

## SEMI-PURITY FOR CYCLES WITH MODULUS

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ABSTRACT. In this paper, we prove a form of purity property for the  $\overline{\square} = (\mathbb{P}^1, \infty)$ -invariant replacement  $h_0^{\overline{\square}}(\mathfrak{X})$  of the Yoneda object  $\mathbb{Z}_{\text{tr}}(\mathfrak{X})$  for a proper modulus pair  $\mathfrak{X} = (\overline{X}, X_\infty)$  over a field  $k$ , consisting of a smooth proper  $k$ -scheme and an effective Cartier divisor on it. As application, we prove the analogue in the modulus setting of Voevodsky’s fundamental theorem on the homotopy invariance of the cohomology of homotopy invariant sheaves with transfers, based on a main result of [20]. This plays an essential role in the development of the theory of motives with modulus, and among other things implies the existence of a homotopy  $t$ -structure on the category  $\mathbf{MDM}^{\text{eff}}(k)$  of Kahn-Saito-Yamazaki.

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## 1. INTRODUCTION

In a series of landmark papers from the 1990’s ([26], [27], [28]), Voevodsky introduced and studied the derived category of effective motives  $\mathbf{DM}^{\text{eff}}(k)$  over a field  $k$ . It is defined as the localization of the derived category  $D(\mathbf{NST})$  of (unbounded) complexes of Nisnevich sheaves on the category of smooth correspondences (i.e. the category of smooth schemes with some extra functoriality, called *transfers structure*, for finite surjective maps) over  $k$  with respect to the projection maps  $\mathbb{A}^1 \times X \rightarrow X$ . This condition leads to the homotopy invariance of every homology theory representable in  $\mathbf{DM}^{\text{eff}}(k)$ , and is a cornerstone of the construction. Explicitly, a sheaf  $F$  is called homotopy invariant if the natural map

$$F(X) \rightarrow F(X \times \mathbb{A}^1)$$

induced by the projection  $X \times \mathbb{A}^1 \rightarrow X$  is an isomorphism. On this note, recall the following

**Theorem 1.0.1** (5.6 [27], 24.1[18]). *Let  $F$  be a homotopy invariant presheaf (of abelian groups) with transfers on  $\mathbf{Sm}(k)$ . Then the Nisnevich sheaf with transfers  $F_{\text{Nis}}$  associated with  $F$  is homotopy invariant. If  $k$  is perfect, the presheaves  $H_{\text{Nis}}^i(-, F_{\text{Nis}})$  have a canonical structure of homotopy invariant presheaves with transfers.*

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The above result is usually called the *homotopy invariance of the cohomology* and plays a fundamental role in the theory of motives. Theorem 1.0.1 implies that  $\mathbf{DM}^{\text{eff}}(k)$  is equivalent to the triangulated subcategory of  $D(\mathbf{NST})$  consisting of those complexes having homotopy invariant cohomology sheaves [28, p. 205] (the perfectness assumption was removed later up to inverting  $p$  in the coefficients. See [24]). Another consequence of 1.0.1 is the fact that the standard  $t$ -structure on the derived category  $D(\mathbf{NST})$  induces a  $t$ -structure on  $\mathbf{DM}^{\text{eff}}(k)$ , whose heart is equivalent to the category of homotopy invariant sheaves. This is the so-called *homotopy  $t$ -structure*. Thus, thanks to Voevodsky’s result, the category of motives, abstractly defined as a Bousfield localization of  $D(\mathbf{NST})$ , admits a fairly concrete description.

Imposing  $\mathbb{A}^1$ -invariance is enough for capturing many geometric and arithmetic properties of smooth schemes. However, it was already noticed by Voevodsky in [28, 2.2] that this is too much to ask in other interesting situations. For example, the Picard group  $\text{Pic}(X)$  of a smooth  $k$ -scheme  $X$  is canonically isomorphic to the motivic cohomology group  $\text{Hom}_{\mathbf{DM}^{\text{eff}}(k)}(M(X), \mathbb{Z}(1)[2])$ , where  $M(X)$  denotes the image of the Yoneda object  $\mathbb{Z}_{\text{tr}}(X)$  of  $X$  in  $\mathbf{DM}^{\text{eff}}(k)$ . On the other hand, we do not have any “motivic” description for  $\text{Pic}(X)$  when  $X$  is a singular variety, since the functor  $X \mapsto \text{Pic}(X)$  considered on  $\mathbf{Sch}(k)$  (rather than on  $\mathbf{Sm}(k)$ ) is not  $\mathbb{A}^1$ -invariant.

But even if one is interested in smooth schemes only, there are natural objects which do not admit a description in Voevodsky’s world. Perhaps the most striking example is given by the abelianized fundamental group  $\pi_1^{ab}(-)$  computed on smooth varieties over positive characteristic fields. In this case, it is well-known that  $\pi_1^{ab}(\mathbb{A}_k^1)$  is far from being trivial, due to the existence of the Artin-Schrier covers  $x \mapsto x - x^p$ . Other examples are given by the sheaves of differential forms  $\Omega^*$ , the de Rham-Witt complexes,  $W_*\Omega^*$  (see e.g. [4]), the commutative algebraic groups (with unipotent part) considered as sheaves with transfers (see e.g. [23]), as well as the natural generalization of Bloch’s higher Chow groups, like additive Chow groups [15], [16] and the higher Chow groups with modulus [2], [13].

With the goal of developing a theory of motives in which non  $\mathbb{A}^1$ -invariant objects can be represented, the second author together with Kahn and Yamazaki introduced a generalized framework in [12], [10], [11], [20], based on the principle that the category of smooth schemes over  $k$  should be replaced by the larger category of *modulus pairs*,  $\mathbf{MCor}(k)$  over a (perfect) field  $k$ . Objects are (as the name suggests) pairs  $\mathfrak{X} = (\overline{X}, X_\infty)$  consisting of a separated  $k$ -scheme of finite type  $\overline{X}$  and an effective (possibly empty) Cartier divisor  $X_\infty$  on it (the *modulus*) such that the complement  $\overline{X} \setminus X_\infty = X$  is smooth. Morphisms are given by finite correspondences between the smooth complements, subject to certain admissibility conditions. Together, they form a symmetric monoidal category. See 2.1 for the precise definition. Write  $\mathbf{MPST}$  for the category of additive presheaves of abelian groups on  $\mathbf{MCor}(k)$ , and  $\mathbf{MPST}$  for the subcategory of presheaves defined on the smaller category  $\mathbf{MCor}$  of *proper* modulus pairs, i.e. pairs  $\mathfrak{X} = (\overline{X}, X_\infty)$  such that the total space  $\overline{X}$  is proper.

The idea behind this is that the pair  $\mathfrak{X}$  looks like a (partial) compactification of the smooth scheme  $X$  in which the scheme structure of the boundary (the divisor  $X_\infty$ ) plays a non-trivial role. This approach is inspired by the theory of cycles with modulus [2], [13], a generalization in higher dimension of the classical theory of Jacobian varieties of Rosenlicht and Serre [22].

In this context, the role played by the affine line in Voevodsky’s world is played by its compactified cousin

$$\overline{\square} = (\mathbb{P}^1, \infty)$$

i.e. the projective line  $\mathbb{P}^1$ , with reduced divisor at infinity (informally called “the cube”). The key property which replaces  $\mathbb{A}^1$ -invariance is the so-called *cube invariance*: a presheaf of abelian groups  $F$  on  $\mathbf{MCor}(k)$  is called  $\overline{\square}$ -invariant if for every (proper) modulus pair  $\mathfrak{X}$  the

canonical map

$$F(\mathfrak{X}) \rightarrow F(\mathfrak{X} \otimes \overline{\square})$$

induced by  $\mathfrak{X} \otimes \overline{\square} \rightarrow \mathfrak{X}$  is an isomorphism. We write  $\mathbf{CI} \subset \mathbf{MPST}$  for the category of  $\overline{\square}$ -invariant presheaves with transfers on modulus pairs. Starting from this, it is possible to define a new category of effective motivic complexes,  $\mathbf{MDM}^{\text{eff}}(k)$ , obtained as localization of the category of complexes of Nisnevich sheaves with transfers on  $\mathbf{MCor}(k)$  (note that the sheaf property is rather subtle in this context, see §2.1 and [10, 3]) with respect to the projection maps  $\mathfrak{X} \otimes \overline{\square} \rightarrow \mathfrak{X}$ . The canonical forgetful functor  $\mathfrak{X} = (\overline{X}, X_\infty) \mapsto \omega(\mathfrak{X}) := X = \overline{X} \setminus X_\infty$  induces then an adjoint pair

$$\omega_{\text{eff}}: \mathbf{MDM}^{\text{eff}}(k) \rightleftarrows \mathbf{DM}^{\text{eff}}(k): \omega^{\text{eff}}$$

which satisfies the property that  $\omega^{\text{eff}}(M(X)) = M(X, \emptyset)$  for every smooth and proper  $k$ -scheme  $X$ . See [10, 7.3]. The functor  $\omega^{\text{eff}}$  is fully faithful, so that Voevodsky's category  $\mathbf{DM}^{\text{eff}}(k)$  can be presented as a further localization of the bigger category  $\mathbf{MDM}^{\text{eff}}(k)$ .

We now have all the ingredients to state one of the main results of the present paper.

**Theorem 1.0.2** (see Theorem 2.2.1). *Let  $F \in \mathbf{CI}^\tau$  be a  $\overline{\square}$ -invariant presheaf with transfers. Then for every  $\mathfrak{X} \in \mathbf{MCor}_{l_s}$ , the map*

$$H^i(\mathfrak{X}_{\text{Nis}}, F_{\text{Nis}}) \rightarrow H^i((\mathfrak{X} \otimes \overline{\square})_{\text{Nis}}, F_{\text{Nis}})$$

*induced by  $\mathfrak{X} \otimes \overline{\square} \rightarrow \mathfrak{X}$  is an isomorphism (see §2.1 for the notation  $F_{\text{Nis}}$  and  $H^q(\mathfrak{X}_{\text{Nis}}, F_{\text{Nis}})$ ).*

The superscript  $\mathbf{CI}^\tau$  denotes the essential image of  $\mathbf{CI}$  under  $\tau: \mathbf{MPST} \rightarrow \mathbf{MPST}$ , i.e. the left Kan extension of a presheaf defined on proper modulus pairs to a presheaf defined on every modulus pair, while the subscript  $\mathbf{MCor}_{l_s}$  stands for the subcategory of *log smooth* modulus pairs, i.e.  $\mathfrak{X} = (\overline{X}, X_\infty)$  such that  $\overline{X}$  is smooth and  $|X_\infty|$  is a strict normal crossing divisor on  $\overline{X}$ .

As in Voevodsky's case, Theorem 1.0.2 implies the existence of a homotopy  $t$ -structure (under an assumption on resolution of singularities, see 2.3, Theorem 2.3.5, [11, Thm. 3.7.1]) on  $\mathbf{MDM}^{\text{eff}}(k)$ , whose heart is equivalent to the category  $\mathbf{CI}_{\text{Nis}}$  of  $\overline{\square}$ -invariant Nisnevich sheaves.

Another consequence is the following result on the representability of the cohomology of  $\overline{\square}$ -invariant sheaves. Let  $\underline{\mathbf{MDM}}^{\text{eff}}(k)$  be the analogue of  $\mathbf{MDM}^{\text{eff}}(k)$  built out of the bigger category  $\mathbf{MCor}$ , and let  $L^{\overline{\square}}: D(\mathbf{MNST}) \rightarrow \underline{\mathbf{MDM}}^{\text{eff}}(k)$  be the localization functor, where  $\mathbf{MNST} \subset \mathbf{MPST}$  is the full subcategory of Nisnevich sheaves.

**Theorem 1.0.3** (see Theorem 2.3.1). *Let  $F^\bullet$  be a complex of Nisnevich sheaves with transfers on  $\mathbf{MCor}$  be such that  $H^i(F^\bullet) \in \mathbf{CI}_{\text{Nis}}^\tau = \tau_! \mathbf{CI}_{\text{Nis}}$  for all  $i \in \mathbb{Z}$ . For  $\mathfrak{X} \in \mathbf{MCor}_{l_s}$ , there exists a natural isomorphism*

$$\text{Hom}_{\underline{\mathbf{MDM}}^{\text{eff}}(k)}(M(\mathfrak{X}), L^{\overline{\square}}(F^\bullet[i])) \cong H_{\text{Nis}}^i(\mathfrak{X}, F^\bullet).$$

As application, we get the following description (see 2.3.3) of relative Suslin homology [19] for proper modulus pairs  $\mathfrak{X}$ :

$$H_i(\overline{X}, X_\infty) = \text{Hom}_{\underline{\mathbf{MDM}}^{\text{eff}}(k)}(\omega^{\text{eff}} M(\text{Spec}(k)), M(\mathfrak{X})[-i])$$

which gives in particular the representability of the Kerz-Saito Chow group of zero cycles with modulus in  $\mathbf{MDM}^{\text{eff}}(k)$ , over any perfect field, as well as a Mayer-Vietorys property for relative Suslin homology (see [11, Thm. 3]. Note that thanks to 1.0.2, the assumption on resolution of singularities in loc.cit. is now removed).

Thanks to [20], Theorem 1.0.2 was previously known to hold under the extra assumption of *semi-purity* of the sheaf  $F_{\text{Nis}}$ . In informal terms, this property can be understood as an analogue of the Gersten property in the modulus setting: let  $(\mathcal{O}, (\Pi))$  be a pair consisting of a regular henselian ring  $\mathcal{O}$  (essentially of finite type over  $k$ ), and  $(\Pi)$  is the ideal generated by a non-zero element  $\Pi$  in the maximal ideal of  $\mathcal{O}$ . We think  $(\mathcal{O}, (\Pi))$  as limit object of  $\underline{\mathbf{MCor}}$  in the usual way. Let  $K$  be the function field of  $\mathcal{O}$ . Then a Nisnevich sheaf  $F$  on  $\underline{\mathbf{MCor}}$  is semipure if the map

$$F(\mathcal{O}, (\Pi)) \rightarrow F(K, \emptyset)$$

is injective for every such pair (this definition is actually slightly stronger than the one used in the paper, see 2.1). Unfortunately, this property is not satisfied by every object in  $\mathbf{CI}_{\text{Nis}}^\tau$ . It is however (somehow surprisingly) satisfied by a large class of objects: the Nisnevich sheaves of the form  $\tau_! h_0^{\square}(\mathfrak{X})_{\text{Nis}}$  for every modulus pair  $\mathfrak{X} = (\overline{X}, X_\infty)$  with  $\overline{X}$  smooth and projective.

The symbol  $h_0^{\square}(-)$  denotes the  $\square$ -replacement functor on  $\mathbf{MPST}$  (i.e. the left adjoint of the inclusion  $\mathbf{CI} \subset \mathbf{MPST}$ , see 2.1), applied then to the representable presheaf  $\mathbb{Z}_{\text{tr}}(\mathfrak{X})$ . This is the analogue of the Suslin-Voevodsky  $h_0(-)$  construction (see [25], [18]). The key result of this paper is then the following

**Theorem 1.0.4** (see Theorem 2.1.3). *For  $\mathfrak{X} \in \mathbf{MCor}^{\text{proj}}$ ,  $\tau_! h_0^{\square}(\mathfrak{X})_{\text{Nis}}$  is semi-pure.*

The proof of this result, which can be reformulated into a problem about algebraic cycles, occupies Sections 3-6. From this, it is possible to prove Theorem 1.0.2 without semi-purity assumptions on  $F$ , together with all the other mentioned corollaries.

We now briefly discuss the contents of the different sections, and we give some ideas about the proofs.

Section 2 contains a recollection of definitions and results from [10], [11] and [20], together with the exact statements of the main theorem and some applications. We then begin with the proof of the semipurity result for modulus pairs of dimension 1 (i.e. for curves). This is achieved by means of a direct and explicit computation in Section 3. In fact, we prove a more general result, namely Theorem 3.1.2. To pass from the case of curves to the case of surfaces, we need to develop some moving techniques for cycles with modulus, and we do so in Section 5 after some preliminaries in Section 4. The idea behind the technical arguments in Section 5 is ultimately simple, and goes back to the Bloch-Quillen's formula, relating the group of 0-cycles on a surface to the cohomology of the sheaf  $\mathcal{K}_2$ . Let us explain it for the reader's convenience.

Let  $X$  be a regular integral surface defined over a field  $F$  and let  $\eta$  be its generic point. Recall that the sheaf  $\mathcal{K}_{2,X}$  has a Gersten resolution

$$(1.1) \quad 0 \rightarrow \mathcal{K}_{2,X} \rightarrow \iota_{\eta,*} K_2(k(\eta)) \xrightarrow{T} \bigoplus_{y \in X^{(1)}} \iota_{y,*} K_1(k(y)) \xrightarrow{\partial} \bigoplus_{x \in X^{(2)}} \iota_{x,*} K_0(k(x))$$

where the first map  $T$  is the tame symbol  $T = (T_y)_{y \in X^{(1)}}$  and  $\partial$  agrees with the divisor map under the identification  $\bigoplus_{x \in X^{(2)}} \iota_{x,*} K_0(k(x)) = \mathcal{Z}_0(X)$ , where the latter denotes the free abelian group of zero cycles on  $X$ . The cokernel of  $\partial$  is then canonically identified with the Chow group of zero cycles  $\text{CH}_0(X)$ . It is a well known fact (first discovered by Bloch [3]) that (1.1) is a flasque resolution of the sheaf  $\mathcal{K}_{2,X}$ , so that  $H^2(X, \mathcal{K}_{2,X}) \cong \text{CH}_0(X)$ . The important remark for us, however, is simply the fact the (1.1) is a complex, so that the composition  $\partial \circ T$  is zero in the free abelian group  $\mathcal{Z}_0(X)$ , and that the image of  $\partial$  is the subgroup of zero cycles rationally equivalent to zero.

In particular, we can add to any class  $\gamma = \sum_y (f_y) \in \bigoplus_{y \in X^{(1)}} K_1(k(y))$  for rational functions  $f_y \in k(y)^\times$ , an element of the form  $T\{a, b\}$  without altering the image under  $\partial$ , i.e. without altering the cycle  $\partial(\gamma)$ , rationally equivalent to zero. We call any such class  $T\{a, b\}$  a *moving symbol*. A careful choice of  $\{a, b\} \in K_2(k(\eta))$  allows us to rearrange in a convenient way any relation in the Chow group.

In geometric terms, this can be summarized as follows. Assume that  $X$  is quasi-projective. Suppose for simplicity that  $\partial(\gamma) = \text{div}_Z(f)$ , for an integral curve  $Z \subset X$ , and that  $Z$  is the zero locus of a global section  $s$  of a very ample line bundle on  $X$ . Choose another section  $t$  of the same line bundle so that  $s/t \in k(\eta)^\times$  is a rational function on  $X$ . Next, choose another rational function  $a/b$ , ratio of two global sections of a (possibly different) line bundle on  $X$  which gives a global lift of the rational function  $f$  on  $Z$ . We now have

$$\partial T(\{s/t, a/b\}) = \text{div}_Z(f) - \text{div}_{(t)}(a/b) + \text{div}_{(a)}(s/t) - \text{div}_{(b)}(s/t) = 0 \in \mathcal{Z}_0(X)$$

so that  $\text{div}_Z(f)$  is now the sum of 3 new cycles. The key observation is that the global sections  $a, b$  and  $t$  can be chosen in a very controlled way by means of suitable applications of Bertini-type theorems, giving a good control on the newly found cycles.

In our application, this simple picture becomes substantially more intricate. The regular surface  $X$  is replaced by the smooth generic fiber  $X = \mathcal{X}_K$  of an integral relative surface  $\mathcal{X}$  over the spectrum of a henselian local ring  $\mathcal{O}$ , which comes equipped with an effective Cartier divisor  $\mathcal{D}$ , surjective over  $S = \text{Spec } \mathcal{O}$ . We briefly mention the three main points that we have to address.

First, we need to impose some extra conditions on the rational functions on the curves on  $X$ , the modulus condition, i.e. we have to replace  $K_1(k(y))$  with suitable relative  $K_1$  groups with respect to the divisor  $\mathcal{D} \times_S K$  on  $X$  (see 5.1). Second, we need to work with a subgroup of the group  $\mathcal{Z}_0(X)$ , generated by closed points in  $X$  whose closure in  $\mathcal{X}$  has a special behavior with respect to the special fiber  $\mathcal{D}_s$  of  $\mathcal{D}$ . We call it the *modulus condition over  $\mathcal{O}$*  (see Definition 3.1.1). Finally, the global sections like  $a, b$  and  $t$  above need to have a model over  $S$  satisfying a (rather weak) good intersection property with respect to  $\mathcal{D}_s$ . This is achieved by means of some new Bertini-type theorems over a local base, which we develop in Section 4.

With a suitable combination of these moving techniques, the global injectivity result, Theorem 2.1.3, can be then reduced from the case of surfaces to the case of curves.

Finally, in Section 6, we complete the proof in arbitrary dimension. This is essentially achieved by reducing to the case of surfaces, using again the Bertini theorems of Section 4.

**Notations and conventions.** Throughout this paper,  $k$  will denote a fixed ground field and  $\mathcal{O}$  a Noetherian local domain, whose residue field is a finite extension of  $k$  (most of the time,  $\mathcal{O}$  will be essentially of finite type over  $k$ ). We write  $K$  for the function field of  $\mathcal{O}$ .

We will use Roman capital letters to denote schemes over  $k$  or over  $K$ , and we follow the convention that Script letters (like  $\mathcal{X}$ ) will denote schemes over  $\mathcal{O}$ . Our main object of interest is the category of *modulus pairs*  $\mathbf{MCor}$  over  $k$  (see Section 2.1), and its variants. Objects of  $\mathbf{MCor}$  will be denote by Gothic letters (like  $\mathfrak{X}$ ).

## 2. MAIN THEOREM AND APPLICATIONS TO $\overline{\square}$ -INVARIANT SHEAVES

**2.1. Review on basic definitions and statement of the main theorem.** We collect some basic definitions and results from [10]. We fix a base field  $k$  which is assumed to be perfect. Let  $\mathbf{Sch}$  be the category of  $k$ -schemes separated and of finite type over  $k$  and  $\mathbf{Sm} \subset \mathbf{Sch}$  be the full subcategory of smooth schemes. Let  $\mathbf{MCor}$  be the category of *modulus pairs*: The objects are pairs  $\mathfrak{X} = (\overline{X}, X_\infty)$  with  $\overline{X} \in \mathbf{Sch}$  and a (possibly empty) effective

Cartier divisor  $X_\infty \subset \overline{X}$  such that  $\overline{X} - |X_\infty| \in \mathbf{Sm}$ . A modulus pair  $\mathfrak{X} = (\overline{X}, X_\infty)$  is *proper* (resp. *projective*) if  $\overline{X}$  is proper (resp. *projective*) over  $k$ . Let  $\mathbf{MCor} \subset \underline{\mathbf{MCor}}$  be the full subcategory of proper modulus pairs and write  $\mathbf{MCor}^{\text{proj}}$  for the subcategory of projective modulus pair. A basic example of an object of  $\mathbf{MCor}$  is the cube

$$\overline{\square} := (\mathbb{P}_k^1, \infty).$$

An *admissible prime correspondence* from  $\mathfrak{X} = (\overline{X}, X_\infty)$  to  $\mathfrak{Y} = (\overline{Y}, Y_\infty)$  is a prime correspondence  $V \in \mathbf{Cor}(X, Y)$  with  $X = \overline{X} - |X_\infty|$  and  $Y = \overline{Y} - |Y_\infty|$  satisfying the following condition

$$X_\infty|_{\overline{V}^N} \geq Y_\infty|_{\overline{V}^N},$$

where  $\overline{V}^N \rightarrow \overline{V} \subset \overline{X} \times \overline{Y}$  is the normalization of the closure of  $V$ . It is called *left proper* if  $\overline{V}$  is proper over  $\overline{X}$ . We denote by  $\underline{\mathbf{MCor}}(\mathfrak{X}, \mathfrak{Y}) \subset \mathbf{Cor}(X, Y)$  the subgroup generated by left proper admissible prime correspondences. By [10, Prop 1.2.3]  $\underline{\mathbf{MCor}}(\mathfrak{X}, \mathfrak{Y})$  is preserved by the composition of finite correspondences so that we can define the category  $\underline{\mathbf{MCor}}$  with objects the modulus pairs and morphisms given by admissible left proper correspondences. We denote by  $\mathbf{MCor}$  the full subcategory with objects the proper modulus pairs.

Let  $\underline{\mathbf{MPST}}$  (resp.  $\mathbf{MPST}$ ) be the category of additive presheaves of abelian groups on  $\underline{\mathbf{MCor}}$  (resp.  $\mathbf{MCor}$ ). For  $F \in \underline{\mathbf{MPST}}$  and  $\mathfrak{X} = (\overline{X}, X_\infty) \in \underline{\mathbf{MCor}}$  write  $F_{\mathfrak{X}}$  for the presheaf on the étale site  $\overline{X}_{\text{ét}}$  over  $\overline{X}$  given by  $U \rightarrow F(\mathfrak{X}_U)$  for  $U \rightarrow \overline{X}$  étale, where  $\mathfrak{X}_U = (U, X_\infty \times_{\overline{X}} U) \in \underline{\mathbf{MCor}}$ . We say  $F$  is a Nisnevich sheaf if so is  $F_{\mathfrak{X}}$  for all  $\mathfrak{X} \in \underline{\mathbf{MCor}}$  (see [10, Section 3]).

We write  $\underline{\mathbf{MNST}} \subset \underline{\mathbf{MPST}}$  for the full subcategory of Nisnevich sheaves. By [10, Prop. 3.5.3] the inclusion  $\underline{\mathbf{MNST}} \rightarrow \underline{\mathbf{MPST}}$  has an exact left adjoint  $\underline{a}_{\text{Nis}}^M$  such that  $(\underline{a}_{\text{Nis}}^M F)_{\overline{X}}$  is the Nisnevich sheafification of  $F_{\mathcal{X}}$  for  $F \in \underline{\mathbf{MPST}}$  and  $\mathcal{X} = (\overline{X}, X_\infty) \in \underline{\mathbf{MCor}}$ . Write  $F_{\text{Nis}} = \underline{a}_{\text{Nis}}^M F$  for  $F \in \underline{\mathbf{MPST}}$ . For  $F \in \underline{\mathbf{MNST}}$ , we write

$$(2.1) \quad H^i(\mathfrak{X}_{\text{Nis}}, F) = H^i(\overline{X}_{\text{Nis}}, F_{\mathfrak{X}}) \text{ for } F \in \underline{\mathbf{MNST}}.$$

We let  $\mathbf{MNST} \subset \mathbf{MPST}$  be the full subcategory of those  $F$  such that  $\tau_1 F \in \underline{\mathbf{MNST}}$ . By [10, Prop. 3.7.3] the inclusion  $\mathbf{MNST} \rightarrow \mathbf{MPST}$  has an exact left adjoint  $a_{\text{Nis}}$  such that  $\underline{a}_{\text{Nis}}^M \tau_1 = \tau_1 a_{\text{Nis}}$ .

By [10, Prop 2.2.1, Prop 2.3.1, Prop 2.4.1] there are three pairs of adjoint functors  $(\omega_!, \omega^*)$ ,  $(\underline{\omega}_!, \underline{\omega}^*)$  and  $(\tau_!, \tau^*)$ :

$$(2.2) \quad \mathbf{PST} \begin{array}{c} \xleftarrow{\omega_!} \\ \xrightarrow{\omega^*} \end{array} \underline{\mathbf{MPST}} \begin{array}{c} \xleftarrow{\tau_!} \\ \xrightarrow{\tau^*} \end{array} \mathbf{MPST} \begin{array}{c} \xleftarrow{\omega_!} \\ \xrightarrow{\omega^*} \end{array} \mathbf{PST},$$

which are given by

$$(2.3) \quad \underline{\omega}^* F(\overline{X}, X_\infty) = F(\overline{X} \setminus |X_\infty|), \quad \omega_! H(X) = H(X, \emptyset),$$

$$(2.4) \quad \omega^* F(\overline{X}, X_\infty) = F(\overline{X} \setminus |X_\infty|), \quad \omega_! G(X) \cong \varinjlim_{\mathfrak{X} \in \mathbf{MSm}(X)} G(\mathfrak{X}),$$

$$(2.5) \quad \tau^* F(\mathfrak{X}) = F(\mathfrak{X}), \quad \tau_! G(\mathfrak{U}) \cong \varinjlim_{\mathcal{X} \in \text{Comp}(\mathfrak{U})} G(\mathfrak{X}),$$

where  $\mathbf{MSm}(X)$  is the subcategory of  $\mathbf{MCor}$  with objects the proper modulus pairs  $(\overline{X}, X_\infty)$  such that  $\overline{X} - |X_\infty| = X$  and the morphisms of modulus pairs which map to the identity in  $\mathbf{Cor}(X, X)$ , and  $\text{Comp}(\mathfrak{U})$  is the category of compactifications of  $\mathfrak{U} = (\overline{U}, U_\infty)$ , namely objects are proper modulus pairs  $\mathfrak{X} = (\overline{X}, \overline{U}_\infty + \Sigma)$ , where  $\overline{U}_\infty$  and  $\Sigma$  are effective Cartier

divisors on  $\overline{X}$  so that  $\overline{X} \setminus |\Sigma| = \overline{U}$  and  $\overline{U}_{\infty|\overline{U}} = U_{\infty}$ , and the morphisms are those which map to the identity in  $\underline{\mathbf{MCor}}(\mathfrak{U}, \mathfrak{U})$ . The functors  $\omega_{!}, \underline{\omega}_{!}, \tau_{!}$  are exact and we have

$$\omega_{!} = \underline{\omega}_{!}\tau_{!} \quad \text{and} \quad \tau_{!}\omega^{*} = \underline{\omega}^{*}.$$

We now introduce two basic properties of  $\underline{\mathbf{MPST}}$  and  $\mathbf{MPST}$ . The first property is *semi-purity*

**Definition 2.1.1.** We say  $F \in \underline{\mathbf{MPST}}$  is *semi-pure* if the unit map  $F \rightarrow \underline{\omega}^{*}\underline{\omega}_{!}F$  is injective.

The second property is an analogue of homotopy invariance exploited by Voevodsky. In order to make sense of it, recall that the categories  $\underline{\mathbf{MCor}}$  and  $\mathbf{MCor}$  enjoy a symmetric monoidal structure defined as follows. For  $\mathfrak{X} = (\overline{X}, X_{\infty})$  and  $\mathfrak{Y} = (\overline{Y}, Y_{\infty})$ , the modulus pair  $\mathfrak{X} \otimes \mathfrak{Y}$  is given as

$$\mathfrak{X} \otimes \mathfrak{Y} = (\overline{X} \times \overline{Y}, X_{\infty} \times \overline{Y} + \overline{X} \times Y_{\infty}).$$

See [10, 1.4]. We can now give the following definition:

**Definition 2.1.2.** We say  $F \in \underline{\mathbf{MPST}}$  (resp.  $F \in \mathbf{MPST}$ ) *cube invariant* if for any  $\mathfrak{X} \in \underline{\mathbf{MCor}}$  (resp.  $\mathfrak{X} \in \mathbf{MCor}$ ), the pullback along  $\mathfrak{X} \otimes \overline{\square} \rightarrow \mathfrak{X}$  induces an isomorphism

$$F(\mathfrak{X}) \cong F(\mathfrak{X} \otimes \overline{\square}).$$

We write  $\mathbf{CI} \subset \mathbf{MPST}$  and  $\underline{\mathbf{CI}} \subset \underline{\mathbf{MPST}}$  for the full subcategories of cube invariant objects, and  $\mathbf{CI}^{\tau} \subset \underline{\mathbf{MPST}}$  for the essential image of  $\mathbf{CI}$  under  $\tau_{!} : \mathbf{MPST} \rightarrow \underline{\mathbf{MPST}}$ . We also write  $\mathbf{CI}_{\text{Nis}}^{\tau} = \mathbf{CI}^{\tau} \cap \underline{\mathbf{MNST}}$ .

By [20, Lem.1.16], we have

$$(2.6) \quad \mathbf{CI}^{\tau} \subset \underline{\mathbf{CI}} \quad \text{and} \quad \mathbf{CI} = \tau_{!}^{-1}(\underline{\mathbf{CI}}).$$

By [11, Prop. 2.2.5] the inclusion  $\mathbf{CI} \rightarrow \mathbf{MPST}$  admits a left adjoint  $h_0^{\overline{\square}}$  given by

$$h_0^{\overline{\square}}(F)(\mathfrak{Y}) = \text{coker}(F(\mathfrak{X} \otimes \overline{\square}) \xrightarrow{i_0^{*} - i_1^{*}} F(\mathfrak{Y})) \quad \text{for } F \in \mathbf{MPST}, \mathfrak{Y} \in \mathbf{MCor},$$

where  $i_{\epsilon} : (\text{Spec } k, \emptyset) \rightarrow \overline{\square}$  is the morphism in  $\mathbf{MCor}$  induced by a  $k$ -rational point  $\epsilon \in \mathbb{P}^1 - \{\infty\}$ . For  $\mathfrak{X} \in \mathbf{MCor}$  we write

$$h_0^{\overline{\square}}(\mathfrak{X}) = h_0^{\overline{\square}}(\mathbb{Z}_{\text{tr}}(\mathfrak{X})),$$

where  $\mathbb{Z}_{\text{tr}}(\mathfrak{X}) = \mathbf{MCor}(-, \mathfrak{X}) \in \mathbf{MPST}$  is the Yoneda object of  $\mathfrak{X}$ . By abuse of notation we write  $h_0^{\overline{\square}}(\mathfrak{X})$  for  $\tau_{!}h_0^{\overline{\square}}(\mathfrak{X})$ . By [10, Lem. 1.8.3] and (2.5) and the exactness of  $\tau_{!}$ , we have

$$h_0^{\overline{\square}}(\mathfrak{X})(\mathfrak{U}) = \text{coker}(\underline{\mathbf{MCor}}(\mathfrak{U} \otimes \overline{\square}, \mathfrak{X}) \xrightarrow{i_0^{*} - i_1^{*}} \underline{\mathbf{MCor}}(\mathfrak{U}, \mathfrak{X})) \quad \text{for } \mathfrak{U} \in \underline{\mathbf{MCor}}.$$

Finally we write  $h_0^{\overline{\square}}(\mathfrak{X})_{\text{Nis}} = \underline{a}_{\text{Nis}}^M h_0^{\overline{\square}}(\mathfrak{X})$ . We now state the main theorem of this paper.

**Theorem 2.1.3.** For  $\mathfrak{X} \in \mathbf{MCor}^{\text{proj}}$ ,  $h_0^{\overline{\square}}(\mathfrak{X})_{\text{Nis}}$  is semi-pure.

**2.2. Strict  $\overline{\square}$ -invariance of  $\overline{\square}$ -invariant sheaves.** Let  $\underline{\mathbf{MCor}}_{l_s} \subset \underline{\mathbf{MCor}}$  be the full subcategory of objects  $(\overline{X}, X_{\infty})$  with  $\overline{X} \in \mathbf{Sm}$  and  $|X_{\infty}|$  a simple normal crossing divisor on  $\overline{X}$ . As an application of Theorem 2.1.3, we prove the following theorem, which plays a fundamental role in theory of motives with modulus.

**Theorem 2.2.1.** For  $F \in \mathbf{CI}^{\tau}$  and  $\mathfrak{X} \in \underline{\mathbf{MCor}}_{l_s}$ , we have an isomorphism

$$\pi^{*} : H^q(\mathfrak{X}_{\text{Nis}}, F_{\text{Nis}}) \simeq H^q((\mathfrak{X} \otimes \overline{\square})_{\text{Nis}}, F_{\text{Nis}})$$

for any integer  $q \geq 0$ , induced by the projection  $\pi : \mathfrak{X} \otimes \overline{\square} \rightarrow \mathfrak{X}$ .

*Proof.* By [20, Theorem 0.6 and 0.8], the assertion holds if we assume further that  $F$  is semi-pure. In general we may write  $F = \tau_! G$  with  $G \in \mathbf{CI}$ . We have the surjective map in **MPST**

$$\bigoplus_{\alpha \in G(\mathfrak{Y})} \mathbb{Z}_{\mathrm{tr}}(\mathfrak{Y}) \rightarrow G,$$

where the direct sum ranges over all  $\mathfrak{Y} \in \mathbf{MCor}$  and  $\alpha \in G(\mathfrak{Y})$ , which induce the Yoneda maps  $\alpha: \mathbb{Z}_{\mathrm{tr}}(\mathfrak{Y}) \rightarrow G$  in **MPST**. By Chow's Lemma, we can in fact assume that the sum ranges over  $\mathfrak{Y} \in \mathbf{MCor}^{\mathrm{proj}}$ . Since  $G \in \mathbf{CI}$ , it factors through a surjective map

$$P := \bigoplus_{\alpha \in G(\mathfrak{Y})} h_0^{\square}(\mathfrak{Y}) \rightarrow G.$$

Let  $H$  be its kernel so that we have an exact sequence in **CI**

$$0 \rightarrow K \rightarrow \bigoplus_{\alpha \in G(\mathfrak{Y})} h_0^{\square}(\mathfrak{Y}) \rightarrow G \rightarrow 0.$$

By the exactness of  $\tau_!$  and  $\underline{a}_{\mathrm{Nis}}^M$  this induces an exact sequence

$$(2.7) \quad 0 \rightarrow \underline{a}_{\mathrm{Nis}}^M \tau_! K \rightarrow H \rightarrow F \rightarrow 0 \quad \text{with } H = \underline{a}_{\mathrm{Nis}}^M(\tau_! P) = \bigoplus_{\alpha \in G(\mathfrak{Y})} h_0^{\square}(\mathfrak{Y})_{\mathrm{Nis}}.$$

By Theorem 2.1.3  $H$  is semi-pure and hence so is  $\underline{a}_{\mathrm{Nis}}^M \tau_! K$  (see [20, Lemma 1.25]). Moreover  $H$  and  $\underline{a}_{\mathrm{Nis}}^M \tau_! K$  are in  $\mathbf{CI}^r$  by [20, Theorem 0.6] and (2.6) and [10, Proposition 3.7.3]. Hence Theorem 2.2.1 holds for  $H$  and  $\underline{a}_{\mathrm{Nis}}^M \tau_! K$ . Now it holds also for  $F$  by the long exact sequence of cohomology groups arising from (2.7).  $\square$

**2.3. Representability of cohomology of  $\square$ -invariant sheaves.** Recall that the category  $\underline{\mathbf{MDM}}^{\mathrm{eff}}(k)$  is defined as the the  $\square$ -localization (see [10, §6.3]):

$$L^{\square} : D(\underline{\mathbf{MNST}}) \rightarrow D(\underline{\mathbf{MNST}})_{\square} := \underline{\mathbf{MDM}}^{\mathrm{eff}}(k).$$

of the (unbounded) derived category of sheaves  $D(\underline{\mathbf{MNST}})$ . For  $\mathfrak{X} \in \mathbf{MCor}$  let  $M(\mathfrak{X}) = L^{\square}(\mathbb{Z}_{\mathrm{tr}}(\mathfrak{X}))$  (the *motive* of  $\mathfrak{X}$ ).

**Theorem 2.3.1.** *Let  $F^{\bullet} \in D(\underline{\mathbf{MNST}})$  be such that  $H^i(F^{\bullet}) \in \mathbf{CI}_{\mathrm{Nis}}^r$  for all  $i \in \mathbb{Z}$ . For  $\mathfrak{X} \in \mathbf{MCor}_{l_s}$ , there exists a natural isomorphism*

$$\mathrm{Hom}_{\underline{\mathbf{MDM}}^{\mathrm{eff}}}(M(\mathfrak{X}), L^{\square}(F^{\bullet}[i])) \cong H^i(\mathfrak{X}_{\mathrm{Nis}}, F^{\bullet}).$$

*Proof.* Let

$$\overline{\square}_{\nu}^n = \ker \left( \bigoplus_i (p_i)_* : \mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{\otimes n}) \rightarrow \bigoplus_{1 \leq i \leq n} \mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{\otimes n-1}) \right),$$

where  $p_i: \overline{\square}^{\otimes n} \rightarrow \overline{\square}^{\otimes n-1}$  is the projection omitting the  $i$ -th component. By [10, 5.6.3] it suffices to show that the natural map

$$F^{\bullet} \rightarrow RC_*(F^{\bullet}) := \underline{\mathrm{Hom}}_{D(\underline{\mathbf{MNST}})}(\overline{\square}_{\nu}^{\bullet}, F^{\bullet})$$

induces an isomorphism  $H^i(\mathfrak{X}_{\mathrm{Nis}}, F) \simeq H^i(\mathfrak{X}_{\mathrm{Nis}}, RC_*(F))$ . By [10, Proposition 3.7.12], the category  $D(\underline{\mathbf{MNST}})$  is left  $t$ -complete, i.e. for any object  $F^{\bullet}$  the canonical map

$$F^{\bullet} \rightarrow \varprojlim_n \tau_{\geq -n} F^{\bullet}$$



is an equivalence. Combining this with the equivalence  $F^\bullet \rightarrow \operatorname{hocolim}_{\rightarrow n} \tau_{\leq n} F^\bullet$  (right completeness), we may assume up to a shift that  $F^\bullet = F[0]$  for some  $F \in \mathbf{CI}_{\text{Nis}}^\tau$ . Thus we are reduced to show that for  $\mathfrak{X} = (\overline{X}, X_\infty) \in \underline{\mathbf{MCor}}_{l_s}$  with  $\overline{X}$  henselian local,

$$H^i(RC_*(F))(\mathfrak{X}) = 0 \text{ for } i \neq 0 \text{ and } F(\mathfrak{X}) \simeq H^0(RC_*(F))(\mathfrak{X}).$$

Let  $F \rightarrow I^\bullet$  with  $I^\bullet \in K(\underline{\mathbf{MNST}})$  be an injective resolution. Then

$$\begin{aligned} H^i(RC_*(F))(\mathfrak{X}) &\simeq H^i(\operatorname{Tot}(\underline{\mathbf{Hom}}_{\underline{\mathbf{MNST}}}(\overline{\square}_\nu^\bullet, I^\bullet)(\mathfrak{X}))) \\ &\simeq H^i(\operatorname{Tot}(\underline{\mathbf{Hom}}_{\underline{\mathbf{MNST}}}(\mathbb{Z}_{\text{tr}}(\mathfrak{X}) \otimes \overline{\square}_\nu^\bullet, I^\bullet))). \end{aligned}$$

Hence we have a spectral sequence

$$E_1^{p,q} = H^q(\underline{\mathbf{Hom}}_{\underline{\mathbf{MNST}}}(\mathbb{Z}_{\text{tr}}(\mathfrak{X}) \otimes \overline{\square}_\nu^{-p}, I^\bullet)) \Rightarrow H^{p+q}(RC_*(F))(\mathfrak{X}).$$

Noting

$$H^q(\underline{\mathbf{Hom}}_{\underline{\mathbf{MNST}}}(\mathbb{Z}_{\text{tr}}(\mathfrak{X}) \otimes \overline{\square}_\nu^{-p}, I^\bullet)) \simeq H^q((\mathfrak{X} \otimes \overline{\square}_\nu^{-p})_{\text{Nis}}, F),$$

we get  $E_2^{p,q} = 0$  unless  $p = q = 0$  by Theorem 2.2.1. This complete the proof.  $\square$

For  $F \in \underline{\mathbf{MNST}}$ , let  $C_*(F)$  be the *Suslin complex with modulus* (see [11, Def.3.2.4]). It is a complex in  $\underline{\mathbf{MNST}}$  whose non-zero terms are in non-positive degrees and whose term in degree  $-n$  for  $n \in \mathbb{Z}_{\geq 0}$  is given by

$$C_n(F)(\mathfrak{Y}) := \underline{\mathbf{Hom}}_{\underline{\mathbf{MNST}}}(\mathbb{Z}_{\text{tr}}(\mathfrak{Y}) \otimes \overline{\square}_\nu^n, F) \quad (\mathfrak{Y} \in \mathbf{MCor}).$$

The following results has been shown in [11] under an assumption of resolution of singularities.

**Corollary 2.3.2.** *For  $\mathfrak{X} \in \underline{\mathbf{MCor}}_{l_s}$  and  $F \in \underline{\mathbf{MNST}}$ , there exists a natural isomorphism*

$$(2.8) \quad \underline{\mathbf{Hom}}_{\underline{\mathbf{MDM}}^{\text{eff}}}(M(\mathfrak{X}), L^{\overline{\square}}(F[i])) \cong H^i(\mathfrak{X}_{\text{Nis}}, \tau_! C_*(F)).$$

*In particular, we have a natural isomorphism*

$$(2.9) \quad \underline{\mathbf{Hom}}_{\underline{\mathbf{MDM}}^{\text{eff}}}(M(\mathfrak{X}), M(\mathfrak{Y})[i]) \simeq H^i(\mathfrak{X}_{\text{Nis}}, \tau_! C_*(\mathfrak{Y})) \text{ for } \mathfrak{Y} \in \mathbf{MCor}.$$

*Proof.* By Lemma 2.3.4 below, the natural map  $F[0] \rightarrow C_*(F)$  induces an isomorphism

$$L^{\overline{\square}}(\tau_! F[0]) \cong L^{\overline{\square}}(\tau_! C_*(F)) \text{ in } \underline{\mathbf{MDM}}^{\text{eff}}.$$

Hence the first assertion follows immediately from Theorem 2.3.1.  $\square$

*Remark 2.3.3.* Let  $\mathfrak{Y} = (\overline{Y}, Y_\infty) \in \mathbf{MCor}$  be a proper modulus pair. Taking  $\mathfrak{X} = (\operatorname{Spec} k, \emptyset)$ , (2.9) implies a natural isomorphism

$$\underline{\mathbf{Hom}}_{\underline{\mathbf{MDM}}^{\text{eff}}}(M(\operatorname{Spec} k, \emptyset), M(\mathfrak{Y})[-i]) \cong H_i(\overline{Y}, Y_\infty),$$

where the right hand side is the Suslin homology considered in [19, Definition 3.1] and generalizing the homology defined by Suslin and Voevodsky in [25] (see [11, Rem. 3.3.2]). In particular we have a natural isomorphism

$$\underline{\mathbf{Hom}}_{\underline{\mathbf{MDM}}^{\text{eff}}}(M(\operatorname{Spec} k, \emptyset), M(\mathfrak{Y})) \cong \operatorname{CH}_0(\overline{Y}, Y_\infty),$$

where the right hand side is the Chow group of zero-cycles with modulus considered in [13].

We consider the  $\overline{\square}$ -localization:

$$\begin{aligned} L^{\overline{\square}, p}: D(\mathbf{MPST}) &\rightarrow D(\mathbf{MPST})_{\overline{\square}}, \\ L^{\overline{\square}}: D(\underline{\mathbf{MNST}}) &\rightarrow D(\underline{\mathbf{MNST}})_{\overline{\square}} := \underline{\mathbf{MDM}}^{\text{eff}}(k), \end{aligned}$$

**Lemma 2.3.4.** *For  $F \in \underline{\mathbf{MNST}}$ ,  $L^{\overline{\square}}(F) \simeq L^{\overline{\square}}(C_*(F))$  in  $\underline{\mathbf{MDM}}^{\text{eff}}(k)$ .*

*Proof.* We have a functor

$$a_{\text{Nis}}: D(\mathbf{MPST})_{\overline{\square}} \rightarrow D(\mathbf{MNST})_{\overline{\square}}$$

induced by the exact functor  $a_{\text{Nis}}: \mathbf{MPST} \rightarrow \mathbf{MNST}$ . By [11, Theorem 2.6.4 and 2.8.1], for  $G \in \mathbf{MPST}$  we have an isomorphism in  $D(\mathbf{MPST})_{\overline{\square}}$ :

$$(2.10) \quad L^{\overline{\square}, p}(G) \simeq L^{\overline{\square}, p}(C_*(G)^p),$$

where  $C_\bullet(G)^p = \underline{\mathbf{Hom}}_{\mathbf{MPST}}(\overline{\square}_\nu^\bullet, G)$ . Hence, for  $F \in \mathbf{MNST}$  we get an isomorphism in  $\mathbf{MDM}^{\text{eff}}(k)$ :

$$\begin{aligned} L^{\overline{\square}}(F) &\simeq L^{\overline{\square}}(a_{\text{Nis}} i_{\text{Nis}} F) \stackrel{(1)}{\simeq} a_{\text{Nis}} L^{\overline{\square}, p}(i_{\text{Nis}} F) \stackrel{(2)}{\simeq} a_{\text{Nis}} L^{\overline{\square}, p}(C_\bullet(i_{\text{Nis}} F)^p) \\ &\stackrel{(3)}{\simeq} L^{\overline{\square}}(a_{\text{Nis}} C_*(i_{\text{Nis}} F)^p) \stackrel{(4)}{\simeq} L^{\overline{\square}}(C_\bullet(F)), \end{aligned}$$

where (1) and (3) follows from the commutativity of the diagram (see [10, Proposition 6.3.2])

$$\begin{array}{ccc} D(\mathbf{MPST}) & \xrightarrow{L^{\overline{\square}, p}} & D(\mathbf{MPST})_{\overline{\square}}, \\ \downarrow a_{\text{Nis}} & & \downarrow a_{\text{Nis}} \\ D(\mathbf{MNST}) & \xrightarrow{L^{\overline{\square}}} & D(\mathbf{MNST})_{\overline{\square}} \end{array}$$

and (2) follows from (2.10), and (4) follows from an isomorphism

$$C_*(F) \simeq a_{\text{Nis}} C_*(i_{\text{Nis}} F)^p$$

which is obtained by applying  $a_{\text{Nis}}$  to [11, (3.5)].  $\square$

As further application, we remind the reader of the following result, which uses Theorem 2.2.1 as geometric input in a crucial way, and that is parallel to one of the main results of [28]. In order to apply the results of [20], we assume that the following condition (which is a form of resolution of singularities) holds.

(RS) For any  $X \in \mathbf{Sm}$ , there exists an open immersion  $X \hookrightarrow \overline{X}$  such that  $\overline{X}$  is smooth and proper over  $k$  and that  $\overline{X} - X$  is the support of a simple normal crossing divisor on  $\overline{X}$ .

**Theorem 2.3.5** (see [11], Theorem 3.7.1). *Assume that  $k$  has resolution of singularities (RS). The standard  $t$ -structure on  $D(\mathbf{MNST})$  induces a  $t$ -structure, the homotopy  $t$ -structure on  $\mathbf{MDM}^{\text{eff}}(k)$  via the inclusion  $\mathbf{MDM}^{\text{eff}}(k) \hookrightarrow D(\mathbf{MNST})$ . Its heart is  $\mathbf{CI}_{\text{Nis}}$ , which is a Serre subcategory of  $\mathbf{MNST}$ . Moreover, we can identify the image of  $\mathbf{MDM}^{\text{eff}}(k)$  in  $D(\mathbf{MNST})$  as*

$$\mathbf{MDM}^{\text{eff}}(k) = \{F^\bullet \in D(\mathbf{MNST}) \mid H^i(F^\bullet) \in \mathbf{CI}_{\text{Nis}} \text{ for every } i \in \mathbb{Z}\}$$

We remark that the assumption (RS) is necessary since Theorem 2.2.1 is proved only for  $\mathfrak{X} \in \mathbf{MCor}_{l_s}$ . See the [11, 3.7].

### 3. SEMI-PURITY FOR RELATIVE CURVES

In this Section, we prove a variant (in fact, a generalization) of Theorem 2.1.3 for relative curves. This generalization is necessary for the reduction of the main theorem from case of relative surfaces to the case of relative dimension 1.

**3.1. Statement and first reductions.** Let  $\mathcal{O}$  be a local normal domain and fix a non-zero element  $\Pi \in \mathcal{O}$ . Let  $\mathcal{C}$  be a normal scheme proper surjective of relative dimension 1 over  $\mathcal{O}$ . Let  $\mathcal{C}_\eta$  be the generic fiber. Note that  $\mathcal{C}_\eta$  is regular. Let  $D \subset \mathcal{C}$  be an effective Cartier divisor which is finite over  $\mathcal{O}$ . Let  $Q(\mathcal{C})$  be the function field of  $\mathcal{C}$ .

**Definition 3.1.1.** We say that a closed point  $x \in \mathcal{C}_\eta$  satisfies the modulus condition over  $\mathcal{O}$  (for short,  $x$  satisfies  $(MC)_\mathcal{O}$ ) if the following holds. Let  $Z$  be the closure of  $x$  in  $\mathcal{C}$ , and let  $Z^N \rightarrow Z$  be its normalization with the natural map  $\nu_Z : Z^N \rightarrow \mathcal{C}$ . Then we have an inequality of Cartier divisors

$$\nu_Z^* D \leq \nu_Z^*(\Pi),$$

where  $(\Pi)$  is the Cartier divisor on  $\text{Spec } \mathcal{O}$  defined by  $\Pi$ .

**Theorem 3.1.2.** Take  $f \in Q(\mathcal{C})$  satisfying the following conditions:

- (i)  $f = 1 + \gamma s$  with  $\gamma \in \mathcal{O}_{\mathcal{C}, D}$ , the semi-localization of  $\mathcal{C}$  at the generic points of  $D$ , and  $s \in \mathcal{O}_{\mathcal{C}, D}$  a local equation of  $D$ .
- (ii) Every prime component of  $\text{div}_{\mathcal{C}_\eta}(f)$  satisfies  $(MC)_\mathcal{O}$ .

Let  $W \subset \mathcal{C} \times_{\mathcal{O}} \mathbb{P}_{\mathcal{O}}^1$  be the closure of the graph  $\Gamma_f \subset \mathcal{C}_\eta \times_\eta \mathbb{P}_\eta^1$  of  $f$ . Then, for any irreducible component  $T$  of  $W$  with its normalization  $T^N$  and the natural maps  $\nu_T : T^N \rightarrow \mathbb{P}_{\mathcal{O}}^1$  and  $\mu_T : T^N \rightarrow \mathcal{C}$ , we have

$$(3.1) \quad \mu_T^*(D) \leq \nu_T^*(1_{\mathcal{O}} + \mathbb{P}_{(\Pi)}^1),$$

where  $1_{\mathcal{O}}$  and  $\mathbb{P}_{(\Pi)}^1$  are Cartier divisor on  $\mathbb{P}_{\mathcal{O}}^1$ .

First we claim that it suffices to prove the theorem assuming  $\mathcal{O}$  is a henselian DVR. Note  $A = \bigcap_{\mathfrak{p} \subset A} A_{\mathfrak{p}}$  for a normal domain  $A$ , where  $\mathfrak{p}$  range over all prime ideals of height one. Thus we may check (3.1) locally at a point  $t$  of codimension one in  $T^N$ . It suffices to consider the case where  $t$  lies on the inverse image of  $T \cap (D \times_{\mathcal{O}} \mathbb{P}_{\mathcal{O}}^1) \subset \mathcal{C} \times_{\mathcal{O}} \mathbb{P}_{\mathcal{O}}^1$ . Since  $D \times_{\mathcal{O}} \mathbb{P}_{\mathcal{O}}^1$  is finite over  $\mathbb{P}_{\mathcal{O}}^1$  by the assumption on  $D$ , the closure  $\overline{\{t\}}$  of  $t$  in  $T^N$  is finite over  $\mathbb{P}_{\mathcal{O}}^1$ . Noting  $\dim(\overline{\{t\}}) = \dim(T) - 1 = \dim(\mathbb{P}_{\mathcal{O}}^1) - 1$ , this implies that  $t$  maps to a point of codimension one in  $\mathbb{P}_{\mathcal{O}}^1$ , and hence maps to a point of codimension  $\leq 1$  in  $\text{Spec } \mathcal{O}$ . Since (3.1) can be checked étale locally, we may replace  $\mathcal{O}$  by its henselization at a point of codimension one and  $\mathcal{C}$  by its base change. This proves the claim.

In what follows  $\mathcal{O}$  is a henselian DVR with a prime element  $\pi$  and  $\Pi = \pi^e$  for an integer  $e > 0$ . Let  $K$  be the quotient field and  $v$  be the normalized valuation of  $\mathcal{O}$ .

Since (3.1) can be checked étale locally, the above theorem follows from the following local version. Let  $A$  be the henselization of the local ring of  $\mathcal{C}$  at a closed point, which is an integral normal local domain essentially of finite type over  $\mathcal{O}$  with  $\dim(A) = 2$ . Let  $s \in A$  be a local equation of  $D$ . By the assumption,  $A/(s)$  is finite flat over  $\mathcal{O}$  and  $A/(\pi^e, s)$  is Artinian. By ZMT the natural map  $\mathcal{O}[s] \rightarrow A$  induces finite map  $\phi : \text{Spec } A \rightarrow \text{Spec } R$  with  $R = \mathcal{O}\{s\}$ . Let  $Q(A)$  be the quotient field of  $A$ .

**Theorem 3.1.3.** Let  $A$  and  $s$  be as above. Take  $f \in Q(A)$  satisfying the following conditions:

- (i)  $f = 1 + \gamma s$  with  $\gamma$  in the semi-localization of  $A$  at the primes lying over  $(s) \subset A$ .
- (ii) We have  $\pi^e/s \in (A/\mathfrak{p})^N$  for all height-one primes  $\mathfrak{p}$  such that  $v_{\mathfrak{p}}(f) \neq 0$  and  $\mathfrak{p}$  does not divide  $(\pi)$ .

Let  $W \subset \text{Spec } A[\tau]$  be the closure of  $(\tau - f = 0) \subset \text{Spec } Q(A)[\tau]$ . Then, for any irreducible component  $T$  of  $W$  with its normalization  $T^N$ , we have

$$(3.2) \quad (\tau - 1)\pi^e/s \in \Gamma(T^N, \mathcal{O}).$$

Note that Theorem 3.1.2 immediately implies Theorem 6.2.2 in dimension 1. Indeed, if  $\mathfrak{X} = (\overline{C}, C_\infty)$  is in **MCor** with  $\dim(\overline{C}) = 1$ , we can apply Theorem 3.1.2 with  $\mathcal{C} = \overline{C} \times_k S$ , and  $D = C_\infty \times_k S$ . Then  $\mathcal{C}_\eta = \overline{C}_K$  and in view of Proposition 6.2.3 and diagram (6.3), Theorem 3.1.2 gives the injectivity of the map

$$j_{\mathcal{O},(\Pi)}: h_0^\square(\mathfrak{X})(\mathcal{O}, (\Pi)) \rightarrow \text{CH}_0(\overline{C}_K, C_{\infty, K})$$

as required.

### 3.2. Valuative criterion for modulus condition on $\mathcal{O}\{s\}$ .

**Definition 3.2.1.** For  $n \in \mathbb{Z}_{>0}$  and  $f = \sum_{i \geq 0} a_i s^i \in R - \{0\}$ , we define

$$\nu_e(f) \geq n \stackrel{\text{def}}{\Leftrightarrow} v(a_i) \geq n - ei \text{ for all } i \geq 0.$$

We easily check the following.

**Lemma 3.2.2.** *Let  $f, g \in R - \{0\}$  and  $n, m \in \mathbb{Z}_{>0}$ .*

- (1) *If  $\nu_e(f) \geq n$  and  $\nu_e(g) \geq n$ , then  $\nu_e(f \pm g) \geq n$ .*
- (2) *If  $\nu_e(f) \geq n$  and  $\nu_e(g) \geq m$ , then  $\nu_e(fg) \geq nm$ .*
- (3)  *$\nu_e(\pi^m f) \geq n + m$  if and only if  $\nu_e(f) \geq n$ .*

**Lemma 3.2.3.** *Let  $\mathfrak{p} \subset R$  be a height-one prime not dividing  $(\pi)$  such that  $\pi^e/s \in (R/\mathfrak{p})^N$ . Then  $\mathfrak{p}$  is generated by an element of the form*

$$g = a_0 + a_1 s + \cdots + a_m s^m \in \mathcal{O}[s]$$

such that  $a_m \in \mathcal{O}^\times$  and  $\nu_e(g) \geq v(a_0)$ .

*Proof.* Since  $s \bmod \mathfrak{p} \in R/\mathfrak{p}$  is finite over  $\mathcal{O}$ , there is a monic irreducible polynomial

$$g = a_0 + a_1 s + \cdots + a_m s^m \in \mathcal{O}[s] \quad (a_m = 1)$$

such that  $g \in \mathfrak{p}$ . This implies  $\mathfrak{p} = (g)$  by the irreducibility of  $g$ . Put  $\theta = \pi^e/s \bmod \mathfrak{p} \in Q(R/\mathfrak{p})$ . From  $g = 0 \in R/\mathfrak{p}$ , we get

$$\theta^m + \sum_{1 \leq i \leq m} \frac{a_i \pi^{ei}}{a_0} \theta^{m-i} = 0 \in Q(R/\mathfrak{p}).$$

Since  $g \in \mathcal{O}[s]$  is irreducible over  $K$ , this gives a minimal equation of  $\theta$  over  $K$ . Since  $\theta$  is finite over  $\mathcal{O}$  by the assumption, this implies  $a_i \pi^{ei}/a_0 \in \mathcal{O}$  for all  $i$ , which implies  $\nu_e(g) \geq v(a_0)$ .  $\square$

**Lemma 3.2.4.** *Let  $f \in R$  and  $a_0 = f \bmod s \in \mathcal{O}$ . Assume*

*(MC)<sub>R</sub> For any height-one prime  $\mathfrak{p}$  dividing  $f$  but not dividing  $(\pi)$ ,  $\pi^e/s \in (R/\mathfrak{p})^N$ .*

*Assume further  $a_0 \neq 0$ . Then  $\nu_e(f) \geq v(a_0)$ .*

*Proof.* Considering the prime decomposition of  $f$  in  $R$ , this follows from Lemmas 3.2.2 and 3.2.3.  $\square$

**3.3. Criterion for modulus condition on  $A$ .** Let the assumption be as in the statement of Theorem 3.1.3. Write  $X = \text{Spec } A$  and  $D = (s) \subset X$ . Let  $\psi: \tilde{X} \rightarrow X$  be the blowup with center in  $(s, \pi^e)$  and  $D' \subset \tilde{X}$  be the proper transform of  $D$ . Write

$$X_+ = \tilde{X} - D' = \text{Spec } A[t]/(st - \pi^e)$$

with the induced map  $\psi_+: X_+ \rightarrow X$ . For a height-one prime  $\mathfrak{p} \subset A$  write  $Z_{\mathfrak{p}} = \overline{\{\mathfrak{p}\}} \subset X$  and let  $Z'_{\mathfrak{p}} \subset \tilde{X}$  be its proper transform.

**Lemma 3.3.1.** *Assume  $\mathfrak{p}$  does not divide  $(\pi)$ . Then the following conditions are equivalent.*

- (i)  $\pi^e/s \in (A/\mathfrak{p})^N$ .
- (ii)  $Z'_\mathfrak{p} \subset X_+ \Leftrightarrow Z'_\mathfrak{p} \cap D' = \emptyset$ .

*Proof.* (ii) is equivalent to that  $Z'_{\mathfrak{p},+} := Z'_\mathfrak{p} \cap X_+$  is finite over  $\mathcal{O}$ . Note that  $Z'_{\mathfrak{p},+}$  is the irreducible component of  $\psi_+^{-1}(Z_\mathfrak{p}) = X_+ \times_X Z_\mathfrak{p}$  which is flat over  $\mathcal{O}$ . Let  $\phi : Z'_\mathfrak{p} = \text{Spec}(A/\mathfrak{p})^N \rightarrow Z_\mathfrak{p}$  be the normalization and  $Z_{\mathfrak{p},+}^N = Z_{\mathfrak{p},+} \times_{Z_\mathfrak{p}} Z_\mathfrak{p}^N$ . Note that  $Z_{\mathfrak{p},+}^N$  is the irreducible component of  $X_+ \times_X Z_\mathfrak{p}^N$  which is flat over  $\mathcal{O}$ . The map  $\phi$  induces a finite surjective map  $Z_{\mathfrak{p},+}^N \rightarrow Z_{\mathfrak{p},+}$ . Hence it suffices to show that (i) is equivalent to that  $Z_{\mathfrak{p},+}^N$  is finite over  $\mathcal{O}$ . Note that the latter condition is equivalent to  $Z_{\mathfrak{p},+}^N = Z_\mathfrak{p}^N$ . Now the desired equivalence follows easily by noting

$$(3.3) \quad X_+ \times_X Z_\mathfrak{p}^N = \text{Spec}(A/\mathfrak{p})^N[t]/(st - \pi^e).$$

Indeed, if (i) holds, there exist  $\theta \in (A/\mathfrak{p})^N$  such that  $s\theta = \pi^e$ . Hence (3.3) implies  $Z_{\mathfrak{p},+}^N \simeq \text{Spec}(A/\mathfrak{p})^N$ . To see the converse, assume  $Z_{\mathfrak{p},+}^N \simeq \text{Spec}(A/\mathfrak{p})^N$ . Note  $(A/\mathfrak{p})^N$  is a DVR and let  $\Pi$  be a prime element of it. Write  $s = u\Pi^n, \pi = v\Pi^m$  in  $A$  with  $m, n \in \mathbb{Z}_{\geq 0}$  and  $u, v \in ((A/\mathfrak{p})^N)^\times$ . It suffices to show  $m \geq n$ . Assume the contrary  $m < n$ . Then

$$Z_{\mathfrak{p},+}^N \simeq \text{Spec}(A/\mathfrak{p})^N[t]/(u\Pi^{n-m}t - v) \simeq \text{Spec} A[1/\Pi],$$

which contradicts the assumption.  $\square$

**Lemma 3.3.2.** *Let  $g \in A$  satisfies the condition:*

$(MC)_A$  *For any height-one prime  $\mathfrak{p} \subset A$  dividing  $(g)$  but not dividing  $(\pi)$ ,  $\pi^e/s \in (A/\mathfrak{p})^N$ .*

*Then there exists  $N > 0$  such that for any  $\alpha \in \mathcal{O}$ ,  $g + \alpha\pi^N \in A$  satisfies  $(MC)_A$ .*

This follows from Lemma 3.3.1 and the following.

**Lemma 3.3.3.** *Let the notation be as in Lemma 3.3.1. Assume that  $g \in A$  satisfies  $(MC)_A$ . Then there exists  $N > 0$  such that for any  $\alpha \in \mathcal{O}$  and any irreducible component  $T$  of  $Z_\alpha := (g + \alpha\pi^N) \subset X = \text{Spec} A$  which is flat over  $\mathcal{O}$ , we have  $T' \cap D' = \emptyset$ , where  $T' \subset \tilde{X}$  is the proper transform of  $T$ .*

*Proof.* Consider the finite map

$$\phi : X \rightarrow Y = \text{Spec} R$$

and put  $W = \phi(Z)$  and  $W_\alpha = \phi(Z_\alpha)$ . Then  $W = (h)$  and  $W_\alpha = (h_\alpha)$  with  $h = N_{A/R}(g)$  and  $h_\alpha = N_{A/R}(g_\alpha)$ . Removing from  $W$  (resp.  $W_\alpha$ ) the component not flat over  $\mathcal{O}$ , we get an effective divisor  $W_{fl} \subset Y$  (resp.  $W_{\alpha,fl} \subset Y$ ). Let  $\psi_Y : \tilde{Y} \rightarrow Y$  be the blowup with center in  $(s, \pi^e)$  with the exceptional divisor  $E \subset \tilde{Y}$ . We have  $\tilde{X} \simeq \tilde{Y} \times_Y X$  with  $\tilde{\phi} : \tilde{X} \rightarrow \tilde{Y}$  the projection. Let  $D'_Y$  (resp.  $W'_{fl}$ , resp.  $W'_{\alpha,fl}$ ) be the proper transform in  $\tilde{Y}$  of  $D_Y = (s) \subset Y$  (resp.  $W_{fl}$ , resp.  $W_{\alpha,fl}$ ). Then we have  $D' = \tilde{\phi}^{-1}(D'_Y)$  and  $T' \subset \tilde{\phi}^{-1}(W'_{\alpha,fl})$  for  $T'$  as in the lemma. Thus it suffices to show  $W'_{\alpha,fl} \cap D'_Y = \emptyset$ . The assumption implies that  $h$  satisfies  $(MC)_R$  and so Lemma 3.3.1 (in case  $A = R$ ) implies  $W'_{fl} \cap D'_Y = \emptyset$ . By the assumption we can write  $h_\alpha = h + \lambda\pi^N$  with  $\lambda \in R$ . Around  $S := D'_Y \cap E$ ,  $\tilde{Y}$  is regular and  $(\sigma, \pi)$  with  $\sigma = s/\pi^e$  is a system of regular parameters. The condition  $W'_{fl} \cap D'_Y = \emptyset$  implies  $h = \pi^m h'$  with  $m \in \mathbb{Z}_{\geq 0}$  and  $h' \in \mathcal{O}_{\tilde{Y}, S}^\times$ . Taking  $N$  so large that  $N > m$ , we get

$$h_\alpha = h + \lambda\pi^N = \pi^m h' + \lambda\pi^N = \pi^m (h' + \lambda\pi^{N-m})$$

and  $h' + \lambda\pi^{N-m} \in \mathcal{O}_{\tilde{Y}, S}^\times$ , which implies  $W'_{\alpha,fl} \cap D'_Y = \emptyset$  as desired.  $\square$

Let  $d = [A : R]$ . For  $f \in A$  and  $\alpha \in R$  we can express the norm of  $\alpha + f$  to  $R$  as

$$(3.4) \quad N_{A/R}(\alpha + f) = \sum_{i=0}^d \alpha^{d-i} \sigma_i(f),$$

where  $\sigma_i(f)$  are symmetric polynomials of homogeneous degree  $i$  in the conjugates  $f_1, \dots, f_r$  of  $f$  over  $R$ , which are independent of  $\alpha$ .

**Lemma 3.3.4.** *Assume that  $g = a + \lambda s \in A$  where  $a \in \mathcal{O} - \{0\}$  and  $\lambda \in A$ , satisfies  $(MC)_A$  from Lemma 3.3.2. Then we have  $\nu_e(\sigma_i(g)) \geq iv(a)$ .*

*Proof.* We may assume  $g = \pi^n + \lambda s \in A$  with  $n \in \mathbb{Z}_{\geq 0}$  and  $\lambda \in A$ . For each  $i \in \mathbb{Z}_{\geq 0}$  there exist integers  $c_{i,j}$  with  $0 \leq j \leq i$  such that for any  $\beta \in \mathcal{O}$ , we have

$$(3.5) \quad \sigma_i(g + \beta) = \sum_{j=0}^i c_{i,j} \sigma_j(g) \beta^{i-j}.$$

Thus it suffices to show  $\nu_e(\sigma_i(\lambda s)) \geq ni$ . By Lemma 3.3.3 there exists an integer  $N > n$  such that for all  $\alpha \in \mathcal{O}$ ,  $g_\alpha = u_\alpha \pi^n + \lambda s \in A$  satisfies  $(MC)_A$ , where  $u_\alpha = 1 + \alpha \pi^{N-n} \in \mathcal{O}^\times$ . By  $\square$  this implies that  $h_\alpha := N_{A/R}(g_\alpha) \in R$  satisfies  $(MC)_R$ . Noting  $h_\alpha \bmod s = u_\alpha^d \pi^{dn}$ , this implies  $\nu_e(h_\alpha) \geq dn$  by Lemma 3.2.4. By (3.4) we have

$$h_\alpha = \sum_{i=0}^d (u_\alpha \pi^n)^{d-i} \sigma_i(\lambda s).$$

By Lemma 3.2.2,  $\nu_e(h_\alpha) \geq dn$  for various choices of  $\alpha$  implies  $\nu_e(\sigma_i(\lambda s)) \geq ni$  as desired.  $\square$

**3.4. Proof of the main theorem for curves.** We now prove Theorem 3.1.3. Let  $\phi : \text{Spec } A[\tau] \rightarrow \text{Spec } R[\tau]$  be the finite map induced by  $\phi : \text{Spec } A \rightarrow \text{Spec } R$  and put  $W_R = \phi(W)$ . It suffices to show (3.2) for any irreducible component  $T$  of  $W_R$ . Let  $\Sigma$  be the set of height-one primes  $\mathfrak{p} \subset A$  not dividing  $(\pi)$  such that  $v_{\mathfrak{p}}(f) \neq 0$ . For each  $\mathfrak{p} \in \Sigma$  take a generator  $h_{\mathfrak{p}}$  of  $\mathfrak{p} \cap R$  as in Lemma 3.2.3. Let  $\Sigma_- \subset \Sigma$  be the subset of  $\mathfrak{p}$  such that  $v_{\mathfrak{p}}(f) < 0$ . By Theorem 3.1.3(i),  $f \in A_{\mathfrak{q}}^\times$  for any prime  $\mathfrak{q}$  dividing  $(s)$ . From this we see that  $\Sigma_-$  coincides with the set of height-one primes  $\mathfrak{p} \subset A$  such that  $v_{\mathfrak{p}}(\gamma) < 0$ . We can choose  $l \in \mathbb{Z}_{\geq 0}$  and  $e_{\mathfrak{p}} \in \mathbb{Z}_{>0}$  for  $\mathfrak{p} \in \Sigma_-$  such that  $h\gamma \in A$  for

$$h = \pi^l \prod_{\mathfrak{p} \in \Sigma_-} h_{\mathfrak{p}}^{e_{\mathfrak{p}}}.$$

Put  $a = h \bmod s \in \mathcal{O}$ . Note  $a \neq 0$  since  $h_{\mathfrak{p}}$  for  $\mathfrak{p} \in \Sigma$  is not divisible by  $s$ . We have

$$g := hf = a + \lambda s \quad \text{with } \lambda = h\gamma + (h - a)/s \in A.$$

By the construction we have

$$W \subset (h\tau - g) \subset \text{Spec } A[\tau],$$

and hence

$$(3.6) \quad W_R \subset (N_{A/R}(h\tau - g)) \subset \text{Spec } R[\tau].$$

Put  $\sigma_i = \sigma_i(h - g)$  and  $t = \tau - 1$ . Since  $h - g \in sA$ , we have

$$(3.7) \quad \sigma_i \in s^i R.$$

By (3.4), we have

$$N_{A/R}(h\tau - g) = N_{A/R}(ht + (h - g)) = \sum_{i=0}^d t^{d-i} h^{d-i} \sigma_i.$$

By the construction,  $h$  (resp.  $g$ ) satisfies the condition  $(MC)_R$  (resp.  $(MC)_A$ ). Hence Lemmas 3.2.4 and 3.3.4 imply

$$\nu_e(h) \geq v(a) \quad \text{and} \quad \nu_e(\sigma_i(g)) \geq iv(a).$$

By (3.5) this implies

$$(3.8) \quad \nu_e(\sigma_i) \geq iv(a) \quad \text{and} \quad \nu_e(h^{d-i} \sigma_i) \geq dv(a).$$

By (3.7) and (3.8), we can write

$$(3.9) \quad h^{d-i} \sigma_i = \sum_{j \geq i} c_{i,j} s^j \quad \text{with} \quad v(c_{i,j}) \geq dv(a) - ej.$$

We then get

$$(3.10) \quad N_{A/R}(h\tau - g) = \sum_{i=0}^d t^{d-i} \left( \sum_{j \geq i} c_{i,j} s^j \right) = \sum_{j \geq 0} s^j \left( \sum_{0 \leq i \leq \min\{j, d\}} c_{i,j} t^{d-i} \right).$$

Write  $n = v(a)$  and take an integer  $N > 0$  such that  $Ne \geq nd$ . Let  $T$  be an irreducible component of  $W_R$  and  $Q(T)$  be its function field. Multiplying (3.10) by  $t^{N-d}/s^N$ , (3.6) implies an equality in  $Q(T)$ :

$$\begin{aligned} 0 &= \sum_{0 \leq j} \frac{1}{s^{N-j}} \left( \sum_{0 \leq i \leq \min\{j, d\}} c_{i,j} t^{N-i} \right) \\ &= \sum_{0 \leq j \leq N} (t/s)^{N-j} \left( \sum_{0 \leq i \leq \min\{j, d\}} c_{i,j} t^{j-i} \right) + \sum_{j \geq N+1} s^{j-N} \left( \sum_{0 \leq i \leq \min\{j, d\}} c_{i,j} t^{N-i} \right). \end{aligned}$$

Put

$$\theta = \pi^e t/s \quad \text{and} \quad \rho_j = \pi^{ej-nd}, \quad \gamma_j = \sum_{0 \leq i \leq \min\{j, d\}} c_{i,j} t^{j-i} \quad \text{for} \quad 1 \leq j \leq N.$$

Multiplying the latter equation by  $\rho_N$ , we get an equality on  $Q(T)$ :

$$\sum_{0 \leq j \leq N} \gamma_j \rho_j \theta^{N-j} + \rho_N \sum_{j \geq N+1} s^{j-N} \left( \sum_{0 \leq i \leq \min\{j, d\}} c_{i,j} t^{N-i} \right) = 0.$$

Note that the second term lies in  $\Gamma(T, \mathcal{O})$  and that  $\gamma_j \rho_j \in \Gamma(T, \mathcal{O})$  for  $0 \leq j \leq N$  in view of (3.9). By definition we have  $\gamma_0 = c_{0,0} = h^d \pmod s = a^d$  and  $\rho_0 = \pi^{-nd}$ . Hence, writing  $a = u\pi^n$  with  $u \in \mathcal{O}^\times$ , we have  $\gamma_0 \rho_0 = u^d \in \mathcal{O}^\times$ . Hence  $\theta$  is integral over  $T$  so that  $\theta \in \Gamma(T^N, \mathcal{O})$  as desired.

#### 4. BERTINI THEOREMS OVER A BASE

In this Section, let  $S$  be the spectrum of a Noetherian local domain  $\mathcal{O}$ . Let  $\mathfrak{m}$  be the maximal ideal of  $\mathcal{O}$ , and write  $k$  for the residue field  $\mathcal{O}/\mathfrak{m}$ . Finally, let  $K$  be the function field of  $\mathcal{O}$ . If  $\mathcal{X}$  is any  $S$ -scheme, write  $\mathcal{X}_\eta$  for its generic fiber and  $\mathcal{X}_s$  for the reduced special fiber. Recall some notations and definitions from e.g. [8] or [21, Section 4].

**Definition 4.0.1.** A *hyperplane*  $H \subset \mathbb{P}_S^N$  is a closed subscheme of the projective space  $\mathbb{P}_S^N$  over  $S$  corresponding to an  $S$ -rational point of the dual  $(\mathbb{P}_S^N)^\vee := \text{Gr}_S(N-1, N)$ .

By definition,  $\mathrm{Gr}_S(N-1, N)(S)$  is the set consisting of surjective maps of  $\mathcal{O}$ -modules,  $q: \mathcal{O}^{\oplus N+1} \rightarrow E$  such that  $E$  is a free  $\mathcal{O}$ -module, of rank  $N$ . Fixing a basis  $\{e_0, \dots, e_N\}$  of  $\mathcal{O}^{\oplus N+1}$ , we can write the kernel of  $q$  as  $\sum_{i=0}^N \langle a_i \rangle e_i \subset \mathcal{O}^{\oplus N+1}$ , for elements  $a_i \in \mathcal{O}$ , where  $\langle a \rangle$  denotes the submodule of  $\mathcal{O}$  generated by  $a$ . Fixing homogeneous coordinates  $X_0, \dots, X_n$  on  $\mathbb{P}_S^N$ , then the hyperplane  $H$  corresponding to  $q$  is the zero locus of the linear polynomial  $q(X) = \sum_{i=1}^N a_i X_i$ . Note that, since the map is surjective, at least one of the  $a_i$ 's is not in the maximal ideal  $\mathfrak{m}$ .

The same polynomial  $q(X)$  defines the hyperplane  $H_\eta \subset \mathbb{P}_K^N$ , corresponding to the  $K$ -point of  $\mathrm{Gr}_S(N-1, N)$  (or equivalently, to a  $K$  point of  $\mathrm{Gr}_K(N-1, N)$ ) given by  $q_K: K^{\oplus N+1} \rightarrow E \otimes_S K$ . Conversely, given a hyperplane  $L \subset \mathbb{P}_K^N$ , defined by a linear polynomial  $p(X) = \sum_{i=1}^N \lambda_i X_i$  with  $\lambda_i \in K$ , its closure  $\overline{L} \subset \mathbb{P}_S^N$  corresponds to the quotient  $\mathcal{O}^{\oplus N+1} \rightarrow \mathcal{O}^{\oplus N+1} / (\sum_{i=1}^N \langle \lambda_i \rangle e_i \cap \mathcal{O}^{\oplus N+1})$ . Explicitly, it is given by the linear polynomial  $q(X)$  obtained by  $p(X)$  by removing the denominators. In general,  $\overline{L}$  might contain the fiber over the closed point  $\mathbb{P}_s^N = \mathbb{P}_S^N \times_S k$ . In this case,  $\overline{L}$  does not define an  $S$ -point of  $\mathrm{Gr}_S(N-1, N)$ , since the  $\mathcal{O}$ -module  $\mathcal{O}^{\oplus N+1} / (\sum_{i=1}^N \langle \lambda_i \rangle e_i \cap \mathcal{O}^{\oplus N+1})$  is not free.

4.0.1. It is convenient to have a coordinate free description. For a free  $\mathcal{O}$ -module of finite rank, write  $\mathbb{P}_S(E)$  for the associated projective bundle over  $S$ . In this case, we will write  $\mathrm{Gr}_S(E)$  for the Grassmannian of hyperplanes in  $\mathbb{P}_S(E)$ . An  $S$ -point of  $\mathrm{Gr}_S(E)$  is a surjective map of  $\mathcal{O}$ -modules  $q: E \rightarrow M$  with  $M$  free and with  $N = \ker(q)$  free of rank 1. Any such  $q$  induces a closed immersion  $H = \mathbb{P}_S(M) \hookrightarrow \mathbb{P}_S(E)$ , and we call  $H$  the hyperplane corresponding to  $q$  (or, equivalently, to  $N$ ).

4.0.2. Let  $E$  be again a free  $\mathcal{O}$ -module of finite rank, and let  $F$  be a submodule of  $E$  such that the quotient  $E/F$  is also free. The inclusion of  $F$  into  $E$  determines a closed immersion  $\mathrm{Gr}_S(F) \hookrightarrow \mathrm{Gr}_S(E)$ , see e.g. [5, III.2.7]. On  $S$ -points, the inclusion

$$\mathrm{Gr}_S(F)(S) \hookrightarrow \mathrm{Gr}_S(E)(S)$$

is explicitly given as follows. Let  $N$  be a 1-dimensional free submodule of  $F$  such that the quotient  $F/N$  is free. Since by assumption the quotient  $E/F$  is free, the quotient  $E/N$  is again free and therefore determines a point  $q: E \rightarrow E/N$  of  $\mathrm{Gr}_S(E)$ . Thus sending  $F \rightarrow F/N$  to  $E \rightarrow E/N$  gives a well defined map on the set of  $S$ -points of the two Grassmannians.

4.1. **Specialization.** If  $S$  is the spectrum of a DVR, we have  $\mathrm{Gr}_S(K) = \mathrm{Gr}_S(S)$  by the valuative criterion of properness. This gives a well defined specialization map

$$(4.1) \quad \begin{aligned} \mathrm{Gr}_S(N-1, N)(K) &\rightarrow \mathrm{Gr}_k(N-1, N)(k), \\ (q: \mathcal{O}^{\oplus N+1} \rightarrow E) &\mapsto (q \otimes_S k: k^{\oplus N+1} \rightarrow E \otimes_S k) \end{aligned}$$

given in coordinates by taking a polynomial  $q(X) = \sum_{i=1}^N a_i X_i$  and reducing it modulo the maximal ideal of  $\mathcal{O}$ . In this case, we can in fact assume that every  $a_i \in \mathcal{O}$  and at least one of the does not belong to  $\mathfrak{m}$  (up to dividing by a suitable power of a uniformizer of  $\mathcal{O}$ ), so that  $\dim_k(E \otimes_S k) = \mathrm{rk}_S(E) = N$ . When  $\dim(S) > 1$ , this is in general not the case, since the closure of a hyperplane  $L$  given by a  $K$ -rational point of  $\mathrm{Gr}_S(N-1, N)$  might contain the whole fiber  $\mathbb{P}_s^N$ , and thus it cannot be specialized to a hyperplane in  $\mathbb{P}_s^N$ . In other words,  $\mathrm{Gr}_S(N-1, N)(K) \neq \mathrm{Gr}_S(N-1, N)(S)$ .

The specialization map is however always defined when we restrict to the set of  $S$ -points

$$\mathrm{sp}: \mathrm{Gr}_S(N-1, N)(S) \rightarrow \mathrm{Gr}_k(N-1, N)(k)$$



or, in coordinate free terms

$$\mathrm{sp}_E: \mathrm{Gr}_S(E)(S) \rightarrow \mathrm{Gr}_k(E)(k), \quad (E \rightarrow M) \mapsto (E \otimes_S k \rightarrow M \otimes_S k)$$

which is always surjective.

The following Lemma is a variant of an argument due to Jannsen and the second author in [8]. If  $S$  is regular and of dimension 1, the proof is easier, and it can be extracted from the proof of [8, Theorem 0.1]. If  $\dim(S) > 1$ , the argument is more delicate.

**Lemma 4.1.1.** *Let  $E$  be a free  $\mathcal{O}$  module of finite rank, and let  $P = \mathrm{Gr}_S(E)$ . Let  $V$  be a Zariski dense open subset of  $P_K$  and let  $U$  be a Zariski dense open subset of  $P_s$ . Suppose that  $k$  is infinite. Then the set*

$$\mathrm{sp}^{-1}(U(k)) \cap V(S) \subset P(S)$$

*is not empty. Here  $V(S) = V(K) \cap P(S) \subset P(K)$ .*

*Proof.* Let  $Z$  be the complement of  $V$  in  $P_K$ , and let  $\overline{Z}$  be the closure of  $Z$  in  $P$ . Let  $\overline{Z}(S)$  be the set of  $S$ -points of  $\overline{Z}$ , i.e.  $\overline{Z}(S) = \overline{Z}(K) \cap P(S)$ . We begin by noting that it is enough to show that  $\mathrm{sp}^{-1}(U(k) \setminus \mathrm{sp}(\overline{Z}(S))) \neq \emptyset$ , i.e. that

$$(4.2) \quad \mathrm{sp}^{-1}(U(k)) \cap V(S) \supset \mathrm{sp}^{-1}(U(k) \setminus \mathrm{sp}(\overline{Z}(S)))$$

and that the latter is not empty. To show the inclusion (4.2), note that

$$\mathrm{sp}^{-1}(U(k) \setminus \mathrm{sp}(\overline{Z}(S))) = \mathrm{sp}^{-1}(U(k)) \setminus \mathrm{sp}^{-1}(\mathrm{sp}(\overline{Z}(S))) = \mathrm{sp}^{-1}(U(k)) \cap \mathrm{sp}^{-1}(\mathrm{sp}(\overline{Z}(S)))^c$$

where the complement of  $\mathrm{sp}^{-1}(\mathrm{sp}(\overline{Z}(S)))$  is taken in  $P(S)$ . It is now easy to see that

$$\mathrm{sp}^{-1}(\mathrm{sp}(\overline{Z}(S)))^c \subset V(S),$$

since  $V(S) = (\overline{Z}(S))^c$  and  $\overline{Z}(S) \subset \mathrm{sp}^{-1}(\mathrm{sp}(\overline{Z}(S)))$ . This proves (4.2).

Since  $\mathrm{sp}$  is surjective, the set  $\mathrm{sp}^{-1}(U(k) \setminus \mathrm{sp}(\overline{Z}(S)))$  is not empty if and only if  $U(k) \setminus \mathrm{sp}(\overline{Z}(S))$  is not empty. Since  $U$  is open and  $P_s$  is an irreducible rational variety over an infinite field  $k$ , the set  $U(k) \setminus \mathrm{sp}(\overline{Z}(S))$  is not empty as long as  $\mathrm{sp}(\overline{Z}(S))$  is nowhere dense in  $P_s$ . Up to shrinking  $V$  further and choosing coordinates on  $P$ , we can assume that  $Z \subset P_K$  is a hypersurface, given by a homogeneous polynomial  $\sum c_I X^I$ , with the obvious multi-index convention. Cleaning the denominators, we get an integral homogeneous equation  $\sum a_I X^I$ , with  $a_I \in \mathcal{O}$ , defining the closure  $\overline{Z}$  of  $Z$  in  $P$ .

We can now divide the proof in two cases. Suppose first that there exists an index  $I$  such that  $a_I \notin \mathfrak{m}$ . Then  $\overline{Z}$  intersects the special fiber  $P_s$  properly, and  $\mathrm{sp}(\overline{Z}(S)) = (\overline{Z} \cap P_s)(k)$ . Since  $\overline{Z} \cap P_s$  is a proper closed subset of the irreducible scheme  $P_s$ , it follows that  $U \setminus (\overline{Z} \cap P_s)$  is open and dense in  $P_s$ , and therefore has a  $k$ -rational point as remarked above.

Suppose now that  $a_I \in \mathfrak{m}$  for every  $I$ . This is the case when  $\overline{Z} \supset P_s$ . Let  $P(S)^\circ \subset P(S)$  be the subset consisting of those points  $(x_0 : \dots : x_N) \in P(S)$  such that  $x_i \notin \mathfrak{m}$  for every  $i = 0, \dots, N$ . Similarly, write  $V(S)^\circ$  for the intersection  $V(S) \cap P(S)^\circ$  and  $Z(S)^\circ = \overline{Z}(S) \cap P(S)^\circ$ . We have an inclusion

$$V(S) \cap \mathrm{sp}^{-1}(U(k)) \supset V(S)^\circ \cap \mathrm{sp}^{-1}(U(k)^\circ) \supset \mathrm{sp}^{-1}(U(F)^\circ \setminus \mathrm{sp}(\overline{Z}(S)))$$

where  $U(F)^\circ$  is the set of  $F$ -points of the open dense subset of  $P_s$  given by  $U \setminus \bigcup_{i=0}^N (X_i = 0)$ . It is clear that the restriction of  $\mathrm{sp}$  to  $P(S)^\circ$  is also surjective, so that in order to show our claim it is enough to prove that  $\mathrm{sp}(Z(S)^\circ)$  is nowhere dense in  $P_s$ . Let

$$n = \min\{m \in \mathbb{Z}_{\geq 0} \mid a_I \in \mathfrak{m}^m \text{ for every } I, \text{ but } a_J \notin \mathfrak{m}^{m+1} \text{ for some } J\} \geq 1$$

and write  $A = \{I \mid a_I \notin \mathfrak{m}^{n+1}\}$ . Let  $L$  be the subspace of the finite dimensional  $k$ -vector space  $\mathfrak{m}^n / \mathfrak{m}^{n+1}$  generated by the set  $\{a_I\}_{I \in A}$ , and choose a basis  $\{b_\lambda\}_{\lambda \in \Lambda}$  for  $L$ . Note that  $V \neq \emptyset$ .

For every  $(x_0 : \dots : x_N) \in Z(S)^o$  (which we can assume to be non empty, since otherwise there is nothing to prove) we have then a non trivial linear relation  $\sum_{J \in \Sigma} a_J x^J = 0$  in  $L \subseteq \mathfrak{m}^n / \mathfrak{m}^{n+1}$ . Spelling this out using the basis  $\{b_\lambda\}_{\lambda \in \Lambda}$ , we get

$$\sum_{J \in \Sigma} a_J x^J = \sum_{\lambda \in \Lambda} \left( \sum_{J \in \Sigma} \nu_{J,\lambda} x^J \right) b_\lambda = 0$$

where  $a_J = \sum_{\lambda \in \Lambda} \nu_{J,\lambda} b_\lambda \in V$ , for  $\nu_{J,\lambda}$  in  $k$ . Since the  $b_\lambda$  are a basis, we have  $(\sum_{J \in \Sigma} \nu_{J,\lambda} x^J) = 0$ . Write  $W_\lambda$  for the proper closed subscheme of  $P_s$  given by  $(\sum_{J \in \Sigma} \nu_{J,\lambda} X^J = 0)$ . It follows from the above discussion that

$$\text{sp}(Z(S)^o) \subset \bigcap_{\lambda \in \Lambda} W_\lambda(F),$$

and since  $\bigcap_{\lambda \in \Lambda} W_\lambda$  is a proper closed subset of the irreducible scheme  $P_s$ , it is nowhere dense as required.  $\square$

*Remark 4.1.2.* The conclusions of Lemma 4.1.1 hold if we replace  $\mathbb{P}_S^N$  or  $\text{Gr}_S(N-1, N)$  with any projective  $S$ -scheme  $P$  such that  $P$  has irreducible fibers,  $P_s = P \otimes_S k$  is a rational variety, the specialization map  $\text{sp}: P(S) \rightarrow P_s(k)$  is surjective and  $U$  and  $V$  are open subsets (dense) in their fibers.

Thanks to the previous Lemma we can parametrize good hyperplanes  $H \subset \mathbb{P}_S^N$  over  $S$  using subsets of the form  $V(S) \cap \text{sp}^{-1}(U(k))$ , for an open subset  $V$  of  $\text{Gr}_K(N-1, N)$ , giving the prescribed behavior on the generic fiber  $H_\eta$ , and an open subset  $U$  of  $\text{Gr}_k(N-1, N)$ , imposing conditions on the special fiber  $H_s$ . We will call a hyperplane  $H \subset \mathbb{P}_S^N$  general if it corresponds to an  $S$ -rational point of a set of the form  $V(S) \cap \text{sp}^{-1}(U(k)) \subset \text{Gr}_S(N-1, N)(S)$ . See [8, Remark 0.2.(i)]

**4.2. Constructing good sections.** We now explain how to apply the previous construction. As in the previous Section,  $S$  will denote the spectrum of a local domain  $\mathcal{O}$ , with function field  $K$  and residue field  $k$ . Let  $X$  be a smooth projective geometrically integral variety over  $K$  and let  $\mathcal{X}$  be a model of  $X$  over  $S$ , i.e. an integral projective  $S$ -scheme which is surjective over  $S$  and such that  $\mathcal{X}_\eta = X$ . Let  $\mathcal{D}$  be an effective Cartier divisor in  $\mathcal{X}$ , and suppose that  $\mathcal{D}$  restricts to an effective Cartier divisor on the special fiber  $\mathcal{X}_s$ . Assume in this section that  $\dim(X) \geq 2$ .

**Theorem 4.2.1.** *Let  $(\mathcal{X}, \mathcal{D})$  be as above, and fix a projective embedding  $\mathcal{X} \subset \mathbb{P}_S^N$ . If  $k$  is infinite, there exists a general (in the sense specified above) hyperplane  $H \subset \mathbb{P}_S^N$  such that the intersection  $H \cdot \mathcal{X} = H \times_{\mathbb{P}_S^N} \mathcal{X}$  is surjective over  $S$ , has smooth geometrically integral generic fiber  $(H \cdot \mathcal{X})_\eta$  and such that  $\mathcal{D} \cdot H$  is an effective Cartier divisor on  $H \cdot \mathcal{X}$  which restricts to an effective Cartier divisor on the special fiber  $(H \cdot \mathcal{X})_s$ .*

*Proof.* By the classical Theorem of Bertini, [9, Theorem 6.3], there exists a dense open subset  $V \subset \text{Gr}_K(N-1, N)$  such that for every  $H \in V(K)$ , the intersection  $H \cap X$  is smooth, geometrically integral, and intersects properly  $D = \mathcal{D}_\eta$ , i.e.  $H \cap D$  is a Cartier divisor in  $H \cap X$ . Similarly, there exists a dense open subset  $U$  of  $\text{Gr}_k(N-1, N)$  such that no hyperplane  $L$  corresponding to a  $k$ -rational point of  $U$  satisfies  $\text{Ass}(L \cap \mathcal{X}_s) \cap |\mathcal{D}_s| \neq \emptyset$ , since  $\mathcal{D}_s$  is a Cartier divisor on  $\mathcal{X}_s$ , and therefore its support  $|\mathcal{D}_s|$  does not contain any associated point of  $\mathcal{X}_s$ . Note that  $L \cap \mathcal{X}_s \neq \emptyset$  for every hyperplane  $L$  over  $k$ , since  $\dim(\mathcal{X}_s) \geq 2$ .

By Lemma 4.1.1, the set  $T = \text{sp}^{-1}(U(k)) \cap V(S)$  is not empty. For  $H \in T$ , we now claim that all the other required properties are satisfied. Since  $(H \cdot \mathcal{X})_s = H_s \cdot \mathcal{X}_s$  is in particular not empty,  $H \cdot \mathcal{X}$  is automatically surjective over  $S$ . Let  $(H \cdot \mathcal{X})^{ns} \subset H \cdot \mathcal{X}$  be the union of

the irreducible components of  $H \cdot \mathcal{X}$  which are not surjective over  $S$ . Note that  $(H \cdot \mathcal{X})_\eta^{ns} = \emptyset$  so that we can then replace  $H \cdot \mathcal{X}$  with the closure in  $\mathcal{X}$  of  $H \cdot \mathcal{X} \setminus ((H \cdot \mathcal{X})^{ns})$  without altering the generic fiber. So we can assume that every component of  $H \cdot \mathcal{X}$  is surjective over  $S$ . But since the generic fiber  $H_\eta \cdot X$  is smooth and geometrically integral,  $H \cdot \mathcal{X}$  is now automatically geometrically irreducible, and generically geometrically reduced. Replacing  $H \cdot \mathcal{X}$  with  $(H \cdot \mathcal{X})_{\text{red}}$ , we can finally assume that  $H \cdot \mathcal{X}$  is integral.

Since  $H_s \in U(k)$ , we have by construction that  $\mathcal{D}$  restricts to an effective Cartier divisor on  $(H \cdot \mathcal{X})_s$ . Finally, note that no component  $\mathcal{D}'$  of  $\mathcal{D}$  can contain the generic point of  $H \cdot \mathcal{X}$ , since otherwise  $\mathcal{D}'_s$  contains an irreducible component of  $H_s \cdot \mathcal{X}_s$  and by assumption  $\text{Ass}(H_s \cdot \mathcal{X}_s) \cap |\mathcal{D}'_s| = \emptyset$ . Thus  $\mathcal{D}$  restricts to a Cartier divisor on  $H \cdot \mathcal{X}$ .  $\square$

We will need a finer version of the previous Theorem, namely an Altman-Kleiman type Bertini theorem for hypersurface sections containing a (closed) subscheme of the generic fiber. We first recall some notation.

Let  $F$  be a field and let  $Y$  be an  $F$ -scheme of finite type. Let  $y$  be a point of  $Y$ . The embedding dimension  $e_y(Y)$  of  $Y$  at  $y$  is defined as the dimension of the  $k(y)$  vector space  $\dim_{k(y)}(\Omega_{Y/F,y}^1 \otimes_{\mathcal{O}_{Y,y}} k(y))$ . Following the convention of [14], for a positive integer  $e > 0$  write  $Y_e$  for the locally closed subscheme of  $Y$  consisting of those  $y \in Y$  such that  $e_y(Y) = e$ .

**Theorem 4.2.2.** *Let  $(\mathcal{X}, \mathcal{D})$  be as above and suppose that  $d = \dim(X) \geq 3$ . Let  $Z$  be a closed subscheme of  $X$ , and suppose that the estimate*

$$(4.3) \quad \max_{e \leq d-1} \{e + \dim(Z_e)\} \leq d - 1$$

*holds. Let  $\mathcal{Z}$  be the closure of  $Z$  in  $\mathcal{X}$ , and suppose moreover that  $\text{Ass}(\mathcal{Z} \times_S k) \cap |\mathcal{D}_s| = \emptyset$ . If  $k$  is infinite, there exists a hypersurface section  $H$  of  $\mathcal{X}$ , of large degree, such that  $H \supset \mathcal{Z}$ , the generic fiber  $H_\eta$  is smooth and geometrically irreducible, and such that  $\mathcal{D} \cdot H$  is an effective Cartier divisor on  $H \cdot \mathcal{X}$  which restricts to an effective Cartier divisor on the special fiber  $(H \cdot \mathcal{X})_s$ .*

*Proof.* Fix an embedding  $\iota: \mathcal{X} \rightarrow \mathbb{P}_S^N$  and let  $\mathcal{O}(n) = i^* \mathcal{O}_{\mathbb{P}_S^N}(n)$  for  $n \geq 1$ . Let  $I_{\mathcal{Z}}$  be the ideal sheaf of  $\mathcal{Z}$  in  $\mathcal{X}$ , and let  $I_{\mathcal{D}}$  be the (locally principal) ideal of  $\mathcal{D}$  in  $\mathcal{X}$ . Write  $I_{\mathcal{Z}_s}$  (resp.  $I_{\mathcal{D}_s}$ ) for the restriction of  $I_{\mathcal{Z}}$  (resp. of  $I_{\mathcal{D}}$ ) to the special fiber  $\mathcal{X}_s$ . Write  $\mathcal{D}_s^1, \dots, \mathcal{D}_s^m$  for the irreducible components of  $\mathcal{D}_s$ . Finally, let  $\mathcal{J}$  be the ideal sheaf of  $\mathcal{Z}_s$  in  $\mathcal{X}$ . By assumption, the restriction  $I_{\mathcal{D}_s} \otimes \mathcal{O}_{\mathcal{Z}_s}$  as well as the restrictions  $I_{\mathcal{D}_s^i} \otimes \mathcal{O}_{\mathcal{Z}_s}$  for  $i = 1, \dots, m$  are the ideal sheaves of a Cartier divisor on  $\mathcal{Z}_s$ . Choose  $n$  sufficiently large so that

$$(4.4) \quad H^1(\mathcal{X}, \mathcal{O}(n)) = H^1(\mathcal{X}, I_{\mathcal{Z}} \otimes \mathcal{O}(n)) = H^1(\mathcal{X}, \mathcal{J} \otimes I_{\mathcal{Z}} \otimes \mathcal{O}(n)) = 0,$$

$$(4.5) \quad H^1(\mathcal{X}, I_{\mathcal{D} \cap \mathcal{Z}}(n)) = H^1(\mathcal{X}_s, I_{\mathcal{D}_s \cap \mathcal{Z}_s}(n)) H^1(\mathcal{X}_s, I_{\mathcal{D}_s^i \cap \mathcal{Z}_s}(n)) = 0$$

for every  $i = 1, \dots, m$ . Write  $E_n$  for the free  $\mathcal{O}$ -module of finite rank  $H^0(\mathcal{X}, \mathcal{O}(n))$  and  $\tilde{E}_n$  for the finitely generated torsion free submodule  $H^0(\mathcal{X}, I_{\mathcal{Z}} \otimes \mathcal{O}(n)) \subset E_n$ . We have a commutative diagram for each  $i$

$$(4.6) \quad \begin{array}{ccc} \tilde{E}_n & \longrightarrow & H^0(\mathcal{X}, I_{\mathcal{Z}}/I_{\mathcal{Z} \cap \mathcal{D}}(n)) \\ \downarrow & & \downarrow \\ H^0(\mathcal{X}, I_{\mathcal{Z}_s}(n)) & \longrightarrow & H^0(\mathcal{X}_s, I_{\mathcal{Z}_s}/I_{\mathcal{Z}_s \cap \mathcal{D}_s^i}(n)) \end{array}$$

where every arrow is surjective (the left vertical one by (4.4) and the horizontal ones by (4.5)). Note that the last term  $H^0(\mathcal{X}_s, I_{\mathcal{Z}_s}/I_{\mathcal{Z}_s \cap \mathcal{D}_s^i}(n))$  is non zero thanks to the assumption on  $\mathcal{Z}_s$  and  $\mathcal{D}_s$ . Choose a section  $s_0 \in \tilde{E}_n$  such that the restrictions  $s_0 \mapsto \bar{s}_0^i \in H^0(\mathcal{X}_s, I_{\mathcal{Z}_s}/I_{\mathcal{Z}_s \cap \mathcal{D}_s^i}(n))$

are non zero for all  $i$ . Let  $F_n$  be a maximal free submodule of  $\tilde{E}_n$  containing  $s_0$  (i.e.  $F_n$  is a free submodule of  $E_n$  containing  $s_0$ , such that its rank  $\text{rk}_S(F_n)$  is maximal i.e.  $\text{rk}_S(F_n) = \dim_K \tilde{E}_n \otimes_S K$ ). Note that this is possible since  $\tilde{E}_n$  is torsion free. We are going to consider  $S$ -points of the Grassmannian  $\text{Gr}_S(F_n)$  to parametrize our good sections. We have

$$\text{Gr}_S(F_n)(S) \subset \text{Gr}_S(F_n \otimes_S K)(K) \xrightarrow{\iota_K} \text{Gr}_K(E_n \otimes_S K)(K)$$

and since  $F_n \otimes_S K = \tilde{E}_n \otimes_S K = H^0(X, I_Z \otimes \mathcal{O}(n))$ , we can identify the image of  $\iota_K$  with the set

$$\text{Gr}_S(F_n \otimes_S K)(K) = \{H_\eta \subset \mathbb{P}(E_n \otimes_S K) \mid H_\eta \supset Z\}$$

i.e. with the set of degree  $n$  hypersurface sections of  $X$  containing  $Z$ . By the Bertini theorem of Altman and Kleiman [14, Theorem 7], the estimate (4.3) implies that a general element of  $\text{Gr}_K(F_n \otimes_S K)(K)$  is smooth and geometrically irreducible. Let  $U$  be such open set of  $\text{Gr}_K(F_n \otimes_S K)$ . Let

$$\text{sp}: \text{Gr}_S(F_n)(S) \rightarrow \text{Gr}_k(F_n)(k)$$

be the specialization map. Let  $M_n$  be the image of  $F_n \otimes_S k$  in  $H^0(\mathcal{X}_s, I_{\mathcal{Z}_s}(n))$  and let  $K_n = \ker F_n \otimes_S k \rightarrow H^0(\mathcal{X}_s, I_{\mathcal{Z}_s}(n))$ . Then  $\text{Gr}_k(K_n)$  is a proper closed subspace of  $\text{Gr}_k(F_n \otimes_S k)$  (see 4.0.2), and the specialization map  $\text{sp}$  restricts to a surjective specialization map

$$\text{sp}: \text{Gr}_S(F_n)(S) \setminus \text{sp}^{-1}(\text{Gr}_k(K_n)(k)) \rightarrow \text{Gr}_k(M_n)(k)$$

Now, the short exact sequence

$$0 \rightarrow H^0(\mathcal{X}_s, I_{\mathcal{Z}_s \cap \mathcal{D}_s^i}(n)) \rightarrow H^0(\mathcal{X}_s, I_{\mathcal{Z}_s}(n)) \rightarrow H^0(\mathcal{X}_s, I_{\mathcal{Z}_s}/I_{\mathcal{Z}_s \cap \mathcal{D}_s^i}(n)) \rightarrow 0.$$

pulls back to a short exact sequence

$$0 \rightarrow V_n^i \cap M_n \rightarrow M_n \rightarrow M_n/(V_n^i \cap M_n) \rightarrow 0$$

where  $V_n^i$  is the subspace  $H^0(\mathcal{X}_s, I_{\mathcal{Z}_s \cap \mathcal{D}_s^i}(n))$  and the last term  $M_n/(V_n^i \cap M_n)$  is non zero, since  $s_0 \in F_n$  restricts by construction to a non zero element of  $H^0(\mathcal{X}_s, I_{\mathcal{Z}_s}/I_{\mathcal{Z}_s \cap \mathcal{D}_s^i}(n))$ . In particular,  $\text{Gr}_k(V_n^i \cap M_n)$  defines a proper closed subspace of  $\text{Gr}_k(M_n)$ , so that the set

$$\Phi = \text{Gr}_S(F_n)(S) \setminus (\text{sp}^{-1}(\text{Gr}_k(K_n)(k)) \cup \bigcup_{i=1}^m \text{sp}^{-1}(\text{Gr}_k(V_n^i \cap M_n)(k)))$$

is not empty. An element of  $\Phi$  corresponds to a hypersurface section  $H$  of  $\mathcal{X}$  containing  $\mathcal{Z}$ , and such that no component of  $\mathcal{D}$  can contain its generic point, since its defining equation is non zero in

$$H^0(\mathcal{X}_s, I_{\mathcal{Z}_s}/I_{\mathcal{Z}_s \cap \mathcal{D}_s^i}(n)) \subset H^0(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}/I_{\mathcal{D}_s^i}(n)).$$

Thus, a general (in the sense specified above) hypersurface section of  $\mathcal{X}$  containing  $\mathcal{Z}$  satisfies the property that  $\mathcal{D} \cdot H$  is an effective Cartier divisor which restricts to a Cartier divisor on the special fiber, as in the proof of Theorem 4.2.1. Now any element in  $U(S) \cap \Phi$  satisfies all the required properties.  $\square$

We now discuss another version of a Bertini-type theorem concerning sections of relative surfaces, imposing very mild conditions on the special fiber. Before that, we introduce the following Definition.

**Definition 4.2.3.** Let  $\mathcal{C}$  be an integral scheme, proper surjective and of relative dimension 1 over  $S$ . Let  $D$  be an effective Cartier divisor on  $\mathcal{C}$ , and suppose that  $D$  is finite over  $S$ . Let  $\mathcal{C}_\eta$  be the generic fiber of  $\mathcal{C}$ . We say that a closed point  $x \in \mathcal{C}_\eta \setminus |D|$  satisfies the strong modulus condition over  $\mathcal{O}$  (for short,  $x$  satisfies  $(\text{SMC})_{\mathcal{O}}$ ) if the closure  $Z = \overline{\{x\}}$  of  $x$  in  $\mathcal{C}$  satisfies  $Z_s \cap |D_s| = \emptyset$ .

We remark that if a closed point  $x$  satisfies  $(\text{SMC})_{\mathcal{O}}$ , then in particular its closure  $Z$  satisfies the weaker modulus condition  $(\text{MC})_{\mathcal{O}}$  in the sense of Definition 3.1.1. Explicitly, for every non-zero element  $\Pi \in \mathcal{O}$  we have

$$\nu_Z^*(D) \leq \nu_Z^*(\Pi)$$

where  $\nu_Z: Z^N \rightarrow Z \rightarrow \mathcal{C}$  is the composition of the normalization morphism with the inclusion  $Z \subset \mathcal{C}$  and  $(\Pi)$  is the Cartier divisor on  $S$  defined by  $\Pi$ , as well as its pullback to  $\mathcal{C}$ . This is clear, since if  $x$  satisfies  $(\text{SMC})_{\mathcal{O}}$ ,  $\nu_Z^*(D) = 0$  as Weil divisor on  $Z^N$ , and  $\nu_Z^*(\Pi)$  is always effective.

**Proposition 4.2.4.** *Let  $(\mathcal{X}, \mathcal{D})$  be as in 4.2, suppose moreover that  $\mathcal{X}$  has dimension 2 over  $S$  and that  $k$  is infinite. Fix an embedding  $\mathcal{X} \hookrightarrow \mathbb{P}_S(E)$  for a free  $\mathcal{O}$ -module  $E$ . Let  $Z$  be a purely 1 dimensional closed subscheme of  $X$ , let  $\mathcal{Z}$  be its closure in  $\mathcal{X}$ , and suppose that  $\text{Ass}(\mathcal{Z} \times_S k) \cap |\mathcal{D}_s| = \emptyset$ . Fix an open subset  $V \subset \text{Gr}_K(E \otimes_S K)$ . Then a general section  $H \in \text{Gr}_S(E)(S)$  satisfies the following conditions*

- i) *The generic fiber  $H_\eta$  belongs to  $V(K)$ , when we see the hyperplane  $H$  as represented by an  $S$ -point of  $\text{Gr}_S(E)$ .*
- ii) *The intersection  $H \cdot \mathcal{X}$  is surjective over  $S$ , and  $\mathcal{D} \cdot H$  is an effective Cartier divisor  $D$  on  $H \cdot \mathcal{X}$  which is finite over  $S$ .*
- iii) *Every  $x \in H_\eta \cap Z$  satisfy  $(\text{SMC})_{\mathcal{O}}$  with respect to  $D$ .*

*Proof.* By assumption,  $\mathcal{D}$  restricts to an effective Cartier divisor on the special fiber  $\mathcal{X}_s$ . Let  $|\mathcal{D}_s|$  be its support. It is a 1 dimensional closed subscheme of  $\mathcal{X}_s$ , and in particular it does not contain any associated point of  $\mathcal{X}_s$ . Thus, there is an open subset  $U' \subset \text{Gr}_k(E \otimes_S k)$  such that for every  $L \in U'(k)$ , we have  $\text{Ass}(L \cap \mathcal{X}_s) \cap |\mathcal{D}_s| = \emptyset$ , as in the proof of Theorem 4.2.1. In particular,  $\mathcal{D}_s$  restricts to a Cartier divisor on  $L \cap \mathcal{X}_s$  for every such  $L$ , and its support is therefore a finite set of closed points.

In a similar way, we have by assumption that  $\mathcal{D}_s$  restricts to a Cartier divisor on  $\mathcal{Z}_s$ . Let  $W \subset \mathcal{Z}_s$  be its support. It consists of finitely many closed points of  $\mathcal{X}_s$ . Let  $U''$  be the subset of  $\text{Gr}_k(E \otimes_S k)$  such that for every  $L \in U''(k)$ ,  $L \cap W = \emptyset$ . Since  $W$  is zero dimensional,  $U''$  is open and dense in  $\text{Gr}_k(E \otimes_S k)$ . Let  $U = U' \cap U''$ . Now, the set  $T = \text{sp}^{-1}(U(k)) \cap V(S)$  is not empty by Lemma 4.1.1 and we claim that every  $H \in T$  satisfies all the required conditions. The item (i) is satisfied by definition. All the properties in (ii) are achieved thanks to Theorem 4.2.1, apart from the finiteness of  $D$  over  $S$ , which follows from Zariski's Main Theorem, noting that the fiber of  $D$  over the closed point of  $S$  is finite and not empty.

Finally, for every  $x \in H_\eta \cap Z$ , note that the closure  $\overline{\{x\}}$  of  $x$  in  $\mathcal{X}$  is contained in  $\mathcal{Z}$ , so that  $\overline{\{x\}} \cap |\mathcal{D}_s| \subset W$ . But  $\overline{\{x\}} \subset H \cdot \mathcal{X}$  as well, and since by choice  $H_s \in U''(k)$ , we must have  $\overline{\{x\}} \cap |\mathcal{D}_s| = \emptyset$  giving the required strong modulus condition over  $\mathcal{O}$ .  $\square$

*Remark 4.2.5.* In this Section, we have assumed in every statement that the residue field  $k$  of  $S$  is infinite to guarantee the existence of  $k$ -rational points in dense open subsets  $U \subset \text{Gr}_k(E \otimes_S k)$  of the restriction to  $k$  of a Grassmannian  $\text{Gr}_S(E)$  for some free  $\mathcal{O}$ -module  $E$  (of finite rank).

If  $k$  is finite, this is the case over the maximal pro  $\ell$ -extension of  $k$  for every prime number  $\ell$ , hence over some extension  $k'/k$  of degree a power of  $\ell$ . If the ring  $\mathcal{O}$  is moreover assumed to be Henselian (and this is the case in our applications), let  $\mathcal{O}'$  be the unramified extension of  $\mathcal{O}$  corresponding to  $k'/k$  and let  $S' = \text{Spec}(\mathcal{O}')$ . We have then a surjective specialization map  $\text{sp}: \text{Gr}_{S'}(E \otimes_S S')(S') \rightarrow \text{Gr}_{k'}(E \otimes_S k')(k')$ , and we can lift  $k'$ -rational points to  $S'$ -rational points.

In other words, we can find good hyperplane sections (in the sense specified above) for  $\mathcal{X}$  over  $S'$ .

## 5. MOVING BY TAME SYMBOLS

This Section contains a key ingredient of the proof of the main theorem in dimension 2. The main idea is to replace an arbitrary finite set of relations in the Chow group of zero cycles with modulus on a surface by a more controlled one, in which “cancellations”, in the appropriate sense, no longer occur. This is achieved by an iterated applications of a moving argument by tame symbols, i.e. by adding cycles which are obtained as boundaries of classes in the  $K_2$  of the function field of the surface. In this form, our argument is inspired by a moving lemma in the context of cycles on a singular variety due to Marc Levine, see [17], [6]. A similar moving technique has been used in [1].

**5.1. Setting.** We now fix the setting for our moving argument. From now until the end of this Section, we fix a Henselian local ring  $\mathcal{O}$ , with fraction field  $K$  and residue field  $k$ . We assume that  $k$  is infinite.

Let  $X$  be a smooth projective  $K$ -surface,  $D$  an effective Cartier divisor on  $X$ . Assume that  $X$  is the generic fiber of an integral relative surface  $\mathcal{X}$  over  $S$ , and that  $D$  is the generic fiber of an effective Cartier divisor  $\mathcal{D}$  on  $\mathcal{X}$  which restricts to an effective Cartier divisor  $\mathcal{D}_s$  on the special fiber  $\mathcal{X}_s$ .

Let  $Z$  be a reduced, purely 1-dimensional closed subscheme of  $X$ , with irreducible components  $Z_1, \dots, Z_n$ . Assume that the closure  $\mathcal{Z}$  of  $Z$  in  $\mathcal{X}$  satisfies

$$\text{Ass}(\mathcal{Z} \times_S k) \cap |\mathcal{D}_s| = \emptyset.$$

Write  $K(Z)$  for the ring of total quotients of  $Z$ . Since  $Z$  is reduced by assumption, we have  $K(Z) = \prod_{i=1}^n K(Z_i)$ , where  $K(Z_i)$  is the field of functions of the (integral) component  $Z_i$  for  $i = 1, \dots, n$ . Each  $f \in K(Z)^\times$  is then determined by the restrictions  $f_i = f|_{Z_i}$ .

Let  $\pi: Z^N = \prod_{i=1}^n Z_i^N \rightarrow Z$  be the normalization morphism. By definition, a function  $f \in K(Z)^\times$  satisfies the modulus condition if for each  $i = 1, \dots, n$ , we have

$$(5.1) \quad f_i \in \bigcap_{x \in \pi^{-1}(Z_i \cap D)} \ker(\mathcal{O}_{Z_i^N, x}^\times \rightarrow (\mathcal{O}_{Z_i^N, x}/I_D \mathcal{O}_{Z_i^N, x})^\times) \subset K(Z_i)^\times = K(Z_i^N)$$

where  $I_D$  denotes the ideal sheaf of  $D$ . Pick up then a function  $f$  satisfying the modulus condition, and write  $\gamma$  for its divisor, i.e.  $\gamma = \text{div}_Z(f) = \sum_{i=1}^n \text{div}_{Z_i}(f_i)$ . By (5.1), we have

$$\gamma \in \hat{F}^{(D)}(X, |D|) \subset \mathcal{Z}_0(X \setminus D).$$

where  $\hat{F}^{(D)}(X, |D|)$  is the subgroup of  $\mathcal{Z}_0(X \setminus D)$  generated by divisors of functions satisfying the modulus condition. See [13]. The quotient

$$\mathcal{Z}_0(X \setminus D)/\hat{F}^{(D)}(X, |D|) =: \text{CH}_0(X|D)$$

is by definition the Chow group of zero cycles with modulus.

Let  $\Sigma$  be the set of closed points in  $X$  where cancellations occur in the expression of  $\gamma$ . Explicitly, let  $\gamma_i = \text{div}_{Z_i}(f_i)$ . Then

$$\Sigma = \{x \in X_{(0)} \mid x \notin |\gamma|, \text{ but } x \in |\gamma_i| \text{ for some } i = 1, \dots, n\}.$$

Note that  $\Sigma \cap D = \emptyset$ .

Our goal in this Section is to rewrite the cycle  $\gamma$  as a sum of divisors of functions on carefully chosen curves so that cancellations in the above sense no longer occur. After this, every term in the sum will be given by points satisfying the modulus condition  $(\text{MC})_\mathcal{O}$ , in the sense of Definition 3.1.1 (this cannot be guaranteed with the expression  $\gamma = \sum_{i=1}^n \text{div}_{Z_i}(f_i)$ , precisely because of the presence of cancellation points). This will allow us to apply directly the results

of Section 3 to each individual term. We summarize the precise statement in the point 5.4.1 below.

**5.2. Good intersection.** Let  $p: \hat{\mathcal{X}} \rightarrow \mathcal{X}$  be a repeated blow up along the closure of closed points lying over points of  $\Sigma$  and points of  $Z \cap D$  such that the following conditions are satisfied:

- (1) Let  $\hat{X} = \hat{\mathcal{X}} \times_S K$  be the generic fiber of  $\hat{\mathcal{X}}$ . Let  $\hat{D}$  be the strict transform of  $D$  in  $\hat{X}$ , and let  $D' = \hat{D} + E_D$  be its total transform. Then the support  $|D'|$  is a strict normal crossing divisor in  $\hat{X}$  (and so are the supports of  $E_D$  and  $\hat{D}$ ).

Write  $E = E_\Sigma + E_D$ , where  $E_D$  is the exceptional divisor lying over points of  $Z \cap D$ , and  $E_\Sigma$  is the exceptional divisor lying over points of  $\Sigma$ .

- (2) Let  $\hat{Z}$  be the strict transform of  $Z$ . Then  $\hat{Z}$  is regular at each point of  $\hat{Z} \cap p^{-1}(\Sigma)$ , and at each point of  $\hat{Z} \cap D'$ , and  $\hat{Z}$  intersects  $|D' + E_\Sigma|$  transversally (which means that  $\hat{Z}$  intersects  $|D' + E_\Sigma|$  only in the regular locus, and transversally).

In particular, for each point  $x$  of  $\hat{Z} \cap p^{-1}(\Sigma)$  and of  $\hat{Z} \cap D'$ , there is exactly one component of  $\hat{Z}$  passing through  $x$ .

The above conditions can be achieved using e.g. [21, Theorem A.1]. By considering, if necessary, further blow-ups, we can also assume that each component  $E_{\Sigma,j}$  of  $E$  lying over a point  $x$  in  $\Sigma$  intersects at most one of the strict transforms  $\hat{Z}_i$  of  $Z_i$ .

For consistency, write  $\hat{\mathcal{D}}$  for the strict transform of  $\mathcal{D}$  in  $\hat{\mathcal{X}}$  and  $\mathcal{D}' = \hat{\mathcal{D}} + \mathcal{E}_{\mathcal{D}}$  for the total transform. Similarly, write  $\mathcal{E} = \mathcal{E}_{\mathcal{D}} + \mathcal{E}_\Sigma$  for the exceptional divisor. Clearly,  $\mathcal{D}'$  restricts to a Cartier divisor on the special fiber  $\hat{\mathcal{X}}_s$ . If we denote by  $\hat{\mathcal{Z}}$  the strict transform of  $\mathcal{Z}$ , note that the condition

$$(5.2) \quad \text{Ass}(\hat{\mathcal{Z}} \times k) \cap |\hat{\mathcal{D}}_s + \mathcal{E}_{\mathcal{D},s}| = \emptyset$$

holds.

Write  $\hat{f} \in K(Z)^\times = K(\hat{Z})^\times$  for the rational function on  $\hat{Z}$  induced by  $f$ , and write similarly  $\hat{f}_i$  for the rational function on  $\hat{Z}_i$  induced by  $f_i$ . By condition (2) above, we have  $\mathcal{O}_{\hat{Z}_i^N, y} \cong \mathcal{O}_{\hat{Z}, y}$  for every  $y \in \hat{Z} \cap D'$  and every  $i = 1, \dots, n$ , where we identify a point  $y \in \hat{Z} \cap D'$ , which lives on at most one component  $\hat{Z}_i$ , with the corresponding point in  $Z_i^N$ . In particular, the modulus condition (5.1) implies that

$$\hat{f} \in 1 + I_{D'} \mathcal{O}_{\hat{Z}, y} \text{ for every } y \in \hat{Z} \cap D'.$$

In other words, the function  $\hat{f}$  satisfies the modulus condition on  $\hat{Z}$  with respect to  $D' = \hat{D} + E_D$ .

Let  $\hat{\gamma} = \sum_{i=1}^n \text{div}_{\hat{Z}_i}(\hat{f}_i)$  be the divisor of  $\hat{f}$  on  $\hat{Z}$ . By construction, it satisfies  $p_*(\hat{\gamma}) = \gamma$ , as cycles on  $X$ . Note that by (2), in the expression of  $\hat{\gamma}$ , cancellations in the above sense no longer occur, and there may be points in  $|\hat{\gamma}|$  which do not satisfy (MC) $_{\mathcal{O}}$  in the sense of Definition 3.1.1. To remedy this, we can carefully choose rational functions  $g_\lambda$  on each exceptional line  $E_{\Sigma, \lambda}$  such that the following holds.

For each  $\sigma \in \Sigma$ , write  $\{\xi_\sigma^1, \dots, \xi_\sigma^{n(\sigma)}\}$  for the set  $p^{-1}(\sigma) \cap \hat{Z}$ . Let  $\gamma_\sigma$  be the 0-cycle

$$\gamma_\sigma = \sum_{i=1}^{n(\sigma)} v_{\hat{Z}_i}(\hat{f})[\xi_\sigma^i]$$

where  $v_{\hat{Z}_i}(\hat{f})$  is the order of  $\hat{f}$  at the point  $\xi_\sigma^i$ . Note that this is well defined, since by construction, each  $\xi_\sigma^i$  lies in exactly one component of  $\hat{Z}$ , and  $\hat{Z}$  is regular there. By definition,

$\gamma_\sigma$  appears in the expression of  $\hat{\gamma}$  but  $p_*(\gamma_\sigma) = 0$  as cycle in  $\mathcal{Z}_0(X \setminus D)$ . Write then  $E_\sigma$  for the exceptional divisor  $p^{-1}(\sigma)$ . It is, by (1), a connected union of rational lines in  $\hat{X}$ , each of which defined over a finite extension  $L_i$  of the residue field  $k(\sigma)$  of  $\sigma$ . We choose then a rational function  $g_i \in L_i(t)$  on each  $E_\sigma^i \cong \mathbb{P}_{L_i}^1$  such that

$$(5.3) \quad v_{\hat{Z}, i}(\hat{f}) = \text{ord}_{\xi_\sigma^i}(g_i), \text{ and } \gamma_\sigma = \sum_{i \in I(\sigma)} \text{div}_{E_\sigma^i}(g_i),$$

where  $I(\sigma)$  is the index set of the lines  $E_\sigma^i$  connecting the points in the set  $\{\xi_\sigma^1, \dots, \xi_\sigma^{n(\sigma)}\}$ .

Note that cancellations appear in the expression of  $\gamma_\sigma$  in (5.3) exactly in the points  $E_\sigma^i \cap E_\sigma^j$  for  $i \neq j$ , but away from  $E_\Sigma \cap \hat{Z}$ . Moreover, since  $\Sigma \cap D = \emptyset$ , we have  $E_\Sigma \cap D' = \emptyset$  and the functions  $g_i$  satisfy tautologically the modulus condition with respect to  $D'$ .

We can then define the cycle

$$\gamma' = \hat{\gamma} - \sum_{\sigma \in \Sigma} \sum_{i \in I(\sigma)} \text{div}_{E_\sigma^i}(g_i)$$

and  $\gamma'$  satisfies  $p_*(\gamma') = p_*(\hat{\gamma}) = \gamma$ . By construction, for each point  $x \in |\gamma'|$ , the map  $p$  induces an isomorphism  $\mathcal{O}_{\hat{X}, x} \cong \mathcal{O}_{X, x}$ , since we have removed from the support of  $\hat{\gamma}$  each point lying above the center of the blow-up, and for every cancellation point  $\xi$  in the above sense, there is exactly one component of  $\hat{Z}$  passing through  $\xi$  and meeting transversally exactly one component of  $E_\Sigma$  there. Changing the notation, write  $Z'$  for the closed subscheme defined by  $\hat{Z} \cup E_\Sigma$ , and  $f'$  for the rational function defined by

$$f'|_{\hat{Z}} = \hat{f}, \quad f'|_{E_\sigma^i} = g_i^{-1} \quad \text{for every } \sigma \in \Sigma, i = 1, \dots, n(\sigma)$$

We will then denote by  $Z'_i$  the integral components of  $Z'$ , and by  $f'_i$  the restriction  $f'|_{Z'_i}$ . We can then write  $\gamma' = \text{div}_{Z'}(f')$ , using the convention introduced above, and note that  $f'$  satisfies the modulus condition on  $Z'$  with respect to  $D'$ . The advantage of passing from  $\gamma$  to  $\gamma'$  will be clear in the next section.

We finally remark that the closed subscheme  $\mathcal{Z}'$  of  $\hat{\mathcal{X}}$  defined by  $\hat{\mathcal{Z}} \cup \mathcal{E}_\Sigma$  is still in good position with respect to  $\mathcal{D}'$  in the sense that

$$\text{Ass}(\mathcal{Z}' \times_S k) \cap |\mathcal{D}'_s| = \emptyset$$

so that no associated point of  $\mathcal{Z}'$  can be a generic point of  $\mathcal{D}'$ . This is clear for the components defined by  $\hat{\mathcal{Z}}$  by (5.2), and it is clear by construction for the components of  $\mathcal{E}_\Sigma$ , using the fact that  $\Sigma \cap D = \emptyset$ .

**5.3. Reduction I.** We keep the notation of the previous section. Recall that a point  $x \in \hat{X}$  is a cancellation point for  $\text{div}_{Z'}(f')$  if  $x \notin |\text{div}_{Z'}(f')|$  (i.e.  $x$  does not appear in the support of the divisor) but there exists  $i \in \{1, \dots, n\}$  such that  $f'_i \in \mathfrak{m}_x \mathcal{O}_{Z'_i, x}$ , i.e.  $x$  appears as a zero of the restriction of  $f$  to one of the components of  $Z'$ . Note that if this happens, there must exist another component  $j \neq i$  such that  $f'_j$  has a pole at  $x$ , with matching multiplicity. Write  $\hat{\Sigma}$  for the set of cancellation points.

We fix the following two sets

$$T_0 = \{x \in \hat{X} | f'_i \in \mathfrak{m}_x \mathcal{O}_{Z'_i, x}, \text{ for some } i, \text{ and } f' \in \mathcal{O}_{Z', x}\},$$

$$T_\infty = \{x \in \hat{X} | f' \notin \mathcal{O}_{Z', x}^\times\} - T_0$$

Informally, we will refer to  $T_0$  as the set of zeros of  $f'$  (note that these are regular points of  $f'$ ), and to  $T_\infty$  as the set of poles of  $f'$ . In particular,  $T_\infty$  contains every point  $x$  of intersection of different irreducible components of  $Z'$  where  $f'$  does not extend to a regular function in the



local ring of  $Z'$  at  $x$ . Note that by condition (2) of 5.2 and the modulus condition of  $f'$ , we have  $T_\infty \cap D' = \emptyset$ . By construction, the set  $T_\infty$  contains also every cancellation point  $x \in \hat{\Sigma}$ .

We also write  $\Lambda$  for the set

$$\Lambda = (T_0 \cup T_\infty) \amalg (D' \cap Z')$$

The argument will involve the choice of several auxiliary objects. We begin by choosing a very ample hypersurface section  $\Gamma \subset \hat{X}$  such that  $\Gamma \cap \Lambda = \emptyset$ . We assume that  $H^0(\hat{X}, \mathcal{O}_{\hat{X}}(\Gamma)) = E \otimes_S K$  for a free  $S$ -module  $E$  of finite rank corresponding to a projective embedding

$$\iota_E: \hat{\mathcal{X}} \hookrightarrow \mathbb{P}_S(E)$$

so that  $\mathcal{O}(\Gamma) = (\iota_E^* \mathcal{O}_{\mathbb{P}_S(E)}(1))_\eta$ .

Choose a global section  $\beta \in H^0(\hat{X}, \mathcal{O}_{\hat{X}}(\Gamma))$  such that the divisors of zeros of  $\beta$ , say  $B = (\beta)$  is integral, satisfies  $B \cap \Lambda = \emptyset$  and intersects  $|D'|$  transversally (which means that  $B$  intersects  $|D'|$  in the regular locus, and transversally). This can be achieved by the classical theorem of Bertini over  $K$ , and we will use our refinement, Theorem 4.2.1 to take care of the model over  $\mathcal{O}$  of the section. In particular,  $B$  has a model  $\mathcal{B}$  over  $S$ , surjective over  $S$  such that  $\mathcal{B}'$  restricts to an effective Cartier divisor on it, finite over  $S$ . If we add to this an application of Proposition 4.2.4, we can moreover assume the following. Let  $V'$  be the set of points in  $Z' \cap B$ . Note that  $V' \cap D' = \emptyset$ . Let  $\mathcal{V}'$  be the closure of  $V'$  in  $\hat{\mathcal{X}}$ . Then  $\mathcal{V}'_s \cap \mathcal{D}_s = \emptyset$ . In other words, every  $x \in Z' \cap B$  satisfies (SMC) $_{\mathcal{O}}$  with respect to  $D'$ .

The following lemma is another application of Bertini's theorem 4.2.1.

**Lemma 5.3.1.** *There exists a hypersurface section  $L \subset \hat{X}$ , of sufficiently large degree, such that the following conditions are satisfied.*

- a)  $L \cap \Lambda = \emptyset$  and  $L$  intersects  $D'$  properly.
- b)  $H^1(\hat{X}, \mathcal{O}_{\hat{X}}(L) \otimes I_{Z' \cup D' \cup B}) = 0$
- c) *There exists a section  $t_0 \in H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L) \otimes I_{Z'})$  such that the divisor of zeros of  $t_0$  satisfies  $(t_0) = Z' \cup Z''$ , with*

$$Z'' \cap \Lambda = Z'' \cap Z' \cap (B \cup D') = Z'' \cap B \cap D' = \emptyset,$$

and  $Z''$  intersects  $|D'| + B$  transversally.

- d) *Let  $V''$  be the set of points in  $Z'' \cap B$ , and let  $\mathcal{V}''$  be the closure of  $V''$  in  $\hat{\mathcal{X}}$ . Then  $\mathcal{V}''_s \cap \mathcal{D}_s = \emptyset$ .*

Moreover,  $L = \mathcal{L}_\eta$  for a hypersurface section  $\mathcal{L}$  of  $\hat{\mathcal{X}}$ , which is surjective over  $S$  and satisfies the property that  $\mathcal{B}' \cap \mathcal{L}$  is an effective Cartier divisor, finite over  $S$ .

*Proof.* Everything follows from Bertini's theorem and its variants. We remark that the final condition d) is achieved by another application of Proposition 4.2.4.  $\square$

Note in particular that if  $V_0 = B \cap (Z' \cup Z'')$ , then every  $x \in V_0$  satisfies (SMC) $_{\mathcal{O}}$ .

As before, we may assume that  $H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L)) = M \otimes_S K$  for a free  $S$ -module  $M$  corresponding to another projective embedding

$$\iota_M: \hat{\mathcal{X}} \hookrightarrow \mathbb{P}_S(M)$$

so that  $\mathcal{O}(L) = (\iota_M^* \mathcal{O}_{\mathbb{P}_S(M)}(1))_\eta$ .

Write  $(t_0) = Z_0$  and write  $f_0$  for the rational function on  $Z_0$  induced by  $f'$  on  $Z'$  and by the constant function 1 on  $Z''$ . Let  $T'_\infty$  be the set of poles of  $f_0$  in the above sense,  $T'_\infty = \{x \in \hat{X} \mid f_0 \notin \mathcal{O}_{Z_0, x}^\times\} - T_0$ . Then  $T'_\infty \supset T_\infty$ , and by construction it is disjoint from  $\Lambda - T_\infty$ . Moreover,  $T'_\infty \cap D' = \emptyset$ , and  $f_0 \in 1 + I_{D'} \mathcal{O}_{Z_0, x}$  for every  $x \in D' \cap Z_0$ , i.e. the function  $f_0$  satisfies the modulus condition on  $Z_0$  with respect to  $D'$ .

**Lemma 5.3.2.** *There exists a lift  $t_\infty \in H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L))$  of the restriction of  $t_0$  to  $D' \cup B$  such that the divisor of zeros  $(t_\infty) = Z_\infty$  of  $t_\infty$  is regular, intersects  $|D'|$  transversally and  $Z_\infty \cap (T_0 \cup T'_\infty) = \emptyset$ .*

*Proof.* Let  $t_1, \dots, t_m$  be sections of  $H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L) \otimes I_{D'} \otimes I_B)$  such that the rational map  $\hat{X} \dashrightarrow \mathbb{P}_K^{m-1}$  that they define is a locally closed immersion on  $\hat{X} \setminus |D' \cup B|$ . We can add  $t_0$  to the  $t_i$ 's so that the rational map  $\psi: \hat{X} \dashrightarrow \mathbb{P}_K^m$  given by the sections  $(t_0, t_1, \dots, t_m)$  of  $H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L))$  is also a locally closed immersion on  $\hat{X} \setminus |D' \cup B|$ , and since the base locus of the linear system associated to  $(t_0, t_1, \dots, t_m)$  is

$$(B \cup D') \cap Z_0 = (D' \cap Z_0) \amalg B \cap Z_0,$$

it is in fact a morphism away from  $(B \cup D') \cap Z_0$ . Thus  $\psi$  is birational, hence separable, and has image of dimension equal to two. By the classical theorem of Bertini, a general divisor  $Z_\infty = (t_\infty)$  in the linear system  $V(t_0, t_1, \dots, t_m)$  is irreducible and generically reduced, and regular away from  $(B \cup D') \cap Z_0$ . Since  $(T_0 \cup T'_\infty) \cap D' = \emptyset$ , and also  $(T_0 \cup T'_\infty) \cap B = \emptyset$ , the set  $(T_0 \cup T'_\infty)$  (which is zero dimensional) is away from the base locus, hence we can assume that  $Z_\infty \cap (T_0 \cup T'_\infty) = \emptyset$ . Note that by construction,

$$Z_0 \cap D' = Z_\infty \cap D', \quad Z_0 \cap B = Z_\infty \cap B.$$

Up to a scalar, the section  $t_\infty \in H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L))$  is of the form  $t_\infty = t_0 + \alpha$ , with  $\alpha$  in  $H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L) \otimes I_{D'} \otimes I_B)$ . By condition c) of the previous Lemma, together with (2) of 5.2, we can assume that at every point  $y \in Z_0 \cap D'$ , the functions  $(t_0, \pi)$  form a regular system of parameters of  $\mathcal{O}_{\hat{X}, y}$ , where  $\pi$  is a local equation for  $|D'|$  (note that  $I_{B, y} \cong \mathcal{O}_{\hat{X}, y}$  since  $y \notin B$  by choice). If  $I_{D', y} = (\pi^e)$ , we can then choose  $\alpha$  satisfying

$$\alpha_y = a_y \pi^e \in I_{D', y} \subset \mathcal{O}_{\hat{X}, y}$$

with  $a_y \in \mathcal{O}_{\hat{X}, y}^\times$ , so that  $t_{\infty, y} = t_0 + a_y \pi^e$  defines a regular divisor at  $y$ , intersecting  $|D'|$  transversally and tangent to  $Z_0$ .

The same argument works if we replace  $D'$  by  $B$ , noting that  $(D' \cap Z_0) \cap B \cap Z_0 = \emptyset$  and that  $Z_0$  intersects  $B$  transversally. Thus we can assume that  $Z_\infty$  is regular at every point of  $Z_\infty \cap B$  as well. Explicitly, let  $b$  be a local equation for  $B$  in a neighborhood of  $y \in Z_0 \cap B$ . As before, note that  $I_{D', y} \cong \mathcal{O}_{\hat{X}, y}$  since this time  $y \notin D'$ . If  $I_{B, y} = (b)$ , we can choose  $\alpha$  satisfying the additional property

$$\alpha_y = c_y b \in I_{B, y}$$

with  $c_y \in \mathcal{O}_{\hat{X}, y}^\times$ , so that  $t_{\infty, y} = t_0 + c_y b$  defines a regular divisor at  $y$ .  $\square$

We now extend the function  $f_0$  to a function  $h$  on  $Z_0 \cup Z_\infty$  by setting  $h = (f_0, 1) \in K(Z_0)^\times \times K(Z_\infty)^\times$ .

**Claim 5.3.3.** *For every  $x \in Z_0 \cap D' = Z_\infty \cap D'$ , we have  $h \in \mathcal{O}_{Z_0 \cup Z_\infty, x}^\times$ .*

*Proof.* In a neighborhood of  $x \in Z_0 \cap D'$ , the scheme  $Z_0 \cup Z_\infty$  is defined by the principal ideal

$$I_{Z_0} I_{Z_\infty} = (t_0(t_0 + a_x \pi^e)),$$

where  $(\pi^e) = I_{D', x}$  and  $a_x \in \mathcal{O}_{\hat{X}, x}^\times$  as in the previous lemma. We have then the following exact sequence of  $\mathcal{O}_{\hat{X}, x}$ -modules

$$0 \rightarrow \mathcal{O}_{\hat{X}, x} / (t_0(t_0 + a_x \pi^e)) \rightarrow \mathcal{O}_{\hat{X}, x} / (t_0) \times \mathcal{O}_{\hat{X}, x} / (t_0 + a_x \pi^e) \rightarrow \mathcal{O}_{\hat{X}, x} / (t_0, a_x \pi^e) \rightarrow 0$$

By assumption, the function  $f_0$  satisfies the modulus condition, i.e.  $f \in 1 + I_{D'} \mathcal{O}_{\hat{X},x}$ . Thus  $f_0 - 1 = 0 \pmod{(t_0, a_x \pi^e)}$ . In other words, the pair  $(f_0, 1)$  determines a regular function

$$h \in \mathcal{O}_{\hat{X},x}/(t_0(t_0 + a_x \pi^e)) = \mathcal{O}_{Z_0 \cup Z_\infty, x},$$

as required. Note that  $h$  is automatically invertible at  $x$ .  $\square$

Write  $\hat{T}_\infty = \{x \in \hat{X} \mid h \notin \mathcal{O}_{Z_0 \cup Z_\infty, x}^\times\} \setminus T_0$ , and write  $\hat{T}_0 = T_0$  coherently (note that  $\hat{T}_0$  is precisely the set of zeros of  $h$ , in the sense discussed above). By Claim 5.3.3, we have that  $\hat{T}_\infty \cap D' = \emptyset$ .

We now choose yet another section of  $\mathcal{O}(L)$ , as in the following lemma.

**Lemma 5.3.4.** *Let  $\hat{\Sigma} \subset \hat{T}_\infty$  be the cancellation set for  $h$  (or, equivalently, for  $f_0$ , or equivalently for  $f'$ ). There exists a section  $s_\infty \in H^0(\hat{X}, \mathcal{O}(L))$  such that its zero locus  $H = (s_\infty)$  satisfies the following properties.*

- i)  $H$  is integral, intersects  $|D'|$  transversally (which means that  $H$  intersects  $|D'|$  only in the regular locus, and transversally).
- ii)  $H$  contains  $\hat{T}_\infty$ , and is regular along each point of  $\hat{\Sigma}$  (note that  $\hat{T}_\infty \supset \hat{\Sigma}$ ).
- iii) The restriction of  $s_\infty$  to  $Z_0 \cup Z_\infty$  determines a global section

$$hs_\infty \in H^0(Z_0 \cup Z_\infty, \mathcal{O}(L))$$

- iv)  $H \cap (\Lambda - \hat{T}_\infty) = H \cap Z_0 \cap D' = H \cap D' \cap L = H \cap D' \cap B = H \cap B \cap Z_0 = H \cap B \cap Z_\infty = \emptyset$ .

Moreover,  $H = \mathcal{H}_\eta$  for a hypersurface section  $\mathcal{H}$  of  $\hat{\mathcal{X}}$ , which is surjective over  $S$  and satisfies the property that  $\mathcal{D}' \cap \mathcal{H}$  is an effective Cartier divisor on  $\mathcal{H}$ , finite over  $S$ .

*Proof.* We first take care of the generic fiber  $H$ . Its model over  $S$ , with the required property, will be obtained using Theorem 4.2.2 applied to the set  $\hat{T}_\infty$  and with respect to the embedding  $\iota_M$  in  $\mathbb{P}_S(M)$ . Finiteness over  $S$  of the restriction of the divisor  $\mathcal{D}'$  is achieved by the same token of Proposition 4.2.4.

Write  $\hat{\Sigma} = \{\xi_1, \dots, \xi_r\}$ , and for  $i = 1, \dots, r$ . Recall that, by construction, we have  $(Z_0 \cup Z_\infty) \cap \hat{\Sigma} = Z_0 \cap \hat{\Sigma} = Z' \cap \hat{\Sigma}$ , and that at each point  $\xi_i \in \hat{\Sigma}$ , there are exactly two regular components of  $Z'$  passing through  $\xi_i$ , and intersecting transversally there. We can then choose a regular system of parameters  $(u_{1,i}, u_{2,i}) \subset \mathcal{O}_{\hat{X}, \xi_i}$  generating the maximal ideal of the local ring of  $\hat{X}$  at  $\xi_i$  such that  $I_{Z_0, \xi_i} = (u_{1,i}, u_{2,i})$ , so that  $u_{1,i}$  and  $u_{2,i}$  are local parameters for the two components  $Z'_{1,i}$  and  $Z'_{2,i}$  of  $Z'$  passing through  $\xi_i$ . Since  $\xi_i$  is a cancellation point, we have

$$\text{ord}_{Z'_{1,i}, \xi_i}(f') + \text{ord}_{Z'_{2,i}, \xi_i}(f') = 0.$$

Write  $\lambda_i = \text{ord}_{Z'_{2,i}, \xi_i}(f')$ , and suppose (up to exchanging the components) that  $\lambda_i > 0$ . We can then write  $f' = \frac{a_i}{u_1^{\lambda_i}} \pmod{u_2}$ , for  $a_i \in \mathcal{O}_{Z'_{2,i}, \xi_i}^\times$  and  $f' = b_i u_2^{\lambda_i} \pmod{u_1}$ , for  $b_i \in \mathcal{O}_{Z'_{1,i}, \xi_i}^\times$ , using that  $\mathcal{O}_{Z'_{1,i}, \xi_i}$  and  $\mathcal{O}_{Z'_{2,i}, \xi_i}$  are DVRs.

Let  $I_{\xi_i} \subset \mathcal{O}_{\hat{X}}$  be the ideal sheaf of the point  $\xi_i$ , and define  $J_{\xi_i} \subset I_{\xi_i}$  to be the subsheaf of  $I_{\xi_i}$  locally generated by  $(J_{\xi_i})_{\xi_i} = (u_{1,i}^{\lambda_i+1}, u_{2,i}) \subset \mathcal{O}_{\hat{X}, \xi_i}$ . Let  $J$  be the product  $\prod_{i=1}^r J_{\xi_i}$  and let  $\hat{J} = JJ_\infty$ , where  $J_\infty \subset \mathcal{O}_{\hat{X}}$  is the ideal sheaf of the points in  $\hat{T}_\infty \setminus \hat{\Sigma}$ .

We now choose  $s_\infty \in H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L) \otimes \hat{J})$  such that  $s_{\infty, \xi_i} \neq 0$  in  $(J_{\xi_i}/J_{\xi_i} \cap I_{\xi_i}^2)_{\xi_i}$ . Note that, if necessary, we could have replaced  $L$  with another hypersurface section of  $\hat{X}$  so to have enough global sections of  $\mathcal{O}_{\hat{X}}(L) \otimes J$  from the beginning (we remark that  $J$  does not depend on the later modifications, such as the choices of  $t_0$  and  $t_\infty$ ).

Write  $H$  for the divisor of zeros of  $s_\infty$ . As remarked above, the classical theorem of Bertini implies that  $H$  is integral, that it intersects transversally  $|D'|$  (since  $|D'| \cap \hat{T}_\infty = \emptyset$ ) and that satisfies condition iv) on the avoidance of the specified set of closed points. In particular, each point of  $H \cap D'$  is a regular point of  $H$ . In a neighborhood of  $\xi \in \hat{\Sigma}$ , we can write  $s_\infty = u_1^{\lambda+1} + \varepsilon u_2$ , with  $\varepsilon \in \mathcal{O}_{\hat{X}, \xi}^\times$  by assumption, so that  $H$  is regular at  $\xi$ . Condition ii) is then achieved.

The function  $h$  is a rational function on  $Z_0 \cup Z_\infty$  (i.e. an element of the ring of total quotients of  $Z_0 \cup Z_\infty$ ), and it's not defined precisely at the finite set of points  $\hat{T}_\infty$ . Consider now the function  $s_\infty h$ . We claim that  $s_\infty$  can be chosen in the previous step so that  $s_\infty h \in \mathcal{O}_{Z_0 \cup Z_\infty, x}$  for every  $x \in \hat{T}_\infty$ . In particular, the function  $s_\infty h$  gives rise to an element of  $H^0(Z_0 \cup Z_\infty, \mathcal{O}(L))$ .

For  $x = \xi \in \hat{\Sigma}$ , this is a direct consequence of the construction. Indeed,  $\mathcal{O}_{Z_0 \cup Z_\infty, x} = \mathcal{O}_{Z'_1 \cup Z'_2, x}$ , where  $Z'_1$  and  $Z'_2$  are the two regular components of  $Z_0 \cup Z_\infty$  passing through  $x$ . We have then a short exact sequence, as in the proof of Claim 5.3.3

$$0 \rightarrow \mathcal{O}_{Z'_1 \cup Z'_2, x} \rightarrow \mathcal{O}_{\hat{X}, x}/(u_2) \times \mathcal{O}_{\hat{X}, x}/(u_1) \rightarrow \mathcal{O}_{\hat{X}, x}/(u_1, u_2) \rightarrow 0$$

and an equality

$$(5.4) \quad hs_\infty = \left(\frac{a}{u_1^\lambda}, bu_2^\lambda\right)(u_1^{\lambda+1} + \varepsilon u_2) = (au_1, \varepsilon bu_2^{\lambda+1}) \in \mathcal{O}_{\hat{X}, x}/(u_2) \times \mathcal{O}_{\hat{X}, x}/(u_1),$$

with  $a$  and  $\varepsilon b$  units, from which it follows that  $(hs_\infty)_{Z'_1 \cap Z'_2} = 0$  in  $\mathcal{O}_{\hat{X}, x}/(u_1, u_2) = k(x)$ , so that  $hs_\infty$  gives rise to an element of  $\mathcal{O}_{Z'_1 \cup Z'_2, x}$ . Note in particular that the required lifting property can be achieved together with the regularity of  $H$ . For  $x \in \hat{T}_\infty \setminus \hat{\Sigma}$ , where the regularity of  $H$  is not required, it is enough to choose  $s_\infty$  such that  $s_{\infty, x} \in \hat{J}_x^N = \mathfrak{m}_x^N \subset \mathcal{O}_{\hat{X}, x}$  for a sufficiently large  $N$  (depending on  $x$ ) so that  $hs_\infty = 0 \pmod{\hat{J}_x}$ . This forces  $hs_\infty$  to be regular on  $Z_0 \cup Z_\infty$  at  $x$ . Note that  $H$  will be, in general, highly singular there.

Finally, since the base locus of the linear system  $|H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L) \otimes \hat{J})|$  is disjoint from the zero dimensional sets  $(\Lambda - \hat{T}_\infty)$ ,  $Z_0 \cap D'$ ,  $D' \cap L$ ,  $D' \cap B$ ,  $B \cap Z_0 = B \cap Z_\infty$ , thanks to the condition  $\hat{T}_\infty \cap D' = \emptyset$ , we can assume that  $H$  satisfies condition iv). Here we are using in particular the fact that  $\hat{T}_\infty \cap D' \cap (Z_\infty \cup L) = \hat{T}_\infty \cap D' \cap (Z_\infty \cup L) = \emptyset$ , which is a consequence of Lemma 5.3.2 and the fact that  $h$  is a regular invertible function on  $Z_0 \cup Z_\infty$  at every  $x \in D' \cap Z_0 = D' \cap Z_\infty$ , which is ensured by Claim 5.3.3.

Similarly, since  $\hat{T}_\infty \cap |D'| = \emptyset$ , condition i) (i.e. the transversality of the intersection) can be achieved as well.  $\square$

Let  $\mathcal{X}_0$  (resp.  $\mathcal{X}_\infty$ ) be the closure of  $Z_0$  (resp.  $Z_\infty$ ) in  $\hat{\mathcal{X}}$ . For every  $N > 0$ , write  $E_N$  for the free  $S$ -module

$$E_N = H^0(\hat{\mathcal{X}}, \iota_E^* \mathcal{O}_{\mathbb{P}_S(E)}(N)).$$

Combining the closed embedding  $\iota_M$  with the  $N$ -fold embedding  $\iota_{E_N}$  and following the result with the Segre embedding (see [7, 4.3.3]), we get a composite embedding

$$\iota: \hat{\mathcal{X}} \hookrightarrow \mathbb{P}_S(M \otimes E_N)$$

For  $N > 0$  sufficiently large, we have that

$$H^1(\hat{X}, \mathcal{O}_{\hat{X}}(L + N\Gamma) \otimes I_{Z_0 \cup Z_\infty}) = H^1(\hat{\mathcal{X}}, \iota_M^* \mathcal{O}_{\mathbb{P}_S(M)}(1) \otimes \iota_E^* \mathcal{O}_{\mathbb{P}_S(E)}(N) \otimes I_{\mathcal{X}_0 \cup \mathcal{X}_\infty}) = 0$$

by Serre's vanishing theorem. This gives in particular a surjection

$$H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L + N\Gamma)) \xrightarrow{\phi} H^0(Z_0 \cup Z_\infty, \mathcal{O}(L + N\Gamma)) \rightarrow 0$$

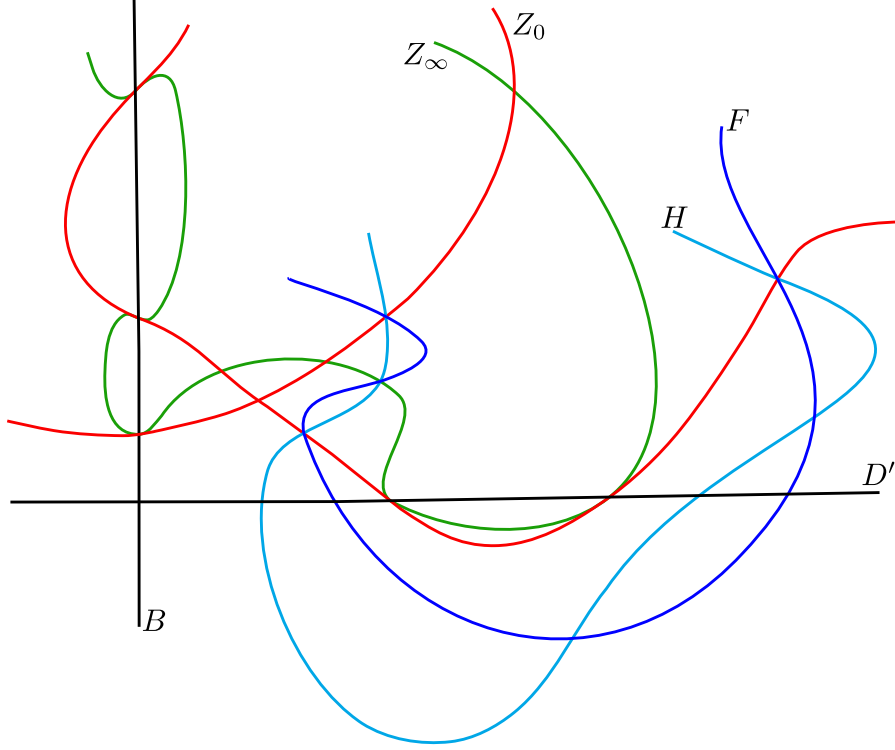


FIGURE 1. The configuration of curves on the generic fiber after Lemma 5.3.5

The restriction of  $\beta^N \in H^0(\hat{X}, \mathcal{O}_{\hat{X}}(N\Gamma))$  to  $Z_0 \cup Z_\infty$  determines a section

$$\beta^N s_\infty h \in H^0(Z_0 \cup Z_\infty, \mathcal{O}(L + N\Gamma)),$$

which we are going to lift in an appropriate way thanks to the following result.

**Lemma 5.3.5.** *There exists a section  $s_0 \in H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L + N\Gamma))$  such that  $\phi(s_0) = \beta^N s_\infty h$  and such that its zero locus  $F = (s_0)$  satisfies the following properties.*

- v)  $F$  is integral, regular along each point of  $\hat{\Sigma}$ .
- vi)  $F \cap D' \cap L = F \cap D' \cap Z_0 = F \cap H \cap D' = \emptyset$ , and  $F$  intersect  $|D'|$  transversally.
- vii)  $F$  intersects  $H \setminus (Z_0 \cup Z_\infty)$  transversally, which means that every intersection point of  $F$  and  $H$  away from  $Z_0 \cup Z_\infty$  is a regular point of both  $F$  and  $H$ , and that they intersect transversally there.

Moreover,  $F = \mathcal{F}_\eta$  for a hypersurface section  $\mathcal{F}$  of  $\hat{\mathcal{X}}$ , which is surjective over  $S$  and satisfies the property that  $\mathcal{D}' \cap \mathcal{F}$  is an effective Cartier divisor on  $\mathcal{F}$ , finite over  $S$ .

*Proof.* We keep the notations of Lemma 5.3.4. Consider the torsion-free submodule

$$M \widetilde{\otimes}_S E_N = H^0(\hat{\mathcal{X}}, \iota_M^* \mathcal{O}_{\mathbb{P}_S(M)}(1) \otimes \iota_E^* \mathcal{O}_{\mathbb{P}_S(E)}(N) \otimes I_{Z_0 \cup Z_\infty})$$

of the free module  $M \otimes_S E_N$ .

Let  $s_1, \dots, s_m$  be a basis for a maximal free submodule  $E'$  of  $M \widetilde{\otimes}_S E_N$ , and denote by the same letters the corresponding sections of  $H^0(\hat{X}, \mathcal{O}(L + N\Gamma) \otimes I_{Z_0 \cup Z_\infty})$ . As in the proof of Lemma 5.3.2, the  $s_i$ 's define a rational map

$$\hat{X} \dashrightarrow \mathbb{P}_K^{m-1}$$

that is locally a closed immersion on  $\hat{X} \setminus (Z_0 \cup Z_\infty)$ . Let  $\tilde{s}_0$  be any lift of  $\beta^N s_\infty h$  to a global section in  $H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L + N\Gamma))$ , and let  $\lambda \in \mathcal{O} - \{0\}$  be a non-zero element such that  $\lambda \tilde{s}_0 \in M \otimes_S E_N$ . Adding  $\lambda \tilde{s}_0$  to the set  $(s_1, \dots, s_m)$  defines another free submodule  $E'' = E' \oplus \langle \lambda \tilde{s}_0 \rangle$  of  $M \otimes_S E_N$ , of rank  $m + 1$ , since  $\tilde{s}_0 \notin H^0(\hat{X}, \mathcal{O}(L + N\Gamma) \otimes I_{Z_0 \cup Z_\infty})$  (so that, a fortiori,  $\lambda s_0$  is  $S$ -linearly independent of the  $s_i$ 's). By construction, we have that the linear system  $V(\tilde{s}_0, s_1, \dots, s_m)$  is nothing but the linear system associated to the sub-vector space  $E'' \otimes_S K$  of the vector space  $(M \otimes_S E_N) \otimes_S K$  or, equivalently to the  $K$ -points of the Grassmannian  $\text{Gr}_S(E'' \otimes_S K)(K)$ .

Next, notice that the base locus of  $V(\tilde{s}_0, s_1, \dots, s_m)$  is zero dimensional, given by  $(\tilde{s}_0) \cap (Z_0 \cup Z_\infty)$ , and disjoint from the sets  $D' \cap L$ ,  $D' \cap Z_0 = D' \cap Z_\infty$ , and  $D' \cap H$ . Indeed,  $D' \cap H \cap (Z_0 \cup Z_\infty) = \emptyset$  by condition iv) of Lemma 5.3.4, while  $D' \cap L \cap (Z_0 \cup Z_\infty) = \emptyset$  by the choice of  $L$  (missing  $\Lambda$ ) and the choice of  $t_\infty$  in Lemma 5.3.1. As for the set  $D' \cap (Z_0 \cup Z_\infty)$ , note that by choice  $\tilde{s}_0$  restricts to  $\beta^N s_\infty h$  on  $Z_0 \cup Z_\infty$ , and we have that  $B \cap Z_0 \cap D' = \emptyset$  by the choice of  $B$  (which forces  $B \cap Z_\infty \cap D' = \emptyset$ , since  $Z_0 \cap D' = Z_\infty \cap D'$ ). In particular,  $\beta^N$  is a unit at each point  $x \in Z_0 \cup Z_\infty \cap D'$ .

We are then left to consider the term  $s_\infty h$  in a neighborhood of any  $x \in D' \cap Z_0 = D' \cap Z_\infty$  in  $Z_0 \cup Z_\infty$  in order to prove that the base locus of  $V(\tilde{s}_0, s_1, \dots, s_m)$  is disjoint from  $D' \cap Z_0$ . Restricting  $\tilde{s}_0/\beta^N$  to the components  $Z_0$  and  $Z_\infty$  we get

$$(5.5) \quad \tilde{s}_0/\beta^N|_{Z_0} = f_0 \cdot s_\infty|_{Z_0}, \quad \tilde{s}_0/\beta^N|_{Z_\infty} = 1 \cdot s_\infty|_{Z_\infty}.$$

The function  $f_0$  satisfies the modulus condition on  $Z_0$  with respect to  $D'$ , so  $f_0$  is a unit at every  $x \in D' \cap Z_0$ . By Lemma 5.3.4,  $H = (s_\infty)$  is disjoint from  $Z_0 \cap D'$ , thus  $s_\infty$  is a unit at every  $x \in D' \cap Z_0$ . The same analysis works for the restriction to  $Z_\infty$ . Thus, independently of the choice of the lift  $\tilde{s}_0$ , the base locus of  $V(\tilde{s}_0, s_1, \dots, s_m)$  is disjoint from the sets in vi).

By the classical theorem of Bertini, there exists an open subset  $U$  of  $\text{Gr}_S(E'' \otimes_S K)$  such that every hypersurface section  $F$  corresponding to a  $K$  point of  $U$  is irreducible and generically reduced, and satisfies both vi) and vii). Chose a section  $s_0 \in E''$  with  $\mathcal{F} = (s_0)$  such that  $F = \mathcal{F} \times_S K$  belongs to  $U$ . We also write  $s_0$  for the corresponding section over  $K$  (so that  $F = (s_0)$ ).

We now turn to the regularity (over  $K$ ) in a neighborhood of each cancellation point  $\xi \in \hat{\Sigma}$ . Thanks to the fact that  $\beta$  is a unit at  $\xi$  (since  $B \cap \hat{\Sigma} = \emptyset$ ) and the fact that  $s_\infty h$  is given by the expression in (5.4), in a neighborhood of  $\xi$  any lifting of  $\beta^N h s_\infty$  is of the form

$$s_0 = \mu_1 u_1 + \mu_2 u_2^{\lambda+1} + \mu_{12} u_1 u_2, \quad \mu_{12} \in \mathcal{O}_{\hat{X}, \xi}, \mu_1, \mu_2 \in \mathcal{O}_{\hat{X}, \xi}^\times$$

in particular,  $F = (s_0)$  is automatically regular at  $\xi$  as required.

Finally, we turn to the last required conditions about the model  $\mathcal{F}$  of  $F$ , given by a point of

$$\text{Gr}_S(E'')(S) \subset \text{Gr}_S(E'' \otimes_S K)(K)$$

This can be achieved via a minor modification of the proof of Theorem 4.2.2. Surjectivity over  $S$  is clear. As for the condition on  $\mathcal{D}'$ , we argue as follows. Let  $\mathcal{D}_s^1, \dots, \mathcal{D}_s^r$  be the irreducible components of  $\mathcal{D}'_s$ . Let  $\mathcal{G}$  be another hypersurface section of  $\hat{\mathcal{X}}$ , surjective over  $S$  and such that the following two conditions are satisfied

- a)  $\mathcal{G} \cap (\mathcal{Z}_0 \cup \mathcal{Z}_\infty) \cap \mathcal{D}_s = \mathcal{G}_K \cap (Z_0 \cup Z_\infty) \cap D' = \emptyset$
- b)  $\mathcal{G} \cap \mathcal{D}_s^i \neq \emptyset$  for each  $i = 1, \dots, r$ .

Both conditions can be easily achieved by the classical theorem of Bertini. Let  $U' \subset \text{Gr}_k(E'' \otimes_S k)$  be the open subset such that for every  $H \in U'(k)$ , we have  $H \cap (\mathcal{G} \cap \mathcal{D}) = \emptyset$  (note that the intersection necessarily takes place in the special fiber). Choosing  $s_0 \in E''$  in the set  $U(S) \cap \text{sp}^{-1}(U'(k))$ , not empty thanks to Lemma 4.1.1, does then the job. Indeed, conditions

a) and b) force to have, as in the proof of 4.2.2, that the restriction  $s_0 \mapsto \overline{s_0}^i$  of  $s_0$  is not zero in  $H^0(\widehat{\mathcal{X}}_s, \mathcal{O}_{\widehat{\mathcal{X}}_s}/I_{\mathcal{D}'_s})$  for all  $i$ . Thus  $\mathcal{D}'_s$  restricts to a Cartier divisor on  $\mathcal{F}_s$ , and its support is therefore a finite set of closed points.  $\square$

We summarize what we achieved so far in the following proposition. We keep the above notations.

**Proposition 5.3.6.** *Let  $\widehat{X}$  be as above. Then there exist integral curves  $H$  and  $F$  such that the following conditions are satisfied*

- a') *Both  $H$  and  $F$  are regular in a neighborhood of  $\widehat{\Sigma}$ .*
- b') *Both  $H$  and  $F$  intersect transversally  $|D'|$  (in particular, they are regular there), and  $H \cap F \cap |D'| = \emptyset$ .*
- c') *Both  $F$  and  $H$  have integral models  $\mathcal{F}$  and  $\mathcal{H}$  over  $S$ , which are surjective over  $S$  and satisfy the property that the restriction of  $\mathcal{D}'$  to them is an effective Cartier divisor, finite over  $S$ .*
- d') *Let  $\tau = t_0/t_\infty \in K(\widehat{X})$  as rational function on  $\widehat{X}$ . Then the restriction of  $\tau$  to  $F$  and to  $H$  satisfy the modulus condition with respect to  $D' \cap F$  and to  $D' \cap H$  respectively.*

Moreover, we have  $\gamma' = \text{div}_F(\tau) - \text{div}_H(\tau)$  as zero cycles on  $\widehat{X}$ .

*Proof.* Properties a') and b') are direct consequences of Lemma 5.3.4 and of Lemma 5.3.5. Since moreover

$$D' \cap (Z_\infty \cup L) \cap H = D' \cap (Z_\infty \cup L) \cap F = D' \cap (Z_0 \cup L) \cap H = D' \cap (Z_0 \cup L) \cap F = \emptyset,$$

thanks to Lemma 5.3.4 part iv) and 5.3.5 part vi), the choice of  $t_\infty$  in Lemma 5.3.2 guarantees that  $\tau \in 1 + I_D \mathcal{O}_{\widehat{X},x}$  for  $x \notin D' \cap Z_\infty$ , so that  $\tau|_F \in 1 + I_{D'} \mathcal{O}_{F,x}$  for every  $x \in F \cap D'$  and that  $\tau|_H \in 1 + I_{D'} \mathcal{O}_{H,x}$  for every  $x \in H \cap D'$  (note that here we are using the property that  $Z_\infty \cap D' \cap (F \cup H) = \emptyset$ ). We can now compute

$$\begin{aligned} 0 &= \text{div}_{Z_\infty}(1) = F \cdot Z_\infty - H \cdot Z_\infty - N(B \cdot Z_\infty), \\ \gamma' &= \text{div}_{Z'}(f') = F \cdot Z_0 - H \cdot Z_0 - N(B \cdot Z_0). \end{aligned}$$

We subtract the two equations and collect to get  $\gamma' = \text{div}_F(\tau) - \text{div}_H(\tau) - N \text{div}_B(\tau)$ . But  $t_0/t_\infty = 1$  on  $B$  thanks to Lemma 5.3.2, thus the term  $N \text{div}_B(\tau)$  vanishes and we get the required expression for  $\gamma'$ .  $\square$

**5.4. Reduction II.** We keep the notations of Section 5.3. In order to further improve the expression of  $\gamma'$ , we introduce some auxiliary divisors. As before, we take care of the models over  $S$  by means of the techniques developed in Section 4.

**Lemma 5.4.1.** *There exists a hypersurface section  $\mathcal{H}' \subset \widehat{\mathcal{X}}$ , such that the following conditions are satisfied. Let  $H' = \mathcal{H}'_\eta$  be the generic fiber.*

- (1)  *$H'$  is integral, regular, and  $H'_{|D'} \geq H_{|D'} + N \cdot B_{|D'}$*
- (2)  *$H' \cap \widehat{T}_0 = H' \cap \widehat{T}_\infty = H' \cap (F \cap H) = H' \cap F \cap D' = H' \cap F \cap B = \emptyset$ .*
- (3)  *$H^1(\widehat{X}, \mathcal{O}_{\widehat{X}}(H') \otimes I_{D'}) = 0$ .*
- (4)  *$\mathcal{H}'$  is integral, surjective over  $S$ . Moreover,  $\mathcal{D}'$  restricts to an effective Cartier divisor on  $\mathcal{H}'$  which is finite over  $S$ .*

*Proof.* Everything can be achieved by using the classical theorem of Bertini over  $K$ , while the condition (4) on the integral model can be achieved by means of Theorem 4.2.1.

More precisely: property (3) can be achieved by taking  $H'$  of sufficiently large degree, and property (2) is clear since all the relevant sets are zero dimensional. As for property (1), notice

that  $H$  intersects  $|D'|$  transversally by construction (by lemma 5.3.4). Similarly,  $B$  intersects  $|D'|$  transversally by choice and away from  $H$  (since  $H \cap D' \cap B = \emptyset$  by Lemma 5.3.4). A local analysis around a neighborhood of any point in  $(B \cap D') \amalg (H \cap D')$  shows that  $H'$  can be chosen to be regular there. Note that  $H'$  can be chosen to intersect  $D'$  transversally at each point  $x \in H \cap D'$ , and will be tangent to  $D'$  at each point  $x \in B \cap D'$ . We can moreover ask that  $H'$  intersects  $|D'|$  transversally at each point of  $(H' \cap D') \setminus (D' \cap (B \cup H))$ .  $\square$

Write  $g = (s_0/s_\infty\beta^N)|_{D'} \in H^0(D', \mathcal{O}(H + N \cdot B))$ . By property (1) above, we can consider the image of  $g$  in  $H^0(D', \mathcal{O}(H'))$ , and choose a lift  $G$  along the surjection (guaranteed by property (3)).

$$H^0(\hat{X}, \mathcal{O}_{\hat{X}}(H')) \rightarrow H^0(D', \mathcal{O}_{D'}(H')) \rightarrow 0$$

Write  $W_0$  for the divisor of zeros of  $G$ . Note that  $\text{div}(G) = W_0 - H' \in \mathcal{Z}^1(\hat{X})$  by construction.

**Claim 5.4.2.**  $W_0$  is regular in a neighborhood of every point  $x$  in  $W_0 \cap D'$  and  $W_0 \cap D' \cap H' = \emptyset$ .

*Proof.* Again by construction, any point  $x$  in  $D'$  such that either  $s_0$  or  $s_\infty\beta$  is not a unit is a regular point of  $|D'|$ . Let then  $\pi$  be a local equation for  $|D'|$  in a neighborhood of  $x$ , so that  $D' = (\pi^e)$ . Any lift  $G$  of  $g$  is then in a neighborhood of  $x$  in  $\hat{X}$  of the form  $G = (s_0 + a\pi^e s_\infty\beta^N)/(s_\infty\beta^N)$  for  $a \in \mathcal{O}_{\hat{X},x}$ , so that  $W_0$  is locally given by  $s_0 + a\pi^e s_\infty\beta^N = 0$ .

This is enough to conclude. In fact, observe that by condition b') in Proposition 5.3.6 and condition (2) of Lemma 5.4.1, for every  $x \in H' \cap |D'|$ , we have that  $s_{0,x} \in \mathcal{O}_{\hat{X},x}^\times$ . In particular,  $s_0 + a\pi^e s_\infty\beta^N$  is not in the maximal ideal of  $\mathcal{O}_{\hat{X},x}$ , showing that  $W_0 \cap D' \cap H' = \emptyset$  as required. As for the regularity, note that if  $x \in D' \cap W_0$ , we have  $G|_{D'} = g = (s_0/s_\infty\beta^N)|_{D'}$  and this time  $s_\infty\beta^N$  is a unit there. Since  $F = (s_0)$  intersects  $|D'|$  transversally, any lift as above is automatically regular.  $\square$

Following the path of Lemmas 5.3.2 and 5.3.4, we can now alter  $G$  by any section of  $H^0(\hat{X}, \mathcal{O}_{\hat{X}}(H') \otimes I_{D'})$  such that

- 1')  $W_0$  is regular,
- 2')  $W_0 \cap F \cap H' = W_0 \cap \hat{T}_0 = W_0 \cap \hat{T}_\infty = \emptyset$

For condition 1'), note that regularity away from  $D'$  can be achieved by standard Bertini. Along  $D'$ , this is guaranteed by Claim 5.4.2. As for condition 2'), it follows from the fact that  $\hat{T}_0 \cap D' = \hat{T}_\infty \cap D' = \emptyset$  and that  $F \cap H' \cap D' = \emptyset$ .

We now proceed as follows. Let  $\hat{\Lambda}$  be the following set

$$\hat{\Lambda} = (F \cap D') \cup (H' \cap D') \cup (W_0 \cap D') \cup (\hat{T}_0 \cup \hat{T}_\infty) \cup (F \cap H') \cup (B \cap D').$$

Choose another hypersurface section  $L'$  of  $\hat{X}$ , of sufficiently large degree, such that  $L' \cap \hat{\Lambda} = \emptyset$ . As before, we may assume that  $H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L')) = M' \otimes_S K$  for a free  $S$ -module  $M'$ , corresponding to a projective embedding  $\iota_{M'}: \hat{\mathcal{X}} \hookrightarrow \mathbb{P}_S(M')$  so that  $\mathcal{O}(L') = (\iota_{M'}^* \mathcal{O}_{\mathbb{P}_S(M')}(1))_\eta$ . We can choose the degree of  $L'$  to be sufficiently large so that

$$(5.6) \quad H^1(\hat{X}, \mathcal{O}_{\hat{X}}(L') \otimes I_{D' \cup W_0}) = H^1(\hat{\mathcal{X}}, \iota_{M'}^* \mathcal{O}(1) \otimes I_{\mathcal{D}' \cup \mathcal{W}_0}) = 0.$$

where  $\mathcal{W}_0$  is the closure of  $W_0$  in  $\hat{\mathcal{X}}$ .

We now let  $\Xi$  be the set of cancellation points in the expression of  $\gamma'$  as  $\text{div}_F(\tau) - \text{div}_H(\tau)$ , which moreover do not satisfy (MC) $_{\mathcal{O}}$ . Explicitly

$$\Xi = \{\xi \in F \cap H \mid \text{ord}_{F,\xi}(\tau) = \text{ord}_{H,\xi}(\tau)\} \cap \{\xi \in F \cap H \mid \xi \text{ does not satisfy (MC)}_{\mathcal{O}}\} \subseteq \hat{\Sigma}$$

Fix  $\xi \in \Xi$ , and let  $\lambda = \lambda_\xi$  be the order of vanishing of  $\tau$  at  $\xi$ , i.e.

$$\text{ord}_{F,\xi}(\tau) = \text{ord}_{H,\xi}(\tau) = \lambda$$



Choose a global section  $l_\infty \in H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L'))$  such that, if  $(l_\infty)$  denotes its divisor of zeros, we have:

- i)  $(l_\infty)$  is integral, and regular.
  - ii)  $\xi \in (l_\infty)$ , and  $(l_\infty) \cap (\Xi \setminus \{\xi\}) = \emptyset$ .
  - iii)  $(l_\infty)$  intersects  $F$ ,  $|D'|$ ,  $H$ ,  $H'$ ,  $B$  and  $W_0$  transversally, which means that  $(l_\infty)$  intersects each divisor in the regular locus, and transversally.
  - iv)  $(l_\infty) \cap D' \cap (F \cup H' \cup B) = (l_\infty) \cap D' \cap W_0 = \emptyset$ , and  $(l_\infty) \cap (\hat{T}_0 \cup \hat{T}_\infty) \setminus \{\xi\} = \emptyset$
  - v) For every closed point  $y \in (l_\infty)$  in the set
- $$(5.7) \quad (((l_\infty) \cap W_0) \cup ((l_\infty) \cap H') \cup ((l_\infty) \cap F) \cup ((l_\infty) \cap H) \cup ((l_\infty) \cap B) \setminus \Xi$$
- we have that  $y$  satisfies  $(SMC)_\mathcal{O}$ .

- vi)  $(l_\infty) = (\mathcal{L}_\infty)_\eta$  for a hypersurface section  $\mathcal{L}_\infty$  of  $\hat{\mathcal{X}}$  which is surjective over  $S$  and satisfies the property that  $\mathcal{D}'$  restricts to an effective Cartier divisor on  $\mathcal{L}_\infty$ , which is finite over  $S$ .

The last two conditions follow from Theorem 4.2.2 (applied to the inclusion of the closure of the point  $\xi$ ) and Proposition 4.2.4 for the embedding  $\iota_{M'}$ . A general choice of  $l_\infty$  satisfies all the other properties, noting that the only closed condition is ii), but by construction  $F$  and  $H$  are regular and tranverse to each other there, so that  $(l_\infty)$  can be itself chosen to be regular at  $\xi$ , and intersecting  $F$  and  $H$  transversally.

By (5.6), we have a surjection

$$H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L')) \rightarrow H^0(D' \cup W_0, \mathcal{O}_{\hat{X}}(L')|_{D' \cup W_0}) \rightarrow 0$$

**Claim 5.4.3.** *There exists a lift  $l_0$  of the restriction of  $l_\infty$  to  $D' \cup W_0$  such that*

- i')  $(l_0) \cap \Xi = \emptyset$ ,  $(l_0)$  is regular and intersect transversally  $|D'|$ .
  - ii')  $(l_0) \cap (\hat{T}_\infty \cup \hat{T}_0) = \emptyset$
  - iii')  $(l_0) \cap D' \cap (F \cup H) = \emptyset$
  - iv') For every closed point  $y \in (l_0)$  in the set
- $$(5.8) \quad ((l_0) \cap F) \cup ((l_0) \cap H) \cup ((l_0) \cap H') \cup ((l_0) \cap B)$$

we have that  $y$  satisfies  $(SMC)_\mathcal{O}$ .

- v')  $(l_0) = (\mathcal{L}_0)_\eta$  for a hypersurface section  $\mathcal{L}_0$  of  $\hat{\mathcal{X}}$  which is surjective over  $S$  and satisfies the property that  $\mathcal{D}'$  restricts to an effective Cartier divisor on  $\mathcal{L}_0$ , which is finite over  $S$ .

*Proof.* Regularity away from  $D' \cup W_0$  is clear, as well as the condition  $(l_0) \cap \Xi = \emptyset$ , since  $\Xi$  is away from  $D'$ , and condition 2') guarantees that  $\Xi$  is away from  $W_0$ . Note that  $D' \cap W_0 \cap (l_\infty) = \emptyset$ , and that  $(l_\infty)$  intersects transversally both  $W_0$  and  $|D'|$ , so that we can choose the lift  $l_0$  to be regular there as well.

Since  $(D' \cup W_0) \cap (\hat{T}_\infty \cup \hat{T}_0) = \emptyset$ , condition ii') is clear. Similarly, since  $(l_\infty) \cap D' \cap (F \cup H) = \emptyset$  by iv), we get condition iii').

As for the last two conditions, we argue as in the proof of Lemma 5.3.5 to further refine the choice of  $l_0$ . More precisely, consider the torsion free-submodule

$$\widetilde{M}' = H^0(\hat{\mathcal{X}}, \iota_{M'}^* \mathcal{O}(1) \otimes I_{\mathcal{D}' \cup W_0})$$

of the free module  $M' = H^0(\hat{\mathcal{X}}, \iota_{M'}^* \mathcal{O}(1))$ .

Let  $l_1, \dots, l_r$  be a basis for a maximal free submodule  $E''$  of  $\widetilde{M}'$ , and denote by the same letters the corresponding sections of  $H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L') \otimes I_{D' \cup W_0})$ . Let  $\tilde{l}_0$  be any lift of the restriction of  $l_\infty$  to  $D' \cup W_0$  to a global section in  $M'_n \otimes_S K$ , and let  $\lambda \in \mathcal{O} - \{0\}$  be a non-zero element such that  $\lambda \tilde{l}_0 \in M'_n$ . We can then add  $\lambda \tilde{l}_0$  to the  $l_i$ 's to get a rank  $r+1$  free  $\mathcal{O}$ -module

$E'''$ . We can now apply Proposition 4.2.4 to the corresponding locally closed immersion to get iv').

Finally, to get v'), we argue again as in the last part of the proof of Lemma 5.3.5. We quickly sketch the argument: a general section of  $\mathrm{Gr}_k(E''' \otimes_S k)(k)$  contains the special fiber  $\mathcal{V}_s$  of the closure  $\mathcal{V}$  of every point in  $(l_\infty) \cap (W_0 \setminus D')$ . But those points satisfy (SMC) $_{\mathcal{O}}$  thanks to v) above, and therefore  $\mathcal{V}_s$  is not contained in any component of  $\mathcal{D}_s$ . But then we can assume that the image of  $l_0$  is not zero in  $H^0(\hat{\mathcal{X}}_s, \mathcal{O}_{\hat{\mathcal{X}}_s}/I_{\mathcal{D}_s^i})$  for each component  $\mathcal{D}_s^i$  of  $\mathcal{D}'_s$ , which means in particular that we can chose it so that  $\mathcal{L}_0$  satisfies the property that  $\mathcal{D}'$  restricts to an effective Cartier divisor on it, finite over  $S$  as required.  $\square$

We summarize for the reader's convenience what we have achieved so far. For simplicity, we focus on the salient properties of the new objects constructed.

**Proposition 5.4.4.** *Let  $\hat{X}$  be as above, and let  $\xi$  be a cancellation point for the cycle  $\gamma'$ . Then there exist integral curves,  $(l_0)$  and  $(l_\infty)$ , divisors of zeros of sections  $l_0, l_\infty \in H^0(\hat{X}, \mathcal{O}_{\hat{X}}(L'))$  of a very ample hypersurface section  $L'$  of  $\hat{X}$  such that the following conditions are satisfied.*

- (1) *Both  $(l_0)$  and  $(l_\infty)$  are regular, and have integral models  $\mathcal{L}_0$  and  $\mathcal{L}_\infty$  over  $S$ , which are surjective over  $S$  and satisfy the property that the restriction of  $\mathcal{D}'$  to them is an effective Cartier divisor, finite over  $S$ .*
- (2) *Both  $(l_0)$  and  $(l_\infty)$  intersect transversally  $|D'|$ .*
- (3)  *$(l_0)$  is disjoint from  $\hat{T}_\infty \cup \hat{T}_0 \cup \Xi$ .*
- (4)  *$(l_\infty)$  passes through  $\xi$ , and intersects both  $F$  and  $H$  and  $H'$  transversally there.*
- (5) *Both  $(l_0)$  and  $(l_\infty)$  are chosen so that every closed points in the sets (5.7) and (5.8) satisfy the strong modulus condition (SMC) $_{\mathcal{O}}$ .*
- (6) *Let  $l_0/l_\infty \in K(\hat{X})$  as rational function on  $\hat{X}$ . Then the restriction of  $l_0/l_\infty$  to  $B, F, H', H$  and  $W_0$  satisfy the modulus condition with respect to  $D'$ .*

The new sections  $(l_0)$  and  $(l_\infty)$  will be used to remove, from the expression of  $\gamma'$ , the specified cancellation point  $\xi$ . This is the content of the next subsection.

5.4.1. *Moving.* We can now compute. Note that by construction each zero cycles appearing is supported on  $\hat{X} \setminus D'$ . Adding to  $\gamma'$  the boundary of  $\left\{ \frac{s_0}{s_\infty \beta^N}, \frac{l_0}{l_\infty} \right\} \in K_2(K(\hat{X}))$  we get

$$\begin{aligned} \gamma' &= \mathrm{div}_F(\tau) - \mathrm{div}_H(\tau) \\ &= \mathrm{div}_F(\tau) - \mathrm{div}_H(\tau) + \lambda \left( \mathrm{div}_F\left(\frac{l_0}{l_\infty}\right) - \mathrm{div}_H\left(\frac{l_0}{l_\infty}\right) \right) + \\ &\quad + \mathrm{div}_{(l_0)}\left(\frac{s_0}{s_\infty \beta^N}\right) - \mathrm{div}_{(l_\infty)}\left(\frac{s_0}{s_\infty \beta^N}\right) - N \mathrm{div}_B\left(\frac{l_0}{l_\infty}\right) \end{aligned}$$

Collecting the terms containing  $F$  and  $H$  gives then

$$(5.9) \quad \gamma' = \mathrm{div}_F\left(\frac{\tau l_0^\lambda}{l_\infty^\lambda}\right) - \mathrm{div}_H\left(\frac{\tau l_0^\lambda}{l_\infty^\lambda}\right) + \lambda \mathrm{div}_{(l_0)}\left(\frac{s_0}{s_\infty \beta^N}\right) - \lambda \mathrm{div}_{(l_\infty)}\left(\frac{s_0}{s_\infty \beta^N}\right) - N \lambda \mathrm{div}_B\left(\frac{l_0}{l_\infty}\right),$$

and each curve appearing in the expression has an integral model over  $S$ , which is surjective and satisfies the property that  $\mathcal{D}'$  restricts to an effective Cartier divisor on it, finite over  $S$ .

In this newly found expression for  $\gamma'$ , note that  $\xi$  does not appear in the support of any of the divisors involved. Indeed,  $\xi$  does not appear in the last 3 terms by construction, while the function  $\frac{\tau l_0^\lambda}{l_\infty^\lambda}$  is a unit at  $\xi$  thanks to the choices of  $l_0$  and  $l_\infty$  (note in particular that we are using the fact that  $(H \cup F) \cap D' \cap (l_\infty) = \emptyset$ , so that the expression  $\frac{\tau l_0^\lambda}{l_\infty^\lambda}$  is indeed a unit at every point of  $F \cap D'$  and at every point of  $H \cap D'$ ).

Since  $l_0/l_\infty = 1$  along  $D'$  and  $(l_\infty) \cap D' \cap B = \emptyset$ , note that the last term  $\operatorname{div}_B\left(\frac{l_0}{l_\infty}\right)$  satisfies the modulus condition. Finally, note that thanks to properties v) and iv') above, every divisor appearing in (5.9) except  $\Xi - \{\xi\}$  satisfies the modulus condition  $(\text{MC})_\emptyset$ .

We are now left to correct the second and the third term of (5.9) to get the modulus condition with respect to  $D'$ . Write  $\gamma''$  for the cycle

$$\gamma'' := \operatorname{div}_F\left(\frac{\tau l_0^\lambda}{l_\infty^\lambda}\right) - \operatorname{div}_H\left(\frac{\tau l_0^\lambda}{l_\infty^\lambda}\right) - N\lambda \operatorname{div}_B\left(\frac{l_0}{l_\infty}\right).$$

Note that, by construction and what remarked above, the cycle  $\gamma''$  already satisfies the modulus condition with respect to  $D'$ . Arguing as above, we add to (5.9) the boundary of  $\{G, \frac{l_0}{l_\infty}\} \in K_2(K(\hat{X}))$  to find

$$\begin{aligned} \gamma' &= \gamma'' + \lambda\left(\operatorname{div}_{(l_0)}\left(\frac{s_0}{s_\infty\beta^N}G_{|(l_0)}^{-1}\right) - \operatorname{div}_{(l_\infty)}\left(\frac{s_0}{s_\infty\beta^N}G_{|(l_\infty)}^{-1}\right) - \operatorname{div}_{(l_0)}(G) + \operatorname{div}_{(l_\infty)}(G)\right) \\ &= \gamma'' + \lambda\left(\operatorname{div}_{(l_0)}\left(\frac{s_0}{s_\infty\beta^N}G_{|(l_0)}^{-1}\right) - \operatorname{div}_{(l_\infty)}\left(\frac{s_0}{s_\infty\beta^N}G_{|(l_\infty)}^{-1}\right) + \operatorname{div}_{W_0}\left(\frac{l_0}{l_\infty}\right) - \operatorname{div}_{H'}\left(\frac{l_0}{l_\infty}\right)\right) \end{aligned}$$

We now note the following

- (1) By construction,  $l_0 = l_\infty \bmod I_{W_0}$ . Thus the term  $\operatorname{div}_{W_0}\left(\frac{l_0}{l_\infty}\right)$  vanishes.
- (2)  $\frac{l_0}{l_\infty} \in 1 + I_{D'}\mathcal{O}_{H',x}$  for every  $x \in H' \cap D'$ . Note that we are in particular using the fact that  $(l_\infty) \cap H' \cap D' = \emptyset$ , which implies that  $l_\infty$  is a unit at every point  $x \in H' \cap D' \supset H \cap D'$ .
- (3) The function  $G$  is constructed as global lift of the restriction of  $\frac{s_0}{s_\infty\beta^N}$  to  $D'$ . Moreover, for every  $x \in (l_0) \cap D'$  we have that  $G_{|(l_0),x} \in \mathcal{O}_{(l_0),x}^\times$ , since  $(H' \cup W_0) \cap D' \cap (l_0) = \emptyset$ . In particular, we have that  $G_{|(l_0),x}^{-1} \in \mathcal{O}_{(l_0),x}^\times$ , so that  $\frac{s_0}{s_\infty\beta^N}G_{|(l_0)}^{-1}$  is regular and invertible at every  $x \in D' \cap (l_0)$ , and it's congruent to 1 mod  $I_{D'}$ . In other words, it satisfies the modulus condition. The same argument applies verbatim to  $\frac{s_0}{s_\infty\beta^N}G_{|(l_\infty)}^{-1}$  on  $(l_\infty)$ .

Thus  $\gamma'$  simplifies as

$$\gamma' = \gamma'' + \lambda\left(\operatorname{div}_{(l_0)}\left(\frac{s_0}{s_\infty\beta^N}G_{|(l_0)}^{-1}\right) - \operatorname{div}_{(l_\infty)}\left(\frac{s_0}{s_\infty\beta^N}G_{|(l_\infty)}^{-1}\right) - \operatorname{div}_{H'}\left(\frac{l_0}{l_\infty}\right)\right)$$

and every term satisfies the modulus condition with respect to  $D'$  as well as every closed point appearing in the expression satisfies  $(\text{MC})_\emptyset$ , with the only exception of the points in  $\Xi - \{\xi\}$ . Repeating the argument using  $\gamma'' + N\lambda \operatorname{div}_B\left(\frac{l_0}{l_\infty}\right)$  instead of  $\gamma'$ , we can remove every other cancellation point  $\xi' \in \Xi$  which do not satisfy  $(\text{MC})_\emptyset$ .

## 6. FURTHER REDUCTIONS AND THE PROOF

**6.1. Relative correspondences.** In order to prove our main result, we need to slightly generalize the notion of admissible correspondence to schemes that are not necessarily smooth and of finite type over a field.

Let  $S$  be the spectrum of an excellent normal local domain  $\mathcal{O}$ , and fix a non-zero element  $\Pi \in \mathfrak{m}$ . Let  $K$  be the field of functions of  $S$ . An  $S$ -modulus pair is a pair  $\mathfrak{X} = (\overline{\mathcal{X}}, \mathcal{X}_\infty)$  consisting of an  $S$ -scheme  $f: \overline{\mathcal{X}} \rightarrow S$ , separated and of finite type over  $S$ , and an effective Cartier divisor  $\mathcal{X}_\infty$  on it such that the complement  $\mathcal{X} = \overline{\mathcal{X}} \setminus \mathcal{X}_\infty$  is generically regular i.e.  $\mathcal{X}_K$  is a regular  $K$ -scheme. Let  $\mu: \overline{\mathcal{X}}^N \rightarrow \overline{\mathcal{X}}$  be the normalization morphism, and let  $\nu: \overline{\mathcal{X}}^N \rightarrow S$  be the composition  $f \circ \mu$ .

We say that  $\mathfrak{X}$  is *admissible* for  $(\mathcal{O}, (\Pi))$  if there is an inequality of Cartier divisors

$$\mu^*(\mathcal{X}_\infty) \geq \nu^*((\Pi))$$

where  $(\Pi)$  is the Cartier divisor on  $S$  defined by  $\Pi$ . A basic example of  $S$ -modulus pair which is admissible for  $(\mathcal{O}, (\Pi))$  is the relative cube

$$\overline{\square}_{\mathcal{O}, (\Pi)} = (\mathbb{P}_\mathcal{O}^1, 1_\mathcal{O} + \mathbb{P}_{(\Pi)}^1).$$

Let  $\mathfrak{X} = (\overline{\mathcal{X}}, \mathcal{X}_\infty)$  and  $\mathfrak{Y} = (\overline{\mathcal{Y}}, \mathcal{Y}_\infty)$  be  $S$ -modulus pairs. Write  $\mathcal{X} = \overline{\mathcal{X}} \setminus \mathcal{X}_\infty$  and  $\mathcal{Y} = \overline{\mathcal{Y}} \setminus \mathcal{Y}_\infty$ . Assume that  $\mathfrak{X}$  is  $(\mathcal{O}, (\Pi))$  admissible and that  $\mathfrak{Y}$  is proper (i.e. that  $\overline{\mathcal{Y}}$  is proper over  $S$ ). We define the group of *admissible  $S$ -correspondences* as the subgroup

$$(6.1) \quad \underline{\mathbf{M}}\mathbf{Cor}_{(\mathcal{O}, (\Pi))}(\mathfrak{X}, \mathfrak{Y}) \subset \mathbf{Cor}_K(\mathcal{X}_K, \mathcal{Y}_K)$$

generated by the finite prime correspondences  $V \in \mathbf{Cor}_K(\mathcal{X}_K, \mathcal{Y}_K)$ , satisfying the condition

$$\nu_V^*(\mathcal{Y}_\infty) \leq \nu_V^*(\mathcal{X}_\infty)$$

where  $\overline{V}$  is the closure of  $V$  in  $\overline{\mathcal{X}} \times_S \overline{\mathcal{Y}}$  and  $\nu_V: \overline{V}^N \rightarrow \overline{\mathcal{X}} \times_S \overline{\mathcal{Y}}$  is the composition of the (finite) normalization map with the inclusion.

*Example 6.1.1.* Let  $\mathfrak{Y} = (\overline{\mathcal{Y}}, \mathcal{Y}_\infty)$  be a proper  $S$ -modulus pair, and let  $Y = \mathcal{Y}_K$  be the generic fiber of the complement of  $\mathcal{Y}_\infty$ . Taking  $\mathfrak{X} = (S, (\Pi))$  in (6.1), we get that

$$\underline{\mathbf{M}}\mathbf{Cor}_{(\mathcal{O}, (\Pi))}(\mathfrak{X}, \mathfrak{Y}) = \underline{\mathbf{M}}\mathbf{Cor}_{(S, (\Pi))}((S, (\Pi)), \mathfrak{Y}) \subset \mathcal{Z}_0(Y)$$

is the subgroup of the group  $\mathcal{Z}_0(Y) = \mathbf{Cor}_K(\mathrm{Spec}(K), Y)$  of zero cycles on  $Y$  generated by closed points  $y \in Y \subset \overline{\mathcal{Y}}$  satisfying the modulus condition  $(\mathrm{MC})_\mathcal{O}$  in the sense of Definition 3.1.1.

Given a proper  $S$ -modulus pair  $\mathfrak{X}$ , we can now define  $h_0^\square(\mathfrak{X})(\mathcal{O}, (\Pi))$  as the cokernel

$$(6.2) \quad \underline{\mathbf{M}}\mathbf{Cor}_{(\mathcal{O}, (\Pi))}((\mathcal{O}, (\Pi)) \otimes \overline{\square}, \mathfrak{X}) \xrightarrow{i_0^* - i_\infty^*} \underline{\mathbf{M}}\mathbf{Cor}_{(\mathcal{O}, (\Pi))}((\mathcal{O}, (\Pi)), \mathfrak{X})$$

where  $(\mathcal{O}, (\Pi)) \otimes \overline{\square} = \overline{\square}_{\mathcal{O}, (\Pi)}$  and the map  $i_0^* - i_\infty^*$  is induced by

$$\mathbf{Cor}_K(\mathbb{P}_K^1 - \{1\}, \mathcal{X}_K) \xrightarrow{i_0^* - i_\infty^*} \mathbf{Cor}_K(\mathrm{Spec}(K), \mathcal{X}_K) = \mathcal{Z}_0(\mathcal{X}_K)$$

Note that the modulus condition is preserved under  $i_0^*$  and  $i_\infty^*$ , thanks to the containment Lemma (see e.g. [16, 2.2]).

If  $\mathfrak{X} = (\overline{X} \times_k S, X_\infty \times_k S)$  for a proper modulus pair  $(\overline{X}, X_\infty) \in \mathbf{M}\mathbf{Cor}$ , this recovers precisely the definition in Section 2.

*Remark 6.1.2.* The notation  $\underline{\mathbf{M}}\mathbf{Cor}_{(\mathcal{O}, (\Pi))}$  is suggesting that the group of admissible  $S$ -correspondences can be taken as group of morphisms in an additive category of  $S$ -modulus pairs (possibly after further restrictions). We do not need to investigate this point further in this paper. In particular, we do not claim that our definition is closed under composition.

**6.2. A reformulation in terms of algebraic cycles.** Let  $k$  be again a perfect field, and  $\mathfrak{X} = (\overline{X}, X_\infty) \in \mathbf{M}\mathbf{Cor}$  a proper modulus pair over  $k$ , and let  $X = \overline{X} \setminus X_\infty$ . Assume that the total space  $\overline{X}$  is projective. Let  $h_0^\square(\mathfrak{X})_{\mathrm{Nis}}$  be the Nisnevich sheaf associated to the presheaf  $h_0^\square(\mathfrak{X}) \in \mathbf{MPST}$ . See again section 2.1 for details. Recall the statement of Theorem 2.1.3.

**Theorem 6.2.1.** *For  $\mathfrak{X}$  as above, the sheaf  $h_0^\square(\mathfrak{X})_{\mathrm{Nis}}$  is semi pure, i.e. that the natural map*

$$h_0^\square(\mathfrak{X})_{\mathrm{Nis}} \rightarrow \omega^* \omega_! h_0^\square(\mathfrak{X})_{\mathrm{Nis}}$$

*is an injective morphism of Nisnevich sheaves.*

This can be checked on stalks, so let  $(\overline{Y}, Y_\infty) \in \mathbf{MCor}$ , and let  $y \in \overline{Y}$  be a point. Let  $\mathcal{O}$  be the henselization of the local ring  $\mathcal{O}_{\overline{Y}, y}$ . Write  $L$  for the residue field of  $\mathcal{O}$ ,  $L = k(y)$ , and  $K$  for its field of fractions. Let  $\Pi \in \mathcal{O}$  be a local equation for  $Y_\infty$  at  $y$ . Theorem 2.1.3 is then equivalent to the following

**Theorem 6.2.2.** *For  $\mathfrak{X} \in \mathbf{MCor}^{\text{proj}}$ , the natural map*

$$j_{\mathcal{O}, (\Pi)}: h_0^{\square}(\mathfrak{X})(\mathcal{O}, (\Pi)) \rightarrow (\omega^* \omega_! h_0^{\square}(\mathfrak{X})_{\text{Nis}})(\mathcal{O}, (\Pi)) \hookrightarrow h_0^{\square}(\mathfrak{X})(K, \emptyset)$$

is injective for every pair  $(\mathcal{O}, (\Pi))$ , where  $k \subset \mathcal{O}$  is an (equicharacteristic) henselian local normal domain with field of fractions  $K$  and  $\Pi \in \mathcal{O}$  is a non zero element. Here, we are using the notation  $(\mathcal{O}, (\Pi))$  to denote the modulus pair  $(\text{Spec}(\mathcal{O}), (\Pi))$ , where  $(\Pi)$  is the effective Cartier divisor on  $\text{Spec}(\mathcal{O})$  defined by  $\Pi$ .

The above statement can be reformulated as follows. Using the definition of  $h_0^{\square}(\mathfrak{X})$ , we have a commutative diagram, with exact columns,

$$(6.3) \quad \begin{array}{ccc} \mathbf{MCor}((\mathcal{O}, (\Pi)) \otimes \overline{\square}, \mathfrak{X}) & \longrightarrow & \mathbf{MCor}(\text{Spec}(K) \otimes \overline{\square}, \mathfrak{X}) \\ \downarrow i_0^* - i_\infty^* & & \downarrow i_0^* - i_\infty^* \\ \mathbf{MCor}((\mathcal{O}, (\Pi)), \mathfrak{X}) & \longrightarrow & \mathcal{Z}_0(X_K) \\ \downarrow & & \downarrow \\ h_0^{\square}(\mathfrak{X})(\mathcal{O}, (\Pi)) & \longrightarrow & h_0^{\square}(\mathfrak{X})(K, \emptyset). \end{array}$$

Here, we have used the identification, which can be easily checked using the definition of admissible correspondences,

$$\mathbf{MCor}(\text{Spec}(K), \mathfrak{X}) = \mathcal{Z}_0(X_K),$$

where the latter denote the group of 0-cycles on  $X_K$ . The subgroup  $\mathbf{MCor}((\mathcal{O}, (\Pi)), \mathfrak{X})$  agrees with the group  $\mathbf{MCor}_{(\mathcal{O}, (\Pi))}((\mathcal{O}, (\Pi)), \mathfrak{X})$  of  $S$  admissible correspondences introduced in the previous paragraph, and is generated by the closed points in  $X_K$  satisfying the modulus condition over  $\mathcal{O}$ . To lighten the notation, we suppress the subscript  $(\mathcal{O}, (\Pi))$  in what follows.

The relationship between the group  $\text{CH}_0(\overline{X}_K | X_{\infty, K})$  and the presheaf  $h_0^{\square}(\mathfrak{X})$  for a modulus pair  $\mathfrak{X}$  is given by the following Proposition.

**Proposition 6.2.3.** *Let  $\mathfrak{X} = (\overline{X}, X_\infty)$  be a proper modulus pair over a field  $F$ , and let  $F \subset K$  be a field extension. Then*

$$h_0^{\square}(\mathfrak{X})(K, \emptyset) = \text{CH}_0(\overline{X}_K, X_{\infty, K})$$

where  $\overline{X}_K$  denotes the extension  $\overline{X} \times_F K$  and  $X_{\infty, K}$  the pullback of  $X_\infty$  to  $\overline{X}_K$ .

*Proof.* See [2, Section 3] for a comparison between the groups  $\text{CH}^r(\overline{Y} | Y_\infty, 0)$ , defined by means of the cubical cycle complex and the relative Chow groups  $\text{CH}^r(\overline{Y} | Y_\infty)$  defined in terms of divisors of rational functions. This, together with the definition of  $h_0^{\square}(\mathfrak{X})(K, \emptyset)$  immediately gives the proof.  $\square$

Theorem 6.2.2 is then implied by the injectivity of the natural map

$$(6.4) \quad j_{\mathcal{O}, (\Pi)}: h_0^{\square}(\mathfrak{X})(\mathcal{O}, (\Pi)) \rightarrow \text{CH}_0(\overline{X}_K, X_{\infty, K})$$

for every pair  $(\mathcal{O}, (\Pi))$  as above and every  $\mathfrak{X} \in \mathbf{MCor}^{\text{proj}}$ .

**6.3. Proof of the main theorem.** In view of the previous discussion, in order to prove the injectivity (6.4) we argue as follows. We keep the same notations as above.

Let  $\alpha \in \ker(j_{\mathcal{O},(\Pi)})$ . According to the Definition (see [13]),  $\alpha = 0$  in  $\mathrm{CH}_0(\overline{X}_K, X_{\infty,K})$  means that there exist integral curves  $C_1, \dots, C_n$  contained in  $\overline{X}_K$ , of finite type over  $K$ , and rational functions  $f_1, \dots, f_n$ , with  $f_i \in G(C_i^N, C_{i,\infty})$  such that

$$\alpha = \sum_{j=1}^N m_j [x_j] = \sum_{i=1}^n \mathrm{div}_{C_i}(f_i) = \sum_{i=1}^n \nu_{i,*} \mathrm{div}_{C_i^N}(f_i)$$

for closed points  $x_j \in \underline{\mathbf{M}}\mathbf{Cor}((\mathcal{O}, (\Pi)), \mathfrak{X}) \subset \mathcal{Z}_0(X_K)$ , i.e. satisfying the modulus condition over  $\mathcal{O}$ . Here  $\nu_i: C_i^N \rightarrow C_i \subset \overline{X}_K$  is the composition of the normalization map with the inclusion. We will consider  $\overline{X} \times_k S = \overline{X}_S$  as projective scheme over  $\mathcal{O}$ , equipped with an effective Cartier divisor  $(X_{\infty})_S = (X_{\infty}) \times_k S$ . Write  $\mathfrak{X}_S$  for the modulus pair over  $S$  given by  $\mathfrak{X}_S = (\overline{X}_S, X_{\infty,S})$ .

Write  $C$  for the union of the  $C_i$ 's and following the convention of Section 5, write  $\alpha = \mathrm{div}_C(f)$ , for  $f \in K(C)^\times = \prod_{i=1}^n K(C_i)^\times$  a meromorphic function restricting to  $f_i$  on the integral component  $C_i$ . Let  $C_S$  be the closure of  $C$  in  $\overline{X}_S$ . Note that  $C_S$  is the union of integral, 2-dimensional subschemes of  $\overline{X}_S$  such that

$$\mathrm{Ass}(C_S \times_S L) \cap |X_{\infty}| = \emptyset$$

where  $X_{\infty,L}$  is seen as effective divisor on the special fiber of  $\overline{X}_S$  (it is the base change  $\overline{X} \times_k L$ ). We divide the proof in the points 6.3.1 - 6.3.3 below.

6.3.1. *Step 1.* We first suppose that  $\dim(X) \geq 3$ . If  $\dim(X) \leq 2$ , pass directly to Step 3. Let  $\Omega = \{x \in (\overline{X}_K)_{(0)} \mid e_x(C) \geq 3\}$ . If  $\Omega = \emptyset$ , pass directly to Step 2. Otherwise, we proceed as follows. Let  $\Omega'$  be the set of singular points of  $C$ . Let  $\rho_S: \overline{X}'_S \rightarrow \overline{X}_S$  be a repeated blow up along the closure of closed points of lying over points of  $\Omega'$  such that the following conditions are satisfied:

- (1) Let  $C'_S$  be the strict transform of  $C_S$ , and let  $C'_K$  be its generic fiber. Then  $C'_K$  is a disjoint union of integral regular curves.
- (2) Let  $X'_{\infty,S}$  be the total transform of  $X_{\infty,S}$ . Then we have  $\mathrm{Ass}(C'_S \times_S L) \cap |X'_{\infty,S}| = \emptyset$ .

Note that the second condition is automatic. Write  $\mathfrak{X}'_S$  for the modulus pair  $(\overline{X}'_S, X'_{\infty,S})$ . We have an induced morphism

$$\rho: \mathfrak{X}'_S \rightarrow \mathfrak{X}_S.$$

Let  $\rho_K: \overline{X}'_K \rightarrow \overline{X}_K$  be the repeated blow up of  $\overline{X}_K$  at the closed points over  $\Omega'$ , and let  $X'_{\infty,K} = X'_{\infty,S} \times_S K$ . Let  $X'_K = \overline{X}'_K \setminus X'_{\infty,K}$ . This yields a commutative diagram

$$\begin{array}{ccc} \underline{\mathbf{M}}\mathbf{Cor}((\mathcal{O}, (\Pi)), \mathfrak{X}'_S) & \xrightarrow{\iota'} & \mathcal{Z}_0(X'_K) \\ \downarrow \rho_* & & \downarrow \rho_{K,*} \\ \underline{\mathbf{M}}\mathbf{Cor}((\mathcal{O}, (\Pi)), \mathfrak{X}_S) & \xrightarrow{\iota} & \mathcal{Z}_0(X_K) \end{array}$$

By our assumption,  $\alpha = \rho_{K,*} \mathrm{div}_{C'_K}(f') \in \mathrm{Im}(\iota)$ , where  $f'$  is the rational function on  $C'_K$  induced by  $f$  on  $C$ . Note that  $K(C)^\times = K(C'_K)^\times$ . A priori,  $\mathrm{div}_{C'_K}(f')$  does not belong to the image of  $\iota'$ , since there may be new cancellations. Namely, there may be distinct closed points  $x$  and  $x'$  in support of  $\mathrm{div}_{C'_K}(f')$  such that  $\rho_K(x) = \rho_K(x')$  and  $\rho_K(x') \notin |\alpha|$ .

To remedy this, we argue as in Section 5.2. First, let  $E_S$  be the exceptional divisor of the blow-up  $\rho_S$ . We can write  $E_S = E_{\infty,S} + E_{\Sigma,S}$ , where  $E_{\infty,S}$  is the exceptional divisor

lying over the closure of points over  $\Omega' \cap X_{\infty, K}$  (and  $E_{\Sigma, S}$  is the exceptional divisor on the complement set). Write  $E_{\Sigma, K} = E_{\Sigma, S} \times_S K$  and similarly  $E_{\infty, K} = E_{\infty, S} \times_S K$ . Next, notice that if  $x, x' \in \overline{X}'_K$  are cancellation points in the above sense, then  $x, x' \in E_{\Sigma, K}$  and not in  $E_{\infty, K}$ , since each rational function  $f_i$  on each component  $C'_{K, i}$  of  $C'_K$  by assumption satisfies the modulus condition. Indeed, we have a commutative diagram

$$\begin{array}{ccc} C_i^N & \xrightarrow{\cong} & C'_{K, i} \\ & \searrow \nu_i & \downarrow \\ & & C_i \end{array}$$

where the vertical map is induced by the blow-up, and the horizontal map is an isomorphism, since by construction  $C'_{K, i}$  is regular. Since  $f_i \in G(C_i^N, C_{i, \infty})$ , the support of  $\text{div}_{C'_K}(f')$  is disjoint from  $C'_K \cap X'_{\infty, K}$ , so a fortiori disjoint from  $E_{\infty, K}$ .

Let  $x \in \Omega \setminus X_{\infty, K}$ , and write  $y_1, \dots, y_r$  for the set  $C'_S \cap \rho_K^{-1}(x)$ . Let  $\gamma_x$  be the 0-cycle on  $X'_K$  given by

$$\gamma_x = \sum_{j=1}^r v_{C'_K}^j(f')[y_j]$$

where  $v_{C'_K}^j(f')$  is the order of  $f'$  at the point  $y_j$ . Note that we can assume that this number is well defined, since each  $y_j$  lies exactly in one component of  $C'_K$ , and  $C'_K$  is regular.

We can now choose a chain of rational lines  $L_\nu \cong \mathbb{P}_{K'}^1 \subset E_{\Sigma, K}$  in the exceptional divisor  $E_{\Sigma, K}$  and rational functions  $g_\nu \in K(L_\nu)^\times = K'(t)$  (for some finite field extension  $K'/K$ ) such that  $\gamma_x = \sum \text{div}_{L_\nu}(g_\nu)$ . By construction, each function  $g_\nu$  automatically satisfies the modulus condition with respect to  $X'_{\infty, K}$ , since  $L_\nu \cap X'_{\infty, K} = \emptyset$ , and

$$\rho_{K, *}( \gamma_x ) = \rho_{K, *}( \sum \text{div}_{L_\nu}(g_\nu) ) = 0.$$

Repeat the argument for every such  $x$ .

Write  $L_{S, \nu}$  for the closure of  $L_\nu$  in  $\overline{X}'_S$ , and replace  $C'_S$  by  $C''_S = C'_S \cup \bigcup_\nu L_{S, \nu}$ . On the generic fiber  $C''_K = C'_K \cup \bigcup_\nu L_\nu$  we can consider the cycle

$$\alpha' = \text{div}_{C'_K}(f) - \sum_x \gamma_x.$$

We now have that  $\alpha' \in \text{Im}(t')$  and its image in  $X_K$  via  $\rho_{K, *}$  agrees with  $\alpha$ . Note that even after adding the  $L_{S, \nu}$ , the total curve  $C''_S$  satisfies  $\text{Ass}(C''_S \times_S L) \cap |X'_{\infty, s}| = \emptyset$ , and for each closed point  $x \in C''_K$ , we have  $e_x(C''_K) \leq 2$ .

At the end of this operation, we can replace  $\overline{X}_S$  with  $\overline{X}'_S$  and  $C_S$  with  $C''_S$  and  $(f_1, \dots, f_n)$  with  $(f_1, \dots, f_n, (g_\nu)_\nu)$  and assume that  $\Omega = \emptyset$ .

**6.3.2. Step 2.** To perform this Step, we assume that for every closed point  $x \in C_K$  we have  $e_x(C_K) \leq 2$ . By an iterated application of Theorem 4.2.2, we can find a relative surface  $H$  over  $S$  (i.e.  $\dim_S(H) = 2$ ), containing  $C_S$  and satisfying the following properties

- (3)  $H$  is integral, surjective over  $S$ , and the generic fiber  $H_K$  is a smooth projective geometrically integral  $K$ -surface.
- (4)  $X_{\infty, S} \cdot H$  is an effective Cartier divisor on  $H$ , which restricts to an effective Cartier divisor on the special fiber  $H_s$ .

This follows directly from Theorem 4.2.2 if the residue field of  $S$  is infinite. If  $L$  is finite, thanks to Remark 4.2.5, the section  $H$  exists after extending the scalars to  $S' = \text{Spec}(\mathcal{O}')$  for the unramified extension  $\mathcal{O}'/\mathcal{O}$  of  $\mathcal{O}$  corresponding to a field extension  $L'/L$  of degree  $\ell^m$  for

some prime number  $\ell$  and some positive integer  $m$ . Since the injectivity of the natural map (6.4) can be checked after base change to  $S'$ , we can replace  $S$  by  $S'$  and assume that  $H$  is defined over  $S$ .

Let  $\mu: H \hookrightarrow \overline{X}_S$  be the inclusion of  $H$  in  $\overline{X}_S$ . Let  $\mathfrak{H} = (H, X_{\infty, S} \cdot H)$  as  $S$ -modulus pair, and let  $H_K^o$  be the open complement of the generic fiber of  $X_{\infty, S} \cdot H$ . We have a commutative diagram

$$\begin{array}{ccc} \underline{\mathbf{M}}\mathbf{Cor}((\mathcal{O}, (\Pi)), \mathfrak{H}) & \xrightarrow{\iota_H} & \mathcal{Z}_0(H_K^o) \\ \downarrow \mu_* & & \downarrow \mu_{K,*} \\ \underline{\mathbf{M}}\mathbf{Cor}((\mathcal{O}, (\Pi)), \mathfrak{X}_S) & \xrightarrow{\iota_X} & \mathcal{Z}_0(\overline{X}_S \times_S K) \end{array}$$

where the right vertical map is the push-forward of 0-cycles from  $H_K^o$  to  $\overline{X}_S \times_S K$ . It induces the left vertical map when restricted to the subgroup of cycles satisfying  $(\mathbf{MC})_{\mathcal{O}}$  thanks to another application of the containment Lemma ([16, 2.2]). The push-forward induces a map

$$\mu_*: h^{\overline{\square}}(\mathfrak{H})(\mathcal{O}, (\Pi)) \rightarrow h^{\overline{\square}}(\mathfrak{X}_S)(\mathcal{O}, (\Pi)).$$

By construction, the cycle  $\alpha$  is in the image of  $\mu_*$ . It is therefore enough to show that it is zero in  $h^{\overline{\square}}(\mathfrak{H})(\mathcal{O}, (\Pi))$  to conclude.

We can then replace  $\overline{X}_S$  by  $H$ , and assume that  $\dim_S(\overline{X}_S) = 2$ .

6.3.3. *Step 3.* At this point, we have the following set of data. We slightly change the notation in order to be more coherent with the one used in Section 5.

- i) An integral projective scheme  $\overline{X}_S = \mathcal{X}$ , of relative dimension 2 and surjective over  $S$ , with smooth, geometrically integral generic fiber  $X = \mathcal{X}_{\eta}$ .
- ii) An effective Cartier divisor  $\mathcal{D}$  on  $\mathcal{X}$ , restricting to an effective Cartier divisor  $\mathcal{D}_s$  on the special fiber  $\mathcal{X}_s$  (this is induced by the divisor  $X_{\infty, S}$  on the original  $\overline{X}_S$ ).
- iii) A zero cycle  $\alpha$  on the generic fiber  $X$ , which satisfy the modulus condition  $(\mathbf{MC})_{\mathcal{O}}$ . In particular, for each point  $x$  in the support of  $\alpha$ , the closure  $\overline{\{x\}}$  intersects  $\mathcal{D}$  in a finite set of closed points.
- iv) A closed subscheme  $\mathcal{C} \subset \mathcal{X}$ , with generic fiber  $Z = \mathcal{C}_{\eta}$  consisting of a finite union of integral curves,  $Z = Z_1 \cup \dots \cup Z_n$ , such that  $D = \mathcal{D}_{\eta}$  intersects  $Z$  properly. The divisor  $\mathcal{D}$  restricts to an effective Cartier divisor on  $\mathcal{C}$ , which is finite over  $S$ .
- v) For each  $i = 1, \dots, n$ , a rational function  $f_i \in K(Z_i)^{\times}$  which satisfy the modulus condition,  $f_i \in G(Z_i^N, \nu_i^* D)$ , where  $\nu_i: Z_i^N \rightarrow X$  the composition of the normalization map  $Z_i^N \rightarrow Z_i$  with the inclusion of  $Z_i$  in  $X$ , satisfying  $\alpha = \sum_{i=1}^n (\nu_i)_* \operatorname{div}_{Z_i^N}(f_i)$ .

We let  $\mathfrak{X} = (\mathcal{X}, \mathcal{D})$  be the corresponding proper modulus pair over  $S$ . To complete the proof it is enough to show that  $\alpha$  is 0 in  $h_0^{\overline{\square}}(\mathfrak{X})(\mathcal{O}, (\Pi))$ . The latter group is defined by means of (6.2).

We are now in the setting of Section 5. With an iterated application of the moving technique 5.4.1 (up to replacing  $\mathcal{X}$  with another blow-up along the closure of closed points of the generic fiber, and possibly after replacing  $S$  by an étale extension  $S'$ ), we can further suppose that for every  $i = 1, \dots, n$ , we have

- vi) Each  $Z_i$  is regular in a neighborhood of every closed point  $x \in Z_i \cap D$
- vii) Every  $x \in |(\nu_i)_* \operatorname{div}_{Z_i^N}(f_i)|$ , satisfies the modulus condition  $(\mathbf{MC})_{\mathcal{O}}$ .

It is then enough to show that each term  $(\nu_i)_* \operatorname{div}_{Z_i^N}(f_i)$  is zero in  $h_0^{\overline{\square}}(\mathfrak{X})(\mathcal{O}, (\Pi))$ .

Theorem 3.1.2 implies then that the closure  $\overline{\Gamma_{f_i}}$  in  $\mathbb{P}_K^1 \times Z_i^N$  of the graph of each  $f_i$ ,

$$\Gamma_{f_i} \in \mathbf{Cor}(\mathbb{P}_K^1 \setminus \{1\}, (Z_i^N \setminus \nu_i^* D))$$



actually lands in the subgroup

$$\mathbf{MCor}((\mathcal{O}, (\Pi)) \otimes \overline{\square}, (\mathcal{Z}_i^N, \nu_i^* \mathcal{D}))$$

of finite correspondences satisfying the admissibility condition (3.1). Here  $\mathcal{Z}_i$  is the model over  $S$  of  $Z_i$  (given by property iv) above) and  $\nu_i: \mathcal{Z}_i^N \rightarrow \mathcal{Z}_i \subset \mathcal{X}$  is the normalization map. We see  $(\mathcal{Z}_i^N, \nu_i^* \mathcal{D})$  as  $S$ -modulus pair in the sense of 6.1.

By definition, this implies that  $\Gamma_{f_i}$ , seen as finite correspondence in  $\mathbf{Cor}(\mathrm{Spec}(K) \otimes \overline{\square}, \mathfrak{X})$  via the map  $\nu_i$  is in the image of the natural map

$$\mathbf{MCor}((\mathcal{O}, (\Pi)) \otimes \overline{\square}, \mathfrak{X}) \subset \mathbf{MCor}(\mathrm{Spec}(K) \otimes \overline{\square}, \mathfrak{X}).$$

Finally, since  $i_0^* - i_\infty^*(\overline{\Gamma_{f_i}}) = \nu_{i,*}(\mathrm{div}_{Z_i^N}(f_i))$ , it follows from the commutative diagram (6.3) (applied to the pair  $\mathfrak{X}$  that we are considering here) that  $(\nu_i)_* \mathrm{div}_{Z_i^N}(f_i) = 0$  in the group  $h_0^{\overline{\square}}(\mathfrak{X})(\mathcal{O}, (\Pi))$ , completing the proof.

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