

ZERO CYCLES WITH MODULUS AND ZERO CYCLES ON SINGULAR VARIETIES

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ABSTRACT. Given a smooth variety X and an effective Cartier divisor $D \subset X$, we show that the cohomological Chow group of 0-cycles on the double of X along D has a canonical decomposition in terms of the Chow group of 0-cycles $\mathrm{CH}_0(X)$ and the Chow group of 0-cycles with modulus $\mathrm{CH}_0(X|D)$ on X . When X is projective, we construct an Albanese variety with modulus and show that this is the universal regular quotient of $\mathrm{CH}_0(X|D)$.

As a consequence of the above decomposition, we prove the Roitman torsion theorem for the 0-cycles with modulus. We show that $\mathrm{CH}_0(X|D)$ is torsion-free and there is an injective cycle class map $\mathrm{CH}_0(X|D) \hookrightarrow K_0(X, D)$ if X is affine. For a smooth affine surface X , this is strengthened to show that $K_0(X, D)$ is an extension of $\mathrm{CH}_1(X|D)$ by $\mathrm{CH}_0(X|D)$.

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1. INTRODUCTION

When X is a smooth quasi-projective scheme over a base field k , the motivic cohomology groups of X admit an explicit description in terms of groups of algebraic cycles, called higher Chow groups, first defined by Bloch [7]. These groups have all the properties that one expects, including Chern classes and a Chern character isomorphism from higher K -groups, as established in [32] and [11], generalizing the well-known relationship between the Chow ring of cycles modulo rational equivalence and the Grothendieck group of vector bundles.

Leaving the safe harbor of smooth varieties leads to a different world, where the picture is substantially less clear. One of the simplest examples of singular varieties is the nilpotent thickening $X_m = X \times_k k[t]/(t^m)$ of a smooth scheme X . For such a scheme, the beautiful correspondence between motivic cohomology, algebraic cycles and K -groups is destroyed, since

one has

$$H_{\mathcal{M}}^*(X, \mathbb{Q}(*)) = H_{\mathcal{M}}^*(X_m, \mathbb{Q}(*))$$

according to the currently available definitions, preventing the existence of a Grothendieck-Riemann-Roch-type formula relating the motivic cohomology groups of X_m with its higher K -groups

With the aim of understanding the algebraic K -theory of the ring $k[t]/(t^2)$ in terms of algebraic cycles, Bloch and Esnault first conceived the idea of algebraic cycles “with modulus” – called additive Chow groups at the time – defined by imposing suitable congruence condition at infinity on admissible cycles. This idea subsequently became the starting point of the discovery of the theory of additive cycle complexes and additive higher Chow groups of schemes in the works of Rülling [40], Park [37] and Krishna-Levine [25].

The additive higher Chow groups are conjectured to give a cycle-theoretic interpretation of the relative K -groups $K_*(X \times_k \mathbb{A}^1, X_m)$ for a smooth scheme X . In recent works of Binda-Saito [5] and Kerz-Saito [20], the construction of the additive higher Chow groups was generalized to develop a theory of higher Chow groups with modulus. These groups, denoted $\mathrm{CH}^*(X|D, *)$, are designed to study the arithmetic and geometric properties of a smooth variety X with fixed conditions along an effective (possibly non-reduced) Cartier divisor D on it, and are supposed to give a cycle-theoretic description of the mysterious relative K -groups $K_*(X, D)$, defined as the homotopy groups of the homotopy fibre of the restriction map $K(X) \rightarrow K(D)$. On the arithmetic side, when X is a smooth variety over a finite field, Kerz and Saito studied the group $\mathrm{CH}_0(\overline{X}|D)$ for \overline{X} an integral compactification of X and D a non-reduced closed subscheme supported on $\overline{X} \setminus D$ (see [20] and 5.1 for the definition), and this has proven to be a fundamental ingredient in the study of wildly ramified class field theory.

Although recently established results by various authors (see [18], [41]) have indicated that the Chow groups with modulus (and, more generally, the relative motivic cohomology groups of [5]) have some of the above expected properties, many questions remain widely open.

In order to provide new evidence that the Chow groups with modulus are the right motivic cohomology groups to compute the relative K -theory of a smooth scheme with respect to an effective divisor, one would like to know if these groups share enough of the known structural properties of the Chow groups without modulus, and to relate them to some geometric or cohomological invariants of the pair (X, D) . This is the subject of this paper. Our interest is to establish these properties and present (an almost complete) picture for the Chow groups of 0-cycles with modulus.

We now state our main results. The precise statement of each of these results and the underlying hypothesis and notations will be explained at appropriate places in this text.

1.1. Albanese variety and Roitman torsion theorem with modulus. One of the most important things known about the ordinary Chow group of 0-cycles of a smooth projective variety is that it admits a universal abelian variety quotient (the Albanese variety) which is useful for studying the question of the representability of the Chow group. The celebrated theorem of Roitman [39] (see also [35] for the case of positive characteristic) says that this quotient map is isomorphism on torsion. This theorem has had profound consequences in the study of the Chow group of 0-cycles. One of the main goals of this paper is to establish these results (under some restrictions in positive characteristic) for the Chow group of 0-cycles with modulus.

Theorem 1.1 (see Theorems 10.3 and 11.3). *Let X be a smooth projective scheme over an algebraically closed field k and let $D \subset X$ be an effective Cartier divisor. Then there is a smooth connected algebraic group $\mathrm{Alb}(X|D)$ and a group homomorphism $\rho_{X|D}: \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \rightarrow \mathrm{Alb}(X|D)$ which is a universal regular quotient of $\mathrm{CH}_0(X|D)_{\mathrm{deg} 0}$.*

Theorem 1.2 (see Theorem 10.4 and 11.5). *Let X be a smooth projective scheme over an algebraically closed field k and let $D \subset X$ be an effective Cartier divisor. Let $n \in \mathbb{N}$ be an integer prime to the characteristic of k . Then $\rho_{X|D}$ induces an isomorphism $\rho_{X|D}: {}_n\mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \xrightarrow{\cong} {}_n\mathrm{Alb}(X|D)$ on the n -torsion subgroups.*

Note that when $k = \mathbb{C}$ and $D_{\mathrm{red}} \subset X$ is a normal crossing divisor, Theorem 1.1 has been proven, using a completely different approach, in [5].

1.2. Bloch's conjecture for 0-cycles with modulus. Let X be a smooth projective surface over \mathbb{C} . Recall that the well-known Bloch conjecture predicts that the Abel-Jacobi map $\rho_X: \mathrm{CH}_0(X)_{\mathrm{deg} 0} \rightarrow \mathrm{Alb}(X)$ is an isomorphism if $H^2(X, \mathcal{O}_X) = 0$. Assuming this, we can show that the analogous statement for the Chow group with modulus also holds. In particular, the Bloch conjecture for 0-cycles with modulus is true if X is not of general type. Remarkably, instead of the vanishing of the second cohomology group of the structure sheaf \mathcal{O}_X , we have to assume the vanishing of the second cohomology group of the ideal sheaf \mathcal{I}_D of D .

Theorem 1.3 (see Theorem 10.8). *Let X be a smooth projective surface over \mathbb{C} and let $D \subset X$ be an effective Cartier divisor. Let \mathcal{I}_D denote the sheaf of ideals defining D . Assume that the Bloch conjecture is true for X . Then the map $\rho_{X|D}: \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \rightarrow \mathrm{Alb}(X|D)$ is an isomorphism if $H^2(X, \mathcal{I}_D) = 0$.*

1.3. Torsion theorem for 0-cycles with modulus on affine schemes. Assume now that X is a smooth affine variety over an algebraically closed field k . One of the consequences of Roitman's theorem is that the Chow group of 0-cycles on X has no torsion, and this itself has had many applications to projective modules on smooth affine varieties. Here comes the extension of this statement to the 0-cycles with modulus.

Theorem 1.4 (see Theorem 14.1). *Let X be a smooth affine scheme of dimension $d \geq 2$ over an algebraically closed field k and let $D \subset X$ be an effective Cartier divisor. Then $\mathrm{CH}_0(X|D)$ is torsion-free.*

In the presence of a modulus, however, the classical argument to deduce Theorem 1.4 from Roitman's Theorem does not go through. For example, the localization sequence for the ordinary Chow groups, which is one of the steps of the proof of the classical case, fails in the modulus setting, as explained in [23, Theorem 1.5]. Our approach is to deduce Theorem 1.4 directly from Theorem 1.9 below.

1.4. Cycle class map to relative K -theory. In the direction of understanding the relation between 0-cycles with modulus and relative K -theory, we have the following results.

Theorem 1.5 (see Theorem 12.4). *Let X be a smooth quasi-projective scheme of dimension $d \geq 1$ over a perfect field k and let $D \subset X$ be an effective Cartier divisor. Then, there is a cycle class map*

$$\mathrm{cyc}_{X|D}: \mathrm{CH}_0(X|D) \rightarrow K_0(X, D).$$

This map is injective if k is algebraically closed and X is affine.

When X has dimension 2, we can prove the following stronger statement which completely describes $K_0(X, D)$ in terms of the Chow groups with modulus.

Theorem 1.6 (see Theorem 14.6 and 14.7). *Let X be a smooth affine surface over an algebraically closed field k and let $D \subset X$ be an effective Cartier divisor. Then, the canonical map $\mathrm{CH}_0(X|D) \rightarrow \mathrm{CH}_0(X|D_{\mathrm{red}})$ is an isomorphism and there is an exact sequence*

$$0 \rightarrow \mathrm{CH}_0(X|D) \rightarrow K_0(X, D) \rightarrow \mathrm{CH}_1(X|D) \rightarrow 0.$$

Finally, for arbitrary quasi-projective surfaces, we prove the following structural result, that we may see as an integral version of a Riemann-Roch-type formula for the relative K_0 -group of the pair (X, D) .

Theorem 1.7 (see Theorem 13.4). *Let X be a smooth quasi-projective surface over an algebraically closed field k and let $D \subset X$ be an effective Cartier divisor. Then, there is a cycle class map $\text{cyc}_{X|D}: \text{CH}_0(X|D) \rightarrow K_0(X, D)$ and a short exact sequence*

$$0 \rightarrow \text{CH}_0(X|D) \rightarrow K_0(X, D) \rightarrow \text{Pic}(X, D) \rightarrow 0.$$

1.5. Bloch's formula. As an application of Theorem 1.7, we get the following Bloch's formula for cycles with modulus on surfaces.

Theorem 1.8. (*Bloch's formula*) *Let X be a smooth quasi-projective surface over an algebraically closed field k . Let $D \subset X$ be an effective Cartier divisor. Then, there are isomorphisms*

$$\text{CH}_0(X|D) \xrightarrow{\cong} H_{\text{zar}}^2(X, \mathcal{K}_{2,(X,D)}^M) \xrightarrow{\cong} H_{\text{nis}}^2(X, \mathcal{K}_{2,(X,D)}^M).$$

1.6. The decomposition theorem. Essentially no case of the above results was previously known, and in order to prove them, we develop a new approach to study the Chow groups with modulus by drawing inspiration from the world of cycles on singular varieties. Given a smooth scheme X with an effective Cartier divisor D , we consider the notion of ‘doubling’ X along D . This idea has previously been used by Milnor [36] to study the patching of projective modules over commutative rings (see [36, Chap. 2]), and also by Levine [32] to study algebraic cycles in a different context. Doubling X along D gives rise to a new scheme, that we denote by $S(X, D)$, which is, in general, highly singular.

The novelty of our approach is the observation that the Chow group of 0-cycles with modulus $\text{CH}_0(X|D)$ can (under some conditions) be suitably realized as a direct summand of the cohomological Chow group of 0-cycles on $S(X, D)$ in the sense of Levine-Weibel [33]. This allows us to transport many of the known statements about the Chow groups of 0-cycles on (possibly singular) schemes to 0-cycles with modulus. The following decomposition theorem can therefore be called the central result of this paper (see Theorem 7.1 for a precise statement).

Theorem 1.9. *Let X be a smooth quasi-projective scheme over a perfect field k . Let $D \subset X$ be an effective Cartier divisor. Then, there is a split short exact sequence*

$$0 \rightarrow \text{CH}_0(X|D) \rightarrow \text{CH}_0(S(X, D)) \rightarrow \text{CH}_0(X) \rightarrow 0.$$

In fact, it turns out that this approach can be taken forward to study the Chow groups with modulus $\text{CH}_*(X|D)$ in any dimension using the theory of Chow groups of singular schemes developed by Levine [29]. This generalization will be studied in a different project. In this paper, we shall show how this approach works for the relative Picard groups, apart from the above result for 0-cycles.

We conclude this Introduction by remarking that an Albanese variety with modulus has been previously constructed by Kato and Russell in [19] and [42]. Their construction uses different techniques and starts from a definition of the Chow group of 0-cycles with modulus that does not agree with the one proposed by Kerz and Saito: as a consequence of this discrepancy, our construction and the Kato-Russell construction are not directly related.

1.7. Outline of the proofs. This paper is organized as follows. Our principal task is to prove the decomposition Theorem 1.9 for the Chow group of 0-cycles. The proof of this takes up the next five sections of this paper. We describe the double construction in §2 and we prove several properties of it that are used throughout the paper.

The proof of Theorem 1.9 requires a non-trivial Bertini-type argument which allows us to give a new description of the Cartier curves in the definition of the Chow group of 0-cycles on the double. We do this first for surfaces and we then explain how to reduce the general case to this one. This is done in §5. To relate the 0-cycles with modulus with the group of 0-cycles on the double, we define a variant of the Levine-Weibel Chow group of 0-cycles on the double and then show that the two definitions agree in as many cases as possible (see Theorem 3.17). This is done in §3.

We construct the Albanese variety with modulus attached to the pair (X, D) in §8 which turns out to be a commutative algebraic group of general type. In characteristic zero, we give an explicit construction of the Albanese variety with modulus using a relative version of Levine's modified Deligne-Beilinson cohomology. We then use Theorem 1.9 to prove the universality of this Albanese and also the Roitman torsion theorem. We use Theorem 1.9 and the main results of [22] to deduce the Bloch conjecture for 0-cycles with modulus in §8. Other applications to affine schemes are obtained in §12, § 13 and § 14.

NOTATIONS

Let k be a field. Since our arguments are geometric in nature, all schemes in this text are assumed to be quasi-projective over k and we shall let \mathbf{Sch}_k denote this category. Let \mathbf{Sm}_k denote the full subcategory of \mathbf{Sch}_k consisting of smooth schemes over k . We shall let $\mathbf{Sch}_k^{\text{ess}}$ denote the category of schemes which are essentially of finite type over k . For a closed subscheme $Z \subset X$, we shall denote the support of Z by $|Z|$. For a scheme X , the notation X_{sing} will mean the singular locus of the associated reduced scheme X_{red} . The nature of the field k will be specified in each section of this paper.

2. THE DOUBLE CONSTRUCTION

The doubling of a scheme along a closed subscheme is the building block of the proofs of our main results of this paper. In this section, we define this double construction and study its many properties. These properties play crucial roles in the later parts of this paper.

2.1. The definition of the double. Recall that given surjective ring homomorphisms $f_i: A_i \rightarrow A$ for $i = 1, 2$, the subring $R = \{(a_1, a_2) \in A_1 \times A_2 \mid f_1(a_1) = f_2(a_2)\}$ of $A_1 \times A_2$ has the property that the diagram

$$(2.1) \quad \begin{array}{ccc} R & \xrightarrow{p_1} & A_1 \\ p_2 \downarrow & & \downarrow f_1 \\ A_2 & \xrightarrow{f_2} & A \end{array}$$

is a Cartesian square in the category of commutative unital rings, where $p_i: R \rightarrow A_i$ is the composite map $R \hookrightarrow A_1 \times A_2 \rightarrow A_i$ for $i = 1, 2$. Using the fact that every morphism $X \rightarrow Y$ in \mathbf{Sch}_k , with Y affine, factors through $X \rightarrow \text{Spec}(\mathcal{O}(X)) \rightarrow Y$, one can easily check that the diagram

$$(2.2) \quad \begin{array}{ccc} \text{Spec}(A) & \xrightarrow{f_1} & \text{Spec}(A_1) \\ f_2 \downarrow & & \downarrow p_1 \\ \text{Spec}(A_2) & \xrightarrow{p_2} & \text{Spec}(R) \end{array}$$

is a Cartesian and co-Cartesian square in \mathbf{Sch}_k .

Let us now assume that $X \in \mathbf{Sch}_k$ and let $\iota: D \hookrightarrow X$ be a closed subscheme. If $f_1 = f_2 = \iota$, we see that the construction of (2.1) is canonical and so it glues (see [15, Ex. II.2.12]) to give us the push-out scheme $S(X, D)$ and a commutative diagram

$$(2.3) \quad \begin{array}{ccc} D & \xrightarrow{\iota} & X \\ \downarrow \iota & & \downarrow \iota_+ \\ X & \xrightarrow{\iota_-} & S(X, D) \end{array} \begin{array}{c} \searrow \text{id} \\ \Delta \\ \searrow \text{id} \end{array} \begin{array}{c} \\ \\ \searrow \text{id} \end{array} \begin{array}{c} \\ \\ X \end{array}$$

One can in fact check, by restricting to affine parts of X and then by using the gluing construction, that the top square in (2.3) is co-Cartesian in \mathbf{Sch}_k . It is also a Cartesian square. The scheme $S(X, D)$ constructed above will be called the *double* of X along D . We shall mostly write $S(X, D)$ in short as S_X if the closed subscheme $D \subset X$ is fixed and remains unchanged in a given context.

Notice that there is a canonical map $\pi: X \amalg X \xrightarrow{(\iota_+, \iota_-)} S(X, D)$ which is an isomorphism over $S(X, D) \setminus D$. Given a map $\nu: C \rightarrow S(X, D)$, we let $C_+ = C \times_{S(X, D)} X_+$, $C_- = C \times_{S(X, D)} X_-$ and $E = C \times_{S(X, D)} D$. Here, X_{\pm} is the component of $X \amalg X$ where π restricts to ι_{\pm} . We then have

$$(2.4) \quad E = C \times_{S(X, D)} D = C_+ \times_X D = C_- \times_X D.$$

More generally, we may often consider the following variant of the double construction.

Definition 2.1. Let $j: D \hookrightarrow X$ be a closed immersion of quasi-projective schemes over k and let $f: T \rightarrow X$ be a morphism of quasi-projective schemes. We shall say that T is a join of T_+ and T_- along D , if there is a push-out diagram

$$(2.5) \quad \begin{array}{ccc} f^*(D) & \xrightarrow{j_+} & T_+ \\ j_- \downarrow & & \downarrow \iota_+ \\ T_- & \xrightarrow{\iota_-} & T \end{array}$$

such that T_{\pm} are quasi-projective schemes and j_{\pm} are closed immersions.

The following lemma related to the double construction will be often used in this text.

Lemma 2.2. *Let $\nu: C \rightarrow S(X, D)$ be an affine morphism. Then the push-out $C_+ \amalg_E C_-$ is a closed subscheme of C . This closed immersion is an isomorphism if C is reduced.*

Proof. There is clearly a morphism $C_+ \amalg_E C_- \rightarrow C$. Showing that this map has the desired properties is a local question on X . So it suffices to verify these properties at the level of rings.

If we set $X = \text{Spec}(A)$, $S(X, D) = \text{Spec}(R)$ and let I be the defining ideal for D , then we have an exact sequence of R -modules

$$(2.6) \quad 0 \rightarrow R \xrightarrow{\phi} A \times A \rightarrow A/I \rightarrow 0.$$

Since ν is affine, we can write $C = \text{Spec}(B)$. Let $J \subset B$ be the ideal defining the closed subscheme E . Tensoring (2.6) with B , we get an exact sequence

$$(2.7) \quad B \xrightarrow{\phi_B} B_+ \times B_- \rightarrow B/J \rightarrow 0$$

and this shows that $C_+ \amalg_E C_- \hookrightarrow C$ is a closed immersion of schemes. It is also clear that this inclusion is an isomorphism in the complement of E . Furthermore, the surjectivity of the map

$C_+ \amalg C_- \rightarrow C$ shows that the above inclusion is also surjective on points. We conclude from this that the closed immersion $C_+ \amalg_E C_- \hookrightarrow C$ induces identity on the underlying reduced schemes. In particular, it is an isomorphism if C is reduced. Equivalently, ϕ_B is injective. \square

2.2. More properties of the double. We now prove some general properties of the double construction that will be used repeatedly in this text. We shall also show that the double shares many of the nice properties of the given scheme if the underlying closed subscheme is an effective Cartier divisor. This will be our case of interest in the sequel.

For $X \in \mathbf{Sch}_k$, let $k(X)$ denote the sheaf of rings of total quotients of X . For a reduced scheme X , let $k_{\min}(X)$ denote the product of the fields of fractions of the irreducible components of X . Note that there are maps of sheaves of rings $\mathcal{O}_X \hookrightarrow k(X) \rightarrow k_{\min}(X)$ and the latter map is an isomorphism if X is reduced and has no embedded primes.

Proposition 2.3. *Let X be a scheme in $\mathbf{Sch}_k^{\text{ess}}$ and let $\iota: D \hookrightarrow X$ be a closed subscheme not containing any irreducible component of X . Then the following hold.*

- (1) *There are finite maps*

$$X \amalg X \xrightarrow{\pi} S(X, D) \xrightarrow{\Delta} X$$

such that $(X \setminus D) \amalg (X \setminus D) \xrightarrow{\pi} S(X, D) \setminus D = \Delta^{-1}(X \setminus D)$ is an isomorphism. In particular, S is affine (projective) if and only if X is so.

- (2) *$S(X, D)$ is reduced if X is so. In this case, one has*

$$k_{\min}(S(X, D) \setminus D) \simeq k_{\min}(X \setminus D) \times k_{\min}(X \setminus D).$$

If D contains no component of X , then $k_{\min}(S(X, D) \setminus D) = k_{\min}(S(X, D))$.

- (3) *The composite map $D \xrightarrow{\iota \pm \circ \iota} \Delta^*(D) \xrightarrow{\Delta} D$ is identity and $|\Delta^*(D)| = |D|$.*
(4) *If $Y \subseteq X$ is a closed (resp. open) subscheme of X and $Y \cap D$ is the scheme-theoretic intersection, then $S(Y, Y \cap D)$ is a closed (resp. open) subscheme of $S(X, D)$. There is an inclusion of subschemes $S(Y, Y \cap D) \hookrightarrow \Delta^*(Y)$ which is an isomorphism if Y is open.*
(5) *Let Y be a subscheme of X . Then $|\Delta^*(Y) \cap D| = |Y \cap D| = |S(Y, Y \cap D) \cap D|$.*
(6) *$S(X, D)_{\text{sing}} = D \cup \Delta^{-1}(X_{\text{sing}})$. In particular, $S(X, D)_{\text{sing}} = D$ if X is non-singular.*
(7) *If $f: Y \rightarrow X$ is a flat morphism, then $S(Y, f^*(D)) \simeq S(X, D) \times_X Y$. In particular, the map $S(f): S(Y, f^*(D)) \rightarrow S(X, D)$ is flat (resp. smooth) if f is so.*
(8) *π is the normalization map and D is a conducting subscheme, if X is normal.*

Proof. To prove the proposition, we can assume that $X = \text{Spec}(A)$ is affine. Let $p_1, p_2: A \times A \rightarrow A$ denote the projections and let $q: A \rightarrow A/I$ be the quotient map. Set $q_i = q \circ p_i$. Set $\psi_i = p_i \circ \phi$ for $i = 1, 2$. Let $\delta: A \hookrightarrow R \hookrightarrow A \times A$ denote the diagonal map. We then have $\psi_i \circ \delta = id_A$ for $i = 1, 2$ and this yields

$$A \times A = \phi \circ \delta(A) \oplus \text{Ker}(p_2);$$

$$(2.8) \quad R = \delta(A) \oplus \text{Ker}(\psi_2) = \delta(A) \oplus I \times \{0\} \simeq A \oplus I.$$

Since $A \times A$ is a finite free A -module and R is an A -submodule, it follows that R is a finite A -module. This proves (1). The item (2) follows immediately from (2.6).

The ideal of D inside $S(X, D)$ is $\text{Ker}(R \xrightarrow{q \circ \psi_i} A/I)$, which is $I \times I$. Since $\delta^*(I) \subseteq I \times I$, we see that $D \subseteq \Delta^*(D)$ and the composite $D \hookrightarrow \Delta^*(D) \xrightarrow{\Delta} D$ is clearly identity. Furthermore, it is clear that $R[(a, b)^{-1}] = A[a^{-1}] \times A[b^{-1}]$ and $\delta^*(I)[(a, b)^{-1}] = R[(a, b)^{-1}]$, whenever $a, b \in I \setminus \{0\}$. Hence, we have $|\Delta^*(D)| = |D|$. This proves (3).

To prove (4), we only need to consider the closed part. Let $A' = A/J$, where J is the ideal defining Y and let $R' = \{(a', b') \in A' \times A' \mid a' - b' \in (I + J)/J\}$. Let \bar{a} denote the residue class

of $a \in A$ modulo J . Suppose there exist $a, b \in A$ such that $\bar{a} - \bar{b} \in (I + J)/J$. This means $a - b \in I + J$ and so we can write $a - b = \alpha + \beta$, where $\alpha \in I$ and $\beta \in J$. We set $a' = a - \beta$ and $b' = b$. This yields $a' - b' = a - b - \beta = \alpha \in I$ and $a' - a = \beta \in J$, $b' - b = 0 \in J$. We conclude that $(a', b') \in R$ and it maps to $(\bar{a}, \bar{b}) \in R'$. Hence $R \twoheadrightarrow R'$.

An element of $\delta^*(J)$ is of the form $(a\alpha, b\alpha)$, where $a, b \in A, \alpha \in J$ and $a - b \in I$. This element clearly dies in R' . Hence $S(Y, Y \cap D) \subseteq \Delta^*(Y)$.

To prove (5), let $S_Y = S(Y, Y \cap D)$. Then

$$|\Delta^*(Y) \cap D| = |\Delta^*(Y) \cap \Delta^*(D)| = |\Delta^*(Y \cap D)|,$$

where the first equality follows from (3). On the other hand, we have

$$|S_Y \cap D| = |\iota_1^Y \circ \iota^Y(Y \cap D)| = |\Delta_Y^*(Y \cap D)| = |\Delta^*(Y \cap D)|,$$

where the second equality follows from (3) with X replaced by Y . The item (5) now follows. The item (6) follows from (1) and the fact that more than one components of $S(X, D)$ meet along D .

To prove (7), let $Y = \text{Spec}(B)$ and tensor (2.6) with B . The flatness of B over A yields the short exact sequence

$$0 \rightarrow R \otimes_A B \xrightarrow{\phi} B \times B \rightarrow B/IB \rightarrow 0$$

and this proves the first part of (7). The second part follows because a base change of a flat (resp. smooth) map is flat (resp. smooth).

The item (8) follows because π is finite and birational and the ideal of $\pi^*(D)$ in $X \amalg X$ is $I_D \times I_D$ which is actually contained in $\mathcal{O}_{S(X, D)}$. So D is a conducting subscheme. \square

2.3. Double along a Cartier divisor. Recall that a morphism $f: X \rightarrow S$ of schemes is called a *local complete intersection* (l.c.i.) at a point $x \in X$ if it is of finite type and if there is an open neighborhood U of x and a factorization

$$\begin{array}{ccc} & & Z \\ & \nearrow i & \downarrow g \\ U & \xrightarrow{f} & S, \end{array}$$

where i is a regular closed immersion and g is a smooth morphism. We say that f is a local complete intersection morphism if it is so at every point of X . We say that f is l.c.i. along a closed subscheme $S' \hookrightarrow S$ if it is l.c.i. at every point in $f^{-1}(S')$.

Proposition 2.4. *Continuing with the notations of Proposition 2.3, assume further that D is an effective Cartier divisor on X . Then the following hold.*

- (1) Δ is finite, flat and $\mathcal{O}_{S(X, D)}$ is a locally free \mathcal{O}_X -module of rank two via Δ .
- (2) $S(X, D)$ is Cohen-Macaulay if X is so.
- (3) If $f: Y \rightarrow X$ is any morphism, then there is a closed immersion of schemes $S(Y, f^*(D)) \hookrightarrow S(X, D) \times_X Y$. This embedding is an isomorphism if f is transverse to $D \hookrightarrow X$.
- (4) If $f: Y \rightarrow X$ is any morphism such that Y is Cohen-Macaulay and $f^*(D)$ does not contain any irreducible component of Y , then the embedding $S(Y, f^*(D)) \hookrightarrow S(X, D) \times_X Y$ is an isomorphism. In this case, $f^*(D)$ is an effective Cartier divisor on Y .
- (5) If $f: Y \rightarrow X$ is l.c.i. along D , then $Y \times_X S(X, D) \rightarrow S(X, D)$ is l.c.i. along D .

Proof. We can again assume that $X = \text{Spec}(A)$ is affine such that $I = (a)$ is a principal ideal such that $a \in A$ is not a zero-divisor. It follows then that I is a free A -module of rank one.

We can now apply (2.8) to conclude (1) as the finiteness of R over A is already shown in Proposition 2.3.

To prove (2), let $\mathfrak{m} \subsetneq R$ be a maximal ideal and let $\mathfrak{n} = \delta^{-1}(\mathfrak{m})$. Then $A_{\mathfrak{n}} \rightarrow R_{\mathfrak{n}}$ is a finite and flat map and hence $A_{\mathfrak{n}} \rightarrow R_{\mathfrak{m}}$ is a faithfully flat local homomorphism of noetherian local rings of same dimension. Since $A_{\mathfrak{n}}$ is Cohen-Macaulay and since this local homomorphism preserves regular sequences, it follows that $\text{depth}(R_{\mathfrak{m}}) \geq \dim(A_{\mathfrak{n}}) = \dim(R_{\mathfrak{m}})$. Hence, $R_{\mathfrak{m}}$ is Cohen-Macaulay.

To prove (3), we let $Y = \text{Spec}(B)$ and tensor (2.6) (over A) with B to get an exact sequence

$$0 \rightarrow \text{Tor}_A^1(A/I, B) \rightarrow R \otimes_A B \xrightarrow{\phi} B \times B \rightarrow B/IB \rightarrow 0$$

and $S(B, IB)$ is (by definition) the kernel of the map $B \times B \rightarrow B/IB$. In particular, we get a surjective map of rings $R \otimes_A B \twoheadrightarrow S(B, IB)$. This proves the first part of (3). The transversality

of B with A/I means precisely that $\text{Tor}_A^1(A/I, B) = 0$ and we get that $R \otimes_A B \xrightarrow{\cong} S(B, IB)$.

This proves (3).

Suppose next that Y is Cohen-Macaulay and no irreducible component of Y is contained in $f^*(D)$. It suffices to show in this case that $\text{Tor}_A^1(A/I, B) = 0$. Let $f: A \rightarrow B$ be the map on the coordinate rings. That no component of Y is contained in $f^*(D)$ means that $f(a)$ does not belong to any minimal prime of B . The Cohen-Macaulay property of B implies that it has no embedded associated prime. In particular, $f(a)$ does not belong to any associated prime and hence is not a zero-divisor in B .

We have a short exact sequence

$$0 \rightarrow A \xrightarrow{a} A \rightarrow A/I \rightarrow 0$$

which says that $\text{Tor}_A^1(A/I, B) = \text{Ker}(B \xrightarrow{f(a)} B)$ and we have just shown that the latter group is zero. We have also shown above that $f(a)$ is not a zero-divisor on B and this implies that $f^*(D)$ is an effective Cartier divisor on Y . This proves (4). The item (5) follows from (1) and an elementary fact that l.c.i. morphisms are preserved under a flat base change. \square

3. CHOW GROUP OF 0-CYCLES ON SINGULAR SCHEMES

In this section, we give a definition of the Chow group of 0-cycles on singular schemes that modifies slightly the one given in [33]. While using the same set of generators, we change the geometric condition imposed on the curves giving the rational equivalence. In many cases, we are able to show that this new definition coincides with the classical one. It turns out that the modified Chow group of 0-cycles has better functorial properties and is more suitable for proving Theorem 1.9.

3.1. Some properties of l.c.i and perfect morphisms. Recall that a finite type morphism $f: X \rightarrow S$ of noetherian schemes is called *perfect* if the local ring $\mathcal{O}_{X,x}$ has finite Tor-dimension as a module over the local ring $\mathcal{O}_{S,f(x)}$ for every point $x \in X$. Equivalently, given any point $x \in X$, there are affine neighborhoods U of x and V of $f(x)$ such that $\mathcal{O}(U)$ is an $\mathcal{O}(V)$ -module of finite Tor-dimension. Recall also the following

Proposition 3.1 (Proposition 5.12, [47]). *Let $f: X \rightarrow S$ be a proper and perfect morphism of noetherian schemes. Then there is a well defined push-forward map $K_0(X) \xrightarrow{f_*} K_0(S)$ between the Grothendieck groups of vector bundles.*

Some known elementary properties of l.c.i and perfect morphisms are recalled in the following lemmas.

Lemma 3.2. (1) *The l.c.i. and perfect morphisms are preserved under flat base change.*

- (2) A flat morphism of finite type is perfect.
- (3) An l.c.i. morphism is perfect.
- (4) l.c.i. and perfect morphisms are closed under composition.
- (5) l.c.i. and perfect morphisms satisfy faithfully flat (fpqc) descent.

Lemma 3.3. *Let $f: X \rightarrow S$ be a finite type morphism of noetherian schemes such that for every $x \in f^{-1}(S_{\text{sing}})$, the map f is l.c.i. at x . Then f is perfect.*

Proof. It follows from the definition of a perfect morphism because if $x \in X$ is such that $s = f(x)$ is a regular point of S , then $\mathcal{O}_{X,x}$ has finite Tor-dimension over $\mathcal{O}_{S,s}$. This property for the points over the singular locus of S follows from the hypothesis of the lemma. \square

3.2. Divisor classes for singular curves. We fix a field k . For X an equidimensional quasi-projective k -scheme and $Y \subsetneq X$ a closed subscheme of X not containing any component of X , write $\mathcal{Z}_0(X, Y)$ for the free abelian group on the closed points of X not in Y .

A curve C will be in what follows a quasi-projective k -scheme of pure dimension 1. We let $k(C)$ denote the ring of total quotients of C . Let $\{\eta_1, \dots, \eta_r\}$ denote the set of generic points of C with closures $\{C_1, \dots, C_r\}$. Let T be a set of closed points of C containing C_{sing} and $Z = T \cup \{\eta_1, \dots, \eta_r\}$. Write $\mathcal{O}_{C,Z}$ for the semi-local ring on the points of T . This yields a sequence of maps

$$(3.1) \quad \mathcal{O}_{C,Z}^\times \hookrightarrow k(C)^\times \rightarrow \prod_{i=1}^r k(C_i)^\times.$$

We let $\theta_{(C,Z)}$ denote the composite map. Letting $k(C, Z)^\times = \mathcal{O}_{C,Z}^\times$, the localization sequence in K -theory yields a natural map

$$(3.2) \quad \partial_{C,Z} : k(C, Z)^\times \rightarrow \prod_{p \in C \setminus Z} G_0(p) = \mathcal{Z}_0(C, Z).$$

If C is a *reduced* curve, it is a Cohen-Macaulay scheme and hence the second map in (3.1) is an isomorphism. Thus the group $\mathcal{O}_{C,Z}^\times$ is the subgroup of $k(C)^\times$ consisting of those f which are regular and invertible in the local rings $\mathcal{O}_{C,x}$ for every $x \in Z$. In this case, the boundary $\partial_{C,Z}(f)$ has a familiar expression: if we let $\theta_{(C,Z)}(f) = \{f_i\}$, then $\text{div}(f) = \sum_i \text{div}(f_i)$, where $\text{div}(f_i)$ is the divisor of the rational function f_i on the integral curve C_i . If C is not reduced, $\partial_{C,Z}$ has a more complicated expression which we do not use in this text.

3.3. A Chow group of 0-cycles on singular schemes. Let X be an equidimensional reduced quasi-projective scheme over k of dimension $d \geq 1$. Let X_{sing} and X_{reg} denote the singular and regular loci of X , respectively. Let $Y \subsetneq X$ be a closed subset containing X_{sing} , but not containing any component of X . Write again $\mathcal{Z}_0(X, Y)$ for the free abelian group on closed points of $X \setminus Y$. We shall often write $\mathcal{Z}_0(X, X_{\text{sing}})$ as $\mathcal{Z}_0(X)$.

Let $f: X' \rightarrow X$ be a proper morphism from another reduced equidimensional scheme over k . Let $Y' \subsetneq X'$ be a closed subset not containing any component of X' such that $f^{-1}(Y) \cup X'_{\text{sing}} \subseteq Y'$. Then there is a push-forward map

$$(3.3) \quad f_* : \mathcal{Z}_0(X', Y') \rightarrow \mathcal{Z}_0(X, Y).$$

This is defined on a closed point $x' \in X' \setminus Y'$ with $f(x') = x$ by $f_*([x']) = [k(x') : k(x)] \cdot [x]$.

Definition 3.4. Let C be a reduced curve in \mathbf{Sch}_k and let $\nu: C \rightarrow X$ be a finite morphism. We shall say that $\nu: (C, Z) \rightarrow (X, Y)$ is a *good curve relative to (X, Y)* if there exists a closed proper subscheme $Z \subsetneq C$ such that the following hold.

- (1) No component of C is contained in Z .
- (2) $\nu^{-1}(Y) \cup C_{\text{sing}} \subseteq Z$.
- (3) ν is locally complete intersection morphism at every point $x \in C$ such that $\nu(x) \in Y$.

Given any good curve (C, Z) relative to (X, Y) , we have a pushforward map as in (3.3)

$$\mathcal{Z}_0(C, Z) \xrightarrow{\nu_*} \mathcal{Z}_0(X, Y).$$

We shall write $\mathcal{R}_0(C, Z, X)$ for the subgroup of $\mathcal{Z}_0(X, Y)$ generated by the set $\{\nu_*(\operatorname{div}(f)) \mid f \in \mathcal{O}_{C,Z}^\times\}$, where $\operatorname{div}(f)$ for a rational function $f \in \mathcal{O}_{C,Z}^\times$ is defined as in (3.2) for reduced curves. Let $\mathcal{R}_0(X, Y)$ denote the subgroup of $\mathcal{Z}_0(X, Y)$ which is the image of the map

$$(3.4) \quad \bigoplus_{\nu: (C,Z) \rightarrow (X,Y) \text{ good}} \mathcal{R}_0(C, Z, X) \rightarrow \mathcal{Z}_0(X, Y).$$

We define the *Chow group of 0-cycles on X relative to Y* to be the quotient

$$(3.5) \quad \operatorname{CH}_0(X, Y) = \frac{\mathcal{Z}_0(X, Y)}{\mathcal{R}_0(X, Y)}.$$

We write $\operatorname{CH}_0(X, X_{\text{sing}})$ as $\operatorname{CH}_0(X)$ for short and call it the *Chow group of 0-cycles on X* .

The following result shows that we can always assume that the morphisms $\nu: C \rightarrow X$ are l.c.i in the definition of our rational equivalence.

Lemma 3.5. *Let (X, Y) be as above. Given any good curve $\nu: (C, Z) \rightarrow (X, Y)$ relative to (X, Y) and any $f \in \mathcal{O}_{C,Z}^\times$, there exists a good curve $\nu': (C', Z') \rightarrow (X, Y)$ relative to (X, Y) and $f' \in \mathcal{O}_{C',Z'}^\times$ such that the following hold.*

- (1) $\nu_*(\operatorname{div}(f)) = \nu'_*(\operatorname{div}(f'))$.
- (2) $\nu': C' \rightarrow X$ is an l.c.i. morphism.

Proof. Let $U_1 \subseteq C$ be an open subset of C containing $S_1 = \nu^{-1}(X_{\text{sing}})$ such that $(C_{\text{sing}} \setminus S_1) \cap U_1 = \emptyset$. This is possible because S_1 is a finite set. Let $\pi: (C \setminus S_1)^N \rightarrow C \setminus S_1$ denote the normalization map. It follows that $\pi: \pi^{-1}(U_1 \setminus S_1) \rightarrow U_1 \setminus S_1$ is an isomorphism. Setting $U_2 = (C \setminus S_1)^N$, we see that U_1 and U_2 glue along $\pi^{-1}(U_1 \setminus S_1)$ to give a unique scheme C' and a unique map $p: C' \rightarrow C$. This scheme has the property that p is finite, $p^{-1}(U_1) \rightarrow U_1$ is an isomorphism and $p^{-1}(C \setminus S_1) = (C \setminus S_1)^N$.

Setting $Z' = p^{-1}(Z)$ and $f' = p^*(f) \in k(C')^\times$, we see that $f' \in \mathcal{O}_{C',Z'}^\times$ and $\operatorname{div}(f) = p_*(\operatorname{div}(f'))$. If we let $\nu' = \nu \circ p$, we get $\nu_*(\operatorname{div}(f)) = \nu'_*(\operatorname{div}(f'))$. Furthermore, $\nu'^{-1}(X_{\text{reg}}) \rightarrow X_{\text{reg}}$ is a finite type morphism of regular schemes and hence is an l.c.i. morphism. Since ν is l.c.i. over X_{sing} and p is an isomorphism in a neighborhood of $\nu^{-1}(X_{\text{sing}})$, we conclude that ν' is an l.c.i. morphism. \square

3.4. The Levine-Weibel Chow group. We now recall the Levine-Weibel (cohomological) Chow group of 0-cycles for singular schemes as defined in [33, Definition 1.2]. Let X be an equidimensional quasi-projective scheme of dimension $d \geq 1$ over k , $X \supseteq Y \supseteq X_{\text{sing}}$ a closed subscheme not containing any component of X .

Definition 3.6. A *Cartier curve on X relative to Y* is a purely 1-dimensional closed subscheme $C \hookrightarrow X$ that has no component contained in Y and is defined by a regular sequence in X at each point of $C \cap Y$.

One example of Cartier curves we shall encounter in this text is given by the following.

Lemma 3.7. *Let X be a connected smooth quasi-projective scheme over k and let $D \subset X$ be an effective Cartier divisor. Let $\nu: C \hookrightarrow X$ be an integral curve which is not contained in D . Assume that C is l.c.i. along D . Let $\Delta: S(X, D) \rightarrow X$ denote the double construction. Then $S(C, \nu^*D)$ is a Cartier curve on $S(X, D)$ relative to D .*

Proof. We write $S(X, D)$ and $S(C, \nu^*(D))$ as S_X and S_C , respectively, in this proof. Since the inclusion $\nu: C \hookrightarrow X$ is l.c.i. along D , it follows Proposition 2.4 that the square

$$(3.6) \quad \begin{array}{ccc} S_C & \xrightarrow{S_\nu} & S_X \\ \Delta_C \downarrow & & \downarrow \Delta_X \\ C & \xrightarrow{\nu} & X \end{array}$$

is Cartesian. It also follows from Proposition 2.4(5) that $S_\nu : S_C \hookrightarrow S_X$ is l.c.i. along D . Moreover, a combination of Proposition 2.4(4) and Proposition 2.3(2) tells us that S_C is reduced with two components, both isomorphic to C . We conclude that $S_C \hookrightarrow S_X$ is a (reduced) Cartier curve relative to D . \square

Given a Cartier curve $\iota : C \hookrightarrow X$ relative to Y , we let $\mathcal{R}_0^{LW}(C, Y, X)$ denote the image of the composite map $k(C, C \cap Y)^\times \xrightarrow{\partial_{C, C \cap Y}} \mathcal{Z}_0(C, C \cap Y) \xrightarrow{\iota_*} \mathcal{Z}_0(X, Y)$. We let $\mathcal{R}_0^{LW}(X, Y)$ denote the subgroup of $\mathcal{Z}_0(X, Y)$ generated by $\mathcal{R}_0^{LW}(C, Z, X)$, where $C \subset X$ runs through all Cartier curves relative to Y .

Definition 3.8. The *Levine-Weibel Chow group of 0-cycles* of X relative to Y is defined as the quotient

$$\mathrm{CH}_0^{LW}(X, Y) = \mathcal{Z}_0(X, Y) / \mathcal{R}_0^{LW}(X, Y).$$

The group $\mathrm{CH}_0^{LW}(X, X_{\mathrm{sing}})$ is often denoted by $\mathrm{CH}_0^{LW}(X)$.

We recall here the following important moving Lemma, due to Levine, that simplifies the set of relations in case X satisfies additional assumptions.

Proposition 3.9 (See Lemma 1.4 [30] and Lemma 2.1 [6]). *Let X be an equidimensional quasi-projective k -scheme and let $X_{\mathrm{sing}} \subset Y \subsetneq X$ be a closed subset of X as above. Assume that X is reduced. Then the subgroup $\mathcal{R}_0^{LW}(X, Y)$ of $\mathcal{Z}_0(X, Y)$ agrees with the subgroup $\mathcal{R}_0^{LW}(X, Y)_{\mathrm{red}}$, generated by divisors of rational functions on reduced Cartier curves on X relative to Y . If X is moreover irreducible, then the Cartier curves generating the rational equivalence can be chosen to be irreducible as well.*

Lemma 3.10. *Let X be a reduced quasi-projective k -scheme. Then there is a canonical surjection*

$$(3.7) \quad \mathrm{CH}_0^{LW}(X, Y) \twoheadrightarrow \mathrm{CH}_0(X, Y).$$

Proof. The map (3.7) is induced by the identity on the set of generators, so we just have to show that it is well defined. Since X is reduced, by Proposition 3.9, we can assume that the Cartier curves defining the rational equivalence on the Levine-Weibel Chow group are reduced. Now, we just note that a reduced Cartier curve is a good curve relative to (X, Y) . \square

Lemma 3.11. *Let X be a reduced quasi-projective scheme over k and let $Y \subsetneq X$ be a closed subset containing X_{sing} and containing no components of X . Let (C, Z) be a good curve relative to (X, Y) . Then there are cycle class maps $\mathrm{cyc}_C : \mathcal{Z}_0(C, Z) \rightarrow K_0(C)$ and $\mathrm{cyc}_X : \mathcal{Z}_0(X, Y) \rightarrow K_0(X)$ making the diagram*

$$(3.8) \quad \begin{array}{ccc} \mathcal{Z}_0(C, Z) & \xrightarrow{\mathrm{cyc}_C} & K_0(C) \\ \nu_* \downarrow & & \downarrow \nu_* \\ \mathcal{Z}_0(X, Y) & \xrightarrow{\mathrm{cyc}_X} & K_0(X) \end{array}$$

commutative.

Proof. Since $\nu^{-1}(Y) \cup X'_{\text{sing}} \subseteq Y'$, we have a push-forward map $\nu_*: \mathcal{Z}_0(C, Z) \rightarrow \mathcal{Z}_0(X, Y)$, given by $\nu_*([x]) = [k(x) : k(\nu(x))] \cdot [\nu(x)]$. Since ν is l.c.i. along X_{sing} , it follows from Lemma 3.3 that the map $\nu: C \rightarrow X$ is perfect. Hence, there is a push-forward map on K_0 -groups $\nu_*: K_0(C) \rightarrow K_0(X)$ by Proposition 3.1.

To construct the cycle class maps and show that the square commutes, let $x \in C \setminus Z$ be a closed point and set $y = \nu(x)$. Let $\iota_x: \text{Spec}(k(x)) \rightarrow C$ and $\iota_y: \text{Spec}(k(y)) \rightarrow X$ be the closed immersions. Since these maps as well as ν are perfect (see Lemma 3.3), we have the induced push-forward maps on Grothendieck groups of vector bundles and a commutative diagram by Proposition 3.1:

$$(3.9) \quad \begin{array}{ccc} \mathbb{Z} = K_0(k(x)) & \xrightarrow{\iota_{x*}} & K_0(C) \\ \nu_* \downarrow & & \downarrow \nu_* \\ \mathbb{Z} = K_0(k(y)) & \xrightarrow{\iota_{y*}} & K_0(X). \end{array}$$

Setting $\text{cyc}_C([x])$ to be $\iota_{x*}(1)$, we get the cycle class maps $\text{cyc}_C: \mathcal{Z}_0(C) \rightarrow K_0(C)$ and $\text{cyc}_X: \mathcal{Z}_0(X) \rightarrow K_0(X)$ such that (3.8) commutes. \square

Lemma 3.12. *Suppose that X is reduced and purely 1-dimensional. Then there is a canonical isomorphism $\text{CH}_0(X, Y) \simeq \text{CH}_0^{LW}(X, Y) \simeq \text{Pic}(X)$.*

Proof. Let $\nu: C \rightarrow X$ be a finite map from a reduced curve and let $Z \subsetneq C$ be a closed subset such that (C, Z) is good relative to (X, Y) . By Lemma 3.11, there is a commutative diagram:

$$(3.10) \quad \begin{array}{ccc} \mathcal{Z}_0(C, Z) & \xrightarrow{\text{cyc}_C} & K_0(C) \\ \nu_* \downarrow & & \downarrow \nu_* \\ \mathcal{Z}_0(X, Y) & \xrightarrow{\text{cyc}_X} & K_0(X). \end{array}$$

Let $f \in \mathcal{O}_{C, Z}^\times$. It follows from [33, Proposition 2.1] that $\text{cyc}_C(\text{div}(f)) = 0$. In particular, we get $\text{cyc}_X \circ \nu_*(\text{div}(f)) = \nu_* \circ \text{cyc}_C(\text{div}(f)) = 0$. It follows again from [33, Proposition 1.4] that $\nu_*(\text{div}(f)) = 0$ in $\text{CH}_0^{LW}(X, Y) \simeq \text{Pic}(X) \hookrightarrow K_0(X)$. We have thus shown that the surjective map $\text{CH}_0^{LW}(X) \twoheadrightarrow \text{CH}_0(X)$ is also injective, hence an isomorphism. \square

Lemma 3.13. *Let X be a reduced quasi-projective scheme of dimension $d \geq 1$ over k and let $Y \subsetneq X$ be a closed subset containing X_{sing} and containing no components of X . Then the cycle class map $\text{cyc}_X: \mathcal{Z}_0(X, Y) \rightarrow K_0(X)$ given by Lemma 3.11 descends to group homomorphisms*

$$\text{cyc}_X: \text{CH}_0(X, Y) \rightarrow K_0(X); \quad \text{cyc}_X^{LW}: \text{CH}_0^{LW}(X, Y) \rightarrow K_0(X)$$

making the diagram

$$(3.11) \quad \begin{array}{ccc} \text{CH}_0^{LW}(X, Y) & \xrightarrow{\text{can}} & \text{CH}_0(X, Y) \\ & \searrow \text{cyc}_X^{LW} & \swarrow \text{cyc}_X \\ & & K_0(X) \end{array}$$

commutative.

Proof. The fact that cyc_X yields a cycle class map $\text{cyc}_X^{LW}: \text{CH}_0^{LW}(X, Y) \rightarrow K_0(X)$ is proved in [33, Proposition 2.1]. To show that cyc_X descends to a map on our modified version of the Chow group, let $\nu: (C, Z) \rightarrow (X, Y)$ be a good curve relative to (X, Y) and let $f \in \mathcal{O}_{C, Z}^\times$. We then have $\text{cyc}_X \circ \nu_*(\text{div}(f)) = \nu_* \circ \text{cyc}_C(\text{div}(f))$ by Lemma 3.11. On the other hand, it follows from Lemma 3.12 that $\text{cyc}_C(\text{div}(f)) = 0$. This shows that cyc_X is defined on the Chow groups. The commutativity of (3.11) is clear from the definitions. \square

3.5. Comparison of two Chow groups in higher dimension. In this section, we prove a comparison theorem for the two Chow groups in higher dimension. More comparison results in positive characteristic will be given in Theorems 7.3 and 9.8.

Suppose that the field k is algebraically closed and let $d = \dim(X)$. Write $F^d K_0(X)$ for the subgroup of $K_0(X)$ generated by the cycle classes of smooth, closed points in X . In [29, Corollary 5.4] (see also [30, Corollary 2.7]), Levine showed the existence of a top Chern class $c_d: F^d K_0(X) \rightarrow \mathrm{CH}_0^{LW}(X)$ such that $c_d \circ \mathrm{cyc}_X^{LW}$ is multiplication by $(d-1)!$. In particular, the kernel of cyc_X^{LW} is torsion. An immediate consequence of Lemma 3.13 is then the following.

Corollary 3.14. *Let X be a reduced quasi-projective scheme over an algebraically closed field k . Then the canonical map $\mathrm{CH}_0^{LW}(X)_{\mathbb{Q}} \rightarrow \mathrm{CH}_0(X)_{\mathbb{Q}}$ is an isomorphism.*

In order to integrally compare the two Chow groups in dimension ≥ 2 , we use the following.

Proposition 3.15. *Let k be an algebraically closed field of characteristic zero and let X be a reduced projective scheme of dimension $d \geq 1$ over k . Then cyc_X^{LW} is injective.*

Proof. Let $\alpha \in \mathrm{CH}_0^{LW}(X)$ be such that $\mathrm{cyc}_X^{LW}(\alpha) = 0$. By Levine's theorem recalled above, we know that α is a torsion class in $\mathrm{CH}_0^{LW}(X)$.

To show that $\alpha = 0$, we can use the Lefschetz principle argument and rigidity of the Chow group of zero-cycles over algebraically closed fields and assume that $k = \mathbb{C}$. Let $\mathbb{H}_{\mathcal{D}^*}^{2d}(X, \mathbb{Z}(d))$ denote the modified Deligne-Beilinson cohomology of X defined in [31, § 2] (see also Section 8 below). There is then a short exact sequence

$$0 \rightarrow A^d(X) \rightarrow \mathbb{H}_{\mathcal{D}^*}^{2d}(X, \mathbb{Z}(d)) \rightarrow H^{2d}(X_{\mathrm{an}}, \mathbb{Z}(d)) \rightarrow 0$$

and it was shown in [31, § 2] that there is a Chern class map $c_{\mathcal{D}^*, X}^d: K_0(X) \rightarrow \mathbb{H}_{\mathcal{D}^*}^{2d}(X, \mathbb{Z}(d))$ which induces an Abel-Jacobi map $\mathrm{AK}_X^d: \mathrm{CH}_0^{LW}(X)_{\mathrm{deg} 0} \rightarrow A^d(X)$ given by $\mathrm{AK}_X^d = c_{\mathcal{D}^*, X}^d \circ \mathrm{cyc}_X^{LW}$, where $\mathrm{CH}_0^{LW}(X)_{\mathrm{deg} 0} := \mathrm{Ker}(\mathrm{CH}_0^{LW}(X) \rightarrow H^{2d}(X_{\mathrm{an}}, \mathbb{Z}(d)))$.

Since $H^{2d}(X_{\mathrm{an}}, \mathbb{Z}(d))$ is torsion-free and α is torsion, it follows that $\alpha \in \mathrm{CH}_0^{LW}(X)_{\mathrm{deg} 0}$. In particular, it is a torsion class in $\mathrm{CH}_0^{LW}(X)_{\mathrm{deg} 0}$. A cycle class map

$$\widetilde{\mathrm{AK}}_X^d: \mathrm{CH}_0^{LW}(X)_{\mathrm{deg} 0} \rightarrow A^d(X)$$

is also constructed in [9, § 2] and it is shown in [22, Lemma 2.2] that $\widetilde{\mathrm{AK}}_X^d = \mathrm{AK}_X^d$ up to a sign. Now, $\mathrm{cyc}_X^{LW}(\alpha) = 0$ implies that $c_{\mathcal{D}^*, X}^d \circ \mathrm{cyc}_X^{LW}(\alpha) = 0$ in $A^d(X)$. We conclude that α is a torsion class in $\mathrm{CH}_0^{LW}(X)_{\mathrm{deg} 0}$ such that $\widetilde{\mathrm{AK}}_X^d(\alpha) = 0$. We now apply [6, Theorem 1.1] to conclude that $\alpha = 0$. This finishes the proof. \square

Remark 3.16. Let X be a projective variety over an algebraically closed field k of exponential characteristic $p \geq 1$ and let $Y \subsetneq X$ be a closed subset of X containing X_{sing} and containing no components of X . When $\mathrm{codim}_X(Y) \geq 2$, the map $\mathrm{cyc}_X^{LW}: \mathrm{CH}_0^{LW}(X) \rightarrow F^d K_0(X)$ is an isomorphism modulo p -torsion by [30, Theorem 3.2]. In particular, this shows directly that for such (X, Y) the canonical map $\mathrm{CH}_0^{LW}(X, Y) \rightarrow \mathrm{CH}_0(X, Y)$ is an isomorphism up to p -torsion. Since in this text we are interested in studying cycles on a double $(S(X, D), D)$, that is not regular in codimension 1, we can't invoke directly Levine's result even in the projective case over a field of characteristic 0, and we need the detour of Proposition 3.15.

We can now deduce our final result comparing the two Chow groups as follows.

Theorem 3.17. *Let X be a reduced quasi-projective scheme of dimension $d \geq 1$ over an algebraically closed field k . Then the canonical map $\mathrm{CH}_0^{LW}(X) \rightarrow \mathrm{CH}_0(X)$ is an isomorphism in the following cases.*

- (1) $d \leq 2$.

- (2) X is affine.
- (3) $\text{char}(k) = 0$ and X is projective.

Proof. In each case, it suffices to show using Lemma 3.13 that cyc_X^{LW} is injective. The case $d \leq 1$ follows from Lemma 3.12. The $d = 2$ case follows from [28, Theorem 7], where it is shown that the map $\text{CH}_0^{LW}(X) \rightarrow F^2K_0(X)$ is an isomorphism. If X is affine, this follows from [24, Corollary 7.3] and [30, Corollary 2.7]. The last case follows from Proposition 3.15. \square

3.6. Some functorial properties of the Chow group of 0-cycles. Recall that any proper map $\phi: X' \rightarrow X$ admits a push-forward map on the Chow groups of 0-cycles when X is smooth. This can not be true if X is singular. But we expect such a push-forward to exist in the singular case provided f is an l.c.i. morphism. Our next goal is to prove this in special cases. We shall use this result later in this text.

Proposition 3.18. *Let X, Y be again as in Lemma 3.13. Let $p: X' \rightarrow X$ be a proper morphism which is l.c.i. over X_{sing} such that X' is reduced. Let $Y' \subsetneq X'$ be a closed subset containing $p^{-1}(Y) \cup X'_{\text{sing}}$ and not containing any component of X' . Then there are push-forward maps $p_*: \text{CH}_0(X', Y') \rightarrow \text{CH}_0(X, Y)$ and $p_*: K_0(X') \rightarrow K_0(X)$ and a commutative diagram*

$$(3.12) \quad \begin{array}{ccc} \text{CH}_0(X', Y') & \xrightarrow{\text{cyc}_{X'}} & K_0(X') \\ p_* \downarrow & & \downarrow p_* \\ \text{CH}_0(X, Y) & \xrightarrow{\text{cyc}_X} & K_0(X). \end{array}$$

Proof. It follows from our assumption and Lemma 3.3 that p is perfect and hence there is a push-forward map $p_*: K_0(X') \rightarrow K_0(X)$. We have seen before that there is also a push-forward map $p_*: \mathcal{Z}_0(X', Y') \rightarrow \mathcal{Z}_0(X, Y)$.

Let us now consider a good curve $\nu': (C, Z) \rightarrow (X', Y')$ relative to (X', Y') . It follows from our assumption that $\nu = p \circ \nu': (C, Z) \rightarrow (X, Y)$ is a good curve relative to (X, Y) . We have the push-forward maps $\mathcal{Z}_0(C, Z) \xrightarrow{\nu'_*} \mathcal{Z}_0(X', Y') \xrightarrow{p_*} \mathcal{Z}_0(X, Y)$ such that $\nu_* = p_* \circ \nu'_*$. This shows that $p_*(\nu'_*(\text{div}(f))) = \nu_*(\text{div}(f))$ for any $f \in \mathcal{O}_{C, Z}^\times$. This implies that p_* descends to a push-forward map on the Chow groups. The commutativity of (3.12) is shown exactly as in the proof of Lemma 3.11. \square

Combining this with Theorem 3.17, we have a similar result for the Levine-Weibel Chow group of 0-cycles. Note that this type of functoriality was not previously known.

Proposition 3.19. *Let X be as in Theorem 3.17. Let $p: X' \rightarrow X$ be a proper morphism between reduced quasi-projective schemes over k . Let Y denote the singular locus of X and let $Y' \subset X'$ be a closed subscheme containing $p^{-1}(Y) \cup X'_{\text{sing}}$ and not containing any component of X' . Assume that p is l.c.i. along Y . Then there is a push-forward map $p_*: \text{CH}_0^{LW}(X', Y') \rightarrow \text{CH}_0^{LW}(X, Y)$.*

Proof. Proposition 3.18 says that there are maps

$$\text{CH}_0^{LW}(X', Y') \rightarrow \text{CH}_0(X', Y') \xrightarrow{p_*} \text{CH}_0(X, Y) \leftarrow \text{CH}_0^{LW}(X, Y)$$

and Theorem 3.17 says that the last map is an isomorphism. The result follows. \square

3.7. Cycles in good position. Let X be a smooth quasi-projective scheme of pure dimension d over a field k . For any closed subset $W \subsetneq X$, let $\mathcal{Z}_0(X, W)$ denote the free abelian group on the set of closed points in $X \setminus W$. Let $\mathcal{R}_0(X)_W$ denote the subgroup of $\mathcal{Z}_0(X, W)$ generated by cycles of the form $\nu_* \text{div}(f)$, where f is a rational function on an integral curve $\nu: C \hookrightarrow X$ such

that $C \not\subset W$ and $f \in \mathcal{O}_{C, C \cap W}^\times$. We denote by $\mathrm{CH}_0(X)_W$ the quotient $\mathcal{Z}_0(X, W)/\mathcal{R}_0(X)_W$. It is the group of 0-cycles that are *in good position* with respect to W (i.e., 0-cycles missing W). We have a canonical map $\mathrm{CH}_0(X)_W \rightarrow \mathrm{CH}_0(X)$. The following result is a consequence of Bloch's moving lemma.

Lemma 3.20. *Let X be a smooth quasi-projective scheme over k . Let $W \subset X$ be a proper closed subscheme of X . Then the map $\mathrm{CH}_0(X)_W \rightarrow \mathrm{CH}_0(X)$ is an isomorphism.*

Proof. We can assume that X is connected. Let $\mathcal{Z}_1(X)_W$ denote the free abelian group on integral curves in $X \times \mathbb{P}_k^1$ which have the following properties.

- (1) $C \cap (X \times \{0, \infty\})$ is finite.
- (2) $C \cap (W \times \mathbb{P}^1)$ is finite.
- (3) $C \cap (W \times \{0, \infty\}) = \emptyset$.
- (4) $C \neq \{x\} \times \mathbb{P}^1$ for any $x \in X$.

Then, the moving lemma of Bloch [7] says that the inclusion of chain complexes

$$(\mathcal{Z}_1(X)_W \xrightarrow{\partial_\infty - \partial_0} \mathcal{Z}_0(X, W)) \rightarrow (\mathcal{Z}_1(X) \xrightarrow{\partial_\infty - \partial_0} \mathcal{Z}_0(X))$$

induces isomorphism on H^0 . In particular, we get exact sequence

$$\mathcal{Z}_1(X)_W \xrightarrow{\partial_\infty - \partial_0} \mathcal{Z}_0(X, W) \rightarrow \mathrm{CH}_0(X) \rightarrow 0.$$

On the other hand, there is an isomorphism $(\partial_\infty - \partial_0)(\mathcal{Z}_1(X)_W) \rightarrow \mathcal{R}_0(X)_W$ given by $\partial([C]) \mapsto \mathrm{div}(N(f))$, where f is the projection map $C \rightarrow \mathbb{P}_k^1$, C' is its image in X and $N(f)$ is the norm of f under the finite map $k(C') \hookrightarrow k(C)$. One checks easily that $N(f) \in \mathcal{O}_{C', C' \cap Y}^\times$. \square

4. THE PULL-BACK MAPS Δ^* AND ι_\pm^*

Let k be a field. Let X be a smooth and connected quasi-projective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. Our goal in this section is to define pull-back maps $\Delta^*: \mathrm{CH}_0(X) \rightarrow \mathrm{CH}_0(S(X, D))$ and $\iota_\pm^*: \mathrm{CH}_0(S(X, D)) \rightarrow \mathrm{CH}_0(X)$. As before, we shall write $S(X, D)$ in short as S_X as long as the divisor D is understood. We shall denote the closed subschemes $\iota_\pm(X)$ of S_X by X_\pm , each of which is a copy of X .

4.1. The map Δ^* . We define the map $\Delta^*: \mathcal{Z}_0(X, D) \rightarrow \mathcal{Z}_0(S_X, D)$ by letting $\Delta^*([x])$ be the 0-cycle on S_X associated to the closed subscheme $\{x\} \times_X S_X$. It follows from Proposition 2.3(1) that $\Delta^*([x]) = [x_+] + [x_-]$, where x_\pm is the point x in $X_\pm \setminus D$. Note also that $D = (S_X)_{\mathrm{sing}}$ by Proposition 2.3(6) since X is non-singular. We show that Δ^* preserves rational equivalences.

Theorem 4.1. *Let X be a smooth quasi-projective scheme of dimension $d \geq 1$ over k and let $D \subsetneq X$ be an effective Cartier divisor. Then $\Delta^*: \mathcal{Z}_0(X, D) \rightarrow \mathcal{Z}_0(S_X, D)$ induces a map*

$$\Delta^*: \mathrm{CH}_0(X) \rightarrow \mathrm{CH}_0(S_X).$$

Proof. In view of Lemma 3.20, we need to show that $\Delta^*((f)_C) \in \mathcal{R}_0(S_X, D)$ for $C \hookrightarrow X$ an integral curve not contained in D and $f \in \mathcal{O}_{C, C \cap D}^\times$. Let $\nu: C^N \rightarrow X$ denote the induced map from the normalization of C and let $E = \nu^*(D)$. We then have $f \in \mathcal{O}_{C^N, E}^\times$ and $(f)_C = \nu_*(\mathrm{div}(f))$. We can thus assume that C is normal and allow the possibility that ν need not be a closed immersion. With this reduction, we now need to show that $\Delta^* \circ \nu_*(\mathrm{div}(f)) \in \mathcal{R}_0(S_X, D)$.

Since Δ is flat, it follows from (3.6) that there is a commutative square (see [12, Proposition 1.7])

$$(4.1) \quad \begin{array}{ccc} \mathcal{Z}_0(C, E) & \xrightarrow{\nu_*} & \mathcal{Z}_0(X, D) \\ \Delta_C^* \downarrow & & \downarrow \Delta_X^* \\ \mathcal{Z}_0(S_C, E) & \xrightarrow{S_{\nu_*}} & \mathcal{Z}_0(S_X, D). \end{array}$$

where the bottom horizontal arrow is well defined since $S_\nu^{-1}(D) \subseteq E$. Note that by Proposition 2.4, the map $S_\nu: S_C \rightarrow S_X$ is l.c.i along D . We also have a commutative square of monomorphisms

$$(4.2) \quad \begin{array}{ccc} \mathcal{O}_{C,E}^\times & \xrightarrow{\iota_C} & k(C)^\times \\ \Delta_C^* \downarrow & & \downarrow \Delta_C^* \\ \mathcal{O}_{S_C,E}^\times & \xrightarrow{\iota_{S_C}} & k(C)^\times \times k(C)^\times. \end{array}$$

Setting $g = \Delta_C^*(f) \in \mathcal{O}_{S_C,E}^\times$, it is then clear that $\iota_{S_C}(g) = (f, f) \in k(C)^\times \times k(C)^\times = k(S_C)^\times$. This yields

$$\operatorname{div}(\Delta_C^*(f)) = \operatorname{div}(f, f) = \operatorname{div}(f) + \operatorname{div}(f) = \Delta_C^*(\operatorname{div}(f)).$$

Combining this with (4.1), we get

$$\Delta_X^* \circ \nu_*(\operatorname{div}(f)) = S_{\nu_*} \circ \Delta_C^*(\operatorname{div}(f)) = S_{\nu_*}(\operatorname{div}(\Delta_C^*(f))) = S_{\nu_*}(\operatorname{div}(g)).$$

Since (S_C, E) is clearly a good curve relative to (S_X, D) , the last term lies in $\mathcal{R}_0(S_X, D)$. This finishes the proof. \square

4.2. The maps ι_\pm^* . Recall that $\iota_\pm: X \hookrightarrow S_X$ denote the two inclusions of X in S_X via the map $\pi: X \amalg X \rightarrow S_X$. We define two pull-back maps $\iota_\pm^*: \operatorname{CH}_0(S_X) \rightarrow \operatorname{CH}_0(X)$. We do this for ι_+ as the other case is identical. By Proposition 2.3, we have that $(S_X)_{\operatorname{sing}} = D$ and $(S_X)_{\operatorname{reg}} = (X \setminus D) \amalg (X \setminus D)$, so that the natural map

$$(4.3) \quad \mathcal{Z}_0(S_X, D) \rightarrow \mathcal{Z}_0(X, D) \oplus \mathcal{Z}_0(X, D)$$

is an isomorphism. We define then $\iota_+^*: \mathcal{Z}_0(S_X, D) \rightarrow \mathcal{Z}_0(X, D)$ to be the first projection of the direct sum in (4.3). Notice that there are push-forward inclusion maps $\iota_{\pm*}: \mathcal{Z}_0(X, D) \rightarrow \mathcal{Z}_0(S_X, D)$ such that $\iota_+^* \circ \iota_{+*} = \operatorname{Id}$ and $\iota_+^* \circ \iota_{-*} = 0$.

Proposition 4.2. *The map $\mathcal{Z}_0(S_X, D) \xrightarrow{(\iota_+^*, \iota_-^*)} \mathcal{Z}_0(X, D) \oplus \mathcal{Z}_0(X, D)$ descends to the pull-back maps*

$$(4.4) \quad \iota_\pm^*: \operatorname{CH}_0(S_X) \rightarrow \operatorname{CH}_0(X)$$

such that $\iota_\pm^* \circ \Delta^* = \operatorname{Id}$.

Proof. For the rational equivalence, we argue as follows: suppose at first that $\nu: C \hookrightarrow S_X$ is an l.c.i curve relative to D and contained in S_X (so, it is in particular a Cartier curve on the double). Set $E = \nu^{-1}(D) \cup C_{\operatorname{sing}}$. Let C'_\pm denote the unique reduced closed subscheme of X_\pm such that $C'_\pm \setminus D = (C \setminus D) \cap X_\pm$. Then C'_+ is a closed subscheme of C with $\dim(C'_+) \leq 1$. It's easy to check that $\dim(C'_+) = 0$ would violate the condition of C being locally defined by a regular sequence at every point of intersection $C \cap D \subset S_X$, so that we can assume that C'_+ (and similarly C'_-) is a union of some irreducible components of C .

Let $f \in \mathcal{O}_{C,E}^\times$. Let $C = C'_+ \cup C'_- = \left(\bigcup_{i=1}^{r_+} C'_+{}^i\right) \cup \left(\bigcup_{j=1}^{r_-} C'_-{}^j\right)$. We can clearly assume that we have $E_{\operatorname{red}} = C'_+ \cap C'_-$ and a commutative square

$$(4.5) \quad \begin{array}{ccc} \mathcal{O}_{C,E}^\times & \xrightarrow{\theta_C} & k(C)^\times \xrightarrow{\cong} k(C'_+)^\times \times k(C'_-)^\times \\ \iota_+^* \downarrow & & \downarrow \iota_+^* \\ \mathcal{O}_{C'_+,E}^\times & \xrightarrow{\theta_{C'_+}} & k(C'_+)^\times \end{array} \quad \begin{array}{c} \swarrow p_+ \\ \end{array}$$

We can write $\theta_C(f) = (f_+, f_-)$ with $f_\pm \in k(C'_\pm)^\times$ and $\operatorname{div}_C(f) = (f)_C = (f_+)_{C_+} + (f_-)_{C_-}$, by definition. We now consider the diagram

$$(4.6) \quad \begin{array}{ccccc} \mathcal{Z}_0(C'_+, E) \oplus \mathcal{Z}_0(C'_-, E) & \xleftarrow{\cong} & \mathcal{Z}_0(C, E) & \xrightarrow{\nu_*} & \mathcal{Z}_0(S_X, D) \xrightarrow{\cong} \mathcal{Z}_0(X, D) \oplus \mathcal{Z}_0(X, D) \\ & \searrow p_+ & \downarrow \iota_+^* & & \downarrow \iota_+^* \\ & & \mathcal{Z}_0(C'_+, E) & \xrightarrow{\nu_{+*}} & \mathcal{Z}_0(X, D). \end{array} \quad \begin{array}{c} \swarrow p_+ \\ \end{array}$$

This diagram is clearly commutative and yields

$$(4.7) \quad \begin{aligned} \iota_+^* \circ \nu_*((f)_C) &= \nu_{+*} \circ \iota_+^*((f)_C) \\ &= \nu_{+*} \circ \iota_+^* \left[\iota_{+*}((f_+)_{C'_+}) + \iota_{-*}((f_-)_{C'_-}) \right] \\ &= \nu_{+*}((f_+)_{C'_+}) + 0 \\ &= \nu_{+*}((f_+)_{C'_+}). \end{aligned}$$

On the other hand, $\nu_{+*}((f_+)_{C'_+}) = \sum_{i=1}^{r_+} \operatorname{div}(f_+^i)$ by definition, where

$$f_+ = (f_+^1, \dots, f_+^{r_+}) \in k(C'_+)^\times = \prod_{i=1}^{r_+} k(C_+^i)^\times.$$

Since each $\operatorname{div}(f_+^i) \in \mathcal{R}_0(X)$, we conclude that $\iota_+^* \circ \nu_*((f)_C) \in \mathcal{R}_0(X)$. In particular, we obtain that the maps ι_\pm^* descend to group homomorphisms $\iota_\pm^*: \operatorname{CH}_0^{LW}(S_X) \rightarrow \operatorname{CH}_0(X)$ from the Levine-Weibel Chow group.

For the general case of a good curve $\nu: C \rightarrow S_X$ which is not necessarily an embedding, we can assume by Lemma 3.5 that $C \rightarrow S_X$ is a finite l.c.i. morphism. We now factor ν as $C \hookrightarrow \mathbb{P}_{S_X}^N \xrightarrow{\pi} S_X$, where π is the projection and $\mu: C \hookrightarrow \mathbb{P}_{S_X}^N$ is a regular embedding. By Proposition 2.3, we can identify $\mathbb{P}_{S_X}^N = \mathbb{P}_{S(X,D)}^N$ with $S(\mathbb{P}_X^N, \mathbb{P}_D^N)$. We have then a commutative diagram

$$(4.8) \quad \begin{array}{ccc} \mathcal{Z}(\mathbb{P}_{S_X}^N, \mathbb{P}_D^N) = \mathcal{Z}(S(\mathbb{P}_X^N, \mathbb{P}_D^N), \mathbb{P}_D^N) & \xrightarrow{\iota_\pm^*} & \mathcal{Z}(\mathbb{P}_X^N) \\ \pi_* \downarrow & & \downarrow \pi_* \\ \mathcal{Z}(S_X, D) & \xrightarrow{\iota_\pm^*} & \mathcal{Z}(X) \end{array}$$

by the definition of ι_\pm^* . For $f \in \mathcal{O}_{C, \nu^*D}^\times$, we have $\iota_\pm^*(\mu_*((f)_C)) \in \mathcal{R}_0(\mathbb{P}_X^N)$ by the embedded case. In particular, we get $\iota_\pm^*(\nu_*((f)_C)) = \iota_\pm^* \pi_* \mu_*((f)_C) = \pi_* \iota_\pm^*(\mu_*((f)_C)) \in \mathcal{R}_0(X)$, completing the proof. \square

5. 0-CYCLES ON S_X AND 0-CYCLES WITH MODULUS ON X

5.1. Overture. The goal of this section is to define the *difference map* τ_X^* from the Chow group of 0-cycles on S_X to the Chow group of 0-cycles with modulus D on X . The heart of this section is the proof that this difference map kills the group of rational equivalences in the Chow group with modulus $\mathrm{CH}_0(X|D)$. We start by recalling the definition of the Chow group with modulus, following Kerz and Saito.

5.2. 0-cycles with modulus. Let k be any field. Given an integral normal curve C over k and an effective divisor $E \subset C$, we say that a rational function f on C has modulus E if $f \in \mathrm{Ker}(\mathcal{O}_{C,E}^\times \rightarrow \mathcal{O}_E^\times)$. Here, $\mathcal{O}_{C,E}$ is the semilocal ring of C at the union of E and the generic point of C . In particular, $\mathrm{Ker}(\mathcal{O}_{C,E}^\times \rightarrow \mathcal{O}_E^\times)$ is just $k(C)^\times$ if $|E| = \emptyset$. Let $G(C, E)$ denote the group of such rational functions.

Let X be an integral scheme of finite type over k and let D be an effective Cartier divisor on X . Let $\mathcal{Z}_0(X \setminus D)$ be the free abelian group on the set of closed points of $X \setminus D$. Let C be an integral normal curve over k and let $\varphi_C: C \rightarrow X$ be a finite morphism such that $\varphi_C(C) \not\subset D$. The push forward of cycles along φ_C gives a well defined group homomorphism

$$\tau_C: G(C, \varphi_C^*(D)) \rightarrow \mathcal{Z}_0(X \setminus D).$$

Recall now the following Definition

Definition 5.1 (Kerz-Saito). We define the Chow group $\mathrm{CH}_0(X|D)$ of 0-cycles of X with modulus D as the cokernel of the homomorphism

$$(5.1) \quad \mathrm{div}: \bigoplus_{\varphi_C: C \rightarrow \bar{X}} G(C, \varphi_C^*(D)) \rightarrow \mathcal{Z}_0(X \setminus D),$$

where the sum is taken over the set of finite morphisms $\varphi_C: C \rightarrow X$ from an integral normal curve such that $\varphi_C(C) \not\subset D$.

It is known that the Chow group of 0-cycles with modulus is covariantly functorial for proper maps: if $f: X' \rightarrow X$ is proper, D and D' are effective Cartier divisors on X and X' respectively such that $f^*(D) \subset D'$, then there is a push-forward map $f_*: \mathrm{CH}_0(X'|D') \rightarrow \mathrm{CH}_0(X|D)$ (see [5, Lemma 2.7] or [26, Proposition 2.10]).

5.3. Setting and goals. From now on, we fix a smooth connected quasi-projective scheme X over k and an effective Cartier divisor $D \subset X$ on it. Let S_X be the double of X along D as defined in § 2. Let $\iota_\pm: X \rightarrow S_X$ denote the closed embeddings of the two components X_+ and X_- of the double. We want to construct maps

$$\tau_X^*: \mathrm{CH}_0(S_X) \rightarrow \mathrm{CH}_0(X|D) \quad \text{and} \quad p_{\pm,*}: \mathrm{CH}_0(X|D) \rightarrow \mathrm{CH}_0(S_X)$$

and prove that $\tau_X^* \circ p_{\pm,*} = \mathrm{Id}$. At the level of the free abelian group $\mathcal{Z}_0(S_X, D)$, the map τ_X^* is simply $\iota_+^* - \iota_-^*$:

$$\begin{aligned} \mathcal{Z}_0(S_X, D) &= \mathcal{Z}_0(X \setminus D) \oplus \mathcal{Z}_0(X \setminus D) \rightarrow \mathcal{Z}_0(X \setminus D), \\ (\alpha_+, \alpha_-) &\mapsto \alpha_+ - \alpha_- = \iota_+^*((\alpha_+, \alpha_-)) - \iota_-^*((\alpha_+, \alpha_-)) \end{aligned}$$

We construct the first map in several steps, starting by considering only embedded l.c.i. curves in the definition of the rational equivalence on the double S_X (or, in other words, by proving the existence of the map τ_X^* for the Levine-Weibel Chow group of zero cycles). The general case is then treated using the same trick as in Proposition 4.2, thanks to the following Lemma.

Lemma 5.2. *Assume that for every smooth connected quasi-projective scheme Y over k and effective Cartier divisor E on Y , the map τ_Y^* given above descends to a well defined homomorphism*

$$\tau_Y^*: \mathrm{CH}_0^{LW}(S(Y, E)) \rightarrow \mathrm{CH}_0(Y|E).$$

Then the map $\mathrm{CH}_0^{LW}(S_X) \rightarrow \mathrm{CH}_0(X|D)$ factors through $\mathrm{CH}_0(S_X)$, giving a well defined homomorphism

$$\tau_X^*: \mathrm{CH}_0(S_X) \rightarrow \mathrm{CH}_0(X|D).$$

Proof. Let $\delta: \mathcal{Z}_0(S_X, D) \rightarrow \mathrm{CH}_0(X|D)$ be the composition

$$\mathcal{Z}_0(S_X, D) \rightarrow \mathrm{CH}_0^{LW}(S_X) \rightarrow \mathrm{CH}_0(X|D).$$

We have to show that δ factors through $\mathrm{CH}_0(S_X)$, defined using good l.c.i. curves as in Definition 3.4. Using again Lemma 3.5, we have to show more precisely that $\delta(\nu_*(\mathrm{div}_C(f))) = 0$ for every $\nu: C \rightarrow S_X$ finite l.c.i. morphism from a reduced curve C that is good relative to (S_X, D) and for every rational function f on C that is regular and invertible along $E = (\nu^{-1}(D) \cup C_{\mathrm{sing}})$. We factor ν as composition $\nu = \pi \circ \mu$, where $\mu: C \hookrightarrow \mathbb{P}_{S_X}^N = S(\mathbb{P}_X^N, P_D^N)$ (using Proposition 2.3) is a regular embedding and $\pi: \mathbb{P}_{S_X}^N \rightarrow S_X$ is the projection. In particular, $\mu(C) = C$ is a Cartier curve on the double $S(\mathbb{P}_X^N, P_D^N)$ relative to \mathbb{P}_D^N .

It follows from (4.8) and the formula $\delta = \iota_+^* - \iota_-^*$ that the square

$$\begin{array}{ccc} \mathcal{Z}_0(\mathbb{P}_{S_X}^N, \mathbb{P}_D^N) & \xrightarrow{\delta_{\mathbb{P}_{S_X}^N}} & \mathrm{CH}_0(\mathbb{P}_X^N|\mathbb{P}_D^N) \\ \pi_* \downarrow & & \downarrow \pi_* \\ \mathcal{Z}_0(S_X, D) & \xrightarrow{\delta} & \mathrm{CH}_0(X|D) \end{array}$$

commutes, where $\delta_{\mathbb{P}_{S_X}^N}$ is the composition $\mathcal{Z}_0(\mathbb{P}_{S_X}^N, \mathbb{P}_D^N) \rightarrow \mathrm{CH}_0^{LW}(S(\mathbb{P}_X^N, \mathbb{P}_D^N)) \rightarrow \mathrm{CH}_0(\mathbb{P}_X^N|\mathbb{P}_D^N)$. That is, $\delta(\nu_*((f)_C)) = \delta(\pi_*(\mu_*((f)_C))) = \pi_*(\delta_{\mathbb{P}_{S_X}^N}(\mu_*((f)_C)))$.

By assumption, we have $\delta_{\mathbb{P}_{S_X}^N}(\mu_*((f)_C)) = 0 \in \mathrm{CH}_0(\mathbb{P}_X^N|\mathbb{P}_D^N)$. Since the push-forward map $\pi_*: \mathrm{CH}_0(\mathbb{P}_X^N|\mathbb{P}_D^N) \rightarrow \mathrm{CH}_0(X|D)$ is well defined, we can conclude. \square

We have therefore reduced the problem to showing that the map τ_X^* is well defined from the Levine-Weibel Chow group of zero cycles.

To begin, we need to keep some control over the Cartier curves on the double S_X which generate $\mathcal{R}_0(S_X, D)^{LW}$. We do this in the next few lemmas.

5.4. The Cartier curves on the double S_X . Our next goal is to use the specific structure of S_X as a join of two smooth schemes to refine the set of Cartier curves used to define $\mathrm{CH}_0^{LW}(S_X, D) = \mathrm{CH}_0^{LW}(S_X)$. In this section, by Cartier curve on S_X (resp. X) we will always mean curve on S_X (resp. X) which is Cartier with respect to D . For the rest of § 5, we shall assume that the ground field k is infinite and perfect.

Notation 5.3. Let S be a quasi-projective k -scheme and let \mathcal{L} be a line bundle on S . For a global section $t \in H^0(S, \mathcal{L})$, we write (t) for the divisor of zeros of s , that we consider as a closed subscheme of S .

Lemma 5.4. *Let X be a connected smooth quasi-projective scheme of dimension $d \geq 3$ over k . The group of rational equivalences $\mathcal{R}_0^{LW}(S_X, D)$ is generated by the divisors of functions on (possibly non-reduced) Cartier curves $C \hookrightarrow S_X$ relative to D , where C satisfies the following.*

(1) *There is a locally closed embedding $S_X \hookrightarrow \mathbb{P}_k^N$ and distinct hypersurfaces*

$$H_1, \dots, H_{d-2} \hookrightarrow \mathbb{P}_k^N$$

such that $Y = S_X \cap H_1 \cap \dots \cap H_{d-2}$ is a complete intersection which is geometrically reduced.

- (2) $X_{\pm} \cap Y = X_{\pm} \cap H_1 \cap \dots \cap H_{d-2} := Y_{\pm}$ *are integral.*
- (3) *No component of Y is contained in D .*
- (4) $C \subset Y$.
- (5) C *is a Cartier divisor on Y .*
- (6) Y_{\pm} *are smooth away from C .*
- (7) C *is Cohen-Macaulay.*

Proof. Let $C \hookrightarrow S_X$ be a reduced Cartier curve relative to D and let $f \in \mathcal{O}_{C, C \cap D}^{\times}$. Since C is Cartier along $D = (S_X)_{\text{sing}}$, it follows that it is Cartier in S_X along each of its generic points.

We will replace C by the curves of desired type using a combination of [30, Lemma 1.3] and [30, Lemma 1.4] as follows. We also refer to [17, Chap. I, § 6] for the Bertini theorem for geometrically reduced schemes and [43, Theorem 12] irreducible schemes.

Since S_X is reduced quasi-projective, X is integral and $C \hookrightarrow S_X$ is Cartier along D , we can apply [30, Lemma 1.3] to find a locally closed embedding $S_X \hookrightarrow \mathbb{P}_k^N$ and distinct hypersurfaces $H_1, \dots, H_{d-2} \hookrightarrow \mathbb{P}_k^N$ such that:

- (1) $Y = S_X \cap H_1 \cap \dots \cap H_{d-2}$ is a complete intersection which is (geometrically) reduced (note that since k is perfect, Y is reduced if and only if it is geometrically reduced).
- (2) $X_{\pm} \cap Y = X_{\pm} \cap H_1 \cap \dots \cap H_{d-2} := Y_{\pm}$ are integral.
- (3) No component of Y is contained in D .
- (4) $C \subset Y$.
- (5) C is locally principal in Y in a neighborhood of the finite set $C \cap D$ and at each generic point of C .

Since $X_{\pm} = X$ are smooth, we can apply [21, Theorem 1] to further assume that Y_{\pm} are smooth away from C_{\pm} . Here C_{\pm} denotes $C \cap X_{\pm} = C \times_{S_X} X_{\pm}$.

Since $Y \hookrightarrow S_X$ is constructed as a complete intersection of hypersurfaces of arbitrarily large degrees (see [30, Lemma 1.4]), we can furthermore find a locally principal closed subscheme C_1 of Y such that

- a) $C \subset C_1$.
- b) $\overline{C_1}$ equals C at every generic point of C .
- c) $(C_1 \setminus C) \cap C \cap D = \emptyset$.

Since Y is geometrically reduced, it follows from Lemma 2.2 that Y is the join of Y_{\pm} along D and there is a short exact sequence

$$(5.2) \quad 0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{Y_+} \times \mathcal{O}_{Y_-} \rightarrow \mathcal{O}_{Y \cap D} \rightarrow 0.$$

Since C_1 is locally principal in Y , it follows that $C_1 \cap X_{\pm} = C_1 \cap Y_{\pm}$ are locally principal in Y_{\pm} . Since Y_{\pm} are integral surfaces, it follows that $C_1 \cap X_{\pm}$ are Cartier divisors in Y_{\pm} . It follows from (5.2) that C_1 is a Cartier divisor in Y . Since S_X is Cohen-Macaulay and $Y \subset S_X$ is a complete-intersection, it follows that Y is also Cohen-Macaulay. Since $C_1 \subset Y$ is a Cartier divisor, we conclude that C_1 is Cohen-Macaulay.

It follows from (c) that f extends to a function $g \in \mathcal{O}_{C_1, C_1 \cap D}^{\times}$ by setting $g = f$ on C and $g = 1$ on $C_1 \setminus C$. In particular, we have $\text{div}(f) = \text{div}(g)$. This finishes the proof. \square

We now further refine the rational equivalence by specifying the shape of the Cartier curves that generate the group of relations.

Lemma 5.5. *Let X be a connected smooth quasi-projective scheme of dimension $d \geq 2$ over k . Let $\nu: C \rightarrow S_X$ be a (possibly non-reduced) Cartier curve relative to $D \subset S_X$. Assume that either $d = 2$ or there are inclusions $C \subset Y \subset S_X$, where Y is a geometrically reduced complete intersection surface and C is a Cartier divisor on Y , as in Lemma 5.4. Let $f \in \mathcal{O}_{C, \nu^*D}^\times \subset k(C)^\times$, where $k(C)$ is the total quotient ring of \mathcal{O}_{C, ν^*D} .*

We can then find two Cartier curves $\nu': C' \hookrightarrow S_X$ and $\nu'': C'' \hookrightarrow S_X$ relative to D satisfying the following.

- (1) *There are very ample line bundles $\mathcal{L}', \mathcal{L}''$ on S_X and sections $t' \in H^0(S_X, \mathcal{L}')$, $t'' \in H^0(S_X, \mathcal{L}'')$ such that $C' = Y \cap (t')$ and $C'' = Y \cap (t'')$ (with the convention $Y = S_X$ if $d = 2$).*
- (2) *C' and C'' are geometrically reduced.*
- (3) *The restrictions of both C' and C'' to X via the two closed immersions ι_\pm are integral curves in X , which are Cartier and smooth along D .*
- (4) *There are functions $f' \in \mathcal{O}_{C', (\nu')^*D}^\times$ and $f'' \in \mathcal{O}_{C'', (\nu'')^*D}^\times$ such that $\nu'_*(\text{div}(f')) + \nu''_*(\text{div}(f'')) = \nu_*(\text{div}(f))$ in $\mathcal{Z}_0(S_X, D)$.*

Proof. In this proof, we shall assume that $Y = S_X$ if $d = 2$. In this case, a Cartier curve on S_X along $D = (S_X)_{\text{sing}}$ must be an effective Cartier divisor on S_X . Hence we can assume that C is an effective Cartier divisor on Y for any $d \geq 2$. Notice that $Y = Y_+ \amalg_D Y_-$, again by Lemma 2.2.

Arguing as in Lemma 5.4, since S_X (and hence Y) is quasi-projective over k , we can find an effective Cartier divisor \tilde{C} on Y such that

- (1) $\overline{C \subset \tilde{C}}$.
- (2) $\tilde{C} \setminus C \cap (C \cap D) = \emptyset$.
- (3) $\mathcal{O}_Y(\tilde{C})$ is a very ample line bundle on Y .

We can extend the function f on C to a function \tilde{f} on \tilde{C} by setting $\tilde{f} = f$ on C and $\tilde{f} = 1$ on $\tilde{C} \setminus C$. Condition (2) guarantees that \tilde{f} is regular and invertible at each point of $\tilde{C} \cap D$, and it is clear by construction that $\text{div}(\tilde{f}) = \text{div}(f)$. Replacing C with \tilde{C} (and changing the notation for simplicity), we can thus assume that C is a Cartier divisor on Y such that the associated line bundle $\mathcal{O}_Y(C)$ is very ample. Choose $t_0 \in H^0(Y, \mathcal{O}_Y(C))$ such that $C = (t_0)$. Since Y is geometrically reduced, standard Bertini (see the proof of [30, Lemma 1.3]) allows us to choose another divisor C_∞ in the linear system $H^0(Y, \mathcal{O}_Y(C))$ such that:

- (1) C_∞ is reduced.
- (2) $C_\infty \cap C \cap D = \emptyset$.
- (3) C_∞ contains no component of C .
- (4) D contains no component of C_∞ .

Denote by t_∞ the section of $\mathcal{O}_Y(C)$ with $C_\infty = (t_\infty)$. As before, we extend the function f on C to a function h on $C_\infty \cup C$ by setting $h = f$ on C and $h = 1$ on C_∞ . Notice that h is meromorphic on $C_\infty \cup C$ and regular invertible in a neighborhood of $(C_\infty \cup C) \cap D$ by (2). Let S denote the finite set of points $S = \{\text{poles of } f\} \cup (C \cap C_\infty)$. Note that $S \cap D = \emptyset$.

Choose now a very ample line bundle $\mathcal{L} = j^*(\mathcal{O}_{\mathbb{P}^r}(1))$ on S_X , corresponding to an embedding $j: S_X \rightarrow \mathbb{P}_k^r$ for $r \gg 0$ of S_X as locally closed subscheme. We can assume that r is sufficiently large so that $\mathcal{L}|_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(C)$ is also very ample on Y . Since k is infinite, there is a dense open subset V_X of the dual projective space $(\mathbb{P}_k^r)^\vee$ such that for $L \in V_X$, the scheme theoretic intersections $L \cdot S_X = L \times_{\mathbb{P}_k^r} S_X$ and $L \cdot Y = L \times_{\mathbb{P}_k^r} Y$ satisfy the following list of properties:

- a) $L \cdot S_X$ and $L \cdot Y$ are geometrically reduced (since both S_X and Y are),
- b) $Y \not\subset L \cdot S_X$,
- c) $L \cdot X_\pm = L \times_{\mathbb{P}_k^r} X_\pm$ and $L \cdot Y_\pm = L \times_{\mathbb{P}_k^r} Y_\pm$ are integral (since X_\pm and Y_\pm are),
- d) $L \cdot Y \cap (C \cup C_\infty) \cap D = \emptyset$,

- e) $L \cdot Y \supset S$,
- f) $L \cdot Y$ intersects $C_\infty \cup C$ in a finite set of points.

The hyperplane L corresponds to a section l of the linear system $H^0(\mathbb{P}_k^r, \mathcal{O}_{\mathbb{P}^r}(1))$. Write s_∞ for the global section of \mathcal{L} that is the restriction of l to S_X . Then $L \cdot S_X = (s_\infty)$. Note that we can assume that s_∞ does not have poles on $(C \cup C_\infty) \cap D$. Let $\overline{S_X}$ be the closure of S_X in \mathbb{P}_k^r and let \mathcal{I} be the ideal sheaf of $\overline{S_X} \setminus S_X$ in \mathbb{P}_k^r . We can find a section s'_∞ of the sheaf $\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}_k^r}(m)$ for some $m \gg 0$, which restricts to a section s_∞ on S_X satisfying the properties (a) - (f) on $S_X = \overline{S_X} \setminus V(\mathcal{I})$. This implies in particular that $S_X \setminus (s_\infty) = \overline{S_X} \setminus (s'_\infty)$ is affine. Thus, up to taking a further Veronese embedding of \mathbb{P}_k^r (replacing s_∞ with s'_∞) we can assume that (a) - (f) as above as well as the following hold

- g) $S_X \setminus (s_\infty) = S_X \setminus L \cdot S_X$ is affine.

We now know that Y_\pm is smooth away from C and d) tells us that (s_∞) intersects Y_\pm along D at only those points which are away from C . It follows that $(s_\infty) \cap Y_\pm \cap D \subset (s_\infty) \cap (Y_\pm)_{\text{reg}}$. We can then use the Bertini theorem of Altman and Kleiman [21, Theorem 1] to ensure that $(s_\infty) \cap (Y_\pm)_{\text{reg}}$ is smooth away from the subscheme $(C \cup C_\infty) \cap X_+$ where we ask the containment condition. In particular, we can assume that $(s_\infty) \cap Y_+$ is smooth along D . The same holds for Y_- as well. We conclude that we can moreover achieve the following property

- h) $(s_\infty) \cap Y_\pm$ are smooth along D .

Consider again the function h on $C \cup C_\infty$. By our choice of S , h is regular on $(C \cup C_\infty) \setminus S$. By g) above, h extends to a regular function H on the affine open $U = S_X \setminus (s_\infty)$. Since H is a meromorphic function on S_X which has poles only along (s_∞) , it follows that for $N \gg 0$ the section Hs_∞^N is an element of $H^0(U, \mathcal{L}^{\otimes N})$ which extends to a section s_0 of $\mathcal{L}^{\otimes N}$ on all S_X . Since h is regular and invertible at each point of $C \cup C_\infty \cap D$ and since s_∞ does not have zeros or poles on $C \cup C_\infty \cap D$, it follows that $(s_0) \cap (C \cup C_\infty) \cap D = \emptyset$ and (using f) above) that (s_0) does not contain any component of $C \cup C_\infty$. Note that up to replacing s_0 by $s_0 s_\infty^i$, we are free to choose N as large as needed.

Write $\mathcal{I}_{C \cup C_\infty}$ for the ideal sheaf of $C \cup C_\infty$ in S_X . We can then find sections s_1, \dots, s_m of $H^0(S_X, \mathcal{L}^{\otimes N} \otimes \mathcal{I}_{C \cup C_\infty})$ such that the rational map $\phi: S_X \dashrightarrow \mathbb{P}_k^{m-1}$ that they define is a locally closed immersion on $S_X \setminus (C \cup C_\infty)$. In particular, there exists an affine open neighbourhood U_x of every $x \in S_X \setminus (C \cup C_\infty)$ where at least one of the s_i is not identically zero and where the k -algebra $k[s_1/s_i, \dots, s_m/s_i]$ generated by $s_1/s_i, \dots, s_m/s_i$ coincides with the coordinate ring of U_x . But then the same must be true for the algebra $k[s_0/s_1, s_1/s_i, \dots, s_m/s_i]$ obtained by adding the element s_0/s_i . Hence, the rational map $\psi: S_X \dashrightarrow \mathbb{P}_k^m$ given by the sections (s_0, s_1, \dots, s_m) of $H^0(S_X, \mathcal{L}^{\otimes N})$ is also a locally closed immersion on $S_X \setminus (C \cup C_\infty)$, and since the base locus of the linear system associated to (s_0, s_1, \dots, s_m) is $(s_0) \cap (C \cup C_\infty)$, it is in fact a morphism away from $(s_0) \cap (C \cup C_\infty)$.

In particular, ψ is birational (hence separable) and has image of dimension at least two, so that the linear system $V = (s_0, s_1, \dots, s_m)$ is not composite with a pencil. By the classical Theorem of Bertini (see, for example, [51, Theorem I.6.3]), a general divisor E in V is generically geometrically reduced. Moreover, E is itself a Cohen-Macaulay scheme (since S_X is Cohen-Macaulay). But a locally Noetherian Cohen-Macaulay scheme that is generically reduced is in fact reduced by [14, Prop. 14.124]. Hence the general divisor E in V is indeed geometrically reduced (hence reduced). We can apply the same argument to $E \cdot Y$, noting that being a complete intersection in S_X , the surface Y is Cohen-Macaulay.

Since Y_\pm and X_\pm are all integral, a general divisor E of V will be moreover irreducible when restricted to X_\pm and Y_\pm . We can therefore assume that there is a global section s'_0 of $H^0(S_X, \mathcal{L}^{\otimes N})$ of the form $s'_0 = s_0 + \alpha$ with $\alpha \in (s_1, \dots, s_m)$, that satisfies the following properties:

- a') (s'_0) and $(s'_0) \cap Y$ are (geometrically) reduced.

- b') $Y \not\subset (s'_0)$.
- c') $(s'_0) \cap X_{\pm}$ and $(s'_0) \cap Y_{\pm}$ are integral.
- d') $(s'_0) \cap (C \cup C_{\infty}) \cap D = \emptyset$.
- e') $C_{\infty} \cup C$ contains no component of $(s'_0) \cap Y$.
- f') $(s'_0) \cap Y_{\pm}$ are smooth along D .

The items a') and c') follow from the previous discussion. Properties b'), d') and e') are clearly open conditions on the space of sections V , and are therefore satisfied by the general divisor. As for the last item f'), it follows from the classical Theorem of Bertini on smoothness applied to Y_{\pm} , the assumption on Y_{\pm} and the item d').

We then have

$$\frac{s'_0}{s_{\infty}^N} = \frac{Hs_{\infty}^N + (\alpha s_{\infty}^{-N})s_{\infty}^N}{s_{\infty}^N} = H + \alpha s_{\infty}^{-N} = H', \text{ (say).}$$

Since α vanishes along $C \cup C_{\infty}$ and s_{∞} does not vanish identically on $U \cap (C \cup C_{\infty})$ by f), it follows that $H'_{|(C \cup C_{\infty}) \cap U} = H_{|(C \cup C_{\infty}) \cap U} = h|_U$. In other words, we have $s'_0/s_{\infty}^N = h$ as rational functions on $C \cup C_{\infty}$. We can now compute:

$$\begin{aligned} \nu_*(\operatorname{div}(f)) &= (s'_0) \cdot C - N(s_{\infty}) \cdot C \\ 0 = \operatorname{div}(1) &= (s'_0) \cdot C_{\infty} - N(s_{\infty}) \cdot C_{\infty}. \end{aligned}$$

Setting $(s_{\infty}^Y) = (s_{\infty}) \cap Y$ and $(s_0^Y) = (s'_0) \cap Y$, we get

$$\begin{aligned} \nu_*(\operatorname{div}(f)) &= (s'_0) \cdot (C - C_{\infty}) - N(s_{\infty})(C - C_{\infty}) \\ &= (s_0^Y) \cdot (\operatorname{div}(t_0/t_{\infty})) - N(s_{\infty}^Y) \cdot (\operatorname{div}(t_0/t_{\infty})) \\ &= \iota_{s_0^Y, *}(\operatorname{div}(f')) - N \iota_{s_{\infty}^Y, *}(\operatorname{div}(f'')), \end{aligned}$$

where $f' = (t_0/t_{\infty})|_{(s_0^Y)} \in \mathcal{O}_{(s_0^Y), D \cap (s_0^Y)}^{\times}$ (by (d')) and $f'' = (t_0/t_{\infty})|_{(s_{\infty}^Y)} \in \mathcal{O}_{(s_{\infty}^Y), D \cap (s_{\infty}^Y)}^{\times}$ (by (d)).

It follows from h) and f') that $(s_0^Y)|_{X_+}$, $(s_0^Y)|_{X_-}$, $(s_{\infty}^Y)|_{X_+}$ and $(s_{\infty}^Y)|_{X_-}$ are all smooth along D . Setting $\mathcal{L}'' = (\mathcal{L}')^N$, $t'' = (s'_0)$ and $t' = (s_{\infty})$, the curves $C' = (t') \cap Y$ and $C'' = (t'') \cap Y$ together with the functions f' and f'' satisfy the conditions of the Lemma. \square

Remark 5.6. It follows from Lemma 2.2 that each of the curves C' and C'' constructed in the proof of the lemma is of the form $C_+ \amalg_E C_-$ for $E = \nu_+^*(D) = \nu_-^*(D)$ and $\nu_{\pm}: C_{\pm} \hookrightarrow X$ are integral Cartier curves in X , which are smooth along D . Moreover, by construction, C_{\pm} are the zero loci of the restrictions t_{\pm} to Y_{\pm} of a global section of a very ample line bundle \mathcal{M} on S_X . It follows that $(t_+)|_D$ and $(t_-)|_D$ agree.

5.5. The map τ_X^* : the case of curves and surfaces. For any smooth quasi-projective scheme X over k with effective Cartier divisor $D \subset X$, we have defined in 5.3 the map

$$(5.3) \quad \tau_X^*: \mathcal{Z}_0(S_X, D) \rightarrow \mathcal{Z}_0(X, D) \quad \text{by} \quad \tau_X^*([x]) = \iota_+^*([x]) - \iota_-^*([x]),$$

where $x \in S_X \setminus D$ is a closed point.

Suppose first that X is a smooth curve. We can clearly assume that X is connected. If $f \in \mathcal{O}_{S_X, D}^{\times}$ and $\theta_X: \mathcal{O}_{S_X, D}^{\times} \rightarrow k(S_X)^{\times} = k(X_+)^{\times} \times k(X_-)^{\times}$ is the natural map, then $\theta_X(f) = (f_+, f_-)$ with $f_{\pm} \in \mathcal{O}_{X_{\pm}, D}^{\times} = \mathcal{O}_{X, D}^{\times}$. It follows from (2.6) that $g := f_+ f_-^{-1} \in \operatorname{Ker}(\mathcal{O}_{X, D}^{\times} \rightarrow \mathcal{O}_D^{\times})$. Moreover, $\tau_X^*(\operatorname{div}(f)) = \iota_+^*(\operatorname{div}(f)) - \iota_-^*(\operatorname{div}(f)) = \operatorname{div}(f_+) - \operatorname{div}(f_-) = \operatorname{div}(g)$. We conclude that τ_X^* descends to a map

$$(5.4) \quad \tau_X^*: \operatorname{CH}_0^{LW}(S_X) = \operatorname{CH}_0(S_X) \rightarrow \operatorname{CH}_0(X|D).$$

Proposition 5.7. *Let X be a smooth connected quasi-projective surface over k with an effective Cartier divisor D . Then the map $\tau_X^*: \mathcal{Z}_0(S_X, D) \rightarrow \mathcal{Z}_0(X|D)$ of (5.3) descends to a group homomorphism $\mathrm{CH}_0^{LW}(S_X) \rightarrow \mathrm{CH}_0(X|D)$.*

Proof. We shall continue to use the notations and various constructions in the proof of Lemma 5.5. We have shown in Lemma 5.5 that in order to prove that τ_X^* preserves the subgroups of rational equivalences, it suffices to show that $\tau_X^*(\mathrm{div}(f')) \in \mathcal{R}_0(X|D)$, where f' is a rational function on a Cartier curve $\nu: C' \hookrightarrow S_X$ that we can choose in the following way.

- (1) There is a very ample line bundle \mathcal{L} on S_X and sections $t \in H^0(S_X, \mathcal{L}), t_{\pm} = \iota_{\pm}^*(t) \in H^0(X, \iota_{\pm}^*(\mathcal{L}))$ such that $C' = (t)$ and $C'_{\pm} = (t_{\pm})$.
- (2) C' is a (geometrically) reduced Cartier curve of the form $C' = C'_+ \amalg_E C'_-$, where $E = \nu^*(D)$ such that C'_{\pm} are integral curves on X , none of which is contained in D and each of which is smooth along D (see Remark 5.6).

Let $\iota_D = \iota_+ \circ \iota = \iota_- \circ \iota: D \hookrightarrow S_X$ denote the inclusion map. Then, it follows from (1) that

$$(5.5) \quad (t_+)_{|D} = \iota_D^*(t) = (t_-)_{|D}.$$

Let (f'_+, f'_-) be the image of f' in $\mathcal{O}_{C'_+, E}^{\times} \times \mathcal{O}_{C'_-, E}^{\times} \hookrightarrow k(C'_+) \times k(C'_-)$. It follows from Lemma 2.2 that there is an exact sequence

$$0 \rightarrow \mathcal{O}_{C', E} \rightarrow \mathcal{O}_{C'_+, E} \times \mathcal{O}_{C'_-, E} \rightarrow \mathcal{O}_E \rightarrow 0.$$

In particular, we have

$$(5.6) \quad (f'_+)_{|E} = (f'_-)_{|E} \in \mathcal{O}_E^{\times}.$$

Let us first assume that $C'_+ = C'_-$ as curves on X . Let C denote this curve and let C^N denote its normalization. Let $\pi: C^N \rightarrow C \hookrightarrow X$ denote the composite map. Since C is regular along E by (2), we get $f'_+, f'_- \in \mathcal{O}_{C^N, E}^{\times}$. Setting $g := f'_+ f'^{-1} \in \mathcal{O}_{C^N, E}^{\times}$, it follows from (5.6) that $g \in \mathrm{Ker}(\mathcal{O}_{C^N, E}^{\times} \rightarrow \mathcal{O}_E^{\times})$. Moreover, $\tau_X^*(\mathrm{div}(f')) = \iota_+^*(\mathrm{div}(f')) - \iota_-^*(\mathrm{div}(f')) = \mathrm{div}(f'_+) - \mathrm{div}(f'_-) = \pi_*(\mathrm{div}(g))$. We conclude from (5.1) that τ_X^* descends to a map $\tau_X^*: \mathrm{CH}_0^{LW}(S_X) \rightarrow \mathrm{CH}_0(X|D)$.

We now assume that $C'_+ \neq C'_-$. Let S denote the set of closed points on $C'_+ \cup C'_-$, where f'_+ or f'_- has a pole. It is clear that $S \cap D = \emptyset$. We now repeat the constructions in the proof of Lemma 5.5 to find a very ample line bundle \mathcal{L} on X and a section $s_{\infty} \in H^0(X, \mathcal{L})$ such that

- a) (s_{∞}) is integral (because X is smooth and connected).
- b) $(s_{\infty}) \cap (C'_+ \cup C'_-) \cap D = \emptyset$.
- c) $(s_{\infty}) \supset S$.
- d) $(s_{\infty}) \not\subset C'_+ \cup C'_-$.
- e) (s_{∞}) is smooth away from S .
- f) $X \setminus (s_{\infty})$ is affine.

It follows that f'_{\pm} extend to regular functions F'_{\pm} on $U = X \setminus (s_{\infty})$. Since F'_{\pm} are meromorphic functions on X which have poles only along (s_{∞}) , it follows that $F'_{\pm} s_{\infty}^N$ are elements of $H^0(U, \mathcal{L}^N)$ which extend to sections $s_{0, \pm}$ of \mathcal{L}^N on all of X , if we choose $N \gg 0$.

Since s_{∞}, F'_+ and F'_- are all invertible on $C'_+ \cup C'_-$ along D , we see that $(s_0)_{\pm}$ are invertible on $C'_+ \cup C'_-$ along D . In particular, $s_{0, \pm} \notin H^0(X, \mathcal{L}^{\otimes N} \otimes \mathcal{I}_{C'_+ \cup C'_-})$. As before, we can moreover find $\alpha_{\pm} \in H^0(X, \mathcal{L}^{\otimes N} \otimes \mathcal{I}_{C'_+ \cup C'_-}) \subset H^0(X, \mathcal{L}^N)$ such that $s'_{0, \pm} := s_{0, \pm} + \alpha_{\pm}$ have the following properties.

- a') $(s'_{0, \pm})$ are integral.
- b') $(s'_{0, \pm}) \not\subset C'_+ \cup C'_-$.
- c') $(s'_{0, \pm}) \cap (C'_+ \cup C'_-) \cap D = \emptyset$.

d') $(s'_{0,\pm})$ are smooth away from $C'_+ \cup C'_-$.

We then have

$$\frac{s'_{0,\pm}}{s_\infty^N} = \frac{F'_\pm s_\infty^N + (\alpha_\pm s_\infty^{-N}) s_\infty^N}{s_\infty^N} = F'_\pm + \alpha_\pm s_\infty^{-N} = H'_\pm, \text{ (say).}$$

Since α_\pm vanish along $C'_+ \cup C'_-$ and s_∞ does not vanish identically along $U \cap (C'_+ \cup C'_-)$, it follows that $(H'_\pm)|_{(C'_+ \cup C'_-) \cap U} = (F'_\pm)|_{(C'_+ \cup C'_-) \cap U}$. Since $(F'_\pm)|_{C'_+ \cup C'_-}$ are invertible regular functions on $C'_+ \cup C'_-$ along D , it follows that $(H'_+)|_{C'_+ \cup C'_-}$ and $(H'_-)|_{C'_+ \cup C'_-}$ are both rational functions on $C'_+ \cup C'_-$ which are regular and invertible along D . In particular, we have

$$(5.7) \quad (H'_+)|_E = (F'_+)|_E = (f'_+)|_E = {}^\dagger (f'_-)|_E = (F'_-)|_E = (H'_-)|_E,$$

where $={}^\dagger$ follows from (5.6).

Since $s'_{0,+}$ and $s'_{0,-}$ are both invertible functions on C'_- in a neighborhood of $C'_- \cap D$ by c'), it follows that the restriction of $s'_{0,-}/s'_{0,+}$ on C'_- is a rational function on C'_- , which is regular and invertible in a neighborhood of $C'_- \cap D$. On the other hand, we have

$$(5.8) \quad \frac{s'_{0,-}}{s'_{0,+}} = \frac{s'_{0,-}/s_\infty^N}{s'_{0,+}/s_\infty^N} = \frac{H'_-}{H'_+},$$

as rational functions on X . Since H'_+ and H'_- are also well-defined non-zero rational functions on C'_- , we conclude that $s'_{0,-}/s'_{0,+}$ and H'_-/H'_+ both restrict to an identical and well-defined rational function on C'_- , which is invertible along D .

We now compute

$$\begin{aligned} \tau_X^*(\text{div}(f')) &= \iota_+^*(\text{div}(f')) - \iota_-^*(\text{div}(f')) \\ &= \text{div}(f'_+) - \text{div}(f'_-) \\ &= [(s'_{0,+}) \cdot C'_+ - (s_\infty^N) \cdot C'_+] - [(s'_{0,-}) \cdot C'_- - (s_\infty^N) \cdot C'_-] \\ &= [(s'_{0,+}) \cdot C'_+ - (s'_{0,+}) \cdot C'_-] - [(s'_{0,-}) \cdot C'_- - (s'_{0,+}) \cdot C'_-] \\ &\quad - [(s_\infty^N) \cdot C'_+ - (s_\infty^N) \cdot C'_-] \\ &= [(s'_{0,+}) \cdot (C'_+ - C'_-)] - [C'_- \cdot (s'_{0,-}) - (s'_{0,+})] - [(s_\infty^N) \cdot (C'_+ - C'_-)] \\ &= (s'_{0,+}) \cdot (\text{div}(t_+/t_-)) - C'_- \cdot (\text{div}(s'_{0,-}/s'_{0,+})) - N(s_\infty) \cdot (\text{div}(t_+/t_-)) \\ &= (s'_{0,+}) \cdot (\text{div}(t_+/t_-)) - C'_- \cdot (\text{div}(H'_-/H'_+)) - N(s_\infty) \cdot (\text{div}(t_+/t_-)). \end{aligned}$$

It follows from b) and c') that t_\pm restrict to regular invertible functions on $(s'_{0,+})$ and (s_∞) along D . We set $h_1 = (\frac{t_+}{t_-})|_{(s'_{0,+})}$, $h_2 = (\frac{H'_-}{H'_+})|_{C'_-}$ and $h_3 = (\frac{t_+}{t_-})|_{s_\infty}$. Let $(s'_{0,+})^N \rightarrow (s'_{0,+})$, $(C'_-)^N \rightarrow C'_-$ and $(s_\infty)^N \rightarrow (s_\infty)$ denote the normalization maps. Let $\nu_1: (s'_{0,+})^N \rightarrow X$, $\nu_2: (C'_-)^N \rightarrow X$ and $\nu_3: (s_\infty)^N \rightarrow X$ denote the composite maps. Since $(s'_{0,+})$, C'_- and (s_∞) are all regular along D by (2), e) and d'), it follows from (5.5) and (5.7) that $h_1 \in G((s'_{0,+})^N, \nu_1^*(D))$, $h_2 \in G((C'_-)^N, \nu_2^*(D))$ and $h_3 \in G((s_\infty)^N, \nu_3^*(D))$. We conclude from (5.1) that $\tau_X^*(\text{div}(f'))$ dies in $\text{CH}_0(X|D)$. In particular, τ_X^* descends to a map $\tau_X^*: \text{CH}_0^{LW}(S_X) \rightarrow \text{CH}_0(X|D)$. This finishes the proof. \square

5.6. The map τ_X^* : the case of higher dimensions. We are left to show that the map $\tau_X^*: \mathcal{Z}_0(S_X, D) \rightarrow \mathcal{Z}_0(X|D)$ of (5.3) descends to a group homomorphism $\text{CH}_0^{LW}(S_X) \rightarrow \text{CH}_0(X|D)$ when X has dimension at least 3 when k is infinite and perfect. This is the content of the following.

Proposition 5.8. *Let X be a smooth connected quasi-projective scheme over k of dimension $d \geq 3$ with an effective Cartier divisor D . Then the map $\tau_X^*: \mathcal{Z}_0(S_X, D) \rightarrow \mathcal{Z}_0(X|D)$ of (5.3) descends to a group homomorphism $\text{CH}_0^{LW}(S_X) \rightarrow \text{CH}_0(X|D)$.*

Proof. Let $\nu: C \hookrightarrow S_X$ be a reduced Cartier curve relative to D and let $f \in \mathcal{O}_{C,E}^\times$, where $E = \nu^*(D)$. By Lemma 5.4, we can assume that there are inclusions $C \hookrightarrow Y \hookrightarrow S_X$ satisfying the conditions (1) - (6) of Lemma 5.4. The only price we pay by doing so is that C may no longer be reduced (but still Cohen-Macaulay). But we solve this by replacing C with a reduced Cartier curve (which we also denote by C) that is of the form given in Lemma 5.5. We shall now continue with the notations of proof of Lemma 5.5.

We write $C = (t) \cap Y$, where $t \in H^0(S_X, \mathcal{L})$ such that \mathcal{L} is a very ample line bundle on S_X . Let $t_\pm = \iota_\pm^*(t) \in H^0(X, \iota_\pm^*(\mathcal{L}))$ and let $C_\pm = (t_\pm) \cap Y = (t_\pm) \cap Y_\pm$. Let $\nu_\pm: C_\pm \hookrightarrow X$ denote the inclusions. It follows from our choice of the section that (t_\pm) are integral. If $C_+ = C_-$, exactly the same argument as in the case of surfaces in Proposition 5.7 applies to show that $\tau_X^*(\text{div}(f)) \in \mathcal{R}_0(X|D)$. So we assume $C_+ \neq C_-$.

Let $\Delta(C) = C_+ \cup C_-$ denote the scheme-theoretic image in X under the finite map Δ . Since X is smooth and connected, the Bertini Theorem of Altman and Kleiman [21, Theorem 1] allows us to find once again a complete intersection integral surface $T \subset X$ satisfying the following.

- (1) $T \supset \Delta(C)$.
- (2) $T \cap (t_\pm)$ are integral curves.
- (3) T is smooth away from $\Delta(C)$.

Set $t_\pm^T = (t_\pm)|_T$. Since C_\pm are integral and contained in $T \cap (t_\pm)$, it follows that

$$(5.9) \quad (t_\pm^T) = C_\pm.$$

Let S be the finite set of points of $\Delta(C)$, where $f_+ = \nu_+^*(f)$ or $f_- = \nu_-^*(f)$ have poles. It is clear that $S \cap D = \emptyset$. We now choose another very ample line bundle \mathcal{M} on X and $s_\infty \in H^0(X, \mathcal{M})$ (see the proof of Lemma 5.5) such that

- i) (s_∞) is integral.
- ii) The intersections $(s_\infty) \cap T$ and $(s_\infty) \cap (t_\pm)$ are proper and integral.
- iii) $(s_\infty) \supset S$.
- iv) $(s_\infty) \cap \Delta(C) \cap D = \emptyset$.
- v) $X \setminus (s_\infty)$ is affine.
- vi) (s_∞) is smooth away from S .
- vii) $(s_\infty) \cap T$ is smooth away from $\Delta(C)$.
- viii) $(s_\infty) \cap T \not\subset \Delta(C)$.

As shown in the proof of Lemma 5.5, it follows from (3), iv) and vii) above that $(s_\infty^T) := (s_\infty) \cap T$ is smooth along D . Using v), we can lift $f_\pm \in k(C_\pm)^\times$ to regular functions F_\pm on $U = X \setminus (s_\infty)$. Using an argument identical to that given in the proof of Proposition 5.7, we can extend the sections $s_{0,\pm} = s_\infty^N F_\pm$ (for some $N \gg 0$) to global sections $s_{0,\pm}$ of \mathcal{M}^N on X so that their zero loci satisfy:

- a) $(s_{0,\pm})$ and $(s_{0,\pm}) \cap T$ are integral.
- b) $(s_{0,\pm}) \cap T \cap \Delta(C) \cap D = \emptyset$.
- c) $(s_{0,\pm}) \cap T \not\subset \Delta(C)$.
- d) $(s_{0,\pm}) \cap T$ are smooth away from $\Delta(C)$.

As we argued in the proof of Lemma 5.5, it follows from iv), vii), c) and d) that (s_∞^T) and $(s_{0,\pm}^T) := (s_{0,\pm}) \cap T$ are smooth along D .

Setting $H_\pm = s_{0,\pm}/s_\infty^N$ and using the argument of the proof of Proposition 5.7, we get $H_\pm \in k(X)^\times$ and they restrict to rational functions on C_+ as well as C_- which are regular and invertible along D . Moreover, we have

$$(5.10) \quad (H_+)|_E = (F_+)|_E = (f_+)|_E = {}^\dagger (f_-)|_E = (F_-)|_E = (H_-)|_E,$$

where $=^\dagger$ follows from our assumption that $f \in \mathcal{O}_{C,E}^\times$. It follows that the restriction of the rational function $s_{0,-}/s_{0,+} = H_-/H_+$ to C_- is an element of $\mathcal{O}_{C_-,C_- \cap D}^\times$ such that $(H_-/H_+)_{|E} = 1$.

We now compute

$$\begin{aligned}
\tau_X^*(\operatorname{div}(f)) &= \iota_+^*(\operatorname{div}(f)) - \iota_-^*(\operatorname{div}(f)) \\
&= \operatorname{div}(f_+) - \operatorname{div}(f_-) \\
&= [(s_{0,+}^T \cdot C_+ - N(s_\infty^T) \cdot C_+) - [(s_{0,-}^T) \cdot C_- - N(s_\infty^T) \cdot C_-]] \\
&= [(s_{0,+}^T) \cdot C_+ - (s_{0,+}^T) \cdot C_-] - [(s_{0,-}^T) \cdot C_- - (s_{0,+}^T) \cdot C_-] \\
&\quad - N[(s_\infty^T) \cdot C_+ - (s_\infty^T) \cdot C_-] \\
&= [((s_{0,+}^T) \cdot (C_+ - C_-))] - [C_- \cdot ((s_{0,-}^T) - (s_{0,+}^T))] - N[(s_\infty^T) \cdot (C_+ - C_-)] \\
&\stackrel{\dagger}{=} [(s_{0,+}^T) \cdot ((t_+^T) - (t_-^T))] - [C_- \cdot ((s_{0,-}^T) - (s_{0,+}^T))] - N[(s_\infty^T) \cdot ((t_+^T) - (t_-^T))] \\
&= (s_{0,+}^T) \cdot (\operatorname{div}(t_+^T/t_-^T)) - C_- \cdot (\operatorname{div}(s_{0,-}/s_{0,+})) - N(s_\infty^T) \cdot (\operatorname{div}(t_+^T/t_-^T)) \\
&= (s_{0,+}^T) \cdot (\operatorname{div}(t_+^T/t_-^T)) - C_- \cdot (\operatorname{div}(H_-/H_+)) - N(s_\infty^T) \cdot (\operatorname{div}(t_+^T/t_-^T)),
\end{aligned}$$

where $=^\dagger$ follows from (5.9).

It follows from iv) and b) that t_+^T/t_-^T restricts to regular and invertible functions on $(s_{0,+}^T)$ and (s_∞^T) along D . Since $t \in H^0(S_X, \mathcal{L})$ and $t_\pm = \iota_\pm^*(t) \in H^0(X, \iota_\pm^*(\mathcal{L}))$, it follows that $(t_+)_|D = \iota_D^*(t) = (t_-)_|D$, where $\iota_D = \iota_+ \circ \iota = \iota_- \circ \iota: D \hookrightarrow S_X$ denotes the inclusion map. In particular, $(t_+^T/t_-^T)_|E = 1$. We have seen before that $(\frac{H_-}{H_+})_{|C_-}$ is a regular and invertible function on C_- along D and $(\frac{H_-}{H_+})_{|E} = 1$.

We set $h_1 = (\frac{t_+^T}{t_-^T})_{|(s_{0,+}^T)}$, $h_2 = (\frac{H_-}{H_+})_{|C_-}$ and $h_3 = (\frac{t_+^T}{t_-^T})_{|s_\infty^T}$. We conclude now using exactly

the same argument as in the proof of Proposition Proposition 5.7. Let $(s_{0,+}^T)^N \rightarrow (s_{0,+}^T)$, $(C_-)^N \rightarrow C_-$ and $(s_\infty^T)^N \rightarrow (s_\infty^T)$ denote the normalization maps. Let $\nu_1: (s_{0,+}^T)^N \rightarrow X$, $\nu_2: (C_-)^N \rightarrow X$ and $\nu_3: (s_\infty^T)^N \rightarrow X$ denote the composite maps. The curves $(s_{0,+}^T)$ and (s_∞^T) are all smooth along D , and C_- is smooth along D by Lemma 5.5. It follows that $h_1 \in G((s_{0,+}^T)^N, \nu_1^*(D))$, $h_2 \in G((C_-)^N, \nu_2^*(D))$ and $h_3 \in G((s_\infty^T)^N, \nu_3^*(D))$. We conclude from (5.1) that $\tau_X^*(\operatorname{div}(f))$ dies in $\operatorname{CH}_0(X|D)$. In particular, τ_X^* descends to a map $\tau_X^*: \operatorname{CH}_0^{LW}(S_X) \rightarrow \operatorname{CH}_0(X|D)$. This finishes the proof. \square

5.7. The maps $p_{\pm,*}$. Let X and D and k be as above. Now that we have constructed the map τ_X^* , we build two maps in the opposite direction.

$$p_{\pm,*}: \mathcal{Z}_0(X|D) \rightrightarrows \mathcal{Z}_0(S_X, D)$$

by $p_{+,*}([x]) = \iota_{+,*}([x])$ (resp. by $p_{-,*}([x]) = \iota_{-,*}([x])$) for a closed point $x \in X \setminus D$. Concretely, the two maps $p_{+,*}$ and $p_{-,*}$ copy a cycle α in one of the two components of the double S_X (the X_+ or the X_- copy). Since α is supported outside D (by definition of $\mathcal{R}_0(X|D)$), the cycles $p_{+,*}(\alpha)$ and $p_{-,*}(\alpha)$ give classes in $\operatorname{CH}_0(S_X)$.

Proposition 5.9. *The maps $p_{\pm,*}$ descend to group homomorphisms $p_{\pm,*}: \operatorname{CH}_0(X|D) \rightarrow \operatorname{CH}_0(S_X)$.*

Proof. Let $\nu: C \rightarrow X$ be a finite map from a normal integral curve such that $\nu(C) \not\subset D$ and let $E = \nu^*(D)$. Since both X and C are smooth, the map ν is automatically a local complete intersection. Let $f \in k(C)^\times$ be a rational function on C such that $f \in G(C, E)$ (in the notations of 5.2).

Since C is smooth, it follows from Proposition 2.3 that $S_C := S(C, E)$ is reduced and is smooth away from E . It follows from Proposition 2.4 that the double map $S_C \rightarrow S_X$ is l.c.i. along D . In particular, the pair (S_C, E) is a good curve relative to (S_X, D) . We now consider

the rational function $h = (h_+, h_-) := (f, 1)$ on S_C . The modulus condition satisfied by f on C guarantees that h is regular and invertible along $E \subset S_C$. It is also easy to see that the divisor of h trivializes $p_{+,*}(\nu_* \operatorname{div}(f))$. The argument for $p_{-,*}$ is symmetric. \square

To summarize, we have shown the following.

Theorem 5.10. *Let X be a smooth connected quasi-projective scheme of dimension $d \geq 1$ over an infinite perfect field k and let $D \subset X$ be an effective Cartier divisor. Then there are maps*

$$(5.11) \quad \tau_X^* : \operatorname{CH}_0(S_X) \rightarrow \operatorname{CH}_0(X|D) \quad \text{and} \quad p_{\pm,*} : \operatorname{CH}_0(X|D) \rightarrow \operatorname{CH}_0(S_X).$$

such that $\tau_X^*(\alpha) = \iota_+^*(\alpha) - \iota_-^*(\alpha)$ and $p_{\pm,*}(\beta) = \iota_{\pm,*}(\beta)$ on cycles.

Proof. This is a combination of Lemma 5.2 and of Propositions 5.7, 5.8 and 5.9. \square

6. REDUCTION TO INFINITE BASE FIELD

In the previous section, our results were based on the assumption that the ground field k is infinite. In order to prove our main theorem for finite fields, we shall use the following descent tricks for cycles on singular varieties and cycles with modulus.

Proposition 6.1. *Let $k \hookrightarrow k'$ be separable algebraic (possibly infinite) extension of fields. Let X be a reduced quasi-projective scheme over k and let $X' = X_{k'} := X \otimes_k k'$. Let $\operatorname{pr}_{k'/k} : X' \rightarrow X$ be the projection map. Then the following hold.*

- (1) *There exist pull-back maps $\operatorname{pr}_{k'/k}^* : \operatorname{CH}_0^{LW}(X) \rightarrow \operatorname{CH}_0^{LW}(X')$ and $\operatorname{pr}_{k'/k}^* : \operatorname{CH}_0(X) \rightarrow \operatorname{CH}_0(X')$ which commute with the canonical map $\operatorname{CH}_0^{LW}(X) \rightarrow \operatorname{CH}_0(X)$.*
- (2) *If there exists a sequence of separable field extensions $k = k_0 \subset k_1 \subset \cdots \subset k'$ with $k' = \cup_i k_i$ such that $X_i := X_{k_i}$ for each $i \geq 1$, then we have $\varinjlim_i \operatorname{CH}_0(X_i) \xrightarrow{\cong} \operatorname{CH}_0(X')$.*

The same holds for $\operatorname{CH}_0^{LW}(-)$ as well.

- (3) *If $k \hookrightarrow k'$ is finite, then there exists a push-forward $(\operatorname{pr}_{k'/k})_* : \operatorname{CH}_0(X') \rightarrow \operatorname{CH}_0(X)$ such that $(\operatorname{pr}_{k'/k})_* \circ \operatorname{pr}_{k'/k}^*$ is multiplication by $[k' : k]$.*

Proof. The proofs of (1) and (2) for $\operatorname{CH}_0(-)$ and $\operatorname{CH}_0^{LW}(-)$ are identical. So we consider only $\operatorname{CH}_0(-)$ in the proof below.

We first note that as $k \hookrightarrow k'$ or $k \hookrightarrow k_i$ is a separable algebraic extension, the scheme X' is reduced and $X'_{\operatorname{sing}} = X_{\operatorname{sing}} \times_k k'$. The same holds for each X_i as well.

Let $x \in X \setminus X_{\operatorname{sing}}$ be a closed point. Since $\operatorname{pr}_{k'/k}$ is flat, it follows from our hypothesis that $\operatorname{pr}_{k'/k}^*([x])$ is a well defined 0-cycle in $\mathcal{Z}_0(X', X'_{\operatorname{sing}})$. We thus have a pull-back map $\operatorname{pr}_{k'/k}^* : \mathcal{Z}_0(X, X_{\operatorname{sing}}) \rightarrow \mathcal{Z}_0(X', X'_{\operatorname{sing}})$.

We show that this map preserves rational equivalences. So let $\nu : (C, Z) \rightarrow (X, X_{\operatorname{sing}})$ be a good curve and let $f : C \dashrightarrow \mathbb{P}_k^1$ be a rational function which is regular in an open neighborhood $Z \subsetneq U \subseteq C$. The base change via $k \hookrightarrow k'$ gives a diagram of Cartesian squares

$$(6.1) \quad \begin{array}{ccccccc} Z_{k'} & \hookrightarrow & U_{k'} & \hookrightarrow & C_{k'} & \xrightarrow{\nu_{k'}} & X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \operatorname{pr}_{k'/k} \\ Z & \hookrightarrow & U & \hookrightarrow & C & \xrightarrow{\nu} & X. \end{array}$$

Since $k \hookrightarrow k'$ is a separable field extension, we see that $C_{k'}$ is reduced and $C_{k'} \setminus Z_{k'} = (C \setminus Z)_{k'}$ is regular. We also have $\nu_{k'}^{-1}(X'_{\operatorname{sing}}) = \nu_{k'}^{-1}(X_{\operatorname{sing}} \times_k k') \subseteq Z_{k'}$. Moreover, the flatness of $\operatorname{pr}_{k'/k}$ ensures that the map $\nu_{k'} : C_{k'} \rightarrow X'$ is l.c.i. over X'_{sing} . It follows that $\nu_{k'} : (C_{k'}, Z_{k'}) \rightarrow (X', X'_{\operatorname{sing}})$ is a good curve. Since $f : C \dashrightarrow \mathbb{P}_k^1$ is regular and invertible

along U , it follows that $f' := f_{k'} : C_{k'} \dashrightarrow \mathbb{P}_{k'}^1$ is a rational function which is regular and invertible along $U_{k'}$.

Let $D \subset C \times_k \mathbb{P}_k^1$ denote the closure of the graph of f with the reduced structure. Note that this graph is already reduced over the regular locus of f . Let $p : C \times_k \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ and $q : C \times_k \mathbb{P}_k^1 \rightarrow C$ denote the projection maps so that $p|_U = f|_U$. We thus have a commutative diagram

$$(6.2) \quad \begin{array}{ccccccc} \mathbb{P}_{k'}^1 & \xleftarrow{p_{k'}} & D_{k'} & \xrightarrow{q_{k'}} & C_{k'} & \xrightarrow{\nu_{k'}} & X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \text{pr}_{k'/k}^* \\ \mathbb{P}_k^1 & \xleftarrow{p} & D & \xrightarrow{q} & C & \xrightarrow{\nu} & X. \end{array}$$

We now have

$$\begin{aligned} \nu_{k'*}(\text{div}(f')) &= \nu_{k'*} \circ q_{k'*}([p_{k'}^*(0)] - [p_{k'}^*(\infty)]) \\ &= \nu_{k'*} \circ q_{k'*}([p_{k'}^* \circ \text{pr}_{k'/k}^*(0) - p_{k'}^* \circ \text{pr}_{k'/k}^*(\infty)]) \\ &= \nu_{k'*} \circ q_{k'*}([p_{k'/k}^* \circ p^*(0) - p_{k'/k}^* \circ p^*(\infty)]) \\ &=^\dagger \nu_{k'*} \circ \text{pr}_{k'/k}^* \circ q_*([p^*(0) - p^*(\infty)]) \\ &= \nu_{k'*} \circ \text{pr}_{k'/k}^*(\text{div}(f)) \\ &=^{\dagger\dagger} \text{pr}_{k'/k}^* \circ \nu_*((\text{div}(f))), \end{aligned}$$

where $=^\dagger$ and $=^{\dagger\dagger}$ follow from [12, Proposition 1.7] because all squares in (6.2) are Cartesian, vertical maps are all flat and q as well as ν is finite. We conclude that $\text{pr}_{k'/k}^* \circ \nu_*((\text{div}(f))) \in \mathcal{R}_0(X', X'_{\text{sing}})$ and this proves (1).

We now consider (2). It is clear that the map $\lim_{\substack{\rightarrow \\ i}} \text{CH}_0(X_i) \rightarrow \text{CH}_0(X')$ is surjective. To show injectivity, suppose there is some $i \geq 0$ and $\alpha \in \mathcal{Z}_0(X_i, (X_i)_{\text{sing}})$ such that $\text{pr}_{k'/k_i}^*(\alpha) \in \mathcal{R}_0(X', X'_{\text{sing}})$. We can replace k by k_i and assume $i = 0$.

Let $\nu^j : (C^j, Z^j) \rightarrow (X', X'_{\text{sing}})$ be good curves and let $f^j : C^j \dashrightarrow \mathbb{P}_{k'}^1$ be a rational function which is regular and invertible in a neighborhood U^j of Z^j for $j = 1, \dots, r$ such that $\text{pr}_{k'/k}^*(\alpha) = \sum_{j=1}^r \nu_*^j(\text{div}(f^j))$.

Since each C^j has a factorization $C^j \hookrightarrow \mathbb{P}_{X'}^{N_j} \rightarrow X'$, we can find some $i \gg 0$ and curves W^j over k_i , a closed subscheme $T^j \subsetneq W^j$, an open neighborhood $V^j \subseteq W^j$ of T^j and invertible regular function $g^j : V^j \rightarrow \mathbb{P}_{k_i}^1$ such that $(C^j, Z^j) \simeq (W^j, T^j)_{k'}$, $U^j = V_{k'}^j$ and $f^j = g_{k'}^j$. Furthermore, we can find a finite map $\delta^j : (W^j, T^j) \rightarrow (X_{k_i}, (X_{k_i})_{\text{sing}})$ such that $(C^j, Z^j) \simeq (W^j, T^j)_{k'}$ and $\nu^j = \delta_{k'}^j$ for each $j = 1, \dots, r$. Since $k_i \hookrightarrow k'$ is separable, it also follows that $W^j \setminus T^j$ is regular.

Since the map $X' \rightarrow X_i$ is faithfully flat, it follows from Lemma 3.2 and our hypothesis on $(X_i)_{\text{sing}}$ that each $W^j \rightarrow X_i$ is l.c.i. along $(X_i)_{\text{sing}}$ and $(\delta^j)^{-1}((X_i)_{\text{sing}}) \subseteq T^j$. It follows that each (W^j, T^j) is a good curve relative to $(X_{k_i}, (X_{k_i})_{\text{sing}})$. Moreover, we have shown in the proof of (1) (with k replaced by k_i) in this situation that $\nu_*^j(\text{div}(f^j)) = \text{pr}_{k'/k_i}^*(\delta_*^j(\text{div}(g^j)))$.

We now set $\alpha_i = \text{pr}_{k_i/k}^*(\alpha)$ and let $\beta = \alpha_i - \sum_{j=1}^r \delta_*^j(\text{div}(g^j)) \in \mathcal{Z}_0(X_i, (X_i)_{\text{sing}})$. It then follows that $\text{pr}_{k'/k_i}^*(\beta) = \text{pr}_{k'/k_i}^*(\alpha_i) - \sum_{j=1}^r \nu_*^j(\text{div}(f^j)) = \text{pr}_{k'/k}^*(\alpha) - \sum_{j=1}^r \nu_*^j(\text{div}(f^j)) = 0$ in $\mathcal{Z}_0(X', X'_{\text{sing}})$. Since the map $\text{pr}_{k'/k_i}^* : \mathcal{Z}_0(X_i, (X_i)_{\text{sing}}) \rightarrow \mathcal{Z}_0(X', X'_{\text{sing}})$ of free abelian groups is clearly injective, we get $\beta = 0$, which means that $\alpha_i \in \mathcal{R}_0(X_i, (X_i)_{\text{sing}})$. This proves (2).

If $k \hookrightarrow k'$ is a finite extension, then $\mathrm{Spec}(k') \rightarrow \mathrm{Spec}(k)$ is an l.c.i. morphism. In particular, it follows from Lemma 3.2 that $\mathrm{pr}_{k'/k} : X' \rightarrow X$ is finite, flat l.c.i. morphism. The push-forward map $(\mathrm{pr}_{k'/k})_* : \mathrm{CH}_0(X') \rightarrow \mathrm{CH}_0(X)$ now follows from our hypothesis on the singular locus of X' and Proposition 3.18. The property $(\mathrm{pr}_{k'/k})_* \circ \mathrm{pr}_{k'/k}^* = [k' : k]$ is obvious. This finishes the proof. \square

Proposition 6.2. *Let $k \hookrightarrow k'$ be a separable algebraic (possibly infinite) extension of fields. Let X be a smooth quasi-projective scheme over k with an effective Cartier divisor D . Let $X' = X_{k'}$ and $D' = D_{k'}$ denote the base change of X and D , respectively. Let $\mathrm{pr}_{k'/k} : X' \rightarrow X$ be the projection map. Then the following hold.*

- (1) *There exists a pull-back $\mathrm{pr}_{k'/k}^* : \mathrm{CH}_0(X|D) \rightarrow \mathrm{CH}_0(X'|D')$.*
- (2) *If there exists a sequence of separable field extensions $k = k_0 \subset k_1 \subset \cdots \subset k'$ with $k' = \bigcup_i k_i$, then we have $\varinjlim_i \mathrm{CH}_0(X_{k_i}|D_{k_i}) \xrightarrow{\cong} \mathrm{CH}_0(X'|D')$.*
- (3) *If $k \hookrightarrow k'$ is finite, then there exists a push-forward $\mathrm{pr}_{k'/k}^* : \mathrm{CH}_0(X'|D') \rightarrow \mathrm{CH}_0(X|D)$ such that $(\mathrm{pr}_{k'/k})_* \circ \mathrm{pr}_{k'/k}^*$ is multiplication by $[k' : k]$.*

Proof. Let $x \in X \setminus D$ be a closed point. Since $\mathrm{pr}_{k'/k}$ is smooth, it follows from our hypothesis that $\mathrm{pr}_{k'/k}^*([x])$ is a well defined 0-cycle in $\mathcal{Z}_0(X'|D')$. We thus have a pull-back map $\mathrm{pr}_{k'/k}^* : \mathcal{Z}_0(X|D) \rightarrow \mathcal{Z}_0(X'|D')$. To show that this map preserves rational equivalence, we shall use a presentation of $\mathcal{R}_0(X|D)$ which is different from the one given in § 5.2 (see [5] or [26]).

Let $C \hookrightarrow X \times_k \mathbb{P}_k^1$ be an integral curve satisfying the following properties.

- (1) $C \cap (D \times_k \mathbb{P}_k^1)$ is finite.
- (2) $C \cap (D \times_k \{0, \infty\}) = \emptyset$.
- (3) The Weil divisor $\nu^*(X \times \{1\}) - \nu^*(D \times \mathbb{P}_k^1)$ is effective, where $\nu : C^N \rightarrow C \hookrightarrow X \times_k \mathbb{P}_k^1$ is the composite finite map.

The group of rational equivalences $\mathcal{R}_0(X|D)$ coincides with the subgroup of $\mathcal{Z}_0(X|D)$ generated by $[C_0] - [C_\infty]$, where C runs over all curves as above.

Let $C \in X \times_k \mathbb{P}_k^1$ be any such curve. It follows again from the smoothness of $\mathrm{pr}_{k'/k}$ that $C' = \mathrm{pr}_{k'/k}^*(C) = C_{k'} \hookrightarrow (X \times_k \mathbb{P}_k^1)_{k'} = X' \times_{k'} \mathbb{P}_{k'}^1$ satisfies conditions (1)-(3) above. In particular, $[C'_0] - [C'_\infty]$ dies in $\mathrm{CH}_0(X'|D')$. However, the flatness of $\mathrm{pr}_{k'/k}$ again shows that $[C'_\infty] = \mathrm{pr}_{k'/k}^*([C_\infty])$ and $[C'_0] = \mathrm{pr}_{k'/k}^*([C_0])$ so that $\mathrm{pr}_{k'/k}^*([C_0] - [C_\infty])$ dies in $\mathrm{CH}_0(X'|D')$. This proves (1).

It is clear that the map $\varinjlim_i \mathrm{CH}_0(X_i|D_i) \rightarrow \mathrm{CH}_0(X'|D')$ is surjective. To show injectivity, suppose there is some $i \geq 0$ and $\alpha \in \mathcal{Z}_0(X_i|D_i)$ such that $\mathrm{pr}_{k'/k_i}^*(\alpha) \in \mathcal{R}_0(X'|D')$. We can replace k by k_i and assume $i = 0$.

Let $C^j \hookrightarrow X' \times_{k'} \mathbb{P}_{k'}^1 = (X \times_k \mathbb{P}_k^1)_{k'}$ for $j = 1, \dots, r$ be a collection of curves as in the proof of (1) so that $\mathrm{pr}_{k'/k}^*(\alpha) = (\sum_{j=1}^r [C_0^j]) - (\sum_{j=1}^r [C_\infty^j])$. Let $\nu^j : C^{j,N} \rightarrow X' \times_{k'} \mathbb{P}_{k'}^1$ denote the maps from the normalizations of the above curves.

We can then find some $i \gg 0$ and integral curves $W^j \hookrightarrow X_i \times_{k_i} \mathbb{P}_{k_i}^1$ such that $C^j = W^j \times_{k_i} k'$ for each $j = 1, \dots, r$. In particular, we have $C_0^j = \mathrm{pr}_{k'/k_i}^*(W_0^j)$ and $C_\infty^j = \mathrm{pr}_{k'/k_i}^*(W_\infty^j)$ for $j = 1, \dots, r$. Since pr_{k'/k_i} is smooth, it follows that $C^{j,N} = W_{k'}^{j,N}$ for each j . Moreover, it follows from [26, Lemma 2.2] that condition (3) above holds on each $W^{j,N}$. It follows that each W^j defines a rational equivalence for 0-cycles with modulus D_i on X_i .

We now set $\alpha_i = \text{pr}_{k_i/k}^*(\alpha)$ and let $\beta = \alpha_i - (\sum_{j=1}^r ([W_0^j] - [W_\infty^j])) \in \mathcal{Z}_0(X_i|D_i)$. It then follows that $\text{pr}_{k'/k_i}^*(\beta) = \text{pr}_{k'/k_i}^*(\alpha_i) - \sum_{j=1}^r ([C_0^j] - [C_\infty^j]) = \text{pr}_{k'/k}^*(\alpha) - \sum_{j=1}^r ([C_0^j] - [C_\infty^j]) = 0$ in $\mathcal{Z}_0(X'|D')$. Since the map $\text{pr}_{k'/k_i}^* : \mathcal{Z}_0(X_i|D_i) \rightarrow \mathcal{Z}_0(X'|D')$ of free abelian groups is clearly injective, we get $\beta = 0$, which means that $\alpha_i \in \mathcal{R}_0(X_i|D_i)$. This proves (2). The existence of push-forward is already known as remarked above and the formula $(\text{pr}_{k'/k})_* \circ \text{pr}_{k'/k}^* = [k' : k]$ is obvious from the definitions. \square

7. THE MAIN RESULTS ON THE CHOW GROUPS OF 0-CYCLES

In this section, we apply the technical results of the previous section to prove our main theorem on the Chow groups of 0-cycles with modulus and the Chow group of 0-cycles on singular varieties. We shall also derive our first set of applications. We shall derive a new presentation of the Chow group of 0-cycles with modulus and prove our main comparison theorem for the two Chow groups of 0-cycles for the double.

Theorem 7.1. *Let k be a field and let X be a smooth quasi-projective scheme of dimension $d \geq 1$ over k with an effective Cartier divisor $D \subset X$.*

Then, there are maps

$$(7.1) \quad \Delta^* : \text{CH}_0(X) \rightarrow \text{CH}_0(S_X); \quad \iota_\pm^* : \text{CH}_0(S_X) \rightarrow \text{CH}_0(X) \quad \text{and} \\ p_{\pm,*} : \text{CH}_0(X|D) \rightarrow \text{CH}_0(S_X)$$

such that $\iota_\pm^ \circ \Delta^* = \text{Id}$ on $\text{CH}_0(X)$.*

If k is perfect, then there is also a map

$$(7.2) \quad \tau_X^* : \text{CH}_0(S_X) \rightarrow \text{CH}_0(X|D)$$

such that $\tau_X^ \circ p_{\pm,*} = \pm \text{Id}$ on $\text{CH}_0(X|D)$. Moreover, the sequences*

$$(7.3) \quad 0 \rightarrow \text{CH}_0(X|D) \xrightarrow{p_{+,*}} \text{CH}_0(S_X) \xrightarrow{\iota_+^*} \text{CH}_0(X) \rightarrow 0$$

and

$$(7.4) \quad 0 \rightarrow \text{CH}_0(X) \xrightarrow{\Delta^*} \text{CH}_0(S_X) \xrightarrow{\tau_X^*} \text{CH}_0(X|D) \rightarrow 0$$

are split exact.

Proof. We can clearly assume that X is connected. All the maps (except (7.2)) involved in the theorem are well defined thanks to the results of the previous sections. On the level of cycles, we clearly have $\iota_\pm^* \circ \Delta^* = \text{Id}$ on $\mathcal{Z}_0(X, D)$ and $\tau_X^* \circ p_{\pm,*} = \pm \text{Id}$ on $\mathcal{Z}_0(X|D)$. We are only thus left with proving (7.2) when k is finite and the exactness of the two sequences in general.

Note that the maps $\tau_X^* : \mathcal{Z}_0(S_X, D) \rightarrow \mathcal{Z}_0(X|D) \rightarrow \text{CH}_0(X|D)$ are defined over any field and for any field extension $k \hookrightarrow k'$, the diagram

$$(7.5) \quad \begin{array}{ccc} \mathcal{Z}_0(S_X, D) & \xrightarrow{\iota_\pm^*} & \mathcal{Z}_0(X|D) \\ \text{pr}_{k'/k}^* \downarrow & & \downarrow \text{pr}_{k'/k}^* \\ \mathcal{Z}_0(S_{X'}, D') & \xrightarrow{\iota_\pm^*} & \mathcal{Z}_0(X'|D') \end{array}$$

commutes, where $X' = X_{k'}$. In particular, we have $\tau_{X'}^* \circ \text{pr}_{k'/k}^* = \text{pr}_{k'/k}^* \circ \tau_X^*$.

We have to show that the composite map $\tau_X^* : \mathcal{Z}_0(S_X, D) \rightarrow \text{CH}_0(X|D)$ kills the rational equivalences, assuming k is finite. Let us therefore assume that $\nu : (C, Z) \rightarrow (S_X, D)$ is a

good curve and that $f : C \dashrightarrow \mathbb{P}_k^1$ is a rational function which is regular and invertible on a neighborhood of Z . Let $\alpha = \nu_*(\text{div}(f))$. We need to show that $\tau_X^*(\alpha) = 0$ in $\text{CH}_0(X|D)$.

We choose two distinct primes ℓ_1 and ℓ_2 different from $\text{char}(k)$ and let k_i denote the pro- ℓ_i extension of k for $i = 1, 2$. Since each k_i is a limit of finite separable extensions of the perfect field k , the hypotheses of Propositions 6.1 and 6.2 are satisfied.

It follows from Proposition 6.1 that $\text{pr}_{k_i/k}^*(\alpha) \in \mathcal{R}_0(S_{X_{k_i}}, D_{k_i})$ for $i = 1, 2$. It follows from the case of infinite perfect fields and (7.5) that $\tau_{X_{k_i}}^*(\alpha_{k_i}) = 0$ for $i = 1, 2$. Equivalently, $\text{pr}_{k_i/k}^* \circ \tau_X^*(\alpha) = 0$ for $i = 1, 2$. Using Proposition 6.2, we can find two finite extensions k'_1 and k'_2 of k of relatively prime degrees such that $\text{pr}_{k'_i/k}^* \circ \tau_X^*(\alpha) = 0$ for $i = 1, 2$. We conclude by applying Proposition 6.2 once again $\tau_X^*(\alpha) = 0$ in $\text{CH}_0(X|D)$. This proves (7.2).

Now we prove the split exactness of the two sequences in the theorem. Since a cycle $\mathcal{Z}_0(X|D)$ does not meet D , it is clear that $\iota_-^* \circ p_{+,*} = 0$. Similarly, $\tau_X^* \circ \Delta^* = 0$ by definitions of these maps and Lemma 3.20. Using the first part of the theorem, we only have to show that both sequences are exact at their middle terms.

Let $\gamma \in \text{CH}_0(S_X)$. We can write $\gamma = \alpha_+ + \beta_-$, where α_+ is a cycle supported on the component $\iota_+(X)$ and β_- is a cycle supported on the component $\iota_-(X)$. We see then that $\gamma = p_{+,*}(\alpha - \beta) + \Delta^*(\beta)$, where α and β are the cycles α_+ and β_- seen in $X \setminus D$ (identifying the two copies of X), so that every element in the kernel of ι_-^* is clearly in the image of $p_{+,*}$. We have therefore shown that the sequence (7.3) is split exact.

Next, suppose $\alpha \in \text{CH}_0(S_X)$ is such that $\tau_X^*(\alpha) = 0$. Since (7.3) is split exact, as we just showed, we can write $\alpha = p_{+,*}(\alpha_1) + \Delta^*(\alpha_2)$. We then have

$$\begin{aligned} \tau_X^*(\alpha) &= 0 \\ \Leftrightarrow \tau_X^* \circ p_{+,*}(\alpha_1) + \tau_X^* \circ \Delta^*(\alpha_2) &= 0 \\ \Leftrightarrow \alpha_1 + 0 &= 0 \\ \Leftrightarrow \alpha &= \Delta^*(\alpha_2). \end{aligned}$$

We have therefore shown that the sequence (7.4) is split exact. \square

7.1. A refinement of the definition of 0-cycles with modulus. As a consequence of Theorem 7.1, we now give the following simplified presentation of the Chow group of 0-cycles with modulus when the ground field is infinite and perfect.

Let X be a smooth quasi-projective scheme of dimension $d \geq 1$ over an infinite perfect field k and let $D \subset X$ be an effective Cartier divisor. Let $\mathcal{R}_0^{\text{mod}}(X|D) \subset \mathcal{Z}_0(X|D)$ be the subgroup generated by $\text{div}_C(f)$, where $C \subset X$ is an integral curve not contained in D and is smooth along D and $f \in \text{Ker}(\mathcal{O}_{C,D}^\times \rightarrow \mathcal{O}_{C \cap D}^\times)$. Here, $\mathcal{O}_{C,D}$ denotes the semi-local ring of C at $(C \cap D) \cup \{\eta\}$ with η being the generic point of C . Set $\text{CH}_0^{\text{mod}}(X|D) = \frac{\mathcal{Z}_0(X|D)}{\mathcal{R}_0^{\text{mod}}(X|D)}$. There is an evident surjection $\text{CH}_0^{\text{mod}}(X|D) \twoheadrightarrow \text{CH}_0(X|D)$.

Corollary 7.2. *Let X be as above. Then the map $\text{CH}_0^{\text{mod}}(X|D) \rightarrow \text{CH}_0(X|D)$ is an isomorphism.*

Proof. Under the given assumption, we get maps

$$\text{CH}_0^{\text{mod}}(X|D) \rightarrow \text{CH}_0(X|D) \xrightarrow{p_{+,*}} \text{CH}_0(S_X) \xrightarrow{\tau_X^*} \text{CH}_0(X|D).$$

The proofs of Propositions 5.7 and 5.8 show that τ_X^* actually factors through the map $\text{CH}_0(S_X) \rightarrow \text{CH}_0^{\text{mod}}(X|D)$. We thus get maps

$$\text{CH}_0^{\text{mod}}(X|D) \rightarrow \text{CH}_0(X|D) \xrightarrow{p_{+,*}} \text{CH}_0(S_X) \xrightarrow{\tau_X^*} \text{CH}_0^{\text{mod}}(X|D),$$

whose composite is clearly the identity. In particular, the map $\text{CH}_0^{\text{mod}}(X|D) \rightarrow \text{CH}_0(X|D)$ is injective. The corollary now follows. \square

7.2. The comparison theorem. Using the modified presentation of $\mathrm{CH}_0(X|D)$ from Corollary 7.2, we prove the following comparison theorem for the two Chow groups of the double.

Theorem 7.3. *Let k be an infinite perfect field and let X be a smooth quasi-projective scheme of dimension $d \geq 1$ over k with an effective Cartier divisor D . Then the canonical map $\mathrm{CH}_0^{LW}(S_X) \rightarrow \mathrm{CH}_0(S_X)$ is an isomorphism.*

Proof. In view of Lemma 3.12, we can assume $d \geq 2$. Recall from § 5.7 that there are two maps $p_{\pm,*}: \mathcal{Z}_0(X|D) \rightrightarrows \mathcal{Z}_0(S_X, D)$. As the first step in the proof of the theorem, we strengthen Proposition 5.9 by showing that these maps descend to group homomorphisms $p_{\pm,*}: \mathrm{CH}_0(X|D) \rightarrow \mathrm{CH}_0^{LW}(S_X)$. To show this, we can use Corollary 7.2 and replace $\mathrm{CH}_0(X|D)$ by $\mathrm{CH}_0^{\mathrm{mod}}(X|D)$.

So let $\nu: C \hookrightarrow X$ be an integral curve not contained in D which is smooth along D and let $f \in \mathrm{Ker}(\mathcal{O}_{C,D}^\times \rightarrow \mathcal{O}_{C \cap D}^\times)$. Since C is smooth along D , the inclusion ν is l.c.i. along D . Since C is reduced, it follows from Proposition 2.3 that $S_C := S(C, E)$ is reduced, where we let $E = \nu^*(D)$. It follows from Proposition 2.4 that the double map $\nu': S_C \hookrightarrow S_X$ is l.c.i. along D . In other words, $S_C \hookrightarrow S_X$ is a Cartier curve.

We now consider the rational function $h = (h_+, h_-) := (f, 1)$ on S_C . The modulus condition satisfied by f on C guarantees that h is regular and invertible along $E \subset S_C$. It is clear as in Proposition 5.9 that the divisor of h trivializes $p_{+,*}(\nu_* \mathrm{div}(f))$. The argument for $p_{-,*}$ is symmetric. We denote the maps $\mathrm{CH}_0(X|D) \rightarrow \mathrm{CH}_0^{LW}(S_X)$ obtained as above by $p_{\pm,*}^{LW}$. It is clear that the composite $\mathrm{CH}_0(X|D) \xrightarrow{p_{\pm,*}^{LW}} \mathrm{CH}_0^{LW}(S_X) \rightarrow \mathrm{CH}_0(S_X) \xrightarrow{\tau_X^*} \mathrm{CH}_0(X|D)$ are the identity maps (up to a sign).

Recall from Lemma 3.20 that $\mathrm{CH}_0(X)$ is the quotient of free abelian group on the closed points of $X \setminus D$ by the subgroup generated by $\mathrm{div}(f)$, where f is a rational function on an integral curve C not contained in D and f is regular invertible along D . Using an easier version of Lemma 5.5, one can now see that the rational equivalences for $\mathrm{CH}_0(X)$ can be defined by further restricting integral curves on X which are smooth along D . In particular, they are l.c.i. on X along D . Using such curves, one can check from the proof of Theorem 4.1 that the map $\Delta^*: \mathrm{CH}_0(X) \rightarrow \mathrm{CH}_0(S_X)$ can actually be lifted to the pull-back map $\Delta^{LW,*}: \mathrm{CH}_0(X) \rightarrow \mathrm{CH}_0^{LW}(S_X)$.

We next consider the composite maps $\mathrm{CH}_0^{LW}(S_X) \rightarrow \mathrm{CH}_0(S_X) \xrightarrow{\iota_\pm^*} \mathrm{CH}_0(X)$, which we denote by $\iota_\pm^{LW,*}$. It is then clear that $\iota_\pm^{LW,*} \circ \Delta^{LW,*} = \mathrm{Id}$ on $\mathrm{CH}_0(X)$. Now, the proof of (7.3) works in verbatim to give a split exact sequence

$$0 \rightarrow \mathrm{CH}_0(X|D) \xrightarrow{p_{+,*}^{LW}} \mathrm{CH}_0^{LW}(S_X) \xrightarrow{\iota_-^{LW,*}} \mathrm{CH}_0(X) \rightarrow 0.$$

We thus have a commutative diagram of split exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathrm{CH}_0(X|D) & \xrightarrow{p_{+,*}^{LW}} & \mathrm{CH}_0^{LW}(S_X) & \xrightarrow{\iota_-^{LW,*}} & \mathrm{CH}_0(X) \rightarrow 0 \\ & & \parallel & & \downarrow \text{can} & & \parallel \\ 0 & \rightarrow & \mathrm{CH}_0(X|D) & \xrightarrow{p_{+,*}} & \mathrm{CH}_0(S_X) & \xrightarrow{\iota_-^*} & \mathrm{CH}_0(X) \rightarrow 0. \end{array}$$

We conclude from this that the map $\mathrm{CH}_0^{LW}(S_X) \rightarrow \mathrm{CH}_0(S_X)$ is an isomorphism. \square

8. ALBANESE WITH MODULUS OVER \mathbb{C}

It is classically known that a smooth projective variety X over an algebraically closed field has an abelian variety, called the Albanese variety $\mathrm{Alb}(X)$ of X , associated to it, which is

Cartier dual to the Picard variety $\text{Pic}^0(X)$. The Albanese comes equipped with a (surjective) Abel-Jacobi map from the Chow group $\text{CH}_0(X)_{\text{deg } 0}$ of zero cycles of degree zero on X , that is universal among regular maps to abelian varieties. When X is a smooth projective curve with an effective Cartier divisor D on it, a universal regular quotient for what we now call the Chow group of zero cycles with modulus, $\text{CH}_0(X|D)_{\text{deg } 0}$, was already constructed by Serre in [45] under the name of the generalized Jacobian variety. Serre showed that this generalized Jacobian is a commutative algebraic group which is an extension of the Jacobian variety of the curve by a linear algebraic group.

If X is now a smooth projective variety of arbitrary dimension over the field of complex numbers \mathbb{C} and $D \subset X$ is an effective Cartier divisor such that D_{red} is a strict normal crossing divisor, a universal regular quotient of $\text{CH}_0(X|D)$ was constructed in [5] as a relative intermediate Jacobian $J_{X|D}^{\dim(X)}$. However, not many properties of this universal regular quotient are known and the techniques used in the construction are not known to generalize to cover the general case.

In this paper, we use our doubling trick to give a direct and explicit construction of the relative Albanese $\text{Alb}(X|D)$ and show that it is the universal regular quotient of the Chow group of 0-cycles with modulus. As a result of our construction, we are able to prove the Roitman torsion theorem for the Chow group of 0-cycles with modulus. In this section, we use the modified Deligne cohomology of Levine [31] to construct the universal regular quotient when the base field is \mathbb{C} . By Theorem 3.17, we can identify the Levine-Weibel Chow group of zero cycles with our modified definition 3.3.

8.1. Relative Deligne cohomology. Let X be a smooth projective connected scheme over \mathbb{C} . Let D be any effective Cartier divisor on it. As before, we write S_X for the double construction applied to the pair (X, D) . We shall frequently refer to the following square:

$$(8.1) \quad \begin{array}{ccc} D & \xrightarrow{j} & X \\ j \downarrow & & \downarrow i_- \\ X & \xrightarrow{i_+} & S_X \end{array}$$

Let $r \geq 1$ be an integer. Following Levine [31], we denote by $\mathbb{Z}(r)_{S_X}^{\mathcal{D}^*}$ (resp. $\mathbb{Z}(r)_D^{\mathcal{D}^*}$) the modified Deligne-Beilinson complex on S_X (resp. on D). Since S_X is projective, both are given by the “naive” Deligne complexes

$$\begin{aligned} \mathbb{Z}(r)_{S_X}^{\mathcal{D}^*} &= \mathbb{Z}(r)_{S_X} \rightarrow \mathcal{O}_{S_X} \xrightarrow{d} \Omega_{S_X}^1 \rightarrow \dots \rightarrow \Omega_{S_X}^{r-1} \\ \mathbb{Z}(r)_D^{\mathcal{D}^*} &= \mathbb{Z}(r)_D \rightarrow \mathcal{O}_D \xrightarrow{d} \Omega_D^1 \rightarrow \dots \rightarrow \Omega_D^{r-1} \end{aligned}$$

on $(S_X)_{\text{an}}$ (resp. on D_{an}), the analytic space associated to S_X (resp. to D).

We can consider the complex $\mathbb{Z}(r)_X^{\mathcal{D}^*}$ on X as well. Since X is smooth and projective, we have by definition $\mathbb{Z}(r)_X^{\mathcal{D}^*} = \mathbb{Z}(r)_X^{\mathcal{D}}$, where $\mathbb{Z}(r)_X^{\mathcal{D}}$ is the classical Deligne complex. Let $i_{\pm}: X \hookrightarrow S_X$ be the inclusions of the two components in the double S_X and let $\Delta: S_X \rightarrow X$ be the natural projection. We denote by $\pi: X|X \rightarrow S_X$ the map (i_+, i_-) from the normalization. Finally, we let $j: D \hookrightarrow X$ denote the inclusion of D in X . Recall from Proposition 2.3(8) that D is a conducting subscheme for π .

We define the following objects in the bounded derived category of complexes of sheaves of abelian groups on $(S_X)_{\text{an}}$ (resp. on X_{an}):

$$\begin{aligned} \mathbb{Z}(r)_{X|D}^{\mathcal{D}^*} &= \text{Cone}(\mathbb{Z}(r)_{S_X}^{\mathcal{D}^*} \xrightarrow{i_-^*} \mathbb{R}i_{-,*}\mathbb{Z}(r)_X^{\mathcal{D}})[-1] \\ \mathbb{Z}(r)_{(X,D)}^{\mathcal{D}^*} &= \text{Cone}(\mathbb{Z}(r)_X^{\mathcal{D}} \xrightarrow{j^*} \mathbb{R}j_*\mathbb{Z}(r)_D^{\mathcal{D}})[-1]. \end{aligned}$$

We define the complexes $\mathbb{Z}(r)_{X|D}$ and $\mathbb{Z}(r)_{(X,D)}$ in a similar fashion.

Note that the maps are all finite, so that the derived functors $\mathbb{R}i_{-,*}$ and $\mathbb{R}j_*$ are superfluous. Since all the maps over which we are taking the cones are actually surjective as maps of complexes, the cones can be computed directly as kernels in the category of complexes of sheaves on S_X . We have the following commutative diagram relating them

$$(8.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathbb{Z}(r)_{X|D}^{\mathcal{D}^*} & \longrightarrow & \mathbb{Z}(r)_{S_X}^{\mathcal{D}^*} & \xrightarrow{i_-^*} & \mathbb{Z}(r)_X^{\mathcal{D}} \rightarrow 0 \\ & & \Phi_{X,D} \downarrow & & \downarrow i_+^* & & \downarrow j^* \\ 0 & \rightarrow & \mathbb{Z}(r)_{(X,D)}^{\mathcal{D}^*} & \longrightarrow & \mathbb{Z}(r)_X^{\mathcal{D}} & \xrightarrow{j^*} & \mathbb{Z}(r)_D^{\mathcal{D}^*} \rightarrow 0. \end{array}$$

8.2. Construction of the Albanese varieties. Let $r = d = \dim(X)$. In order to simplify the notation, we denote by $\overline{S_X} \xrightarrow{\pi} S_X$ the normalization of S_X , i.e., the disjoint union of two copies of X . This gives an induced map $\pi_D: \overline{D} = D \amalg D \rightarrow D$. We shall use the relative complexes $\mathbb{Z}(d)_{X|D}^{\mathcal{D}^*}$ and $\mathbb{Z}(d)_{(X,D)}^{\mathcal{D}^*}$ to define in the usual way two natural receptors for the group of algebraic cycles with modulus that are homologically trivial. Coherently with the excision Theorem 7.1 in the projective case over the complex numbers, we will show that the two, a priori different constructions, give rise to the same invariant naturally attached to the pair (X, D) , that we call *the Albanese variety with modulus*.

8.2.1. The Albanese with modulus. By construction, the map $\Delta: S_X \rightarrow X$ satisfies $\Delta \circ i_{\pm} = \text{Id}_X$, and therefore induces a splitting of the restriction maps i_{\pm}^* from the cohomology of S_X to the cohomology of X . This gives the commutative diagram of the analytic cohomology groups

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{H}^{2d}(S_X, \mathbb{Z}(d)_{X|D}^{\mathcal{D}^*}) & \longrightarrow & \mathbb{H}^{2d}(S_X, \mathbb{Z}(d)_{S_X}^{\mathcal{D}^*}) & \xleftarrow[\Delta^*]{i_-^*} & \mathbb{H}^{2d}(X, \mathbb{Z}(d)_X^{\mathcal{D}}) \rightarrow 0 \\ & & \epsilon_{X|D} \downarrow & & \downarrow \epsilon_{S_X} & & \downarrow \epsilon_X \\ 0 & \rightarrow & H^{2d}(S_X, \mathbb{Z}(d)_{X|D}) & \longrightarrow & H^{2d}(S_X, \mathbb{Z}(d)_{S_X}) & \xleftarrow[\Delta^*]{i_-^*} & H^{2d}(X, \mathbb{Z}(d)_X) \rightarrow 0, \end{array}$$

where the vertical maps are induced by the natural projections $\mathbb{Z}(d)^{\mathcal{D}^*} \rightarrow \mathbb{Z}(d)$.

Since X is smooth projective and connected, we have $H^{2d}(X, \mathbb{Z}(d)_X) = \mathbb{Z}$ and $H^{2d}(S_X, \mathbb{Z}(d)_{S_X}) = \mathbb{Z} \oplus \mathbb{Z}$, one copy for each component of S_X . In particular, the restriction map i_-^* sends $(0, 1)$ to 1 and we can identify $H^{2d}(S_X, \mathbb{Z}(d)_{X|D})$ with the other copy of \mathbb{Z} . Under this identification, we denote by $A^d(X|D)$ the kernel of the natural map

$$0 \rightarrow A^d(X|D) \rightarrow \mathbb{H}^{2d}(S_X, \mathbb{Z}(d)_{X|D}^{\mathcal{D}^*}) \xrightarrow{\epsilon_{X|D}} \mathbb{Z}$$

and call it the *Albanese variety of X with modulus D* .

By construction, we have an isomorphism

$$A^d(X|D) = \mathbb{H}^{2d-1}(S_X, \Omega_{(S_X, X_-)}^{<d}) / H^{2d-1}(S_X, \mathbb{Z}(d)_{X|D}),$$

where $\Omega_{(S_X, X_-)}^{<d}$ denotes $\text{Cone}(\Omega_{S_X}^{<d} \xrightarrow{i_-^*} \Omega_X^{<d})[-1]$. Following [9], we define the generalized Albanese variety of S_X , denoted $A^d(S_X)$, as the kernel of the map

$$\mathbb{H}^{2d}(S_X, \mathbb{Z}(d)_{S_X}^{\mathcal{D}^*}) \rightarrow H^{2d}(S_X, \mathbb{Z}(d)_{S_X}).$$

This gives a split short exact sequence

$$(8.3) \quad 0 \rightarrow A^d(X|D) \xrightarrow{p_{+,*}} A^d(S_X) \xrightleftharpoons[\Delta^*]{i_-^*} A^d(X) \rightarrow 0,$$

where $A^d(X) = \text{Alb}(X)$ is the usual Albanese variety of X , namely, $\mathbb{H}^{2d-1}(X, \Omega_X^{<d}) / H^{2d-1}(X, \mathbb{Z}(d)_X)$. Write $J^d(S_X)$ for the quotient

$$J^d(S_X) = \frac{H^{2d-1}(S_X, \mathbb{C}(d))}{F^0 H^{2d-1}(S_X, \mathbb{C}(d)) + \text{image} H^{2d-1}(S_X, \mathbb{Z}(d))}.$$

It is a semi-abelian variety by a result of Deligne. By [9, Lemma 3.1], there is a natural surjection

$$\psi: A^d(S_X) \rightarrow J^d(S_X)$$

whose kernel is a \mathbb{C} -vector space. Moreover, there is a unique structure of algebraic group on $A^d(S_X)$ making ψ a morphism of algebraic groups, with unipotent kernel.

Notice that there is a short exact sequence

$$(8.4) \quad 0 \rightarrow G \rightarrow A^d(S_X) \xrightarrow{\pi^*} A^d(X) \times A^d(X) \rightarrow 0$$

where G is simply defined as the kernel of the map π^* , that is surjective since $H^{2d}(S_X, \mathbb{Z}(d)) \simeq H^{2d}(\overline{S_X}, \mathbb{Z}(d))$ and $\mathbb{H}^{2d-1}(S_X, \Omega_{S_X}^{<d}) \rightarrow \mathbb{H}^{2d-1}(\overline{S_X}, \Omega_{\overline{S_X}}^{<d})$ because of GAGA and the fact that π is finite and birational. A combination of (8.3) and (8.4) gives then the short exact sequence

$$(8.5) \quad 0 \rightarrow G \rightarrow A^d(X|D) \xrightarrow{\phi} A^d(X) \rightarrow 0,$$

where the (surjective) forgetful map ϕ is same as the map $i_+^* \circ p_{+,*}: A^d(X|D) \rightarrow A^d(X)$.

8.2.2. *The relative Albanese.* Taking cohomology of the bottom exact sequence in (8.2), we get exact sequence

$$\mathbb{H}^{2d-1}(D, \mathbb{Z}(d)_D^{\mathcal{D}^*}) \rightarrow \mathbb{H}^{2d}(X, \mathbb{Z}(d)_{(X,D)}^{\mathcal{D}^*}) \rightarrow \mathbb{H}^{2d}(X, \mathbb{Z}(d)_X^{\mathcal{D}}) \rightarrow 0,$$

where the term $\mathbb{H}^{2d}(D, \mathbb{Z}(d)_D^{\mathcal{D}^*}) = 0$ for dimension reasons.

We denote by $A^d(X, D)$ the kernel of the natural map

$$\mathbb{H}^{2d}(X, \mathbb{Z}(d)_{(X,D)}^{\mathcal{D}^*}) \rightarrow \mathbb{H}^{2d}(X, \mathbb{Z}(d)_{(X,D)}) \simeq H^{2d}(X, \mathbb{Z}(d)_X) = \mathbb{Z},$$

and call it the *relative Albanese variety of the pair* (X, D) . It fits in a short exact sequence

$$0 \rightarrow \frac{\mathbb{H}^{2d-1}(D, \mathbb{Z}(d)_D^{\mathcal{D}^*})}{\mathbb{H}^{2d-1}(X, \mathbb{Z}(d)_X^{\mathcal{D}})} \rightarrow A^d(X, D) \rightarrow A^d(X) \rightarrow 0$$

The vertical maps in (8.2) induce then the following diagram

$$(8.6) \quad \begin{array}{ccccccc} 0 & \longrightarrow & A^d(X|D) & \xrightarrow{p_{+,*}} & A^d(S_X) & \xrightarrow{i_-^*} & A^d(X) \rightarrow 0 \\ & & \downarrow \phi_{X|D}^d & \searrow \varphi & \downarrow i_+^* & & \downarrow \\ 0 \rightarrow & \frac{\mathbb{H}^{2d-1}(D, \mathbb{Z}(d)_D^{\mathcal{D}^*})}{\mathbb{H}^{2d-1}(X, \mathbb{Z}(d)_X^{\mathcal{D}})} & \longrightarrow & A^d(X, D) & \longrightarrow & A^d(X) & \longrightarrow 0. \end{array}$$

Proposition 8.1. *Let X be a smooth projective \mathbb{C} -scheme of dimension $d \geq 1$. Then the natural map $\phi_{X|D}^d: A^d(X|D) \rightarrow A^d(X, D)$ of (8.6) is an isomorphism.*

Proof. If $d = 1$, it follows from [33, Proposition 1.4] and Proposition 12.2 that $A^1(X|D) \simeq \text{Pic}(X, D)$. On the other hand, we have

$$A^1(X, D) = \mathbb{H}^2(X, \mathbb{Z}(2)_{(X,D)}^{\mathcal{D}^*}) \simeq H^1(X, (1 + \mathcal{I}_D)^\times),$$

where \mathcal{I}_D is the ideal sheaf of $D \subset X$. It follows from [48, Lemma 2.1] that $H^1(X, (1 + \mathcal{I}_D)^\times) \simeq \text{Pic}(X, D)$ and the proposition follows. We can thus assume that $d \geq 2$.

We can assume that X is connected. Since the singular cohomology satisfies the Mayer-Vietoris property with respect to the square (8.1), we are left to show that

$$(8.7) \quad \mathbb{H}^{2d-1}(S_X, \Omega_{(S_X, X_-)}^{<d}) \xrightarrow{i_+^*} \mathbb{H}^{2d-1}(X, \Omega_{(X,D)}^{<d})$$

is bijective. Here, $\Omega_{(X,D)}^{<d} = \text{Cone}(\Omega_X^{<d} \rightarrow \Omega_D^{<d})[-1]$ and the morphism between the cohomology groups is induced by the restriction along $i_+ : X \hookrightarrow S_X$.

Using GAGA, it is equivalent to prove the bijectivity for the cohomology of the associated Zariski sheaves of differential forms. It is easy to check from (2.6) that i_*^* induces an isomorphism $\mathcal{I}_{X_-} \xrightarrow{\simeq} \mathcal{I}_D$, where \mathcal{I}_D is the ideal sheaf for $D \subset X_+$. Furthermore, $(S_X \setminus D) = (X_- \setminus D) \amalg (X_+ \setminus D)$. This information can be used to get a commutative diagram of exact sequences

$$(8.8) \quad \begin{array}{ccccccc} 0 & \rightarrow & \Omega_{(S_X, X_-)}^{[1, d-1]}[-1] & \rightarrow & \Omega_{(S_X, X_-)}^{<d} & \rightarrow & \mathcal{I}_{X_-} \rightarrow 0 \\ & & \downarrow i_+^* & & \downarrow i_+^* & & \downarrow \simeq \\ 0 & \rightarrow & \Omega_{(X,D)}^{[1, d-1]}[-1] & \rightarrow & \Omega_{(X,D)}^{<d} & \rightarrow & \mathcal{I}_D \rightarrow 0. \end{array}$$

We therefore have to prove that the map of Zariski cohomology groups

$$(8.9) \quad \mathbb{H}^{2d-2}(S_X, \Omega_{(S_X, X_-)}^{[1, d-1]}) \xrightarrow{i_+^*} \mathbb{H}^{2d-2}(X, \Omega_{(X,D)}^{[1, d-1]})$$

is bijective.

We start by proving surjectivity. Letting $\mathcal{F}_1[-1]$ and $\mathcal{F}_2[-1]$ denote the kernel and the cokernel of the left vertical arrow in (8.8), respectively, we have an exact sequence

$$(8.10) \quad 0 \rightarrow \mathcal{F}_1 \rightarrow \Omega_{(S_X, X_-)}^{[1, d-1]} \xrightarrow{i_+^*} \Omega_{(X,D)}^{[1, d-1]} \rightarrow \mathcal{F}_2 \rightarrow 0.$$

Setting $\mathcal{G} = \Omega_{(S_X, X_-)}^{[1, d-1]}/\mathcal{F}_1$, we obtain a diagram of exact sequences

$$\begin{array}{ccccccc} \mathbb{H}^{2d-2}(\mathcal{F}_1) & \longrightarrow & \mathbb{H}^{2d-2}(S_X, \Omega_{(S_X, X_-)}^{[1, d-1]}) & \longrightarrow & \mathbb{H}^{2d-2}(\mathcal{G}) & \longrightarrow & \mathbb{H}^{2d-1}(\mathcal{F}_1) \\ & & & & \parallel & & \\ \mathbb{H}^{2d-3}(\mathcal{F}_2) & \longrightarrow & \mathbb{H}^{2d-2}(\mathcal{G}) & \longrightarrow & \mathbb{H}^{2d-2}(X, \Omega_{(X,D)}^{[1, d-1]}) & \longrightarrow & \mathbb{H}^{2d-2}(\mathcal{F}_2). \end{array}$$

Since $d \geq 2$ and \mathcal{F}_i are complexes of coherent sheaves supported on D , a standard spectral sequence argument shows that $\mathbb{H}^{2d-2+j}(\mathcal{F}_i) = 0$ for $i = 1, 2$ and $j \geq 0$. Analyzing the spectral sequence computing the cohomology group $H^{2d-3}(\mathcal{F}_2)$, we see that thanks to the same dimension argument used above, the only surviving term is given by $H^{d-1}(\Omega_{(X,D)}^{d-1}/\Omega_{(S_X, X_-)}^{d-1})$. We thus get an exact sequence

$$(8.11) \quad 0 \rightarrow \frac{H^{d-1}(\Omega_{(X,D)}^{d-1}/\Omega_{(S_X, X_-)}^{d-1})}{\mathbb{H}^{2d-3}(X, \Omega_{(X,D)}^{[1, d-1]})} \rightarrow \mathbb{H}^{2d-2}(S_X, \Omega_{(S_X, X_-)}^{[1, d-1]}) \xrightarrow{i_+^*} \mathbb{H}^{2d-2}(X, \Omega_{(X,D)}^{[1, d-1]}) \rightarrow 0$$

and this proves the surjectivity of i_+^* .

We now prove injectivity. Write D_{sing} for $|D|_{\text{sing}}$. Let $U = X \setminus D_{\text{sing}} \hookrightarrow X$ and write D_U for the restriction of D to U . We note that $(D_U)_{\text{red}}$ is a smooth (possibly non connected) divisor on U . We claim that the restriction to U of the sheaf $\mathcal{F}_3 = \Omega_{(X,D)}^{d-1}/\Omega_{(S_X, X_-)}^{d-1}$ is zero. Note that since \mathcal{F}_3 is anyway supported on D , our claim actually implies that \mathcal{F}_3 is supported on $Y = D_{\text{sing}}$. Since $\dim(Y) \leq d-2$, this will give the desired vanishing of the cohomology group $H^{d-1}(\Omega_{(X,D)}^{d-1}/\Omega_{(S_X, X_-)}^{d-1})$. Since the first term of the sequence (8.11) is a quotient of $H^{d-1}(\Omega_{(X,D)}^{d-1}/\Omega_{(S_X, X_-)}^{d-1})$, this will complete the proof of the proposition.

We now prove the claim. Since all the components of $(D_U)_{\text{red}}$ are regular and disjoint, we can assume that at a point y of U , a local equation for D_U is given by x^n , where (x, x_1, \dots, x_{d-1}) is a regular system of parameters in $A = \mathcal{O}_{X,y}$. Write $A' = A/(x^n)$, $B = A/(x)$ and let R be the double construction applied to the pair $(A, (x^n))$. Write $\Omega_{(R, A_-)}^{d-1}$ for the kernel of the second projection $i_-^* : \Omega_{R/\mathbb{C}}^{d-1} \rightarrow \Omega_{A/\mathbb{C}}^{d-1}$ and write $\Omega_{(A, A')}^{d-1}$ for the kernel of the restriction map $j^* : \Omega_{A/\mathbb{C}}^{d-1} \rightarrow \Omega_{A'/\mathbb{C}}^{d-1}$. The claim is equivalent to showing that the induced map

$$(8.12) \quad i_+^* : \Omega_{(R, A_-)}^{d-1} \rightarrow \Omega_{(A, A')}^{d-1}$$

is surjective.

Since A is a regular local ring of dimension d , B is a regular local ring of dimension $d-1$ and the infinitesimal lifting property (see [16, Proposition 4.4]) gives a splitting of the projection $A' \rightarrow B$. Since the relative embedding dimension of A' in A is 1, this gives an isomorphism $A' \simeq B[x]/(x^n)$. We can then compute the following modules

$$\begin{aligned} \Omega_{A'/\mathbb{C}}^1 &= (\Omega_{B/\mathbb{C}}^1 \otimes_B A') \oplus \left(\bigoplus_{j=0}^{n-2} Bx^j dx \right) \quad \text{as } B\text{-module;} \\ \Omega_{A'/\mathbb{C}}^{d-1} &= (\Omega_{B/\mathbb{C}}^{d-2} \otimes_B A/(x^{n-1})dx) \oplus (\Omega_{B/\mathbb{C}}^{d-1} \otimes_B A') \\ &= \bigoplus_{i=1}^{d-1} A/(x^{n-1})dx \wedge dx_1 \wedge \dots \wedge dx_i \wedge \dots \wedge dx_{d-1} \oplus A'dx_1 \wedge \dots \wedge dx_{d-1}; \\ \Omega_{A/\mathbb{C}}^{d-1} &= \bigoplus_{i=1}^{d-1} (Adx \wedge dx_1 \wedge \dots \wedge dx_i \wedge \dots \wedge dx_{d-1}) \oplus Adx_1 \wedge \dots \wedge dx_{d-1}. \end{aligned}$$

One can easily check from the above expressions that $\Omega_{(A, A')}^{d-1}$ is generated by the forms $x^n dx_1 \wedge \dots \wedge dx_{d-1}$ and $(x^{n-1} dx \wedge dx_1 \wedge \dots \wedge dx_i \wedge \dots \wedge dx_{d-1})_{i=1}^{d-1}$ as A -module.

We now consider the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \Omega_{(R, A_-)}^{d-1} & \longrightarrow & \Omega_{R/\mathbb{C}}^{d-1} & \xrightarrow{i_-^*} & \Omega_{A/\mathbb{C}}^{d-1} \rightarrow 0 \\ & & \downarrow i_+^* & & \downarrow i_+^* & & \downarrow \\ 0 & \rightarrow & \Omega_{(A, A')}^{d-1} & \longrightarrow & \Omega_{A/\mathbb{C}}^{d-1} & \longrightarrow & \Omega_{A'/\mathbb{C}}^{d-1} \rightarrow 0. \end{array}$$

We can lift (up to multiplication by elements in \mathbb{C}^\times) the generators $x^n dx_1 \wedge \dots \wedge dx_{d-1}$ and $(x^{n-1} dx \wedge dx_1 \wedge \dots \wedge dx_i \wedge \dots \wedge dx_{d-1})_{i=1}^{d-1}$ of $\Omega_{(A, A')}^{d-1}$ via the projection i_+^* to elements

$$\begin{aligned} &(x^n, 0)d(x_1, x_1) \wedge \dots \wedge d(x_{d-1}, x_{d-1}), \\ &(d(x^n, 0) \wedge d(x_1, x_1) \wedge \dots \wedge d(x_i, x_i) \wedge \dots \wedge d(x_{d-1}, x_{d-1}))_{i=1}^{d-1} \quad \text{in } \Omega_{R/\mathbb{C}}^{d-1} \end{aligned}$$

and we immediately see that they go to zero via the second projection i_-^* , so that they lift to elements in $\Omega_{(R, A_-)}^{d-1}$, proving that (8.12) is surjective and therefore completing the proof of the proposition. \square

As a consequence of Proposition 8.1, (8.4) and (8.5), it follows that $A^d(X|D)$ is an extension of the abelian variety $A^d(X)$ by the linear group $G = \frac{\mathbb{H}^{2d-1}(D, \mathbb{Z}(d)_D^{\mathcal{P}^*})}{\mathbb{H}^{2d-1}(X, \mathbb{Z}(d)_X^{\mathcal{P}^*})}$.

Remark 8.2. If X is a surface, then one checks that $\mathbb{Z}(2)_{(X,D)}^{\mathcal{P}^*}$ is quasi-isomorphic to the complex $j_!(\mathbb{Z}) \rightarrow \mathcal{I}_D \rightarrow \Omega_{(X,D)}^1$ (see [31, § 4]), where $j: X \setminus D \hookrightarrow X$ is the open inclusion. When D_{red} is a strict normal crossing divisor, the complex $j_!(\mathbb{Z}) \rightarrow \mathcal{I}_D \rightarrow \Omega_{(X,D)}^1$ is used in [5] to construct a universal regular quotient of $\text{CH}_0(X|D)_{\text{deg } 0}$. One consequence of Proposition 8.1 is that it provides a cohomological proof that the Albanese variety with modulus of § 8.2.1 coincides with the one constructed in [5] when X is a surface and D_{red} is strict normal crossing. The universality Theorem 10.3 will tell us, more generally, that the two constructions agree in higher dimension as well, whenever D_{red} is strict normal crossing divisor.

9. AN INTERLUDE ON REGULAR HOMOMORPHISMS

Let k be an algebraically closed field and let Y be a projective reduced scheme over k . Let $\text{CH}_0^{LW}(Y) = \text{CH}_0^{LW}(Y, Y_{\text{sing}})$ and $\text{CH}_0(Y) = \text{CH}_0(Y, Y_{\text{sing}})$ be the groups of zero-cycles associated to Y . Let U be a dense open subscheme of Y_{reg} and choose a base point x_i in every irreducible component U_i of U . The following Definition is taken from [9].

Definition 9.1. Let G be smooth commutative algebraic group over k . We say that a group homomorphism $\rho': \text{CH}_0^{LW}(Y)_{\text{deg } 0} \rightarrow G$ of abstract groups is a regular homomorphism if the map $\pi: U \rightarrow G$ with $\pi|_{U_i}(x) = \rho'([x] - [x_i])$ is a morphism of schemes (i.e., there exists a morphism of schemes whose restriction to closed points coincides with π).

The same definition allow us to talk about regular homomorphisms from the Chow group $\text{CH}_0(Y)_{\text{deg } 0}$ instead.

Remark 9.2. In [9, Definition 1.14], there are other equivalent definitions of regular map from the Levine-Weibel Chow group of 0-cycles on a singular projective variety to a smooth commutative algebraic group. We will not need this explicitly, but we recall one of them for the reader who wishes to remove a reference to the base points.

Let U be an open dense in Y_{reg} . Let U_1, \dots, U_s be the irreducible components of U . Consider the map

$$\gamma^{(-)}: \Pi_U = \bigcup_{i=1}^s U_i \times U_i \rightarrow \text{CH}_0^{LW}(Y)_{\text{deg } 0}$$

defined by $\gamma^{(-)}(u, u') = [u] - [u']$. We have then:

Proposition 9.3 (Corollary 1.13, [9]). *Let G be a smooth commutative algebraic group. Let $\rho': \text{CH}_0^{LW}(Y)_{\text{deg } 0} \rightarrow G$ be a group homomorphism. Then the following conditions are equivalent.*

- i) *The composition $\rho' \circ \gamma^{(-)}: \Pi_{Y_{\text{reg}}} \rightarrow G$ is a morphism of scheme.*
- ii) *The morphism ρ' is regular in the sense of Definition 9.1.*

As above, the expression “the map ϕ is a morphism of schemes”, stands for “there exists a morphisms of schemes whose restriction to closed points coincides with ϕ ”.

9.1. The case of the double. Let X be now a smooth connected projective k -variety, equipped with an effective Cartier divisor D and let X_+ and X_- denote as above the irreducible components of the double S_X of X along D . Given any dense open subset $V \subset (S_X)_{\text{reg}}$, we denote by V_+ and V_- the intersection of V with X_+^o and X_-^o respectively. We adapt the definition of regular homomorphism recalled above to the particular geometry of the double S_X .

Definition 9.4. Let G be a smooth commutative algebraic group over k . A homomorphism

$$\rho' : \text{CH}_0(S_X)_{\text{deg } 0} \rightarrow G$$

is called a regular homomorphism if given base points $x_{0,\pm}$ on each irreducible components V_{\pm} of some open dense subscheme V of $(S_X)_{\text{reg}}$, the composition of ρ' with the map

$$\pi_{x_{0,\pm}}^V : V \rightarrow \text{CH}_0(S_X)_{\text{deg } 0}, \quad x \mapsto [x] - [x_{0,\theta(x)}],$$

where $\theta(x) = \pm$ according to $x \in V_{\pm}$, is a morphism of schemes. In a similar fashion, one can define the notion of regular homomorphism using the Levine-Weibel Chow group of 0-cycles on the double S_X .

It follows from [9, Lemma 1.4] that the image of a regular homomorphism is a connected algebraic closed subgroup of G . The initial object (whose underlying map is necessarily surjective) in the category of regular maps $\text{CH}_0(S_X)_{\text{deg } 0} \rightarrow G$ is called the universal regular quotient of $\text{CH}_0(S_X)_{\text{deg } 0}$. It was shown in [9] that the universal regular quotient of $\text{CH}^{LW}(Y)_{\text{deg } 0}$ always exists for any projective variety Y over k . We state this theorem below for S_X .

Theorem 9.5. ([9, Theorem 1]) *There exists a smooth connected algebraic group $\text{Alb}(S_X)$, together with a regular surjective homomorphism $\rho_{S_X} : \text{CH}_0^{LW}(S_X)_{\text{deg } 0} \rightarrow \text{Alb}(S_X)$ such that ρ_{S_X} is universal among regular homomorphisms from $\text{CH}_0^{LW}(S_X)_{\text{deg } 0}$ to smooth commutative algebraic groups. When $k = \mathbb{C}$, then $\text{Alb}(S_X)$ coincides with the Albanese variety $A^d(S_X)$ introduced in (8.3).*

9.2. The universal semi-abelian quotient of $\text{CH}_0(Y)_{\text{deg } 0}$. Let Y be a reduced projective scheme of dimension $d \geq 1$ over an algebraically closed field k . Let us assume the characteristic of k to be positive in this subsection. In this case, do not know if the canonical map $\text{CH}_0^{LW}(Y) \rightarrow \text{CH}_0(Y)$ is an isomorphism. A weaker question is if the Albanese map $\rho_Y : \text{CH}_0^{LW}(Y)_{\text{deg } 0} \rightarrow \text{Alb}(Y)$ factors through $\text{CH}_0(Y)_{\text{deg } 0}$. We expect this to be true, but we do not yet know how to verify this either. The reason for this is that, one does not know any description of the Albanese variety in positive characteristic except its existence. We however show in this section that the semi-abelian Albanese variety of Y indeed has this property. We shall use this to prove a comparison result for the two Chow groups in positive characteristic.

The following description of the semi-abelian Albanese variety of Y is recalled from [34, § 2]. Let $\pi : Y^N \rightarrow Y$ denote the normalization map. Let $\text{Cl}(Y^N)$ and $\text{Pic}_W(Y^N)$ denote the class group and the Weil Picard group of Y^N . Recall (see e.g. [50]) that $\text{Pic}_W(Y^N)$ is the subgroup of $\text{Cl}(Y^N)$ consisting of Weil divisors which are algebraically equivalent to zero in the sense of [12, Chap. 19]. Let $\text{Div}(Y)$ denote the free abelian group of Weil divisors on Y . Let $\Lambda_1(Y)$ denote the subgroup of $\text{Div}(Y^N)$ generated by the Weil divisors which are supported on $\pi^{-1}(Y_{\text{sing}})$. This gives us a map $\iota_Y : \Lambda_1(Y) \rightarrow \frac{\text{Cl}(Y^N)}{\text{Pic}_W(Y^N)}$.

Let $\Lambda(Y)$ denote the kernel of the canonical map

$$(9.1) \quad \Lambda_1(Y) \xrightarrow{(\iota_Y, \pi_*)} \frac{\text{Cl}(Y^N)}{\text{Pic}_W(Y^N)} \oplus \text{Div}(Y).$$

The semi-abelian Albanese variety of Y is the Cartier dual of the 1-motive

$$[\Lambda(Y) \rightarrow \text{Pic}_W(Y^N)]$$

and is denoted by $J^d(Y)$. It follows from [34, § 4] that $J^d(Y)$ is the universal semi-abelian quotient of the Esnault-Srinivas-Viehweg Albanese variety $\text{Alb}(Y)$. Let $\rho_Y^{\text{semi}} : \text{CH}_0^{LW}(Y)_{\text{deg } 0} \rightarrow J^d(Y)$ denote the universal regular homomorphism.

Lemma 9.6. *Let $f : Z \rightarrow Y$ be a smooth projective morphism of relative dimension r . Then there is a push-forward map $f_*^{\text{semi}} : J^{d+r}(Z) \rightarrow J^d(Y)$ and a commutative diagram*

$$(9.2) \quad \begin{array}{ccc} \mathcal{Z}_0(Z, Z_{\text{sing}}) & \xrightarrow{f_*} & \mathcal{Z}_0(Y, Y_{\text{sing}}) \\ \rho_Z^{\text{semi}} \downarrow & & \downarrow \rho_Y^{\text{semi}} \\ J^{d+r}(Z) & \xrightarrow{f_*} & J^d(Y). \end{array}$$

Proof. Since f is smooth and projective, it follows that $f^N : Z^N \rightarrow Y^N$ is also smooth and projective. It is clear that the flat pull-back $f^{N,*}$ takes integral Weil divisors to integral Weil divisors. It is also clear that this map preserves Weil divisors which are algebraically equivalent to zero. Since $Z_{\text{sing}} = f^{-1}(Y_{\text{sing}})$, we see that $f^{N,*}(\Lambda_1(Y)) \subset \Lambda_1(Z)$. Furthermore, it follows from [12, Proposition 1.7] that $f^{N,*}(\text{Ker}(\Lambda_1(Y) \rightarrow \text{Div}(Y))) \subset \text{Ker}(\Lambda_1(Z) \rightarrow \text{Div}(Z))$. We conclude that f^* induces a morphism of 1-motives $f^* : [\Lambda(Y) \rightarrow \text{Pic}_W(Y^N)] \rightarrow [\Lambda(Z) \rightarrow \text{Pic}_W(Z^N)]$ and hence a map $f_* : J^{d+r}(Z) \rightarrow J^d(Y)$.

To show the commutative diagram (9.2), we need to observe that $J^d(Y)$ is a quotient of the Cartier dual $J_{\text{Serre}}^d(Y)$ of the 1-motive $[\Lambda_1(Y) \rightarrow \text{Pic}_W(Y^N)]$ and this dual semi-abelian variety is the universal object in the category of morphisms from Y_{reg} to semi-abelian varieties (see [44]). Since $Z_{\text{reg}} = f^{-1}(Y_{\text{reg}})$, it follows from this universality of $J_{\text{Serre}}^d(Y)$ that there is a commutative diagram as in (9.2) if we replace $J^d(Y)$ by $J_{\text{Serre}}^d(Y)$. The commutative diagram

$$\begin{array}{ccc} J_{\text{Serre}}^{d+r}(Z) & \xrightarrow{f_*} & J_{\text{Serre}}^d(Y) \\ \downarrow & & \downarrow \\ J^{d+r}(Z) & \xrightarrow{f_*} & J^d(Y) \end{array}$$

now finishes the proof. \square

Proposition 9.7. *The semi-abelian variety $J^d(Y)$ is the universal regular semi-abelian variety quotient of $\text{CH}_0(Y)_{\text{deg } 0}$.*

Proof. Let S denote the singular locus of Y . It is enough to show that $\rho_Y^{\text{semi}} : \text{CH}_0^{LW}(Y)_{\text{deg } 0} \rightarrow J^d(Y)$ factors through the canonical map $\text{CH}_0^{LW}(Y)_{\text{deg } 0} \rightarrow \text{CH}_0(Y)_{\text{deg } 0}$. Let then $\nu : C \rightarrow Y$ be a finite l.c.i map from a good curve C relative to $S \subset Y$ and let $f \in \mathcal{O}_{C,E}^\times$ for $E = C_{\text{sing}} \cup \nu^{-1}(S)$.

As in Lemma 5.2, we factor ν as composition $\nu = \pi \circ \mu$, where $\mu : C \hookrightarrow \mathbb{P}_Y^N$ is a regular embedding and $\pi : \mathbb{P}_Y^N \rightarrow Y$ is the projection. By Lemma 9.6, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_0(\mathbb{P}_Y^N, \mathbb{P}_S^N)_{\text{deg } 0} & \xrightarrow{\pi_*} & \mathcal{Z}_0(Y, S)_{\text{deg } 0} \\ \bar{\rho}_{\mathbb{P}_Y^N}^{\text{semi}} \downarrow & & \downarrow \bar{\rho}_Y^{\text{semi}} \\ J^{d+N}(\mathbb{P}_Y^N) & \xrightarrow{\pi_*} & J^d(Y) \end{array}$$

where $\tilde{\rho}^{\text{semi}}$ denotes on both vertical sides of the diagram the composition of the universal map ρ^{semi} with the canonical map $\mathcal{Z}_0(-)_{\text{deg } 0} \rightarrow \text{CH}_0^{LW}(-)_{\text{deg } 0}$. Since $\tilde{\rho}_{\mathbb{P}_Y^N}^{\text{semi}}(\mu_*((\text{div})_C(f))) = 0$ in $J^{d+N}(\mathbb{P}_Y^N)$, the result follows. \square

Using Proposition 9.7 and Theorem 11.1, we obtain the following comparison between the torsion parts of the Levine-Weibel Chow group and our modified definition.

Theorem 9.8. *Let Y be an equidimensional reduced projective scheme of dimension $d \geq 1$ over an algebraically closed field k of exponential characteristic p . Then the canonical map $\text{CH}_0^{LW}(Y)\{l\} \xrightarrow{\cong} \text{CH}_0(Y)\{l\}$ between the l -primary torsion subgroups, is an isomorphism for every prime $l \neq p$.*

Proof. Let L denote the kernel of the canonical surjective map $\text{CH}_0^{LW}(Y) \rightarrow \text{CH}_0(Y)$. We first show that L is a p -primary group of bounded exponent. It follows from Proposition 9.7 that the semi-Albanese map ρ_Y^{semi} from $\text{CH}_0^{LW}(Y)_{\text{deg } 0}$ to $J^d(Y)$ factors through $\text{CH}_0(Y)_{\text{deg } 0}$. Since ρ_Y^{semi} is an isomorphism on n -torsion subgroups for n prime to p by Theorem 11.1, we immediately deduce that L (which is same as $\text{Ker}(\text{CH}_0^{LW}(Y)_{\text{deg } 0} \rightarrow \text{CH}_0(Y)_{\text{deg } 0})$ is a p -torsion group. We now show that L has a bounded exponent.

We first note from Lemma 3.13 that there is a factorization

$$\text{CH}_0^{LW}(Y) \xrightarrow{\text{can}} \text{CH}_0(Y) \xrightarrow{\text{cyc}_Y} K_0(Y)$$

of the cycle class map cyc_Y^{LW} from the Levine-Weibel Chow group to $K_0(Y)$. On the other hand, it follows from [29, Corollary 5.4] (see also [30, Corollary 2.7]) that the kernel of the map cyc_Y^{LW} is a group of exponent $N_d := (d-1)!$. We conclude that $N_d \cdot L = 0$.

We write $N_d = p^a q$, where $a \geq 0$ and $p \nmid q$. We fix a cycle $\alpha \in L$. Since L is a p -primary group, we can write $p^n \alpha = 0$ for some $n \gg a$. We then have an identity $xq + yp^{n-a} = 1$ for some $x, y \in \mathbb{Z}$. This yields

$$p^a \alpha = (xq + yp^{n-a})p^a \alpha = (xqp^a + yp^n)\alpha = xN_d \alpha + yp^n \alpha = 0.$$

Since $a \in \mathbb{Z}$ depends only on N_d and not on α , we get $p^a \cdot L = 0$.

It is easy to see using 5-lemma that for every prime $l \neq p$, there is an exact sequence

$$0 \rightarrow L\{l\} \rightarrow \text{CH}_0^{LW}(Y)\{l\} \rightarrow \text{CH}_0(Y)\{l\} \rightarrow L \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l.$$

Since L is a p -primary group of bounded exponent and $\mathbb{Q}_l/\mathbb{Z}_l$ is p -divisible, we must have $L\{l\} = 0 = L \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l$. In particular, $\text{CH}_0^{LW}(Y)\{l\} \xrightarrow{\cong} \text{CH}_0(Y)\{l\}$ \square

Corollary 9.9. *Let k be an algebraically closed field of exponential characteristic p . Let Y be an equidimensional reduced projective scheme of dimension $d \leq p$. Then the canonical map $\text{CH}_0^{LW}(Y) \rightarrow \text{CH}_0(Y)$ is an isomorphism.*

9.3. Regular homomorphism from Chow group with modulus. We now turn to the definition of a regular morphism for the Chow group of 0-cycles with modulus. Let (X, D) be as in 9.1. We write X° for the open complement of D in X .

Definition 9.10. We fix a closed point $x_0 \in X^\circ$. For $U \subset X^\circ$ open with $x_0 \in U$, we define the map of sets

$$\pi_{x_0}^U: U \rightarrow \text{CH}_0(X|D)_{\text{deg } 0}, \quad x \mapsto [x] - [x_0],$$

where $[x]$ and $[x_0]$ denote the classes of x and x_0 respectively in $\text{CH}_0(X|D)$. For a commutative algebraic group G over k , we say that a homomorphism of abelian groups

$$\rho: \text{CH}_0(X|D)_{\text{deg } 0} \rightarrow G$$

is regular if there exists an open subset U of X° and a closed point $x_0 \in U$ such that $\rho \circ \pi_{x_0}^U: U \rightarrow G$ is a morphism of algebraic varieties.

10. THE ABEL-JACOBI MAP WITH MODULUS AND ITS UNIVERSALITY

Let X be a smooth projective scheme of dimension $d \geq 1$ over \mathbb{C} and let $D \subset X$ be an effective Cartier divisor. In this section, we show that $A^d(X|D)$ constructed in § 8.2.1 has a natural structure of a connected commutative algebraic group and it is the universal regular quotient of $\mathrm{CH}_0(X|D)_{\mathrm{deg} 0}$ via an Abel-Jacobi map. Since we are working over \mathbb{C} , we keep identifying $\mathrm{CH}_0^{LW}(Z)$ with $\mathrm{CH}_0(Z)$ in this section (using Theorem 3.17) for any projective scheme Z .

10.1. The Abel-Jacobi map. Let (X, D) be as above. We write again X° for the open complement of D in X and S_X for the double construction applied to the pair (X, D) . Recall from [10] (see also [9, Lemma 2.1]) that given $x \in (S_X)_{\mathrm{reg}} = X^\circ \amalg X^o$, there is a unique element $[x] \in \mathbb{H}_{\{x\}}^{2d}(S_X, \mathbb{Z}(d)_{S_X}^{\mathcal{D}^*})$ mapping to the topological cycle class of x in $H_{\{x\}}^{2d}(S_X, \mathbb{Z}(d))$ as well as to the de Rham cycle class of x in $\mathbb{H}_{\{x\}}^{2d}(S_X, \Omega_{S_X}^{\geq d})$. Using the canonical forget support map $\mathbb{H}_{\{x\}}^{2d}(S_X, \mathbb{Z}(d)_{S_X}^{\mathcal{D}^*}) \rightarrow \mathbb{H}^{2d}(S_X, \mathbb{Z}(d)_{S_X}^{\mathcal{D}^*})$ and extending linearly, this gives rise to a well defined map

$$\mathrm{cyc}_{S_X}^{\mathcal{D}} : \mathcal{Z}_0(S_X, D) \rightarrow \mathbb{H}^{2d}(S_X, \mathbb{Z}(d)_{S_X}^{\mathcal{D}^*})$$

that composed with $\mathbb{H}^{2d}(S_X, \mathbb{Z}(d)_{S_X}^{\mathcal{D}^*}) \rightarrow H^{2d}(S_X, \mathbb{Z}(d)_{S_X}) = \mathbb{Z} \oplus \mathbb{Z}$ coincides with the degree map. The same construction on X gives rise to the diagram

$$(10.1) \quad \begin{array}{ccc} \mathcal{Z}_0(S_X, D) & \xrightarrow{i_-^*} & \mathcal{Z}_0(X, D) \subset \mathcal{Z}_0(X) \\ \mathrm{cyc}_{S_X}^{\mathcal{D}} \downarrow & & \downarrow \mathrm{cyc}_X^{\mathcal{D}} \\ \mathbb{H}^{2d}(S_X, \mathbb{Z}(d)_{S_X}^{\mathcal{D}^*}) & \xrightarrow{i_-^*} & \mathbb{H}^{2d}(X, \mathbb{Z}(d)_X^{\mathcal{D}}) \end{array}$$

whose commutativity is easily checked. We also note that $\mathrm{cyc}^{\mathcal{D}}$ commutes with the map

$$\Delta^* : \mathcal{Z}_0(X, D) \rightarrow \mathcal{Z}_0(S_X, D).$$

By [9, Lemma 2.6], the cycle class map $\mathrm{cyc}_{S_X}^{\mathcal{D}}$ factors through the Chow group $\mathrm{CH}_0(S_X)$ and therefore determines a homomorphism

$$\rho_{S_X} : \mathrm{CH}_0(S_X)_{\mathrm{deg} 0} \rightarrow A^d(S_X),$$

where $\mathrm{CH}_0(S_X)_{\mathrm{deg} 0}$ denotes the kernel of the degree map $\mathrm{deg} : \mathrm{CH}_0(S_X) \rightarrow \mathbb{Z} \oplus \mathbb{Z}$, that is the generalized Abel-Jacobi map of [9]. Since the cycle class map to Deligne cohomology anyway factors through the usual Chow group of 0-cycles for smooth projective varieties (see [10]), we have then the following commutative diagram, with split exact rows

$$(10.2) \quad \begin{array}{ccccccc} 0 \rightarrow \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} & \xrightarrow{p_{+,*}} & \mathrm{CH}_0(S_X)_{\mathrm{deg} 0} & \xrightleftharpoons[\Delta^*]{i_-^*} & \mathrm{CH}_0(X)_{\mathrm{deg} 0} & \rightarrow & 0 \\ \rho_{X|D} \downarrow & & \downarrow \rho_{S_X} & & \downarrow \rho_X & & \\ 0 \rightarrow A^d(X|D) & \xrightarrow{p_{+,*}} & A^d(S_X) & \xrightleftharpoons[\Delta^*]{i_-^*} & A^d(X) & \rightarrow & 0, \end{array}$$

where $\rho_X : \mathrm{CH}_0(X)_{\mathrm{deg} 0} \rightarrow A^d(X)$ is the usual Abel-Jacobi map and

$$\rho_{X|D} : \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \rightarrow A^d(X|D)$$

is the induced map on the kernels. Note that thanks to the existence of the splitting Δ^* of i_-^* and its compatibility with the degree maps, the exactness of the first row follows immediately from our main Theorem 7.1. We shall call $\rho_{X|D}$ the *Abel-Jacobi map with modulus*.

10.2. Regularity of $\rho_{X|D}$. We now fix base points $x_{0,\pm} = x_0$ on each irreducible components of $(S_X)_{\text{reg}}$ and consider the diagram

$$(10.3) \quad \begin{array}{ccccc} X_+^o \amalg X_-^o & \xrightarrow{\pi_{x_{0,\pm}}^{(S_X)_{\text{reg}}}} & \text{CH}_0(S_X)_{\text{deg } 0} & \twoheadrightarrow & A^d(S_X) \\ \psi \downarrow & & \downarrow i_-^* & & \downarrow \widetilde{i}_-^* \\ X^o & \xrightarrow{\pi_{x_0}^{X^o}} & \text{CH}_0(X)_{\text{deg } 0} & \twoheadrightarrow & A^d(X), \end{array}$$

where $\psi|_{X_-} = \text{Id}_X$ and $\psi|_{X_+} = x_0$ so that the left square commutes.

ψ is a morphism of schemes and the composite map on the bottom is a regular morphism of schemes. It follows that the composite map

$$X_+^o \amalg X_-^o \rightarrow \text{CH}_0(S_X)_{\text{deg } 0} \rightarrow \text{CH}_0(X)_{\text{deg } 0} \rightarrow A^d(X)$$

is a morphism of schemes. It follows from the universal property of $A^d(S_X)$ that there is a unique regular homomorphism of algebraic groups $\widetilde{i}_-^* : A^d(S_X) \rightarrow A^d(X)$ such that the right square commutes.

On the other hand, the right square in (10.2) also commutes. Since $\text{CH}_0(S_X)_{\text{deg } 0} \rightarrow A^d(S_X)$ is surjective, it follows that $\widetilde{i}_-^* = i_-^*$. We conclude that the map i_-^* on the bottom row of (10.2) is a regular homomorphism of connected commutative algebraic groups. Since $A^d(X|D)$ is the inverse image of the identity element under this homomorphism, it follows that $A^d(X|D)$ is a commutative algebraic group. We have thus shown that $A^d(S_X)$ is an extension of the abelian variety $A^d(X)$ by the connected commutative algebraic group $A^d(X|D)$.

Lemma 10.1. *The group homomorphism $\rho_{X|D} : \text{CH}_0(X|D)_{\text{deg } 0} \rightarrow A^d(X|D)$ is regular and surjective, making $A^d(X|D)$ a regular quotient of $\text{CH}_0(X|D)_{\text{deg } 0}$.*

Proof. The surjectivity of the generalized Abel-Jacobi map $\rho_{X|D}$ is a consequence of the definition. Indeed, ρ_{S_X} is surjective by Theorem 9.5 while ρ_X is classically known to be surjective. The surjectivity of $\rho_{X|D}$ follows then from the existence of the splitting

$$\Delta^* : \text{CH}_0(X)_{\text{deg } 0} \rightarrow \text{CH}_0(S_X)_{\text{deg } 0}$$

that makes the induced map $\text{Ker}(\rho_{S_X}) \rightarrow \text{Ker}(\rho_X)$ surjective (see (10.2)).

For the regularity, let V be an open dense subset of $(S_X)_{\text{reg}}$ such that $\rho_{S_X} \circ \pi_{x_0}^V$ is regular. By Theorem 9.5, such V exists. Up to shrinking V further, we can assume that V is of the form $U \amalg U$, for $U \subset X^o$ open (dense) subset of X disjoint from D . Let $i_{U,+}$ denote the inclusion $U \rightarrow U \amalg U$ of the first component. Then we clearly have

$$\rho_{S_X} \circ p_{+,*} \circ \pi_{x_0}^U = \rho_{S_X} \circ \pi_{x_{0,\pm}}^{U \amalg U} \circ i_{U,+}$$

so that the composition $U \rightarrow \text{CH}_0(X|D)_{\text{deg } 0} \rightarrow A^d(X|D) \hookrightarrow A^d(S_X)$ is a morphism of schemes. Since $A^d(X|D) \hookrightarrow A^d(S_X)$ is a closed immersion, we get the claim. \square

10.3. Universality of $\rho_{X|D}$. Our next goal is to prove that the Abel-Jacobi map with modulus $\rho_{X|D} : \text{CH}_0(X|D)_{\text{deg } 0} \rightarrow A^d(X|D)$ makes $A^d(X|D)$ the universal regular quotient of $\text{CH}_0(X|D)_{\text{deg } 0}$.

Lemma 10.2. *The homomorphism $\Delta^* : A^d(X) \rightarrow A^d(S_X)$ of (10.2) is a morphism of schemes.*

Proof. We have seen in the proof of Lemma 10.1 that there is a dense open subset $U \subset X^o$ and a closed point $x_0 = x_{0,\pm} \in U$ such that the composition $U \amalg U \xrightarrow{\pi_{x_{0,\pm}}^{U \amalg U}} \text{CH}_0(S_X)_{\text{deg } 0} \rightarrow A^d(S_X)$ is a morphism of schemes. Let $i_{U,\pm} : U \hookrightarrow U \amalg U$ denote the inclusions into the first and the

second component, respectively. Again from the proof of Lemma 10.1, we have that the maps $\rho_{S_X} \circ \pi_{x_0, \pm}^{UIU} \circ i_{U, \pm}$ are both morphisms of schemes. In particular, the composition

$$(10.4) \quad \theta_U: U \xrightarrow{\psi} A^d(S_X) \times A^d(S_X) \xrightarrow{+} A^d(S_X)$$

is also a morphism of schemes, where $\psi = (\rho_{S_X} \circ \pi_{x_0, \pm}^{UIU} \circ i_{U, +}, \rho_{S_X} \circ \pi_{x_0, \pm}^{UIU} \circ i_{U, -})$ and the second arrow in (10.4) is the addition.

We now consider a diagram

$$(10.5) \quad \begin{array}{ccccc} U & \xrightarrow{\pi_{x_0}^U} & \mathrm{CH}_0(X)_{\mathrm{deg} 0} & \xrightarrow{\rho_X} & A^d(X) \\ \parallel & & \downarrow \Delta^* & & \downarrow \\ U & \longrightarrow & \mathrm{CH}_0(S_X)_{\mathrm{deg} 0} & \xrightarrow{\rho_{S_X}} & A^d(S_X), \end{array}$$

where the first arrow on the bottom is $(\pi_{x_0, \pm}^{UIU} \circ i_{U, +}) + (\pi_{x_0, \pm}^{UIU} \circ i_{U, -})$. It is clear from the definition of Δ^* in the middle that the left square of (10.5) commutes. The composite map on the bottom is same as θ_U , which we just showed above to be a morphism of schemes. We conclude that the map $\rho_{S_X} \circ \Delta^*$ is a regular homomorphism. It follows from the universality of $A^d(X)$ that there is a unique morphism of algebraic groups $\widetilde{\Delta}^*: A^d(X) \rightarrow A^d(S_X)$ such that the right square of (10.5) commutes.

On the other hand, we have seen in (10.2) that right square also commutes if we replace $\widetilde{\Delta}^*$ by Δ^* . Since ρ_X is surjective, we must have $\widetilde{\Delta}^* = \Delta^*$. In particular, Δ^* is morphism of schemes. \square

10.3.1. To prove the universality of $A^d(X|D)$, we begin with the following construction. Consider the homomorphism

$$\tau^*: A^d(S_X) \rightarrow A^d(S_X), \quad \tau^* = \mathrm{Id}_{S_X} - \Delta^* \circ i_-^*.$$

Note that Δ^* is a morphism of schemes by Lemma 10.2 and we have shown in § 10.2 that i_-^* is also a morphism of schemes. We conclude that τ^* is morphism of algebraic groups.

Note that τ^* uniquely factors through $A^d(X|D)$, since $i_-^* \circ \tau^* = 0$ and we have already identified $A^d(X|D)$ with the fiber over the identity element of $A^d(X)$ via i_-^* . The map τ^* gives then an explicit isomorphism of algebraic groups

$$(10.6) \quad (\tau^*, i_-^*): A^d(S_X) \xrightarrow{\cong} A^d(X|D) \times A^d(X).$$

Moreover, since $i_*^* \circ \Delta^* = \mathrm{Id}_{A^d(X)}$, we get an extension of commutative algebraic groups

$$(10.7) \quad 0 \rightarrow A^d(X) \xrightarrow{\Delta^*} A^d(S_X) \xrightarrow{\tau^*} A^d(X|D) \rightarrow 0.$$

We now claim that the diagram

$$(10.8) \quad \begin{array}{ccc} \mathrm{CH}_0(S_X)_{\mathrm{deg} 0} & \xrightarrow{\tau_X^*} & \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \\ \rho_{S_X} \downarrow & & \downarrow \rho_{X|D} \\ A^d(S_X) & \xrightarrow{\tau^*} & A^d(X|D) \end{array}$$

commutes. To see this, we can write, using Theorem 7.1, any element $\alpha \in \mathrm{CH}_0(S_X)_{\mathrm{deg} 0}$ as $\alpha = p_{+,*}(\alpha_1) + \Delta^*(\alpha_2)$. Since $\tau_X^* \circ \Delta^* = 0$, by definitions, we get $\rho_{X|D} \circ \tau_X^*(\alpha) = \rho_{X|D}(\alpha_1)$ by (10.2).

On the other hand, $\tau^* \circ \Delta^* = 0$ and $\tau^* \circ p_{+,*} = \text{Id}_{A^d(X|D)}$ so that

$$\begin{aligned} \tau^* \circ \rho_{S_X}(\alpha) &= \tau^* \circ \rho_{S_X} \circ p_{+,*}(\alpha_1) + \tau^* \circ \rho_{S_X} \circ \Delta^*(\alpha_2) \\ &\stackrel{\dagger}{=} \tau^* \circ p_{+,*} \circ \rho_{X|D}(\alpha_1) + \tau^* \circ \Delta^* \circ \rho_X(\alpha_2) \\ &= \rho_{X|D}(\alpha_1), \end{aligned}$$

where $\stackrel{\dagger}{=}$ follows from (10.2). This proves the commutativity of (10.8).

Theorem 10.3. *The Abel-Jacobi map $\rho_{X|D}: \text{CH}_0(X|D)_{\text{deg } 0} \rightarrow A^d(X|D)$ makes $A^d(X|D)$ the universal regular quotient of $\text{CH}_0(X|D)_{\text{deg } 0}$.*

Proof. We only need to prove the universality. Let G be a commutative algebraic group and let $\psi: \text{CH}_0(X|D)_{\text{deg } 0} \rightarrow G$ be a regular homomorphism. Let $U \subset X^o$ be an open dense subset so that the composite $\psi \circ \pi_{x_0}^U$ is a morphism of schemes, for $x_0 \in U$ a base point. Let $V = U \amalg U$. We claim that $\delta = \psi \circ \tau_X^* \circ \pi_{x_0, \pm}^V$ is a morphism of schemes, where $\tau_X^*: \text{CH}_0(S_X)_{\text{deg } 0} \rightarrow \text{CH}_0(X|D)_{\text{deg } 0}$ is splitting the map $p_{+,*}$ of Theorem 7.1. Indeed, it is actually enough to show that the restriction of δ to the two components U_{\pm} of V is a morphism of schemes. But we have, for $x \in U_+$:

$$\psi \circ \tau_X^* \circ \pi_{x_0, \pm}^V(x) = \psi \circ \tau_X^*([x]_+ - [x_0, +]) = \psi([x] - [x_0]) = \psi \circ \pi_{x_0}^U(x),$$

where $[x]_+$ denotes the class in $\text{CH}_0(S_X)$ of the closed point x in the component X_+ of S_X . Since $\psi \circ \pi_{x_0}^U$ is by assumption a morphism of schemes, this proves the claim for the restriction to U_+ . Similarly for $x \in U_-$, we have

$$\psi \circ \tau_X^* \circ \pi_{x_0, \pm}^V(x) = \psi \circ \tau_X^*([x]_- - [x_0, -]) = \psi(-[x] + [x_0]) = -\psi \circ \pi_{x_0}^U(x)$$

that is also a morphism of schemes, since G is an algebraic group.

By the claim and Theorem 9.5, there is then a unique morphism of algebraic groups $\tilde{\psi}: A^d(S_X) \rightarrow G$ such that there is a commutative square

$$\begin{array}{ccc} \text{CH}_0(S_X)_{\text{deg } 0} & \xrightarrow{\tau_X^*} & \text{CH}_0(X|D)_{\text{deg } 0} \\ \rho_{S_X} \downarrow & & \downarrow \psi \\ A^d(S_X) & \xrightarrow{\tilde{\psi}} & G. \end{array}$$

We now claim that the composition

$$A^d(X) \xrightarrow{\Delta^*} A^d(S_X) \xrightarrow{\tilde{\psi}} G$$

is equal to the constant map $A^d(X) \rightarrow 0$. Indeed, since the left square of the diagram

$$\begin{array}{ccccc} \text{CH}_0(X)_{\text{deg } 0} & \xrightarrow{\Delta^*} & \text{CH}_0(S_X)_{\text{deg } 0} & \xrightarrow{\tau_X^*} & \text{CH}_0(X|D)_{\text{deg } 0} \\ \rho_X \downarrow & & \rho_{S_X} \downarrow & & \downarrow \psi \\ A^d(X) & \xrightarrow{\Delta^*} & A^d(S_X) & \xrightarrow{\tilde{\psi}} & G \end{array}$$

also commutes by (10.2) and since ρ_X is surjective, it is enough to show that $\tilde{\psi} \circ \Delta^* \circ \rho_X = 0$.

But $\tilde{\psi} \circ \Delta^* \circ \rho_X = \tilde{\psi} \circ \rho_{S_X} \circ \Delta^* = \psi \circ (\tau_X^* \circ \Delta_X^*) = 0$ since $\tau_X^* \circ \Delta_X^* = 0$. This proves the claim. Using this fact, the exact sequence (10.7) and the commutative diagram (10.8), it follows immediately that there exists a unique morphism of algebraic groups $\psi_G: A^d(X|D) \rightarrow G$ such that $\psi_G \circ \rho_{X|D} = \psi$. This finishes the proof. \square

10.4. Roitman's theorem for 0-cycles with modulus. The first application of our approach to study algebraic cycles with modulus was already given in Theorem 14.1. As second application, we now prove the following Roitman torsion theorem for 0-cycles with modulus on smooth projective schemes over \mathbb{C} . This will be generalized to positive characteristic in the next section.

Theorem 10.4. *Let X be a smooth projective variety of dimension $d \geq 1$ over \mathbb{C} and let $D \subset X$ be an effective Cartier divisor. Then the Abel-Jacobi map $\rho_{X|D}: \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \rightarrow A^d(X|D)$ induces an isomorphism on the torsion subgroups.*

Proof. We have the following commutative diagram with split exact rows

$$(10.9) \quad \begin{array}{ccccccc} 0 \rightarrow \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} & \xrightarrow{p_{+,*}} & \mathrm{CH}_0(S_X)_{\mathrm{deg} 0} & \xrightarrow{i_-^*} & \mathrm{CH}_0(X)_{\mathrm{deg} 0} & \rightarrow & 0 \\ & & \rho_{S_X} \downarrow & & \rho_X \downarrow & & \\ 0 \rightarrow A^d(X|D) & \xrightarrow{p_{+,*}} & A^d(S_X) & \xrightarrow{i_-^*} & A^d(X) & \rightarrow & 0. \end{array}$$

Since the maps i_-^* on the top and the bottom rows are split by Δ^* , the two sequences remain exact even after passing to the torsion subgroups. The statement then follows from the theorem of Roitman [39] for ρ_X and the theorem of Biswas-Srinivas [6] for ρ_{S_X} . \square

10.5. Bloch's conjecture for 0-cycles with modulus. Let X be a reduced projective surface over \mathbb{C} . Recall that the Chow group of 0-cycles $\mathrm{CH}_0(X)$ is said to be *finite dimensional* if the Abel-Jacobi map $\rho_X: \mathrm{CH}_0(X)_{\mathrm{deg} 0} \rightarrow A^2(X)$ is an isomorphism. Recall that the famous Bloch conjecture about 0-cycles on surfaces says the following.

Conjecture 10.5. (Bloch) *Let X be a smooth projective surface over \mathbb{C} such that $H^2(X, \mathcal{O}_X) = 0$. Then $\mathrm{CH}_0(X)$ is finite dimensional.*

This conjecture is known to be true for surfaces of Kodaira dimension less than two [8] and is open in general. It has been shown by Voisin [49] that a generalized version of this conjecture in higher dimensions is very closely related to the Hodge conjecture.

Let X be a smooth projective surface over \mathbb{C} and let $D \subset X$ be an effective Cartier divisor. Let \mathcal{I}_D denote the subsheaf of ideals in \mathcal{O}_X defining D . We shall say that $\mathrm{CH}_0(X|D)$ is finite dimensional if the map $\rho_{X|D}: \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \rightarrow A^d(X|D)$ is an isomorphism. We can now state the following analogue of the Bloch conjecture for 0-cycles with modulus.

Conjecture 10.6. *Let X be a smooth projective surface over \mathbb{C} . Let $D \subset X$ be an effective Cartier divisor such that $H^2(X, \mathcal{I}_D) = 0$. Then $\mathrm{CH}_0(X|D)$ is finite dimensional.*

Remark 10.7. As explained in [5, 2.1.2], the Chow groups with modulus can be used to define a notion of Chow groups with compact support for the complement $X^\circ = X \setminus |D|$. In this perspective, we can view Conjecture 10.6 as an analogue of Bloch's conjecture for the open surface X° .

As an application of Theorem 7.1, we can show the following.

Theorem 10.8. *Let X be a smooth projective surface over \mathbb{C} . Let $D \subset X$ be an effective Cartier divisor such that $H^2(X, \mathcal{I}_D) = 0$. Assume that Conjecture 10.5 holds for X . Then $\mathrm{CH}_0(X|D)$ is finite dimensional.*

Proof. We first observe that $\pi: \overline{S_X} := X \amalg X \rightarrow S_X$ is the normalization map and hence the Bloch conjecture for X is same as that for $\overline{S_X}$. Since $H^2(X, \mathcal{I}_D) \rightarrow H^2(X, \mathcal{O}_X)$, it follows that $H^2(X, \mathcal{O}_X) = 0$. In particular, $\mathrm{CH}_0(X)$ and $\mathrm{CH}_0(\overline{S_X})$ are finite dimensional.

Since the map $\mathcal{I}_{X_-} \rightarrow \mathcal{I}_D$ is an isomorphism (see (8.8)), there is an exact sequence

$$H^2(X, \mathcal{I}_D) \rightarrow H^2(S_X, \mathcal{O}_{S_X}) \rightarrow H^2(X, \mathcal{O}_X) \rightarrow 0.$$

We conclude that $H^2(S_X, \mathcal{O}_{S_X}) = 0$. We now apply [22, Theorem 1.3] to conclude that $\mathrm{CH}_0(S_X)$ is finite dimensional. We remark here that the statement of this cited result assumes validity of Conjecture 10.5 for all smooth surfaces, but its proof (see [22, § 7]) only uses the fact that the normalization of the surface (which is already smooth in our case) is finite-dimensional. We now use (10.9) to conclude that $\mathrm{CH}_0(X|D)$ is finite dimensional. \square

A combination of Theorem 10.8 and [8] yields the following.

Corollary 10.9. *Let X be a smooth projective surface over \mathbb{C} of Kodaira dimension less than two. Let $D \subset X$ be an effective Cartier divisor such that $H^2(X, \mathcal{I}_D) = 0$. Then $\mathrm{CH}_0(X|D)$ is finite dimensional.*

Infinite-dimensionality of $\mathrm{CH}_0(X|D)$: The following result provides examples of smooth projective surfaces X with an effective Cartier divisor $D \subset X$ such that $\mathrm{CH}_0(X)$ is finite-dimensional but $\mathrm{CH}_0(X|D)$ is not. In particular, it provides a partial converse to Theorem 10.8.

Theorem 10.10. *Let X be a smooth projective surface over \mathbb{C} . Let $D \subset X$ be an effective Cartier divisor such that $H^2(X, \mathcal{I}_D) \neq 0$. Assume that X is regular with $p_g(X) = 0$ and that the Bloch conjecture is true for X . Then $\mathrm{CH}_0(X|D)$ is not finite-dimensional.*

Proof. The exact sequence

$$H^1(X, \mathcal{O}_X) \rightarrow H^2(X, \mathcal{I}_D) \rightarrow H^2(S_X, \mathcal{O}_{S_X}) \rightarrow H^2(X, \mathcal{O}_X) \rightarrow 0$$

and our assumption together imply that $H^2(S_X, \mathcal{O}_{S_X}) \neq 0$. We claim that the map

$$\rho_{S_X}: \mathrm{CH}_0(S_X)_{\mathrm{deg} 0} \rightarrow A^2(S_X)$$

is not injective. Suppose on the contrary, that ρ_{S_X} is injective. But then it must be an isomorphism. It follows then from [9, Theorem 7.2] (see its proof on pg. 657) that there are finitely many reduced Cartier curves $\{C_1, \dots, C_r\}$ on X such that the map $\bigoplus_{i=1}^r \mathrm{CH}_0(C_i)_{\mathrm{deg} 0} \rightarrow \mathrm{CH}_0(S_X)_{\mathrm{deg} 0}$ is surjective. However, as $H^2(S_X, \mathcal{O}_{S_X}) \neq 0$, [46, Theorem 5.2] tells us that this is not possible. This proves the claim. Our assumption says that the map ρ_X in (10.9) is an isomorphism. It follows that $\mathrm{Ker}(\rho_{X|D}) \neq 0$. \square

If we let $D \subset \mathbb{P}_{\mathbb{C}}^2$ be a smooth hypersurface of degree 3, we have

$$H^2(\mathbb{P}_{\mathbb{C}}^2, \mathcal{I}_D) \simeq H^2(\mathbb{P}_{\mathbb{C}}^2, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(-D)) \simeq H^2(\mathbb{P}_{\mathbb{C}}^2, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(-3)) \simeq \mathbb{C}.$$

All conditions of Theorem 10.10 are clearly satisfied and we get an example of a (smooth) pair (X, D) such that $\mathrm{CH}_0(X)_{\mathrm{deg} 0} = 0$ but $\mathrm{CH}_0(X|D)$ is infinite-dimensional.

11. ALBANESE WITH MODULUS IN ARBITRARY CHARACTERISTIC

In this section, we generalize the results of § 10 for 0-cycles with modulus on smooth projective varieties over an arbitrary algebraically closed field. Most of the arguments are straightforward copies of those in § 10. So we keep the discussion brief. We fix an algebraically closed field k of exponential characteristic $p \geq 1$.

Let again Y be a projective reduced k -variety of dimension d and write $\mathrm{Alb}(Y)$ for the Esnault-Srinivas-Viehweg Albanese variety. While there is an explicit description of $\mathrm{Alb}(Y) = A^d(Y)$ over \mathbb{C} (as recalled in 8.2) using Hodge theory, in positive characteristic, [9] gives only an existence statement and little is known on the properties of $\mathrm{Alb}(Y)$ (some pathological properties of $\mathrm{Alb}(Y)$ for specific singular varieties are studied in [9, § 3]).

In [34], V. Mallick proves the following Roitman-style theorem.

Theorem 11.1. ([34, Theorem 16]) *For any reduced projective variety Y of dimension d over k and for n coprime with the characteristic of k , the map ρ_Y induces an isomorphism on n -torsion subgroups*

$$\rho_Y: {}_n\mathrm{CH}_0^{LW}(Y)_{\mathrm{deg} 0} \xrightarrow{\cong} {}_n\mathrm{Alb}(Y) = {}_nJ^d(Y).$$

For the rest of this section, the Albanese variety $\mathrm{Alb}(Y)$ will be denoted by $A^d(Y)$ to keep consistency with the notations of the previous sections.

11.1. Albanese with modulus and its universality in any characteristic. Let X be a smooth projective scheme of dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. Write as usual S_X for the double construction applied to the pair (X, D) . Using Theorem 7.3, we shall again identify the two Chow groups $\mathrm{CH}_0^{LW}(S_X)$ and $\mathrm{CH}_0(S_X)$ throughout this section.

Let U be an open dense subset contained in $X^\circ = X \setminus D$, $x_0 \in U$ a closed point and $V = U \amalg U \subset (S_X)_{\mathrm{reg}}$ such that the compositions

$$\pi_{x_0, \pm}^V: V \rightarrow \mathrm{CH}_0(S_X)_{\mathrm{deg} 0} \xrightarrow{\rho_{S_X}} A^d(S_X), \quad \pi_{x_0}^U: U \rightarrow \mathrm{CH}_0(X)_{\mathrm{deg} 0} \xrightarrow{\rho_X} A^d(X)$$

are morphisms of schemes (see Definition 9.10 for the notation $\pi_{x_0, \pm}^V$ and $\pi_{x_0}^U$).

Theorem 7.1 gives the familiar split short exact sequence on the group of zero cycles

$$(11.1) \quad 0 \rightarrow \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \xrightarrow{p_{+, *}} \mathrm{CH}_0(S_X)_{\mathrm{deg} 0} \xrightarrow{\iota_-^*} \mathrm{CH}_0(X)_{\mathrm{deg} 0} \rightarrow 0$$

and there is a homomorphism $\Delta^*: \mathrm{CH}_0(X)_{\mathrm{deg} 0} \rightarrow \mathrm{CH}_0(S_X)_{\mathrm{deg} 0}$ such that $\iota_-^* \circ \Delta^* = \mathrm{Id}$.

It follows from the discussion in § 10.2 and the proof of Lemma 10.2 that the homomorphisms $\rho_X \circ i_-^*: \mathrm{CH}_0(S_X)_{\mathrm{deg} 0} \rightarrow A^d(X)$ and $\rho_{S_X} \circ \Delta^*: \mathrm{CH}_0(X)_{\mathrm{deg} 0} \rightarrow A^d(S_X)$ are both regular. It follows from the universality of $A^d(X)$ and $A^d(S_X)$ that there are unique homomorphisms of algebraic groups $i_-^*: A^d(S_X) \rightarrow A^d(X)$ and $\Delta^*: A^d(X) \rightarrow A^d(S_X)$ such that the diagram

$$(11.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} & \xrightarrow{p_{+, *}} & \mathrm{CH}_0(S_X)_{\mathrm{deg} 0} & \xleftarrow[\Delta^*]{i_-^*} & \mathrm{CH}_0(X)_{\mathrm{deg} 0} \rightarrow 0 \\ & & & & \downarrow \rho_{S_X} & & \downarrow \rho_X \\ & & & & A^d(S_X) & \xleftarrow[\Delta^*]{i_-^*} & A^d(X) \rightarrow 0 \end{array}$$

is commutative.

It also follows from this commutative diagram that $i_-^* \circ \Delta^*: A^d(X) \rightarrow A^d(X)$ is the identity. Indeed, any $\alpha \in A^d(X)$ can be written as $\alpha = \rho_X(\beta)$ and then

$$(11.3) \quad i_-^* \circ \Delta^* \circ \rho_X(\beta) = i_-^* \circ \rho_{S_X} \circ \Delta^*(\beta) = \rho_X \circ i_-^* \circ \Delta^*(\beta) = \rho_X(\beta).$$

Definition 11.2. We define $A^d(X|D)$ to be the closed algebraic subgroup of $A^d(S_X)$ given by the inverse image of the identity element of $A^d(X)$ via ι_-^* . As before, we denote the inclusion $A^d(X|D) \hookrightarrow A^d(S_X)$ by $p_{+, *}$. It follows from (11.2) that there is a unique surjective homomorphism $\rho_{X|D}: \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \twoheadrightarrow A^d(X|D)$ such that $\rho_{S_X} \circ p_{+, *} = p_{+, *} \circ \rho_{X|D}$.

The reader can now check easily that using (11.2) and (11.3), the proofs of Lemma 10.1 and Theorem 10.3 go through mutatis mutandis to give the following generalization of the latter.

Theorem 11.3. *The group homomorphism $\rho_{X|D}: \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \rightarrow A^d(X|D)$ is regular and surjective and makes $A^d(X|D)$ the universal regular quotient of $\mathrm{CH}_0(X|D)_{\mathrm{deg} 0}$.*

Remark 11.4. Since X is smooth and projective, the Albanese variety $A^d(X)$ is an abelian variety over k . By [9], the generalized Albanese $A^d(S_X)$ is a smooth commutative algebraic group of general type, i.e., an extension of an abelian variety by a linear algebraic group. An immediate consequence of the definition and of Theorem 11.3 is that the linear part of $A^d(S_X)$ coincides with the linear part of the Albanese with modulus $A^d(X|D)$.

11.2. Roitman theorem in arbitrary characteristic. Using Theorem 11.1, we can now generalize Theorem 10.4 over any algebraically closed field as follows. We keep the notations and the assumptions of 11.1.

Theorem 11.5. *Let X be a smooth projective variety of dimension $d \geq 1$ over an algebraically closed field k and let $D \subset X$ be an effective Cartier divisor. Let n be an integer prime to the exponential characteristic of k . Then the Albanese map $\rho_{X|D}: \mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \rightarrow A^d(X|D)$ induces an isomorphism on n -torsion subgroups*

$$\rho_{X|D}: {}_n\mathrm{CH}_0(X|D)_{\mathrm{deg} 0} \xrightarrow{\cong} {}_nA^d(X|D).$$

Proof. We consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & {}_n\mathrm{CH}_0(X|D)_{\mathrm{deg} 0} & \xrightarrow{p_{+,*}} & {}_n\mathrm{CH}_0(S_X)_{\mathrm{deg} 0} & \xrightarrow{i_-^*} & {}_n\mathrm{CH}_0(X)_{\mathrm{deg} 0} \rightarrow 0 \\ & & \rho_{X|D} \downarrow & & \rho_{S_X} \downarrow \simeq & & \downarrow \rho_X \\ 0 & \rightarrow & {}_nA^d(X|D) & \xrightarrow{p_{+,*}} & {}_nA^d(S_X) & \xrightarrow{i_-^*} & {}_nA^d(X) \rightarrow 0 \end{array}$$

It follows from (11.2) that the top and the bottom rows are compatibly split exact. The original Roitman's theorem [39] says that the vertical arrow on the left is an isomorphism. It follows from Theorems 7.3 and 11.1 that the middle vertical arrow is an isomorphism. The theorem follows. \square

Remark 11.6. We note that the linear part of $A^d(X|D)$ (that coincides with the linear part of $A^d(S_X)$) depends heavily on the geometry of D and will have, in general, both a unipotent and a torus part. For example, if D is a smooth divisor inside a smooth surface, the presence of a \mathbb{G}_m part in $A^d(X|D)$ depends on the class of D in the Neron-Severi group of X .

12. CYCLE CLASS MAP TO RELATIVE K -THEORY

As we mentioned in § 1, one of the motivations for studying cycles with modulus is to find a cohomology theory which can describe relative K -theory of divisors in a scheme in terms of algebraic cycles. If the higher Chow groups with modulus are indeed the right objects which serve this purpose, there must be a cycle class map from the Chow groups with modulus to relative K -groups. Moreover, this map must describe the Chow group as a part of relative K -groups in most of the cases. Our goal in this section is to use our double construction to answer these questions for the 0-cycles with modulus. We first consider the case of line bundles in this setting.

12.1. Vector bundles on the double and relative Picard groups. Let k be any field and let X be a smooth quasi-projective scheme over k with an effective Cartier divisor D . Let X° be the open complement $X \setminus D$. We denote as above by S_X the double $S(X, D)$.

12.1.1. Vector bundles on the double. Let $\mathcal{P}_S = \mathcal{P}_{S(X, D)}$ denote the category of locally free sheaves of finite rank on S_X . Since X is quasi-projective, S_X is quasi-projective as well and therefore it admits an ample family of line bundles. Thus we can replace Thomason spectrum $K(S_X)$ built out of perfect complexes on S_X with $\Omega BQ\mathcal{P}_S$, at least for computing the groups $K_p(S_X)$ for $p \geq 0$, and similarly for X and D . By construction, the category \mathcal{P}_S is equivalent to the category of triples (E, E', ϕ) , where E and E' are locally free sheaves on

X and $\phi: \iota_D^* E \rightarrow \iota_D^* E'$ is a fixed isomorphism on the restriction to D (see [32], but also [36, Theorem 2.1]). This description gives us the following.

Lemma 12.1. *The composite map of spectra*

$$(12.1) \quad K(S_X) \xrightarrow{\iota_1^*, \iota_2^*} K(X) \amalg K(X) \xrightarrow{f} K(D)$$

is homotopy trivial, where

$$f = \iota_D^* \oplus -\iota_D^*: K(X) \amalg K(X) \xrightarrow{(id, -id)} K(X) \amalg K(X) \xrightarrow{\iota_D^* + \iota_D^*} K(D).$$

Notice that the maps in the definition of f make sense because they are defined in the homotopy category of spectra which is an additive category.

12.1.2. *Relative Picard group.* We denote by $\text{Pic}(X, D)$ the group of isomorphism classes of pairs (\mathcal{L}, σ) consisting of a line bundle \mathcal{L} on X together with a fixed trivialization $\sigma: \mathcal{L}|_D \xrightarrow{\cong} \mathcal{O}_D$ along D , under tensor product operation. It is called the relative Picard group of the pair (X, D) . Write G for the group of isomorphism classes of triples $\{(\mathcal{L}_+, \mathcal{L}_-, \phi)\}$ for \mathcal{L}_\pm line bundles on X with a given isomorphism $\phi: \mathcal{L}_+|_D \xrightarrow{\cong} \mathcal{L}_-|_D$ along D . The above description of the category of vector bundles on S_X gives in particular two maps, one inverse to the other

$$\theta: \text{Pic}(S_X) \rightarrow G, \quad \eta: G \rightarrow \text{Pic}(S_X)$$

defined by

$$\theta(\mathcal{L}) = (\mathcal{L}_+ = \iota_+^*(\mathcal{L}), \mathcal{L}_- = \iota_-^*(\mathcal{L}), \phi: \iota_D^* \iota_+^* \mathcal{L} \simeq \iota_D^* \iota_-^* \mathcal{L}), \quad \eta((\mathcal{L}_+, \mathcal{L}_-, \phi)) = \mathcal{L}_+ \times_\phi \mathcal{L}_-$$

where $\mathcal{L}_+ \times_\phi \mathcal{L}_-$ is the gluing of \mathcal{L}_+ and \mathcal{L}_- along ϕ .

We will then identify the group $\text{Pic}(S_X)$ with G . In this way we can easily define maps

$$p_{\pm, *}: \text{Pic}(X, D) \rightrightarrows \text{Pic}(S_X), \quad \tau_X^*: \text{Pic}(S_X) \rightarrow \text{Pic}(X, D)$$

using formally the same definitions that we gave for 0-cycles in § 5. Explicitly, we have

$$\tau_X^*((\mathcal{L}_+, \mathcal{L}_-, \phi)) = (\mathcal{L}_+ \otimes \mathcal{L}_-^{-1}, \phi \otimes \text{id}_{\iota_D^* \mathcal{L}_-^{-1}})$$

for $\phi \otimes \text{id}_{\iota_D^* \mathcal{L}_-^{-1}}: \iota_D^*(\mathcal{L}_+ \otimes \mathcal{L}_-^{-1}) = \iota_D^* \mathcal{L}_+ \otimes_{\mathcal{O}_D} \iota_D^* \mathcal{L}_-^{-1} \xrightarrow{\phi \otimes 1} \iota_D^*(\mathcal{L}_-) \otimes \iota_D^*(\mathcal{L}_-)^{-1} \xrightarrow{\text{can}} \mathcal{O}_D$, and $p_{+, *}(\mathcal{L}, \sigma) = \mathcal{L} \times_\sigma \mathcal{O}_X = (\mathcal{L}, \mathcal{O}_X, \sigma)$ (and similarly for $p_{-, *}$).

It is immediate to check that $p_{\pm, *}$ are injective, splitting τ_X^* . Moreover, we have maps

$$\iota_\pm^*: \text{Pic}(S_X) \rightrightarrows \text{Pic}(X), \quad \Delta_X^*: \text{Pic}(X) \rightarrow \text{Pic}(S_X)$$

given on isomorphism classes by $\iota_+^*((\mathcal{L}_+, \mathcal{L}_-, \phi)) = \mathcal{L}_+$, $\iota_-^*((\mathcal{L}_+, \mathcal{L}_-, \phi)) = \mathcal{L}_-$ and $\Delta_X^*(\mathcal{L}) = (\mathcal{L}, \mathcal{L}, \text{id})$. And one clearly has that the composition $\iota_\pm^* \circ \Delta_X^*$ is the identity.

We summarize the result in the following Proposition, that is the analogue of Theorem 7.1 for line bundles and is used in § 8.

Proposition 12.2. *Let X be a smooth quasi-projective scheme over k and let $D \subset X$ be an effective Cartier divisor. Then there are maps*

$$\Delta^*: \text{Pic}(X) \rightarrow \text{Pic}(S_X); \quad \text{and} \quad \iota_\pm^*: \text{Pic}(S_X) \rightarrow \text{Pic}(X);$$

$$\tau_X^*: \text{Pic}(X) \rightarrow \text{Pic}(X, D); \quad \text{and} \quad p_{\pm, *}: \text{Pic}(X, D) \rightarrow \text{Pic}(S_X)$$

such that $\iota_\pm^* \circ \Delta^* = \text{Id}$ on $\text{Pic}(X)$ and $\tau_X^* \circ p_{\pm, *} = \pm \text{Id}$ on $\text{Pic}(X, D)$. Moreover, the sequences

$$0 \rightarrow \text{Pic}(X, D) \xrightarrow{p_{+, *}} \text{Pic}(S_X) \xrightarrow{\iota_-^*} \text{Pic}(X) \rightarrow 0;$$

$$0 \rightarrow \text{Pic}(X) \xrightarrow{\Delta^*} \text{Pic}(S_X) \xrightarrow{\tau_X^*} \text{Pic}(X, D) \rightarrow 0$$

are split exact.

12.2. Cycle class map for 0-cycles with modulus. The goal of this subsection is the proof of Theorem 1.5. In order to do define the cycle class map from the Chow group with modulus to relative K -group and prove its injectivity, we need to have an analogue of Theorem 7.1 for the relative K -theory. Our strategy for proving this is to first construct a variant of relative K -theory, which we call the K -theory with modulus, for which Theorem 7.1 is immediate. We then show that this K -theory with modulus coincides with the known relative K -theory in as many cases as possible.

Recall that for any map of schemes $f: X \rightarrow Y$, the relative K -theory of the pair (X, Y) is the spectrum defined as the homotopy fiber of the map $f^*: K(Y) \rightarrow K(X)$. Let X be a smooth quasi-projective scheme over k and let $D \subset X$ be an effective Cartier divisor. Let $K(X|D)$ denote the homotopy fiber of the restriction map $i_-^*: K(S_X) \rightarrow K(X)$. It is clear that $K(X|D)$ is another notation for the relative K -theory $K(S_X, X_-)$. We call $K(X|D)$ the K -theory with modulus. We have $i_-^* \circ \Delta^* = \text{Id}_{K(X)}$ and a commutative diagram of homotopy fiber sequences

$$(12.2) \quad \begin{array}{ccccc} K(X|D) & \xrightarrow{p_{+,*}} & K(S_X) & \xrightarrow{i_-^*} & K(X) \\ \phi \downarrow & & \downarrow i_+^* & & \downarrow i_-^* \\ K(X, D) & \longrightarrow & K(X) & \xrightarrow{i_+^*} & K(D). \end{array}$$

We have the following analogue of Proposition 8.1 for affine schemes.

Proposition 12.3. ([36, Theorem 6.2, Lemma 4.1]) *Let $X = \text{Spec}(A)$ be an affine scheme and let I be the ideal defining $D \subset X$. Then the map ϕ defines isomorphisms*

$$\phi_i: K_i(X|D) \xrightarrow{\sim} K_i(X, D) \quad \text{for } i = 0, 1.$$

We do not know if Proposition 12.3 is true for non-affine schemes. But we can show that this is indeed the case for curves and surfaces when $i = 0$. The case of curves follows directly from Proposition 12.2. We shall prove this for surfaces in Proposition 13.2.

We can now prove the main result of this section:

Theorem 12.4. *Let X be a smooth quasi-projective scheme of dimension $d \geq 1$ over a perfect field k and let $D \subset X$ be an effective Cartier divisor. Then, there is a cycle class map*

$$\text{cyc}_{X|D}: \text{CH}_0(X|D) \rightarrow K_0(X, D).$$

This map is injective if k is algebraically closed and X is affine.

Proof. We have a commutative diagram of short exact sequences

$$(12.3) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{CH}_0(X|D) & \xrightarrow{p_{+,*}} & \text{CH}_0(S_X) & \xrightarrow{i_-^*} & \text{CH}_0(X) \rightarrow 0 \\ & & \downarrow & & \downarrow \text{cyc}_{S_X} & & \downarrow \text{cyc}_X \\ 0 & \rightarrow & K_0(X|D) & \xrightarrow{p_{+,*}} & K_0(S_X) & \xrightarrow{i_-^*} & K_0(X) \rightarrow 0 \\ & & \downarrow \phi_0 & & \downarrow i_+^* & & \downarrow i_-^* \\ & & K_0(X, D) & \longrightarrow & K_0(X) & \longrightarrow & K_0(D). \end{array}$$

The maps cyc_X and cyc_{S_X} are the cycle class maps of Lemma 3.13 (where $Y = (S_X)_{\text{sing}}$ for cyc_{S_X} and $Y = \emptyset$ for cyc_X . See also [33, § 2]). The above diagram uniquely defines a cycle class map $\text{map } \widetilde{\text{cyc}}_{X|D}: \text{CH}_0(X|D) \rightarrow K_0(X|D)$ such that $p_{+,*} \circ \widetilde{\text{cyc}}_{X|D} = \text{cyc}_{S_X} \circ p_{+,*}$. We set $\text{cyc}_{X|D} = \phi_0 \circ \widetilde{\text{cyc}}_{X|D}$.

If k is algebraically closed and X is affine, the map cyc_{S_X} is injective by [24, Corollary 7.3]. It follows that $\widetilde{cyc}_{X|D}$ is injective. We conclude proof of the injectivity of $cyc_{X|D}$ using Proposition 12.3. \square

Remark 12.5. There are (at least) two general constructions of a cycle class map

$$cyc_{X|D}^{d+n,n}: \text{CH}^{d+n}(X|D, n) \rightarrow K_n(X, D)$$

from higher Chow groups with modulus in the sense of [5] to higher relative K -groups (here $d = \dim X$). See [3, I.4] for one of them. We will study the properties of this map in a different work.

13. THE CASE OF SURFACES

In this section, we shall apply Theorem 7.1 to establish the relation between cycles with modulus and relative K -theory for surfaces. In particular, we prove a modulus version of Bloch's formula. Before we do this, we need to prove a generalization of Proposition 12.3 for non-affine surfaces.

Lemma 13.1. *Let X be a reduced quasi-projective surface over any field k containing at least three elements. Let $F^2K_0(X)$ be the subgroup of $K_0(X)$ generated by the cycle classes of regular points of X . Then, there is a canonical isomorphism $H^2(X, \mathcal{K}_{2,X}) \xrightarrow{\cong} F^2K_0(X)$.*

Proof. By the Thomason-Trobaugh spectral sequence for K -theory, we get an exact sequence

$$K_1(X) \rightarrow H^0(X, \mathcal{K}_{1,X}) \xrightarrow{\partial} H^2(X, \mathcal{K}_{2,X}) \rightarrow K_0(X).$$

On the other hand, it is well known that the map $K_1(X) \rightarrow H^0(X, \mathcal{K}_{1,X})$ is surjective (see, for example, [23, § 2]). It follows that $H^2(X, \mathcal{K}_{2,X}) \hookrightarrow K_0(X)$.

We are only left to show that $H^2(X, \mathcal{K}_{2,X}) \rightarrow F^2K_0(X)$. But this follows by the results of Levine [28], because he shows that there is a surjective map $\mathcal{Z}_0(X, X_{\text{sing}}) = \coprod_{x \in X_{\text{reg}}} \mathbb{Z} \rightarrow H^2(X, \mathcal{K}_{2,X})$ such that the composite $\mathcal{Z}_0(X, X_{\text{sing}}) \rightarrow H^2(X, \mathcal{K}_{2,X}) \rightarrow K_0(X)$ is the cycle class map. We only need to remark here that at the outset of [28], Levine assumes the ground field to be infinite. But the surjectivity of the map $\mathcal{Z}_0(X, X_{\text{sing}}) \rightarrow H^2(X, \mathcal{K}_{2,X})$ does not require this assumption. The assumption on the cardinality of k is required to use Matsumoto's presentation of Quillen K_2 of a field. \square

Proposition 13.2. *Let X be a smooth quasi-projective surface over a field k containing at least three elements. Let $D \subset X$ be an effective Cartier divisor. Then the canonical map $\phi_0: K_0(X|D) \rightarrow K_0(X, D)$ is an isomorphism.*

Proof. We have convergent spectral sequences

$$(13.1) \quad H^p(S_X, \mathcal{K}_{q,X|D}) \Rightarrow \pi_{-p-q}K(S_X, X_-) = \pi_{-p-q}K(X|D);$$

$$(13.2) \quad H^p(X, \mathcal{K}_{q,(X,D)}) \Rightarrow \pi_{-p-q}K(X, D)$$

induced by the Thomason-Trobaugh spectral sequence for the K -theory of S_X , X and D . Here, $\mathcal{K}_{q,X|D}$ is the Zariski sheaf associated to $U \mapsto K_q(U|D \cap U)$ and $\mathcal{K}_{q,(X,D)}$ denotes the Zariski sheaf associated to $U \mapsto K_q(U, U \cap D)$. We denote by $F^iK_0(X, D)$ the filtration on $K_0(X, D)$ induced by (13.1).

Since $\mathcal{K}_{0,X|D} = \text{Ker}(\mathcal{K}_{0,S_X} = \mathbb{Z} \rightarrow \mathcal{K}_{0,X} = \mathbb{Z}) = 0$, we have

$$0 \rightarrow F^1K_0(X|D) \rightarrow K_0(X|D) \rightarrow H^0(S_X, \mathcal{K}_{0,X|D}) = 0$$

and therefore $F^1K_0(X|D) \xrightarrow{\cong} K_0(X|D)$. Next, we have the exact sequence

$$0 \rightarrow F^2K_0(X|D) \rightarrow F^1K_0(X|D) = K_0(X|D) \rightarrow H^1(S_X, \mathcal{K}_{1,X|D}) \rightarrow 0$$

and a canonical map $H^2(S_X, \mathcal{K}_{2,X|D}) \rightarrow F^2K_0(X|D)$. Since X is a smooth surface, the corresponding map $H^2(X, \mathcal{K}_{2,X}) \rightarrow F^2K_0(X)$ is an isomorphism and the same holds for the map $H^2(S_X, \mathcal{K}_{2,S_X}) \rightarrow F^2K_0(S_X)$ by Lemma 13.1. By construction, the sequence of sheaves

$$(13.3) \quad 0 \rightarrow \mathcal{K}_{i,X|D} \rightarrow \mathcal{K}_{i,S_X} \rightarrow \mathcal{K}_{i,X} \rightarrow 0$$

is split exact and the space $K(X|D)$ is a retract of $K(S_X)$, so that by taking cohomology and comparing with the F^2 -piece of the filtration on the π_0 groups, we obtain that $H^2(S_X, \mathcal{K}_{2,X|D}) \rightarrow F^2K_0(X|D)$ is indeed an isomorphism. In particular, we have a short exact sequence

$$(13.4) \quad 0 \rightarrow H^2(S_X, \mathcal{K}_{2,X|D}) \rightarrow K_0(X|D) \rightarrow H^1(S_X, \mathcal{K}_{1,X|D}) \rightarrow 0.$$

Consider now the second spectral sequence (13.2). Since $\mathcal{K}_{0,(X,D)} = 0$ as well, we have another exact sequence

$$0 \rightarrow H^2(X, \mathcal{K}_{2,(X,D)}) \rightarrow K_0(X, D) \rightarrow H^1(X, \mathcal{K}_{1,(X,D)}) \rightarrow 0$$

where the first map is injective by [23, Lemma 2.1]. The natural map $\phi: K(X|D) \rightarrow K(X, D)$ induces then a commutative diagram, with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & H^2(S_X, \mathcal{K}_{2,X|D}) & \longrightarrow & K_0(X|D) & \longrightarrow & H^1(S_X, \mathcal{K}_{1,X|D}) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & H^2(X, \mathcal{K}_{2,(X,D)}) & \longrightarrow & K_0(X, D) & \longrightarrow & H^1(X, \mathcal{K}_{1,(X,D)}) \rightarrow 0. \end{array}$$

As $H^1(S_X, \mathcal{K}_1) \simeq \text{Pic}(S_X)$ and $H^1(X, \mathcal{K}_1) \simeq \text{Pic}(X)$, applying cohomology to (13.3) with $i = 1$ gives

$$0 \rightarrow H^1(S_X, \mathcal{K}_{1,X|D}) \rightarrow \text{Pic}(S_X) \xrightarrow{\iota_*^*} \text{Pic}(X) \rightarrow 0$$

and by Proposition 12.2, we have an identification $H^1(S_X, \mathcal{K}_{1,X|D}) = \text{Pic}(X, D)$. Similarly, we have $H^1(X, \mathcal{K}_{1,(X,D)}) = \text{Pic}(X, D)$ by [48, Lemma 2.1] and hence the right vertical map is an isomorphism. To finish the proof of the proposition, we are now left with proving the following Lemma. \square

Lemma 13.3. *The map $H^2(S_X, \mathcal{K}_{2,X|D}) \rightarrow H^2(X, \mathcal{K}_{2,(X,D)})$ is an isomorphism.*

Proof. Given an open subset $W \subset D$, let $U = S_X \setminus (D \setminus W)$ be the open subset of S_X . Let $\mathcal{K}_{i,(S_X, X_-, D)}$ be the sheaf on D associated to the presheaf $W \mapsto K_i(U, X_+ \cap U, X_- \cap U) = \text{hofib}((K(U, X_- \cap U) \xrightarrow{i_+^*} K(X_+ \cap U, D \cap U))$ (see [38, Proposition A.5]). There is an exact sequence of K -theory sheaves

$$\iota_*(\mathcal{K}_{2,(S_X, X_-, D)}) \rightarrow \mathcal{K}_{2,(S_X, X_-)} \rightarrow \mathcal{K}_{2,(X_+, D)} \rightarrow \iota_*(\mathcal{K}_{1,(S_X, X_-, D)}),$$

where $\iota: D \hookrightarrow S_X$ is the inclusion. We have $\mathcal{K}_{1,(S_X, X_-, D)} = \mathcal{I}_D/\mathcal{I}_D^2 \otimes_D \Omega_{D/X}^1$ by [13, Theorem 1.1] and the latter term is zero. Since $\mathcal{K}_{2,(S_X, X_-)} = \mathcal{K}_{2,X|D}$ by definition, we get then an exact sequence

$$\iota_*(\mathcal{K}_{2,(S_X, X_-, D)}) \rightarrow \mathcal{K}_{2,X|D} \rightarrow \mathcal{K}_{2,(X_+, D)} \rightarrow 0.$$

Since $H^2(S_X, \iota_*(\mathcal{K}_{2,(S_X, X_-, D)})) = H^2(D, \mathcal{K}_{2,(S_X, X_-, D)}) = 0$, the lemma follows. \square

We now prove our main result on cycles with modulus on surfaces.

Theorem 13.4. *Let X be a smooth quasi-projective surface over an algebraically closed field k and let $D \subset X$ be an effective Cartier divisor. Then the following hold.*

(1) The cycle class map $cyc_{X|D} : \mathrm{CH}_0(X|D) \rightarrow K_0(X, D)$ induces a short exact sequence

$$0 \rightarrow \mathrm{CH}_0(X|D) \rightarrow K_0(X, D) \rightarrow \mathrm{Pic}(X, D) \rightarrow 0$$

and the image of $\mathrm{CH}_0(X|D)$ agrees with $F^2K_0(X|D)$.

(2) There are isomorphisms

$$\mathrm{CH}_0(X|D) \xrightarrow{\simeq} H_{\mathrm{zar}}^2(X, \mathcal{K}_{2,(X,D)}^M) \xrightarrow{\simeq} H_{\mathrm{nis}}^2(X, \mathcal{K}_{2,(X,D)}^M).$$

Proof. The injectivity of the cycle class map follows exactly like Theorem 12.4 by using Proposition 13.2 instead of Proposition 12.3. Next, (13.4) and Proposition 13.2 show that the exactness of (1) is equivalent to showing that $\mathrm{CH}_0(X|D) \simeq F^2K_0(X|D)$. But this follows again by observing that cyc_{S_X} and cyc_X become isomorphisms (using [28, Main Theorem]) if we replace the middle row of (12.3) by $F^2K_0(-)$ which keeps the row exact.

To prove (2), recall that $\mathcal{K}_{2,(X,D)}^M$ is, by definition, the kernel of the map of Milnor K -theory sheaves $\mathcal{K}_{2,X}^M \rightarrow \mathcal{K}_{2,D}^M$. If we let $\mathcal{K}_{2,X|D}^M = \mathrm{Ker}(\Delta_*(\mathcal{K}_{2,S_X}^M) \rightarrow \mathcal{K}_{2,X}^M)$, then the top square of (12.3) has factorization

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathrm{CH}_0(X|D) & \xrightarrow{p_{+,*}} & \mathrm{CH}_0(S_X) & \xrightarrow{i_-^*} & \mathrm{CH}_0(X) \rightarrow 0 \\ & & \downarrow & & \downarrow cyc_{S_X} & & \downarrow cyc_X \\ 0 & \rightarrow & H^2(S_X, \mathcal{K}_{2,X|D}^M) & \xrightarrow{p_{+,*}} & H^2(S_X, \mathcal{K}_2^M) & \xrightarrow{i_-^*} & H^2(X, \mathcal{K}_2^M) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & K_0(X|D) & \xrightarrow{p_{+,*}} & K_0(S_X) & \xrightarrow{i_-^*} & K_0(X) \rightarrow 0. \end{array}$$

The commutative square

$$\begin{array}{ccc} \mathrm{CH}_0(X|D) & \rightarrow & H^2(S_X, \mathcal{K}_{2,X|D}^M) \rightarrow K_0(X|D) \\ & & \downarrow i_+^* \quad \downarrow \phi_0 \\ & & H^2(X, \mathcal{K}_{2,(X,D)}^M) \rightarrow K_0(X, D) \end{array}$$

now shows that there is a factorization $\mathrm{CH}_0(X|D) \rightarrow H^2(X, \mathcal{K}_{2,(X,D)}^M) \rightarrow K_0(X, D)$ of the map $cyc_{X|D}$. It follows from (1) that the first map is injective. On the other hand, it follows from [23, Theorem 1.2] that this map is surjective. We conclude that the map $\mathrm{CH}_0(X|D) \rightarrow H_{\mathrm{zar}}^2(X, \mathcal{K}_{2,(X,D)}^M)$ is an isomorphism. Furthermore, [23, Lemma 2.1] implies that

$$H_{\mathrm{zar}}^2(X, \mathcal{K}_{2,(X,D)}^M) \xrightarrow{\simeq} H_{\mathrm{nis}}^2(X, \mathcal{K}_{2,(X,D)}^M)$$

as required. \square

14. 0-CYCLES WITH MODULUS ON AFFINE SCHEMES

In Theorem 12.4, we gave our first application of Theorem 7.1 to 0-cycles with modulus on affine schemes. In this section, we deduce more applications of Theorem 7.1 for such schemes.

14.1. Affine Roitman torsion for 0-cycles with modulus. For affine schemes, our second application is the following Roitman torsion theorem for 0-cycles with modulus.

Theorem 14.1. *Let X be a smooth affine scheme of dimension $d \geq 2$ over an algebraically closed field k and let $D \subset X$ be an effective Cartier divisor. Then $\mathrm{CH}_0(X|D)$ is torsion-free.*

Proof. The proof is immediate from Theorem 7.1 and [24, Theorem 1.1], using the comparison given by Theorem 7.3 (or Theorem 3.17). \square

Remark 14.2. An independent proof of the vanishing of the prime-to- p torsion part of $\mathrm{CH}_0(X|D)$ for affine varieties (where p denotes the exponential characteristic of k) can be found in [4]. The argument in *loc.cit.* does not rely on our decomposition Theorem 7.1, but follows instead closely the approach of Levine in [30].

14.2. Vanishing theorems. As another application of Theorem 7.1, we get the following vanishing theorems.

Corollary 14.3. *Let X be a smooth affine scheme of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$ and let $D \subset X$ be an effective Cartier divisor. Then $\mathrm{CH}_0(X|D) = 0$.*

Proof. Using Theorems 3.17 and 7.1, it is enough to know that $\mathrm{CH}_0(S_X) = 0$. But this follows from [27, Theorem 6.4.1]. \square

The same argument, using [27, Theorem 6.4.2], shows the following.

Corollary 14.4. *Let $X = \mathrm{Spec}(A)$ be a smooth affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{Q}}$. Assume that $A = \bigoplus_{n \geq 0} A_n$ is a graded algebra with $A_0 = \overline{\mathbb{Q}}$. Assume moreover that $D \subset X$ is a divisor on X defined by a homogeneous element of A . Then $\mathrm{CH}_0(X|D) = 0$.*

14.3. Decomposition of $K_0(X, D)$. Let X be a smooth quasi-projective scheme over a field k and let $D \subset X$ be an effective Cartier divisor. Let $\mathcal{Z}^1(X|D)$ denote the free abelian group on integral closed subschemes $Z \subset X$ of codimension one such that $D \cap Z = \emptyset$. Let $\mathcal{R}^1(X|D) = \varinjlim_U \mathrm{Ker}(\mathcal{O}^\times(U) \rightarrow \mathcal{O}^\times(D))$, where U ranges over open subsets of X containing

D . Note that if D has an affine open neighbourhood in X then this limit is same as the limit taken over $X \setminus Z$, where $Z \subset X$ is a divisor disjoint from D and is principal in an affine neighbourhood of D .

Recall from [5, § 3] that the Chow group of codimension one cycles with modulus $\mathrm{CH}^1(X|D)$ is the cokernel of the map $\mathcal{R}^1(X|D) \xrightarrow{\mathrm{div}} \mathcal{Z}^1(X|D)$. We often write $\mathrm{CH}^1(X|D)$ as $\mathrm{CH}_{d-1}(X|D)$ if $\dim(X) = d$. If X is a smooth affine surface, we can refine Theorem 13.4 to completely describe $K_0(X, D)$ in terms of the Chow groups with modulus. In order to do this, we need the following elementary result from commutative algebra. For any commutative noetherian ring A and $a \in A$, let M_a denote the localization $M[a^{-1}]$ if M is an A -module.

Lemma 14.5. *Let A be a commutative noetherian ring and let $I \subset A$ be an ideal. Let P be a projective A -module of rank one and let $\phi : P/IP \xrightarrow{\sim} A/I$ be a given A/I -linear isomorphism. Then we can find an element $a \in A$ such that $a \equiv 1 \pmod{I}$ and an isomorphism $\tilde{\phi} : P_a \xrightarrow{\sim} A_a$ and a commutative diagram*

$$\begin{array}{ccc} P_a & \xrightarrow{\tilde{\phi}} & A_a \\ \downarrow & & \downarrow \\ P/IP & \xrightarrow{\phi} & A/I. \end{array}$$

Proof. As P is projective, we do have an A -linear map $\tilde{\phi} : P \rightarrow A$ and a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\phi}} & A \\ \downarrow & & \downarrow \\ P/IP & \xrightarrow[\cong]{\phi} & A/I. \end{array}$$

Letting $M = \text{Coker}(\tilde{\phi})$, our assumption says that $IM = M$. But this implies by the Nakayama lemma that there exists $a \in A$ such that $a \equiv 1 \pmod{I}$ and $M_a = 0$. It follows that $\tilde{\phi} : P_a \rightarrow A_a$ is surjective and hence an isomorphism. \square

Theorem 14.6. *Let X be a smooth affine surface over an algebraically closed field k and let $D \subset X$ be an effective Cartier divisor. Then, there is a short exact sequence*

$$(14.1) \quad 0 \rightarrow \text{CH}_0(X|D) \rightarrow K_0(X, D) \rightarrow \text{CH}_1(X|D) \rightarrow 0.$$

Proof. In view of Theorem 13.4, we only need to show that $\text{CH}_1(X|D) \simeq \text{Pic}(X, D)$. Since $H^1(X, \mathcal{K}_{1,(X,D)}) \simeq \text{Pic}(X, D)$ as we have seen above, we need to show that

$$\text{CH}_1(X|D) = \text{CH}^1(X|D) \simeq H^1(X, \mathcal{K}_{1,(X,D)}).$$

For a closed subset $Z \subset X$ of dimension one with $Z \cap D = \emptyset$, there is an exact sequence

$$(14.2) \quad H^0(X \setminus Z, \mathcal{K}_{1,(X \setminus Z, D)}) \rightarrow H^1_Z(X, \mathcal{K}_{1,(X,D)}) \rightarrow H^1(X, \mathcal{K}_{1,(X,D)}) \rightarrow H^1(X \setminus Z, \mathcal{K}_{1,(X \setminus Z, D)}).$$

The excision theorem says that $H^1_Z(X, \mathcal{K}_{1,(X,D)}) = H^1_Z(X \setminus D, \mathcal{K}_{1,(X,D)}) = H^1_Z(X \setminus D, \mathcal{K}_{1,X \setminus D})$ and it follows easily from the Gersten resolution of $\mathcal{K}_{1,X \setminus D}$ and the Thomason-Trobaugh spectral sequence $H^p_Z(X \setminus D, \mathcal{K}_{q,X \setminus D}) \Rightarrow K_{q-p}(Z)$ that $H^1_Z(X, \mathcal{K}_{1,X \setminus D})$ is the free abelian group on the irreducible components of Z . Using the isomorphism $H^1(X \setminus Z, \mathcal{K}_{1,(X \setminus Z, D)}) \simeq \text{Pic}(X \setminus Z, D)$, Lemma 14.5 tells us precisely that $\varinjlim_Z H^1(X \setminus Z, \mathcal{K}_{1,(X \setminus Z, D)}) = 0$.

Taking the colimit (14.2) over all closed subschemes Z as above, we thus get an exact sequence

$$(14.3) \quad \mathcal{R}^1(X|D) \rightarrow \mathcal{Z}^1(X|D) \rightarrow H^1(X, \mathcal{K}_{1,(X,D)}) \rightarrow 0.$$

It follows by a direct comparison of this exact sequence with the similar sequence for $\mathcal{K}_{1,X}$ that the arrow on the left is just the divisor map. We conclude that there is a canonical isomorphism $\text{CH}^1(X|D) \xrightarrow{\simeq} H^1(X, \mathcal{K}_{1,(X,D)})$. \square

14.4. Nil-invariance of 0-cycles with modulus. As another application of Theorem 7.1, we can prove the following result showing that the Chow group of 0-cycles with modulus on a smooth affine surface depends only on the support of the underlying Cartier divisor.

Theorem 14.7. *Let X be a smooth affine surface over a perfect field k and let $D \subset X$ be an effective Cartier divisor. Then the canonical map*

$$\text{CH}_0(X|D) \rightarrow \text{CH}_0(X|D_{\text{red}})$$

is an isomorphism.

Proof. Using Theorem 13.4, it suffices to show that $F^2K_0(X|D) \rightarrow F^2K_0(X|D_{\text{red}})$ is an isomorphism.

Write $X = \text{Spec}(A)$, $D' = D_{\text{red}} = \text{Spec}(A/I)$. Write $S_n = S(X, nD') = \text{Spec}(R_n)$ for the double construction applied to the pair (A, I^n) . Then we have a chain of inclusions of rings

$$\dots \subset R_{n+1} \subset R_n \subset \dots \subset R_1 \subset R_0 = A \times A$$

where $R_n = \{(a, b) \in R_0 \mid a - b \in I^n\}$. This gives a corresponding sequence of maps of schemes

$$S_0 = X \amalg X \rightarrow S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_n \rightarrow \dots \xrightarrow{\Delta} X.$$

By [1, Theorem 3.3] and the fact that $SK_1(B) = SK_1(B_{\text{red}})$ for any commutative ring B (see [2, Chap IX, Propositions 1.3, 1.9]), we have for every $n \geq 1$ an exact sequence

$$0 \rightarrow \frac{SK_1(D') \oplus SK_1(D')}{SK_1(D') \oplus SK_1(X \amalg X)} \longrightarrow F^2K_0(S_n) \longrightarrow F^2K_0(S_0) \rightarrow 0.$$

(Apply [1, Theorem 3.3] to $X = S_n$, $\tilde{X} = S_0$ and $Y = D_n = \text{Spec}(A/I^n)$. Note that $F^2K_0(S_n) = SK_0(S_n)$ and $F^2K_0(S_0) = SK_0(S_0)$ in the notations of [1]). The natural maps $\rho_n: K_0(S_n) \rightarrow K_0(S_1)$ give then a commutative diagram with exact rows (for $n \geq 1$)

$$(14.4) \quad \begin{array}{ccccccc} 0 & \rightarrow & \frac{SK_1(D') \oplus SK_1(D')}{SK_1(D') \oplus SK_1(X \amalg X)} & \longrightarrow & F^2K_0(S_n) & \longrightarrow & F^2K_0(S_0) \rightarrow 0 \\ & & \parallel & & \downarrow \rho_n & & \parallel \\ 0 & \rightarrow & \frac{SK_1(D') \oplus SK_1(D')}{SK_1(D') \oplus SK_1(X \amalg X)} & \longrightarrow & F^2K_0(S_1) & \longrightarrow & F^2K_0(S_0) \rightarrow 0. \end{array}$$

It follows that $F^2K_0(S_n) \simeq F^2K_0(S_1)$. Combining this with Proposition 12.3 and the commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & F^2K_0(X, nD') & \longrightarrow & F^2K_0(S_n) & \xrightarrow{i_-^*} & F^2K_0(X) \rightarrow 0 \\ & & \downarrow & & \downarrow \rho_n & & \parallel \\ 0 & \rightarrow & F^2K_0(X, D') & \longrightarrow & F^2K_0(S_1) & \xrightarrow{i_-^*} & F^2K_0(X) \rightarrow 0, \end{array}$$

we get $F^2K_0(X, nD') \xrightarrow{\simeq} F^2K_0(X, D')$ for every $n \geq 1$. Finally, we have inclusions $D' \subset D \subset nD'$ for $n \gg 1$ and hence a sequence of maps $F^2K_0(X, nD') \rightarrow F^2K_0(X, D) \rightarrow F^2K_0(X, D')$. It is easy to see using the isomorphism $\text{CH}_0(X|nD') \simeq F^2K_0(X, nD')$ that two maps in this sequence are surjective. Since the composite map is injective, the theorem follows. \square

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