Upward Book Embeddings of st-Graphs

Carla Binucci
Università degli Studi di Perugia, Perugia, Italy
carla.binucci@unipg.it

Giordano Da Lozzo
Roma Tre University, Rome, Italy
giordano.dalozzo@uniroma3.it

Emilio Di Giacomo
Università degli Studi di Perugia, Perugia, Italy
emilio.digiacomo@unipg.it

Walter Didimo
Università degli Studi di Perugia, Perugia, Italy
walter.didimo@unipg.it

Tamara Mchedlidze
Karlsruhe Institute of Technology, Karlsruhe, Germany
mched@iti.uka.de

Maurizio Patrignani
Roma Tre University, Rome, Italy
maurizio.patrignani@uniroma3.it

Abstract

We study *k-page upward book embeddings* (kUBEs) of *st*-graphs, that is, book embeddings of single-source single-sink directed acyclic graphs on k pages with the additional requirement that the vertices of the graph appear in a topological ordering along the spine of the book. We show that testing whether a graph admits a kUBE is NP-complete for $k \geq 3$. A hardness result for this problem was previously known only for $k = 6$ [Heath and Pemmaraju, 1999]. Motivated by this negative result, we focus our attention on $k = 2$. On the algorithmic side, we present polynomial-time algorithms for testing the existence of 2UBEs of planar *st*-graphs with branchwidth β and of plane *st*-graphs whose faces have a special structure. These algorithms run in $O(f(\beta) \cdot n + n^3)$ time and $O(n)$ time, respectively, where f is a singly-exponential function on β. Moreover, on the combinatorial side, we present two notable families of plane *st*-graphs that always admit an embedding-preserving 2UBE.

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Related Version Details for the omitted and sketched proofs can be found in the full version of the paper [23], which is available at `https://arxiv.org/abs/1903.07966`.

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1 Introduction

A *k-page book embedding* $\langle \pi, \sigma \rangle$ of an undirected graph $G = (V, E)$ consists of a vertex ordering $\pi : V \leftrightarrow \{1, 2, \dots, |V|\}$ and of an assignment $\sigma : E \rightarrow \{1, \dots, k\}$ of the edges of G to one of k sets, called *pages*, so that for any two edges (a, b) and (c, d) in the same page, with $\pi(a) < \pi(b)$ and $\pi(c) < \pi(d)$, we have neither $\pi(a) < \pi(c) < \pi(b) < \pi(d)$ nor $\pi(c) < \pi(a) < \pi(d) < \pi(b)$. From a geometric perspective, a k-page book embedding can be associated with a *canonical drawing* $\Gamma(\pi, \sigma)$ of G where the k pages correspond to k half-planes sharing a vertical line, called the *spine*. Each vertex v is a point on the spine with y-coordinate $\pi(v)$; each edge e is a circular arc on the $\sigma(e)$-th page, and the edges in the same page do not cross.

For k-page book embeddings of directed graphs (digraphs), a typical requirement is that all the edges are oriented in the upward direction. This implies that G is acyclic and that all the vertices appear along the spine in a topological ordering. This type of book embedding for digraphs is called an *upward k-page book embedding* of G (for short, $k\,UBE$). Note that, when $k = 2$ and the two pages are coplanar, drawing $\Gamma(\pi, \sigma)$ is an *upward planar drawing* of G, i.e., a planar drawing where all the edges monotonically increase in the upward direction. The study of upward planar drawings is a most prolific topic in the theory of graph visualization [6, 7, 20, 21, 25, 26, 28, 31, 33, 35, 51, 69].

The *page number* of a (di)graph G (also called *book thickness*) is the minimum number k such that G admits a (upward) k-page book embedding. Computing the page number of directed and undirected graphs is a widely studied problem, which finds applications in a variety of domains, including VLSI design, fault-tolerant processing, parallel process scheduling, sorting networks, parallel matrix computations [29, 53, 68], computational origami [2], and graph drawing [22, 37, 52, 76]. See [42] for additional references.

Book embeddings of undirected graphs. Seminal results on book embeddings of undirected graphs are described in the paper of Bernhart and Kainen [19]. They prove that the graphs with page number one are exactly the outerplanar graphs, while graphs with page number two are the sub-Hamiltonian graphs. This second result implies that it is NP-complete to decide whether a graph admits a 2-page book embedding [75]. Yannakakis [77] proved that every planar graph has a 4-page book embedding, while the fascinating question whether the page number of planar graphs can be reduced to three is still open. The aforementioned works have inspired several papers about the page number of specific families of undirected graphs (e.g., [16, 18, 29, 46]) and about the relationship between the page number and other graph parameters (e.g., [43, 50, 60, 61]). Different authors studied constrained versions of k-page book embeddings where either the vertex ordering π is (partially) fixed [8, 30, 62, 73, 74] or the page assignment σ for the edges is given [10, 9, 11, 57]. Relaxed versions of book embeddings where edge crossings are allowed (called *k-page drawings*) or where edges can cross the spine (called *topological book embeddings*) have also been considered (e.g., [1, 14, 24, 27, 36, 44, 45]). Finally, 2-page (topological) book embeddings find applications to point-set embedding and universal point set (e.g., [12, 13, 39, 40, 47, 59]).

Book embeddings of directed graphs. As for undirected graphs, there are many papers devoted to the study of upper and lower bounds on the page number of directed graphs. Heath et al. [56] show that directed trees and unicyclic digraphs have page number one and two, respectively. Alzohairi and Rival [4], and later Di Giacomo et al. [37] with an improved linear-time construction, show that series-parallel digraphs have page number two. Mchedlidze and Symvonis [63] generalize this result and prove that N-free upward planar digraphs, which contain series-parallel digraphs, also have page number two (a digraph is upward planar if it admits an upward planar drawing). Frati et al. [49] give several conditions under which upward planar triangulations have bounded page number. Overall, the question asked by Nowakowski and Parker [66] almost 30 years ago, of whether the page number of upward planar digraphs is bounded, remains open. Several works study the page number of acyclic digraphs in terms of posets, i.e., the page number of their Hasse diagram (e.g., [3, 66]).

About the lower bounds, Nowakowski and Parker [66] give an example of a *planar st-graph* that requires three pages for an upward book embedding (see Fig. 9a). A planar st-graph is an upward planar digraph with a single source s and a single sink t. Hung [58] shows an upward planar digraph with page number four, while Heath and Pemmaraju [54] describe an acyclic planar digraph (which is not upward planar) requiring $\lfloor n/2 \rfloor$ pages. Syslo [72] provides a lower bound on the page number of a poset in terms of its bump number.

Besides the study of upper and lower bounds on the page number of digraphs, several papers concentrate on the design of testing algorithms for the existence of kUBEs. The problem is NP-complete for $k = 6$ [55]. For $k = 2$, Mchedlidze and Symvonis [65] give linear-time testing algorithms for outerplanar and planar triangulated st-graphs. An $O(w^2 n^w)$-time testing algorithm for 2UBEs of planar st-graphs whose width is w is given in [63], where the *width* is the minimum number of directed paths that cover all the vertices. Heath and Pemmaraju [55] describe a linear-time algorithm to recognize digraphs that admit 1UBEs.

Finally, as for the undirected case, constrained or relaxed variants of kUBEs for digraphs are studied [2, 38, 52], as well as applications to the point-set embedding problem [37, 52].

Contribution. Our paper is motivated by the gap present in the literature about the computation of upward book embeddings of digraphs: Polynomial-time algorithms are known only for one page or for two pages and subclasses of planar digraphs, while NP-completeness is known only for exactly 6 pages. We shrink this gap and address the research direction proposed by Heath and Pemmaraju [55]: Identification of graph classes for which the existence of kUBEs can be solved efficiently. Our results are as follows:

- We prove that testing whether a digraph G admits a kUBE is NP-complete for every $k \geq 3$, even if G is an st-graph (Section 3). An analogous result was previously known only for the constrained version in which the page assignment is given [2].
- We describe another meaningful subclass of upward planar digraphs that admit a 2UBE (Section 4). This class is structurally different from the N-free upward planar digraphs, the largest class of upward 2-page book embeddable digraphs previously known.
- We give algorithms to test the existence of a 2UBE for notable families of planar st-graphs. First, we give a linear-time algorithm for plane st-graphs whose faces have a special structure (Section 5). Then, we describe an $O(f(\beta) \cdot n + n^3)$-time algorithm for n-vertex planar st-graphs of branchwidth β, where f is a singly-exponential function (Section 6). The algorithm works for both variable and fixed embedding. This result also implies a sub-exponential-time algorithm for general planar st-graphs.

2 Preliminaries

We assume familiarity with basic definitions on graph connectivity and planarity (see [15, 23]). We only consider (di)graphs without loops and multiple edges, and we denote by $V(G)$ and $E(G)$ the sets of vertices and edges of a (di)graph G.

A digraph G is a *planar st-graph* if and only if: (i) it is acyclic; (ii) it has a single source s and a single sink t; and (iii) it admits a planar embedding $\mathcal{E}$ with s and t on the outer face. A graph G together with $\mathcal{E}$ is a *planar embedded st-graph*, also called a *plane st-graph*.

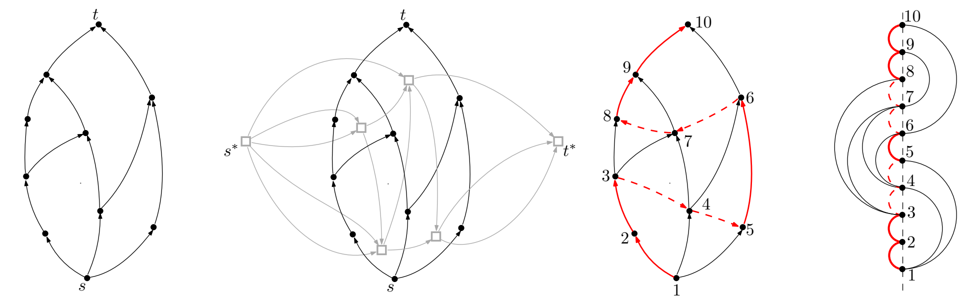
Let G be a plane st-graph and let $e = (u, v)$ be an edge of G. The *left face* (resp. *right face*) of e is the face to the left (resp. right) of e while moving from u to v. The boundary of every face f of G consists of two directed paths p_l and p_r from a common source s_f to a common sink t_f. The paths p_l and p_r are the *left path* and the *right path* of f, respectively. The vertices s_f and t_f are the *source* and the *sink* of f, respectively. If f is the outer face, p_l (resp. p_r) consists of the edges for which f is the left face (resp. right face); in this case p_l and p_r are also called the *left boundary* and the *right boundary* of G, respectively. If f is an internal face, p_l (resp. p_r) consists of the edges for which f is the right face (resp. left face).

The *dual graph* G^* of a plane st-graph G is a plane st-graph (possibly with multiple edges) such that: (i) G^* has a vertex associated with each internal face of G and two vertices s^* and t^* associated with the outer face of G, that are the source and the sink of G^*, respectively; (ii) for each internal edge e of G, G^* has a dual edge from the left to the right face of e; (iii) for each edge e in the left boundary of G, there is an edge from s^* to the right face of e; (v) for each edge e in the right boundary of G, there is an edge from the left face of e to t^*.

Consider a planar st-graph G and let $\overline{G}$ be a planar st-graph obtained by augmenting G with directed edges in such a way that it contains a directed Hamiltonian st-path $P_{\overline{G}}$. The graph $\overline{G}$ is an *HP-completion* of G. Consider now a plane st-graph G and let $\mathcal{E}$ be a planar embedding of G. Let $\overline{G}$ be an embedded HP-completion of G whose embedding $\overline{\mathcal{E}}$ is such that its restriction to G is $\mathcal{E}$. We say that $\overline{G}$ is an *embedding-preserving HP-completion* of G.

Bernhart and Kainen [19] prove that an undirected planar graph admits a 2-page book embedding if and only if it is *sub-Hamiltonian*, i.e., it can be made Hamiltonian by adding edges while preserving its planarity. Theorem 1 is an immediate consequence of the result in [19] for planar digraphs (see also Fig. 1); when we say that a 2UBE $\langle \pi, \sigma \rangle$ is *embedding-preserving* we mean that the drawing $\Gamma(\pi, \sigma)$ preserves the planar embedding of G.

▶ **Theorem 1.** *A planar (plane) st-graph G admits a (embedding-preserving) 2UBE $\langle \pi, \sigma \rangle$ if and only if G admits a (embedding-preserving) HP-completion $\overline{G}$. Also, the order π coincides with the order of the vertices along $P_{\overline{G}}$.*



(a) **(b)** **(c)** **(d)**

Figure 1 (a) A plane st-graph G. (b) The dual of G is shown in gray. (c) An embedding-preserving HP-completion of G. (d) An embedding-preserving 2UBE Γ of G corresponding to (c).

3 NP-Completeness for kUBE ($k \geq 3$)

We prove that the kUBE Testing problem of deciding whether a digraph G admits an upward k-page book embedding is NP-complete for each fixed $k \geq 3$. The proof uses a reduction from the Betweenness problem [67].

Betweenness
Instance: A finite set S of elements and a set $R \subseteq S \times S \times S$ of triplets.
Question: Does there exist an ordering $\tau : S \to \mathbb{N}$ of the elements of S such that for any element $(a, b, c) \in R$ either $\tau(a) < \tau(b) < \tau(c)$ or $\tau(c) < \tau(b) < \tau(a)$?

We incrementally define a set of families of digraphs and prove some properties of these digraphs. Then, we use the digraphs of these families to reduce a generic instance of Betweenness to an instance of 3UBE Testing, thus proving the hardness result for $k = 3$. We then explain how the proof can be easily adapted to work for $k > 3$.

For a digraph G, we denote by $u \rightsquigarrow v$ a directed path from a vertex u to a vertex v in G. Let $\gamma = \langle \pi, \sigma \rangle$ be a 3UBE of G. Two edges (u, v) and (w, z) of G *conflict* if either $\pi(u) < \pi(w) < \pi(v) < \pi(z)$ or $\pi(w) < \pi(u) < \pi(z) < \pi(v)$. Two conflicting edges cannot be assigned to the same page. The next property will be used in the following; it is immediate from the definition of book embedding and from the pigeonhole principle.

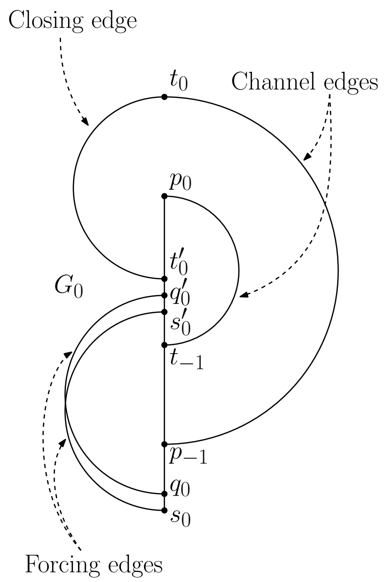
▶ **Property 1.** *In a 3UBE there cannot exist* 4 *edges that mutually conflict.*

Shell digraphs. The first family that we define are the *shell digraphs*, recursively defined as follows. Digraph G_0, depicted in Fig. 2a, consists of a directed path P with 8 vertices denoted as s_0, q_0, p_{-1}, t_{-1}, s'_0, q'_0, t'_0, and p_0 in the order they appear along P. Besides the edges of P, the following directed edges exists in G_0: (s_0, s'_0), (q_0, q'_0), (t_{-1}, p_0). Finally, there is a vertex t_0 connected to P by means of the two directed edges (p_{-1}, t_0) and (t'_0, t_0). Graph G_h is obtained from G_{h-1} with additional vertices and edges as shown in Fig. 2b. A new directed path of two vertices s_h and q_h is connected to G_{h-1} with the edge (q_h, s_{h-1}); a second path of four vertices s'_h, q'_h, t'_h, and p_h is connected to G_h with the edge (t_{h-1}, s'_h). The following edges exist between these new vertices: (s_h, s'_h), (q_h, q'_h), (t_{h-1}, p_h). Finally, there is a vertex t_h connected to the other vertices by means of the two directed edges (p_{h-1}, t_h) and (t'_h, t_h). For any $h \geq 0$, the edges (s_h, s'_h) and (q_h, q'_h) are called the *forcing edges* of G_h; the edges (p_{h-1}, t_h) and (t_{h-1}, p_h) are the *channel edges* of G_h; the edge (t'_h, t_h) is the *closing edge* of G_h. The vertices and edges of $G_h \setminus G_{h-1}$ are the *exclusive vertices and edges* of G_h. The following lemma establishes some basic properties of the shell digraphs.

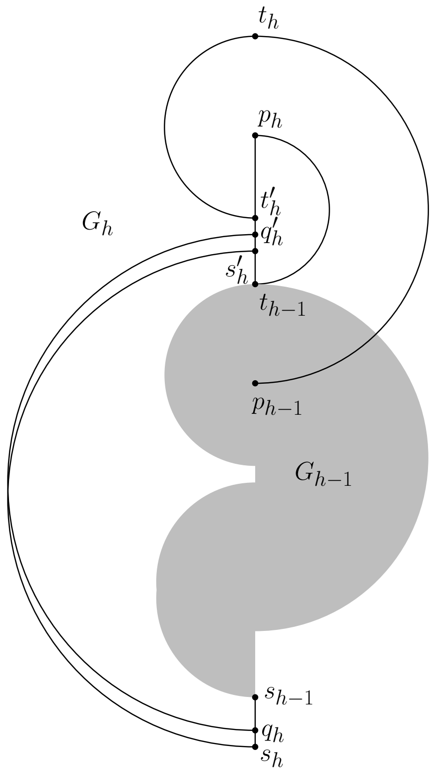
▶ **Lemma 2.** *Every shell digraph G_h for $h \geq 0$ admits a 3UBE. In any 3UBE $\gamma = \langle \pi, \sigma \rangle$ of G_h the following conditions hold for every $i = 0, 1, \ldots, h$:*
S1 *all vertices of G_i are between s_i and t_i in π;*
S2 *the channel edges of G_i are in the same page;*
S3 *if $i > 0$, the channel edges of G_i and those of G_{i-1} are in different pages.*

Note that Condition S1 uniquely defines the vertex ordering of G_h in every 3UBE. Namely, the path $s_h \rightsquigarrow p_0$ precedes each path $t_{i-1} \rightsquigarrow p_i$ (for $i = 1, \ldots, h$), and each path $t_{i-1} \rightsquigarrow p_i$ precedes the path $t_i \rightsquigarrow p_{i+1}$ (for $i = 1, \ldots, h-1$) (see Fig. 3a for an example with $h = 2$).

Filled shell digraphs. Let G_h be a shell digraph. A *filled shell digraph* $H_{h,s}$ (for $h \geq 0$ and $s \geq 1$) is obtained from G_h by adding $h+2$ groups $\alpha_{-1}, \alpha_0, \ldots, \alpha_h$ of s vertices each; see Fig. 3b for an illustration. The vertices of group α_i are denoted as $v_{i,1}, v_{i,2}, \ldots v_{i,s}$. These vertices will be used to map the elements of the set S of an instance of Betweenness to an instance of 3UBE Testing. For each vertex $v_{-1,j}$ of the set α_{-1} there is a directed edge $(p_{-1}, v_{-1,j})$ and a directed edge $(v_{-1,j}, t_{-1})$. For each vertex $v_{i,j}$ of the set α_i with $i \geq 0$ and i even, there is a directed edge $(p_i, v_{i,j})$. Finally, for each vertex $v_{i,j}$ of the set α_i with $i \geq 0$, there is a directed edge $(v_{i-1,j}, v_{i,j})$.



(a) G_0



(b) G_k

Figure 2 Definition of shell digraphs. Edges are oriented from bottom to top.

▶ **Lemma 3.** *Every filled shell digraph $H_{h,s}$ for $s > 0$ and even $h \geq 0$ admits a 3UBE. In any 3UBE $\gamma = \langle \pi, \sigma \rangle$ of $H_{h,s}$ the following conditions hold for every $i = -1, 0, 1, \ldots, h$:*

F1 *the vertices of the group α_i are between p_i and t_i in π;*

F2 *if $i \geq 0$ the vertices of α_i are in reverse order with respect to those of α_{i-1} in π;*

F3 *if $i \geq 0$ each edge $(v_{i-1,j}, v_{i,j})$ is in the page of the channel edges of G_i (for $j = 1, \ldots, s$).*

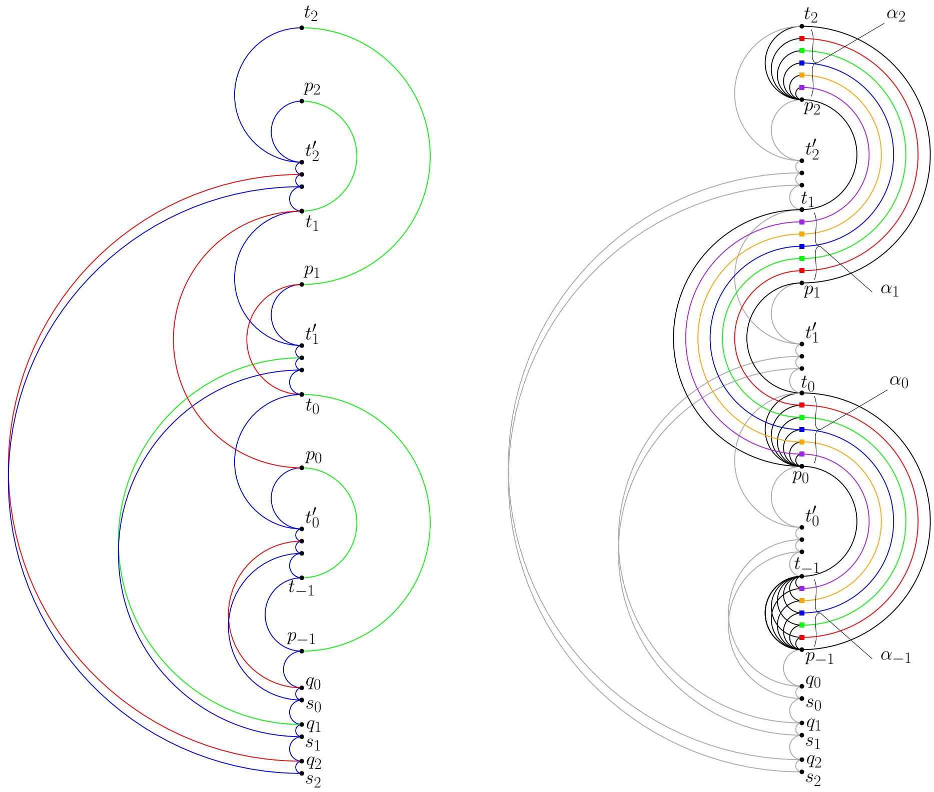
Observe that, by Condition F2, all groups α_i with even index have the same ordering in π and all groups with odd index have the opposite order. As mentioned above the vertices in the groups α_i will correspond to the elements of the set S of an instance of Betweenness in the reduced instance of 3UBE Testing. If the reduced instance admits a 3UBE, the order of the groups in π will give the desired order for the instance of Betweenness.

Λ-filled shell digraphs and hardness proof. Starting from a filled shell digraph $H_{h,s}$, a Λ-*filled shell digraph* $\widehat{H}_{h,s}$ is obtained by replacing some edges with a gadget that has two possible configurations in any 3UBE of $\widehat{H}_{h,s}$. More precisely, we replace each edge (t'_i, p_i) of $H_{h,s}$ for i odd with the gadget shown in Fig. 4a. The gadget replacing (t'_i, p_i) will be denoted as Λ_i. Notice that, this replacement preserves Conditions F1–F3 of Lemma 3.

▶ **Lemma 4.** *Every Λ-filled shell digraph $\widehat{H}_{h,s}$ for $s > 0$ and even $h \geq 0$ admits a 3UBE. In any 3UBE $\gamma = \langle \pi, \sigma \rangle$ of $\widehat{H}_{h,s}$ the following conditions hold for every $i = 1, 3, \ldots, h-1$:*

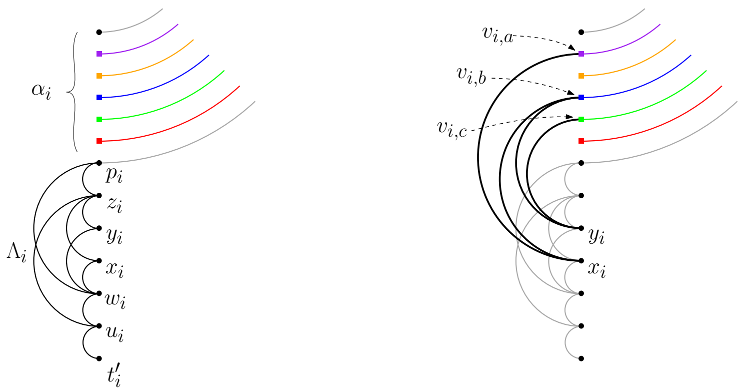
G1 *the vertices of the gadget Λ_i are between t'_i and p_i in π;*

G2 *the vertices x_i and y_i are between w_i and z_i in π and there exists a 3UBE $\gamma' = \langle \pi', \sigma' \rangle$ of $\widehat{H}_{h,s}$ where the order of x_i and y_i is exchanged in π'.*



(a) **(b)**

Figure 3 (a) A 3UBE of the shell digraph G_2; the colors of the edges represent the pages. (b) Definition of $H_{h,s}$ for $h = 2$ and $s = 5$. In both figures edges are oriented from bottom to top.



(a) **(b)**

Figure 4 (a) A gadget Λ_i (black edges). (b) The triplet edges of G_i (bold edges).

▶ **Theorem 5.** 3UBE TESTING *is NP-complete even for st-graphs.*

Proof sketch. 3UBE TESTING is clearly in NP. To prove the hardness we describe a reduction from BETWEENNESS. From an instance $I = \langle S, R \rangle$ of BETWEENNESS we construct an instance G_I of 3UBE TESTING that is an *st*-graph; we start from the Λ-filled shell digraph $\widehat{H}_{h,s}$ with $h = 2|R|$ and $s = |S|$. Let $v_1, v_2, \ldots, v_s$ be the elements of S. They are represented in $\widehat{H}_{h,s}$

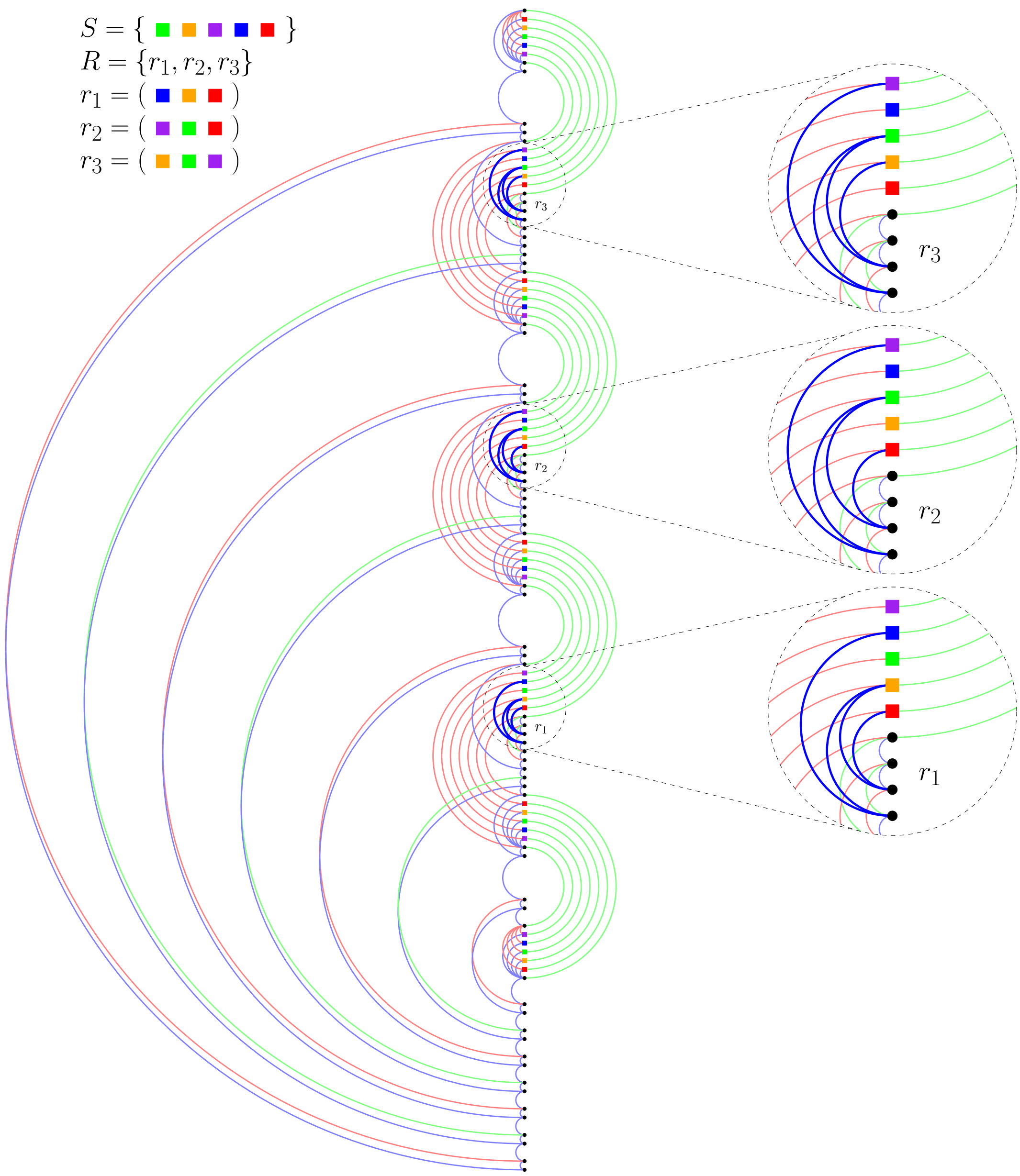


Figure 5 A 3UBE of the st-graph G_I reduced from a positive instance $I = \langle S, R \rangle$ of Betweenness; the edge colors represent the corresponding pages. Edges are oriented from bottom to top.

by the vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,s}$ of the groups α_i, for $i = -1, 0, 1, \ldots, h$. In the reduction each group α_i with odd index is used to encode one triplet and, in a 3UBE of G_I, the order of the vertices in these groups (which is the same by Condition F2) corresponds to the desired order of the elements of S for the instance I. Number the triplets of R from 1 to $|R|$ and let (v_a, v_b, v_c) be the j-th triplet. We use the group α_i and the gadget Λ_i with $i = 2j - 1$ to encode the triplet (v_a, v_b, v_c). More precisely, we add to $\widehat{H}_{h,s}$ the edges $(x_i, v_{i,a})$, $(x_i, v_{i,b})$, $(y_i, v_{i,b})$, and $(y_i, v_{i,c})$ (see Fig. 4b). These edges are called *triplet edges* and are denoted as T_i. In any 3UBE of G_I the triplet edges are forced to be in the same page and this is possible if and only if the constraints defined by the triplets in R are respected. The digraph obtained

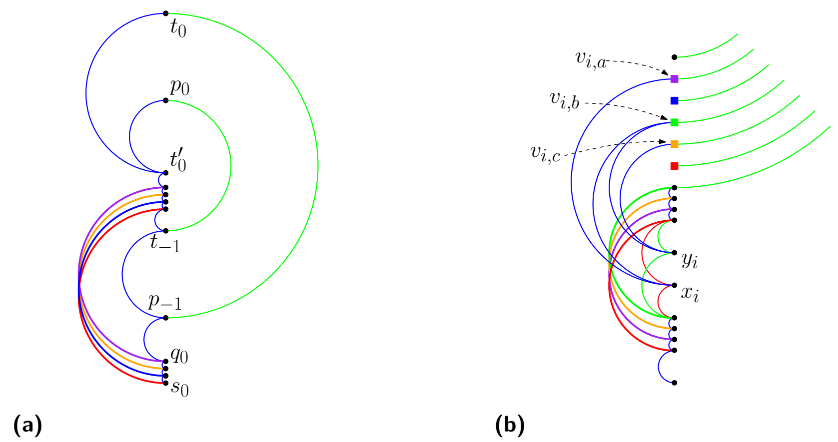


Figure 6 Reduction for kUBE Testing (example with $k = 5$). (a) Replacement of the forcing edges. (b) Replacement of the gadget Λ_i. In both figures colors represent the pages.

by the addition of the triplet edges is not an *st*-graph because the vertices of the last group α_h are all sinks. The desired instance G_I of 3UBE Testing is the *st*-graph obtained by adding the edges $(v_{h,j}, t_h)$ (for $j = 1, 2, \ldots, s$). Fig. 5 shows a 3UBE of the *st*-graph G_I reduced from a positive instance I of Betweenness. ◀

For $k > 3$, the reduction from an instance I of Betweenness to an instance G_I of kUBE Testing is similar. In the shell digraph every pair of forcing edges is replaced by a bundle of $k-1$ edges that mutually conflict (see Fig. 6a). The edges in each such bundle require $k-1$ pages and force all edges that conflict with them to use the k-th page. Analogously, the two edges (u_i, z_i) and (w_i, p_i) of the gadget Λ_i are replaced by a bundle of $k-1$ edges that mutually conflict (see Fig. 6b); this forces the triplet edges to be in the k-th page.

▶ **Corollary 6.** *kUBE Testing is NP-complete for every $k \geq 3$, even for st-graphs.*

4 Existential Results for 2UBE

Let f be an internal face of a plane *st*-graph, and let p_l and p_r be the left and the right path of f; f is a *generalized triangle* if either p_l or p_r is a single edge (i.e., a transitive edge), and it is a *rhombus* if each of p_l and p_r consists of exactly two edges (see Figs. 7a and 7b).

Let G be a plane *st*-graph. A *forbidden configuration* of G consists of a transitive edge $e = (u, v)$ shared by two internal faces f and g such that $s_f = s_g = u$ and $t_f = t_g = v$ (i.e., two generalized triangles sharing the transitive edge); see Fig. 7c. The absence of forbidden configurations is a necessary condition for the existence of an embedding-preserving 2UBE. If G is triangulated, the absence of forbidden configurations is also a sufficient condition [65].

▶ **Theorem 7.** *Any plane st-graph such that the left and the right path of every internal face contain at least two and three edges, respectively, admits an embedding-preserving 2UBE.*

Proof sketch. We prove how to construct an embedding-preserving HP-completion. The idea is to construct $\overline{G}$ by adding a face of G per time from left to right, according to a topological ordering of the dual graph of G. When a face f is added, its right path is attached to the right boundary of the current digraph. We maintain the invariant that at least one edge e in the left path of f belongs to the Hamiltonian path of the current digraph. The

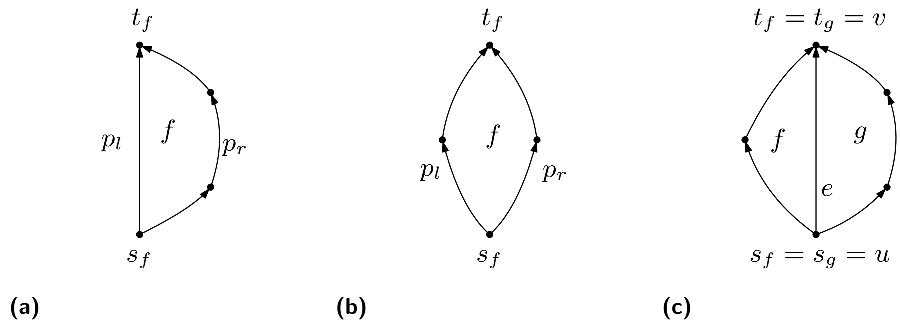


Figure 7 (a) A generalized triangle G. (b) A rhombus (c) A forbidden configuration.

Hamiltonian path is extended by replacing e with a path that traverses the vertices of the right path of f. To this aim, dummy edges are suitably inserted inside f. When all faces are added, the resulting graph is an HP-completion $\overline{G}$ of G. The idea is illustrated in Fig. 8. ◀

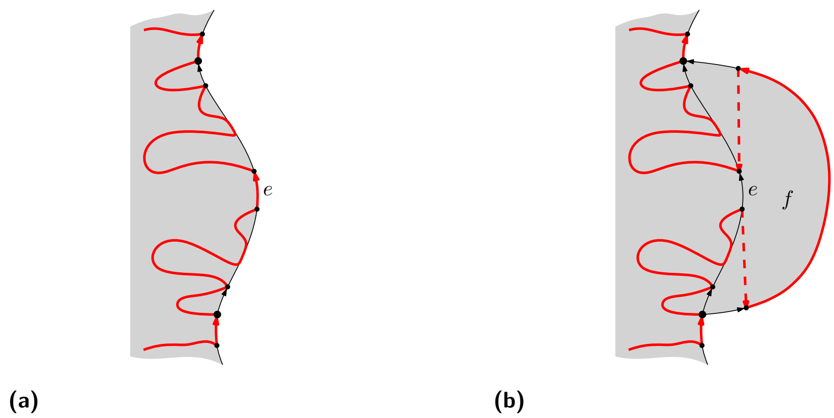


Figure 8 Idea of the construction in the proof of Theorem 7. Dummy edges are dashed.

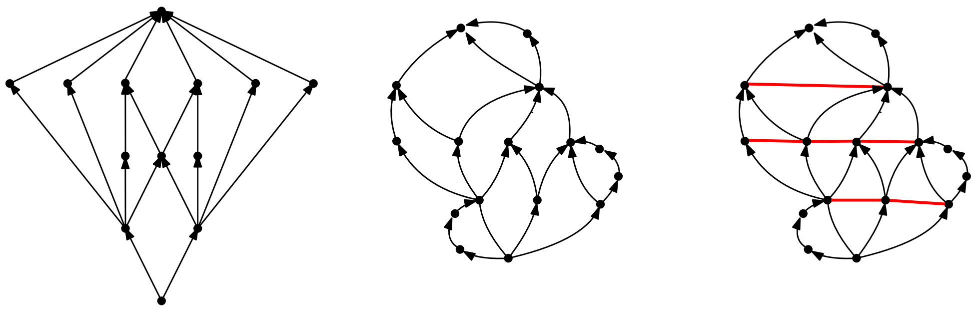
The next theorem is proved with a construction similar to that of Theorem 7.

▶ **Theorem 8.** *Let G be a plane st-graph such that every internal face of G is a rhombus. Then G admits an embedding-preserving 2UBE.*

5 Testing 2UBE for Plane Graphs with Special Faces

By Theorem 7, if all internal faces of a plane st-graph G are such that their left and right path contain at least two and at least three edges, respectively, G admits an embedding-preserving 2UBE. If these conditions do not hold, an embedding-preserving 2UBE may not exist (see Fig. 9a). We now describe an efficient testing algorithm for a plane st-graph $G = (V, E)$ whose internal faces are generalized triangles or rhombi (see Fig. 9b). We construct a *mixed* graph $G_M = (V, E \cup E_U)$, where E_U is a set of undirected edges and $(u, v) \in E_U$ if u and v are the two vertices of a rhombus face f distinct from s_f and t_f (red edges in Fig. 9c). For a rhombus face f, the graph obtained from G by adding the directed edge (u, v) inside f is still a plane st-graph (see, e.g. [17, 32]). Since there is only one edge of E_U inside each rhombus face of G, this implies the following property.

▶ **Property 2.** *Every orientation of the edges in E_U transforms G_M into an acyclic digraph.*



(a) **(b)** **(c)**

Figure 9 (a) A plane *st*-graph that does not admit a 2UBE [66]. (b) A plane *st*-graph G whose faces are generalized triangles or rhombi. (b) The mixed graph $G_M = (V, E, E_U)$.

▶ **Theorem 9.** *Let G be a plane st-graph such that every internal face of G is either a generalized triangle or a rhombus. There is an $O(n)$-time algorithm that decides whether G admits an embedding-preserving 2UBE, and which computes it in the positive case.*

Proof. The edges of E_U are the only edges that can be used to construct an embedding-preserving HP-completion of G. This, together with Theorem 1, implies that G admits a 2UBE if and only if the undirected edges of G_M can be oriented so that the resulting digraph $\overrightarrow{G_M}$ has a directed Hamiltonian path from s to t. By Property 2, any orientation of the undirected edges of G_M gives rise to an acyclic digraph. On the other hand an acyclic digraph is Hamiltonian if and only if it is unilateral (see, e.g. [5, Theorem 4]); we recall that a digraph is *unilateral* if each pair of vertices is connected by a directed path (in at least one of the two directions) [64]. Testing whether the undirected edges of G_M can be oriented so that the resulting digraph $\overrightarrow{G_M}$ is unilateral, and computing such an orientation if it exists, can be done in time $O(|V| + |E| + |E_U|) = O(n)$ [64, Theorem 4]. A Hamiltonian path of $\overrightarrow{G_M}$ is given by a topological ordering of its vertices. ◀

6 Testing Algorithms for 2UBE Parameterized by the Branchwidth

In this section, we show that the 2UBE Testing problem is fixed-parameter tractable with respect to the branchwidth of the input *st*-graph both in the fixed and in the variable embedding setting. Since the treewidth $tw(G)$ and the branchwidth $bw(G)$ of a graph G are within a constant factor from each other (i.e., $bw(G) - 1 \leq tw(G) \leq \lfloor \frac{3}{2} bw(G) \rfloor - 1$ [70]), our FPT algorithm also extends to graphs of bounded treewidth. Previously, the complexity of this problem was settled only for graphs of treewidth at most 2 in the variable embedding setting[1] [37].

We use the SPQR-tree data structure [34] to efficiently handle the planar embeddings of the input digraphs and sphere-cut decompositions [71] to develop a dynamic-programming approach on the skeletons of the rigid components. For the definition of the *SPQR-tree* $\mathcal{T}$ of a biconnected graph and the related concepts of *skeleton* skel(μ) and *pertinent graph* pert(μ) of a node μ of $\mathcal{T}$, *types* of the nodes of $\mathcal{T}$ (namely, *S-,P-,Q-, and R-nodes*), and *virtual edges*

[1] To our knowledge, no efficient algorithm was known for treewidth 2 in the fixed embedding setting.

of a skeleton, see [23]. To ease the description, we can assume that each S-node has exactly two children [41] and that the skeleton of each node μ does not contain the virtual edge representing the parent of μ. In particular, we will exploit the following property of $\mathcal{T}$ when G is an *st*-graph containing the edge $e = (s,t)$ and $\mathcal{T}$ is rooted at the Q-node of e.

▶ **Property 3** ([34]). *Let $\mu \in \mathcal{T}$ with poles u and v. Without loss of generality, assume that the directed paths connecting u and v in G are oriented from u to v. Then,* $\mathrm{pert}(\mu)$ *is a uv-graph.*

For the definition of *branchwidth* and *sphere-cut decomposition*, and for the related concepts of *middle set* $\mathrm{mid}(e)$ and *noose* $\mathcal{O}_e$ of an arc e of the decomposition, and *length* of a noose, see [23]. We denote a sphere-cut decomposition of a plane graph $G = (V,E)$ by the triple $\langle T, \xi, \Pi = \bigcup_{a \in E(T)} \pi_a \rangle$, where T is a ternary tree whose leaves are in a one-to-one correspondence with the edges of G, which is defined by a bijection $\xi : \mathcal{L}(T) \leftrightarrow E(G)$ between the leaf set $\mathcal{L}(T)$ of T and the edge set E, and where π_a is a circular order of $\mathrm{mid}(a)$, for each arc a of T. In particular, we will exploit the property that each of the two subgraphs that lie in the interior and in the exterior of a noose is connected and that the set of nooses forms a laminar set family, that is, any two nooses are either disjoint or nested.

Without loss of generality, we assume that the input *st*-graph G contains the edge (s,t), which guarantees that G is biconnected. In fact, in any 2UBE of G vertices s and t have to be the first and the last vertex of the spine, respectively. Thus, either (s,t) is an edge of G or it can be added to any of the two pages of the spine of a 2UBE of G to obtain a 2UBE $\langle \pi, \sigma \rangle$ of $G \cup (s,t)$. Clearly, the edge (s,t) will be incident to the outer face of $\Gamma(\pi,\sigma)$.

Overview. Our approach leverages on the classification of the embeddings of each triconnected component of the biconnected graph G. Intuitively, such classification is based on the visibility of the spine that the embedding "leaves" on its outer face. We show that the planar embeddings of a triconnected component that yield a 2UBE of the component can be partitioned into a finite number of equivalence classes, called *embedding types*. By visiting the SPQR-tree $\mathcal{T}$ of G bottom-up, we describe how to compute all the *realizable embedding types* of each triconnected component, that is, those embedding types that are allowed by some embedding of the component. To this aim we will exploit the realizable embedding types of its child components. If the root of $\mathcal{T}$, which represents the whole *st*-graph G, admits at least one planar embedding belonging to some embedding type, then G admits a 2UBE. The most challenging part of this approach is handling the triconnected components that correspond to the P-nodes, where the problem is reduced to a maximum flow problem on a capacitated flow network with edge demands, and to the R-nodes, where a sphere-cut decomposition of bounded width is used to efficiently compute the feasible embedding types.

Embedding Types. Given a 2UBE $\langle \pi, \sigma \rangle$, the two pages will be called the *left page* (the one to the left of the spine) and the *right page* (the one to the right of the spine), respectively. We write $\sigma(e) = L$ (resp. $\sigma(e) = R$) if the edge e is assigned to the left page (resp. right page). A point p of the spine is *visible from the left (right) page* if it is possible to shoot a horizontal ray originating from p and directed leftward (rightward) without intersecting any edge in $\Gamma(\pi,\sigma)$. Let μ be a node of the SPQR-tree $\mathcal{T}$ of G rooted at (s,t). Recall that, by Property 3, since $\mathcal{T}$ has been rooted at (s,t), the pertinent graph $\mathrm{pert}(\mu)$ and the skeleton $\mathrm{skel}(\mu)$ of μ are $s't'$-graphs, where s' and t' are the poles of μ . We denote by s_μ (by t_μ) the pole of μ that is the source (the sink) of $\mathrm{pert}(\mu)$ and of $\mathrm{skel}(\mu)$. Let $\langle \pi_\mu, \sigma_\mu \rangle$ be a 2UBE of $\mathrm{pert}(\mu)$ and let $\mathcal{E}_\mu$ be the embedding of $\Gamma(\pi_\mu, \sigma_\mu)$. We say that $\mathcal{E}_\mu$ has *embedding type* (or is *of Type*) $\langle s\_vis, spine\_vis, t\_vis \rangle$ with $s\_vis, t\_vis \in \{L, R, N\}$ and $spine\_vis \in \{L, R, B, N\}$ where:

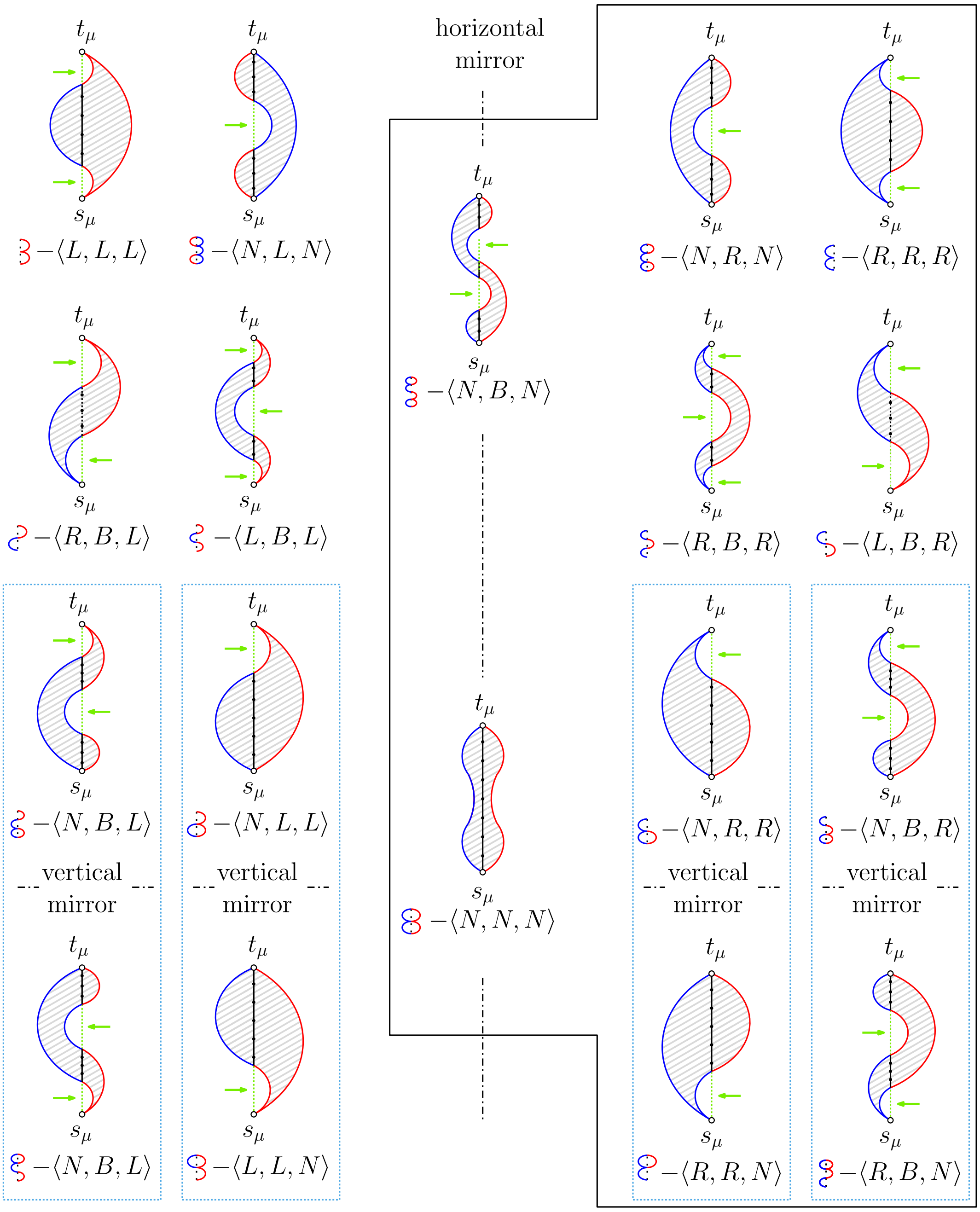
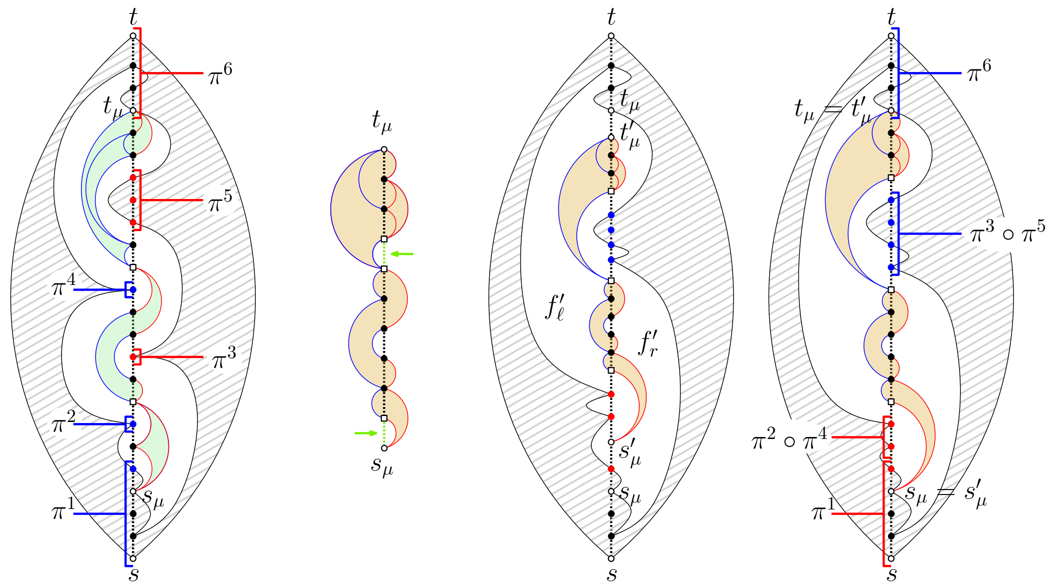


Figure 10 Illustrations of the possible embedding types of a node μ with poles s_μ and t_μ; the portion of the spine that is visible from the left or from the right is green. Pairs of embedding types in the same dotted box are one the vertically-mirrored copy of the other. Embedding types on the top are the horizontally-mirrored copy of the ones on the bottom. Embedding types -$\langle N, B, N\rangle$ and -$\langle N, N, N\rangle$ are the horizontal and vertical mirrored copies of themselves.



(a) Drawing $\Gamma(\pi, \sigma)$ of G **(b)** Drawing $\Gamma(\pi'_\mu, \sigma'_\mu)$ of pert(μ) **(c)** Drawing Γ^* of $G_{\overline{\mu}} \cup$ pert(μ) **(d)** Drawing Γ' of G

Figure 11 Illustrations for Lemma 11. The 2UBEs of pert(μ) are of Type -$\langle L, B, N\rangle$.

1. *s_vis* is L (resp., R) if in $\mathcal{E}_\mu$ there is a portion of the spine incident to s and between s and t that is visible from the left page (resp., from the right page); otherwise, *s_vis* is N.
2. *t_vis* is L (resp., R) if in $\mathcal{E}_\mu$ there is a portion of the spine incident to t and between s and t that is visible from the left page (resp., from the right page); otherwise, *t_vis* is N.
3. *spine_vis* is L (resp., R) if in $\mathcal{E}_\mu$ there is a portion of the spine between s and t that is visible from the left page (resp., from the right page); *spine_vis* is B if in $\mathcal{E}_\mu$ there is a portion of the spine between s and t that is visible from the left page, and a portion of the spine between s and t that is visible from the right page; otherwise, *spine_vis* is N.

We also say that a node μ and pert(μ) *admits Type* $\langle x, y, z\rangle$ if pert(μ) admits an embedding of Type $\langle x, y, z\rangle$. We have the following lemma.

▶ **Lemma 10.** *Let μ be a node of $\mathcal{T}$, let $\langle \pi_\mu, \sigma_\mu\rangle$ be a 2UBE of* pert(μ) *and let $\mathcal{E}_\mu$ be a planar embedding of $\Gamma(\pi_\mu, \sigma_\mu)$. Then $\mathcal{E}_\mu$ has exactly one embedding type, where the possibile embedding types are the 18 depicted in Fig. 10.*

Let $\langle \pi, \sigma\rangle$ be a 2UBE of G, let μ a node of $\mathcal{T}$, and let $\langle \pi_\mu, \sigma_\mu\rangle$ be the restriction of $\langle \pi, \sigma\rangle$ to pert(μ). Further, let $\langle \pi'_\mu, \sigma'_\mu\rangle \neq \langle \pi_\mu, \sigma_\mu\rangle$ be a 2UBE of pert(μ).

▶ **Lemma 11.** *If $\langle \pi'_\mu, \sigma'_\mu\rangle$ and $\langle \pi_\mu, \sigma_\mu\rangle$ have the same embedding type, then G admits a 2UBE whose restriction to* pert(μ) *is $\langle \pi'_\mu, \sigma'_\mu\rangle$.*

Proof sketch. First, insert a possibly squeezed copy of $\Gamma(\pi'_\mu, \sigma'_\mu)$ (Fig. 11b) inside $\Gamma(\pi, \sigma)$ (Fig. 11a) in the interior of the face f_μ of the plane digraph $G_{\overline{\mu}}$ resulting from removing pert(μ) (except its poles) from $\Gamma(\pi, \sigma)$. Second, suitably move parts of the boundary of f_μ along portions of the spine incident to the inserted drawing of pert(μ) (Fig. 11c). Then, continuously move the copies of the poles of μ inside f_μ towards their copies in $\Gamma(\pi, \sigma)$, without intersecting any edge, to obtain a drawing Γ' of G (Fig. 11d). ◀

Recall that, for each node μ of $\mathcal{T}$, $\mathrm{pert}(\mu)$ may have exponentially many embeddings, given by the permutations of the children of the P-nodes and by the flips of the R-nodes. Lemma 11 is the reason why we only need to compute a single embedding for each embedding type realizable by $\mathrm{pert}(\mu)$, i.e., a constant number of embeddings instead of an exponential number.

We first describe an algorithm to decide if G admits a 2UBE and its running time. The same procedure can be easily refined to actually compute a 2UBE of G, with no additional cost, by decorating each node $\mu \in \mathcal{T}$ with the embedding choices performed at μ, for each of its $O(1)$ possible embedding types.

Testing Algorithm. The algorithm is based on computing, for each non-root node μ of $\mathcal{T}$, the set of embedding types realizable by $\mathrm{pert}(\mu)$, based on whether μ is an S-, P-, Q-, or an R-node. Since, by Lemmas 10 and 11, G admits a 2UBE if and only if the pertinent graph of the unique child of the root Q-node admits an embedding of at least one of the 18 possible embedding types, this approach allows us to solve the 2UBE Testing problem for G.

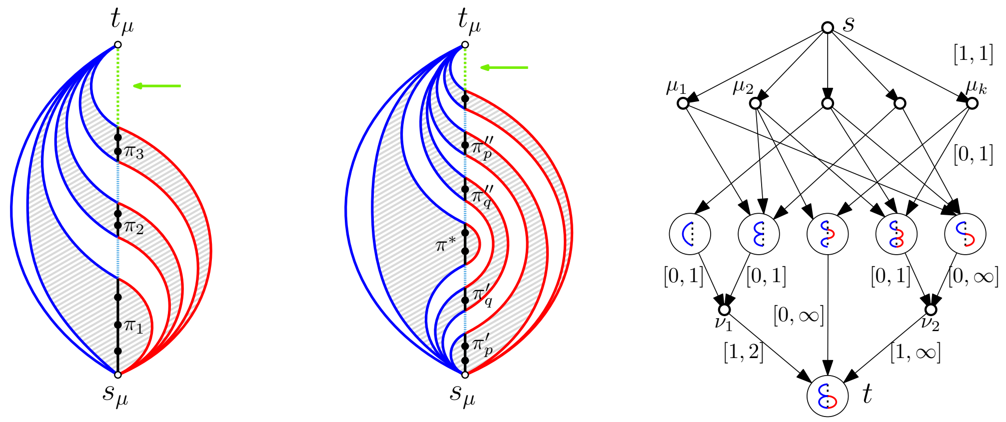
Recall that the only possible embedding choices for G happen at P- and R-nodes. While the treatment of Q- and S-nodes does not require any modification when considering the variable and the fixed embedding settings, for P- and R-nodes we will discuss how to compute the embedding types that are realizable by $\mathrm{pert}(\mu)$ in both such settings. In particular, in the fixed embedding scenario the above characterization needs to additionally satisfy the constraints imposed by the fixed embedding on the skeletons of the P- and R-nodes in $\mathcal{T}$.

Note that a leaf Q-node only admits embeddings of type -$\langle L, L, L\rangle$ or -$\langle R, R, R\rangle$. Also, combining 2UBEs of the two children of an S-node μ always yields a 2UBE of $\mathrm{pert}(\mu)$, whose embedding type can be easily computed. In [23] we prove the following.

▶ **Lemma 12.** *Let μ be an S-node. The set of embedding types realizable by $\mathrm{pert}(\mu)$ can be computed in $O(1)$ time, both in the fixed and in the variable embedding setting.*

P-nodes. Let μ be a P-node with poles s_μ and t_μ. Recall that an embedding for a P-node is obtained by choosing a permutation for its children and an embedding type for each child. Our approach to compute the realizable types of $\mathrm{pert}(\mu)$ consists of considering one type at a time for μ. For each embedding type, we check whether the children of μ, together with their realizable embedding types, can be arranged in a finite number of families of permutations (which we prove to be a constant number) so to yield an embedding of the considered type. In order to ease the following description, consider that the arrangements of the children for obtaining some embedding types can be easily derived from the arrangements to obtain the (horizontally) symmetric ones by (i) reversing the left-to-right sequence of the children in the construction and (ii) by taking, for each child, the horizontally-mirrored embedding type; for instance, the arrangements to construct an embedding of Type -$\langle L, L, N\rangle$ can be obtained from the ones to construct an embedding of Type -$\langle R, R, N\rangle$, and vice versa. Moreover, two embedding types, namely Type -$\langle N, B, N\rangle$ and Type -$\langle N, N, N\rangle$, are (horizontally) self-symmetric. As a consequence, in order to consider all the embedding types that are realizable by $\mathrm{pert}(\mu)$ we describe how to obtain only 10 "relevant" embedding types (enclosed by a solid polygon in Fig. 10): -$\langle R, R, R\rangle$, -$\langle N, R, N\rangle$, -$\langle R, B, R\rangle$, -$\langle L, B, R\rangle$, -$\langle N, R, R\rangle$, -$\langle N, B, R\rangle$, -$\langle R, R, N\rangle$, -$\langle R, B, N\rangle$, -$\langle N, B, N\rangle$, and -$\langle N, N, N\rangle$.

Next, we give necessary and sufficient conditions under which the pertinent graph of a P-node admits an embedding of Type -$\langle N, R, R\rangle$. Then, we show how to test these conditions efficiently by exploiting a suitably defined flow network. The conditions for the remaining types, given in [23], can be tested with the same algorithmic strategy.



(a) -$\langle N, R, R\rangle$; Case 1 **(b)** -$\langle N, R, R\rangle$; Case 2 **(c)** Network $\mathcal{N}$ for Case 2

Figure 12 Case 1 (a) and Case 2 (b) of Lemma 13 for a P-nodes μ of Type -$\langle N, R, R\rangle$. The spine is colored either green, blue, or black. The green part is the portion of the spine that is visible from the right, the black parts correspond to the bottom-to-top sequences of the internal vertices of pert(μ) inherited from the 2UBEs of the children of μ, the blue parts join sequences inherited from different children. (c) Capacitated flow network $\mathcal{N}$ with edge demands corresponding to (b).

▶ **Lemma 13** (Type -$\langle N, R, R\rangle$). *Let μ be a P-node. Type -$\langle N, R, R\rangle$ is admitted by μ in the variable embedding setting if and only if at least one of two cases occurs.* ***(Case 1)*** *The children of μ can be partitioned into two parts: The first part consists either of a Type--$\langle R, R, R\rangle$ Q-node child, or of a Type--$\langle N, R, R\rangle$ child, or both. The second part consists of any number, even zero, of Type--$\langle L, B, R\rangle$ children.* ***(Case 2)*** *The children of μ can be partitioned into three parts: The first part consists either of a Type--$\langle R, R, R\rangle$ Q-node child, or of a non-Q-node Type--$\langle R, R, R\rangle$ child, or both. The second part consists of any number, even zero, of Type--$\langle R, B, R\rangle$ children. The third part consists of any positive number of Type--$\langle N, B, R\rangle$ or Type--$\langle L, B, R\rangle$ children, with at most one Type--$\langle N, B, R\rangle$ child.*

Regarding the time complexity of testing the existence of a Type--$\langle N, R, R\rangle$ embedding of pert(μ), we show that deciding if one of **(Case 1)** or **(Case 2)** of Lemma 13 applies can be reduced to a network flow problem on a network $\mathcal{N}$ with edge demands. The network for **(Case 2)** is depicted in Fig. 12c. The details of this construction are given in [23].

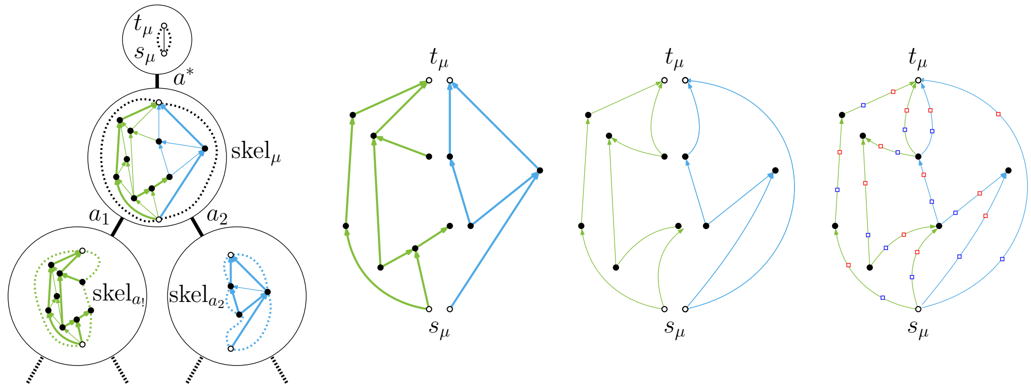
▶ **Lemma 14.** *Let μ be a P-node with k children. The set of embedding types realizable by* pert(μ) *can be computed in $O(k^2)$ time in the variable embedding setting.*

The fixed embedding scenario for a P-node μ can be addressed by processing the children of μ in the left-to-right order defined by the given embedding of G. The details of such an approach are given in [23], where the following is proven.

▶ **Lemma 15.** *Let μ be a P-node with k children. The set of embedding types realizable by* pert(μ) *can be computed in $O(k)$ time in the fixed embedding setting.*

Lemmas 12 and 15 yield a counterpart, in the fixed embedding setting, of the linear-time algorithm by Di Giacomo et al. [37] to compute 2UBEs of series-parallel graphs.

▶ **Theorem 16.** *There exists an $O(n)$-time algorithm to decide whether an n-vertex series-parallel st-graph admits an embedding-preserving 2UBE.*



(a) A sphere-decomposition of $\mathrm{skel}(\mu) \cup (s_\mu, t_\mu)$ **(b)** Outer faces of skel_{a_1} and skel_{a_2} **(c)** Directed paths of skel_{a_1} and skel_{a_2} **(d)** Graph $A = A_1 \cup A_2$

Figure 13 Illustrations for the R-node case. (a) A partial sphere-cut decomposition of the graph $\mathrm{skel}(\mu) \cup (s_\mu, t_\mu)$ rooted at the edge (s_μ, t_μ), where μ is an R-node. The nooses are dotted curves.

R-nodes. Let μ be an R-node and (T, ξ, Π) be a sphere-cut decomposition of $\mathrm{skel}(\mu) \cup (s_\mu, t_\mu)$ of width β, rooted at the node ρ with $\xi(\rho) = (s_\mu, t_\mu)$; refer to Fig. 13a. Each arc a of T is associated with a subgraph skel_a of $\mathrm{skel}(\mu)$ and with a subgraph pert_a of $\mathrm{pert}(\mu)$, both bounded by the noose of a. Let a_1 and a_2 be the two arcs leading to a from the bottom of T. Intuitively, our strategy to compute the embedding types of $\mathrm{pert}(\mu)$ is to visit T bottom-up maintaining a succinct description of size $O(\beta)$ of the properties of the 2UBEs of pert_a. To this aim, we construct cycles composed of directed edges that are in one-to-one correspondence with maximal directed paths along the outer face of skel_{a_1} and skel_{a_2} (Figs. 13b and 13c), which we use to define an auxiliary graph A whose 2UBEs concisely represent the possible 2UBEs of pert_a obtained by combining the 2UBEs of pert_{a_1} and pert_{a_2} (Fig. 13d). When we reach the arc a^* incident to ρ with $\mathrm{skel}_{a^*} = \mathrm{skel}(\mu)$, we use the computed properties to determine the embedding types realizable by $\mathrm{pert}(\mu)$. We provide full details in [23].

▶ **Lemma 17.** *Let μ be an R-node whose skeleton $\mathrm{skel}(\mu)$ has k children and branchwidth β. The set of embedding types realizable by $\mathrm{pert}(\mu)$ can be computed in $O(2^{O(\beta \log \beta)} \cdot k)$ time, both in the fixed and in the variable embedding setting, provided that a sphere-cut decomposition $\langle T_\mu, \xi_\mu, \Pi_\mu \rangle$ of width β of $\mathrm{skel}^+(\mu)$ is given.*

By Lemmas 12, 14, 15 and 17 and since $\mathcal{T}$ has $O(|G|)$ size [34, 41], we get the following.

▶ **Theorem 18.** *There exists an $O(2^{O(\beta \log \beta)} \cdot n + n^2 + g(n))$-time algorithm to decide if an n-vertex planar (plane) st-graph of branchwidth β admits a (embedding-preserving) 2UBE, where $g(n)$ is the computation time of a sphere-cut decomposition of an n-vertex plane graph.*

Observe that $g(n)$ is $O(n^3)$ by the result in [71]. Thus, we get the following.

▶ **Corollary 19.** *There exists an $O(2^{O(\beta \log \beta)} \cdot n + n^3)$-time algorithm to decide whether an n-vertex planar (plane) st-graph of branchwidth β admits a (embedding-preserving) 2UBE.*

Since the branchwidth of a planar graph G is at most $2.122\sqrt{n}$ [48], Corollary 19 immediately implies that the 2UBE Testing problem can be solved in sub-exponential time.

▶ **Corollary 20.** *There exists an $O(2^{O(\sqrt{n} \log \sqrt{n})} + n^3)$-time algorithm to decide whether an n-vertex planar (plane) st-graph admits a (embedding-preserving) 2UBE.*

7 Conclusion and Open Problems

Our results provide significant advances on the complexity of the kUBE Testing problem. We showed NP-hardness for $k \geq 3$; we gave FPT- and polynomial-time algorithms for relevant families of planar st-graphs when $k = 2$. We point out that our FPT-algorithm can be refined to run in $O(n^2)$ time for st-graphs of treewidth at most 3, by constructing in linear time a sphere-cut decomposition of their rigid components. We conclude with some open problems.

- The main open question is about the complexity of the 2UBE Testing problem, which has been conjectured to be NP-complete in the general case [55].
- The digraphs in our NP-completeness proof are not upward planar. Since there are upward planar digraphs that do not admit a 3UBE [58], it would be interesting to study whether the problem remains NP-complete for three pages and upward planar digraphs.
- Finally, it is natural to investigate other families of planar digraphs for which a 2UBE always exists or polynomial-time testing algorithms can be devised.

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