted as  $O[S]$ , is the set of all letters from the adjacencies in  $CS^*$  which are half-occupied by  $(a, \bullet)$  via  $a$  or by  $(\bullet, b)$  via  $b$ , i.e.,  $O[S] = O[(a, \bullet)] \cup O[(\bullet, b)]$ .

The **breaking set** of a common strip  $S$  in  $CS$ , denoted by  $B[S]$ , is the set of all letters from adjacencies in  $CS^*$ , which are broken by adjacencies of  $S$ , i.e.,  $B[S] = \cup_{(a,b) \subseteq S} B[(a, b)]$ . It follows from the definition that, for each common strip  $S$  in  $CS$ ,  $O[S] \cap B[S] = \emptyset$ .

The **auxiliary set** of a common strip  $S$  in  $CS$ , denoted by  $U[S]$ , is the set of all letters such that (1) not in  $CS$ , (2) appearing in between adjacencies of  $CS^*$ , which are broken or half-occupied by adjacencies of  $S$ .

Let  $CS_{\geq 4}^*$  be the set of letters from  $p$ -adjacencies of  $CS^*$  with  $p \geq 4$ , which are not broken or half-occupied by any common strip of  $CS$ . Let  $U_{\geq 4}^*$  be the set of letters which appear in between adjacencies formed by letters in  $CS_{\geq 4}^*$ . Viewing  $CS$  and  $CS^*$  as set of letters, we have  $ALG \cup CS = OPT \cup CS^*$ . Consequently

$$ALG - OPT = CS^* - CS = \cup_{S \in CS} (B[S] \cup O[S]) \bigcup CS_{\geq 4}^*.$$

From the definition, we know that the union of the auxiliary sets are letters in  $OPT \cap ALG$  which appear in between adjacencies of  $CS^* - CS_{\geq 4}^*$ , i.e.,

$$OPT \cap ALG \supseteq (\cup_{S \in CS} U[S]) \bigcup U_{\geq 4}^*.$$

We also have,

$$OPT - ALG = CS - CS^* = \cup_{S \in CS} (S - CS^*).$$

The approximation factor can be described as

$$\frac{|ALG|}{|OPT|} = \frac{|ALG - OPT| + |OPT \cap ALG|}{|OPT - ALG| + |OPT \cap ALG|} \leq 2$$

which is equivalent to,

$$\begin{aligned} \frac{|ALG|}{|OPT|} &\leq \frac{|ALG - OPT| + |OPT \cap ALG|}{|OPT - ALG|} \\ &\leq \frac{|\cup_{S \in CS} (B[S] \cup O[S]) \bigcup CS_{\geq 4}^*| + |(\cup_{S \in CS} U[S]) \bigcup U_{\geq 4}^*|}{|\cup_{S \in CS} (S - CS^*)|} \\ &= \frac{|\cup_{S \in CS} (B[S] \cup O[S])| + |\cup CS_{\geq 4}^*| + |(\cup_{S \in CS} U[S]) \bigcup U_{\geq 4}^*|}{|\cup_{S \in CS} (S - CS^*)|} \leq 2 \end{aligned} \quad (1)$$

Since each letter in  $OPT \cap ALG$  appears in  $G_1$  and  $G_2$  exactly once respectively, we have,

$$|(\cup_{S \in CS} U[S]) \bigcup U_{\geq 4}^*| = \frac{\sum_{S \in CS} |U[S]| + |U_{\geq 4}^*|}{2}.$$

Then, it is sufficient to show that

$$(1) = \frac{|\cup_{S \in CS} (B[S] \cup O[S])| - \frac{\sum_{S \in CS} |U[S]|}{2} + |\cup CS_{\geq 4}^*| - \frac{|U_{\geq 4}^*|}{2}}{|\cup_{S \in CS} (S - CS^*)|} \leq 2 \quad (2)$$



## 5:8 Approximation Algorithm for the CMSR Problem

Note that a letter may belong to more than one breaking and half-occupying sets. If we assign a weight  $\omega(-)$  to each letter of each set, then (2) can be rewritten as

$$(2) = \frac{\sum_{S \in CS} (\omega(B[S]) + \omega(O[S]) - \frac{|U[S]|}{2}) + |\bigcup CS_{\geq 4}^*| - \frac{|U_{\geq 4}^*|}{2}}{\sum_{S \in CS} |S - CS^*|} \leq 2 \quad (3)$$

► **Theorem 6.** *The approximation factor will not be greater than two, if the following two conditions hold:*

1.  $|\bigcup CS_{\geq 4}^*| - |U_{\geq 4}^*|/2 \leq 0$ , and
2. for each  $S \in CS$ ,  $(\omega(B[S]) + \omega(O[S]) - |U[S]|/2)/|S - CS^*| \leq 2$ .

The former condition holds from the following lemma straightforwardly.

► **Lemma 7.** *Let  $(x, y)$  be a  $p$ -adjacency of  $CS^*$  with  $p \geq 4$ , which is not broken or half-occupied by any adjacency of  $CS$ , being not able to return  $(x, y)$  as an adjacency will not result in an approximation factor greater than 2.*

**Proof.** Since the optimal solution includes the  $p$  letters appearing in between  $x$  and  $y$ , besides these  $p$  letters, the approximated solution also includes  $x$  and  $y$ . Then we have  $2 - p/2 \leq 0$ , in light of  $p \geq 4$ . ◀

It remains to assign weights to the letter of the breaking and half-occupying sets. Note that a letter cannot appear in two half-occupying sets.

### Weight Assignment.

- (I) For each  $S \in CS_0$ , each letter of  $B[S]$  bears a weight of 1; each letter of  $O[S]$  bears a weight of 1 if it does not appear in any other set, and bears a weight of 0 if it also appears in some other breaking sets.
- (II) If a letter appears in exactly one breaking set, then this appearance bears a weight of 1; if it appears in two or more breaking sets, each of its appearance bears a weight of  $\frac{1}{2}$ .
- (III) If a letter appears in exactly one half-occupying set, then this appearance bears a weight of 1.
- (IV) If a letter appears in the half-occupying set  $O[S]$  ( $S \notin CS_0$ ), as well as in the breaking set  $B[S']$ , then  $O[S]$  bears a weight of  $-\frac{1}{2}$ , and  $B[S']$  bears a weight of  $3/2$ .
- (V) If a letter appears in the half-occupying set  $O[S]$  ( $S \notin CS_0$ ), as well as two breaking sets  $B[S']$  and  $B[S'']$ , then  $O[S]$  bears a weight of  $-\frac{1}{2}$ , and  $B[S']$  bears a weight of 1 and  $B[S'']$  bears a weight of  $\frac{1}{2}$ .

► **Lemma 8.**  $\sum_{S \in CS} (\omega(B[S]) + \omega(O[S])) \geq |\bigcup_{S \in CS} (B[S] \cup O[S])|$ .

**Proof.** It can be verified that, under the above weight assignment, each letter of  $\bigcup_{S \in CS} (B[S] \cup O[S])$  has a total weight of at least 1. ◀

Next, we show that condition (2) is satisfied. We say that a common strip  $S \in CS$  is *safe*, if  $\frac{|B[S]|}{|S - CS^*|} \leq 2$ . An algorithm is *safe* if all the common strips generated at the end of the algorithm are safe.



#### 4.1 Algorithm 1 is Safe

We first try to show that all the common strips founded by Algorithm 1 are safe.

► **Lemma 9.** *The existing common strips are safe.*

**Proof.** Assume that  $S$  is an existing common strip of  $G_1$  and  $G_2$ . If  $S \subseteq CS^*$ , we are done. If not, according to Lemma 3, it satisfies that  $|S - CS^*| = |S| \geq 2$ . In this case, reinserting  $S$  back into  $G_1^*$  and  $G_2^*$ , would break at most 2 adjacencies in  $CS^*$ , thus  $|B(S)| \leq 4$ . then we have  $|B(S)|/|S - CS^*| \leq 4/2 = 2$ . ◀

The following example shows that (keeping) the existing common strips would be safe even for the worst case. Let  $G_1 = 1ab2 \cdot 34$  and  $G_2 = -2 - 1 \cdot -4ab - 3$ . Keeping  $S = ab$  would imply deleting the four letters  $\{1, 2, 3, 4\}$ , while the optimal solution is to delete  $\{a, b\}$ . Note that in this example,  $B[S] = \{1, 2, 3, 4\}$ ,  $S - CS^* = \{a, b\}$ , hence  $|B[S]| = 4$  and  $|S - CS^*| = 2$ ; moreover,

$$|B[S]|/|S - CS^*| = 2.$$

► **Lemma 10.** *Every 3-candidate subsequence is safe.*

**Proof.** Let  $T = abc$  be a 3-candidate subsequence. From **Definition 1**, both  $(a, b)$  and  $(b, c)$  are 1-candidate adjacencies. Assume that the letter  $x$  appears in between  $a$  and  $b$  in  $G_1$  or  $G_2$ , and the letter  $y$  appears in between  $b$  and  $c$  in  $G_1$  or  $G_2$ . There are 8 cases according to whether or not  $a, b, c$  belong to  $OPT$ .

1.  $a, b, c \notin OPT$ .  $[a, b, c]$  becomes a common strip after deleting  $x$  and  $y$  from both  $G_1$  and  $G_2$ , according to Lemma 9,  $[a, b, c]$  is safe.
2.  $\{a, b, c\} \subseteq OPT$ . Reinserting  $\{a, b, c\}$  back into  $G_1^*$  and  $G_2^*$  would break at most 4 adjacencies, thus  $|B([a, b, c])| \leq 6$ , and  $|B([a, b, c])|/|(\{a, b, c\} - CS^*)| \leq 6/3=2$ .
3.  $\{a, b\} \subseteq OPT$ . Reinserting  $[a, b]$  back into  $G_1^*$  and  $G_2^*$  would break at most 2 adjacencies, thus  $|B([a, b, c])| \leq 4$ , and  $|B([a, b, c])|/|(\{a, b, c\} - CS^*)| \leq 4/2=2$ .
4. The case when  $\{b, c\} \subseteq OPT$  is symmetric to the case  $\{a, b\} \subseteq OPT$ .
5.  $a \in OPT$ , then  $y \in OPT$ . Reinserting  $a$  back into  $G_1^*$  and  $G_2^*$  would break at most one adjacency, thus  $|B([a, b, c])| \leq 2$ , and  $|B([a, b, c])|/|(\{a, b, c\} - CS^*)| \leq 2/1=2$ .
6. The case when  $c \in OPT$  is symmetric to the case  $a \in OPT$ .
7.  $b \in OPT$ . Reinserting  $b$  back into  $G_1^*$  and  $G_2^*$  would break at most one adjacency  $(x, y)$ , thus  $|B([a, b, c])| \leq 2$ , and  $|B([a, b, c])|/|(\{a, b, c\} - CS^*)| \leq 2/1=2$ .
8.  $\{a, c\} \subseteq OPT$  but not  $b$ , then there exists a  $p$ -adjacency  $(b, d) \in CS^*$ . Since  $In_1(b, d) \cup In_2(b, d) \supseteq In_1(b, c) \cup In_2(b, c) = \{y\}$ , we could obtain another optimal solution by replacing  $c$  by  $d$  in  $CS^*$ . ◀

► **Lemma 11.** *Generate a common strip  $S' = [a, \dots, b, c]$  by attaching a letter  $c$  to a safe common strip  $S = [a, \dots, b]$  via a 1-candidate adjacency  $(b, c)$ , then  $S'$  is safe.*

**Proof.** Since  $S = [a, \dots, b]$  is safe, we have  $\frac{|B(S)|}{|S - CS^*|} \leq 2$ . Assume that the letter  $x$  appears in between  $b$  and  $c$  in  $G_1$  or  $G_2$ . There are 2 cases according to whether or not  $c \in OPT$ :

- (1)  $c \in OPT$ . Reinserting  $c$  back into  $G_1^*$  and  $G_2^*$  would break at most one adjacency  $(x, y)$ , which is not broken by  $a, b$ . Thus,  $|B(S') - B(S)| \leq 2$ . Note that  $|S' - CS^*| = |S - CS^*| + 1$ , then  $\frac{|B(S')|}{|S' - CS^*|} \leq \frac{|B(S)| + 2}{|S - CS^*| + 1} \leq 2$ .
- (2)  $c \notin OPT$ . In this case, either  $x \in OPT$  or  $x \in B(S)$ . In either case,  $B(S') = B(S)$  and  $S - CS^* = S' - CS^*$ . ◀

From Lemma 9,10,11, we know that all the common strips of  $CS_0$  are safe.

► **Lemma 12.** For each  $S \in CS_0$ ,  $\frac{\omega(B[S]) + \omega(O[S]) - |U[S]|/2}{|S - CS^*|} \leq 2$ .

**Proof.** It suffices to show that  $\omega(O[S]) - |U[S]|/2 \leq 0$ . Assume that  $S = [a, \dots, b]$ ,  $(b, c)$  is an adjacency in  $CS^*$ , which means  $c \in O[S]$ . By the **Weight Assignment** scheme, if  $(b, c)$  is broken by some other adjacencies in  $CS$ , then  $c$  bears a weight of 0 in  $O[S]$ . If  $(b, c)$  is not broken by any other adjacency of  $CS$ , then  $c$  bears a weight of 1 in  $O[S]$ . According **Algorithm 1**,  $(b, c)$  is a  $p$ -candidate adjacency with  $p \geq 2$ , and these  $p$  letters are all in  $U[S]$ . Thus,  $1 - p/2 \leq 0$ . ◀

## 4.2 Algorithm 2 is Safe

Aside from the common strips in  $CS_0$ , we now focus on the other common strips in  $CS$  which are of length either 2 or 3. If it is of length 2 and it is also an  $i$ -adjacency, we say that it is an  $i$ -common strip; if its length is 3 and it has two consecutive adjacencies: an  $i$ -adjacency and a  $j$ -adjacency, we say that it is an  $i \bowtie j$ -common strip. From Property 1, there are five types of common strips: 1-common strip, 2-common strip, 3-common strip,  $1 \bowtie 2$ -common strip,  $2 \bowtie 2$ -common strip.

Before showing that the common strips found by **Algorithm 2** fulfills condition 2, we first show some properties. The key idea is that, when **Algorithm 2** terminates, the value of  $F$  will not be increased by applying more 0,1,2-substitutions.

► **Lemma 13.** At the termination of **Algorithm 2**, a 1-candidate adjacency will either become an adjacency in  $CS$  or be broken by some 1-adjacency in  $CS$ .

**Proof.** Firstly, a 1-adjacency cannot be half-occupied by a 1-adjacency in  $CS$ . The reason is that, in this case, the two 1-adjacencies form a 3-candidate subsequence, which would have been handled by **Algorithm 1**.

Then we show that a 1-adjacency cannot only be half-occupied by adjacencies in  $CS$ . Assume that  $S = [a, \dots, b] \in CS$  and  $(b, c)$  is a 1-adjacency. Obviously,  $S \notin CS_0$ , thus  $|S| = 2$  or  $|S| = 3$ , and  $(\bullet, b)$  is a  $p$ -adjacency with  $p \geq 2$ . If  $|S|=2$ , then left-attaching  $c$  to  $S$  will increase the value of  $F$  definitely, if  $|S|=3$ , then substituting  $S$  with  $[a, \dots]$  and  $[b, c]$  will also increase the value of  $F$ . In case that  $S' = [c, \dots, d] \in CS$ , then  $|S'| = 2$  or  $|S'| = 3$ , and  $(c, \bullet)$  is a  $q$ -adjacency with  $q \geq 2$ . keeping the common strip  $[b, c]$  will increase the value of  $F$ .

If a 1-adjacency  $(b, c)$  is half-occupied by some adjacency  $A$  in  $CS$  and is also broken by another  $t$ -adjacencies  $A'$  of  $CS$ , then for  $t \geq 2$ , keeping the common strips  $[b, c]$  will increase the value of  $F$  by at least  $15 - 2 \times 5 = 5$ . ◀

From Lemma 13 and the **Weight Assignment** scheme, if an adjacency is multiply broken, then each common strip breaking it bears a weight of 1 from it. If an adjacency is singly broken by  $S \in CS$  and is also half-occupied by  $S' \in CS$ , then the common strip breaking it bears a weight of  $3/2$  from it, and the common strip half-occupying it bears a weight of  $-1/2$  from it; moreover, this adjacency must be a  $p$ -adjacency with  $p \geq 2$ , which means that there could be a letter in  $U[S]$ .

► **Lemma 14.** At the termination of **Algorithm 2**, a 2-adjacency will (1) either become an adjacency of  $CS$ , (2) or be broken by some adjacencies of  $CS$ , (3) or be half-occupied by two adjacencies of  $CS$ , (4) or be half-occupied by only one 1-adjacency of  $CS$ , which is in a common strip of  $CS_0$  or a  $1 \bowtie 2$ -common strip.

**Proof.** From Lemma 5, a 2-adjacency will either become an adjacency of  $CS$  or be broken or be half-occupied. Note that an adjacency can be half-occupied at most twice, then it is

sufficient to show that (4) holds. The reason is that if a 2-adjacency is only half-occupied by a 1-common strip, 2-common strip, 3-common strip, or a 2-adjacency in either a  $1 \bowtie 2$ -common strip or a  $2 \bowtie 2$ -common strip, then value of  $F$  will be increased by keeping it.  $\blacktriangleleft$

► **Lemma 15.** *At the termination of **Algorithm 2**, if a  $p$ -adjacency in  $CS^*$  is broken by a  $q$ -adjacency in  $CS$ , where  $q \leq 3$ , then either  $p \geq q$  or it is also broken or half-occupied by some other adjacencies in  $CS$ .*

**Proof.** Since otherwise, **Algorithm 2** will keep running as the current solution can be improved.  $\blacktriangleleft$

► **Lemma 16.** *A  $p$ -common strip  $S = [a, b]$ , where  $p = 1, 2, 3$ , guarantees that  $\frac{\omega(B[S]) + \omega(O[S]) - |U[S]|/2}{|S - CS^*|} \leq 2$ .*

**Proof.** There are three cases:  $|S - CS^*| = 2$ ,  $|S - CS^*| = 1$ ,  $|S - CS^*| = 0$ .

(1)  $|S - CS^*| = 2$ .  $|B[S]| \leq p + 4$  and  $|O[S]| = 0$ .

(1.1)  $p = 1$ , there are  $p + 2 = 3$  adjacencies broken by  $S$ . Since  $S$  is a common strip at the termination of **Algorithm 2** (i.e., no more local improvement is possible), at most one of these three adjacencies is a 1-candidate adjacency, which is singly broken by  $S$ . The other adjacencies must either be broken or half-occupied by some other adjacency in  $CS$ , or be  $p$ -candidate adjacencies with  $p \geq 4$ . For each of them, if it is half-occupied,  $|B[S]| + |O[S]| \leq 4$ . According to the **Weight Assignment** scheme,  $B[S]$  bears a weight of  $3/2$ , which means some other adjacency in  $CS$  bears a weight of  $-1/2$ . Moreover, if it is broken,  $B[S]$  bears a weight of 1; if it is a  $p$ -candidate adjacency with  $p \geq 4$ , then three letters are added to  $U[S]$ . In the worst case, we have  $\frac{\omega(B[S]) + \omega(O[S]) - |U[S]|/2}{2} \leq \frac{p+4-p}{2} = 2$ .

(1.2)  $p = 2$ , there are at least  $p + 1 = 3$  candidate adjacencies broken by  $S$ . As no local improvement is available, at most one of them could be a 2-candidate adjacency, which is singly broken by  $S$ . The others must either be broken or half-occupied by other adjacencies in  $CS$ , or be  $p$ -candidate adjacencies with  $p \geq 4$ . By an argument similar to (1.1), in the worst case, we have  $\frac{\omega(B[S]) + \omega(O[S]) - |U[S]|/2}{2} \leq \frac{p+4-p}{2} = 2$ .

(1.3)  $p = 3$ , there are at least  $p + 2 = 5$  candidate adjacencies broken by  $S$ . Similar to the above argument, as no local improvement is available, at least  $p + 1$  candidate adjacencies must either be broken or half-occupied by other adjacencies in  $CS$ , or be  $p$ -candidate adjacencies with  $p \geq 4$ . Similar to the previous arguments, we have  $\frac{\omega(B[S]) + \omega(O[S]) - |U[S]|/2}{2} \leq \frac{p+4-(p+1)}{2} \leq 2$ .

(2)  $|S - CS^*| = 1$ .  $|B[S]| \leq p + 2$  and  $|O[S]| = 1$ . In this case, according to the **Weight Assignment** scheme,  $O[S]$  bears a weight of  $-1/2$ .

(2.1)  $p = 1$ , then we are done, since  $|B[S]| \leq 2$ .

(2.2)  $p = 2$ , there are two candidate adjacencies broken by  $S$ . As local improvement is not possible, at most one of them could be a 2-candidate adjacency. The 2-candidate adjacency bears a weight 2, while the other one bears a weight 1; moreover, the 2-candidate adjacency also implies that a letter is in  $U[S]$ . So we have,  $\frac{\omega(B[S]) + \omega(O[S]) - |U[S]|/2}{2} \leq \frac{p+2-1/2-1-1/2}{1} = 2$ .

(2.3)  $p = 3$ , there are three candidate adjacencies broken by  $S$ . As local improvement cannot be performed further, at most one of them could be a 3-candidate adjacency. The 3-candidate adjacency bears a weight 2, while the other one bears a weight 1; moreover, the 3-candidate adjacency also implies that two letters are in  $U[S]$ . So we have,  $\frac{\omega(B[S]) + \omega(O[S]) - |U[S]|/2}{2} \leq \frac{p+2-1/2-1-1-1/2}{1} = 2$ .

- (3)  $|S - CS^*| = 0$ .  $|B[S]| \leq p$  and  $|O[S]| = 2$ . In this case, according to the **Weight Assignment** procedure,  $O[S]$  bears a weight of  $-1$ .
- (3.1)  $p = 1$ , then we are done, since  $\omega(B[S]) + \omega(O[S]) \leq 0$ .
- (3.2)  $p = 2$ , there is one candidate adjacency broken by  $S$ , which could be multiply broken or be a  $p$ -candidate adjacency with  $p \geq 2$ . The former means that  $B[S]$  bears a weight of  $1$ , the latter means that there are two letters in  $U[S]$ . So we have,  $\omega(B[S]) + \omega(O[S]) - |U[S]|/2 \leq 0$ .
- (3.3)  $p = 3$ , if there is one candidate adjacency broken by  $S$ , then it becomes the case (3.2). If there are two candidate adjacencies composed of three letters broken by  $S$ , then they could be multiply broken or be a  $p$ -candidate adjacency with  $p \geq 3$ . For each of them, if it is also broken by some common strips of  $CS_0$ , then  $B[S]$  bears a weight of  $0$ , and we are done; if it is a  $p$ -candidate adjacency with  $p \geq 3$ , then there are three letters in  $U[S]$ . If both of them are also broken by common strips not in  $CS_0$ , both adjacencies are not 1-candidate adjacencies (since otherwise, they become a 3-candidate subsequence). Thus, a new letter appears in  $U[S]$ . So we have,  $\omega(B[S]) + \omega(O[S]) - |U[S]|/2 \leq 0$ . ◀

► **Lemma 17.** *A  $p \bowtie q$ -common strip  $S = [a, b, c]$ , where  $p = 1, 2$  and  $q = 2$ , guarantees that  $\frac{\omega(B[S]) + \omega(O[S]) - |U[S]|/2}{|S - CS^*|} \leq 2$ .*

**Proof.** Similar to Lemma 16, hence the details are omitted. ◀

We summarize the main result of this paper as follows.

► **Theorem 18.** *The algorithm CMSR by Nonoblivious-Local-Search approximates CMSR with a factor of 2, and it runs in  $O(n^2)$  time.*

## 5 Concluding Remarks

We show a non-trivial application of non-oblivious local search for the CMSR problem. The local update step does not always increase the objective function. The difficulty has been assigning different weights for some potential common strips. We hope that this technique might be useful to other optimization problems in computational biology.

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