Bounded Degree Conjecture Holds Precisely for c-Crossing-Critical Graphs with $c \leq 12$

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Abstract

We study c-crossing-critical graphs, which are the minimal graphs that require at least c edge-crossings when drawn in the plane. For every fixed pair of integers with $c \geq 13$ and $d \geq 1$, we give first explicit constructions of c-crossing-critical graphs containing a vertex of degree greater than d. We also show that such unbounded degree constructions do not exist for $c \leq 12$, precisely, that there exists a constant D such that every c-crossing-critical graph with $c \leq 12$ has maximum degree at most D. Hence, the bounded maximum degree conjecture of c-crossing-critical graphs, which was generally disproved in 2010 by Dvořák and Mohar (without an explicit construction), holds true, surprisingly, exactly for the values $c \leq 12$.

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1 Introduction

Minimizing the number of edge-crossings in a graph drawing in the plane (the *crossing number* of the graph, see Definition 2.1) is considered one of the most important attributes of a "nice drawing" of a graph. In the case of classes of dense graphs (those having superlinear number of edges in terms of the number vertices), the crossing number is necessarily very high – see the famous Crossing Lemma [1, 13]. However, within sparse graph classes (those having only linear number of edges), we may have planar graphs at one end and graphs with up to quadratic crossing number at the other end. In this situation, it is natural to study the "minimal obstructions" for low crossing number, with the following definition.

Let c be a positive integer. A graph G is called *c-crossing-critical* if the crossing number of G is at least c, but every proper subgraph has crossing number smaller than c. We say that G is *crossing-critical* if it is c-crossing-critical for some positive integer c.

Since any non-planar graph contains at least one crossing-critical subgraph, the understanding of the properties of the crossing-critical graphs is a central part of the theory of crossing numbers.

In 1984, Širáň gave the earliest construction of nonsimple c-critical-graphs for every fixed value of $c \geq 2$ [18]. Three years later, Kochol [11] gave an infinite family of c-crossing-critical, simple, 3-connected graphs, for every $c \geq 2$. Another early result on c-crossing-critical graphs was reported in the influential paper of Richter and Thomassen [17], who proved that c-crossing-critical graphs have bounded crossing number in terms of c. They also initiated research on degrees in c-crossing-critical graphs by showing that, if there exists an infinite family of r-regular, c-crossing-critical graphs for fixed c, then $r \in \{4, 5\}$. Of these, 4-regular 3-critical graphs were constructed by Pinontoan and Richter [16], and 4-regular c-critical graphs are known for every $c \geq 3$, $c \neq 4$ [3]. Salazar observed that the arguments of Richter and Thomassen could be applied to average degree as well, showing that an infinite family of c-crossing-critical graphs of average degree d can exist only for $d \in (3, 6]$, and established their existence for $d \in [4, 6)$. Nonexistence of such families with $d = 6$ was established much later by Hernández, Salazar, and Thomas [9], who proved that, for each fixed c, there are only finitely many c-crossing-critical simple graphs of average degree at least six. The existence of such families with $d \in [\frac{7}{2}, 4]$ was established by Pinontoan and Richter [16], whereas the whole possible interval was covered by Bokal [2], who showed that, for sufficiently large crossing number, both the crossing number c and the average degree $d \in (3, 6)$ could be prescribed for an infinite family of c-crossing critical graphs of average degree d.

In 2003, Richter conjectured that, for every positive integer c, there exists an integer $D(c)$ such that every c-crossing-critical graph has maximum degree less than $D(c)$ [14]. Reflecting upon this conjecture, Bokal in 2007 observed that the known 3-connected crossing-critical graphs of that time only had degrees $3, 4, 6$, and asked for existence of such graphs with arbitrary other degrees, possibly appearing arbitrarily many times. Hliněný augmented his construction of c-crossing-critical graphs with pathwidth linear in c to show the existence of c-crossing-critical graphs with arbitrarily many vertices of every set of even degrees. Only a recent paper by Bokal, Bračič, Derňár, and Hliněný [3] provided the corresponding result for odd degrees, showing in addition that, for sufficiently high c, all the three parameters – crossing number c, rational average degree d, and the set of degrees $D \subseteq \mathbb{N} \setminus \{1, 2\}$ that appear arbitrarily often in the graphs of the infinite family – can be prescribed. They also analysed the interplay of these parameters for 2-crossing-critical graphs that were recently completely characterized by Bokal, Oporowski, Richter, and Salazar [5].

Despite all this research generating considerable understanding of the behavior of degrees in known crossing-critical graphs as well as extending the construction methods of such graphs, the original conjecture of Richter was not directly addressed in the previous works. It was, however, disproved by Dvořák and Mohar [8], who showed that, for each integer $c \geq 171$, there exist c-crossing-critical graphs of arbitrarily large maximum degree. Their counterexamples, however, were not constructive, as they only exhibited, for every such c, a graph containing sufficiently many critical edges incident with a fixed vertex and argued that those edges belong to every c-crossing-critical subgraph of the exhibited graph. On the other hand, as a consequence of [5] it follows that, except for possibly some small examples, the maximum degree in a large 2-crossing-critical graph is at most 6, implying that Richter's conjecture holds for $c = 2$. In view of these results, and the fact that 1-crossing-critical graphs (subdivisions of K_5 and $K_{3,3}$) have maximum degree at most 4, this leaves Richter's conjecture unresolved for each $c \in \{3, 4, \ldots, 170\}$.

The richness of c-crossing-critical graphs is restricted for every c by the result of Hliněný that c-crossing-critical graphs have bounded path-width [10]; this structural result is complemented by a recent classification of all large c-crossing-critical graphs for arbitrary c by Dvořák, Hliněný, and Mohar [7]. We use these results in Section 3 to show that Richter's conjecture holds for $c \leq 12$. The result is stated below. It is both precise and surprising and shows how unpredictable are even the most fundamental questions about crossing numbers.

▶ **Theorem 1.1.** *There exists an integer D such that, for every positive integer $c \leq 12$, every c-crossing-critical graph has maximum degree at most D.*

In fact, one can separately consider in Theorem 1.1 twelve upper bounds D_c for each of the values $c \in \{1, 2, \ldots, 12\}$. For instance, $D_1 = 4$ and the optimal value of D_2 (we know $D_2 \geq 8$) should also be within reach using [5] and continuing research. On the other hand, due to the asymptotic nature of our arguments, we are currently not able to give any "nice" numbers for the remaining upper bounds, and we leave this aspect to future investigations.

We cover the remaining values of $c \geq 13$ in the gap with the following:

▶ **Theorem 1.2.** *For every positive integer d, there exists a 3-connected 13-crossing-critical graph $G(d)$, whose maximum degree is at least d.*

▶ **Corollary 1.3.** *For every two integers $c \geq 13$ and $d \geq 1$, there exists a 3-connected c-crossing-critical graph $G(c, d)$, whose maximum degree is at least d.*

We also address the related question about the structure of c-crossing-critical graphs with more than one vertex of large degree. We show the following:

▶ **Corollary 1.4.** *For any integers $c \geq 13$, $i \geq 1$ and i, where $1 \leq i \leq c/13$, there exists a 3-connected c-crossing-critical graph $G(c, d, i)$, which contains i vertices of degree greater than d.*

Note that, without the 3-connectivity assumption, Corollary 1.4 is established simply by taking disjoint or vertex-identified copies of the graphs from Corollary 1.3.

The paper is structured as follows. The preliminaries, needed to help understanding the structure of large c-crossing critical graphs are defined in Section 2. We prove Theorem 1.1 in Section 3, and Theorem 1.2 in Section 4. This construction is combined with a new technical operation called 4-to-3 expansion and zip product to establish Corollaries 1.3 and 1.4 in Section 5. We conclude with some remarks and open problems in Section 6.

2 Graphs and the crossing number

In this paper, we consider multigraphs by default, even though we could always subdivide parallel edges (while sacrificing 3-connectivity) in order to make our graphs simple. We follow basic terminology of topological graph theory, see e.g. [15].

A *drawing* of a graph G in the plane is such that the vertices of G are distinct points and the edges are simple (polygonal) curves joining their end vertices. It is required that no edge passes through a vertex, and no three edges cross in a common point. A *crossing* is then an intersection point of two edges other than their common end. A *face* of the drawing is a maximal connected subset of the plane minus the drawing. A drawing without crossings in the plane is called a *plane drawing* of a graph, or shortly a *plane graph*. A graph having a plane drawing is *planar*.

The following are the core definitions used in this work.

▶ **Definition 2.1** (crossing number). *The* crossing number $\mathrm{cr}(G)$ *of a graph G is the minimum number of crossings of edges in a drawing of G in the plane. An* optimal drawing *of G is every drawing with exactly* $\mathrm{cr}(G)$ *crossings.*

▶ **Definition 2.2** (crossing-critical). *Let c be a positive integer. A graph G is c*-crossing-critical *if* $\mathrm{cr}(G) \geq c$*, but every proper subgraph G' of G has* $\mathrm{cr}(G') < c$*.*

Let us remark that a c-crossing-critical graph may have no drawing with only c crossings (for $c = 2$, such an example is the Cartesian product of two 3-cycles, $C_3 \square C_3$).

Suppose G is a graph drawn in the plane with crossings. Let G' be the plane graph obtained from this drawing by replacing the crossings with new vertices of degree 4. We say that G' is the plane graph associated with the drawing, shortly the *planarization* of (the drawing of) G, and the new vertices are the *crossing vertices* of G'.

Preliminaries. In some of our constructions, we will have to combine crossing-critical graphs as described in the next definition.

▶ **Definition 2.3.** *Let $d = 2$ or 3. For $i \in \{1, 2\}$, let G_i be a graph and let $v_i \in V(G_i)$ be a vertex of degree d that is only incident with simple edges, such that $G_i - v_i$ is connected. Let u_i^j, $j \in \{1, \dots, d\}$ be the neighbors of v_i. The zip product of G_1 and G_2 at v_1 and v_2 is obtained from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ by adding the edges $u_1^j u_2^j$, for each $j \in \{1, \dots, d\}$.*

Note that, for different labellings of the neighbors of v_1 and v_2, different graphs may result from the zip product. However, the following has been shown:

▶ **Theorem 2.4** ([4]). *Let G be a zip product of G_1 and G_2 as in Definition 2.3. Then,* $\mathrm{cr}(G) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$*. Furthermore, if for both $i = 1$ and $i = 2$, G_i is c_i-crossing-critical, where $c_i = \mathrm{cr}(G_i)$, then G is $(c_1 + c_2)$-crossing-critical.*

For vertices of degree 2, this theorem was established already by Leaños and Salazar in [12].

Dvořák, Hliněný, and Mohar [7] recently characterized the structure of large c-crossing-critical graphs. From their result, it can be derived that in a crossing-critical graph with a vertex of large degree, there exist many internally vertex-disjoint paths from this vertex to the boundary of a single face, see the following corollary for a more precise formulation. To keep our contribution self-contained, we also give a simple independent proof of this claim in the full version of the paper.

▶ **Corollary 2.5.** *There exists a function $f_{2.5} : \mathbb{N}^2 \to \mathbb{N}$ such that the following holds. Let $c \geq 1$ and $t \geq 3$ be integers and let G be an optimal drawing of a 2-connected c-crossing-critical graph. If G has maximum degree greater than $f_{2.5}(c,t)$, then there exists a path Q contained in the boundary of a face of G and internally vertex-disjoint paths $P_1, \ldots, P_t$ starting in the same vertex not in Q and ending in distinct vertices appearing in order on Q (and otherwise disjoint from Q), such that no crossings of G appear on P_1, P_t, nor in the face of $P_1 \cup P_t \cup Q$ that contains $P_2, \ldots, P_{t-1}$.*

3 Crossing-critical graphs with at most 12 crossings

We now use Corollary 2.5 to prove the following "redrawing" lemma.

▶ **Lemma 3.1.** *Let G be a 2-connected c-crossing-critical graph. If G has maximum degree greater than $f_{2.5}(c, 6c+1)$, then there exist integers $r \geq 2$ and $k \geq 0$ such that $kr \leq c-1$ and G has a drawing with at most $c - 1 - kr + \binom{k}{2}$ crossings.*

Proof. Consider an optimal drawing of G. Let $P_1, \ldots, P_{6c+1}$ be paths obtained using Corollary 2.5 and v their common end vertex. For $2 \leq i \leq 6c-1$, let T_i denote the 2-connected block of $G - ((V(P_{i-1}) \cup V(P_{i+2})) \setminus \{v\})$ containing P_i and P_{i+1}, and let C_i denote the cycle bounding the face of T_i containing P_{i-1}. Note that if $2 \leq i$ and $i+3 \leq j \leq 6c-1$, then $G - V(T_i \cup T_j)$ has at most three components: one containing $P_{i+2} - v$, one containing $P_1 - v$, and one containing $P_{6c+1} - v$, where the latter two components can be the same.

Let e be the edge of P_{3c+1} incident with v and let G' be an optimal drawing of $G - e$. Since G is c-crossing-critical, G' has at most $c-1$ crossings. Hence, there exist indices i_1 and i_2 such that $2 \leq i_1 \leq 3c-1$, $3c+2 \leq i_2 \leq 6c-1$, and none of the edges of T_{i_1} and T_{i_2} is crossed. Let us set $L = T_{i_1}$, $C_L = C_{i_1}$, $R = T_{i_2}$, and $C_R = C_{i_2}$. Let M, S_1, and S_2 denote the subgraphs of G consisting of the components of $G - V(L \cup R)$ containing $P_{3c+1} - v$, $P_1 - v$, and $P_{6c+1} - v$, respectively, together with the edges from these components to the rest of G and their incident vertices (where possibly $S_1 = S_2$). Let S_L and M_L be subpaths of C_L of length at least one intersecting in v such that $V(S_1 \cap C_L) \subseteq V(S_L)$ and $V(M \cap C_L) \subseteq V(M_L)$. Analogously, let S_R and M_R be subpaths of C_R of length at least one intersecting in v such that $V(S_2 \cap C_R) \subseteq V(S_R)$ and $V(M \cap C_R) \subseteq V(M_R)$. See Figure 1.

We can assume without loss of generality (by circle inversion of the plane if necessary) that neither C_L nor C_R bounds the outer face of $C_L \cup C_R$ in the drawings inherited from G and from G'. Let $e_{M_L}, e_{S_L}, e_{S_R}, e_{M_R}$ be the clockwise cyclic order of the edges of $C_L \cup C_R$ incident with v in the drawing G, where $e_Q \in E(Q)$ for every $Q \in \{M_L, S_L, S_R, M_R\}$. By the same argument, we can assume that the clockwise cyclic order of these edges in the drawing of G' is either the same or $e_{M_L}, e_{S_L}, e_{M_R}, e_{S_R}$.

In G, L is drawn in the closed disk bounded by C_L, R is drawn in the closed disk bounded by C_R, and M, S_1, and S_2 together with all the edges joining them to v are drawn in the outer face of $C_L \cup C_R$. Since C_L and C_R are not crossed in the drawing G', we can if necessary rearrange the drawing of G' without creating any new crossings[1] so that the same holds for the drawings of L, R, M, S_1, and S_2 in G'. Let $r \geq 1$ denote the maximum number of pairwise

[1] As G is not 3-connected, it is possible that some 2-connected components or some edges of L, R are drawn in the exterior of the disk bounded by C_L, C_R. However, these can be flipped into the interior of C_L, C_R, and after such *rearranging*, C_L, C_R bound the outer face of the drawings of L, R. Similarly, if $S_1 \neq S_2$, either of them could be in the interior of C_L, C_R, and we flip them into the exterior, so that the interior of C_L, C_R contains only drawings of L, R, respecitvely.

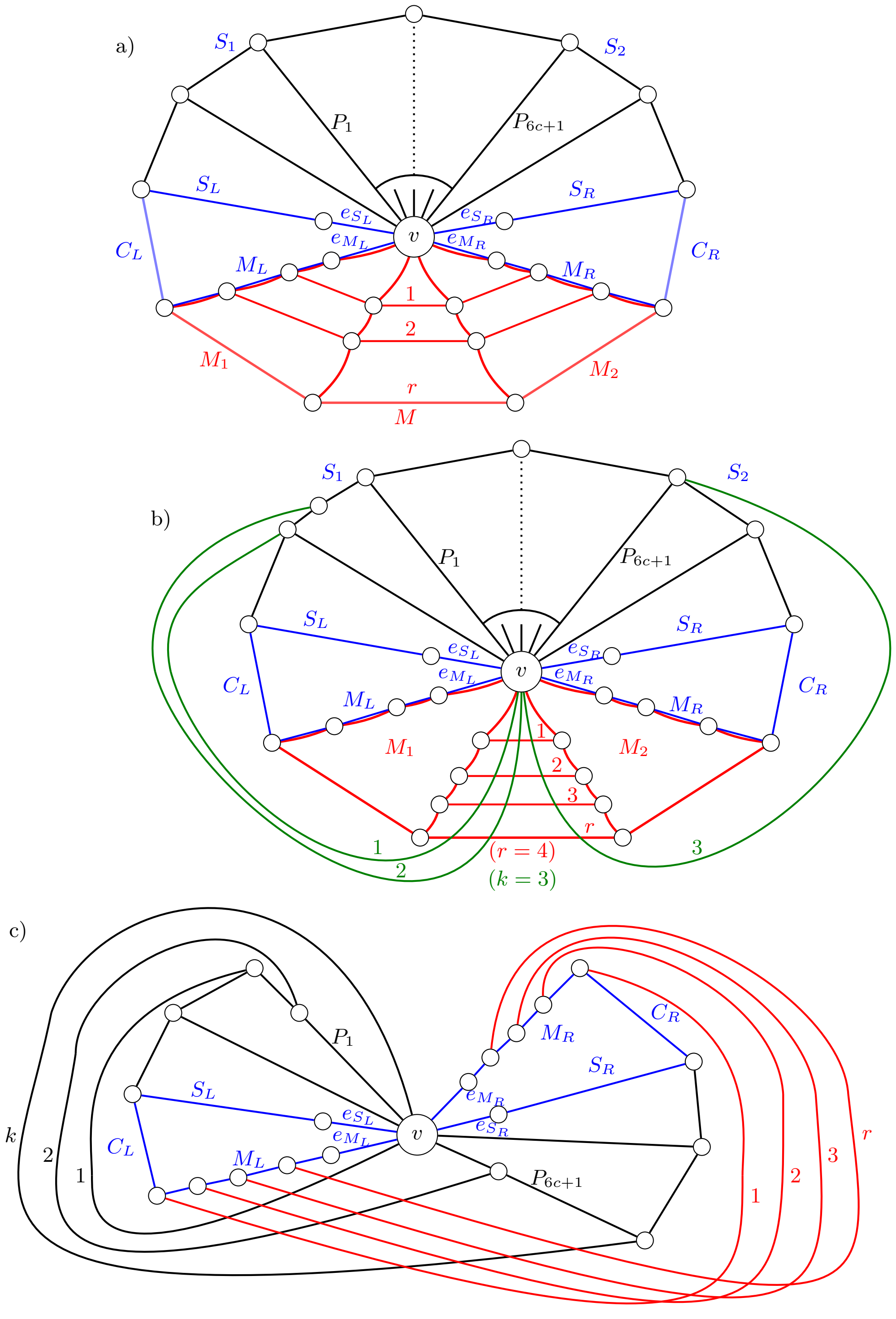

Figure 1 An illustration of the proof of Lemma 3.1. a) The original optimal drawing of G, with subdrawings of M_1 and M_2 (red) that will be glued into the drawing of G_0 from an optimal drawing of $G - e$. b) A drawing of G with at most $c - 1$ crossings, obtained from G_0 (black, blue, green) and M_1, M_2 (red). c) A drawing of G with at most $\binom{c-1-kr+k}{2}$ crossings, obtained from G_0 (black, blue) and M_1, M_2 (red).

edge-disjoint paths in $M - v$ from $V(M \cap C_L - v)$ to $V(M \cap C_R - v)$. By Menger's theorem, $M - v$ has disjoint induced subgraphs M_1' and M_2' such that $V(M - v) = V(M_1') \cup V(M_2')$, $V(M \cap C_L - v) \subseteq V(M_1')$, $V(M \cap C_R - v) \subseteq V(M_2')$, and G contains exactly r edges with one end in M_1' and the other end in M_2'. For $i \in \{1,2\}$, let M_i be the subgraph of M induced by $V(M_i') \cup \{v\}$. Let F be a path in $M - v$ from $V(M \cap C_L - v)$ to $V(M \cap C_R - v)$ that has in the drawing G' the smallest number of intersections with the edges of $S_1 \cup S_2$, and let k denote the number of such intersections. Let G_0 denote the drawing $G' - (V(M) \setminus V(M_L \cup M_R))$. Since $M - v$ contains r pairwise edge-disjoint paths from $V(M \cap C_L - v)$ to $V(M \cap C_R - v)$ and each of them crosses $S_1 \cup S_2$ at least k times, we conclude that G' has at least kr crossings (and thus $kr \leq c - 1$) and G_0 has at most $c - 1 - kr$ crossings.

Suppose first that edges of $C_L \cup C_R$ incident with v are in G' drawn in the same clockwise cyclic order as in G. We construct a new drawing of the graph G in the following way: Start with the drawing of G_0. Take the plane drawings of M_1 and M_2 as in G, "squeeze" them and draw them very close to M_L and M_R, respectively, so that they do not intersect any edges of G_0. Finally, draw the r edges between M_1 and M_2 very close to the curve tracing F (as drawn in G'), so that each of them is crossed at most k times. This gives a drawing of G with at most $(c - 1 - kr) + kr < c$ crossings, contradicting the assumption that G is c-crossing-critical.

Hence, we can assume that the edges of $C_L \cup C_R$ incident with v are in G' drawn in the clockwise order e_{M_L}, e_{S_L}, e_{M_R}, e_{S_R}. If $r = 1$, then proceed analogously to the previous paragraph, except that a mirrored version [2] of the drawing of M_2 is inserted close to M_R; as there is only one edge between M_1 and M_2, this does not incur any additional crossings, and we again conclude that the resulting drawing of G has fewer than c crossings, a contradiction. Therefore, $r \geq 2$.

Consider the drawing G', and let q be a closed curve passing through v, following M_L slightly outside C_L till it meets F, then following F almost till it hits M_R, then following M_R slightly outside C_R till it reaches v. Note that q only crosses G_0 in v and in relative interiors of the edges, and it has at most k crossings with the edges. Shrink and mirror the part of the drawing of G_0 drawn in the open disk bounded by q, keeping v at the same spot and the parts of edges crossing q close to q; then reconnect these parts of the edges with their parts outside of q, creating at most $\binom{k}{2}$ new crossings in the process. Observe that in the resulting re-drawing of G_0, the path $M_L \cup M_R$ is contained in the boundary of a face (since q is drawn close to it and nothing crosses this part of q), and thus we can add M planarly (as drawn in G) to the drawing without creating any further crossings. Therefore, the resulting drawing has at most $c - 1 - kr + \binom{k}{2}$ crossings. ◀

It is now easy to prove Theorem 1.1.

Proof of Theorem 1.1. We prove by induction on c that, for every positive integer $c \leq 12$, there exists an integer Δ_c such that every c-crossing-critical graph has maximum degree at most c. The only 1-crossing-critical graphs are subdivisions of K_5 and $K_{3,3}$, and thus we can set $\Delta_1 = 4$. Suppose now that $c \geq 2$ and the claim holds for every smaller value. We define $\Delta_c = \max(2\Delta_{c-1}, f_{2.5}(c, 6c + 1))$. Let G be a c-crossing-critical graph and suppose for a contradiction that $\Delta(G) > \Delta_c$.

[2] *Mirrored version of a drawing* is the drawing obtained by reversing the vertex rotations of edges around every vertex and every crossing, and embedding the edges and the vertices accordingly. The name explains that this is homeomorphic to the original drawing seen in a mirror.

If G is not 2-connected, then it contains induced subgraphs G_1 and G_2 such that $G_1 \neq G \neq G_2$, $G = G_1 \cup G_2$, and G_1 intersects G_2 in at most one vertex. Then $c \leq \mathrm{cr}(G) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$, and for every edge $e \in E(G_1)$ we have $c > \mathrm{cr}(G - e) = \mathrm{cr}(G_1 - e) + \mathrm{cr}(G_2)$. Hence, $\mathrm{cr}(G_1) \geq c - \mathrm{cr}(G_2)$ and $\mathrm{cr}(G_1 - e) < c - \mathrm{cr}(G_2)$ for every edge $e \in E(G_1)$, and thus G_1 is $(c - \mathrm{cr}(G_2))$-crossing-critical. Similarly, G_2 is $(c - \mathrm{cr}(G_1))$-crossing-critical. Since $\mathrm{cr}(G_1) \geq 1$ and $\mathrm{cr}(G_2) \geq 1$, it follows by the induction hypothesis that $\Delta(G_i) \leq \Delta_{c-1}$ for $i \in \{1, 2\}$, and thus $\Delta(G) \leq \Delta_c$, which is a contradiction.

Hence, G is 2-connected. By Lemma 3.1, there exist integers $r \geq 2$ and $k \geq 0$ such that $kr \leq c - 1$ and $c - 1 - kr + \binom{k}{2} \geq c$, and thus $\binom{k}{2} \geq kr + 1 \geq 2k + 1$. This inequality is only satisfied for $k \geq 6$, and thus the first inequality implies $c \geq kr + 1 \geq 13$. This is a contradiction. Hence, the maximum degree of G is at most Δ_c. ◀

4 Explicit 13-crossing-critical graphs with large degree

We define the following family of graphs, which is illustrated in Figure 2. To simplify the terminology and the pictures, we introduce "thick edges": for a positive integer t, we say that uv is a *t-thick edge*, or an edge of *thickness* t, if there is a bunch of t parallel edges between u and v. Naturally, if a t_1-thick edge crosses a t_2-thick edge, then this counts as $t_1 t_2$ ordinary crossings. By routing every parallel bunch of edges along the "cheapest" edge of the bunch, we get the following important folklore claim:

▷ Claim 4.1. For every graph G, there exists an optimal drawing $\mathcal{D}$ of G, such that every bunch of parallel edges is drawn as one thick edge in $\mathcal{D}$.

▶ **Definition 4.2** (Critical family $\{G_{13}^k\}$). *Let $k \geq 2$ be an integer. Let C_u be a 6-cycle on the vertex set $\{x, u_1, u_2, u_3, u_4, u_5\}$ with (thick) edges xu_1, u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5x which are of thickness $7, 5, 4, 4, 4, 1$ in this order. Analogously, let C_v be a 6-cycle on the vertex set $\{x, v_1, v_2, v_3, v_4, v_5\}$ isomorphic to C_u in this order of vertices. We denote by B the graph obtained from the union $C_u \cup C_v$ (identifying their vertex x) by adding edges u_2v_3 and u_3v_2, and 2-thick edges u_1v_4 and u_4v_1.*

Let D_i, for $i \in \{1, \ldots, k\}$, denote the graph on the vertex set $\{x, w_1^i, w_2^i, w_3^i, w_4^i\}$ with the edges xw_1^i, xw_4^i, $w_1^iw_4^i$, $w_2^iw_3^i$ and the 2-thick edges $w_1^iw_2^i$ and $w_3^iw_4^i$. From the union $B \cup D_1 \cup \ldots \cup D_k$ (again identifying their vertex x), we obtain obtain the graph G_{13}^k via

- *identifying u_5 with w_2^1 and w_3^k with v_5, and*
- *for $i = 2, 3, \ldots, k$, identifying w_3^{i-1} with w_2^i.*

This definition is illustrated in Figure 2. For reference, we will call the graph B the bowtie *of G_{13}^k, and the graph D_i the i-th* wedge *of G_{13}^k.*

▶ **Observation 4.3.**

a) *For every $k \geq 2$, the graph G_{13}^k is a 3-connected and non-planar.*

b) *The degree of the vertex x in G_{13}^k equals $2k + 16$.*

c) *There exists an automorphism of G_{13}^k exchanging u_i with v_i, for $i = 1, 2, 3, 4, 5$.*

In order to prove Theorem 1.2, e.g. for $G(d) = G_{13}^{\lfloor d/2 \rfloor}$, it suffices to show two claims; that $\mathrm{cr}(G_{13}^k) \geq 13$ for $k \geq 2$, and that, for every edge e of G_{13}^k, we get $\mathrm{cr}(G_{13}^k - e) \leq 12$. (We also remark that $\mathrm{cr}(G_{13}^1) \leq 12$, and for this reason we assume $k \geq 2$.)

▶ **Lemma 4.4.** $\mathrm{cr}(G_{13}^2) = \mathrm{cr}(G_{13}^3) = 13$.

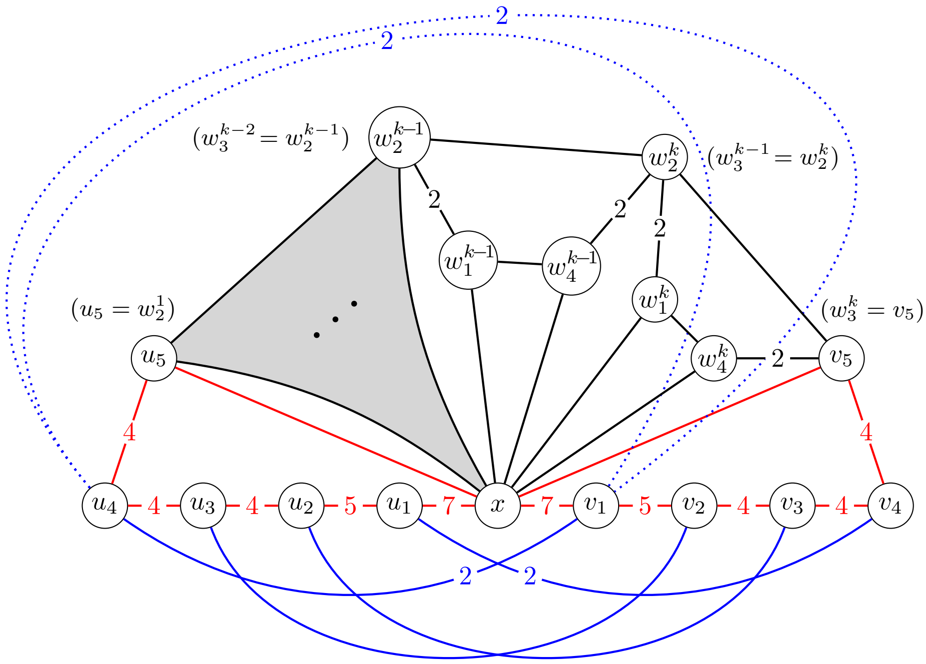

Figure 2 The graph G^k_{13} of Definition 4.2, drawn with 13 crossings. The thick edges of this graph have their thickness written as numeric labels, and all the unlabeled edges are of thickness 1. The bowtie part of this graph is drawn in red and blue (where blue edges are those between u_i and v_j vertices), and the wedges are drawn in black. Only the $(k-1)$-th and k-th wedges are detailed, while the remaining wedges $1, \dots, k-2$ analogously span the grey shaded area. Dotted lines show possible alternate routings of the edge v_1u_4 (which preserve the number of 13 crossings).

Proof. For every $k \geq 2$, Figure 2 shows a drawing of G^k_{13} with 13 crossings. For the lower bounds, we use the computer tool *Crossing Number Web Compute* [6] which uses an ILP formulation of the crossing number problem (based on Kuratowski subgraphs), and solves it via a branch-and-cut-and-price routine. Moreover, this computer tool generates machine-readable proofs[3] of the lower bound, which (roughly) consist of a branching tree in which every leaf holds an LP formulation of selected Kuratowski subgraphs certifying that, in this case, the crossing number must be greater than 12. ◀

▶ **Lemma 4.5.** *For every $k \geq 2$, $\mathrm{cr}(G^k_{13}) \geq 13$.*

Proof. We proceed by induction on k, where the base cases $k = 2, 3$ are proved in Lemma 4.4. Hence, we may assume that $k \geq 4$.

Consider a drawing of G^k_{13} with $c = \mathrm{cr}(G^k_{13})$ crossings. Let $1 \leq i \leq k-1$, and recall that $w^i_3 = w^{i+1}_2$. By Claim 4.1, we may assume that all thick edges are drawn together in a bunch. We now distinguish three cases based on the cyclic order of edges leaving the vertices w^i_3 (the orientation is not important):

[3] See the computation results at `http://crossings.uos.de/job/PZPmFDmDEKsgxLpftZmlXw` ($k = 2$) and `http://crossings.uos.de/job/EDsMIoyqrgonXEeDOplqdg` ($k = 3$). Vertex x is labeled 0. Cycle C_u uses vertices $0, 1, 2, 3, 4, 10$. Cycle C_v uses vertices $0, 5, 6, 7, 8$, and 14 for $k = 2$ (resp. 17 for $k = 3$). We remark that the proof file for $k = 3$ is particularly large, it spans 2150 cases with 717 Kuratowski constraints per case on average. For $k = 2$, there are just 1300 cases with about 180 constraints each.

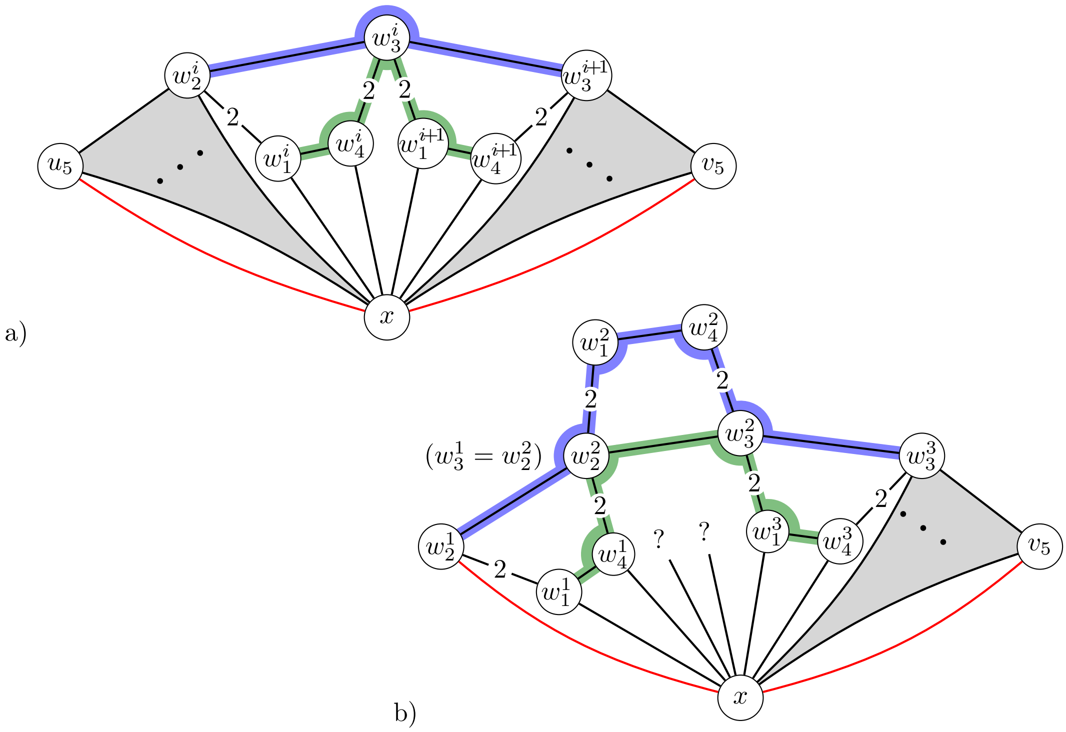

Figure 3 Two cases of the induction step in the proof of Lemma 4.5. (a) We "shrink" two wedges into one by drawing new edges $w_1^i w_4^{i+1}$ (green) and $w_2^i w_3^{i+1}$ (blue) along the depicted paths. (b) We likewise "shrink" three wedges into one by drawing the depicted new edges $w_1^1 w_4^3$ and $w_2^1 w_3^3$ (this picture does not specify how w_1^2 and w_4^2 connect to x since it is not important for us).

- There exists $i \in \{1, \ldots, k-1\}$, such that the edges incident to $w_3^i = w_2^{i+1}$, in a small neighbourhood of w_3^i, have the cyclic order $w_3^i w_4^i$, $w_3^i w_1^{i+1}$, $w_3^i w_3^{i+1}$, $w_3^i w_2^i$. See in Figure 3 a, where this cyclic order is anti-clockwise. In this case, we draw a new edge $w_1^i w_4^{i+1}$ along the path $(w_1^i, w_4^i, w_3^i, w_1^{i+1}, w_4^{i+1})$, and another new edge $w_2^i w_3^{i+1}$ along the path $(w_2^i, w_3^i, w_3^{i+1})$ (both new edges are of thickness 1). Then we delete the vertices w_4^i, w_3^i, w_1^{i+1} together with incident edges. The resulting drawing represents a graph which is clearly isomorphic to G_{13}^{k-1} – the wedges i and $i+1$ have been replaced with one wedge.
  Moreover, thanks to the assumption, we can avoid crossing between $w_1^i w_4^{i+1}$ and $w_2^i w_3^{i+1}$ in the considered neighbourhood of former w_3^i. Therefore, every crossing of the new drawing (including possible crossings of each of the new edges $w_1^i w_4^{i+1}$ and $w_2^i w_3^{i+1}$ among themselves or with other edges) existed already in the original drawing of G_{13}^k, and so $\mathrm{cr}(G_{13}^{k-1}) \leq c$. However, $\mathrm{cr}(G_{13}^{k-1}) \geq 13$ by the induction assumption, and so $c \geq 13$ holds true in this case.
- The same proof as above works if the cyclic order around w_3^i is $w_3^i w_4^i$, $w_3^i w_1^{i+1}$, $w_3^i w_2^i$, $w_3^i w_3^{i+1}$.
- For all $i \in \{1, \ldots, k-1\}$, in a small neighbourhood of w_3^i, the edges incident to $w_3^i = w_2^{i+1}$ have (up to orientation reversal) the cyclic order $w_3^i w_4^i$, $w_3^i w_3^{i+1}$, $w_3^i w_1^{i+1}$, $w_3^i w_2^i$. See Figure 3 b. We will use this assumption only for $i = 1, 2$ as follows.
  We draw a new edge $w_1^1 w_4^3$ along the path $(w_1^1, w_4^1, w_3^1, w_3^2, w_1^3, w_4^3)$, and another new edge $w_2^1 w_3^3$ along the path $(w_2^1, w_3^1, w_1^2, w_4^2, w_3^2, w_3^3)$ (both new edges are of thickness 1). Then we delete the vertices $w_4^1, w_3^1, w_1^2, w_4^2, w_3^2, w_1^3$ together with incident edges. The resulting

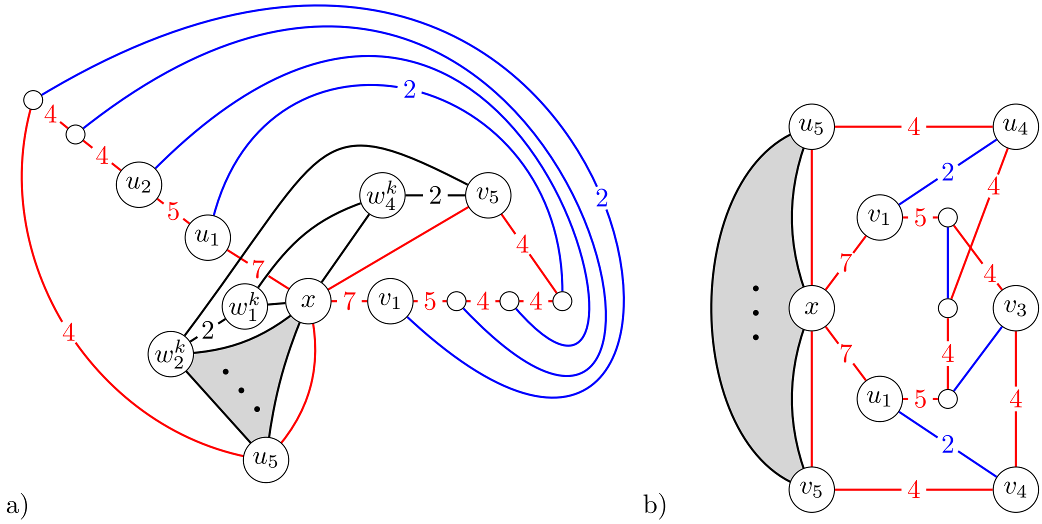

Figure 4 Two drawings of the graph G_{13}^k, having (a) 14 and (b) 16 crossings. These drawings are used to argue criticality of some of the bowtie (red) edges of G_{13}^k. The grey areas span the crossing-free wedges of G_{13}^k which are not detailed in the pictures, similarly as in Figure 2.

drawing represents a graph which is now isomorphic to G_{13}^{k-2} – the wedges 1, 2 and 3 have been replaced with one wedge.

As in the previous case, we can avoid crossing between $w_1^1 w_4^3$ and $w_2^1 w_3^3$ in the considered neighbourhoods of former w_3^1 and w_3^2. Therefore, analogously, $\mathrm{cr}(G_{13}^{k-2}) \leq c$. However, $k-2 \geq 2$ and so $\mathrm{cr}(G_{13}^{k-2}) \geq 13$ by the induction assumption, and hence $c \geq 13$ holds true also in this case.

By induction, for every integer $k \geq 2$, $\mathrm{cr}(G_{13}^k) = c \geq 13$ holds. ◀

▶ **Lemma 4.6.** *For every edge e of G_{13}^k, we have $\mathrm{cr}(G_{13}^k - e) \leq 12$.*

Proof. On a high level, our proof strategy can be described as follows. We provide a collection of drawings of G_{13}^k, such that each edge of G_{13}^k in some of the drawings, when deleted, exhibits a "drop" of the crossing number below 13. Note that, for thick edges, we are deleting only one edge of the multiple bunch.

In particular, the drawing of Figure 2 proves the claim for $e \in \{u_1v_4, u_2v_3, u_3v_2, u_4v_1\}$ (the blue edges of the bowtie subgraph); whenever any one of these edges is removed, we save at least one crossing from the optimal number 13. Likewise, Figure 2 proves the claim for $e \in \{xv_5, v_4v_5\}$ (see the dotted routings of the edge v_1u_4), and hence also for $e \in \{xu_5, u_4u_5\}$ by symmetry (the automorphism from Observation 4.3).

To proceed with the remaining edges e of the bowtie graph $B \subseteq G_{13}^k$ (the red edges), we resort to drawings that may have more than 13 crossings but still provide a drawing of $G_{13}^k - e$ with at most 12 crossings. Consider first the drawing obtained from the one of Figure 2 by "flipping" the right-hand part of the picture along the line through x and u_5. See Figure 4 a. In this drawing (with 14 crossings), the only crossings occur between the 7-thick edge xu_1 and the edges $w_1^k w_4^k$, $w_2^k v_5$. By a slight shift of the latter two edges, we obtain another drawing with only 14 crossing between the 5- and 2-thick edges u_1u_2, u_1v_4 and again the edges $w_1^k w_4^k$, $w_2^k v_5$. Removing an edge of the bunch xu_1 (respectively, one edge of u_1u_2) drops the crossing number down to 12. Second, there is a drawing with 16 crossing, which occur only between the 4-thick edges v_2v_3 and u_3u_4; see Figure 4 b. Deleting any one

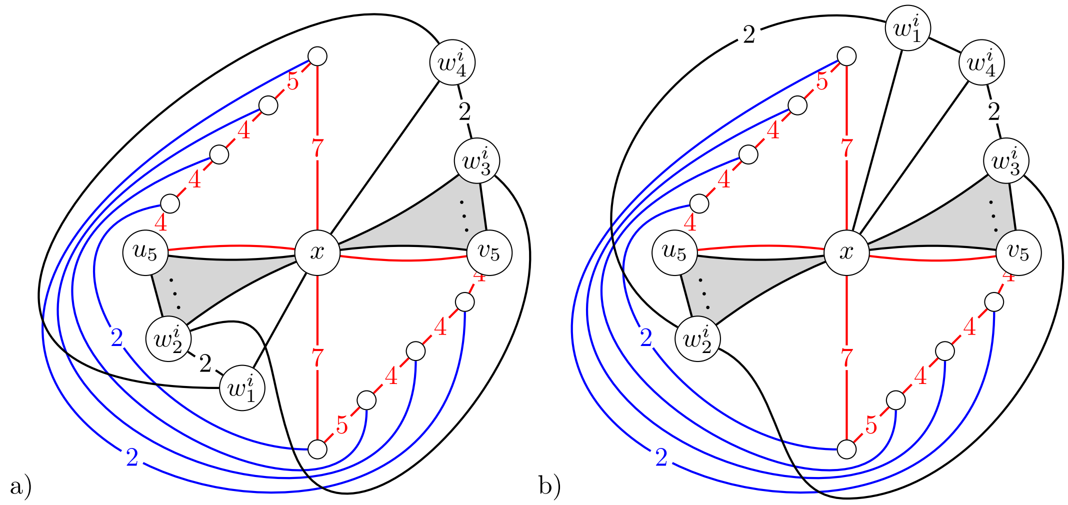

Figure 5 Two drawings of the graph G_{13}^k, having (a) 13 and (b) 18 crossings. These drawings are used to argue criticality of edges of the i-th wedge. The grey areas span the crossing-free wedges of G_{13}^k which are not detailed in the pictures, similarly as in Figure 4.

edge of v_2v_3, u_3u_4 again drops the crossing number down to at most 12. Consequently, our claim holds for $e \in \{xu_1, u_1u_2, v_2v_3, u_3u_4\}$ and, by symmetry, for $e \in \{xv_1, v_1v_2, u_2u_3, v_3v_4\}$.

We are left with the last, and perhaps most interesting, cases in which e is an edge in the i-th wedge D_i. Imagine we "disconnect" D_i by removing vertices w_1^i and w_4^i and the edge $w_2^iw_3^i$, and then twist the bowtie graph B together with the adjacent strips of wedges $D_1 \cup \ldots \cup D_{i-1}$ and $D_{i+1} \cup \ldots \cup D_k$, removing all crossings. If we introduce the vertices and edges of D_i back, we get a drawing in Figure 5 a with 13 crossings which are only between the edges xw_1^i, $w_2^iw_3^i$, $w_1^iw_4^i$ and the six blue bowtie edges. For every edge $e \in \{xw_1^i, w_2^iw_3^i, w_1^iw_4^i\}$ its removing from G_{13}^k thus decreases the crossing number to at most 12. Lastly, we may move the vertex w_1^i to obtain another drawing as in Figure 5 b. The latter drawing has 18 crossings between the edges $w_1^iw_2^i$, $w_2^iw_3^i$ and the six blue bowtie edges. Removing one edge of $w_1^iw_2^i$ thus drops the crossing number down to 12, again. With help of symmetry, the claim is thus proved also for every edge $e \in E(D_i)$, for each $i \in \{1, 2, \ldots, k\}$. ◀

For $d \geq 4$, Theorem 1.2 is then established with $G(d) := G_{13}^{\lfloor d/2 \rfloor}$.

5 Extended crossing-critical constructions

In the previous section, we have constructed an infinite family of 13-crossing-critical graphs with unbounded maximum degree. There are two further natural questions to be asked; (a) what about analogous c-crossing-critical families for $c > 13$, and (b) what about constructing c-crossing-critical graphs with more than one high-degree vertex?

Clearly, disjoint union of a graph G_{13}^k with $c-13$ disjoint copies of $K_{3,3}$ yields a (disconnected) c-crossing-critical graph with maximum degree greater than d, for every $c \geq 14$. Similarly, concerning (b), we can consider disjoint union of t copies of G_{13}^k to get a $13t$-crossing-critical graph with t vertices of arbitrarily high degree. Though, our aim is to preserve also the 3-connectivity property of the resulting graphs.

First, to motivate the coming construction, we recall that the zip product of Definition 2.3 requires a vertex of degree 3 in the considered graphs. However, the graph G_{13}^k of Definition 4.2 has no such vertex, and so we come with the following modification.

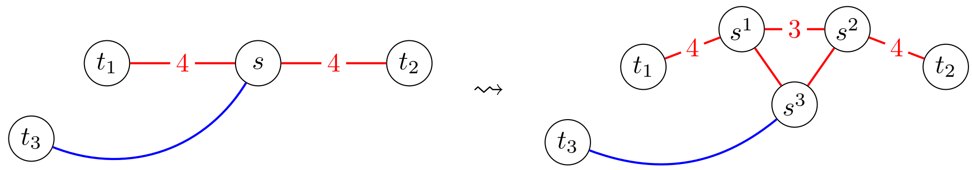

Figure 6 An illustration of the 4-to-3 expansion of the vertex s in a graph G. Clearly, for every optimal drawing of G respecting Claim 4.1, this "split" construction can be preformed in a small neighbourhood of s without introducing additional crossings.

▶ **Definition 5.1** (Critical family $\{H_{13}^k\}$)**.** *Let G be a graph and $s \in V(G)$ be a vertex incident exactly with two 4-thick edges st_1, st_2, and one ordinary edge st_3. We call a* 4-to-3 expansion *of s the following operation: in G, remove the vertex s with its incident edges, and add three new vertices s^1, s^2, s^3, two 4-thick edges s^1t_1, s^2t_2, one 3-thick edge s^1s^2, and three ordinary edges s^3t_1, s^3t_2, s^3t_3. See the sketch in Figure 6.*

Now, for $k \geq 2$, the graph H_{13}^k is constructed from the graph G_{13}^k of Definition 4.2 by a 4-to-3 expansion of the vertex v_3 and of the vertex u_3 (cf. Figure 2).

The proof of the following technical Lemma is present in the full paper.

▶ **Lemma 5.2.** *Let G be a 13-crossing-critical graph, and let $s \in V(G)$ be a vertex incident exactly with two 4-thick edges and one ordinary edge in G. If G_1 is a graph obtained by a 4-to-3 expansion of s, then G_1 is also 13-crossing-critical.*

▶ Remark 5.3. The number 13 of crossings in Lemma 5.2 is rather special and the claim cannot be easily generalized to other numbers of crossings. For instance, one can construct a graph of crossing number 14, such that one of its 4-to-3 expansions has crossing number only 13.

▶ **Corollary 5.4.** *For every $k \geq 2$, the graph H_{13}^k is 13-crossing-critical.*

Proof. We start with Theorem 1.2, and apply Lemma 5.2 to the vertices v_3 and u_3 of G_{13}^k. ◀

Proof of Corollary 1.3. For $c = 13$, we set $G(13, d) := H_{13}^{\lfloor d/2 \rfloor}$ from Corollary 5.4. Note that $G(13, d)$ contains two vertices of degree 3. For $c > 13$, we proceed by induction, assuming that we have already constructed the graph $G(c-1, d)$ and it contains a vertex of degree 3. Theorem 2.4 establishes that $G(c, d)$, as a zip product of $G(c-1, d)$ with $K_{3,3}$, is c-crossing-critical (and it still contains the same vertex x of high degree as H_{13}^k does). Furthermore, $G(c, d)$ contains a vertex of degree 3 coming from the $K_{3,3}$ part. ◀

Proof of Corollary 1.4. The proof proceeds in a manner similar to the proof of Corollary 1.3. This time we inductively zip together (cf. Theorem 2.4) $i \leq c/13$ copies of the graph $H_{13}^{\lfloor d/2 \rfloor}$ and $c - 13i$ copies of $K_{3,3}$, which results in a c-crossing-critical graph with i vertices (one per each copy) of degree greater than d. Note that we never "run out" of degree-3 vertices in the construction since each copy of $H_{13}^{\lfloor d/2 \rfloor}$ has two such vertices. ◀

6 Concluding remarks and open problems

While our contribution closes the questions related to the validity of the bounded maximum degree conjecture, the following problems remain open:

▶ **Problem 6.1.** *For each $c \leq 12$, determine the least integer $D(c)$ bounding maximum degree of c-crossing-critical graphs.*

▶ **Problem 6.2.** *Develop a theory of wedges that parallels the theory of tiles for constructively establishing c-crossing-criticality of graphs with large maximum degrees.*

Furthermore, the requirement in Corollary 1.4 that the number i of large degree vertices is at most $c/13$ could possibly be weakened. The following is a natural question. For an integer $i \geq 0$, let c_i denote the largest possible positive integer c for which there exists an integer D such that every 3-connected c-crossing-critical graph has at most i vertices of degree larger than D. From the previous, we easily see that $12 \leq c_i \leq 13i + 12$, and we have exactly determined the value $c_0 = 12$ in Theorems 1.1 and 1.2.

▶ **Problem 6.3.** *Determine c_i as a function of i.*

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