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On the Lebesgue constant of the trigonometric Floater-Hormann rational interpolant at equally spaced nodes

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Abstract

It is well known that the classical polynomial interpolation gives bad approximation if the nodes are equispaced. A valid alternative is the family of barycentric rational interpolants introduced by Berrut in [4], analyzed in terms of stability by Berrut and Mittelmann in [5] and their extension done by Floater and Hormann in [8]. In this paper firstly we extend them to the trigonometric case, then as in the Floater-Hormann classical interpolant, we study the growth of the Lebesgue constant on equally spaced points. We show that the growth is logarithmic providing a stable interpolation operator.

Keywords: trigonometric interpolation, barycentric rational interpolation, Lebesgue constant.
AMS Subject classification (2010): 42A05, 42A10, 42A15.

1 Introduction

Given a continous function f defined in the interval $[a,b] \subset \mathbb{R}$, $n+1$ points (*nodes*), $X = \{a = x_0 < \cdots < x_n = b\}$, and the corresponding function values $F = \{f_i = f(x_i),\ i = 0,\ldots,n\}$, the classical univariate interpolation problem consists in finding a function g, in the Banach space of continous function on $[a,b]$ equipped with the sup-norm, $C([a,b],\|.\|_\infty)$, such that $g_X = F$, namely $g(x_i) = f_i$, $\forall\, i$. Letting b_k the Lagrange elementary polynomials of total degree n such that $b_k(x_j) := \delta_{kj}$ we may rewrite the interpolant in Lagrange form, at every point $x \in [a,b]$, as

$$g(x) = \sum_{k=0}^{n} b_k(x) f_k. \tag{1}$$

For stability purposes we need to understand the behaviour of the *Lebesgue function* $\lambda_n(x) = \sum_{k=0}^{n} |b_k(x)|$ in particular the value of its maximum

$$\Lambda_n := \max_{x\in[a,b]} \lambda_n(x) \tag{2}$$

which is the so-called *Lebesgue constant*.

It is well known that the Lebesgue constant represents an index of stability for the interpolation operator. Bounds for the Lebesgue constant on different sets of nodes, are presented in various papers. The reader can consider for this purpose the interesting survey by L. Brutman [6], that contains many fundamental results and a wide literature on the topic. The role of the interpolation nodes is crucial and the rule is to choose them in order to have a logarithmic growth of the Lebesgue constant, which represents the optimal growth in polynomial interpolation with nodes having the so-called arccosine distribution (like Chebyshev points) while in the rational case this growth holds when we choose equally spaced points (cf. [5, 9, 10]).

In the classical setting the equispaced nodes do not garantee the stability and the growth turns out to be exponential. This is one of the reasons why other interpolation methods should be used in this case. One alternative is represented by the Floater-Hormann rational interpolant, shortly FHRI, that has a good order of approximation and the growth of the Lebesgue constant is logarithmic also with equispaced nodes.

The paper is organized as follows: in Section 2 we recall the construction of the FHRI and show how we can extend it to the trigonometric case, in Section 3 we provide upper and lower bounds for the Lebesgue constant of the trigonometric interpolant previously introduced and finally in Section 4 we report some numerical experiments that confirm the goodness of the theoretical results.

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2 From the classical to the trigonometric FHRI

Let X the set of nodes and F the set of values, as above. The Floater-Hormann Rational Interpolant (FHRI), is a family of rational interpolants that depends on a *blending* parameter d. Following [8] the construction runs as follows. Choose an integer $0 \le d \le n$, for each $i = 0, 1, \dots, n-d$, let $p_i(x)$ be the unique polynomial of degree at most d that interpolates the given continous function f at the $d+1$ points $x_i, \dots, x_{i+d}$. Then the FHRI at x is given by the ratio

$$r(x) = \frac{\sum_{i=0}^{n-d} s_i(x) p_i(x)}{\sum_{i=0}^{n-d} s_i(x)} \tag{3}$$

with the *blending functions*

$$s_i(x) = \frac{(-1)^i}{(x-x_i)\cdots(x-x_{i+d})}. \tag{4}$$

The interpolant can be also written in barycentric form at every point x as (cf. [8])

$$r(x) = \frac{\sum_{k=0}^{n} \frac{w_k}{x-x_k} f_k}{\sum_{k=0}^{n} \frac{w_k}{x-x_k}} \tag{5}$$

where

$$w_k = \sum_{i \in J_k} (-1)^i \prod_{j=i, j \ne k}^{i+d} \frac{1}{x_k - x_j} \tag{6}$$

with $J_k = \{i \in \{0, 1, 2, \dots, n-d\}\ \textit{such that}\ k-d \le i \le k\}$.

As in the classical case (cf. [2, 3, 11, 12]), the trigonometric polynomials p_i can be written in Lagrange form. That is, consider the points $x_i, \dots, x_{i+d}$ then

$$p_i(x) = \sum_{k=i}^{i+d} \ell_k^{(i)}(x) f_k$$

where $\{\ell_k^{(i)}\}_{k=i,\dots,i+d}$ are the *Lagrange trigonometric "polynomials"*

$$\ell_k^{(i)} = a_{k,i} \ell^{(i)}(x) \left(\operatorname{cst}\left(\frac{\omega}{2}(x-x_k) \right) + c_i \right), \tag{7}$$

$$a_{k,i} = \prod_{j=i,\ j \ne k}^{i+d} \frac{1}{\sin\left(\frac{\omega}{2}(x_k - x_j)\right)}, \quad \ell^{(i)}(x) = \prod_{j=i}^{i+d} \sin\left(\frac{\omega}{2}(x-x_j) \right),$$

$$\operatorname{cst}(x) := \begin{cases} \csc(x) = \frac{1}{\sin(x)} & \text{if } d \text{ even} \\ \cot(x) = \frac{\cos(x)}{\sin(x)} & \text{if } d \text{ odd} \end{cases}, \quad c_i := \begin{cases} 0 & \text{if } d \text{ even} \\ \cot\left(\frac{\omega}{2} \sum_{j=i}^{i+d} x_j\right) & \text{if } d \text{ odd} \end{cases}$$

and the pulsation ω that must be appropriately chosen (see next section). Now, letting

$$s_k^t(x) = \frac{(-1)^k}{\sin\left(\frac{\omega}{2}(x-x_k)\right) \cdots \sin\left(\frac{\omega}{2}(x-x_{k+d})\right)} \tag{8}$$

the *Trigonometric Floater-Hormann Rational Interpolant*, shortly TFHRI, can be defined as

$$r^t(x) := \frac{\sum_{i=0}^{n-d} s_i^t(x) \left(\sum_{k=i}^{i+d} a_{k,i} \ell^{(i)}(x) \left(\operatorname{cst}\left(\frac{\omega}{2}(x-x_k) \right) + c_i \right) f_k \right)}{\sum_{i=0}^{n-d} s_i^t(x)} \tag{9}$$

Noting that

$$\sum_{k=i}^{i+d} a_{k,i} \ell^{(i)}(x) \left(\operatorname{cst}\left(\frac{\omega}{2}(x-x_k) \right) + c_i \right) = 1$$

letting $q_k(x) = w_k^t \operatorname{cst}\left(\frac{\omega}{2}(x-x_k) \right) + \alpha_k$ where

$$w_k^t = \sum_{i \in J_k} (-1)^i a_{k,i} = \sum_{i \in J_k} (-1)^i \prod_{j=i, j \ne k}^{i+d} \frac{1}{\sin\left(\frac{\omega}{2}(x_k - x_j)\right)} \tag{10}$$

with J_k as above, and

$$\alpha_k := \begin{cases} 0 & \text{if } d \text{ even} \\ \sum_{i \in J_k} (-1)^i a_{k,i} c_i & \text{if } d \text{ odd} \end{cases}$$

the TFHRI in barycentric form is

$$r^t(x) = \sum_{k=0}^{n} \underbrace{\left(\frac{q_k(x)}{\sum_{j=0}^{n} q_j(x)} \right)}_{b_k^t(x)} f_k \,. \tag{11}$$

For illustration, in Figure 2 (Left) we provide the plot of b_k^t on $n = 5$ equally spaced points of $[0, 1]$.

3 On the Lebesgue constant of the trigonometric FHRI

Suppose for the moment that d is an *even* positive integer and $I = [a, b]$. The TFHRI in (11) has now basis functions

$$b_k^t(x) = \frac{\dfrac{w_k^t}{\sin\left(\frac{\omega}{2}(x - x_k)\right)}}{\displaystyle\sum_{j=0}^{n} \frac{w_k^t}{\sin\left(\frac{\omega}{2}(x - x_j)\right)}} \,. \tag{12}$$

For the well-posedness of the interpolant, ω has to satisfy what we call **Condition C1**. That is, for every $x \in I$ and every k, $\frac{\omega}{2}|x - x_k| \le \frac{\omega}{2}(b - a) < \frac{\pi}{2}$, that is

$$0 < \omega < \frac{\pi}{(b - a)} \tag{13}$$

Now, consider the function $g(x) = \frac{1}{\mathrm{sinc}(x)}$ (sinc being the unnormalized `sinc` function) whose plot in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is shown in Figure 1.

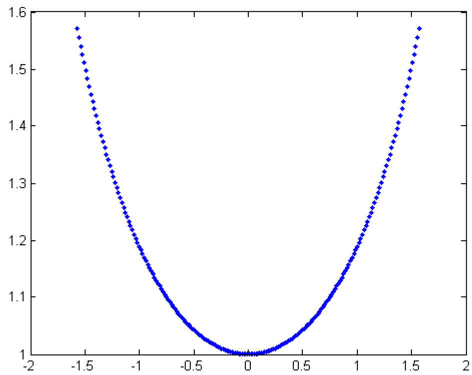

Figure 1: Plot of $g = 1/\mathtt{sinc}$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

The function g is positive, greater than 1, symmetric with respect to the origin, but not defined at the point set $X_k = \{x_k = k\pi\ , \forall k \in \mathbb{Z} \setminus \{0\}\}$. For these reasons we may restrict our investigations to $I = [0, 1]$ so that $0 < \omega < \pi$.

Moreover, letting $M_\omega = \dfrac{1}{\mathrm{sinc}\left(\frac{\omega}{2}\right)}$ then, for any $x, x_k \in [0, 1]$ the following hold:

$$1 \le g\left(\frac{\omega}{2}(x - x_k)\right) \le M_\omega \,, \tag{14}$$

$$(x - x_k)(x_{k+1} - x) \le \frac{1}{4n^2} . \tag{15}$$

For later use, we also recall the below inqualities for the partial sums of Leibnitz's and harmonic series, namely $\forall n \in \mathbb{N}$

$$\frac{\pi}{4} - \frac{1}{2n+3} \le \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \le \frac{\pi}{4} + \frac{1}{2n+3} \tag{16}$$

and

$$\ln(n+1) \le \sum_{k=0}^{n} \frac{1}{k} \le \ln(2n+1) \,. \tag{17}$$

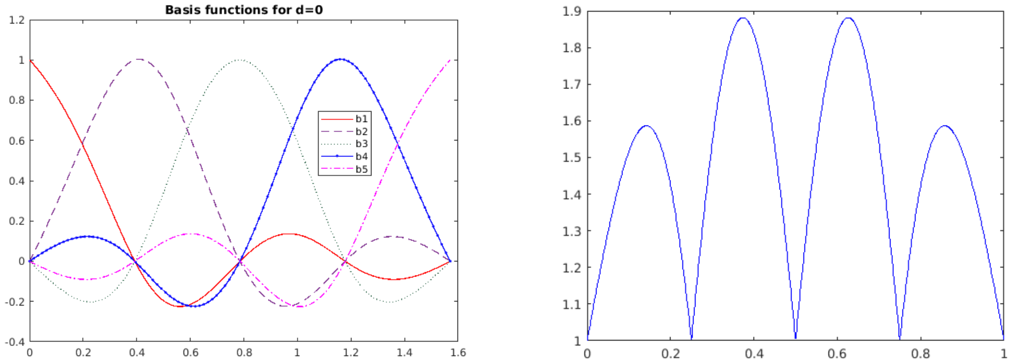

Figure 2: Basis functions (left) and the corresponding Lebesgue function (right) for $n = 5, d = 0$ on $x_0 = 0,\ x_1 = \pi/4,\ x_2 = \pi/2$ and $\omega = \pi/2$.

In particular, from (17), we get

$$\sum_{k=0}^{n} \frac{1}{2k+1} = \sum_{k=1}^{2n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{2k} \geq \ln(2n+2) - \frac{1}{2}\ln(2n+1) \geq \frac{1}{2}\ln(2n+3) \tag{18}$$

3.1 Case $d = 0$

The case $d = 0$ is of particular interest in order to understand how to bound the Lebesgue constant. It corresponds to Berrut's trigonometric interpolant with weights in (10) given by $w_k^t = (-1)^k$.

Theorem 3.1. *The Lebesgue constant associated with the TFHRI in* $[0, 1]$ *at equidistant nodes* $\{x_j = \frac{j}{n},\ \ j = 0, \ldots, n\}$*, for* $d = 0$ *and pulsation* ω *satisfying* ***Condition C1****, has upper bound*

$$\Lambda_n \leq \frac{M_\omega}{2 - M_\omega}(2 + \ln n) \tag{19}$$

Proof. If $x = x_k$ for any k then $\lambda_n(x) = 1$. Let us take $x_k < x < x_{k+1}$ for $k = 0, \ldots, n-1$, the Lebesgue function can be written as

$$\lambda_n(x) = \frac{(x - x_k)(x_{k+1} - x)\displaystyle\sum_{j=0}^{n} \frac{1}{\sin(\frac{\omega}{2}|x - x_j|)}}{(x - x_k)(x_{k+1} - x)\left|\displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x - x_j))}\right|} = \frac{N(x)}{D(x)}. \tag{20}$$

Following [9, 10], we start bounding the numerator N from above and the denominator D from below. That is

$$\begin{aligned} N(x) \quad &= (x - x_k)(x_{k+1} - x)\tfrac{2}{\omega}\textstyle\sum_{j=0}^{n} \dfrac{g(\frac{\omega}{2}|x - x_j|)}{|x - x_j|} \\ &\leq \frac{2M_\omega}{\omega}(x - x_k)(x_{k+1} - x)\textstyle\sum_{j=0}^{n} \dfrac{1}{|x - x_j|} \\ &\leq \frac{2M_\omega}{\omega}\left(\frac{1}{n} + \frac{\ln(n)}{2n}\right). \end{aligned} \tag{21}$$

where we use (14),(15) and the last inequality comes from [9, Th. 1] with $d = 0$.

Now we look at the denominator and split the proof into 4 cases.

Case 1: k **and** n **both even.**

$$D(x) = (x - x_k)(x_{k+1} - x)\left|\sum_{j=0}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x - x_j))}\right|.$$

Thanks to **Condition C1** we observe

$$\sum_{j=0}^{k} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} - \sum_{j=k+1}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))} \tag{22}$$
$$= \frac{1}{\sin(\frac{\omega}{2}(x-x_0))} + \left(\frac{1}{\sin(\frac{\omega}{2}(x-x_2))} - \frac{1}{\sin(\frac{\omega}{2}(x-x_1))}\right) + \cdots +$$
$$+ \left(\frac{1}{\sin(\frac{\omega}{2}(x-x_k))} - \frac{1}{\sin(\frac{\omega}{2}(x-x_{k-1}))}\right) + \left(\frac{1}{\sin(\frac{\omega}{2}(x_{k+1}-x))} - \frac{1}{\sin(\frac{\omega}{2}(x_{k+2}-x))}\right) +$$
$$+ \cdots + \left(\frac{1}{\sin(\frac{\omega}{2}(x_{n-1}-x))} - \frac{1}{\sin(\frac{\omega}{2}(x_n-x))}\right) > 0$$

so that we can ignore the absolute value in the denominator D. Following again [9] and using also (14), we obtain

$$D(x) = (x-x_k)(x_{k+1}-x) \sum_{j=0}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} \tag{23}$$
$$= (x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} + \frac{g(\frac{\omega}{2}(x-x_k))}{\frac{\omega}{2}(x-x_k)}\right.$$
$$\left. + \frac{g(\frac{\omega}{2}(x_{k+1}-x))}{\frac{\omega}{2}(x_{k+1}-x)} - \sum_{j=k+2}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))}\right)$$
$$\geq (x-x_k)(x_{k+1}-x)\left(\frac{1}{\frac{\omega}{2}(x-x_k)} + \frac{1}{\frac{\omega}{2}(x_{k+1}-x)}\right.$$
$$\left. + \sum_{j=0}^{k-1} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} - \sum_{j=k+2}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))}\right)$$
$$= \frac{2}{\omega n} + (x-x_k)(x_{k+1}-x)S(x). \tag{24}$$

Now,

$$S(x) = \sum_{j=0}^{k-1} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j)} - \sum_{j=k+2}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))} \tag{25}$$
$$= \frac{1}{\sin(\frac{\omega}{2}(x-x_0))} + \sum_{s=1}^{\frac{k-2}{2}} \overbrace{\left(\frac{1}{\sin(\frac{\omega}{2}(x-x_{2s}))} - \frac{1}{\sin(\frac{\omega}{2}(x-x_{2s-1}))}\right)}^{>0}$$
$$- \frac{1}{\sin(\frac{\omega}{2}(x-x_{k-1}))} - \frac{1}{\sin(\frac{\omega}{2}(x_{k+2}-x))}$$
$$+ \sum_{s=2}^{\frac{n-k}{2}} \overbrace{\left(\frac{1}{\sin(\frac{\omega}{2}(x_{k+2s-1}-x))} - \frac{1}{\sin(\frac{\omega}{2}(x_{k+2s}-x))}\right)}^{>0}$$
$$\geq -\left(\frac{1}{\sin(\frac{\omega}{2}(x_k-x_{k-1}))} + \frac{1}{\sin(\frac{\omega}{2}(x_{k+2}-x_{k+1}))}\right)$$
$$= g\left(\frac{\omega}{2n}\right)\frac{2(-2n)}{\omega} \geq M_\omega \frac{2(-2n)}{\omega} = \frac{-4nM_\omega}{\omega}.$$

Hence, by using (15),

$$D(x) = \frac{2}{\omega n} + (x-x_k)(x_{k+1}-x)S(x) \tag{26}$$
$$\geq \frac{2}{\omega n} - (x-x_k)(x_{k+1}-x)\frac{4nM_\omega}{\omega}$$
$$\geq \frac{2}{\omega n} - \frac{1}{4n^2}\frac{4nM_\omega}{\omega} = \frac{2-M_\omega}{\omega n}.$$

Putting together (21) and (26) for computing the Lebesgue function as

$$\Lambda_n = \max_{k=0,\ldots,n-1}\left(\max_{x_k<x<x_{k+1}} \lambda_n(x)\right) \tag{27}$$

we get the bound (19).

Case 2: k **even** n **odd**. In this case we add a single positive term $(\sin\left(\frac{\omega}{2}(x_n-x)\right))^{-1}$ to S. Therefore, all the previous bounds are still true.

Case 3: k **odd and** n **even**.

With similar calculations as before, we notice

$$\sum_{j=0}^{k}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))}+\sum_{j=k+1}^{n}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))}<0.$$

Let $\hat{D}$ be the denominator D without the absolute value.

$$\begin{aligned}\hat{D}(x)&=(x-x_k)(x_{k+1}-x)\sum_{j=0}^{n}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))}=(x-x_k)(x_{k+1}-x)\Bigg(\sum_{j=0}^{k-1}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))}-\\&-\frac{g(\frac{\omega}{2}(x-x_k))}{\frac{\omega}{2}(x-x_k)}-\frac{g(\frac{\omega}{2}(x_{k+1}-x))}{\frac{\omega}{2}(x_{k+1}-x)}-\sum_{j=k+2}^{n}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))}\Bigg)\\&=-(x-x_k)(x_{k+1}-x)\left(\frac{g(\frac{\omega}{2}(x-x_k))}{\frac{\omega}{2}(x-x_k)}+\frac{g(\frac{\omega}{2}(x_{k+1}-x))}{\frac{\omega}{2}(x_{k+1}-x)}\right)\\&+(x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))}-\sum_{j=k+2}^{n}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))}\right)\\&=-\frac{2}{\omega}\left((x_{k+1}-x)g\left(\frac{\omega}{2}(x-x_k)\right)+(x-x_k)g\left(\frac{\omega}{2}(x_{k+1}-x)\right)\right)+(x-x_k)(x_{k+1}-x)S(x).\end{aligned}$$

Pairing the positive and negative terms as follows, we get

$$\begin{aligned}S(x)&=\sum_{j=0}^{k-1}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))}-\sum_{j=k+2}^{n}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))}\\&=\sum_{s=0}^{\frac{k-3}{2}}\overbrace{\left(\frac{1}{\sin(\frac{\omega}{2}(x-x_{2s}))}-\frac{1}{\sin(\frac{\omega}{2}(x-x_{2s+1}))}\right)}^{<0}+\frac{1}{\sin(\frac{\omega}{2}(x-x_{k-1}))}+\frac{1}{\sin(\frac{\omega}{2}(x_{k+2}-x))}\\&+\sum_{s=2}^{\frac{n-k-1}{2}}\overbrace{\left(\frac{1}{\sin(\frac{\omega}{2}(x_{k+2s}-x))}-\frac{1}{\sin(\frac{\omega}{2}(x_{k+2s-1}-x))}\right)}^{<0}-\frac{1}{\sin(\frac{\omega}{2}(x_n-x))}\\&\le\frac{1}{\sin(\frac{\omega}{2}(x-x_{k-1}))}+\frac{1}{\sin(\frac{\omega}{2}(x_{k+2}-x))}\le\frac{2M_\omega}{\omega}\left(\frac{1}{x_k-x_{k-1}}+\frac{1}{x_{k+2}-x_{k+1}}\right)=\frac{4nM_\omega}{\omega}.\end{aligned}\tag{28}$$

We return to $\hat{D}$ and use (15)

$$\begin{aligned}\hat{D}(x)&\le-\frac{2}{\omega}\left((x_{k+1}-x)g\left(\frac{\omega}{2}(x-x_k)\right)+(x-x_k)g\left(\frac{\omega}{2}(x_{k+1}-x)\right)\right)+(x-x_k)(x_{k+1}-x)S(x)\\&\le-\frac{2}{\omega}\left((x_{k+1}-x)g\left(\frac{\omega}{2}(x-x_k)\right)+(x-x_k)g\left(\frac{\omega}{2}(x_{k+1}-x)\right)\right)+(x-x_k)(x_{k+1}-x)\frac{4nM_\omega}{\omega}\\&\le\frac{2}{\omega}\left(-\frac{1}{n}+\frac{M_\omega}{2n}\right).\end{aligned}$$

Hence by passing to the modulus since now $M_\omega<g(\frac{\pi}{2})<2$, we can bound D below as

$$D(x)\ge\frac{2}{\omega}\left(\frac{|-2+M_\omega|}{2n}\right)=\frac{2-M_\omega}{n\omega}.\tag{29}$$

Again (19) follows.

Case 4: k **and** n **both odd**.

In S the terms with $j\ge k+3$ are an even number therefore we can group them all in the second sum. So we get again the bounds of the case 3).

This concludes the proof. □

Theorem 3.2. *The Lebesgue constant associated with the TFHRI in* $[0,1]$, *with a pulsation* ω *satisfying* ***Condition C1*** *at equally spaced nodes* $\{x_j=\frac{j}{n},\ j=0,\dots,n\}$ *has lower bound*

$$\Lambda_n\ge\frac{2n}{M_\omega}\frac{\ln(n)}{4+n\pi}\tag{30}$$

Proof. Following [9] and our notations, the Lebesgue function at equally spaced points is

$$\lambda_n(x) = \frac{\displaystyle\sum_{j=0}^{n} \frac{g(\frac{\omega}{2}|x-\frac{j}{n}|)}{|2nx-2j|}}{\left|\displaystyle\sum_{j=0}^{n} \frac{(-1)^j g(\frac{\omega}{2}(x-\frac{j}{n}))}{(2nx-2j)}\right|} =: \frac{N(x)}{D(x)}.$$

1. Let n be even, say $n=2k$, with points enumerated as $x_0,\ldots,x_n$. The local maxima of λ_n are taken at the mid point of each subinterval and at $x^* = \dfrac{n+1}{2n}$ the maximum is attained (see Figure 3, Left) . Then, taking the numerator N at x^* and using (14), we obtain

$$N\left(\frac{n+1}{2n}\right) = \sum_{j=0}^{n} \frac{g(\frac{\omega}{2}|\frac{n+1-2j}{2n}|)}{|n+1-2j|} \geq \sum_{j=0}^{n} \frac{1}{|2(k-j)+1|} \geq \ln(n+1) \geq \ln(n).$$

where the last inequality is described in [9, Th. 1].

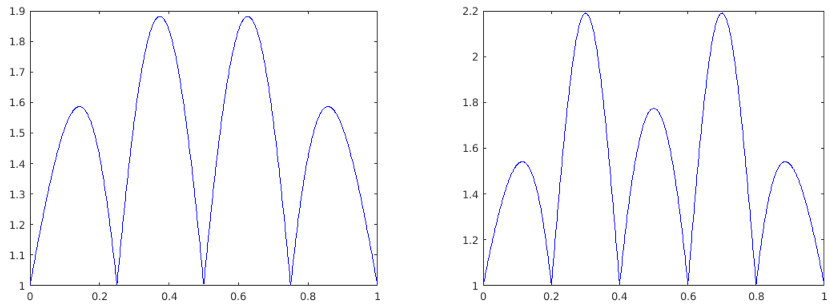

Figure 3: Lebesgue functions on equally spaced points. Left: $n=4$. Right: $n=5$

Now we consider the denominator also at x^*. Recalling [9],

$$\begin{aligned} D\left(\frac{n+1}{2n}\right) &= \left|\sum_{j=0}^{2k} \frac{g(\frac{\omega}{2}(\frac{2(k-j)+1}{2n}))(-1)^j}{2(k-j)+1}\right| \\ &\leq \left|\sum_{j=0}^{k} \frac{g(\frac{\omega}{2}(\frac{2j+1}{4k}))(-1)^j}{2j+1}\right| + \left|\sum_{j=0}^{k-1} \frac{g(\frac{\omega}{2}(\frac{2j+1}{4k}))(-1)^j}{2j+1}\right| =: |A| + |B|\,. \end{aligned}$$

Keeping in mind (16) and thanks to Condition 1 that make the sine function increasing, it is easy to prove that A (and therefore B) is positive and so we can ignore the absolute value.
Then

$$A = \sum_{j=0}^{k} \frac{g\left(\frac{\omega}{2}(\frac{2j+1}{4k})\right)(-1)^j}{2j+1} \leq M_\omega \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \leq M_\omega \left(\frac{\pi}{4} + \frac{1}{n+3}\right).$$

With similar computations it may be seen that

$$|B| = B \leq M_\omega \left(\frac{\pi}{4} + \frac{1}{n+1}\right).$$

Finally

$$|A| + |B| \leq M_\omega \left(\frac{\pi}{4} + \frac{1}{n+3} + \frac{\pi}{4} + \frac{1}{n+1}\right) \leq M_\omega \left(\frac{\pi}{2} + \frac{2}{n+1}\right) \leq M_\omega \left(\frac{\pi}{2} + \frac{2}{n}\right)$$

obtaining the required result

$$\Lambda_n \geq \frac{\ln(n)}{M_\omega\left(\frac{\pi}{2} + \frac{2}{n}\right)} = \frac{2n\ln(n)}{M_\omega(n\pi+4)}. \tag{31}$$

2. In the case n odd, $n=2k+1$, we consider $x^* = \frac{n+2}{2n}$ and evaluate both the numerator and the denominator at this point. We follow the same ideas of the proof 1.
First we analyze the numerator and by using (14), (18), we get

$$
\begin{aligned}
N(x^*) &= \sum_{j=0}^{n} \frac{g(\frac{\omega}{2}|\frac{n+2-2j}{2n}|)}{|n+2-2j|} \geq \sum_{j=0}^{n} \frac{1}{|2(k-j+1)+1|} = \sum_{j=0}^{k+1} \frac{1}{2(k-j+1)+1} - \sum_{j=k+2}^{n} \frac{1}{2(k-j+1)+1} \\
&= \sum_{j=0}^{k+1} \frac{1}{2(k-j+1)+1} + \sum_{j=k+2}^{n} \frac{1}{2(j-k-1)-1} \\
&= \sum_{j=0}^{k+1} \frac{1}{2(k-j+1)+1} + \sum_{l_2=0}^{n-k-2} \frac{1}{2l_2+1} \quad (l_2 = j-k-2) \\
&= \sum_{l_1=0}^{k+1} \frac{1}{2l_1+1} + \sum_{l_2=0}^{n-k-2} \frac{1}{2l_2+1} \quad (l_1 = k-j+1) \\
&\geq \frac{1}{2}\ln(2(k+1)+3) + \frac{1}{2}\ln(2(n-k-2)+3) \\
&= \frac{1}{2}(\ln(n+4)+\ln(n)) \geq \frac{1}{2}(\ln(n)+\ln(n)) = \ln(n).
\end{aligned}
$$

We turn now to the denominator. As in the previous proof, it is easy to see that A, B are positive. So by using the index substitutions as above and (16), it follows that

$$
\begin{aligned}
D(x^*) &\leq \left| \sum_{j=0}^{2k+1} \frac{(-1)^j g(\frac{\omega}{2}|\frac{n+2-2j}{2n}|)}{2(k-j+1)+1} \right| \leq M_\omega \left(\sum_{l=0}^{k+1} \frac{(-1)^l}{2l+1} + \sum_{l=0}^{k-1} \frac{(-1)^l}{2l+1} \right) \\
&\leq M_\omega \left(\frac{\pi}{4} + \frac{1}{n+4} + \frac{\pi}{4} + \frac{1}{n} \right) \leq M_\omega \left(\frac{\pi}{2} + \frac{2}{n} \right).
\end{aligned}
$$

We then obtain

$$
\lambda_n(x^*) = \frac{N(x^*)}{D(x^*)} \geq \frac{\ln(n)}{M_\omega\left(\frac{\pi}{2}+\frac{2}{n}\right)} = \frac{2n\ln(n)}{M_\omega(n\pi+4)}.
$$

and conclude

$$
\Lambda_n \geq \frac{2n\ln(n)}{M_\omega(4+n\pi)}. \tag{32}
$$

This proves the result.

□

In the next two subsections we study the growth of the Lebesgue constant for $d > 0$, providing upper and lower bounds for the TFHRI (11), separating the case of d even and d odd.

3.2 Case $d > 0$

We start by observing that the weights w_k^t in (10), in the case of equispaced interpolation nodes and $d > 0$, satisfy

$$
\frac{2^d}{\omega^d}|w_k| \leq |w_k^t| \leq \frac{2^d M_\omega^d}{\omega^d}|w_k|. \tag{33}
$$

As in [10] we assume that $n \geq 2d$ and recall that in the classical Floater-Hormann case

$$
(-1)^d h^d d! w_k = (-1)^k h^d d! |w_k| = (-1)^k \beta_k \tag{34}
$$

where

$$
\beta_k = \sum_{i=d}^{n} \binom{d}{i-k} = \begin{cases} \sum_{i=0}^{k} \binom{d}{k}, & \text{if } k \leq d, \\ 2^d, & \text{if } d \leq k \leq n-d, \\ \beta_{n-k}, & \text{if } k \geq n-d \end{cases} \tag{35}
$$

and h is the separation between the points, which is indeed constant in our setting.

3.3 Case d even

In what follows, $d \geq 2$ and d even.

Theorem 3.3 (Upper bound)**.** *The Lebesgue constant associated with TFHRI in* $[0,1]$*, with* ω *satisfying* **Condition C1***, at equispaced nodes* $\{x_j = \frac{j}{n},\ j = 0,\dots,n\}$ *and basis functions (12) has the upper bound*

$$\Lambda_n \leq M_\omega^{d+1} 2^{d-1}\big(2 + \ln(n)\big). \tag{36}$$

Proof. As above, consider $x_k < x < x_{k+1}$ and the corresponding Lebesgue function

$$\lambda_n(x) = \frac{\displaystyle\sum_{j=0}^{n} \frac{|w_j^t|}{\sin(\frac{\omega}{2}|x-x_j|)}}{\left|\displaystyle\sum_{j=0}^{n} \frac{w_j^t}{\sin(\frac{\omega}{2}(x-x_j))}\right|} = \frac{h^d d!(x-x_k)(x_{k+1}-x)\displaystyle\sum_{j=0}^{n} \frac{|w_j^t|}{\sin(\frac{\omega}{2}|x-x_j|)}}{\left|h^d d!(x-x_k)(x_{k+1}-x)\displaystyle\sum_{j=0}^{n} \frac{w_j^t}{\sin(\frac{\omega}{2}(x-x_j))}\right|} := \frac{N(x)}{D(x)}.$$

Using the same line of proof for the case $d = 0$, for the numerator we have

$$\begin{aligned}
N(x) &= \frac{2}{\omega} h^d d!(x-x_k)(x_{k+1}-x) \sum_{j=0}^{n} \frac{g(\frac{\omega}{2}|x-x_j|)|w_j^t|}{|x-x_j|} \\
&\leq \frac{2^{d+1}M_\omega^d}{\omega^{d+1}} h^d d!(x-x_k)(x_{k+1}-x) \sum_{j=0}^{n} \frac{g(\frac{\omega}{2}|x-x_j|)|w_j|}{|x-x_j|} \\
&\leq \frac{2^{d+1}M_\omega^{d+1}}{\omega^{d+1}} (x-x_k)(x_{k+1}-x) \sum_{j=0}^{n} \frac{|w_j| h^d d!}{|x-x_j|} \\
&= \frac{2^{d+1}M_\omega^{d+1}}{\omega^{d+1}} (x-x_k)(x_{k+1}-x) \sum_{j=0}^{n} \frac{\beta_j}{|x-x_j|} \leq \frac{2^{d+1}}{\omega^{d+1}} M_\omega^{d+1} 2^d \left(\frac{1}{n} + \frac{1}{2n}\ln(n)\right),
\end{aligned}$$

where the last inequality has been taken from [10, Th. 1].

In order to find out a bound for D and thanks to (34) and (33), we notice that

$$(-1)^d h^d d! \sum_{j=0}^{n} \frac{w_j^t}{\sin(\frac{\omega}{2}(x-x_j))} = (-1)^d h^d d! \sum_{j=0}^{n-d} s_j^t(x),$$

with s_j^t as in (8) such that, $\forall j \in \{0,\dots,n-d\}$,

$$s_j^t(x) = \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))\dots\sin(\frac{\omega}{2}(x-x_{j+d}))} = \frac{2^{d+1}}{\omega^{d+1}} s_j(x) \prod_{i=j}^{j+d} g\left(\frac{\omega}{2}(x-x_i)\right) \geq s_j(x)\left(\frac{2}{\omega}\right)^{d+1}.$$

Assuming $k \leq n-d$ and keeping in mind the proof in [10, Th. 1], we have

$$\begin{aligned}
D(x) &= h^d d!(x-x_k)(x_{k+1}-x)\left|\sum_{j=0}^{n} s_j^t(x)\right| \geq h^d d!(x-x_k)(x_{k+1}-x)|s_k^t(x)| \\
&\geq \left(\frac{2}{\omega}\right)^{d+1} h^d d!(x-x_k)(x_{k+1}-x)|s_k(x)| \geq \left(\frac{2}{\omega}\right)^{d+1}\frac{1}{n},
\end{aligned}$$

with the last inequality coming from the results in [10].

If $k > n-d$ a similar reasoning leads to the same lower bound for D by considering s_{k-d+1} instead of s_k. Recalling formula (27), finally we obtain

$$\Lambda_n \leq M_\omega^{d+1} 2^{d-1}\big(2 + \ln(n)\big).$$

□

Theorem 3.4 (Lower bound)**.** *Under the same assumptions of Theorem 3.3, we have*

$$\Lambda_n \geq \frac{1}{M_\omega^{d+1} 2^{d+2}} \binom{2d+1}{d} \ln\left(\frac{n}{d} - 1\right). \tag{37}$$

Proof.

$$\lambda_n(x) = \frac{h^d d! \sum_{j=0}^{n} \frac{|w_j^t|}{\sin(\frac{\omega}{2}|x-x_j|)}}{\left| h^d d! \sum_{j=0}^{n} \frac{w_j^t}{\sin(\frac{\omega}{2}(x-x_j))} \right|} := \frac{N(x)}{D(x)}.$$

We consider $x^* = \frac{x_1 - x_0}{2} = \frac{1}{2n}$. By using (14) and the bounds in (33),

$$N(x^*) = h^d d! \sum_{j=0}^{n} \frac{|w_j^t|}{\sin(\frac{\omega}{2}|x^*-x_j|)} \geq h^d d! \frac{2^{d+1}}{\omega^{d+1}} \sum_{j=0}^{n} \frac{g(\frac{\omega}{2}(x^*-x_j))|w_j|}{|x^*-x_j|} \geq \frac{2^{d+1}}{\omega^{d+1}} \sum_{j=0}^{n} \frac{\beta_j}{|x^*-x_j|}$$
$$\geq \frac{2^{d+1}}{\omega^{d+1}} n 2^d \ln\left(\frac{n}{d} - 1\right),$$

where the last inequality has been taken from [9].

We turn to the denominator

$$D(x^*) = h^d d! \left| \sum_{j=0}^{n-d} s_j^t(x^*) \right|.$$

We notice that $s_0^t(x^*)$ e $s_1^t(x^*)$ have the same sign and that $s_j^t(x^*)$ oscillate in sign and decrease in absolute value. Keeping in mind [9] and thank to (14) we get

$$D(x^*) \leq h^d d! (|s_0^t(x^*)| + |s_1^t(x^*)|)$$
$$= d! h^d \frac{2^{d+1}}{\omega^{d+1}} \left(|s_0(x^*)| \prod_{i=0}^{d} g(\frac{\omega}{2}(x^*-x_i)) + |s_1(x^*)| \prod_{i=1}^{d+1} g(\frac{\omega}{2}(x^*-x_i)) \right)$$
$$\leq \frac{2^{d+1}}{\omega^{d+1}} M_\omega^{d+1} \left(|s_0(x^*)| + |s_1(x^*)| \right) \leq \frac{2^{d+1}}{\omega^{d+1}} M_\omega^{d+1} n \frac{2^{2d+2}}{\binom{2d+1}{d}}.$$

Finally we obtain

$$\Lambda_n \geq \frac{1}{M_\omega^{d+1} 2^{d+2}} \binom{2d+1}{d} \ln\left(\frac{n}{d} - 1\right).$$

□

3.4 Case d odd

The interpolant has now the form

$$r^t(x) = \sum_{k=0}^{n} \left(\frac{w_k^t \cot\left(\frac{\omega}{2}(x-x_k)\right) + \alpha_k}{\sum_{k=0}^{n} w_k^t \cot\left(\frac{\omega}{2}(x-x_k)\right) + \sum_{k=0}^{n} \alpha_k} \right) f_k \tag{38}$$

where

$$w_k^t = \sum_{i \in J_k} (-1)^i a_{k,i} = \sum_{i \in J_k} (-1)^i \prod_{j=i, j \neq k}^{i+d} \frac{1}{\sin\left(\frac{\omega}{2}(x_k - x_j)\right)}, \quad \alpha_k = \sum_{i \in J_k} (-1)^i a_{k,i} c_i \tag{39}$$

with $J_k = \{i \in \{0,1,2,...,n-d\} \textit{ such that } k-d \leq i \leq k\}$.

We stress the fact that if there exists an index $\bar{j}$ such that $\sum_{j=\bar{j}}^{\bar{j}+d} x_j = 0$ then for this set of nodes the construction of the interpolant is impossible. Otherwise we will choose ω so that r^t is well-defined and interpolates every set of data $\{x_k, f_k\}$. For sure ω must be chosen such that, for every sequence of $d+1$ adjacent nodes,

$$\frac{\omega}{2} \sum_{j=k}^{k+d} x_j \neq \nu\pi \ \forall \nu \in \mathbb{Z} \ \forall k \in \{0, \ldots, n-d\}.$$

In general the check of this property might be a little nasty, so in the following we restrict on $I = [0,1]$ and replace **Condition C1** with **Condition C2**

$$\omega \sum_{j=k}^{k+d} x_j < \pi \quad \forall k \in \{0, \ldots, n-d\}. \tag{40}$$

Since $\sum_{k=0}^{n} \alpha_k = \sum_{i=0}^{n-d} (-1)^i c_i \sum_{k=i}^{i+d} a_{k,i}$, we prove that this sum is equal to zero showing that $\forall i \ \sum_{k=i}^{i+d} a_{k,i} = 0$ (cf. [1, Prop. 2.4.1]).

Proposition 3.5. *If d is odd and the nodes $\{x_k\}$ are equispaced in $[0,1]$ then*

$$\sum_{k=i}^{i+d}\left(\prod_{j=i\ j\neq k}^{i+d}\frac{1}{\sin\left(\frac{\omega}{2}(x_k-x_j)\right)}\right)=0\,. \tag{41}$$

Proof. We notice that it is enough to prove that

$$\prod_{j=i\ j\neq k}^{i+d}\sin\left(\frac{\omega}{2}(x_k-x_j)\right)=-\prod_{j=i\ j\neq i+d-k}^{i+d}\sin\left(\frac{\omega}{2}(x_{i+d-k}-x_j)\right)\ \ \forall k\in\{i,\dots,i+\frac{d+1}{2}-1\}.$$

Fix k and let $h=|x_{i+1}-x_i|=\frac{1}{n}$. Then,

$$\begin{aligned}\prod_{j=i\ j\neq k}^{i+d}\sin\left(\frac{\omega}{2}(x_k-x_j)\right)&=\prod_{j<k}\sin\left(\frac{\omega}{2}(x_k-x_j)\right)\prod_{j>k}\sin\left(\frac{\omega}{2}(x_k-x_j)\right)=\\&=\prod_{j<k}\sin\left(\frac{\omega}{2}|x_k-x_j|\right)(-1)^{i+d-k}\prod_{j>k}\sin\left(\frac{\omega}{2}|x_k-x_j|\right)=\\&=(-1)^{i+d-k}\prod_{j=i}^{k-1}\sin\left(\frac{\omega(k-j)h}{2}\right)\prod_{j=k+1}^{i+d}\sin\left(\frac{\omega(j-k)h}{2}\right)\\&=(-1)^{i+d-k}\prod_{l=1}^{k-i}\sin\left(\frac{\omega}{2}lh\right)\prod_{l=1}^{i+d-k}\sin\left(\frac{\omega}{2}lh\right)\end{aligned}$$

and

$$\begin{aligned}\prod_{\substack{j=i\\ j\neq 2i+d-k}}^{i+d}\sin\left(\frac{\omega}{2}(x_{2i+d-k}-x_j)\right)&=\prod_{j<(i+d)-(k-i)}^{i+d}\sin\left(\frac{\omega}{2}(x_{2i+d-k}-x_j)\right)\prod_{j>(i+d)-(k-i)}^{i+d}\sin\left(\frac{\omega}{2}(x_{2i+d-k}-x_j)\right)\\&=\prod_{j<(i+d)-(k-i)}^{i+d}\sin\left(\frac{\omega}{2}|x_{2i+d-k}-x_j|\right)(-1)^{k-i}\prod_{j>(i+d)-(k-i)}^{i+d}\sin\left(\frac{\omega}{2}|x_{2i+d-k}-x_j|\right)\\&=(-1)^{k-i}\prod_{j=i}^{2i+d-k-1}\sin\left(\frac{\omega}{2}(2i+d-k-j)h\right)\prod_{j=2i+d-k+1}^{i+d}\sin\left(\frac{\omega}{2}(j-2i-d+k)h\right)\\&=(-1)^{k-i}\prod_{l=1}^{i+d-k}\sin\left(\frac{\omega}{2}lh\right)\prod_{l=1}^{k-i}\sin\left(\frac{\omega}{2}lh\right).\end{aligned}$$

Since d is odd and $k-i+(i+d-k)=d$, it follows that k and $i+d-(k-i)$ can not be both even or odd. So $(-1)^{i+d-k}=-(-1)^{k-i}$ and the conclusion holds. □

Thanks to the previous result TFHRI can be written as

$$r^t(x)=\sum_{k=0}^{n}b_k^t(x)f_k,$$

where

$$b_k^t(x)=\frac{w_k^t\cot\left(\frac{\omega}{2}(x-x_k)\right)+\alpha_k}{\sum_{k=0}^{n}w_k^t\cot\left(\frac{\omega}{2}(x-x_k)\right)}\,. \tag{42}$$

For what follows, we need the trigonometric identity

$$\cot(x)+\cot(y)=\frac{\sin(x+y)}{\sin(x)\sin(y)}\ \ \forall x,y\neq k\pi\ k\in\mathbb{Z}. \tag{43}$$

and a stronger condition than **Condition C2**, that is ω must satisfy the **Condition C3**

$$\omega\left(2+\sum_{j=n-d+1}^{n-1}x_j\right)<\pi\,. \tag{44}$$

Theorem 3.6 (Upper bound)**.** *The Lebesgue constant associated with TFHRI in* $[0,1]$*, with* ω *satisfying* **Condition C3***, at equispaced nodes* $\{x_j = \frac{j}{n},\ j = 0,\dots,n\}$ *and basis functions (42) has the upper bound*

$$\Lambda_n \le \bar{C}_1 M_\omega^{d+1} 2^{d-1}\big(2+\ln(n)\big). \tag{45}$$

with $\bar{C}_1$ *a suitable positive constant.*

Proof. As above, consider $x_k < x < x_{k+1}$ and the corresponding Lebesgue function

$$\lambda_n(x) = \frac{\displaystyle\sum_{s=0}^{n}\left| w_j^t \cot\left(\frac{\omega}{2}(x-x_s)\right) + \alpha_s\right|}{\displaystyle\left|\sum_{s=0}^{n} w_s^t \cot\left(\frac{\omega}{2}(x-x_s)\right)\right|} = \frac{N(x)}{D(x)}$$

and define

$$\tilde{z}_{s,i}(x) = \frac{\sin(\frac{\omega}{2}(x + \sum_{j=i,\ j\neq s}^{i+d} x_j))}{\sin(\frac{\omega}{2}\sum_{j=i}^{i+d} x_j)}.$$

Since the terms $a_{s,i}$ oscillate in sign, under the assumptions on ω, for fixed s the terms in parentheses have the same sign. Therefore, $\tilde{z}_{s,i}(x)$ is greater than 0. By using (43) we can rewrite

$$\begin{aligned} N(x) &= \sum_{s=0}^{n}\left|\sum_{i\in J_s}(-1)^i a_{s,i}\left(\cot\left(\frac{\omega}{2}(x-x_s)\right)+c_i\right)\right| \\ &= \sum_{s=0}^{n}\left(\sum_{i\in J_s}|a_{s,i}|\tilde{z}_{s,i}(x)\right)\frac{1}{\sin(\frac{\omega}{2}|x-x_s|)} \\ &\le Z_k(x)\sum_{s=0}^{n}\frac{|w_s^t|}{\sin(\frac{\omega}{2}|x-x_s|)} \end{aligned}$$

with $Z_k(x) = \max\limits_{(s,i)\in S}\tilde{z}_{s,i}(x)$, $\quad S = \{(s,i)|\ 0\le s\le n,\ i\in J_s\}$.

We notice that for fixed x, $\tilde{z}_{s,i} > \tilde{z}_{s+1,i}$, with s such that $i \in J_s$. It then follows that

$$Z_k(x) = \max_{(s,i)\in\bar{S}}\tilde{z}_{s,i}(x),$$

where $\bar{S} = \{(s,i)|i\in\{0,\dots,n-d\},\ s=\min\{t|t\in J_i\}\} = \{(i,i)|i\in\{0,\dots,n-d\}\}$.
Hence,

$$\lambda_n(x) \le \frac{h^d d!(x-x_k)(x_{k+1}-x)Z_k(x)\displaystyle\sum_{s=0}^{n}\frac{|w_s^t|}{\sin(\frac{\omega}{2}|x-x_s|)}}{h^d d!(x-x_k)(x_{k+1}-x)\left|\displaystyle\sum_{j=0}^{n} w_j^t\cot\left(\frac{\omega}{2}(x-x_j)\right)\right|}.$$

Using the same proof for d even, we get

$$\lambda_n(x) \le Z_k(x) M_\omega^{d+1}2^{d-1}(2+\log(n)).$$

Since $\tilde{z}_{i,i}$ is an increasing function in x for every i, then $Z_k(x) \le \max\limits_{i\in\{0,\dots,n-d\}}\tilde{z}_{i,i}(1) := \bar{C}_1$, we can conclude by taking the maximum on all subintervals

$$\Lambda_n = \max_{k=0,\dots,n-1}\left(\max_{x_k<x<x_{k+1}}\lambda_n(x)\right) \le \bar{C}_1 M_\omega^{d+1}2^{d-1}(2+\log(n)).$$

□

Theorem 3.7 (Lower bound)**.** *Under the same hypoteses of the previous theorem,*

$$\Lambda_n \ge \frac{\bar{C}_2}{M_\omega^{d+1}2^{d+2}}\binom{2d+1}{d}\ln\left(\frac{n}{d}-1\right).$$

Proof. Now,

$$\lambda_n(x) = \frac{h^d d! \sum_{s=0}^{n} \left| w_s^t \cot\left(\frac{\omega}{2}(x - x_s)\right) + \alpha_s \right|}{\left| h^d d! \sum_{s=0}^{n} w_s^t \cot\left(\frac{\omega}{2}(x - x_s)\right) \right|} := \frac{N(x)}{D(x)}.$$

We consider $x^* = \frac{x_1 - x_0}{2} = \frac{1}{2n}$.
We first investigate the numerator, using (14) and the bounds on the weights w_s^t, we get

$$N(x^*) = h^d d! \sum_{s=0}^{n} \left(\sum_{i \in J_s} |a_{s,i}| \frac{\tilde{z}_{s,i}}{\sin(\frac{\omega}{2}|x^* - x_s|)} \right) \geq C h^d d! \sum_{s=0}^{n} \frac{|w_s^t|}{\sin(\frac{\omega}{2}|x^* - x_s|)}$$

with $\tilde{z}_{s,i} := \dfrac{\sin(\frac{\omega}{2}(x^* + \sum_{j=i,\ j\neq s}^{i+d} x_j))}{\sin(\frac{\omega}{2}\sum_{j=i}^{i+d} x_j)}$ and $C := \min_{(s,i)\in S} \tilde{z}_{s,i}$ with S as in the previous proof.
Using the same inequality for the lower bound in the case d even, finally we have

$$N(x^*) \geq C \frac{2^{d+1}}{\omega^{d+1}} n 2^d \ln(\frac{n}{d} - 1).$$

As before, noting that

$$\tilde{z}_{s,i} > \tilde{z}_{s+1,i}$$

we have

$$C = \bar{C}_2 = \min_{(s,i)\in \bar{S}} \tilde{z}_{s,i}$$

where $\bar{S} = \{(s,i) | i \in \{0, \ldots, n-d\},\ s = \max\{t | t \in J_i\}\}$.
The inequality becomes

$$N(x^*) \geq \bar{C}_2 \frac{2^{d+1}}{\omega^{d+1}} n 2^d \ln(\frac{n}{d} - 1). \tag{46}$$

For the denominator, in the case d even, the inequality is still true

$$D(x^*) \leq h^d d! \left| \sum_{s=0}^{n-d} s_s^t(x^*) \right| \leq \frac{2^{d+1}}{\omega^{d+1}} M_\omega^{d+1} n \frac{2^{2d+2}}{\binom{2d+1}{d}}.$$

Finally, we get

$$\Lambda_n \geq \frac{\bar{C}_2}{M_\omega^{d+1} 2^{d+2}} \binom{2d+1}{d} \ln\left(\frac{n}{d} - 1\right),$$

as claimed. □

4 Numerical experiments

Before going to the purely numerical part, we make several observations about the implementations. First of all, we noticed that the expression of the basis functions b_k in the case d even in (12) (similarly for the d odd case), is numerically unstable as ω is closer to 0. For instance, the aforementioned formula is

$$b_k(x) = \frac{w_k^t \prod_{s=0, s\neq k}^{n} \sin(\frac{\omega}{2}(x - x_s))}{\sum_{j=0}^{n} w_j^t \prod_{s=0, s\neq j}^{n} \sin(\frac{\omega}{2}(x - x_s))}.$$

The reason why can be easily understood, for example, by noting that, if $\omega = 10^{-8}$, $n = 50$ and I=[0,1],

$$\left| \prod_{s=0,s\neq k}^{n} \sin\left(\frac{\omega}{2}(x - x_s)\right) \right| \leq \prod_{s=0,s\neq k}^{n} \sin\left(\frac{\omega}{2}\right) \approx \omega^n$$

so that $\omega^{50} \approx 10^{-400}$, that is numerically approximated by 0. With the aim to improve the formula and make it more stable, we have modified the expression of the weights and of the total b_k both, turning out to be

$$\tilde{b}_k^t(x) = \frac{\frac{2^d}{\omega^d} \frac{\tilde{w}_k^t g\left(\frac{\omega}{2}(x-x_k)\right)}{\frac{\omega}{2}(x-x_k)}}{\sum_{j=0}^{n} \frac{2^d}{\omega^d} \frac{\tilde{w}_j^t g\left(\frac{\omega}{2}(x-x_j)\right)}{\frac{\omega}{2}(x-x_j)}} = \frac{\frac{\tilde{w}_k^t g\left(\frac{\omega}{2}(x-x_k)\right)}{x-x_k}}{\sum_{j=0}^{n} \frac{\tilde{w}_j^t g\left(\frac{\omega}{2}(x-x_j)\right)}{x-x_j}} = \frac{\tilde{w}_k^t g\left(\frac{\omega}{2}(x-x_k)\right) \prod_{s=0,s\neq k}^{n}(x-x_s)}{\sum_{j=0}^{n} \tilde{w}_j^t g\left(\frac{\omega}{2}(x-x_j)\right) \prod_{s=0,s\neq j}^{n}(x-x_s)} \tag{47}$$

where

$$\tilde{w}_k^t = \sum_{i\in J_k}(-1)^i \prod_{j=i,j\neq k}^{i+d} \frac{g\left(\frac{\omega}{2}(x_k-x_j)\right)}{x_k-x_j} \tag{48}$$

$J_k = \{i \in \{0,1,2,\ldots,n-d\}\ s.\ t.\ k-d \le i \le k\}$
so that we can stably compute the Lebesgue function and so the Lebesgue constant.

4.1 Lebesgue constant

In order to verify the goodness of the results in the previous sections and to analyze the behaviour of the Lebesgue constant of trigonometric Floater-Hormann interpolants, we have performed some numerical experiments. We computed the Lebesgue contant with different d by evaluating the Lebesgue function λ_n on a set of 1000 points in $[0,1]$.

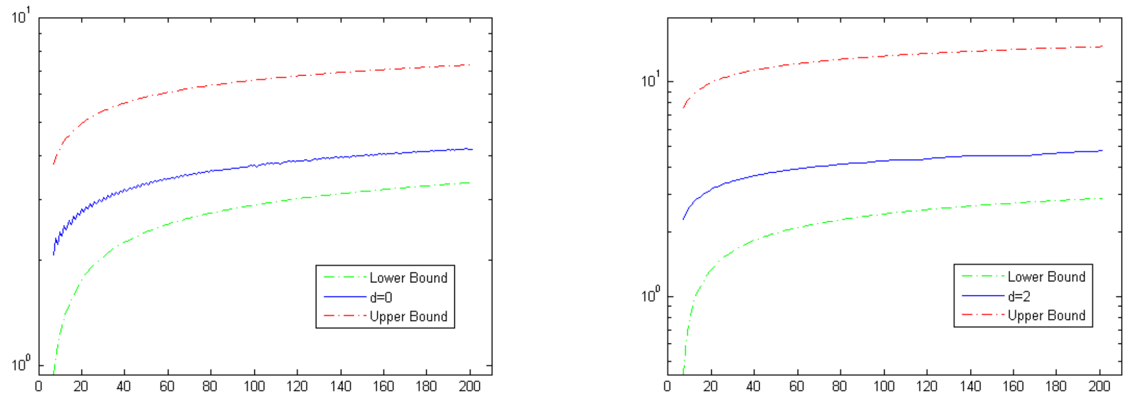

Figure 4: Lebesgue constants for $d=0$ (left) $d=2$ (right), compared with their upper and lower bounds, for $6 \le n \le 200$ equally spaced in $[0,1]$, $\omega = 0.1$

We first consider the even case with $d = 0, 2$. When $d = 0$ we notice a zig-zag behaviour for λ_{2k} and λ_{2k+1} as happend for the Berrut's rational interpolant and as in that case we report only the values λ_{2k} (cf. [9]). Figure 4 in semilogarithmic scale reveals that with $\omega = 0.1$ the Lebesgue constants satisfy the bounds.

For d odd, we take $d = 1, 3$ and ω in such a way that for every $6 \le n \le 200$ **Condition C3** is satisfied. A good choice is $\omega = 0.1$ as before. The results are reported in Figure 5, where it can be noticed that the constants are between the bounds.

For the sake of completeness, Figure 6 and Figure 7 show the Lebesgue functions with $d = 2$, $d = 3$ and $n = 10, 20, 40$. We note that when $d = 2$, as in the classical FHRI case, the Lebesgue function is symmetric with respect to the middle of the interval and that its maximum, is obtained in the most external subintervals, $[x_0, x_1]$ and $[x_{n-1}, x_n]$. Instead when $d = 3$ the Lebesgue function in not symmetric and its maximum is in $[x_{n-1}, x_n]$.

We have made more experiments on the Lebesgue constant with varying ω. We rescrict our analysis on the case $d = 0$ and $d = 1$ but the same is still true for other values of d. According to the theoretical Conditions, we have choosen different values of the pulsation near the maximum admissible number up to 0. It always turns out that there are not no significant differences for the increasing of the constant (Figure 8).

4.2 Interpolation

We consider in $[0,1]$ the following functions:

$$\begin{aligned} f_1(x) &= \frac{1}{25x^2+1}, \quad (Runge\ function) \\ f_2(x) &= x^3 - x^2 - 1, \quad (polynomial) \\ f_3(x) &= \sin(x) + \cos(x), \quad (trigonometric). \end{aligned}$$

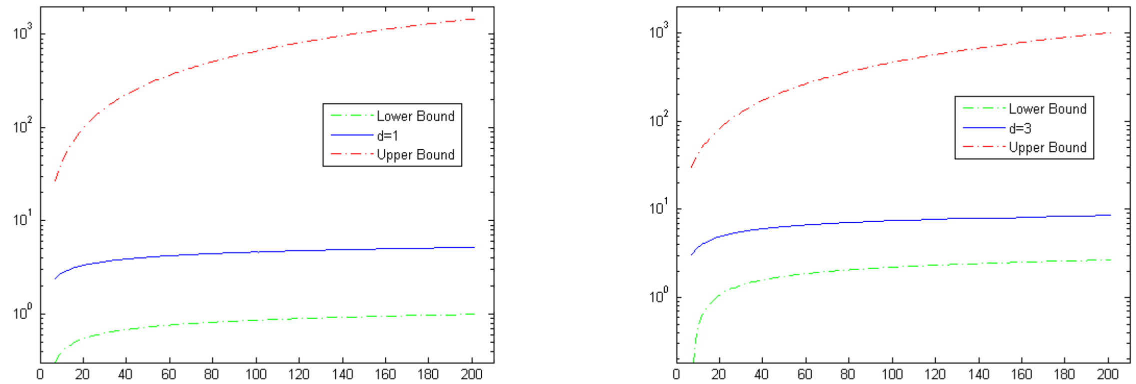

Figure 5: Lebesgue constants for $d = 1$ (left) $d = 3$ (right), compared with their upper and lower bounds, for $6 \le n \le 200$ in $[0, 1]$, $\omega = 0.1$.

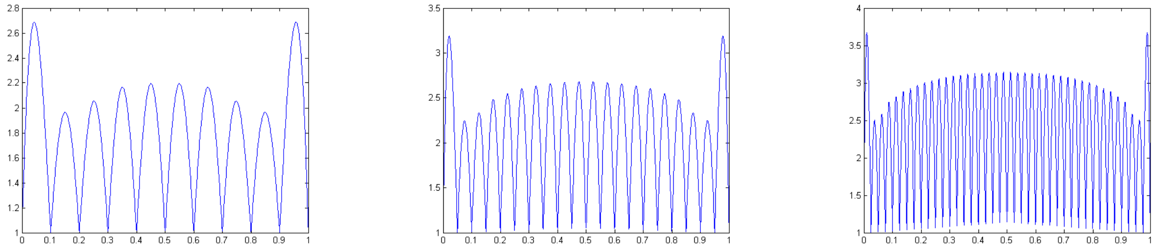

Figure 6: Lebesgue functions with equally spaced nodes in $[0, 1]$, $n = 10, 20, 40$, $\omega = 0.1$ and $d = 2$

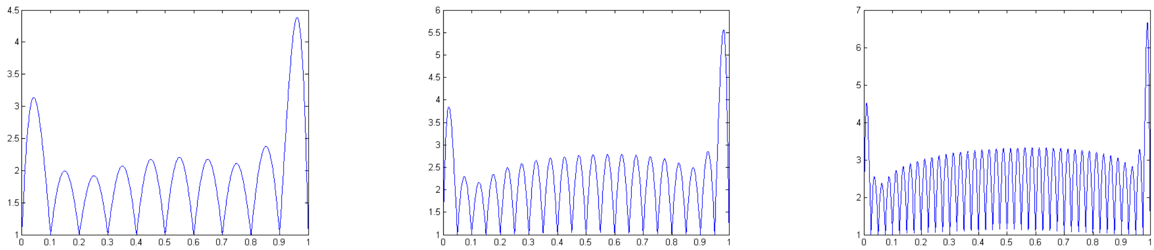

Figure 7: Lebesgue functions with equally spaced nodes in $[0, 1]$, $n = 10, 20, 40$, $\omega = 0.1$ and $d = 3$

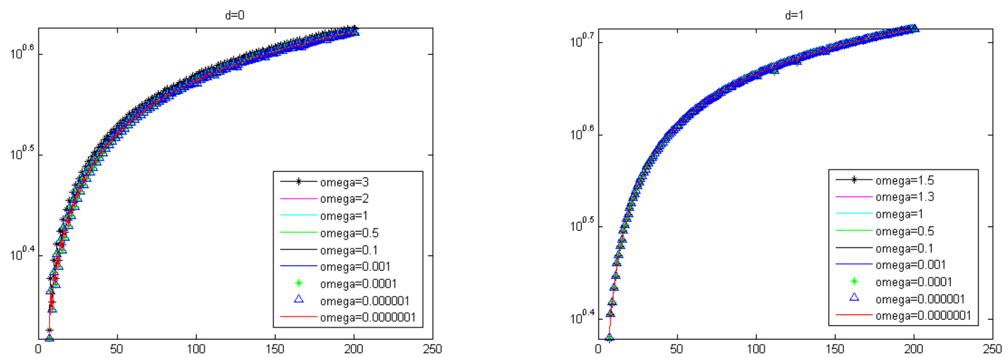

Figure 8: Lebesgue constant as $\omega \to 0$ for $d = 0, 1$

Using formula (3), the first experiment considers two different d, even and odd, on a fixed number $n = 20$ of equally spaced interpolation nodes and ω satisfying the Condition 1. In Table 1 we report the results of the relative errors (the evaluation points are 200 equispaced points in $[0,1]$). In the case of f_2 and f_3 (i.e. a polynomial and a trigonometric polynomial) we have also

	$d=0$	$d=2$	$d=1$	$d=3$
f_1	$1.9e-2$	$1.3e-3$	$7.8e-3$	$1.0e-3$
f_2	$3.3e-3$	$2.7e-6$	$2.9e-4$	$3.7e-6$
f_3	$3.2e-3$	$1.8e-6$	$3.7e-3$	$1.6e-7$

Table 1: Relative interpolation errors

compared the behaviour of the relative errors on varying n from $n=10$ to $n=160$ every 10 degrees. We observe that similar results can be obtained for f_1 but, as evident from the first line of Table 1, the errors decrease slowly.

In Figures 9 and 10 we show the results for the cases $d=2$ and $d=3$, respectively.

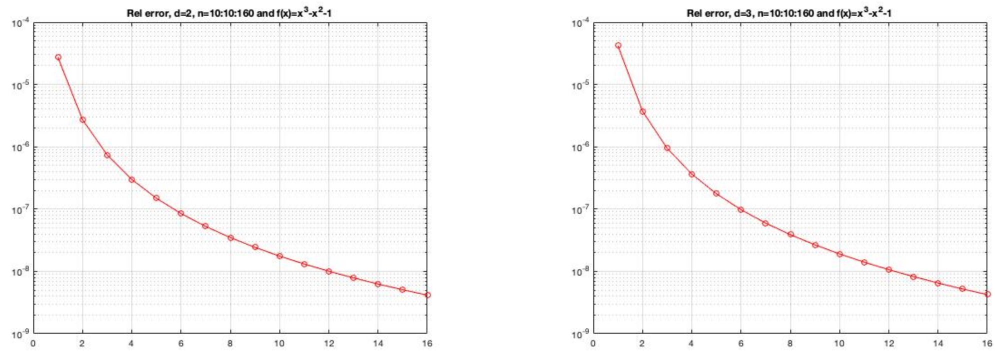

Figure 9: Relative interpolation errors for the polynomial f_2

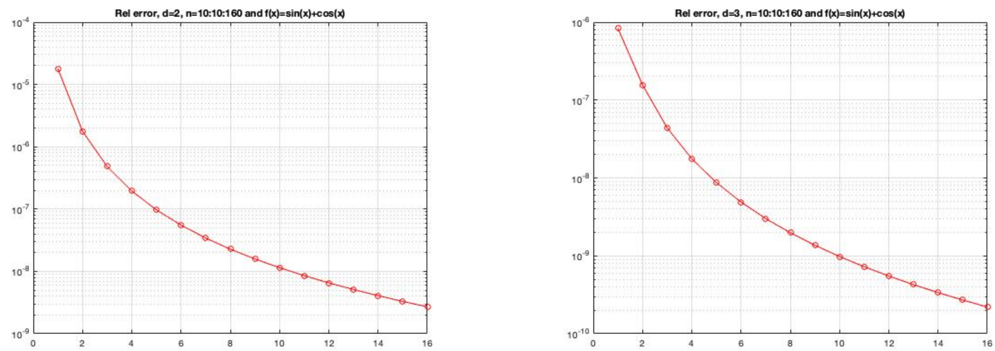

Figure 10: Relative interpolation errors for the trigonometric function f_3

As final experiments, we kept n fixed and varied d. In Figures 11 and 12 we show the behavior of the relative interpolation errors again for f_2 and f_3. In both cases we observed that the errors decrease to almost the machine precision for relatively small values of d (usually $d \le 15$), then they start to increase. This is consistent with the upper bound in Theorem 3.6 which is logarithmic in n and exponential in d.

5 Conclusion and future work

In this work we have introduced the Trigonometric Floater-Hormann Rational Interpolant (TFHRI). This interpolant, as for the classical case, is a blend of trigonometric functions. The construction depends indeed on n, the number of points, a *blending parameter* d for constructing the blending trigonometric "polynomials" (see above §2) and a parameter $\omega \in (0, \frac{\pi}{(b-a)})$ necessary for the well-posedness of the interpolant.

We have studied the Lebesgue constant providing upper and lower bounds in the case of equispaced points. It turns out that our bounds have logarithmic growth in the number n of interpolation points, which are slightly different in case d even or odd. All bounds depend on the constant $M_\omega = 1/\mathtt{sinc}(\omega/2)$ and, when $M_\omega = 1$, the bounds correspond to those of the classical Floater-Hormann rational interpolant on equispaced nodes.

In Section 4, we have presented the plots of the Lebesgue constants on varying n with even or odd values of d. The bounds are not yet optimal, but they represent quite well the growth of the true Lebesgue constants.

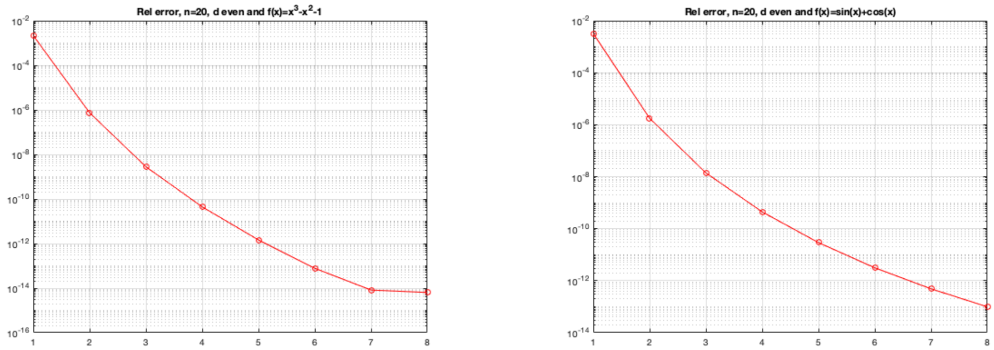

Figure 11: Relative interpolation errors for even $d \leq 14$

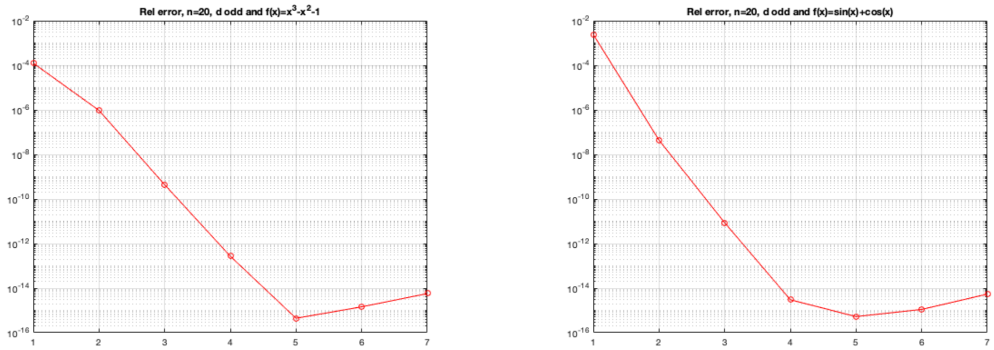

Figure 12: Relative interpolation errors for odd $d \leq 13$

What remains to do is the error analysis which will be also quite technical. In particular we need to know if the interpolant is exact on polynomials up to certain degree or/and if it is exact on trigonometric functions. Another question will be how the smoothness of the function will influence the convergence. The convergence seems guaranteed by the numerical evidence but theoretical results must be provided.

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