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On the subset combinatorics of G-spaces

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Abstract. Let G be a group and let X be a transitive G-space. We classify the subsets of X with respect to a translation invariant ideal J in the Boolean algebra of all subsets of X, introduce and apply the relative combinatorical derivations of subsets of X. Using the standard action of G on the Stone-Čech compactification βX of the discrete space X, we characterize the points $p \in \beta X$ isolated in Gp and describe a size of a subset of X in terms of its ultracompanions in βX. We introduce and characterize scattered and sparse subsets of X from different points of view.

1. Introduction

Let G be a group and let X be a transitive G-space with the action $G \times X \to X$, $(g, x) \mapsto gx$. If $X = G$ and gx is a product of g and x then X is called the *left regular G-space*.

A family J of subsets of X is called an ideal in the Boolean algebra $\mathcal{P}_X$ of all subsets of X if $X \notin J$ and $A, B \in J$, $C \subset A$ imply $A \cup B \in J$ and $C \in J$. The ideal of all finite subsets of X is denoted by $[X]^{<\omega}$. An ideal J is *translation invariant* if $gA \in J$ for all $g \in G$, $A \in J$, where $gA = \{ga : a \in A\}$. If X is finite then $J = \{\varnothing\}$ so in what follows all G-spaces are supposed to be infinite.

Now we fix a translation invariant ideal J in $\mathcal{P}_X$ and say that a subset A of X is

- *J-large* if $X = FA \cup I$ for some $F \in [G]^{<\omega}$ and $I \in J$;

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- J-*small* if $L \setminus A$ is J-large for every J-large subset L of X;
- J-*thick* if $Int_F(A) \notin J$ for each $F \in [G]^{<\omega}$, where $Int_F(A) = \{a \in A : Fa \subseteq A\}$;
- J-*prethick* if FA is thick for some $F \in [G]^{<\omega}$.

If $J = \varnothing$ we omit the prefix J and get a well-known classification of subsets of a G-spaces by their combinatorial size (see the survey [11]).

In the case of the left regular G-spaces, the notions of J-large and J-small subsets appeared in [1].

We say that a mapping $\Delta_J : \mathcal{P}_X \to \mathcal{P}_G$ defined by

$$\Delta_J(A) = \{g \in G : gA \cap A \notin J\}$$

is a *combinatorial derivation relatively to the ideal* J. If X is the left regular G-space and $J = [X]^{<\infty}$, the mapping Δ_J was introduced in [12] under the name combinatorial derivation and studied in [13].

In Section 2 we prove that if a subset A of X is not J-small then $\Delta_J(A)$ is large in G. For the left regular G-space X and $J = [X]^{<\omega}$, this statement was proved in [6].

We endow X with the discrete topology and take the points of βX, the Stone-Čech compactification of X, to be the ultrafilters on X, with the points of X identified with the principal ultrafilters on X. The topology on βX can be defined by stating that the set of the form $\overline{A} = \{p \in \beta X : A \in p\}$, where A is a subset of X, form a base for the open sets. We note the sets of this form are clopen and that for any $p \in \beta X$ and $A \subset X$, $A \in p$ if and only if $p \in \overline{A}$. We denote $A^* = \overline{A} \cap X^*$, where $X^* = \beta X \setminus X$. The universal property of βX states that every mapping $f : X \to Y$, where Y is a compact Hausdorff space, can be extended to the continuous mapping $f^\beta : \beta X \to Y$.

Now we endow G with the discrete topology and, using the universal property of βG, extend the group multiplication from G to βG (see [8, Chapter 4]), so βG becomes a compact right topological semigroup.

We define the action of βG on βX in two steps. Given $g \in G$, the mapping

$$x \mapsto gx : \ X \to \beta X$$

extends to the continuous mapping

$$p \mapsto gp : \ \beta X \to \beta X.$$

Then, for each $p \in \beta X$, we extend the mapping $g \mapsto gp : \ G \to \beta X$ to the continuous mapping

$$q \mapsto qp : \ \beta G \to \beta X.$$

Let $q \in \beta G$ and $p \in \beta X$. To describe a base for the ultrafilter $qp \in \beta X$, we take any element $Q \in q$ and, for every $g \in Q$ choose some element $P_x \in p$. Then $\bigcup_{g \in Q} gP_x \in qp$, and the family of subsets of this form is a base for the ultrafilter qp.

Given a subset A of X and an ultrafilter $p \in X^*$ we define a *p-companion* of A by

$$\triangle_p(A) = A^* \cap Gp = \{gp : g \in G, A \in gp\},$$

and say that a subset S of X^* is an *ultracompanion* of A if $S = \triangle_p(A)$ for some $p \in X^*$.

In Section 3 we characterze the subsets of X of different types in terms of their ultracompanions. For example a subset A of X is J-large if and only if $\triangle_p(A) \neq \varnothing$ for each $p \in \check{J}$, where $\check{J} = \{p \in X^* : X \setminus I \in p$ for every $I \in J\}$. For the left regular X and $J = \{\varnothing\}$, these characterizations are obtained in [15].

In Section 4 we describe the points $p \in \beta X$ isolated in Gp and introduce the piecewise shifted FP-sets in X to characterize the subsets $A \subseteq X$ such that $\triangle_p(A)$ is discrete for each $p \in X^*$.

In Section 5 we extend the notions scattered and sparse subsets from groups [3] to G-space and characterize these subsets from different points of view.

2. Relative combinatorial derivations

Let X be a transitive G-space and let J be a translation invariant ideal in $\mathcal{P}_X$.

Lemma 2.1. *For a subset A of X, the following statements are equivalent*

(i) A is J-small;

(ii) $G \setminus FA$ is J-large for each $F \in [G]^{<\omega}$;

(iii) A is not J-prethick.

Proof. Apply the arguments proving Theorem 2.1 in [1]. □

The next lemma follows directly from the definition of J-small subsets.

Lemma 2.2. *The family of all J-small subsets of X is a translation invariant ideal in $\mathcal{P}_X$.*

Lemma 2.3. *Let L be a J-large subset of X. Then given a partition $L = A \cup B$, either $\Delta_J(A)$ is large or B is J-large.*

Proof. We take $F \in [G]^{<\omega}$ and $I \in J$ such that $G = F(A \cup B) \cup I$. Assume that $G \neq F\Delta_J(A)$ and show that B is J-large.

Let $F = \{f_1, ..., f_k\}$. We take $g \in G \setminus F\Delta_J(A)$ and put $I_i = f_i^{-1}gA \cap A$, $i \in \{1, ..., k\}$. Since $g \notin f_i\Delta_J(A)$, we have $I_i \in J$ and $f_i^{-1}gx \notin A$ for each $x \in A \setminus I_i$.

If $x \in X$ and $F^{-1}gx \cap L = \varnothing$ then $gx \notin FL$ so $gx \in I$ and $x \in g^{-1}I$. We put

$$I' = I_1 \cup ... \cup I_k \cup g^{-1}I.$$

If $x \in A \setminus I'$ then there is $i \in \{1, ..., k\}$ such that $f_i^{-1}gx \in A \cup B$. Since $f_i^{-1}gx \notin A$, we have $f_i^{-1}gx \in B$. Hence, $A \setminus I' \subseteq F^{-1}gB$ and

$$G = F(A \setminus I') \cup FI' \cup FB \cup I = FF^{-1}gB \cup FB \cup (FI' \cup I),$$

and we conclude that B is J-large. □

Theorem 2.4. *If a subset A of X is J-prethick then $\Delta_J(A)$ is large.*

Proof. By Lemma 2.1, A is not J-small. We take a J-large subset L such that $L \setminus A$ is not J-large. Since $L = (L \cap A) \cup (L \setminus A)$, by Lemma 2.3, $\Delta_J(L \cap A)$ is large so $\Delta_J(A)$ is large. □

Corollary 2.5. *If an J-prethick subset A of X is finitely partitioned $A = A_1 \cup ... A_n$ then $\Delta_J(A_i)$ is large for some $i \in \{1, ..., n\}$*

Proof. By Lemma 2.2 some cell A_i is prethick. Apply Theorem 2.4. □

Remark 2.6. Given a translation invariant ideal J in $\mathcal{P}_X$, there is a function $\Phi_J : \mathbb{N} \to \mathbb{N}$ such that, for any n-partition $X_1 \cup ... \cup X_n$ of X, there exists A_i and $F \in [G]^{<\omega}$ such that $G = F\Delta_J(A_i)$ and $|F| \leqslant \Phi_J(n)$. These functions are intensively studied in [2] and [4].

3. Ultracompanions

Given a translation invariant ideal J in $\mathcal{P}_X$, we denote

$$\check{J} = \{p \in X^* : X \setminus I \in p \text{ for each } I \in J\},$$

and observe that $\check{J}$ is closed in X^* and $gp \in \check{J}$ for all $g \in G$ and $p \in \check{J}$.

Theorem 3.1. *For a subset A of X, the following statements hold* *(i)* *A is J-large if and only if $\triangle_p(A) \neq \varnothing$ for each $p \in \check{J}$;*

(ii) *A is J-thick if and only if there exists $p \in \check{J}$ such that $\triangle_p(A) = Gp$;*

(iii) *A is J-prethich if and only if there exists $p \in \check{J}$ and $F \in [G]^{<\omega}$ such that $\triangle_p(FA) = Gp$;*

(iv) *A is J-small if and only if for every $p \in \check{J}$ and every $F \in [G]^{<\omega}$, we have $\triangle_p(A) \neq Gp$;.*

Proof. (i) Suppose that A is J-large and choose $F \in [G]^{<\omega}$ and $I \in J$ such that $X = FA \cup I$. We take an arbitrary $p \in \check{J}$ and choose $g \in F$ such that $gA \in p$ so $A \in g^{-1}p$ and $\triangle_p(A) \neq \varnothing$

Assume that $\triangle_p(A) \neq \varnothing$ for each $p \in J$. Given $p \in J$, we choose $g_p \in G$ such that $A \in g_p p$. Then we consider a covering of $\check{J}$ by the subsets $\{g_p^{-1}A^* : p \in \check{J}\}$ and choose its finite subcovering $g_{p_1}^{-1}A^*, ..., g_{p_n}^{-1}A^*$ We take $I \in J$ and $H \in [X]^{<\omega}$ such that $X \setminus (g_{p_1}^{-1}A^* \cup ... \cup g_{p_n}^{-1}A^*) = I \cup H$. At last, we choose $F \in [G^{<\omega}]$ such that $\{g_{p_1}^{-1}, ..., g_{p_n}^{-1}\} \subseteq F$ and $H \subseteq FA$. Then $X = FA \cup I$ and A is J-large.

(ii) We note that A is J-thick if and only if $X \setminus A$ is not J-large and apply (i).

(iii) follows from (ii).

(iv) follows from (iii) and Lemma 2.1. □

We suppose that $J \neq \{\varnothing\}$ and say that a subset A of X is J-thin if, for every $F \in [G]^{<\omega}$, there exists $I \in J$ such that $|Fa \cap A| \leqslant 1$ for each $a \in A \setminus I$.

Theorem 3.2. *A subset A of X is I-thin if and only if $\triangle_p(A) \leqslant 1$ for each $p \in J$.*

Proof. Suppose that A is not J-thin and choose $F \in [G]^{<\omega}$ such that, for each $I \in J$, there is $a_I \in A \setminus I$ satisfying $Fa_I \cap A \neq \{a_I\}$. We pick $g_I \in F$ and $b_I \in A$ such that $g_I a_I = b_I$ and $b_I \in A$. Then we put $A_I = \{a_{I'} : I \subseteq I', I' \in J\}$ and take $p \in \check{J}$ such that $A_I \in p$ for each $I \in J$. Since p is an ultrafilter, there exists $g \in F$ such that $gp \neq p$ and $A \in gp$. Hence $\{p, gp\} \subseteq \triangle_p(A)$ and $|\triangle_p(A)| > 1$.

Assume that $|\triangle_p(A)| > 1$ for some $p \in J$. We pick distinct $g_1p, g_2p \in \triangle_p(A)$ and put $F = \{g_2 g_1^{-1}\}$. Since $A \setminus I \in g_1p \cap g_2p$ for each $I \in J$, there is $a_I \in A \setminus I$ such that $g_2^{-1}g_1 a_I \in A \setminus \{a_I\}$. Hence, A is not J-thin. □

Remark 3.3. We say that a non-empty subset S of βX^* is invariant if $gS \subseteq S$ for each $g \in G$. It is easy to see that each closed invariant subset

S of X contains a minimal by inclusion closed invariant subset M and $M = cl(Gp)$ for each $p \in M$. By analogy with Theorem 4.39 from [8], we can prove that for $p \in X^*$ the subset $cl(Gp)$ is minimal if and only if, for every $P \in p$, there exists $F \in [G]^\omega$ such that $Gp \subseteq (FP)^*$.

Remark 3.4. Given a translation invariant ideal J in $\mathcal{P}_X$, we denote

$$K(\check{J}) = \bigcup\{M : M \text{ is a minimal closed invariant subset of } \check{J}\}.$$

By analogy with Theorem 4.40 from [8], we can prove that $p \in cl(K(\check{J}))$ if and only if each subset $P \in p$ is J-prethick. It is worth to be mentioned that each closed invariant subset S of X^* is of the form $S = \check{J}$ for some translation invariant ideal J in $\mathcal{P}_X$.

Remark 3.5. By Theorem 6.30 from [8], for every infinite group of cardinality $\varkappa$, there exists $2^{2^\varkappa}$ distinct minimal closed invariant subsets of G^*. We show that this statement fails to be true for G-spaces. Let $X = \omega$ and G be the group of all permutations of X. If S is a closed invariant subset of X^* then $S = X^*$.

Remark 3.6. We describe a relationship between ultracompanions and relative combinatorial derivations. Let J be a translation invariant ideal in $\mathcal{P}_X$, $A \subseteq X$, $p \in \check{J}$. We denote $A_p = \{g \in G : A \in gp\}$ so $\triangle_p(A) = A_p p$. Then

$$\Delta_J(A) = \bigcap\{A_p^{-1} : p \in \check{J}, A \in p\}.$$

4. Isolated points

Given any $p \in X^*$, we put

$$St(p) = \{g \in G : gp = p\},$$

and note that, by [8, Lemma 3.33], $gp = p$ if and only if there exists $P \in p$ such that $gx = x$ for each $x \in P$.

Theorem 4.1. *For every $p \in X^*$, the following statements are equivalent*

(i) p is not isolated in Gp;

(ii) there exists $q \in (G \setminus St(p))^*$ such that $qp = p$;

(iii) there exists $\varepsilon \in (G \setminus St(p))^*$ such that $\varepsilon\varepsilon = \varepsilon$ and $\varepsilon p = p$.

Proof. The implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are evident.

$(ii) \Rightarrow (iii)$. In view of Theorem 2.5 from [8], it suffices to show that the set

$$S = \{q \in (G \setminus St(p))^* : qp = p\}$$

is a subsemigroup of G^*. Let $q, r \in S$, $Q \in q$. For each $x \in Q$, we choose $R_x \in r$ such that $x^{-1}St(p) \cap R_x = \varnothing$. Then $xy \notin St(p)$ for each $y \in R_x$. We put

$$P = \bigcup_{x \in Q} xR_x,$$

and note that $P \in qr$ and $P \cap St(p) = \varnothing$. Hence $qr \in S$. □

Remark 4.2. For each $g \in G$, the mapping $p \mapsto gp : \ \beta X \to \beta X$ is a homeomorphism. It follows that Gp has an isolated point if and only if Gp is discrete.

Let $(g_n)_{n\in\omega}$ be sequence in G and let $(x_n)n \in \omega$ be a sequence in X such that

(1) $\{g_0^{\varepsilon_0} ... g_n^{\varepsilon_n} x_n : \varepsilon_i \in \{0,1\}\} \cap \{g_0^{\varepsilon_0} ... g_n^{\varepsilon_m} x_m : \varepsilon_i \in \{0,1\}\} = \varnothing$ for all distinct $m, n \in \omega$;

(2) $|\{g_0^{\varepsilon_0} ... g_n^{\varepsilon_n} x_n : \varepsilon_i \in \{0,1\}\}| = 2^{n+1}$ for every $n \in \omega$.

We say that a subset Y of X is a *piecewise shifted FP-set* if there exist $(g_n)_{n\in\omega}$, $(x_n)_{n\in\omega}$ satisfying (1) and (2) such that

$$Y = \{g_0^{\varepsilon_0} ... g_n^{\varepsilon_n} x_n : \varepsilon_i \in \{0,1\}, n \in \omega\}.$$

For definition of an FP-set in a group see [8, p. 108].

Theorem 4.3. *Let p be an ultrafilter from X^* such that Gp is not discrete. Then every subset $P \in p$ contains a piecewise shifted FP-set.*

Proof. We choose $g_0 \in G$ such that $p \neq g_0 p$, $P \in g_0 p$ and take $P_0 \subseteq P$, $P_0 \in p$ such that $g_0 P_0 \cap P_0 = \varnothing$. We pick an arbitrary $x_0 \in P_0$.

Suppose that the elements $g_0, ..., g_n$ from G and $x_0, ..., x_n$ from X have been chosen so that

(3) $g_0^{\varepsilon_0} ... g_k^{\varepsilon_k} x_k \in P$ for all $\varepsilon_i \in \{0,1\}$ and $k \leqslant n$;

(4) $\{g_0^{\varepsilon_0} ... g_k^{\varepsilon_k} x_k : \varepsilon_i \in 0,1\} \cap \{g_0^{\varepsilon_0} ... g_m^{\varepsilon_m} x_m : \varepsilon_i \in \{0,1\}\} = \varnothing$ for all $k < m \leqslant n$;

(5) $|\{g_0^{\varepsilon_0} ... g_k^{\varepsilon_k} x_k : \varepsilon_i \in 0,1\}| = 2^{k+1}$ for all $k \leqslant n$;

(6) $P \in g_0^{\varepsilon_0}...g_k^{\varepsilon_k}p$ for all $\varepsilon_i \in \{0,1\}$ and $k \leqslant n$;

(7) $|\{g_0^{\varepsilon_0}...g_k^{\varepsilon_k}p : \varepsilon_i \in 0,1\}| = 2^{k+1}$ for all $k \leqslant n$.

Since p is not isolated in Gp, we use (6) and (7) to choose $g_{n+1} \in G$ such that $P \in g_0^{\varepsilon_0}...g_{n+1}^{\varepsilon_{n+1}}p$ for all $\varepsilon_i \in \{0,1\}$ and $|\{g_0^{\varepsilon_0}...g_{n+1}^{\varepsilon_{n+1}}p : \varepsilon_i \in \{0,1\}\}| = 2^{n+2}$.

Then we choose $P_{n+1} \in p$ such that $g_0^{\varepsilon_0}...g_{n+1}^{\varepsilon_{n+1}}P_{n+1} \subseteq P$ for all $\varepsilon_i \in \{0,1\}$ and

$$g_0^{\varepsilon_0}...g_{n+1}^{\varepsilon_{n+1}}P_{n+1} \cap g_0^{\delta_0}...g_{n+1}^{\delta_{n+1}}P_{n+1} = \varnothing$$

for all distinct $(\varepsilon_0, ..., \varepsilon_{n+1})$ and $(\delta_0, ..., \delta_{n+1})$ from $\{0,1\}^{n+2}$

We pick $x_{n+1} \in P_{n+1}$ so that

$$\{g_0^{\varepsilon_0}...g_{n+1}^{\varepsilon_{n+1}}x_{n+1} : \varepsilon_i \in \{0,1\}\} \cap \{g_0^{\varepsilon_0}...g_k^{\varepsilon_k}x_k : \varepsilon_i \in \{0,1\}\} = \varnothing$$

for each $k \leqslant n$.

After ω steps, we get the sequences $(g_n)_{n\in\omega}$ and $(x_n)_{n\in\omega}$ which define the desired FP-set in P. □

Theorem 4.4. *For an infinite subset A of a G-space X, the following statements are equivalent*

(i) Gp is discrete for each $p \in A^*$;

(ii) A contains no piecewise shifted FP-sets.

Proof. The implication $(ii) \Rightarrow (i)$ follows from Theorem 4.3. To prove $(i) \Rightarrow (ii)$, we suppose that A contains a piecewise shifted FP-set Y defined by the sequence $(g_n)_{n\in\omega}$ and $(x_n)_{n\in\omega}$. By [8, Theorem 5.12], there is an idempotent $\varepsilon \in G^*$ such that, for each $m \in \omega$,

$$\{g_m^{\varepsilon_m}...g_n^{\varepsilon_n} : \varepsilon_i \in \{0,1\}, m < n < \omega\} \in \varepsilon.$$

We take an arbitrary $q \in A^*$ such that $\{x_n : n \in \omega\} \in q$. Put $p = \varepsilon q$. Since $Y \subseteq A$, we have $p \in A^*$. Clearly, $\varepsilon p = p$. We note that $g_m^{\varepsilon_m}...g_n^{\varepsilon_n} \in St(p)$ if and only if $\varepsilon_m = ... = \varepsilon_n = 0$. Hence $G \setminus St(p) \in \varepsilon$ and, applying Theorem 4.1, we conclude that p is not isolated in Gp. □

5. Scattered and sparse subsets of G-spaces

Given $F \in [G]^{<\omega}$ and $x \in X$, we denote $B(x, F) = Fx \cup \{x\}$ and say that $B(x, F)$ is a *ball of radius* F *around* x. For subset Y of X and $y \in Y$, we denote $B_Y(y, F) = B(y, F) \cap Y$.

A subset A of X is called

- *scattered* if, for every infinite subset Y of X, there exists $H \in [G]^{<\omega}$ such that, for every $F \in [G]^{<\omega}$ there is $y \in Y$ such that $B_Y(y, F) \cap B_Y(y, H) = \varnothing$;
- *sparse* if, for every infinite subset Y of X, there exists $H \in [G]^{<\omega}$ such that, for every $F \in [G]^{<\omega}$ there is $y \in Y$ such that $B_A(y, F) \cap B_A(y, H) = \varnothing$.

Clearly, each sparse subset is scattered. The sparse subsets of groups were introduced in [7] and studied in [9] [10]. From the asymptotic point of view [16], the scattered subsets of G-spaces can be considered as counterparts of the scattered subspaces of topological spaces.

Proposition 5.1. *A subset A of a G-space X is sparse if and only if $\triangle_p(A)$ is finite for each $p \in X^*$.*

Proof. Repeat the arguments proving Theorem 10 in [14]. □

Proposition 5.2. *A subset A of a G-space X is scattered if and only if, for every infinite subset Y of X, there exists $p \in Y^*$ such that $\triangle_p(Y)$ is finite.*

Proof. Repeat the arguments proving Proposition 1 in [3]. □

To formulate further results, we need some asymptology (see [16, Chapter 1]). Let G_1, G_2 be groups, X_1 be a G_1-space, X_2 be a G_2-space, $Y_1 \subseteq X_1$, $Y_2 \subseteq X_2$. A mapping $f : Y_1 \to Y_2$ is called a $\prec$*-mapping* if, for every $F \in [G_1]^{<\omega}$, there exists $H \in [G_2]^{<\omega}$ such that, for every $y \in Y_1$

$$f(B_{Y_1}(y, F)) \subseteq B_{Y_2}(f(y), H).$$

If f is a bijection such that f and f^{-1} are $\prec$-mappings, we say that f is an *asymorphism*. The subset subsets Y_1 and Y_2 are *coarsely equivalent* if there exist asymorphic subsets $Z_1 \subseteq Y_1$, $Z_2 \subseteq Y_2$ such that $Y_1 = B_{Y_1}(Z_1, F)$, $Y_2 = B_{Y_2}(Z_2, H)$ for some $F \in [G_1]^{<\omega}$, $H \in [G_2]^{<\omega}$. We say that a property $\mathcal{P}$ of subsets of G-spaces is *coarse* if $\mathcal{P}$ is stable under coarse equivalent, and note that "sparse" and "scattered" are coarse properties.

In asymptology, the group $\oplus_\omega \mathbb{Z}_2$ is known under name the Cantor macrocube, for its coarse characterization see [5].

Theorem 5.3. *A subset A of a G-space X is sparse if and only if A has no subsets asymorphic to the subset $W_2 = \{g \in \oplus_\omega \mathbb{Z}_2 : supt g \leqslant 2\}$ of the Cantor macrocube.*

Proof. Apply arguments from [14, Proof of Theorem 3]. □

Theorem 5.4. *For a subset A of a G-space X, the following statements are equivalent*

- *(i) A is scattered;*
- *(ii) $\triangle_p(A)$ is discrete for each $p \in X^*$;*
- *(iii) A contains no piecewise shifted FP-sets;*
- *(iv) A contains no subsets coarsely equivalent to the Cantor macrocube.*

Proof. The equivalence $(ii) \Rightarrow (iii)$ follows from Theorem 4.4. To prove $(i) \Rightarrow (iii)$, repeat the arguments from [3, Proof of Theorem 1].

$(ii) \Rightarrow (i)$. Let Y be an infinite subset of A. We denote by $\mathcal{F}$ the family of all closed invariant subsets of X^* and put $\mathcal{F}_Y = \{F \cap Y^* : F \in \mathcal{F}\}$. By the Zorn's lemma, there exists minimal by inclusion element $M \in \mathcal{F}_Y$. We take an arbitrary $p \in M$ and show that $\triangle_p(Y)$ is finite. Assume the contrary. Then the set $\triangle_p(Y)$ has a limit point q. Since M is minimal and $p \in M$, there exists $r \in \beta G$ such that $p = rq$. By the definition of the action of βG on βX, for every $P \in p$, there exists $Q \in q$ and $g \in G$ such that $gQ \subseteq P$. It follows that p is a limit point of $\triangle_p(Y)$. Hence, $\triangle_p(Y)$ is not discrete and we get a contradiction.

The implication $(i) \Rightarrow (iv)$ is evident because the Cantor macrocube is not scattered. To prove $(iv) \Rightarrow (i)$, we use the characterization of the Cantor macrocube from [5] and the arguments from [3, Proof of the Proposition 3]. □

Remark 5.5. Let G be an amenable group, A be scattered subset of G. By [3, Theorem 2], $\mu(A) = 0$ for each left invariant Banach measure μ on G. This statement cannot be extended to all G-spaces. As a counterexample, we take $X = \omega$ and G is a group of all permutations of X with finite supports. In this case, each subset of X is scattered.

Let X be a G-space, J be a translation invariant ideal in $\mathcal{P}_X$. We say that a subset A of X is

- J*-sparse* if $\triangle_p(A)$ is finite for each $p \in \check{J}$;
- J*-scattered* if, for every subset Y of A, $Y \notin \check{J}$, there is $p \in \check{J} \cap Y^*$ such that $\triangle_p(Y)$ is finite.

In this context, sparse and scattered subsets coincide with $[X]^{<\omega}$-sparse and $[X]^{<\omega}$-scattered subsets respectively.

The arguments proving $(ii) \Rightarrow (i)$ in Theorem 5.4 witness that A is scattered provided that each point $p \in \check{J} \cap A^*$ is isolated in X^*.

Question 5.6. *Assume that A is J-scattered. Is every point $p \in \check{J} \cap A^*$ isolated in X^*?*

If a subset A of X has a subset $Y \notin J$ coarsely equivalent to $\oplus_\omega \mathbb{Z}_2$ then A is not J-scattered.

Question 5.7. *Assume that a subset A of X has no subsets $Y \notin J$ coarsely equivalent to $\oplus_\omega \mathbb{Z}_2$. Is A J-scattered?*

We note that the families $\sigma(J)$ and $\partial(J)$ of all J-sparse and J-scattered subsets of X are translation invariant ideals in $\mathcal{P}_X$ and say that J is σ*-complete* (resp. ∂*-complete*) if $\sigma J = J$ (resp $\partial(J) = J$). We denote by $\sigma^*(J)$ (resp. $\partial^*(J)$) the intersection of all σ-complete (resp ∂-complete) ideals containing J. Clearly, $\sigma^*(J)$ and $\partial^*(J)$ are the smallest σ-complete and ∂-complete ideals such that $J \subseteq \sigma^*(J)$ and $J \subseteq \partial^*(J)$. We say that $\sigma^*(J)$ and $\partial^*(J)$ are the σ*-completion* and ∂*-completion* of J respectively.

We define a sequence $(\sigma^n(J))_{n<\omega}$ by the recursion: $\sigma^0(J) = J$, $\sigma^{n+1}(J) = \sigma(\sigma^n(J))$, and note that $\bigcup_{n\in\omega} \sigma^n(J) \subseteq \sigma^*(J)$. If X is left regular, by [10, Theorem 4(1)], $\sigma^*(J) = \bigcup_{n\in\omega} \sigma^n(J)$ and by [10, Theorem 4(2)], $\sigma^{n+1}([G]^{<\omega}) \neq \sigma^n([G]^{<\omega})$ for each $n \in \omega$.

Question 5.8. *Is $\sigma^* J) = \bigcup_{n\in\omega} \sigma^n(J)$ for each translation invariant ideal J in an arbitrary G-space X?*

In contrast to σ-completion, for each translation invariant ideal J in $\mathcal{P}_X$, we have $\partial^*(J) = \partial(J)$. In particular the ideal $\partial([X]^{<\omega})$ of all sparse subsets of X is ∂-complete. Indeed, assume that $A \notin \partial(J)$ and choose $Y \subseteq A$, $Y \notin J$ such that $\triangle_p(Y)$ is infinite for each $p \in \check{J} \cap Y^*$. Then $Y \notin \partial(Y)$ and $A \notin \partial^2(J)$. Hence, $\partial^2(J) = \partial(J)$ so $\partial^*(J) = \partial(J)$.

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