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MEASURE-VALUED SOLUTIONS TO A NONLINEAR FOURTH-ORDER REGULARIZATION OF FORWARD-BACKWARD PARABOLIC EQUATIONS*

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Abstract. We introduce and analyze a new, nonlinear fourth-order regularization of forward-backward parabolic equations. In one space dimension, under general assumptions on the potentials, which include those of Perona–Malik type, we prove existence of Radon measure-valued solutions under both natural and essential boundary conditions. If the decay at infinity of the nonlinearities is sufficiently fast, we also exhibit examples of local solutions whose atomic part arises and/or persists (in contrast to the linear fourth-order regularization) and even disappears within finite time (in contrast to pseudoparabolic regularizations).

Key words. forward-backward parabolic equations, fourth-order parabolic equations, Radon measures, Perona–Malik equation

AMS subject classifications. 35D99, 35K35, 35K55, 35M13, 35R25, 28A33

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1. Introduction.

1.1. Motivation and goals.

The *Perona–Malik equation* is a nonlinear diffusion equation which is of *forward-backward* parabolic character. For example, in one space dimension it reduces to

$$w_t = [\varphi(w_x)]_x,$$

and the diffusion mechanism is expressed by the natural condition that $s\varphi(s) > 0$ for $s \neq 0$. Typically $\varphi(\pm\infty) = \varphi(0) = 0$, $\varphi' > 0$ in an interval $(-\overline{u}, \overline{u})$ where the equation is forward parabolic, and $\varphi' < 0$ in the intervals $(-\infty, -\overline{u})$ and $(\overline{u}, \infty)$ where the equation is backward parabolic.

The Perona–Malik equation arises in various applications [3, 26, 28]. In a model for temperature (or salinity) stratification in turbulent shear flow in the ocean [3], the equation is intrinsically one-dimensional (x is the vertical component), w_x is nonnegative (deeper in the ocean, water is colder) and backward parabolicity models the decrease in magnitude of turbulent temperature fluxes for large temperature gradients. In the multidimensional case, the Perona–Malik equation arises in the context of image processing [28]. We refer the reader to [19, 25] for recent referenced discussions on the analytical theory developed so far.

Due to its ill-posedness, it is natural to introduce suitable regularizations of the Perona-Malik equation (see, e.g., [1, 3, 4, 5, 6, 16, 24, 25] and references therein).

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However, because of the intrinsically unstable character of backward parabolic equations, one may reasonably expect the qualitative behavior of solutions to strongly depend on the type of regularization. Let us briefly discuss two types of regularization, both of which are local and of higher order.

In the context of temperature stratification in the ocean, modeling considerations [3] suggest adopting a nonlinear *pseudoparabolic regularization*,

$$w_t = [\varphi(w_x) + \tau[\psi(w_x)]_t]_x \,, \tag{1.1}$$

where τ is a small parameter and ψ is an increasing but bounded function. The boundedness of ψ makes the equation strongly degenerate for large values of the gradient. Adding to (1.1) appropriate initial-boundary conditions leads to a well-posed initial-boundary value problem, but it was observed for the first time in [3] that, due to the strong degeneracy of the equation, there are solutions which develop spatial discontinuities (jumps) within finite time, whose amplitude then increases with time; in particular, jumps cannot disappear.

In [5], a *linear* fourth-order regularization was analyzed for two prototypical forms of φ,

$$w_t = \left[\varphi(w_x) - \varepsilon^2 w_{xxx}\right]_x \,, \tag{1.2}$$

where ε is a small parameter. Observe that (1.2), say in a spatial interval $\Omega = (a,b) \subset \mathbb{R}$, has a natural gradient flow structure with respect to the functional

$$\int_\Omega \left(\Phi(w_x) + \frac{\varepsilon^2}{2} w_{xx}^2\right) dx \,, \qquad \Phi(u) := \int_0^u \varphi(s)\, ds \,.$$

The H^2-control encoded by the functional implies, in contrast to (1.1), that solutions to (1.2) remain smooth.

Concerning the qualitative behavior of solutions, interesting numerical experiments, strengthened by various analytical observations, have been performed on (1.2) [5] as well as on other regularizations [24, 25] of the Perona–Malik equation. These experiments suggest mechanisms analogous to spinodal decomposition and coarsening, characterized by three different time scales: an initial, short time-scale where staircase-type or ramp-type microstructures (also referred to as wrinkles) develop in regions where $\{\varphi' < 0\}$ in order to reduce the energy in the backward regime; an intermediate time-scale during which diffusion operates in regions where $\{\varphi' > 0\}$, whereas in $\{\varphi' < 0\}$ microstructures evolve into macroscopic steps; and a final, long-time scale where solutions are close to piecewise constant functions, with neighboring plateaus colliding and merging. In particular, coarsening occurs at both intermediate and long time-scales, with disappearance of jump discontinuities.

Notable efforts have been devoted to the qualitative description of such behavior, e.g., in terms of time-scales and finite-dimensional reduction [5, 6, 18, 19]. In this regard, however, the regularization mechanisms in (1.1) and (1.2) suffer from one disadvantage. On one hand, (1.2) has continuous solutions, in which jumps are replaced by diffuse interfaces with high (but finite) gradients; on the other hand, though (1.1) does allow for discontinuous solutions, jumps cannot disappear; hence the aforementioned coarsening phenomenon cannot be modeled by (1.1).

The main motivation of the present paper is to introduce a new regularization mechanism which overcomes both disadvantages, allowing for discontinuous solutions with jumps that can both appear (as is assumed to occur in the early-stage development of microstructures) and disappear (as is assumed to occur in the subsequent coarsening): it consists of a *nonlinear* fourth-order regularization, which maintains the gradient flow structure of (1.2) but strongly degenerates for large values of w_x. More precisely, consider the functional

$$\int_\Omega \left(\Phi(w_x) + \frac{\varepsilon^2}{2}[\psi(w_x)]_x^2 \right) dx\,, \qquad \Phi(u) = \int_0^u \varphi(s)\,ds\,, \tag{1.3}$$

where ψ is again an odd, bounded, and increasing function. The corresponding regularization of the Perona–Malik equation is

$$w_t = \left[\varphi(w_x) - \varepsilon^2 \psi'(w_x)[\psi(w_x)]_{xx}\right]_x . \tag{1.4}$$

In the present paper, we will first prove an existence result, introduced in section 1.3, which covers the case of discontinuous initial data with bounded variation. Then, through the construction of special solutions introduced in section 1.4 and elaborated in section 6, we will argue that the new regularization (1.4) does what we designed it for; that is,

- in contrast to (1.2), solutions to (1.4) are *not* necessarily smooth: jump discontinuities persist for some time and may also appear within finite time; and
- in contrast to (1.1), jump discontinuities may not only appear but also *disappear* within finite time, at least when Φ is bounded at infinity.

We see these properties as the basic advantage of (1.4), which might permit a better understanding and a quantitative description of the aforementioned staircasing and coarsening phenomena. For instance, by extending the methods used in section 6, it might be possible (though highly nontrivial, due to the higher-order character of the equation) to construct self-similar solutions which exhibit a coarsening phenomenon, in the sense that one jump discontinuity grows at the expense of a "train" of other jumps which decrease. As a by-product, such a construction would provide information about the typical space and time-scales for the coarsening phenomenon. We hope to investigate these features, as well as other qualitative properties of solutions to (1.4), in the future.

To conclude this subsection, we stress that our analysis is heavily based on the fact that $\Omega \subseteq \mathbb{R}$. In the multidimensional case, the most natural extension of (1.4) would amount to studying the formal gradient flow of

$$\int_\Omega \left(\Phi(|\nabla w|) + \frac{\varepsilon^2}{2} \left|\operatorname{div}(\boldsymbol{\psi}(\nabla w))\right|^2 \right) dx$$

with Φ as in (1.3) and $\boldsymbol{\psi}(\boldsymbol{\xi}) = \frac{\psi(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}\boldsymbol{\xi}$. However, such a generalization will definitely require new tools.

1.2. Setting. Since jumps of w correspond to Dirac masses for w_x, the meaning of $\Phi(w_x)$ is unclear. To highlight this mathematical difficulty, we find it convenient to differentiate the equation with respect to x and to describe the problem in terms of the *Radon measure*

$$u = w_x$$

(in the one-dimensional framework, it is of course equivalent to work with w or its distributional derivative). This leads to the equation

$$u_t = v_{xx} \quad \text{in } Q := \Omega \times (0,\infty), \qquad v := \varphi(u) - \varepsilon^2 \psi'(u)[\psi(u)]_{xx}, \tag{1.5}$$

which we complement with either natural boundary conditions,

(1.6a) $$[\psi(u)]_x = v_x = 0 \quad \text{in } \partial\Omega \times (0,\infty), \qquad u = u_0 \quad \text{in } \Omega \times \{0\},$$

or essential boundary conditions,

(1.6b) $$\psi(u) = v = 0 \quad \text{in } \partial\Omega \times (0,\infty), \qquad u = u_0 \quad \text{in } \Omega \times \{0\}\,.$$

Here ε is a positive constant, $\Omega \equiv (a,b) \subset \mathbb{R}$ is a bounded interval, and u_0 is a Radon measure on Ω. Typical examples of the functions φ and ψ are

(1.7) $$\varphi(u) = \frac{u}{(1+u^2)^{\frac{\alpha+1}{2}}} \text{ with } \alpha > 0 \quad \text{and} \quad \psi(u) = \int_0^u \frac{ds}{(1+s^2)^{\frac{\sigma+1}{2}}} \text{ with } \sigma > 0,$$

(1.8) $$\varphi(u) = u\, e^{-\alpha\sqrt{1+u^2}} \text{ with } \alpha > 0 \quad \text{and} \quad \psi(u) = \int_0^u e^{-\sigma\sqrt{1+s^2}} ds \text{ with } \sigma > 0\,.$$

In both (1.7) and (1.8), φ is of "Perona–Malik type" (see [3, 26, 28]). In case (1.7),

(1.9) $$|\varphi(u)| \sim |u|^{-\alpha} \quad \text{and} \quad |\psi(u)| \sim \gamma - \frac{|u|^{-\sigma}}{\sigma} \quad \text{as } |u| \to \infty\,,$$

with $\gamma = \psi(\infty)$. However, the assumptions under which problem (1.5)–(1.6) will be addressed hold for a much wider class of nonlinearities and include the case in which φ does not vanish at infinity (see subsection 3.1).

1.3. Existence results. The first purpose of the present paper is to construct Radon measure-valued solutions of problem (1.5)–(1.6) (see Theorem 3.8). Two a priori estimates are at the core of both the solution concept and the existence theory.

The first estimate reflects the gradient flow structure of (1.5) with respect to the energy

(1.10) $$E[u] := \int_\Omega \left(\Phi(u) + \frac{\varepsilon^2}{2} [\psi(u)]_x^2 \right) dx$$

(cf. (1.3)) and may be formally obtained as follows:

$$\begin{aligned} \frac{d}{dt} E[u(t)] &= \int_\Omega \left(\varphi(u) u_t + \varepsilon^2 [\psi(u)]_x [\psi(u)]_{xt} \right) dx \\ &\overset{(1.6)}{=} \int_\Omega \left(\varphi(u) - \varepsilon^2 [\psi(u)]_{xx} \psi'(u) \right) u_t dx \overset{(1.5),(1.6)}{=} - \int_\Omega v_x^2 dx\,. \end{aligned}$$

As we shall elaborate in subsection 3.2, nontrivial initial measures "with finite energy" exist provided ψ' is sufficiently degenerate at infinity ($\sigma > 1/2$ in case (1.7)): their singular part concentrates on the set where the regular part blows up (see Figure 1). In this case, the energy estimate guarantees that such a property is preserved for later times, additionally providing a uniform control on the "flux" v_x.

The second a priori estimate is of entropy type: letting Ψ be a primitive of ψ, we see that, formally,

$$\frac{d}{dt} \int_\Omega \Psi(u(t)) dx = \int_\Omega \psi(u) u_t dx \overset{(1.5),(1.6)}{=} \int_\Omega [\psi(u)]_{xx} \varphi(u) - \varepsilon^2 \int_\Omega \psi'(u) [\psi(u)]_{xx}^2 dx.$$

Since ψ is increasing, the second integral has negative sign, whereas the first may be either absorbed into or controlled by the energy, depending on suitable relations between the behavior of φ and ψ (cf. (H_3) below): in the prototype case (1.7), (H_3) holds for $\sigma \le 2\alpha$. The entropy estimate provides uniform controls on both the "pressure" v and, since Ψ has linear growth at infinity, the L^1-norm of $u(\cdot,t)$.

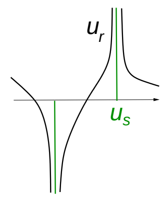

Fig. 1.

1.4. Evolution of singularities. The second purpose of the present paper is to give examples of local solutions of (1.5) (namely, solutions defined in a subset of Ω) whose singular part increases or decreases in time. In Theorem 3.9 we prove that such local solutions exist in a particular case of (1.9) if both φ and ψ' decay sufficiently fast: $\alpha > 1$ and $\sigma > 1/2$. These solutions have the form of a Dirac mass concentrated at a point and surrounded by a spatially constant regular part which blows up at that point (cf. (3.13)–(3.14)).

The spontaneous appearance of singularities confirms that solutions of problem (1.5)–(1.6) are, like those of (1.1), intrinsically Radon measure-valued (see [8, 9, 10, 11, 12, 13, 14, 31]). In both cases this phenomenon is caused by the negative sign of $\varphi'(u)$ and the smallness of $\psi'(u)$ for large values of u.

Theorem 3.9 also provides examples of local measure-valued solutions for which the singular part of the measure *disappears* in finite time. As we mentioned already, this phenomenon is new and does not occur in the case of (1.1), where singular parts can only grow.

1.5. Plan. The paper is organized as follows. In section 2 we collect some preliminary facts and characterize measures with finite energy. In section 3 we state precise assumptions, give the definition of global and local solutions of problem (1.5)–(1.6), and state the main results of the paper. In section 4 we address solutions of approximating problems by which in section 5 we prove the general existence result for measure-valued initial data (see Theorem 3.8). In section 6 we provide examples of local solutions of (1.5) with growing or decreasing singular part. Finally, in the appendix we prove some facts concerning measures with finite energy, and we adapt to the present case the standard proof of existence when ψ is nondegenerate and u_0 is a function, which is needed to deal with the approximating problems.

2. Preliminaries. For every measurable function f defined on Ω, the positive and negative parts of f are $f^{\pm} := \max\{\pm f, 0\}$. For the sake of brevity, we write ∞ instead of $+\infty$.

2.1. Radon measures. We denote by $\mathcal{M}(\Omega)$ the space of finite Radon measures on Ω and by $\mathcal{M}^+(\Omega)$ and the cone of nonnegative measures in $\mathcal{M}(\Omega)$. If $\mu_1, \mu_2 \in \mathcal{M}(\Omega)$, we write $\mu_1 \le \mu_2$ if $\mu_2 - \mu_1 \in \mathcal{M}^+(\Omega)$. For $\mu \in \mathcal{M}(\Omega)$, $\mu = \mu^+ - \mu^-$ is the Jordan decomposition into positive and negative parts, and $\mu = \mu_{ac} + \mu_s$ is the Lebesgue decomposition into absolutely continuous and singular parts with respect to the Lebesgue measure on Ω; $\mu_r \in L^1(\Omega)$ denotes the density of μ_{ac}. The Jordan decomposition is minimal, in the sense that $\mu = \mu^{(1)} - \mu^{(2)}$ with $\mu^{(1)}, \mu^{(2)} \in \mathcal{M}^+(\Omega)$ implies $\mu^+ \le \mu^{(1)}$, $\mu^- \le \mu^{(2)}$. Given $\mu \in \mathcal{M}(\Omega)$ and a Borel set $F \subseteq \Omega$, the restriction $\mu \llcorner F \in \mathcal{M}(\Omega)$ is defined by $(\mu \llcorner F)(G) := \mu(F \cap G)$ for any Borel set $G \subseteq \Omega$. Similar notation is used for the spaces of finite Radon measures on $Q_T = \Omega \times (0, T)$.

We denote by $\langle \cdot, \cdot \rangle_\Omega$ the duality map between $\mathcal{M}(\Omega)$ and the space $C_c(\Omega)$ of continuous functions with compact support in Ω. By abuse of notation, we also set

$$\langle \mu, \rho \rangle_\Omega := \int_\Omega \rho(x)\, d\mu(x) \quad \text{if } \mu \in \mathcal{M}(\Omega) \text{ and } \rho \in L^1(\Omega, \mu).$$

For $T > 0$ we denote by $L^\infty(0,T;\mathcal{M}(\Omega))$ the set of measures $z \in \mathcal{M}(Q_T)$ for which for a.e. $t \in (0,T)$ there is a measure $z(\cdot,t) \in \mathcal{M}(\Omega)$ such that

(i) for all $\zeta \in C_c(Q_T)$ the map $t \mapsto \langle z(\cdot,t), \zeta(\cdot,t)\rangle_\Omega$ belongs to $L^1(0,T)$ and

$$\langle z, \zeta \rangle_{Q_T} = \int_0^T \langle z(\cdot,t), \zeta(\cdot,t)\rangle_\Omega \, dt\,; \tag{2.1}$$

(ii) the map $t \mapsto \|z(t)\|_{\mathcal{M}(\Omega)}$ belongs to $L^\infty(0,T)$.

Accordingly, we use the notation

$$\|z\|_{L^\infty(0,T;\mathcal{M}(\Omega))} := \operatorname{ess}\sup_{t\in(0,T)} \|z(t)\|_{\mathcal{M}(\Omega)} \quad \text{for } z \in L^\infty(0,T;\mathcal{M}(\Omega))\,.$$

Observe that by the above definition the map $t \mapsto \langle z(\cdot,t), \rho\rangle_\Omega$ is measurable for all $\rho \in C_c(\Omega)$, and thus the map $z : (0,T) \to \mathcal{M}(\Omega)$ is weakly* measurable. For simplicity we use the notation $L^\infty(0,T;\mathcal{M}(\Omega))$ instead of the more correct $L^\infty_{w*}(0,T;\mathcal{M}(\Omega))$.

If $z \in L^\infty(0,T;\mathcal{M}(\Omega))$, for every Borel set $F \subseteq Q_T$ there holds

$$z(F) = \int_0^T z(\cdot,t)\,(F_t)\, dt\,,$$

where $F_t := \{x \in \Omega \,|\, (x,t) \in F\}$ is the t-section of F. It is easily seen that z is concentrated on a Borel set $F \subseteq Q_T$ if and only if $z(\cdot,t)$ is concentrated on the section F_t for a.e. $t \in (0,T)$ (see [31, Proposition 4.2]).

If $z \in L^\infty(0,T;\mathcal{M}(\Omega))$, then also $z_{ac}, z_s \in L^\infty(0,T;\mathcal{M}(\Omega))$ and, by (2.1),

$$\langle z_{ac}\,, \zeta\rangle_{Q_T} = \iint_{Q_T} z_r\, \zeta\, dxdt\,, \quad \langle z_s, \zeta\rangle_{Q_T} = \int_0^T \langle z_s(\cdot,t), \zeta(\cdot,t)\rangle_\Omega\, dt \quad \text{for } \zeta \in C_c(Q_T)\,. \tag{2.2}$$

One easily checks that

$$z_{ac}(\cdot,t) = [z(\cdot,t)]_{ac}\,, \quad z_s(\cdot,t) = [z(\cdot,t)]_s \quad \text{for a.e. } t \in (0,T)\,. \tag{2.3}$$

In particular,

$$z_r(\cdot,t) = [z(\cdot,t)]_r \quad \text{for a.e. } t \in (0,T)\,, \tag{2.4}$$

where $[z(\cdot,t)]_r$ denotes the density of the measure $[z(\cdot,t)]_{ac}$: for $\zeta \in C(\overline{\Omega})$,

$$\langle [z(\cdot,t)]_{ac}, \zeta\rangle_\Omega = \int_\Omega [z(\cdot,t)]_r \zeta\, dx = \int_\Omega z_r(\cdot,t)\, \zeta\, dx \quad \text{for a.e. } t \in (0,T)\,.$$

In view of (2.2)–(2.4), we always identify the quantities which appear on either side of equalities (2.3)–(2.4).

2.2. Function spaces. In addition to the standard Sobolev spaces $H^k(\Omega)$ ($k \in \mathbb{N}$), the following spaces are convenient to define in order to deal with different boundary conditions:

$$C^1_*(\Omega) := \begin{cases} C^1(\overline{\Omega}) & \text{in case (1.6a)}, \\ C^1_c(\Omega) & \text{in case (1.6b)}, \end{cases} \qquad H^1_*(\Omega) := \begin{cases} H^1(\Omega) & \text{in case (1.6a)}, \\ H^1_0(\Omega) & \text{in case (1.6b)}, \end{cases}$$

$$H^3_*(\Omega) := \begin{cases} \{u \in H^3(\Omega) \,|\, u_x = 0 \text{ on } \partial\Omega\} & \text{in case (1.6a)}, \\ \{u \in H^3(\Omega) \,|\, u = u_{xx} = 0 \text{ on } \partial\Omega\} & \text{in case (1.6b)}. \end{cases}$$

By $\langle\cdot,\cdot\rangle_*$ we denote the duality map between the spaces $(H^1_*(\Omega))'$ and $H^1_*(\Omega)$. We use the following well-known result [30, 21]; cf. the discussion in [17].

LEMMA 2.1 (Aubin–Lions–Dubinskiĭ–Simon). *Let $T>0$, and let X, Y, Z be reflexive Banach spaces, such that X is compactly embedded into Y and Y is continuously embedded into Z. Then the following embeddings are compact:*

$$\left\{u \in L^2(0,T;X)\,|\,u_t \in L^2(0,T;Z)\right\} \hookrightarrow L^2(0,T;Y),$$
$$\left\{u \in L^\infty(0,T;X)\,|\,u_t \in L^2(0,T;Z)\right\} \hookrightarrow C([0,T];Y).$$

We also make use of the following result, which is a particular case of Proposition 3.1 and Remark 3.2 in [7].

LEMMA 2.2. *Let $X \subset L^2(\Omega)$ be dense with continuous embedding. If $z \in L^2(0,T;X)$ with $z_t \in L^2(0,T;X')$, then $z \in C([0,T];L^2(\Omega))$ and*

$$\frac{1}{2}\int_\Omega z^2(x,t_2)\,dx - \frac{1}{2}\int_\Omega z^2(x,t_1)\,dx = \int_{t_1}^{t_2} \langle z_t(\cdot,t), z(\cdot,t)\rangle_{X',X}\,dt\,.$$

3. Main results.

3.1. Assumptions. Throughout the paper we assume that φ and ψ satisfy the following conditions (H_1)–(H_3). On φ we assume that

$$(H_1)\qquad \begin{cases} \text{(i)} & \varphi \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}), \quad \varphi(0)=0;\\ \text{(ii)} & \text{a function } \Phi:\mathbb{R}\to\mathbb{R} \text{ and a constant } k_0>0 \text{ exist such that}\\ & \qquad \Phi'(u)=\varphi(u)\,, \quad 0 \le \Phi(u) \le k_0|u| \quad \text{for all } u\in\mathbb{R}. \end{cases}$$

Note that the assumptions on φ are very weak: in particular, we do not assume symmetry, and φ' may change sign arbitrarily many times. The assumption $\varphi(0)=0$ is needed to treat essential boundary conditions, the sign condition on Φ provides a lower bound on the energy, and the growth condition on Φ allows one to consider measure-valued initial data.

As to ψ, we assume that $k_1>0$, $\gamma>0$ exist such that

$$(H_2)\qquad \begin{cases} \text{(i)} & \psi \in C^3(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R}),\ \psi \text{ odd},\ \psi'>0 \text{ in } \mathbb{R}\,,\ \lim_{u\to\infty}\psi(u)=\gamma;\\ \text{(ii)} & |\psi''(u)| \le k_1\,\psi'(u) \text{ for any } u\in\mathbb{R}\,. \end{cases}$$

Note that by (H_2)(i),

$$(3.1)\qquad 0 \le \Psi(u) \le \gamma\,|u| \quad \text{for all } u\in\mathbb{R}\,, \quad \text{where} \quad \Psi(u) := \int_0^u \psi(s)\,ds.$$

The functions φ and ψ are related by the following assumption: there exists $k_2 \ge 0$ such that

$$(H_3)\qquad \begin{cases} \text{either} & \text{(i) } [\varphi(u)]^2 \le k_2\,\psi'(u)[1+\Psi(u)] \quad \text{for all } u\in\mathbb{R}\,,\\ \text{or} & \text{(ii) } |\varphi'(u)| \le k_2|\psi'(u)| \quad \text{for all } u\in\mathbb{R}\,. \end{cases}$$

Generally speaking, (H_3) relates the behavior of φ to that of ψ. (H_3)(i) covers the prototypal cases (1.7) and (1.8): indeed, a straightforward computation shows that in these cases (H_3)(i) holds if $0<\sigma\le 2\alpha$, whereas (H_3)(ii) gives the stronger constraint $0<\sigma\le\alpha$. In fact, (H_3)(ii) is better suited (by appropriately choosing the regularization ψ) to be applied when $\varphi(-\infty)<\varphi(\infty)$.

Remark 3.1. The oddness assumption in (H_2)(i) is made only for convenience and could be omitted provided $\psi(0)=\psi''(0)=0$, which are used to treat essential boundary conditions.

3.2. Admissible initial data. We consider Radon measures "with finite energy" in the following sense.

DEFINITION 3.2. *Let* (H_1) *and* (H_2)(i) *hold. A measure* $u \in \mathcal{M}(\Omega)$ *has* finite energy *if there exists a sequence* $\{u_n\} \subset H^1(\Omega)$ *such that*

$$u_n \overset{*}{\rightharpoonup} u \qquad \text{in } \mathcal{M}(\Omega)\,, \tag{3.2}$$

$$\sup_n E[u_n] \le E_0 \quad \text{for some } E_0 > 0\,. \tag{3.3}$$

Note that by (3.2) we have

$$\sup_n \|u_n\|_{L^1(\Omega)} \le M_0 \quad \text{for some } M_0 > 0\,. \tag{3.4}$$

Measures with finite energy are characterized as follows.

PROPOSITION 3.3. *Assume* (H_1) *and* (H_2)(i). *A measure* $u \in \mathcal{M}(\Omega)$ *has finite energy if and only if* $u \in \mathcal{A}(\Omega)$, *where*

$$\mathcal{A}(\Omega) := \left\{u \in \mathcal{M}(\Omega) \mid \psi(u_r) \in H^1(\Omega),\ u_s^{\pm} = u_s^{\pm} \llcorner \mathcal{S}^{\pm}(\Omega)\right\},$$
$$\mathcal{S}^{\pm}(\Omega) := \left\{x \in \overline{\Omega} \mid \psi(u_r(x)) = \pm\gamma\right\}.$$

The proof of Proposition 3.3 is essentially contained in [15] (see section A.1), where it is also shown that if $u \in \mathcal{A}(\Omega)$, the sequence can be chosen such that $\{\psi(u_n)\}$ converges strongly to $\psi(u_r)$ in $H^1(\Omega)$.

Remark 3.4. It follows immediately from Proposition 3.3 that a Dirac mass is not an element of $\mathcal{A}(\Omega)$. However, if the degeneracy of ψ' is sufficiently strong, the set $\mathcal{A}(\Omega)$ contains the sum of a Dirac mass concentrated at any $x_0 \in \Omega$ with a nonnegative function having an integrable, yet suitably strong, singularity at x_0. In the prototypal case (1.9), Proposition 3.3 immediately implies that the measure $u = z + \delta_{x_0}$, $z(x) := |x - x_0|^{-\beta}$ belongs to $\mathcal{A}(\Omega)$ for any $\sigma > 1/2$ and any $\beta \in \left(\frac{1}{2\sigma}, 1\right)$.

3.3. The notion of solution. Let us first define local (in time and possibly in space) solutions of (1.5).

DEFINITION 3.5. *Let* $\Omega_0 \subseteq \Omega$ *be open, and let* $T \in (0,\infty)$. *A* local solution *of* (1.5) *in* $Q_{0T} := \Omega_0 \times (0,T)$ *with initial datum* $u_0 \in \mathcal{M}(\Omega_0)$ *is a pair* (u,v) *such that*

(i) $u \in L^\infty(0,T;\mathcal{M}(\Omega_0))$, $\psi(u_r) \in C(\overline{Q}_{0T}) \cap L^\infty(0,T;H^1(\Omega_0))$, *and* $[\psi(u_r)]_{xx} \in L^2_{loc}(\overline{Q}_{0T} \setminus \mathcal{S}(Q_{0T}))$, *where*

$$\mathcal{S}(Q_{0T}) := \left\{(x,t) \in \overline{Q}_{0T} \,\middle|\, |\psi(u_r)(x,t)| = \gamma\right\}; \tag{3.5}$$

(ii) $v \in L^2(0,T;H^1(\Omega_0))$ *and*

$$v = \varphi(u_r) - \varepsilon^2\psi'(u_r)[\psi(u_r)]_{xx} \quad \text{a.e. in } Q_{0T}\,; \tag{3.6}$$

(iii) $u(\cdot,t) \in \mathcal{A}(\Omega_0)$ *for a.e.* $t \in (0,T)$;

(iv) *for all* $\zeta \in C^1([0,T];C^1_c(\Omega_0))$ *such that* $\zeta(\cdot,T) = 0$ *in* Ω_0, *there holds*

$$\int_0^T \langle u(\cdot,t), \zeta_t(\cdot,t)\rangle_{\Omega_0}\,dt - \iint_{Q_{0T}} v_x\,\zeta_x\,dxdt = -\langle u_0, \zeta(\cdot,0)\rangle_{\Omega_0}\,. \tag{3.7}$$

Remark 3.6. By (i), $\psi(u_r)(\cdot,t)\in H^1(\Omega_0)$ for a.e. $t\in(0,T)$, so (iii) is equivalent to

$$u_s^{\pm}(\cdot,t)=u_s^{\pm}(\cdot,t)\llcorner\mathcal{S}_t^{\pm}(\Omega_0)\quad\text{for a.e. } t\in(0,T),\quad \mathcal{S}_t^{\pm}(\Omega_0):=\left\{x\in\overline{\Omega}_0\,|\,\psi(u_r)(x,t)=\pm\gamma\right\}. \tag{3.8}$$

Observe that the set $\mathcal{S}_t(\Omega_0):=\mathcal{S}_t^{+}(\Omega_0)\cup\mathcal{S}_t^{-}(\Omega_0)$ is the t-section of the set in (3.5):

$$\mathcal{S}_t(\Omega_0)=\left\{x\in\overline{\Omega}_0\,|\,|\psi(u_r)(x,t)|=\gamma\right\}=\left\{x\in\overline{\Omega}_0\,|\,(x,t)\in\mathcal{S}(Q_{0T})\right\}.$$

We now define global (in time and space) solutions to (1.5)–(1.6).

Definition 3.7. *Let $u_0\in\mathcal{M}(\Omega)$. By a* solution *of problem* (1.5)–(1.6) *we mean a pair (u,v) such that*

(i) *(u,v) is a local solution to* (1.5) *in $\Omega\times(0,T)$ for all $T>0$, $\psi(u_r)\in L^\infty(0,\infty;H^1(\Omega))$, and $v_x\in L^2(Q)$;*

(ii) *for all $T>0$,* (3.7) *holds with $\Omega_0=\Omega$ for all $\zeta\in C^1([0,T];C^1_*(\Omega))$ such that $\zeta(\cdot,T)=0$ in Ω;*

(iii)$_a$ *in case* (1.6a), *$[\psi(u_r)]_x=0$ on $\partial\Omega\times(0,\infty)$ in the sense that*

$$\iint_Q[\psi(u_r)]_{xx}\,\zeta\,dxdt=-\iint_Q[\psi(u_r)]_x\,\zeta_x\,dxdt \tag{3.9}$$

for all $\zeta\in C^1_c(\overline{Q}\setminus\mathcal{S}(Q))$;

(iii)$_b$ *in case* (1.6b), *$\psi(u_r)\in L^\infty(0,\infty;H^1_0(\Omega))$ and $v\in L^2_{loc}([0,\infty);H^1_0(\Omega))$.*

3.4. The existence result. The first main result of the paper is the following.

Theorem 3.8. *Assume (H_1)–(H_3). Then for every $u_0\in\mathcal{A}(\Omega)$ there exists a solution (u,v) of problem* (1.5)–(1.6).

The proof is based on an approximating procedure. For $u_0\in\mathcal{A}(\Omega)$, let $\{u_{0n}\}$ be a sequence given by Definition 3.2, and let

$$\psi_n(u):=\psi(u)+\frac{1}{\kappa_n}\quad(u\in\mathbb{R}),\quad\text{where}\quad\kappa_n:=\max\left\{n,\frac{1}{2}\|u_{0n}\|_{L^\infty(\Omega)}\right\} \tag{3.10}$$

(the choice of κ_n is motivated by the proof of Lemma 4.4). Clearly, $\psi_n\to\psi$ pointwise in $\mathbb{R}$. We consider the following *nondegenerate* approximating problems associated with (1.5)–(1.6):

$$u_t=v_{xx}\quad\text{in } Q,\qquad v:=\varphi(u)-\varepsilon^2\psi_n'(u)[\psi_n(u)]_{xx}, \tag{3.11}$$

with either natural or essential homogeneous boundary conditions,

$$u_x=v_x=0\quad\text{in }\partial\Omega\times(0,\infty),\qquad u=u_{0n}\quad\text{in }\Omega\times\{0\}, \tag{3.12a}$$

$$u=v=0\quad\text{in }\partial\Omega\times(0,\infty),\qquad u=u_{0n}\quad\text{in }\Omega\times\{0\}, \tag{3.12b}$$

respectively. Thanks to nondegeneracy, existence of strong solutions to (3.11)–(3.12) may be proved by a Galerkin-type argument (see Proposition 4.1), which is relatively standard in the framework of nonlinear higher-order equations (see, e.g., [2, 20, 22, 23]). Note, however, that the nonlinearity of the first-order term in the energy forces us to use a two-step argument, requiring a preliminary local existence result based on auxiliary interpolation estimates. Then Theorem 3.8 is proved by considering limiting points of the sequence $\{u_n\}$ in suitable topologies.

3.5. Formation and disappearance of singularities. The second main result shows that, in contrast to the pseudoparabolic regularization, the singular part of solutions to (1.5) may either appear for $t > 0$ (starting from an L^1-initial datum) or disappear in finite time. Let φ, ψ be such that

$$(H_4) \qquad \varphi(u) = u^{-\alpha}, \quad \psi(u) = \gamma - \frac{1}{\sigma} u^{-\sigma} \quad \text{for all } u \geq \bar{u} \quad \text{for some } \alpha, \sigma, \bar{u} > 0$$

(in this connection see (1.9)), and let δ_0 denote the Dirac mass at $x = 0$.

THEOREM 3.9. *Assume* (H_4)*, and let* $\alpha > 1$*,* $\sigma > \frac{1}{2}$*. For any* $t_0 > 0$ *and* $V \geq 0$ *there exist two four-parameter families of local solutions to* (1.5) *of the form*

$$u_1(x,t) = \tilde{u}(x) + V\delta_0(x)t\,, \quad t > 0, \tag{3.13}$$

$$u_2(x,t) = \tilde{u}(x) + V\delta_0(x)(t_0 - t)\,, \quad t \in (0, t_0), \tag{3.14}$$

with $x \in \Omega_0 := (-\xi, \xi)$ *for some* $\xi > 0$*. In both cases,* $\tilde{u} \in L^1(\Omega_0) \cap C^4(\Omega_0 \setminus \{0\})$ *and for some* $c \in \mathbb{R} \setminus \{0\}$ *there holds*

$$|x|^{2/(2\sigma+1)}\tilde{u}(x) \to c \quad \text{as } |x| \to 0\,.$$

For solutions of type (3.13) the singular part appears for $t > 0$, whereas (3.14) provides an example of a Dirac mass which disappears at time t_0. The number of free parameters is consistent with that of boundary conditions in (1.6); thus it is reasonable to expect that a local solution of (1.5) may be turned into a global solution of (1.5)–(1.6) by tuning these parameters. Based on our experience, however, there may be technical obstructions in achieving this goal by, e.g., a shooting argument, due to the lack of straightforward monotonicity properties in nonlinear higher-order ODEs. In any event, since the formation/disappearance of singularities is a local phenomenon, we prefer to put the issue aside in this first investigation. Finally, note that in (3.13) $u(\cdot, 0) = \tilde{u}$ is unbounded: we do not know whether there exist *bounded* initial data for which the solution develops a nontrivial singular part within finite time.

4. A priori estimates. According to (3.10), we define *approximating energies*

$$E_n[u(\cdot,t)] := \int_\Omega \left(\Phi(u) + \frac{\varepsilon^2}{2} \left[\psi_n(u)\right]_x^2 \right)(x,t)\,dx\,. \tag{4.1}$$

In this section and the following one we assume that the initial datum u_0 is a measure with finite energy.

In the appendix, a general existence result for problem (1.5)–(1.6) under a non-degeneracy assumption on ψ is proven (see Theorem A.2). This result applies in particular to problem (3.11)–(3.12) and provides the following starting point of our analysis.

PROPOSITION 4.1. *Assume* (H_1)*–*(H_2)*. For* $u_0 \in \mathcal{A}(\Omega)$*, let* $\{u_{0n}\}$ *be as in Definition* 3.2*. Let* $\kappa_n \geq 1$ *and* ψ_n *be defined by* (3.10) $(n \in \mathbb{N})$*. Then there exists a strong solution* u_n *to* (3.11)*–*(3.12)*, in the sense that for all* $T > 0$ *there holds* $u_n \in L^2(0,T; H^3_*(\Omega)) \cap C([0,T]; H^1_*(\Omega))$*,* $u_{nt} \in L^2(0,T; \left(H^1_*(\Omega)\right)')$*,* $u_n(\cdot, 0) = u_{0n}$*,*

$$\int_0^T \langle u_{nt}, \zeta \rangle_* dt = -\iint_{Q_T} v_{nx}\zeta_x\,dxdt \quad \textit{for all } \zeta \in L^2(0,T; H^1_*(\Omega)), \tag{4.2}$$

where

$$v_n := \varphi(u_n) - \varepsilon^2 \psi_n'(u_n)[\psi_n(u_n)]_{xx}, \tag{4.3}$$

and

$$E_n[u_n(\cdot,t_2)] + \int_{t_1}^{t_2}\int_\Omega v_{nx}^2\,dxdt = E_n[u_n(\cdot,t_1)] \quad \textit{if } 0 \le t_1 < t_2 < \infty. \tag{4.4}$$

Our goal is to prove a few a priori estimates on strong solutions of problem (3.11)–(3.12). Throughout the section $C_T > 0$ (resp., $C > 0$) denotes a generic positive constant independent of n (resp., of n and T).

LEMMA 4.2. *Let $\{u_n\}$ be given by Proposition* 4.1. *Then the following holds:*

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} E_n[u_n(\cdot,t)] + \iint_Q v_{nx}^2\,dxdt \le C\,, \tag{4.5}$$

$$\int_\Omega u_n(x,t)\,dx = \int_\Omega u_{0n}(x)\,dx \quad \textit{for all } t>0 \textit{ in case } (3.12a), \tag{4.6}$$

$$\|u_{nt}\|^2_{L^2(0,\infty;(H^1_*(\Omega))')} \le C\,. \tag{4.7}$$

Proof. Inequality (4.5) follows from (4.4) and (3.3). In case (3.12a), (4.6) follows immediately from (4.2). Concerning (4.7), from (4.2) for a.e. $t \in (0,\infty)$ we get

$$|\langle u_{nt}(\cdot,t),\zeta\rangle_*| \le \left(\int_\Omega v_{nx}^2(x,t)\,dx\right)^{1/2} \|\zeta\|_{H^1_*(\Omega)} \quad \text{for all } \zeta \in H^1_*(\Omega),$$

and thus (4.7) follows from (4.5). ∎

The following lemma shows that estimate (4.5), together with (4.6) or (3.12b), implies boundedness and equicontinuity of the sequence $\{\psi_n(u_n)\}$. Here assumption (H_2)(ii) plays an essential role.

LEMMA 4.3. *Let $\{u_n\}$ be given by Proposition* 4.1. *Then*
(i) *there holds*

$$\|\psi_n(u_n)\|_{L^\infty(Q)} \le C\,; \tag{4.8}$$

(ii) *for all $t \in (0,\infty)$, $x_1, x_2 \in \Omega$,*

$$|\psi_n(u_n)(x_1,t) - \psi_n(u_n)(x_2,t)| \le C\,|x_1-x_2|^{1/2}\,; \tag{4.9}$$

(iii) *for all $x \in \Omega$, $0 \le t_1 \le t_2 < \infty$,*

$$|\psi_n(u_n)(x,t_1) - \psi_n(u_n)(x,t_2)| \le C[(t_2-t_1)^{1/8} + (t_2-t_1)^{1/6}]\,. \tag{4.10}$$

In particular, the sequence $\{\psi_n(u_n)\}$ is bounded and equicontinuous in $\overline{Q}$.

Proof. Inequality (4.9) follows from (4.5) and the definition of E_n (see (4.1)). In case (3.12b), (4.8) follows immediately from (4.9) and the equality $\psi_n(u_n(\cdot,t)) = 0$ on $\partial\Omega$. In case (3.12a), by (3.10), (4.6), and (3.4), we obtain

$$\begin{aligned}\left|\int_\Omega \psi_n(u_n(x,t))\,dx\right| &\le \left|\int_\Omega \psi(u_n(x,t))\,dx\right| + \frac{1}{\kappa_n}\left|\int_\Omega u_n(x,t)\,dx\right| \\ &= \left|\int_\Omega \psi(u_n)(x,t)\,dx\right| + \frac{1}{\kappa_n}\left|\int_\Omega u_{0n}(x)\,dx\right| \le \gamma\,|\Omega| + M_0.\end{aligned}$$

Hence the mean value of $\psi_n(u_n(\cdot,t))$ is uniformly controlled: together with (4.9), this implies (4.8). To prove (4.10), choose a regularizing family $\rho_\eta(x) = \frac{1}{\eta}\rho\left(\frac{x}{\eta}\right)$ ($x \in \mathbb{R}, \eta > 0$), where $\rho \in C_c^\infty(\mathbb{R})$ is a standard mollifier with unit mass. Set

$$\psi_\eta^n(x,t) := \int_\Omega \rho_\eta(x-y)[\psi_n(u_n)](y,t)\,dy\,.$$

Then for every fixed $\bar{x} \in \Omega$ and $t_1, t_2 \in (0,T)$, $t_1 \le t_2$, there holds

$$\begin{aligned}|\psi_n(u_n)(\bar{x},t_1) - \psi_n(u_n)(\bar{x},t_2)| &\le |\psi_n(u_n)(\bar{x},t_1) - \psi_\eta^n(\bar{x},t_1)| \\ &+ |\psi_\eta^n(\bar{x},t_1) - \psi_\eta^n(\bar{x},t_2)| + |\psi_\eta^n(\bar{x},t_2) - \psi_n(u_n)(\bar{x},t_2)|\,.\end{aligned} \tag{4.11}$$

By (4.9), there exists $C > 0$ such that for all $\overline{x}$, t_1, t_2 as above, $\eta > 0$, and $i = 1,2$, there holds

$$\begin{aligned}|\psi_n(u_n)(\bar{x},t_i) - \psi_\eta^n(\bar{x},t_i)| &\le \int_\Omega \rho_\eta(\bar{x}-y)|\psi_n(u_n)(\bar{x},t_i) - \psi_n(u_n)(y,t_i)|\,dy \\ &\le C\int_\Omega \rho_\eta(\bar{x}-y)\,|\bar{x}-y|^{1/2}\,dy \le C\,\eta^{1/2}.\end{aligned} \tag{4.12}$$

To estimate the second term in the right-hand side of (4.11) observe that, by assumption (H_2)(ii),

$$|[\psi_n'(u_n)]_x| = |\psi''(u_n)\,u_{nx}| \le k_1\,\psi'(u_n)|u_{nx}| \le k_1\,|[\psi_n(u_n)]_x|\,.$$

Using (4.2) and the above inequality we get

$$\begin{aligned}|\psi_\eta^n(\bar{x},t_1) - \psi_\eta^n(\bar{x},t_2)| &= \left|\int_\Omega \rho_\eta(\bar{x}-y)[\psi_n(u_n)(y,t_1) - \psi_n(u_n)(y,t_2)]\,dy\right| \\ &= \left|\int_{t_1}^{t_2} \langle u_{nt}(\cdot,t), \psi_n'(u_n)(\cdot,t)\rho_\eta(\bar{x}-\cdot)\rangle_*\,dt\right| \\ &= \left|\int_{t_1}^{t_2}\int_\Omega v_{nx}(x,t)\,\{\psi_n'(u_n)(x,t)\,\rho_\eta(\bar{x}-x)\}_x\,dxdt\right| \\ &\le k_1\int_{t_1}^{t_2}\int_\Omega |v_{nx}\,[\psi_n(u_n)]_x|\,(x,t)\,\rho_\eta(\bar{x}-x)\,dxdt \\ &\quad + \int_{t_1}^{t_2}\int_\Omega [|v_{nx}|\,\psi_n'(u_n)](x,t)\,\left|\rho_\eta'(\bar{x}-x)\right|dxdt =: k_1\,I_{1n} + I_{2n}.\end{aligned} \tag{4.13}$$

It follows from (4.5) that

$$\begin{aligned}I_{n1} &\le \frac{\|\rho\|_{L^\infty(\mathbb{R})}}{\eta}\int_{t_1}^{t_2} \|v_{nx}(\cdot,t)\|_{L^2(\Omega)}\|\left[\psi_n(u_n)\right]_x(\cdot,t)\|_{L^2(\Omega)}\,dt \\ &\le \frac{\sqrt{2}\,\|\rho\|_{L^\infty(\mathbb{R})}}{\varepsilon\,\eta}\left(\operatorname*{ess\,sup}_{t\in(0,\infty)} E_n[u_n(\cdot,t)]\right)^{1/2}\left(\int_0^\infty \|v_{nx}(\cdot,t)\|_{L^2(\Omega)}^2\,dt\right)^{1/2}(t_2-t_1)^{1/2} \\ &\le \frac{\sqrt{2}\,C\,\|\rho\|_{L^\infty(\mathbb{R})}}{\varepsilon\,\eta}\,(t_2-t_1)^{1/2}\,.\end{aligned}$$

Moreover, by (4.5) there holds

$$
\begin{aligned}
I_{n2} &\le (\|\psi'\|_{L^\infty(\mathbb{R})}+1)\int_{t_1}^{t_2}\int_\Omega |v_{nx}|(x,t)\left|\rho'_\eta(\bar{x}-x)\right| dxdt \\
&\le \frac{(\|\psi'\|_{L^\infty(\mathbb{R})}+1)\|\rho'\|_{L^\infty(\mathbb{R})}}{\eta^{3/2}}\int_{t_1}^{t_2}\|v_{nx}(\cdot,t)\|_{L^2(\Omega)}\,dt \\
&\le \frac{(\|\psi'\|_{L^\infty(\mathbb{R})}+1)\|\rho'\|_{L^\infty(\mathbb{R})}}{\eta^{3/2}}\left(\int_0^\infty \|v_{nx}(\cdot,t)\|^2_{L^2(\Omega)}\,dt\right)^{1/2}(t_2-t_1)^{1/2} \\
&\le \frac{(\|\psi'\|_{L^\infty(\mathbb{R})}+1)\,C^{1/2}\|\,\rho'\|_{L^\infty(\mathbb{R})}}{\eta^{3/2}}\,(t_2-t_1)^{1/2}\,.
\end{aligned} \tag{4.14}
$$

By (4.11)–(4.14) there exists $K>0$ such that

$$
|\psi_n(u_n)(\bar{x},t_1)-\psi_n(u_n)(\bar{x},t_2)| \le K\left[\eta^{1/2}+\frac{(t_2-t_1)^{1/2}}{\eta}\left(1+\frac{1}{\eta^{1/2}}\right)\right] \tag{4.15}
$$

for all $\bar{x}$, t_1, t_2 as above and $\eta>0$. It is easily seen that minimizing the right-hand side of (4.15) with respect to η in $(0,\infty)$ gives (4.10) (cf. subsection A.3). Hence the result follows. ∎

Let

$$
\Psi_n(u) := \int_0^u \psi_n(s)\,ds = \Psi(u)+\frac{u^2}{2\kappa_n} \qquad (u\in\mathbb{R})\,. \tag{4.16}
$$

The next entropy-type estimate is the base for obtaining uniform controls on the norms of u_n and v_n. Here the choice of κ_n and assumption (H_3) are essential ingredients in the proof.

LEMMA 4.4. *Let $\{u_n\}$ be given by Proposition 4.1. For any $T>0$ there exists $C_T>0$ such that for all $n\in\mathbb{N}$,*

$$
\sup_{t\in(0,T)}\int_\Omega \Psi_n(u_n(x,t))\,dx + \iint_{Q_T}\left\{\psi'_n(u_n)[\psi_n(u_n)]^2_{xx}\right\}dxdt \le C_T. \tag{4.17}
$$

Proof. First, by (4.16) and the definition of κ_n (cf. (3.10)) we have

$$
|\Psi_n(u_{0n})| = \Psi_n(|u_{0n}|) \le \gamma|u_{0n}|+\frac{u_{0n}^2}{2\kappa_n} \le (\gamma+1)|u_{0n}|\,,
$$

whence by (3.4),

$$
\|\Psi_n(u_{0n})\|_{L^1(\Omega)} \le (\gamma+1)M_0. \tag{4.18}
$$

Choosing $\psi_n(u_n)(\cdot,t)$ as test functions in (4.2), we have

$$
I_n(t) := \int_\Omega [\Psi_n(u_n(x,t))-\Psi_n(u_{0n}(x))]\,dx = -\iint_{Q_t} v_{nx}[\psi_n(u_n)]_x\,dxds\,.
$$

In view of the boundary conditions $\psi_n(u_n)_x=0$ or $v_n=0$ on $\partial\Omega$, we may integrate by parts once more, obtaining

$$
I_n(t) = \iint_{Q_t}\varphi(u_n)[\psi_n(u_n)]_{xx}\,dxds-\varepsilon^2\iint_{Q_t}\psi'_n(u_n)[\psi_n(u_n)]^2_{xx}\,dxds\,. \tag{4.19}
$$

We now distinguish two cases.

(i) If (H_3)(i) holds, by the definition of ψ_n, Ψ_n we have

$$[\varphi(u)]^2 \le k_2\psi'(u)\,[1+\Psi(u)] \le k_2\psi_n'(u)\,[1+\Psi_n(u)] \quad \text{for all } u\in\mathbb{R},\ n\in\mathbb{N}\,. \tag{4.20}$$

On the other hand, by Young's inequality,

$$\iint_{Q_t}\varphi(u_n)[\psi_n(u_n)]_{xx}\,dxds \le \frac{1}{2\varepsilon^2}\iint_{Q_t}\frac{[\varphi(u_n)]^2}{\psi_n'(u_n)}dxds + \frac{\varepsilon^2}{2}\iint_{Q_t}\psi_n'(u_n)[\psi_n(u_n)]_{xx}^2 dxds,$$

whence by (4.19),

$$I_n(t) \le \frac{1}{2\varepsilon^2}\iint_{Q_t}\frac{[\varphi(u_n)]^2}{\psi_n'(u_n)}\,dxds - \frac{\varepsilon^2}{2}\iint_{Q_t}\psi_n'(u_n)[\psi_n(u_n)]_{xx}^2\,dxds\,. \tag{4.21}$$

Therefore, by (4.20)–(4.21),

$$I_n(t) \le \frac{k_2}{2\varepsilon^2}\iint_{Q_t}[1+\Psi_n(u_n)]\,dxds - \frac{\varepsilon^2}{2}\iint_{Q_t}\psi_n'(u_n)[\psi_n(u_n)]_{xx}^2\,dxds\,,$$

whence the conclusion follows by a Gronwall argument and inequality (4.18).

(ii) If (H_3)(ii) holds, then in view of the boundary conditions $\psi_n(u_n)_x = 0$ or $\varphi(u_n) = 0$ on $\partial\Omega$ we may integrate by parts the first integral in (4.19), obtaining

$$\begin{aligned} I_n(t) &= -\iint_{Q_t}[\varphi(u_n)]_x[\psi_n(u_n)]_x\,dxds - \varepsilon^2\iint_{Q_t}\psi_n'(u_n)[\psi_n(u_n)]_{xx}^2\,dxds \\ &\le k_2\iint_{Q_t}[\psi_n(u_n)]_x^2\,dxds - \varepsilon^2\iint_{Q_t}\psi_n'(u_n)[\psi_n(u_n)]_{xx}^2\,dxds\,, \end{aligned}$$

whence the conclusion follows by (4.5) and (4.18). □

Lemmas 4.2 and 4.4 combine into a few additional a priori bounds.

LEMMA 4.5. *Let $\{u_n\}$ be given by Proposition* 4.1. *For any $T>0$ there exists $C_T>0$ such that for all $n\in\mathbb{N}$,*

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \le C_T\,, \tag{4.22}$$

$$\|v_n\|_{L^2(0,T;H^1_*(\Omega))} \le C_T\,. \tag{4.23}$$

Proof. To prove (4.22), let $\bar u_n := \psi_n^{-1}(\gamma/2)$. Then

$$\Psi_n(u) - \Psi_n(\bar u_n) = \int_{\bar u_n}^u \psi_n(s)ds \ge \frac{\gamma}{2}(u-\bar u_n) \quad \text{for all } u\ge\bar u_n\,,$$

whence

$$|u| \le \bar u_n + \frac{2}{\gamma}\Psi_n(u) \quad \text{for all } |u|\ge\bar u_n\,.$$

Therefore, since $0<\bar u_n<\psi^{-1}\left(\frac{\gamma}{2}\right)$ for all $n\in\mathbb{N}$,

$$\int_\Omega |u_n(x,t)|dx \le \bar u_n\,|\Omega| + \frac{2}{\gamma}\int_\Omega\Psi_n(u_n(x,t))dx \le \psi^{-1}\left(\frac{\gamma}{2}\right)|\Omega| + \frac{2}{\gamma}\int_\Omega\Psi_n(u_n(x,t))dx,$$

and the claim follows from (4.17). Concerning (4.23), by (4.17) for any $T>0$ we have

$$\begin{aligned} \|v_n\|^2_{L^2(Q_T)} &\le 2\|\varphi(u_n)\|^2_{L^2(Q_T)} + 2\varepsilon^4\|\psi_n'(u_n)[\psi_n(u_n)]_{xx}\|^2_{L^2(Q_T)} \\ &\le 2\|\varphi\|^2_{L^\infty(\mathbb{R})}|Q_T| + 2\varepsilon^4\left(\|\psi'\|_{L^\infty(\mathbb{R})}+1\right)\iint_{Q_T}\psi_n'(u_n)[\psi_n(u_n)]_{xx}^2\,dxdt \le C_T\,. \end{aligned}$$

Combined with (4.5), the above estimate yields (4.23). □

5. Proof of existence. We first collect some preliminary results about the limiting points of the sequence $\{u_n\}$ introduced in Proposition 4.1. From the previous uniform estimates we obtain the following.

PROPOSITION 5.1. *Let $\{u_n\}$ be given by Proposition 4.1. Then there exist a subsequence of $\{u_n\}$ (not relabeled), $u^{(1)}, u^{(2)} \in L^\infty_{loc}([0,\infty); \mathcal{M}^+(\Omega))$, and $v \in L^2_{loc}([0,\infty); H^1_*(\Omega))$ with $v_x \in L^2(Q)$, such that*

$$u_n^+ \overset{*}{\rightharpoonup} u^{(1)}, \quad u_n^- \overset{*}{\rightharpoonup} u^{(2)}, \quad u_n \overset{*}{\rightharpoonup} u := u^{(1)} - u^{(2)} \quad \text{in } \mathcal{M}(Q), \tag{5.1}$$

$$u_{nt} \rightharpoonup u_t \ \text{ in } \ L^2(0,\infty; (H^1_*(\Omega))'), \tag{5.2}$$

$$v_n \rightharpoonup v \ \text{ in } \ L^2_{loc}([0,\infty); H^1_*(\Omega)). \tag{5.3}$$

Proof. Fix any $T_1 \in (0,\infty)$. By inequality (4.22) the sequence $\{u_n\}$ is bounded in $L^1(Q_{T_1})$, and thus the same holds for the sequences $\{u_n^\pm\}$. Hence there exist a subsequence $\{u_k\} \equiv \{u_{n_k}\} \subseteq \{u_n\}$ and $u_1^{(1)}, u_1^{(2)} \in \mathcal{M}^+(Q_{T_1})$ such that

$$u_k^+ \overset{*}{\rightharpoonup} u_1^{(1)}, \quad u_k^- \overset{*}{\rightharpoonup} u_1^{(2)} \quad \text{in } \mathcal{M}(Q_{T_1}). \tag{5.4}$$

Arguing as in [29, Proposition 5.3] one proves that $u_1^{(1)}, u_1^{(2)} \in L^\infty(0, T_1; \mathcal{M}^+(\Omega))$.

Further, let $T_2 \in (T_1, \infty)$. By (4.22) there exist a subsequence $\{u_j\} \equiv \{u_{k_j}\} \subseteq \{u_k\}$ and $u_2^{(1)}, u_2^{(2)} \in \mathcal{M}^+(Q_{T_2})$ such that

$$u_j^+ \overset{*}{\rightharpoonup} u_2^{(1)}, \quad u_j^- \overset{*}{\rightharpoonup} u_2^{(2)} \quad \text{in } \mathcal{M}(Q_{T_2}). \tag{5.5}$$

By (5.4)–(5.5) there holds $u_1^{(1)} = u_2^{(1)}$, $u_1^{(2)} = u_2^{(2)}$ in $\mathcal{M}(Q_{T_1})$. As before, one proves that $u_2^{(1)}, u_2^{(2)} \in L^\infty(0, T_2; \mathcal{M}^+(\Omega))$. Iterating the procedure, by a diagonal argument there exist a subsequence of $\{u_n\}$ (not relabeled for simplicity) and $u^{(1)}, u^{(2)} \in L^\infty_{loc}([0,\infty); \mathcal{M}^+(\Omega))$ such that

$$u_n^+ \overset{*}{\rightharpoonup} u^{(1)}, \quad u_n^- \overset{*}{\rightharpoonup} u^{(2)} \quad \Rightarrow \quad u_n \overset{*}{\rightharpoonup} u := u^{(1)} - u^{(2)} \quad \text{in } \mathcal{M}(Q_T) \text{ for all } T > 0.$$

Then $u \in L^\infty_{loc}([0,\infty); \mathcal{M}(\Omega))$ and the limits in (5.1) hold.

To prove (5.2) observe that, by (4.7), there exist a subsequence of $\{u_{nt}\}$ and $z \in L^2(0,\infty; (H^1_*(\Omega))')$ such that

$$u_{nt} \rightharpoonup z \ \text{ in } \ L^2(0,\infty; (H^1_*(\Omega))'). \tag{5.6}$$

Plainly, by (5.1) and (5.6) there holds $z = u_t$, and thus (5.2) follows. The convergence in (5.3) follows similarly from (4.23). ∎

PROPOSITION 5.2. *Let $\{u_n\}$ be as in Proposition 5.1. Then there exist a subsequence (not relabeled) and a function $w \in C(\overline{Q}) \cap L^\infty(0,\infty; H^1_*(\Omega))$ such that*

$$\psi_n(u_n) \to w \ \ \text{uniformly in } \overline{Q}, \tag{5.7}$$

$$\psi_n(u_n) \overset{*}{\rightharpoonup} w \ \ \text{in } L^\infty(0,\infty; H^1(\Omega)), \tag{5.8}$$

$$|w| \le \gamma \qquad \text{in } Q. \tag{5.9}$$

Proof. The sequence $\{\psi_n(u_n)\}$ is bounded and equicontinuous in $\overline{Q}$ by Lemma 4.3, and hence the convergence in (5.7) follows immediately. By (4.5) and (4.8) there exists $C > 0$ such that for all $n \in \mathbb{N}$,

$$\|\psi_n(u_n)\|_{L^\infty(0,\infty; H^1_*(\Omega))} \le C,$$

and thus it is easily seen that $w \in L^\infty(0,\infty;H^1_*(\Omega))$ and (5.8) holds. By (4.22), for any $T>0$ there exists $C_T>0$ such that for all $n \in \mathbb{N}$,

$$\|\psi(u_n)-\psi_n(u_n)\|_{L^\infty(0,T;L^1(\Omega))} \le \frac{1}{n}\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \le \frac{C_T}{n},$$

whence by (5.7),

$$\psi(u_n) \to w \text{ in } L^\infty(0,T;L^1(\Omega)).$$

Hence (possibly up to a subsequence) there holds $\psi(u_n) \to w$ a.e. in Q_T. Since $|\psi(u)| \le \gamma$ for all $u \in \mathbb{R}$, inequality (5.9) follows. ∎

PROPOSITION 5.3. *Let $\{u_n\}$ and w be as in Proposition 5.2, and let*

$$\tilde{\mathcal{S}}^\pm := \left\{(x,t) \in \overline{Q} \,\middle|\, w(x,t) = \pm\gamma\right\}, \quad \tilde{\mathcal{S}} := \tilde{\mathcal{S}}^+ \cup \tilde{\mathcal{S}}^-. \tag{5.10}$$

Then $\tilde{\mathcal{S}}$ has zero Lebesgue measure and

$$u_n^\pm \to \left[\psi^{-1}(w)\right]^\pm \text{ uniformly in every compact } K \subset \overline{Q} \setminus \tilde{\mathcal{S}}^\pm. \tag{5.11}$$

Proof. Let $T>0$ be fixed. For any $\eta \in (0,\gamma/2)$ set

$$\tilde{\mathcal{S}}_{\eta,T} := \left\{(x,t) \in \overline{Q}_T \,\middle|\, |w(x,t)| > \gamma - \eta\right\}.$$

Then $\tilde{\mathcal{S}} \cap \overline{Q}_T \subseteq \tilde{\mathcal{S}}_{\eta,T}$, and by (5.7) $|\psi_n(u_n)| \ge \gamma - 2\eta$ in $\tilde{\mathcal{S}}_{\eta,T}$ for all n sufficiently large. Observe that

$$|\psi_n(u_n(x,t))| \ge \gamma - 2\eta \quad \Leftrightarrow \quad |u_n(x,t)| \ge \psi_n^{-1}(\gamma-2\eta)$$

since the function ψ_n is strictly increasing and odd. Hence by inequality (4.22),

$$|\tilde{\mathcal{S}} \cap \overline{Q}_T| \cdot \psi_n^{-1}(\gamma-2\eta) \le |\tilde{\mathcal{S}}_{\eta,T}| \cdot \psi_n^{-1}(\gamma-2\eta) \le \iint_{Q_T} |u_n(x,t)|dxdt \le C_T\,T,$$

whence letting $n \to \infty$ we get

$$|\tilde{\mathcal{S}} \cap \overline{Q}_T| \le \frac{C_T\,T}{\psi^{-1}(\gamma-2\eta)}.$$

Letting $\eta \to 0$ in the above inequality proves that $|\tilde{\mathcal{S}} \cap \overline{Q}_T| = 0$ for all $T>0$, and thus $|\tilde{\mathcal{S}}| = 0$ by the arbitrariness of T.

To prove (5.11) fix any compact subset $K \subset \overline{Q} \setminus \tilde{\mathcal{S}}^+$ (observe that the sets $\tilde{\mathcal{S}}^\pm$, $\tilde{\mathcal{S}}$ are closed since w is continuous). By (5.7), (5.9), and (5.10) there exists $\eta \in (0,\gamma)$ such that $\psi_n(u_n) \le \gamma - \eta$ in K for any $n \in \mathbb{N}$ sufficiently large. For every such n there holds

$$0 \le \psi(u_n^+) = \psi_n(u_n^+) - \frac{u_n^+}{\kappa_n} \le \gamma - \eta \text{ in } K,$$

whence $u_n^+ \le \psi^{-1}(\gamma-\eta)$ in K.

By assumption (H_2)(i) there holds

$$m := \min_{|u| \le \psi^{-1}(\gamma-\eta)} \psi'(u) > 0. \tag{5.12}$$

Therefore, since ψ is odd, we get

$$m\left\|u_n^+ - \left[\psi^{-1}(w)\right]^+\right\|_{C(K)} \le \|\psi(u_n^+) - w^+\|_{C(K)} \le \left\|[\psi_n(u_n)]^+ - w^+\right\|_{C(\overline{Q})} + \frac{\psi^{-1}(\gamma-\eta)}{n}.$$

By (5.7), letting $n \to \infty$ in the above inequality we obtain the claim with "+" in (5.11). The claim with "−" is proven similarly. ∎

PROPOSITION 5.4. *Let u, w, and $\tilde{\mathcal{S}}^\pm$ be as in Propositions* 5.1–5.3. *Then*

$$u_s^\pm = u_s^\pm \llcorner \tilde{\mathcal{S}}^\pm, \qquad u_r = [\psi^{-1}(w)] \quad a.e.\ in\ Q. \tag{5.13}$$

Proof. By (5.1) and (5.11), for any $\zeta \in C_c(\overline{Q} \setminus \tilde{\mathcal{S}}^+)$,

$$< u^{(1)}, \zeta >_Q = \lim_{n\to\infty} < u_n^+, \zeta >_Q = \lim_{n\to\infty} \iint_Q u_n^+ \zeta\, dxdt = \iint_Q \left[\psi^{-1}(w)\right]^+ \zeta\, dxdt,$$

namely,

$$< u_s^{(1)}, \zeta >_Q = \iint_Q \left\{\left[\psi^{-1}(w)\right]^+ - u_r^{(1)}\right\} \zeta\, dxdt.$$

By the above equality, for any compact subset $K \subset \overline{Q} \setminus \tilde{\mathcal{S}}^+$ the measure $u_s^{(1)} \llcorner K$ is absolutely continuous with respect to the Lebesgue measure, and thus $u_s^{(1)} \llcorner K = 0$. Therefore,

$$< u_s^{(1)}, \zeta >_Q = \iint_Q \left\{\left[\psi^{-1}(w)\right]^+ - u_r^{(1)}\right\} \zeta\, dxdt = 0$$

for any ζ as above. Hence we obtain

$$u_s^{(1)} = u_s^{(1)} \llcorner \tilde{\mathcal{S}}^+, \quad u_r^{(1)} = \left[\psi^{-1}(w)\right]^+ \quad \text{a.e. in } Q$$

since $|\tilde{\mathcal{S}}| = 0$ (see Proposition 5.3). It is similarly seen that

$$u_s^{(2)} = u_s^{(2)} \llcorner \tilde{\mathcal{S}}^-, \quad u_r^{(2)} = \left[\psi^{-1}(w)\right]^- \quad \text{a.e. in } Q,$$

with $u^{(2)}$ as in (5.1). Then it holds that $u_r = \psi^{-1}(w)$ a.e. in Q by Proposition 5.1 and $u_s^{(1)}(\overline{Q} \setminus \tilde{\mathcal{S}}^+) = u_s^{(2)}(\overline{Q} \setminus \tilde{\mathcal{S}}^-) = 0$. By the minimality of the Jordan decomposition, it follows that $u_s^\pm(\overline{Q} \setminus \tilde{\mathcal{S}}^\pm) = 0$, and thus $u_s^\pm(F \setminus \tilde{\mathcal{S}}^\pm) = 0$ for every Borel set $F \subseteq \overline{Q}$. Hence the result follows. ∎

Remark 5.5. In view of the second equality in (5.13), hereafter we identify $w \in C(\overline{Q})$ with the continuous representative of $\psi(u_r)$.

PROPOSITION 5.6. *Let $\{u_n\}$, u, v, w, and $\tilde{\mathcal{S}}$ be as in Propositions* 5.1–5.3. *Then*

(i) $u_r \in C(\overline{Q} \setminus \tilde{\mathcal{S}})$ *and*

$$u_n \to u_r \text{ uniformly in every compact } K \subset \overline{Q} \setminus \tilde{\mathcal{S}}; \tag{5.14}$$

(ii) $w_{xx} \in L^2_{loc}(\overline{Q} \setminus \tilde{\mathcal{S}})$,

$$[\psi_n(u_n)]_{xx} \rightharpoonup w_{xx} \text{ in } L^2_{loc}(\overline{Q} \setminus \tilde{\mathcal{S}}), \tag{5.15}$$

and in case (1.6a), *for all* $\zeta \in C_c^1(\overline{Q} \setminus \tilde{\mathcal{S}})$,

$$\iint_Q w_{xx}\, \zeta\, dxdt = -\iint_Q w_x\, \zeta_x\, dxdt; \tag{5.16}$$

(iii) *there holds*

$$v = \varphi(u_r) - \varepsilon^2 \psi'(u_r)[\psi(u_r)]_{xx} \quad a.e.\ in\ Q. \tag{5.17}$$

Proof. The convergence in (5.14) and the claim that $u_r \in C(\overline{Q} \setminus \tilde{\mathcal{S}})$ follow from (5.11) and Remark 5.5.

To prove (5.15), fix any compact subset $K \subset \overline{Q} \setminus \tilde{\mathcal{S}}$, and observe that $K \subset \overline{Q}_T \setminus \tilde{\mathcal{S}}$ for some $T > 0$. Arguing as in the proof of (5.11) and using (4.17), there exists $C_K > 0$ such that for all $n \in \mathbb{N}$ sufficiently large,

$$\iint_K [\psi_n(u_n)]_{xx}^2 \, dxdt \le \frac{1}{m} \iint_K \psi_n'(u_n)[\psi_n(u_n)]_{xx}^2 \, dxdt \le C_K \,,$$

with m given by (5.12). Hence there exist a subsequence $\{u_{n_k}\} \subseteq \{u_n\}$ and $h \in L^2(K)$ such that

$$[\psi_{n_k}(u_{n_k})]_{xx} \rightharpoonup h \quad \text{in } L^2(K) \text{ as } k \to \infty \,.$$

Together with (5.7), this implies that

$$\begin{aligned} \iint_K h\,\zeta \, dxdt &= \lim_{k\to\infty} \iint_K [\psi_{n_k}(u_{n_k})]_{xx}\,\zeta \, dxdt = \lim_{k\to\infty} \iint_K \psi_{n_k}(u_{n_k})\,\zeta_{xx} \, dxdt \\ &= \iint_K w\,\zeta_{xx}\, dxdt \quad \text{for any } \zeta \in C_c^2(K). \end{aligned}$$

Therefore $h = w_{xx} \in L^2(K)$, whence $w_{xx} \in L^2_{loc}(\overline{Q} \setminus \tilde{\mathcal{S}})$ by the arbitrariness of K, and

$$[\psi_{n_k}(u_{n_k})]_{xx} \rightharpoonup w_{xx} \ \text{ in } L^2(K) \,.$$

Since the above limit is the same for every convergent subsequence of $\{[\psi_n(u_n)]_{xx}\}$, the whole sequence weakly converges to w_{xx} in $L^2(K)$, and (5.15) follows by the arbitrariness of K. Also, in case (1.6a) by (5.8) and (5.15) there holds

$$\begin{aligned} &\iint_Q w_{xx}\,\zeta \, dxdt = \lim_{n\to\infty} \iint_Q [\psi_n(u_n)]_{xx}\,\zeta \, dxdt \\ &= -\lim_{n\to\infty} \iint_Q [\psi_n(u_n)]_x\,\zeta_x \, dxdt = -\iint_Q w_x\,\zeta_x \, dxdt \quad \text{for any } \zeta \in C_c^1(\overline{Q} \setminus \tilde{\mathcal{S}}) \end{aligned}$$

since $\psi_n(u_n)(\cdot,t) \in H^3_*(\Omega)$ for a.e. $t \in (0,T)$ (see Proposition 4.1). This proves (5.16).

Finally, let us prove (5.17). Since $|\tilde{\mathcal{S}}| = 0$, by (5.14) and the regularity of φ,

$$\varphi(u_n) \to \varphi(u_r) \quad \text{uniformly in every compact } K \subset \overline{Q} \setminus \tilde{\mathcal{S}}. \tag{5.18}$$

By (5.3) and (5.18) we get

$$-\varepsilon^2 \psi_n'(u_n)[\psi_n(u_n)]_{xx} = v_n - \varphi(u_n) \rightharpoonup v - \varphi(u_r) \ \text{ in } L^2_{loc}(\overline{Q} \setminus \tilde{\mathcal{S}}) \,. \tag{5.19}$$

On the other hand, by (5.14)–(5.15) there holds

$$\psi_n'(u_n)[\psi_n(u_n)]_{xx} \rightharpoonup \psi'(u_r)[\psi(u_r)]_{xx} \quad \text{in } L^2_{loc}(\overline{Q} \setminus \tilde{\mathcal{S}}) \,. \tag{5.20}$$

From (5.19)–(5.20) we obtain that $v - \varphi(u_r) = -\varepsilon^2 \psi'(u_r)[\psi(u_r)]_{xx}$ a.e. in $Q \setminus \tilde{\mathcal{S}}$ and thus a.e. in Q since $|\tilde{\mathcal{S}}| = 0$. ∎

Now we can prove Theorem 3.8.

Proof of Theorem 3.8. By Propositions 5.1–5.6 and Remark 5.5, the pair (u, v) given by Proposition 5.1 satisfies all requirements of Definitions 3.5–3.7 with $\mathcal{S}(Q) = \tilde{\mathcal{S}}$

(in particular, equalities (3.8) follow from (5.13)), apart from equality (3.7), which we now show. For any $\zeta \in C([0,\infty); C^1_*(\Omega))$ and $n \in \mathbb{N}$ by (4.2), there holds

$$\iint_{Q_T} u_n \zeta_t \, dxdt - \iint_{Q_T} v_{nx} \zeta_x \, dxdt = -\int_\Omega u_{0n}(x)\zeta(x,0)\, dx\,.$$

By (3.2) and (5.1)–(5.3), letting $n \to \infty$ in the above equality, we obtain (3.7). Hence the result follows. ∎

6. Formation and disappearance of singularities. In this section we prove Theorem 3.9. Namely, we construct some explicit local solutions of (1.5) if

$$\varphi(u) = u^{-\alpha}, \quad \psi(u) = \gamma - \frac{u^{-\sigma}}{\sigma} \quad \text{for } u \ge \overline{u} > 0 \qquad \left(\alpha > 1,\ \sigma > \frac{1}{2}\right). \tag{6.1}$$

Let $\Omega_0 = (-\xi, \xi)$ for some $\xi > 0$ to be chosen, and let $t_0 > 0$. We look for solutions in $Q_{0t_0} = \Omega_0 \times [0, t_0)$ of the form

$$u(x,t) = \tilde{u}(x) + \delta_0(x) A(t)\,, \tag{6.2}$$

where δ_0 is the Dirac mass concentrated at the origin, $A \in C^1([0,t_0])$ is nonnegative, and $\tilde{u} \in L^1(\Omega_0) \cap C^4(\Omega_0 \setminus \{0\})$ is such that

$$\tilde{u} \ge \bar{u} \quad \text{and} \quad \tilde{u}(x) \to \infty \text{ as } |x| \to 0\,. \tag{6.3}$$

We require the regular part $\tilde{u}$ of u to be a stationary solution:

$$\left\{\varphi(\tilde{u}) - \varepsilon^2 \psi'(\tilde{u})[\psi(\tilde{u})]_{xx}\right\}_{xx} = 0 \quad \text{in } \Omega_0 \setminus \{0\}\,. \tag{6.4}$$

In view of (6.1), it is natural to set

$$v := \tilde{u}^{-\alpha} - \varepsilon^2 \tilde{u}^{-(\sigma+1)}[\tilde{u}^{-(\sigma+1)}\tilde{u}']' \quad \text{in } \Omega_0 \setminus \{0\}\,. \tag{6.5}$$

Then (6.4), together with the constraint $v \in H^1(\Omega_0)$, implies that v must be a continuous piecewise linear function in Ω_0,

$$v = \begin{cases} -D - F_- x & \text{if } -\xi < x \le 0, \\ -D - F_+ x & \text{if } 0 < x < \xi, \end{cases} \tag{6.6}$$

for some constants $D > 0$ and $F_\pm \in \mathbb{R}$.

By symmetry, we may restrict the construction of functions $\tilde{u}$ which satisfy (6.3) and (6.5)–(6.6) to the interval $(0,\xi)$. Setting $y := x/\varepsilon$, $U(y) := \tilde{u}(\varepsilon y)$, $F := \varepsilon F_+$, $\eta := \xi/\varepsilon$, the problem becomes

$$\begin{cases} U^{-\alpha} - U^{-(\sigma+1)}[U^{-(\sigma+1)}U']' = -D - Fy & \text{in } (0,\eta), \\ U(0^+) = \infty. \end{cases} \tag{6.7}$$

The next lemma provides the desired family of solutions to (6.7).

Lemma 6.1. *Let $\alpha > 1$ and $\sigma > \frac{1}{2}$. For any $D > 0$ and $F, C \in \mathbb{R}$, there exist $\eta > 0$ and a solution $U \in L^1(0,\eta)$ such that $U > \bar{u}$ in $(0,\eta)$, and as $y \to 0^+$,*

$$\begin{aligned} &y^{-\frac{2}{2\sigma+1}}\left[\frac{U^{-\frac{2\sigma+1}{2}}(y)}{y} - \sqrt{\frac{D}{2}}(2\sigma+1)\right] \to C\,, \\ &y^{-\frac{2}{2\sigma+1}}\left[\frac{yU'(y)}{U(y)} + \frac{2}{2\sigma+1}\right] \to -\frac{4C}{(2\sigma+1)^3}\sqrt{\frac{2}{D}}\,. \end{aligned} \tag{6.8}$$

Proof. Set

$$z := U^{-\lambda}, \qquad w := \frac{yU'}{U} = -\frac{yz'}{\lambda z}$$

for some $\lambda > 0$ to be chosen below. Then

$$zz'' = -\left(\frac{\sigma}{\lambda} - 1\right)(z')^2 - \lambda(D + Fy)z^{-\frac{2\sigma+1}{\lambda}+2} - \lambda z^{-\frac{2\sigma-\alpha+1}{\lambda}+2},$$

$$yw' = y\left(-\frac{z'}{\lambda z} + \frac{y(z')^2}{\lambda z^2} - \frac{yz''}{\lambda z}\right) = w + \sigma w^2 + (D + Fy)y^2 z^{-\frac{2\sigma+1}{\lambda}} + y^2 z^{-\frac{2\sigma-\alpha+1}{\lambda}}.$$

Now we rescale z by y and change the independent variable:

$$s := \log y, \qquad \tilde{z}(s) := e^{-s}z(e^s), \qquad \tilde{w}(s) := w(e^s).$$

Then we get

$$\frac{d\tilde{z}}{ds} = -\left(\frac{2\sigma+1}{2}\tilde{w} + 1\right)\tilde{z}, \quad \frac{d\tilde{w}}{ds} = \tilde{w}+\sigma\tilde{w}^2+(D+Fe^s)\tilde{z}^{-\frac{2\sigma+1}{\lambda}} e^{\frac{2\lambda-2\sigma-1}{\lambda}s} + \tilde{z}^{-\frac{2\sigma-2\alpha+1}{\lambda}} e^{\frac{2\lambda-2\sigma+\alpha-1}{\lambda}s}.$$

One easily checks that this system has an equilibrium as $s \to -\infty$ with $\tilde{z} \in \mathbb{R} \setminus \{0\}$ if and only if

$$\lambda = \frac{2\sigma+1}{2}.$$

In this case, the system reads as

$$\frac{d\tilde{z}}{ds} = -\left(\frac{2\sigma+1}{2}\tilde{w} + 1\right)\tilde{z}, \quad \frac{d\tilde{w}}{ds} = \tilde{w} + \sigma\tilde{w}^2 + (D + Fe^s)\tilde{z}^{-2} + e^{\frac{2\alpha}{2\sigma+1}s}\tilde{z}^{-2+\frac{2\alpha}{2\sigma+1}},$$

and $(\tilde{z}, \tilde{w}) \equiv (\sqrt{\frac{D}{2}}(2\sigma+1), -\frac{2}{2\sigma+1})$ is an equilibrium as $s \to -\infty$. Set $z_0 := \sqrt{\frac{D}{2}}(2\sigma+1)$ and

$$Z(s) := e^{-ms}[\tilde{z}(s) - z_0], \qquad W(s) := e^{-ms}\left[\tilde{w}(s) + \frac{2}{2\sigma+1}\right]$$

with $m > 0$ to be chosen below. Then

$$\frac{dZ}{ds} = -mZ - \frac{2\sigma+1}{2}W\left[z_0 + Ze^{ms}\right]$$

and, using also the definition of z_0,

$$\begin{aligned}\frac{dW}{ds} &= -\left(m + \frac{2\sigma-1}{2\sigma+1}\right)W + \sigma W^2 e^{ms} \\ &\quad - \frac{DZz_0^{-2}(2z_0 + Ze^{ms}) + Fe^{(1-m)s} - e^{(\frac{2\alpha}{2\sigma+1}-m)s}(z_0 + Ze^{ms})^{\frac{2\alpha}{2\sigma+1}}}{(z_0+Ze^{ms})^2}.\end{aligned}$$

If $m < 1$ and $m < \frac{2\alpha}{2\sigma+1}$, asymptotic equilibria as $s \to -\infty$ are determined by the equations

$$\frac{2\sigma+1}{2}Wz_0 = -mZ, \qquad \left(m + \frac{2\sigma-1}{2\sigma+1}\right)Wz_0 = -2Dz_0^{-2}Z = -\frac{4}{(2\sigma+1)^2}Z,$$

whence

$$\left[\frac{2\sigma+1}{2m} - \left(m + \frac{2\sigma-1}{2\sigma+1}\right)\frac{(2\sigma+1)^2}{4}\right]Wz_0 = 0.$$

To have nontrivial equilibria we choose $m > 0$ such that

$$\frac{2\sigma+1}{2m} = \left(m + \frac{2\sigma-1}{2\sigma+1}\right)\frac{(2\sigma+1)^2}{4} \Leftrightarrow (2\sigma+1)m^2 + (2\sigma-1)m - 2 = 0 \Rightarrow m = \frac{2}{2\sigma+1}.$$

(Since by assumption $\alpha > 1$ and $\sigma > \frac{1}{2}$, there holds $m < 1$ and $m < \frac{2\alpha}{2\sigma+1}$.) Hence there is a one-parameter family of asymptotic equilibria,

$$(Z, W) = \left(C, -\frac{4}{(2\sigma+1)^2 z_0} C\right) \qquad (C \in \mathbb{R}).$$

To make the above system autonomous we set

$$t := e^{\frac{s}{(2\sigma+1)k}}, \qquad \frac{1}{k} := \min\{2, 2\sigma-1, 2\alpha-2\} > 0.$$

Then

$$\text{(6.9)} \qquad \begin{cases} \frac{dt}{ds} = \frac{1}{(2\sigma+1)k} t, \\ \frac{dZ}{ds} = -\frac{2}{2\sigma+1} Z - \frac{2\sigma+1}{2} z_0 W + \frac{2\sigma+1}{2} W Z t^{2k}, \\ \frac{dW}{ds} = -W + \sigma W^2 t^{2k} - \frac{2}{(2\sigma+1)^2} Z(2z_0 + Zt^{2k})(z_0 + Zt^{2k})^{-2} \\ \qquad -F t^{(2\sigma-1)k}(z_0 + Zt^{2k})^{-2} + t^{2(\alpha-1)k}(z_0 + Zt^{2k})^{\frac{2(\alpha-1)}{2\sigma+1}}. \end{cases}$$

Linearization around $(t, Z, W) = (0, C, -\frac{4}{(2\sigma+1)^2 z_0} C)$ leads to the matrix

$$\begin{pmatrix} \frac{1}{(2\sigma+1)k} & 0 & 0 \\ \cdots & -\frac{2}{2\sigma+1} & -\frac{2\sigma+1}{2} z_0 \\ \cdots & -\frac{4}{(2\sigma+1)^2 z_0} & -1 \end{pmatrix},$$

with eigenvalues $\frac{1}{(2\sigma+1)k} > 0$, 0, and $-\frac{2\sigma+3}{2\sigma+1} < 0$ (the eigenvalue 0 occurs because there is a continuum of equilibria). By the center manifold theorem (see e.g., [27, section 2.7]), there exists a one-dimensional manifold which is stable as $s \to -\infty$ (i.e., corresponding to the positive eigenvalue). Therefore, for any $C \in \mathbb{R}$ there exists a solution $(t(s), Z(s), W(s))$ of system (6.9) in an interval $(-\infty, s_C)$ such that $t = e^{\frac{1}{(2\sigma+1)k} s}$ and $(t(s), Z(s), W(s)) \to (0, C, -\frac{4C}{(2\sigma+1)^2 z_0})$ as $s \to -\infty$. Translating this solution into the original variables, with $0 < y < \eta_+ := e^{s_C}$, we obtain that

$$\begin{aligned} y^{-1} U^{-\frac{2\sigma+1}{2}}(y) = \tilde{z} &= \sqrt{\frac{D}{2}(2\sigma+1) + Z(\log y) y^{\frac{2}{2\sigma+1}}} \\ &= \sqrt{\frac{D}{2}(2\sigma+1) + [C + o(1)] y^{\frac{2}{2\sigma+1}}} \quad \text{as } y \to 0^+. \end{aligned}$$

This proves the first convergence in (6.8); the other follows similarly from the definition of w and $\tilde{w}$ and the asymptotical behavior of W as $y \to 0^+$. ∎

Now we can complete the proof of Theorem 3.9.

Proof of Theorem 3.9. Given $D > 0$ and $F_\pm, C_\pm \in \mathbb{R}$, let $U_\pm$ be the solutions, defined in $(0, \eta_\pm)$, of (6.7) with $F = \varepsilon F_+$ and $C = C_+$ (resp., $F = -\varepsilon F_-$ and $C = C_-$). Then let $\xi := \varepsilon \min\{\eta_+, \eta_-\}$, $\Omega_0 = (-\xi, \xi)$, and

$$\text{(6.10)} \qquad \tilde{u}(x) := \begin{cases} U_+(x/\varepsilon) & 0 < x < \xi, \\ U_-(-x/\varepsilon) & -\xi < x < 0, \end{cases}$$

and define u, v through (6.2) and (6.5).

Let us check that (u,v) is a local solution in Q_{0t_0} in the sense of Definition 3.5. By the very definition of u there holds $u \in L^\infty(0,t_0;\mathcal{M}(\Omega_0))$, and it is easily seen that $u_r = \tilde{u} \in C(\overline{\Omega}_0)$, $v \in H^1(\Omega_0)$ (see (6.10), (6.8), and (6.6)). To show that $[\psi(\tilde{u})]_x \in L^2(\Omega_0)$, observe that by (6.8),

$$U(y) \sim cy^{-\frac{2}{2\sigma+1}} \quad \text{and} \quad U'(y) \sim -\frac{2}{2\sigma+1}\frac{U(y)}{y}$$

for some positive constant c. Then

$$[\psi(U)]_y^2 \sim \frac{4}{(2\sigma+1)^2}\frac{U(y)^{-2\sigma}}{y^2} \sim \frac{4c^{-2\sigma}}{(2\sigma+1)^2}y^{-\frac{2}{2\sigma+1}} \quad \text{as } y \to 0^+,$$

which is integrable around zero for $\sigma > 1/2$. Therefore, since $\mathcal{S}_t(\Omega_0) = \{0\}$ (see (6.1) and (6.3)), there holds $u(\cdot,t) \in \mathcal{A}(\Omega_0)$ for all $t \in (0,t_0)$.

By standard density arguments it suffices to check equality (3.7) with $\zeta(x,t) = \rho(x)h(t)$, $\rho \in C_c^2(\Omega_0)$, $h \in C^1([0,t_0])$ such that $h(t_0) = 0$, in which case it simply reads

$$\rho(0)\int_0^{t_0}[A'(t) - (F_- - F_+)]\,h(t)\,dt = 0\,.$$

By the arbitrariness of ρ and h, this equality is satisfied if and only if there holds $A' = F_- - F_+$ in $[0,t_0]$, namely if and only if

$$A(t) = A_0 - (F_+ - F_-)t \quad \text{for } 0 \le t \le t_0$$

with some $A_0 \in \mathbb{R}$. If $F_+ < F_-$, we choose $A_0 = 0$ and $V = F_- - F_+$ in (6.2), which provides a solution of type (3.13). If $F_+ > F_-$, we choose $V = F_+ - F_-$ and $A_0 = Vt_0$, thus exhibiting a solution of type (3.14). In both cases the four free parameters are $(F_- + F_+)$, D, C_+, and C_-. This completes the proof. ∎

Appendix A.

A.1. Proof of Proposition 3.3. The fact that any $u \in \mathcal{A}(\Omega)$ has finite energy follows by [15, Theorem A.2]. It remains to show that $u \in A(\Omega)$ if u has finite energy. Let $\{u_n\} \subset H^1(\Omega)$ be as in Definition 3.2. Since $|\psi(u_n)| \le \gamma$ and (3.3) holds, the sequence $\{\psi(u_n)\}$ is bounded and equicontinuous. Hence there exists a subsequence (not relabeled) such that

$$\psi(u_n) \rightharpoonup w \quad \text{in } H^1(\Omega), \quad \psi(u_n) \to w \quad \text{in } C(\overline{\Omega}).$$

Let

$$\tilde{\mathcal{S}}^\pm(\Omega) := \left\{x \in \overline{\Omega} \mid w(x) = \pm\gamma\right\}.$$

Note that the sets $\tilde{\mathcal{S}}^\pm$ are closed since w is continuous. As in the proof of Propositions 5.3–5.4, one shows that $|\tilde{S}^\pm(\Omega)| = 0$ and

$$u_s^\pm = u_s^\pm \llcorner \tilde{S}^\pm(\Omega), \quad u_r = [\psi^{-1}(w)] \quad \text{a.e. in } \Omega\,.$$

Hence $w = \psi(u_r)$ a.e. in Ω, $\tilde{S}^\pm(\Omega) = \mathcal{S}^\pm(\Omega)$, and the conclusion follows.

A.2. Nondegenerate problems. In this subsection we consider problem (1.5)–(1.6) under the following assumptions: $u_0 \in H^1(\Omega)$,

$$(H_1') \qquad \varphi \in C^1(\mathbb{R})\,, \quad \varphi' \in L^\infty(\mathbb{R})\,, \quad \varphi(0) = 0,$$

and

$$(H_2') \qquad \begin{cases} \text{(i)} \quad \psi \in C^3(\mathbb{R}),\ \psi' \in W^{2,\infty}(\mathbb{R}),\ \psi \text{ odd;} \\ \text{(ii)} \quad \text{there exists } c_0 > 0 \text{ such that } \psi'(u) \ge c_0 \text{ for all } u \in \mathbb{R}. \end{cases}$$

Observe that under assumption (H_2')(ii) problem (1.5)–(1.6) is nondegenerate. In this case, a stronger notion of solutions to (1.5)–(1.6) may be given.

Definition A.1. *Let $u_0 \in H^1(\Omega)$ and $\tau > 0$. A* strong solution *of problem* (1.5)–(1.6) *in $\Omega \times (0,T)$ is a function $u \in L^2(0,T;H^3_*(\Omega)) \cap C([0,T];H^1_*(\Omega))$ such that $u_t \in L^2(0,T;\left(H^1_*(\Omega)\right)')$, $u(\cdot,0) = u_0$,*

$$\text{(A.1)} \qquad \int_0^T \langle u_t, \zeta\rangle_* dt = -\iint_{Q_T} v_x \zeta_x \, dxdt \quad \textit{for all } \zeta \in L^2(0,T;H^1_*(\Omega)),$$

where $v := \varphi(u) - \varepsilon^2 \psi'(u)[\psi(u)]_{xx}$, and

$$\text{(A.2)} \qquad E[u(\cdot,t_2)] + \int_{t_1}^{t_2}\int_\Omega v_x^2 \, dxdt = E[u(\cdot,t_1)] \quad \textit{for all } 0 \le t_1 < t_2 \le T.$$

We will prove the following.

Theorem A.2. *Assume (H_1')–-(H_2'). Then for any $u_0 \in H^1(\Omega)$ there exists a function $u : \Omega \times [0,\infty) \to \mathbb{R}$ that is a strong solution to* (1.5)–(1.6) *in $\Omega \times (0,T)$ for all $T > 0$.*

We will use the following well-known interpolation inequalities.

Lemma A.3. *Let $\Omega \subset \mathbb{R}$ be open and connected. A universal constant $K_1 > 0$ and a constant $K_2 > 0$ depending on Ω exist, such that*

$$\text{(A.3)} \qquad \int_\Omega f^6 \, dx \le K_1 \left(\int_\Omega f_{xx}^2 \, dx\right)^{1/2} \left(\int_\Omega f^2 \, dx\right)^{5/2} + K_2 \left(\int_\Omega f^2 \, dx\right)^3,$$

$$\text{(A.4)} \qquad \int_\Omega |f_x|^3 \, dx \le K_2 \left(\int_\Omega f_{xx}^2 \, dx\right)^{7/8} \left(\int_\Omega f^2 \, dx\right)^{5/8} + K_2 \left(\int_\Omega f^2 \, dx\right)^{3/2}$$

for any $f \in H^2(\Omega)$. Furthermore, $K_2 = 0$ if $f = 0$ somewhere in $\overline{\Omega}$.

The proof of Theorem A.2 is based on the following local existence result, which will be proven by the Galerkin method.

Lemma A.4. *Assume (H_1')–(H_2'). Then for any $u_0 \in H^1(\Omega)$ there exist $\tau > 0$ and a strong solution u of* (1.5)–(1.6) *in $\Omega \times (0,\tau)$.*

Proof. Let $\{\phi_j\}_{j\in\mathbb{N}} \subset C^\infty(\Omega)$ be the eigenfunctions of the Laplace operator with appropriate boundary conditions,

$$\text{(A.5)} \qquad \begin{cases} -\phi_j'' = \lambda_j \phi_j & \text{in } \Omega \\ \phi_j' = 0 & \text{on } \partial\Omega \end{cases} \text{in case (1.6a)}, \quad j \ge 0, \qquad \begin{cases} -\phi_j'' = \lambda_j \phi_j & \text{in } \Omega \\ \phi_j = 0 & \text{on } \partial\Omega \end{cases} \text{in case (1.6b)}, \quad j \ge 1,$$

normalized so that they are orthonormal in $L^2(\Omega)$ (and orthogonal in $H^1_*(\Omega)$). Observe that the eigenfunctions ϕ_j are smooth, $\lambda_0 = 0$, and ϕ_0 is constant.

(1) *Local existence of approximating solutions.* Let $u_0 \in H^1(\Omega)$. We seek approximate solutions of (1.5)–(1.6) of the form

$$u_m(x,t) := \sum_{j=0}^{m} w_{jm}(t)\phi_j(x)\,, \quad v_m := \varphi(u_m) - \varepsilon^2\psi'(u_m)[\psi(u_m)]_{xx}\,. \tag{A.6}$$

Since $\varphi(0) = \psi(0) = \psi''(0) = 0$, we have

$$0 = u_{mx} = u_{mxxx}\,, \quad \text{hence} \quad v_{mx} = 0 \quad \text{on } \partial\Omega \quad \text{in case (1.6a)}, \tag{A.7}$$

$$0 = u_m = u_{mxx}\,, \quad \text{hence} \quad v_m = 0 \quad \text{on } \partial\Omega \qquad \text{in case (1.6b)}. \tag{A.8}$$

The coefficients w_{jm} are to be determined by requiring that

$$w'_{km}(t) = (u_{mt}(\cdot,t),\phi_k)_{L^2(\Omega)} = -(v_{mx}(\cdot,t),\phi'_k)_{L^2(\Omega)} \qquad (t>0)\,, \tag{A.9}$$

$$w_{km}(0) = (u_m(\cdot,0),\phi_k)_{L^2(\Omega)} = (u_0,\phi_k)_{L^2(\Omega)} \tag{A.10}$$

for any $k = 0,\dots,m$ and any $t \in (0,T)$. Equalities (A.9)–(A.10) form a system of ordinary differential equations for the unknowns $w_{1m},\dots,w_{mm}$. By (H'_1) and (H'_2)(i), the right-hand side of (A.9) depends continuously on $w_{1m},\dots,w_{mm}$, and thus a solution of (A.9)–(A.10) exists in some interval $(0,T_m)$.

(2) *Uniform estimate of T_m.* By straightforward calculations, it follows from (A.5)–(A.6) and (A.9) that

$$I'_m = \int_\Omega v_{mx}\, u_{mxxx}\, dx\,, \quad \text{where} \quad I_m(t) := \frac{1}{2}\int_\Omega u^2_{mx}(x,t)\, dx.$$

Recalling the definition of v_m (cf. (A.6)), by Young's inequality and assumptions (H'_1)–(H'_2) it is easily seen that for all $m \in \mathbb{N}$ and $t \in (0,T_m)$ there holds

$$I'_m(t) + \frac{\varepsilon^2}{2c_0^2}\int_\Omega u^2_{mxxx}(x,t)\, dx \le k_1 \int_\Omega \left[u^2_{mx} + u^6_{mx} + |u_{mxx}|^3\right](x,t)\, dx\,, \tag{A.11}$$

where the constant c_0 is defined in (H'_2)(ii) and $k_1 > 0$ is independent of m. Applying (A.3) and (A.4) with $f = u_{mx}$, by (A.11) and Young's inequality we obtain

$$I'_m(t) + k_2\int_\Omega u^2_{mxxx}(x,t)\, dx \le k_3\left[I_m(t) + I^5_m(t)\right] \tag{A.12}$$

for all $m \in \mathbb{N}$ and $t \in (0,T_m)$, with $k_2, k_3 > 0$ independent of m. Since $I_m(0) \le \frac{1}{2}\|u_0\|^2_{H^1(\Omega)}$, from (A.12) we get

$$I_m(t) \le F_0(t) := \frac{\|u_0\|^2_{H^1(\Omega)} e^{k_3 t}}{\left(16 - \|u_0\|^8_{H^1(\Omega)}\left(e^{4k_3 t} - 1\right)\right)^{1/4}} \quad \text{for } t < \tau_0 := \frac{1}{4k_3}\log\left(\frac{16 + \|u_0\|^8_{H^1(\Omega)}}{\|u_0\|^8_{H^1(\Omega)}}\right)$$

for all $m \in \mathbb{N}$. Fix any $\tau \in (0,\tau_0)$. By integrating (A.12) on $[0,\tau]$, by the above inequality we get for all $m \in \mathbb{N}$,

$$\sup_{t\in[0,\tau]}\int_\Omega u^2_{mx}(x,t)dx + k_2\iint_{Q_\tau} u^2_{mxxx}\, dxdt \le J_\tau := \int_0^\tau \left[F_0(t) + F_0^5(t)\right] dt\,. \tag{A.13}$$

In case (1.6b), by (A.13) and (A.8) there exists $C_\tau > 0$ such that

$$\|u_m\|_{L^\infty(0,\tau;H^1(\Omega))} \le C_\tau \quad \text{for all } m \in \mathbb{N}\,. \tag{A.14}$$

In case (1.6a), we need an additional estimate on the mean of u_m. Since ϕ_0 is constant, it follows from (A.9) that $w'_{0m} \equiv 0$ for all m. Hence, by (A.5)–(A.6) for all $m \in \mathbb{N}$ and $t \in (0, T_m)$ we have that

$$\frac{d}{dt} \int_\Omega u_m(x,t) dx = \sum_{j=0}^{m} w'_{jm}(t) \int_\Omega \phi_j \, dx = w'_{0m}(t)\, \phi_0 \, |\Omega| = 0 \, ,$$

whence by (A.6) and (A.10),

$$\left| \int_\Omega u_m(x,t) \, dx \right| \le \|u_0\|_{L^1(\Omega)} \quad \text{for all } m \in \mathbb{N} \text{ and } t \in (0, T_m). \tag{A.15}$$

Using (A.15) and (A.13), we obtain (A.14) also in case (1.6a). By estimate (A.14), in both cases $w_{1m}, \dots, w_{mm}$ are uniformly bounded in $[0, \tau]$, and thus solutions u_m of (A.9)–(A.10) are defined up to time τ for any $\tau < \tau_0$; in particular, $T_m \ge \tau_0$ for all $m \in \mathbb{N}$.

(3) *Estimates of* $\{u_m\}$, $\{v_m\}$ *in* $[0, \tau]$ *for any* $\tau < \tau_0$. From (A.13)–(A.14) we also obtain for every $m \in \mathbb{N}$,

$$\|u_m\|_{L^2(0,\tau;H^3(\Omega))} \le C_\tau \tag{A.16}$$

(hereafter $C_\tau > 0$ denotes a generic constant independent of m). It follows from (A.14) and (A.16) that

$$\|v_m\|_{L^2(0,\tau;H^1(\Omega))} \le C_\tau \, , \tag{A.17}$$

$$\|\varphi(u_m)\|_{L^\infty(0,\tau;H^1(\Omega))} \le C_\tau \, , \tag{A.18}$$

$$\|\psi(u_m)\|_{L^2(0,\tau;H^3(\Omega))} \le C_\tau \, . \tag{A.19}$$

Let us prove that

$$\|u_{mt}\|_{L^2\left(0,\tau;(H^1_*(\Omega))'\right)} \le C_\tau \, , \tag{A.20}$$

$$\|u_{mtx}\|_{L^2(0,\tau;X'_*(\Omega))} \le C_\tau \, , \tag{A.21}$$

where

$$X_*(\Omega) := \begin{cases} \{u \in H^2(\Omega) \mid u \in H^1_0(\Omega) \} & \text{in case (1.6a)}, \\ \{u \in H^2(\Omega) \mid u_x \in H^1_0(\Omega) \} & \text{in case (1.6b)}. \end{cases}$$

We fix any $\zeta \in L^2(0, \tau; H^1_*(\Omega))$ and denote by P_m the projection of $L^2(\Omega)$ onto *span* $\{\phi_0, \dots, \phi_m\}$ (in case (1.6a)) or *span* $\{\phi_1, \dots, \phi_m\}$ (in case (1.6b)). Then by (A.9) and (A.17) we get

(A.22)

$$\begin{aligned} \left| \iint_{Q_\tau} u_{mt} \zeta \, dxdt \right| &= \left| \iint_{Q_\tau} u_{mt} P_m \zeta \, dxdt \right| = \left| \iint_{Q_\tau} v_{mx} [P_m \zeta]_x \, dxdt \right| \\ &\le \left(\iint_{Q_\tau} v_{mx}^2 \, dxdt \right)^{1/2} \left(\iint_{Q_\tau} [P_m \zeta]_x^2 \, dxdt \right)^{1/2} \le C_\tau \|\zeta_x\|_{L^2(Q_\tau)} \, , \end{aligned}$$

whence (A.20) follows. Inequality (A.21) also follows from (A.22):

$$\left| \iint_{Q_\tau} u_{mtx} \zeta dxdt \right| = \left| \iint_{Q_\tau} u_{mt} \zeta_x dxdt \right| \le C_\tau \|\zeta_{xx}\|_{L^2(Q_\tau)}$$

for all $\zeta \in L^2(0, \tau; X_*(\Omega))$.

(4) *The limit* $m \to \infty$. By estimates (A.14), (A.16), and (A.20), there exist a subsequence $\{u_m\}$ (not relabeled) and a function $u \in L^\infty(0,\tau;H^1(\Omega)) \cap L^2(0,\tau;H^3(\Omega))$, with $u_t \in L^2\left(0,\tau;(H^1(\Omega))'\right)$, such that as $m \to \infty$,

$$u_m \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\left(0,\tau;H^1_*(\Omega)\right), \tag{A.23}$$

$$u_m \rightharpoonup u \quad \text{in } L^2\left(0,\tau;H^3_*(\Omega)\right), \tag{A.24}$$

$$u_{mt} \rightharpoonup u_t \quad \text{in } L^2\left(0,\tau;(H^1_*(\Omega))'\right), \tag{A.25}$$

$$u_{mtx} \rightharpoonup u_{tx} \quad \text{in } L^2\left(0,\tau;X'_*(\Omega)\right). \tag{A.26}$$

By Lemma 2.1 we also have

$$u_m \to u \quad \text{in } L^2\left(0,\tau;H^2(\Omega)\right) \cap C([0,\tau];L^2(\Omega)) \quad \text{and a.e. in } Q_\tau. \tag{A.27}$$

In view of (A.24) and (A.26), applying Lemma 2.2 with $X = X_*(\Omega)$ and $z = u_x$, we see that $u_x \in C([0,\tau];L^2(\Omega))$. Combined with (A.27), this yields that

$$u \in C([0,\tau];H^1_*(\Omega)). \tag{A.28}$$

By (A.27), (A.18)–(A.19), and the properties of φ and ψ we get (possibly up to a subsequence)

$$\varphi(u_m) \rightharpoonup \varphi(u) \quad \text{in } L^2\left(0,T;H^1(\Omega)\right), \tag{A.29}$$

$$[\psi(u_m)]_{xx} \rightharpoonup [\psi(u)]_{xx} \quad \text{in } L^2\left(0,T;H^1(\Omega)\right), \tag{A.30}$$

$$v_m \rightharpoonup v := \varphi(u) - \varepsilon^2\psi'(u)[\psi(u)]_{xx} \quad \text{in } L^2\left([0,\tau);H^1_*(\Omega)\right). \tag{A.31}$$

To prove (A.1) observe that, for all $m,n \in \mathbb{N}$, $m \geq n$, and $\zeta \in L^2(0,T;H^1_*(\Omega))$, there holds

$$\iint_{Q_\tau} u_{mt}\, P_n\zeta\, dxdt = -\iint_{Q_\tau} v_{mx}[P_n\zeta]_x\, dxdt\,.$$

As $m \to \infty$, by (A.25) and (A.31) we obtain that

$$\int_0^T \langle u_t, P_n\zeta\rangle_* dt = -\iint_{Q_T} v_x[P_n\zeta]_x\, dxdt\,,$$

whence (A.1) follows by the arbitrariness of n. Therefore, the function u is a strong solution of (1.5)–(1.6) in $\Omega \times (0,\tau)$.

(5) *Energy identity.* It remains to prove equality (A.2). For all $m \in \mathbb{N}$ and $\zeta \in L^2(0,\tau;X_*(\Omega))$, we have

$$\iint_{Q_\tau} [\psi(u_m)]_{xt}\, \zeta\, dxdt = -\iint_{Q_\tau} \psi'(u_m)\, u_{mt}\, \zeta_x\, dxdt\,. \tag{A.32}$$

Observe that by (A.14) there holds

$$\int_0^\tau \|\psi'(u_m)(\cdot,t)\,\zeta_x(\cdot,t)\|^2_{H^1_*(\Omega)}\, dt \leq C_\tau \int_0^\tau \|\zeta_x(\cdot,t)\|^2_{H^1_*(\Omega)}\, dt$$

for some $C_\tau > 0$. From (A.20), (A.32), and the above inequality we obtain

$$\left|\iint_{Q_\tau} [\psi(u_m)]_{xt}\, \zeta\, dxdt\right| \leq C_\tau \|\zeta\|_{L^2(0,\tau;X_*(\Omega))}\,.$$

Hence $\{[\psi(u_m)]_{xt}\}$ converges weakly in $L^2(0,\tau;X'_*(\Omega))$ (possibly up to a subsequence), and by (A.27) the weak limit is identified with $[\psi(u)]_{xt}$, and thus

$$[\psi(u_m)]_{xt} \rightharpoonup [\psi(u)]_{xt} \quad \text{in } L^2(0,\tau;X'_*(\Omega)). \tag{A.33}$$

By (A.25)–(A.27) and (A.33), letting $m\to\infty$ in (A.32) gives

$$\iint_{Q_\tau} \langle [\psi(u)]_{xt}, \zeta\rangle_{X'_*(\Omega),X_*(\Omega)} dt = -\iint_{Q_\tau} \langle u_t, \psi'(u)\zeta_x\rangle_*\, dt. \tag{A.34}$$

In view of (A.33), since $[\psi(u)]_x \in L^2(0,\tau;X_*(\Omega))$, by Lemma 2.2 with $X = X_*(\Omega)$ and $z=[\psi(u)]_x$ there holds $[\psi(u)]_x \in C([0,\tau];L^2(\Omega))$, and

$$\int_\Omega [\psi(u)]_x^2\, dx\Big|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2} \langle [\psi(u)]_{xt}, [\psi(u)]_x\rangle_{X'_*(\Omega),X_*(\Omega)}\, dt \tag{A.35}$$

for all $t_1,t_2\in[0,\tau)$. By (A.28), we also have that $\psi(u)\in C([0,\tau];H^1(\Omega))$ and $E[u(\cdot,t)]\in C([0,\tau])$. Therefore, using (A.34)–(A.35),

$$\begin{aligned} E[u(\cdot,t_2)] - E[u(\cdot,t_1)] &= \int_{t_1}^{t_2} \left\{\langle u_t, \varphi(u)\rangle_* + \langle [\psi(u)]_{xt}, [\psi(u)]_x\rangle_{X'_*(\Omega),\,X_*(\Omega)}\right\} dt \\ &= \int_{t_1}^{t_2} \langle u_t, \varphi(u) - \psi'(u)[\psi(u)]_{xx}\rangle_*\, dt \end{aligned}$$

for all $0\le t_1<t_2\le\tau$, which implies (A.2). ◻

We are now ready to prove Theorem A.2.

Proof of Theorem A.2. Let $u_0\in H^1(\Omega)$. Set

$$\tau_M := \sup\{\tau>0 \mid \exists \text{ a strong solution } u \text{ of (1.5)–(1.6) in } \Omega\times(0,\tau)\}.$$

By Lemma A.4, τ_M is well defined and $\tau_M>0$. Hence there exists a function u which is a strong solution of (1.5)–(1.6) in $\Omega\times(0,\tau)$ for all $\tau<\tau_M$. It follows from (A.1) that

$$\int_\Omega u(x,t)\, dx = \int_\Omega u_0(x)\, dx \quad \text{for all } t<\tau_M \text{ in case (1.6a)} \tag{A.36}$$

(no control on the mean is needed in case (1.6b) since $u(\cdot,t)\in H^1_0(\Omega)$). Using the lower bound on ψ', it follows from (A.2) that

$$\sup_{t\in(0,\tau_M)} \int_\Omega u_x^2(x,t)dx + \iint_{Q_{\tau_M}} v_x^2 \le C, \tag{A.37}$$

where $C\ge1$ denotes a generic constant independent of τ_M. Arguing as in the proof of (A.20) and (A.21) in the proof of Lemma A.4, (A.36)–(A.37) yield

$$u\in L^\infty(0,\tau_M;H^1_*(\Omega)),\quad u_t\in L^2(0,\tau_M;(H^1_*(\Omega))'),\quad u_{tx}\in L^2(0,\tau_M;X'_*(\Omega)). \tag{A.38}$$

Assume by contradiction that $\tau_M<\infty$. Using Young's inequality, the boundedness of u, and (H_2'), we see that

$$C^{-1}u_{xxx}^2 \le \varepsilon^4(\psi'(u))^2 u_{xxx}^2 \le v_x^2 + C\left(u_x^2+u_x^6+|u_{xx}|^3\right).$$

Recalling (A.3) and (A.4) and using once more Young's inequality, for all $\tau < \tau_M$ we have

$$\iint_{Q_\tau} u_{xxx}^2\,dxdt \le C\iint_{Q_\tau}(v_x^2+u_x^2)\,dxdt+\frac{1}{2}\iint_{Q_\tau}u_{xxx}^2\,dxdt+C\int_0^\tau\left(\int_\Omega u_x^2(x,t)dx\right)^5 dt$$

$$\overset{(A.37)}{\le} C+C\tau_M+\frac{1}{2}\iint_{Q_\tau}u_{xxx}^2\,dxdt. \tag{A.39}$$

By (A.38) and (A.39), $u\in L^2_{loc}([0,\tau_M);H^3_*(\Omega))$ is upgraded to $u\in L^2(0,\tau_M;H^3_*(\Omega))$. Applying twice Lemma 2.2, in view of (A.38) we obtain $u\in C([0,\tau_M];H^1_*(\Omega))$. In particular,

$$u(\cdot,t)\to u(\cdot,\tau_M)=:u_M \quad \text{in } H^1_*(\Omega) \text{ as } t\to\tau_M^-.$$

Applying Lemma A.4 with u_M as initial datum at $t=\tau_M$ and patching solutions together, we obtain a strong solution to (1.5)–(1.6) in $(0,\tau_M+\tilde\tau)\times\Omega$ for some $\tilde\tau>0$, which contradicts the definition of τ_M. Thus $\tau_M=\infty$, and the result follows. ∎

A.3. Estimating the right-hand side of (4.15). Setting $y=\eta^{1/2}$ and $A=|t_2-t_1|^{1/2}$, we will equivalently estimate $f(y)=y+Ay^{-2}(1+y^{-1})$ for $y>0$. Differentiating, we see that the unique minimum point y_m satisfies $2y_m^{-3}+3y_m^{-4}=A^{-1}$. We now distinguish two cases. If $y_m\le 1$, then $y_m^{-3}\le y_m^{-4}$, so that $3A\le y_m^4\le 5A$ and, therefore,

$$\min_{y\in(0,\infty)} f(y)\le\left(5^{1/4}+2\cdot 3^{-3/4}\right)A^{1/4} \quad \text{if } y_m\le 1.$$

If, on the other hand ,$y_m\ge 1$, then $y_m^{-4}\le y_m^{-3}$, so that $2A\le y_m^3\le 5A$ and, therefore,

$$\min_{y\in(0,\infty)} f(y)\le\left(5^{1/3}+2\cdot 2^{-2/3}\right)A^{1/3} \quad \text{if } y_m\ge 1.$$

Combining the two inequalities, we conclude that

$$\min_{y\in(0,\infty)} f(y)\le\left(5^{1/4}+2\cdot 3^{-3/4}\right)A^{1/4}+\left(5^{1/3}+2^{1/3}\right)A^{1/3}.$$

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