Wave Propagation

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under the supervision of

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DECLARATION

This thesis entitled WAVE PROPAGATION submitted by me to The Indian Institute of Technology, Hyderabad for the award of the degree of Master of Science in Mathematics contains a literature survey of the work done by some authors in this area. The work presented in this thesis has been carried out under the supervision of Dr. Anantha Lakshmi Narayana P, Department of Mathematics, Indian Institute of Technology, Hyderabad, Telangana.

I hereby declare that, to the best of my knowledge, the work included in this thesis has been taken from the journal, lecture and books, "(1) Lecture notes of a course of about twenty four lectures given at the T.I.F.R. centre, Indian Institute of Science, Bangalore, in January and February1978 by G.B. Whitham ", (2) Linear and Nonlinear Waves by G.B. Whitham, (3) An Introduction to the Mathematical Theory of Waves by Roger Knobel, - American Mathematical Society, etc.
No new results have been created in this thesis. The definitions, notations and results in Wave Propagation are learnt from the above mentioned sources and are presented here. I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

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Approval Sheet

This Thesis entitled **Wave Propagation** by **Tanay Kumar Karmakar** is approved for the degree of Master of Science from IIT Hyderabad.

(Dr. Anantha Lakshmi Narayana P) Advisor

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IITH

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I also thankful to **Dr.G.B.Whitham** whose lecture series, book and discoveries helped me to do this work.

I hope that I have incorporated all the corrections suggested by my advisor. I will be solely held responsible for any mistake found in this thesis, if any, and not my advisor.

I would also like to thank all the teachers of the department of Mathematics for their constant support and encouragement and imparting knowledge.

Tanay Kumar Karmakar

Dedication

Dedication to Dr. Anantha Lakshmi Narayana P who supervised, motivated me to do this work.

Abstract

Here main topic of discussion is on Linear and non-linear wave propagation and it's application. At first we discussed about some concept of quasi-linear hyperbolic PDE, conservation law and their analysis. Then we used all those things on shallow water theory.

Application:- Traffic Flow, Flood waves in rivers, Chemical exchange process, Glaciers, Erosion, Dam break problem, Piston wave maker problem etc.

Here I also discussed about some different types of shallow water waves.

Contents

Chapter 1

Derivation of the Wave Equation

In these notes we apply Newton's law to an elastic string. Consider a tiny element of the string.

The basic notation is

u(x, t)= vertical displacement of the string from the x axis at position x and time t

$\theta(x,t)$ = angle between the string and a horizontal line at position x and time t

T(x, t) = tension in the string at position x and time t

ρ= mass density of the string at position x The forces acting on the tiny element of string are

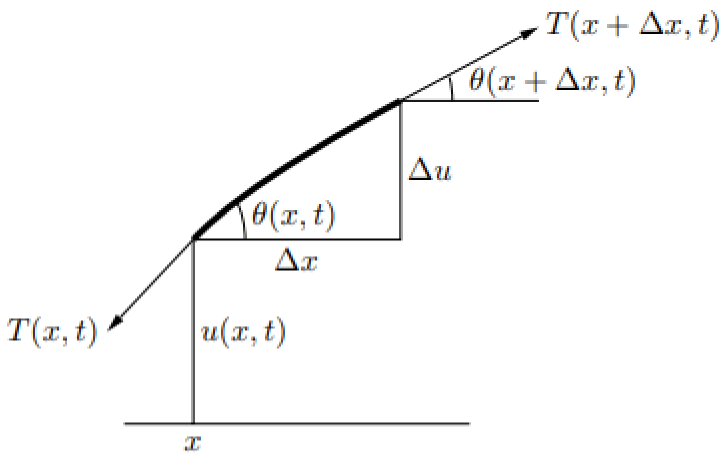


Figure 1.1

(a) tension pulling to the right, which has magnitude T(x $+\Delta$x, t)and acts at an angle θ(x $+\Delta$x, t) above horizontal,

(b) tension pulling to the left, which has magnitude T(x, t) and acts at an angle θ(x, t) below horizontal and, possibly,

(c) various external forces, like gravity. We shall assume that all of the external forces act vertically and we shall denote by F(x, t)Δx the net magnitude of the external force acting on the element of string. The mass of the element of string is essentially ρ(x)$\sqrt{(\Delta x)^2 + (\Delta u)^2}$ so the vertical component of Newton's law says that

$$\rho(x)\sqrt{(\Delta x)^2 + (\Delta u)^2}\frac{\partial^2 u}{\partial t^2} = T(x+\Delta x, t)\sin(\theta(x+\Delta x, t) - T(x,t)\sin\theta(x,t) + F(x,t)\Delta x \tag{1.1}$$

Dividing by Δx and taking the limit as Δx $\to 0$ gives

ρ(x)$\sqrt{1 + (\frac{\partial u}{\partial x})^2}\,\frac{\partial^2 u}{\partial t^2}$(x,t) $= \frac{\partial}{\partial x}$[T(x,t)sin($\theta$(x,t))]+F(x,t) $= \frac{\partial T}{\partial x}$(x,t)sin($\theta$(x,t))+ T(x,t)cos($\theta$(x,t))$\frac{\partial \theta}{\partial x}$

We can dispose of all the θ's by observing from the figure that

$\tan(\theta(x,t)) = \lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}$(x,t)

which implies, using the figure on the right below, that

sin(θ(x,t)) $= \frac{\frac{\partial u}{\partial x}(x,t)}{\sqrt{1+(\frac{\partial u}{\partial x}(x,t))^2}}$

cos(θ(x,t)) $= \frac{1}{\sqrt{1+(\frac{\partial u}{\partial x}(x,t))^2}}$

θ(x,t) $= \tan^{-1}\frac{\partial u}{\partial x}$(x,t)

$\frac{\partial \theta}{\partial x}$(x,t) $= \frac{\frac{\partial^2 u}{\partial x^2}(x,t)}{1+(\frac{\partial u}{\partial x}(x,t))^2}$

Substituting these formulae into (1.1). However, we can get considerable simplification by looking only at small vibrations. By a small vibration, we mean that

$$\mid \theta(x,t) \mid << 1$$

for all x and t. This implies that

$$| \tan\theta(x,t) | << 1$$

hence that

$$| \frac{\partial u}{\partial x}(x,t) | << 1$$

and hence that

$$\begin{cases} \sqrt{1+(\frac{\partial u}{\partial x})^2} \approx 1 \\ sin\theta(x,t) \approx \frac{\partial u}{\partial x}(x,t) \\ \cos\theta(x,t) \approx 1 \\ \frac{\partial \theta}{\partial x}(x,t) \approx \frac{\partial^2 u}{\partial x^2}(x,t) \end{cases} \tag{1.2}$$

substituting these into equation (1.1) give

$$\rho(x)\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial T}{\partial x}(x,t)\frac{\partial u}{\partial x}(x,t) + T(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) + F(x,t) \tag{1.3}$$

which is indeed relatively simple, but still exhibits a problem. This is one equation in the two unknowns u and T.

Fortunately there is a second equation lurking in the background, that we haven't used. Namely, the horizontal component of Newton's law of motion. As a second simplification, we assume that there are only transverse vibrations. Our tiny string element moves only vertically. Then the net horizontal force on it must be zero. That is,

$$T(x+\Delta x,t)\cos(x+\Delta x,t) - T(x,t)\cos\theta(x,t) = 0$$

Dividing by Δx and taking the limit as Δx tends to zero gives

$$\frac{\partial}{\partial x}[T(x,t)\cos\theta(x,t)] = 0$$

For small amplitude vibrations, cos is very close to one and

$$\frac{\partial T}{\partial x}(x,t)$$

is very close to zero. In other words T is a function of t only, which is determined by how hard you are pulling on the ends of the string at time t. So for small, transverse vibrations, (1.3) simplifies further to

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T(t)\frac{\partial^2 u}{\partial x^2}(x,t) + F(x,t) \tag{1.4}$$

In the event that the string density is a constant, independent of x, the string tension T(t) is a constant independent of t and there are no external forces F we end up with

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t)$$

where

$$c = \sqrt{\frac{T}{\rho}}$$

Chapter 2

Introduction Waves and First Order Equation

We are starting the detailed discussion of hyperbolic waves with a study of first order equations. The simplest wave equation is

$$\rho_t + c_0\rho_x = 0, \tag{2.1}$$

c_0=constant. with initial curve Γ, ρ(x,0)=f(x)

When this equation arises, the dependent variable is usually the density of something so we now use the symbol.

Parametrize the initial curve Γ, i.e. write

x(0)=a;

t(0)=0;

z(0)=f(a);

the system of ODE initial value problems

$\frac{dx}{ds}$=c_0, x(0)=a;

$\frac{dt}{ds}$=1, t(0)=0;

$\frac{dz}{ds}$=0, z(0)=f(a);

Then the general solution is $\rho = f(x - c_0 t)$, where f(x) is an arbitrary function, and the solution of any particular problem consists merely of matching the function f to initial or boundary values. It clearly describes a wave motion since an initial profile f(x) would be translated unchanged in shape a distance $c_0 t$ to the right at time t.

Although this linear case is almost trivial, the nonlinear counterpart

$$\rho_t + c(\rho)\rho_x = 0 \tag{2.2}$$

where c(ρ) is a given function of ρ, is certainly not and a study of it leads to most of the essential ideas for nonlinear hyperbolic waves.

2.1 Continuous Solution

One approach to the solution of (2.2) is to consider the function p(x,t) at each point of the (x,t) plane and to note that $\rho_t + c(\rho)\rho_x$ is the total derivative of ρ along a curve which has slope

$$\frac{dx}{dt} = c(\rho) \tag{2.3}$$

at every point of it. For along any curve in the (x,t) plane, we may consider x and ρ to be functions of t, and the total derivative of ρ is $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x}$. The total derivative notation should be sufficient to indicate when x and ρ are being treated as functions of t on a certain curve; the introduction of new symbols each time this is done eventually becomes confusing. We now consider a curve C in the (x,t) plane which satisfies (2.3). Of course such a curve cannot be determined explicitly in advance since the defining equation (2.3) involves the unknown values of ρ on the curve. However, its consideration will lead us to a simultaneous determination of a possible curve C, and the solution ρ on it. On C we deduce from the total derivative relation and from (2.2) that

$$\frac{d\rho}{dt} = 0, \frac{dx}{dt} = c(\rho). \tag{2.4}$$

We first observe that ρ remains constant on C. It then follows that c(ρ) remains constant on C, and therefore that the curve C must be a straight line in the (x,t) plane with slope c(ρ). Thus the general solution of (2.2) depends on the construction of a family of straight lines in the (x, t) plane, each line with slope c(ρ) corresponding to the value of ρ on it. This is easily done in any specific problem.

Let us take for example the initial value problem

ρ=f(x), t=0, -$\infty < x < \infty$

and refer to the (x,t) diagram in Fig. 2.1. If one of the curves C intersects t= 0 at x =ξ then $\rho = f(\xi)$ on the whole of that curve. The corresponding slope of the curve is c (/(£)), which we will denote by F(£); it is a known function of ξ calculated from the function c(ρ) in the equation and the given initial function f(ξ). The equation of the curve then is

x=$\xi + F(\xi)t$, Here ξ is a parameter.

This detemines one typical curve and the value of ρ on it is f(ξ). Allowing ξ to vary, we obtain the whole family:

$$\rho = f(\xi), c = F(\xi) = c(f(\xi)) \tag{2.5}$$

on

$$x = \xi + F(\xi)t \tag{2.6}$$

We may now change the property and use (2.5) and (2.6) as an analytic expression for the solution, free of the particular construction. That is, ρ is given by (2.5) where $\xi(x,t)$ is defined implicitly by (2.6). Let us check that this gives the solution. From (2.5),

ρ_t=$\frac{df(\xi)}{d\xi}\xi_t$, $\rho_x = \frac{df(\xi)}{d\xi}\xi_x$

and from the t and x derivatives of (2.6)

0=$F(\xi) + (1 + \frac{dF(\xi)}{d\xi}t)\xi_t$

$1 = (1 + \frac{dF(\xi)}{d\xi})\xi_x$

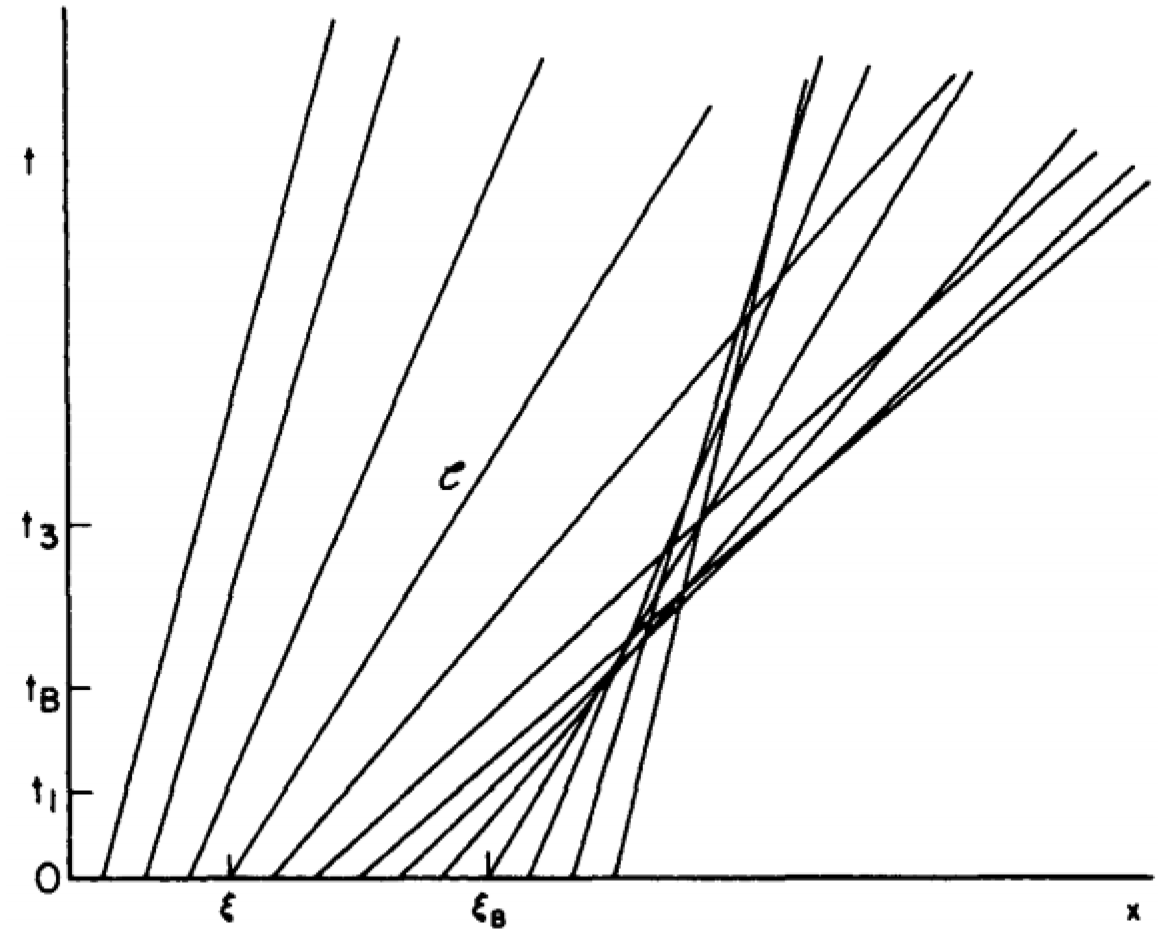


Figure 2.1: Characteristic diagram for nonlinear waves.

Therefore

$$\rho_t = -\frac{F(\xi)\frac{df(\xi)}{d\xi}}{1+\frac{dF(\xi)}{d\xi}t}, \rho_x = \frac{\frac{df(xi)}{d\xi}}{1+\frac{dF(\xi)}{d\xi}t} \tag{2.7}$$

and we see that

$\rho_t + c(\rho)\rho_x = 0$, since $c(\rho) = F(\xi)$. The initial condition $\rho = f(x)$ is satisfied because $\xi = x$ when t=0.

The curves used in the construction of the solution are the characteristic curves for this special problem. Similar characteristics play an important role in all problems involving hyperbolic differential equations. In general, characteristic curves do not have the property that the solution remains constant along them. This happens to be true in the special case of (2.2); it is not the defining property of characteristics. The general definitions will be considered later, but it will be convenient now to refer to the curves defined by (2.3) as characteristics.

The basic idea of wave propagation is that some recognizable feature of the disturbance moves with a finite velocity.

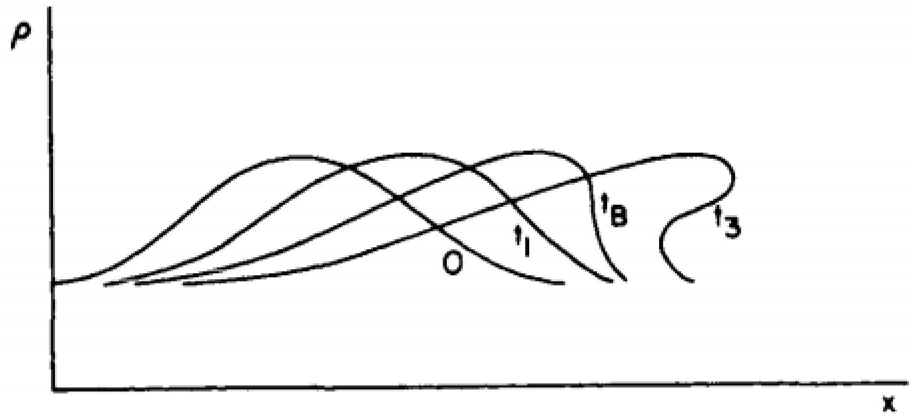


Figure 2.2: Breaking wave: successive profiles corresponding to the times 0,t_1, t_B, t_3in Fig. 2.1.

The mathematical statement in (2.4) may be given this type of property by saying that different values of ρ "propagate" with velocity $c(\rho)$. Indeed, the solution at time t can be constructed by moving each point on the initial curve ρ=f(x) a distance c(ρ)t to the right; the distance moved is different for the different values of ρ. This is shown in Fig. 2.2 for the case $c'(\rho)$ >0; the corresponding time levels are indicated in Fig. 2.1.

The dependence of c on ρ produces the typical nonlinear distortion of the wave as it propagates. When $c'(\rho) > 0$, higher values of ρ propagate faster than lower ones. When $c'(\rho) < 0$, higher values of ρ propagate slower.

For the linear case(see fig 2.3), c is constant and the profile is translated through a distance ct without any change of shape.

Any compressive part of the wave, where the propagation velocity is a decreasing function of x(since, when velocity is decreasing, then $c'(\rho) < 0$), ultimately "breaks" to give a triple-valued solution for ρ(x,t). The breaking starts at the time indicated

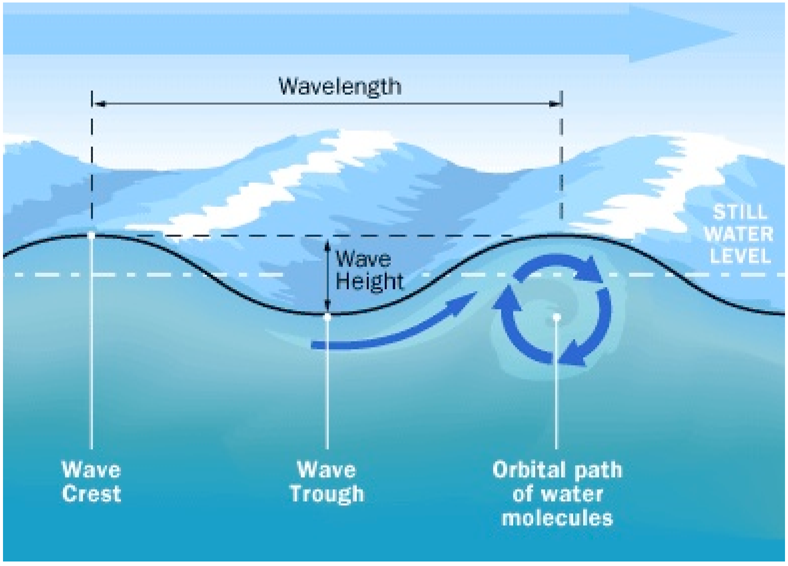


Figure 2.3: Linear Water Wave

by t $= t_B$ in Fig. 2.2, when the profile of ρ first develops an infinite slope. The analytic solution (2.7) confirms this and allows us to determine the breaking time t_B. On any characteristic for which $F'(\xi) < 0$, P_x and p_t become infinite, i.e 1+F'(ξ)=0 $\Rightarrow$t=-$\frac{1}{F'(\xi)}$

Therefore breaking first occurs on the characteristic $\xi = \xi_B$ for which F'(ξ) <0 and $\mid F'(\xi) \mid$ is a maximum; the time of first breaking is

$$t_B = -\frac{1}{F'(\xi)} \tag{2.8}$$

$0 < t < \frac{1}{\max\limits_{F'(\xi)<0} |F'(\xi)|}$

This development can also be followed in the (x,t) plane. A compressive part of the wave with $F'(\xi)$ <0 has converging characteristics; since the characteristics are straight lines, they must eventually overlap to give a region where the solution is multivalued, as in Fig. 2.1. This region may be considered as a fold in the (x,t) plane made up of three sheets, with different values of ρ on each sheet. The family of characteristics is given by (2.6) with ξ as parameter. The condition that two neighboring characteristics ξ, $\xi + \delta\xi$ intersect at a point (x,t) is that

x=$\xi + F(\xi)t$,

and

x=$\xi + \delta\xi + F(\xi + \delta\xi)t$

hold simultaneously. In the limit $\delta\xi$ →0, these give

x=$\xi + F(\xi)t$,

And, x-x=$\xi + \delta\xi + tF(\xi + \delta\xi) - (\xi + tF(\xi))$

$\Rightarrow t(F(\xi + \delta\xi) - F(\xi)) = -\delta\xi$

Taking limit at both side as $\delta\xi \to 0$

$\Rightarrow \lim_{\delta\xi \to 0} t\frac{F(\xi+\delta\xi)-F(\xi)}{\delta\xi} = -1$

$\Rightarrow 1 + F'(\xi)$t$= 0$.

★ *Uniqueness*

Let, ψ(x,t) and Ξ(x,t) are different solution of

$\phi_t + c(\phi)\phi_x = 0, t > 0, -\infty < x < \infty$

$t = 0 : \phi = f(x), -\infty < x < \infty$ Then on $x = \xi + tF(\xi)$

$\psi(x, t) = \psi(\xi, 0) = f(\xi) = \Xi(\xi, 0) = \Xi(x, t)$ (From 2.5)

Hence $\psi = \Xi$

★ *Theorem*

The initial value problem

$\phi_t + c(\phi)\phi_x = 0, t > 0, -\infty < x < \infty$

$t = 0 : \phi = f(x), -\infty < x < \infty$ has a unique solution in

$0 < t < \frac{1}{\max\limits_{F'(\xi)<0} |F'(\xi)|}$

if $f \in C^1()$, c $\in$C^1()

where $F(\xi) = c(f(\xi))$

The solution is given in the parametric form:

$x = \xi + tF(\xi)$

$\rho(x, t) = f(\xi)$

★ *Remark*

When $c(\rho) = c_0$, apositive constant, equation(2.2) becomes a linear wave equation: which is (2.1)

The characteristic curve are $x = c_0 t + \xi$, and ρ is given by

$\rho(x,t) = f(\xi) = f(x - c_0 t)$

2.2 Expansion Wave

Consider the problem

$\phi_t + c(\phi)\phi_x = 0$, on $t > 0, -\infty < x < \infty$

t=0:$\phi = f(x), -\infty < x < \infty$

where

$$\text{f(x)} = \begin{cases} \phi_2, & \text{if } x \leq 1, \\ \text{monotonic increasing ,} & \text{if } 0 \leq x \leq L, \\ \phi_1, & \text{ifx } \geq L \end{cases}$$

with $\phi_1 > \phi_2$ and $c'(\phi) > 0$

We shall let $c_1 = c(\phi_1)$,$c_2 = c(\phi_2)$

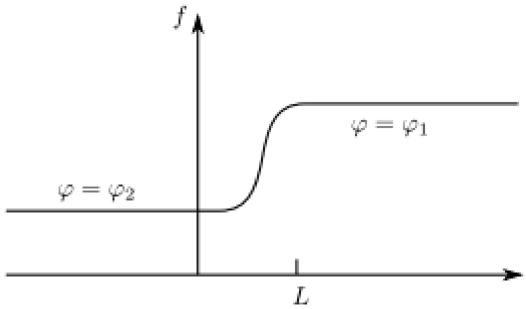


Figure 2.4: Expansion Wave

We recall the solution of the problem:

$\phi = f(\xi)$

$x = \xi + tF(\xi)$

where

$F(\xi) = c(f(\xi))$

Let us consider the characteristics of this problem. For,$\xi \leq 0$

$F(\xi) = c(f(\xi)) = c(\phi_2) = c_2$

Therefore the charecteristic through $\xi(\leq 0)$ are straight lines with constant slope $\frac{1}{c_2}$

For $\xi \geq L, F(\xi) = c(f(\xi)) = c(\phi_1) = c_1$ Hence, the characteristics through $\xi(\geq L)$ are also straight lines, with constant slope $\frac{1}{c_1}$. For $0 \leq \xi \leq L$, the characteristics through ξ are straight lines having slopes $\frac{1}{F(\xi)}$ with $\frac{1}{c_1} \leq \frac{1}{F(\xi)} \leq \frac{1}{c_2}$

Since $0 \leq \xi \leq L$

$\Rightarrow f(0) \leq f(\xi) \leq f(L)$, [f is m.i]

$\Rightarrow \phi_2 \leq f(\xi) \leq \phi_1$, $[C'(\phi) > 0]$

$\Rightarrow c(\phi_2) \leq c(f(\xi)) \leq c(\phi_1)$

$\Rightarrow c_2 \leq F(\xi) \leq c_1$

$\Rightarrow \frac{1}{c_1} \leq \frac{1}{F(\xi)} \leq \frac{1}{c_2}$

Since the characteristics do not intersect, (and this corresponds $1 + tF'(\xi) \neq 0$) we obtain as a single valued function. The behavior of the solution can be explained geometrically as shown in the figures 1.3(a), 1.3(b).

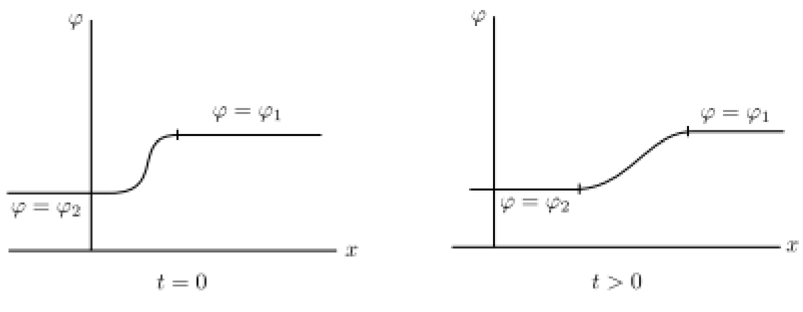


Figure 2.5: Expansion Waves

Every point $(\xi, \phi(\xi))$ at t = 0 will move parallel to the x-axis through a distance ct_1 in time t_1. Since $c'(\phi) > 0, \phi_2 < \phi_1$, the points $(\xi, \phi_1)(\xi \geq L)$ move faster than the points $(\xi, \phi_2)(\xi \leq 0)$. Hence, the graph of ϕ at t = 0 is stretched as the time increases.

The analytic details can be carried out most easily by working entirely with c as the dependent variable.

★Equation for C:

Consider the equation

$\phi_t + c(\phi)\phi_x = 0$, on $t > 0, -\infty < x < \infty$

t=0:$\phi = f(x), -\infty < x < \infty$

We have found that $c(\phi)$ is the "propagation speed", and in constructing solutions we have to deal with two functions, namely, ϕ and c. But by multiplying the equation by $c'(\phi)$ we obtain

$$\begin{cases} C_t + CC_x = 0, t > 0, -\infty < x < \infty. \\ t = 0, C = F(x), -\infty < x < \infty \end{cases} \quad (2.9)$$

where C(x,t)=c($\phi(x,t)$) and $F(\xi) = c(f(\xi))$

This equation involves only the unknown function C(x, t) = c (ϕ(x, t)); we can recover ϕ from C afterwards. The solution of the problem in (2.9) is

$$\text{C(x,0)} = \begin{cases} c_2, & \text{if } x \leq 0, \\ c_2 + \frac{c_1 - c_2}{L}x, & \text{if } 0 \leq x \leq L, \\ c_1, & \text{ifx} \geq L \end{cases}$$

In $0 \leq x \leq L$ the equation of the straight line is

$\frac{C - c_2}{x - 0} = \frac{c_2 - c_1}{0 - L}$

The x-t diagram is shown below in fig 2.6

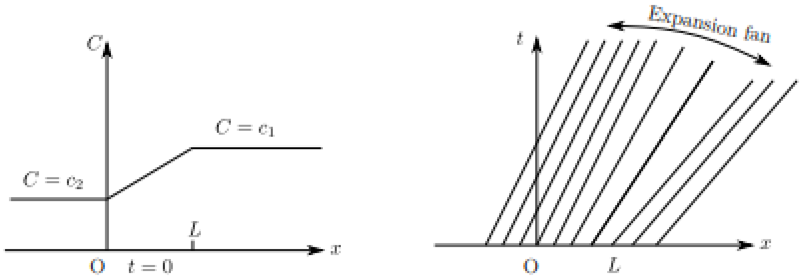


Figure 2.6: Expansion Wave and Expansion fan

2.3 Centered Expansion Wave

We now consider the limiting case of the above problem, as L→0. In the limit the interval $[c_2, c_1]$ is associated with the origin. In the limit we will have the characteristics

$x = \xi + tc_2$, if $\xi < 0$

$x = \xi + tc_1$, if $\xi > 0$

x=Ct, if $\xi = 0, c_2 \leq C \leq c_1$

The collection of characteristics x = Ct : C $[c_2, c_1]$ through the origin is called a 'Centred fan' and we have C = $\frac{x}{t}$. In this case the full solution is

$$C = \begin{cases} c_2, & \text{if} x \leq c_2 t \\ \frac{x}{t}, & \text{if} c_2 t < x < c_1 t \\ c_1, & \text{if} x \geq c_1 t \end{cases} \tag{2.10}$$

We shall discuss later how to solve this type of wave equation in later.

★*Theorem :*

The intial value problem

$C_t + CC_x = 0, t > 0, -\infty < x < \infty,$

t=0:C=$\begin{cases} c_2, \text{ if } \xi < 0, \\ c_1, \text{ if } \xi > 0 \end{cases}$

and C conrtinuous for $t > 0$, has a unique solution given by (2.10)

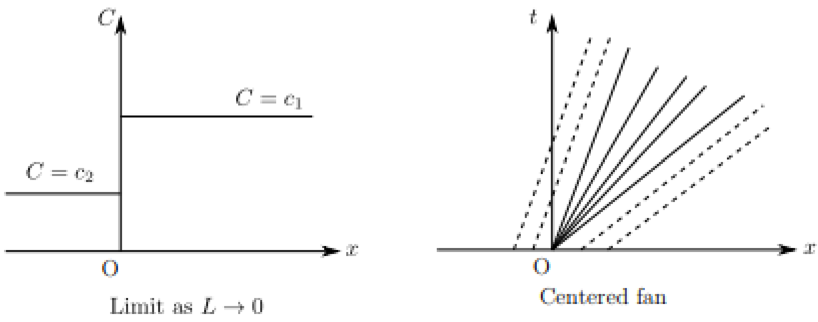


Figure 2.7: Centred Expansion Wave and Centered Fan

2.4 Breaking

We consider again the geometrical intepretation of the solution of the equations

$\phi_t + c(\phi)\phi_x = 0, t > 0, -\infty < x < \infty$

$t = 0 : \phi = f(x), -\infty < x < \infty$

We assume that $c'(\phi) > 0$. The graph of ϕ at time t = 0 is the graph of f . Since

$\phi(\xi + tF(\xi), t) = f(\xi)$,

we find that the point $(\xi, f(\xi))$ moves parallel to x-axis in the positive direction through a distance tF(ξ) = ct. It is important to note that the distance moved depends on ξ; this is typical of non-linear phenomena. (In the linear case the curve moves parallel to x-axis with constant velocity c_0).

After some time t =t_B, the graph of the curve ϕ may become many valued as shown in the below figure 2.8. This phenomenon is called "breaking". It could at least make physical sense in the case of water waves (although the equations are in fact not valid), but in most cases a three valued solution would not make sense. We have to reconsider our approximations and assumptions.

We have seen that if $tF'(\xi) + 1 \neq 0$ then breaking will not occur. A necessary and sufficient condition for breaking to occur is that $F'(\xi) < 0$ for some ξ. (We assume c'(ϕ) > 0).

x=$\xi + F(\xi)t$,

$1 + F'(\xi)$t= 0.

If we assume that F'(ξ) is minimum only at ξ_B and $F'(\xi_B) < 0$, the first breaking time will be

$t_B = \frac{1}{\max|F'(\xi_B)|}$

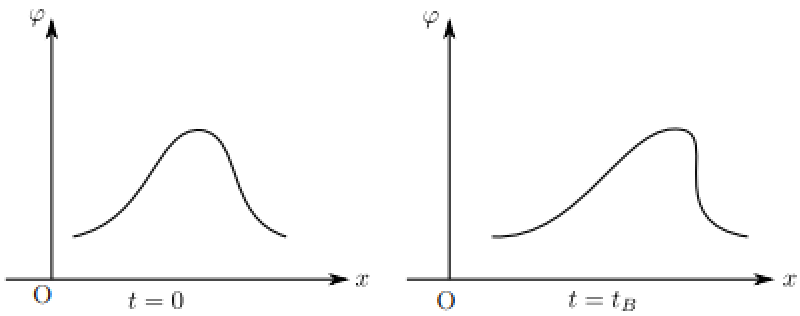


Figure 2.8

In the x, t, plane the breaking can be seen as follows: since $F'(\xi_B) < 0$. F is a decreasing function in a neighbourhood of ξ_B will have increasing slopes and therefore will converge giving a multivalued region.

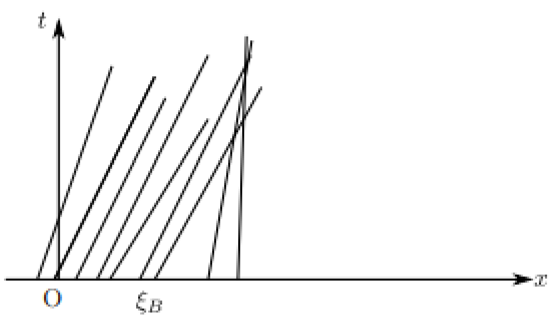


Figure 2.9

From the equations $\rho_t = -\frac{F(\xi)\frac{df(\xi)}{d\xi}}{1+\frac{dF(\xi)}{d\xi}t}, \rho_x = \frac{\frac{df(xi)}{d\xi}}{1+\frac{dF(\xi)}{d\xi}t}$

We see that ϕ_t, ϕ_x will become infinite at the time of breaking.

For physical meaning of breaking we have to look at specific physical problem.

Figure 2.10: Plunging breaker

Figure 2.11: Large wave breaking, from Menlo Park, USA

2.5 Gradient Catastrophes and Breaking Times

We know that the solution of a conservation law $u_t + \phi_x= 0$ could be constructed at the point (x, t) by following a characteristic curve from (x,t) back to a point (x_0,0). An implicit assumption in this method is that there is exactly one characteristic extending from the x—axis to (x,t) in the xt—plane. In nonlinear conservation laws, however, it is possible for two (or more) characteristics to intersect at (x,t):

Such an occurrence can cause the solution u(x,t) to break down with an event called a gradient catastrophe. In this chapter we will describe the cause of gradient

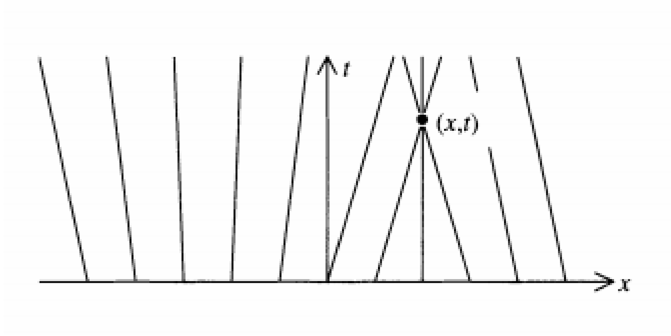


Figure 2.12

catastrophes and predict the time at which they occur. As will be discussed in the next chapter, gradient catastrophes are a precursor to shock waves

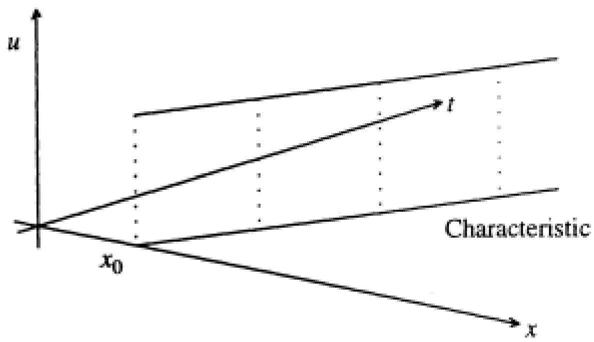


Figure 2.13: Constant value of u(x,t) along a characteristic

2.5.1 Gradient catastrophe

We know that the characteristic curves of the initial value problem

$$\begin{cases} u_t + c(u)u_x = 0, -\infty < x < \infty, t > 0, \\ u(x,0) = u_0(x) \end{cases} \tag{2.11}$$

are lines x = $c(u_0(x_0))$t+x_0 along which the value of u is constant. When viewed in the xtu—diagram shown in Figure 2.13, each characteristic is a line in the xt—plane, and the height of the surface represented by u(x,t) is constant along that line.

In the special case where cu) is constant (the advection equation), the characteristic lines x — ct + x_0 are parallel. By following these characteristics, we see that an initial profile u(x, 0) in the xu—plane has the appearance of being translated along the characteristics as t increases, forming a traveling wave (Figure 2.14).

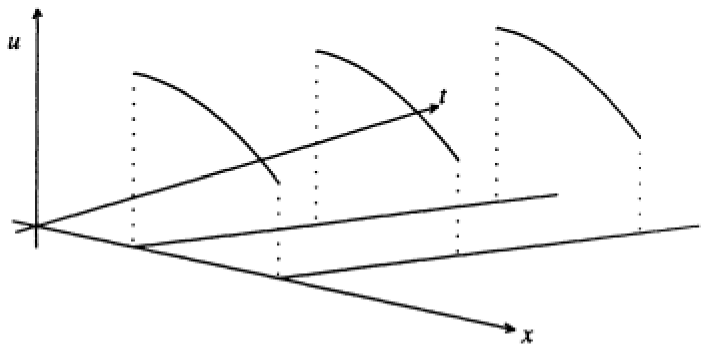


Figure 2.14: Parallel characteristics translate the initial profile in time.

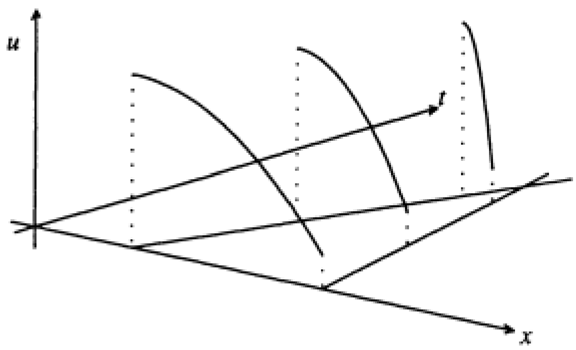


Figure 2.15: Crossing characteristics can result in infinite slope u_x

When c(u) is not constant, however, the characteristic lines x $=c(u_0(x_0))t+x_0$ are not necessarily parallel and may cross. The value of u nevertheless remains constant along each individual characteristic line. As shown in Figure 2.15, if two characteristic lines intersect and the value of u is different along each line, then the slope u_x(x, t) in the x—direction becomes infinite as t approaches the time corresponding to the intersection of the lines. The formation of an infinite slope u_x in the solution u is called a Gradient Catastrophe.

The gradient catastrophe can also be seen in the animation of u(x,t). When viewing Figure 2.13 facing the xu—plane, the point (x(t),t,u(x(t),t)) on the surface u

= u(x,t) above the characteristic curve is projected onto the xu—plane as the point (x(t),u(x(t),t)). As t increases, this point appears to move in the xu—plane at a constant height u, since (x(t),t) is following along a characteristic curve.

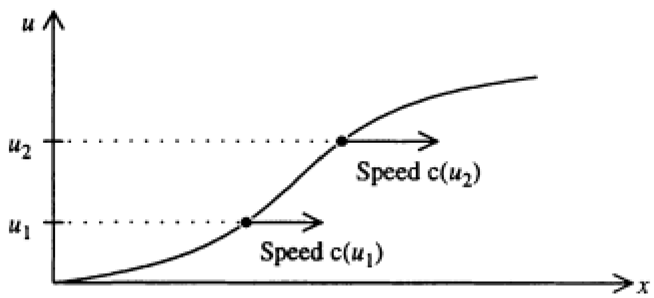


Figure 2.16: Horizontal velocity of a point on the profile of u(x, t) is c(u)

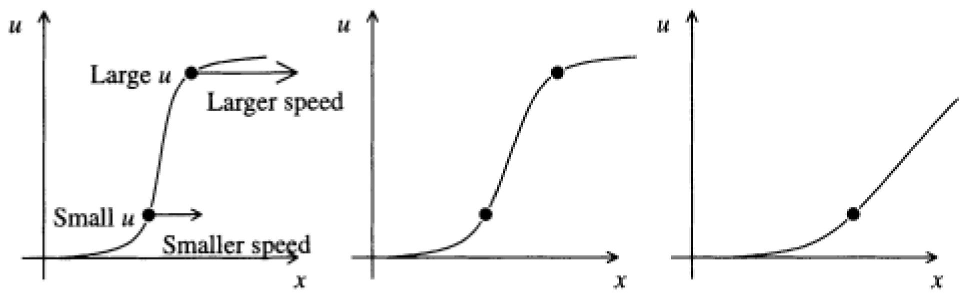


Figure 2.17: Top part of the profile of u(x, t) moves with greater speed than the lower part when $u_t + uu_x = 0$.

The velocity at which this point moves in the x direction is $\frac{dx}{dt}$, which by construction of the characteristic curve is dx/dt=c(u(x,t)). Thus the function c(u) represents the velocity at which a point at height u in the xu—plane animation moves horizontally (Figure 2.16).

Now suppose that c(u) is an increasing function of u, such as c(u)= u. In this case, larger values of $u \leq 0$ give larger speeds c, and so the upper part of the profile of u(x, t) (larger values of u) will appear to move to the right faster than the lower part (smaller values of u). As shown in Figure 2.17, if the profile of u(x, t) at one time is an increasing function of x, then at later times t the profile of u(x, t) will appear to have "thinned out" or rarefied.

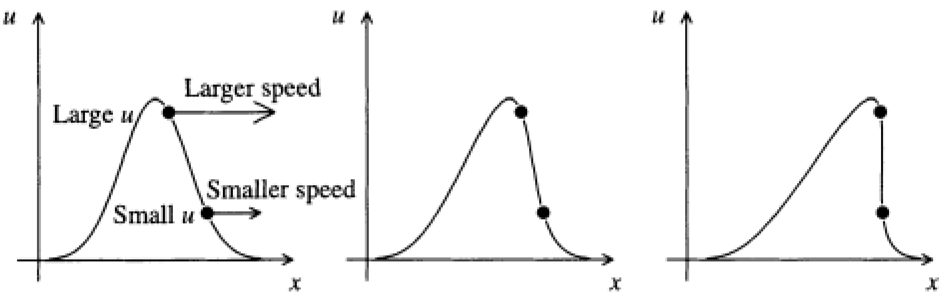


Figure 2.18: Top part of the profile of u(x, t) can catch up to the lower part, forming a gradient catastrophe.

On the other hand, if a profile of u(x, t) looks more like a pulse, then the top part of the profile of u(x, t) catches up with the slower moving lower part of the profile (Figure 2.18). This forms an infinite slope u_x, creating a gradient catastrophe. If time were to continue beyond this point, the top part of the profile would appear to overtake the lower part and u(x, t) would fail to be a function.

2.5.2 Breaking Time

The earliest time $t_b \geq 0$ at which a gradient catastrophe occurs in a solution of a conservation law is called the breaking time.

★Example

Consider the following initial value problem for the inviscid Burgers equation:

$$\begin{cases} u_t + uu_x = 0, -\infty < x < \infty, t > 0, \\ u(x, 0) = e^{-x^2} \end{cases} \tag{2.12}$$

With the speed c(u) = u and initial profile $u_0(x) = e^{-x^2}$, the characteristic starting at (x_0,0) is

$x = c(u_0(x_0))t + x_0 = e^{-x_0^2}t + x_0$

A diagram of characteristics with different starting points (x_0,0) is displayed in Figure 2.19 and shows that there are characteristics which intersect. From the figure,

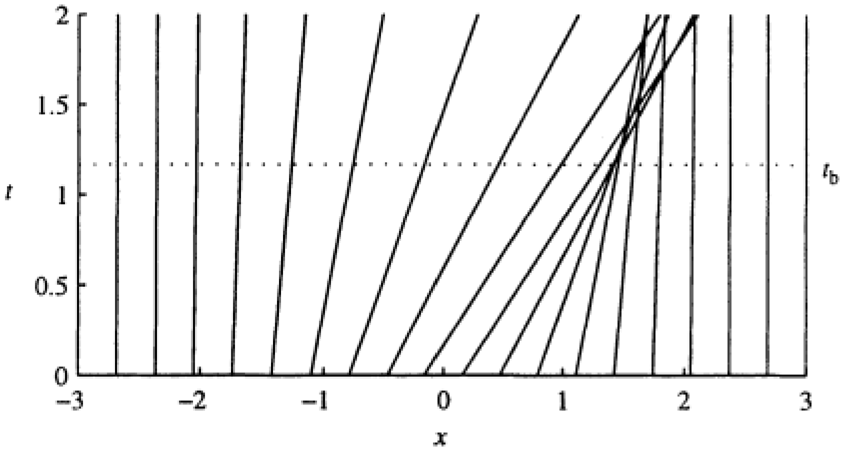


Figure 2.19: Characteristics $x = e^{-x_0^2}t + x_0$ of Equation 2.12

the earliest time at which characteristics cross appears to be at a breaking time of approximately $t_b = 1.2$.

we will discuss how the breaking time t_b can be computed by calculating u_x(x,t) and finding the first time t_b at which u_x becomes infinite.

By the method of characteristics, the value of the solution u of

$$\begin{cases} u_t + c(u)u_x = 0, -\infty < x < \infty, t > 0, \\ u(x,0) = u_0(x) \end{cases} \tag{2.13}$$

at the point (x,t) is u(x,t) $= u_0(x_0)$, where $x_0 = x_0$(x,t) determines the starting point (x_0,0) of the characteristic passing through (x,t).

The derivative u_x is then

$$u_x(x,t) = u_0'(x_0)\frac{\partial x_0}{\partial x} \tag{2.14}$$

by the chain rule.

The value of x_0 which determines the starting point (x_0, 0) of the characteristic through (x, t) is defined implicitly by the equation

x = $c(u_0(x_0))$t+x_0

The derivative of x_0 with respect to x can be found from this equation by implicit differentiation. Taking the partial derivative of both sides with respect to x gives

$\frac{\partial}{\partial x} = \frac{\partial}{\partial x}[c(u_0(x_0))\text{t}+x_0]$

$1 = t\frac{d}{dx_0}[c(u_0(x_0))]\frac{\partial x_0}{\partial x} + \frac{\partial x_0}{\partial x}$

Solving for $\frac{\partial x_0}{\partial x}$ then shows that

$\frac{\partial x_0}{\partial x} = \frac{1}{1+t\frac{d}{dx_0}c(u_0(x_0))}$

Substituting this into (2.14) expresses the derivative of u(x,t) with respect to x as

$$u_x(x,t) = \frac{u_0'(x_0)}{1+t\frac{d}{dx_0}c(u_0(x_0))} \tag{2.15}$$

The problem of determining when ux becomes infinite is now reduced to a problem of determining when the denominator of (2.15) approaches zero.

If $\frac{d}{dx_0}c(u_0(x_0))\ geq0$ for all initial points (x_0,0), then the denominator in (2.15) never approaches 0 as t increases from zero. In this case, no gradient catastrophe occurs. On the other hand, if $\frac{d}{dx_0}c(u_0(x_0))$ is negative for some x_0, then a gradient catastrophe can occur since the denominator in (2.15) will approach 0 as t approaches $\frac{-1}{\frac{d}{dx_0}c(u_0(x_0))}$. The value of x_0 which produces the earliest blowup time t is the value of x_0 which makes $\frac{d}{dx_0}c(u_0(x_0))$ the most negative.

Using this value of x_0 , the breaking time is then

$$\boxed{t_b = \frac{-1}{\frac{d}{dx_0}c(u_0(x_0))}} \tag{2.16}$$

★Example

Returning to the initial value problem in previous Example, the expression (2.16) will be used to compute the breaking time t_b in Figure 2.19. With the speed c(u) = u and initial profile $u_0(x) = e^(- x^2)$ from that example, the speed of the characteristic starting at (x_0,0) is

$c(u_0(x_0)) = c(e^{-x_0^2}) = e^{-x_0^2}$

The breaking time t_b in (2.16) requires finding the most negative value of

$F(x_0) = \frac{d}{dx_0}c(u_0(x_0) = \frac{d}{dx_0}e^(- x_0^2) = -2x_0e^{-x_0^2}$

The derivative $F'(x_0) = (-2 + 4x_0^2)e^{-x_0^2}$ shows that F(x_0) has critical points at x_0 $=\pm\frac{1}{\sqrt{2}}$, with $x_0 = \frac{1}{\sqrt{2}}$ yielding the most negative value of F(x_0). The breaking time (2.16) with $x_0 = \frac{1}{\sqrt{2}}$ is then

$t_b = \frac{-1}{-2x_0e^{-x_0^2}} = \frac{1}{\sqrt{2}e^{-1/2}} = \sqrt{\frac{e}{2}}$

The value of $t_b = \sqrt{\frac{e}{2}}$ is approximately 1.16 and is shown earlier in Figure 2.19

★ ★ An extreme case of breaking arises when the initial distribution has a discontinuous step with the value of c(ϕ) behind the discontinuity greater than that ahead. If we have the initial functions

$$\text{f(x)} = \begin{cases} \phi_2, \text{ if, } x > 0 \\ \phi_1, \text{ if, } x < 0 \end{cases}$$

and

$$\text{F(x)} = \begin{cases} c_1 = c(\phi_1), \text{ if } x > 0 \\ c_2 = c(\phi_2), \text{ if } x < 0 \end{cases}$$

with $c_2 > c_1$, then the breaking occurs immediately. This is shown in fig.(2.13) for the case $c'(\phi) > 0$,($\phi_2 > \phi_1$).

The characteristic lines will be

$$\text{x} = \begin{cases} tc_1 + a, \text{ if } a > 0 \\ tc_2 + a, \text{ if } a < 0 \end{cases}$$

The multivalued region starts right at the origin and is bounded by the characteristics x = c_1t and $x = c_2t$. F and its derivatives are not continuous.

On the other hand, if the initial step function is expansive with $c_2 < c_1$ there is a perfectly good continuous solution. Since The characteristic lines will be

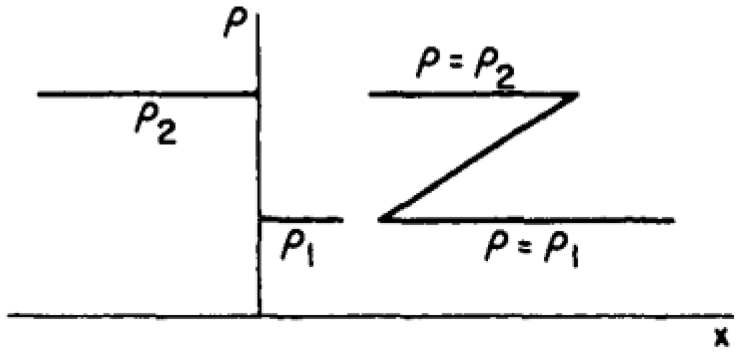


Figure 2.20: Initial Solution at t=0, Here $\phi = \rho, \phi_1 = \rho_1, \phi_2 = \rho_2$

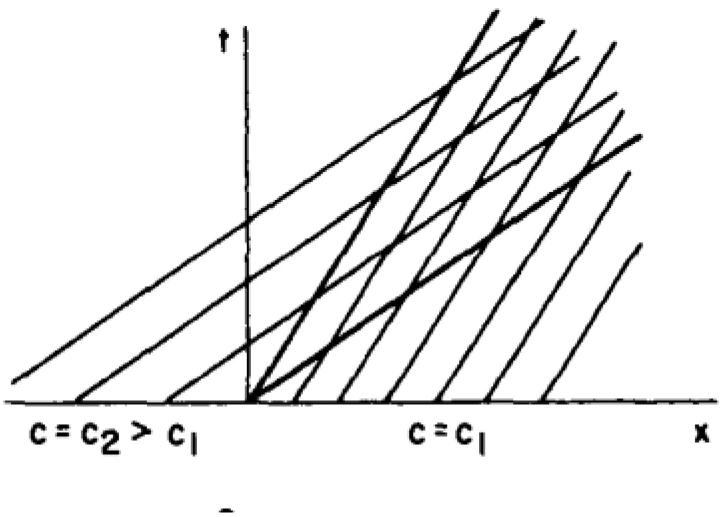


Figure 2.21: Centered compression wave with overlap

$$x=\begin{cases} tc_1 + a, \text{ if } a < 0 \\ tc_2 + a, \text{ if } a > 0 \end{cases}$$

It may be obtained as the limit of (2.5) and (2.6) in which all the values of F between c_2 and c_l are taken on characteristics through the origin ξ=0 . This corresponds to a fan of characteristics in the (x,t) plane as in Fig. 2.14. Each member of the fan has a different slope F but the same ξ. The function F is a step function but we use all the values of F between c_2 and c_1, on the face of the step and take them all to correspond to ξ=0.

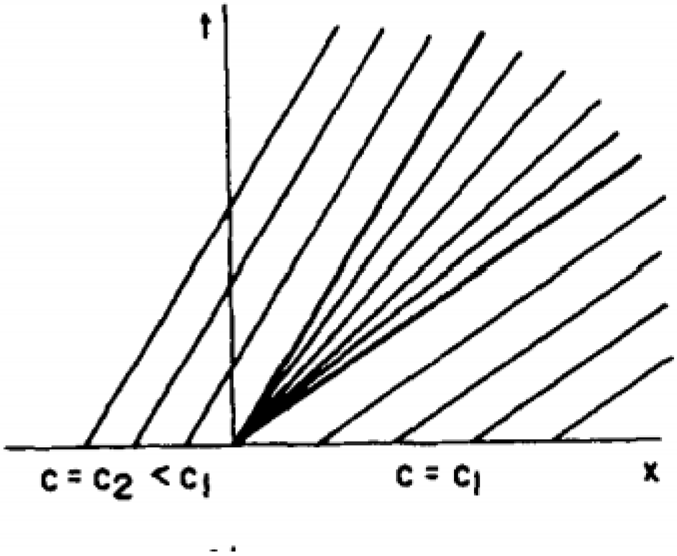


Figure 2.22: Rarefaction Waves

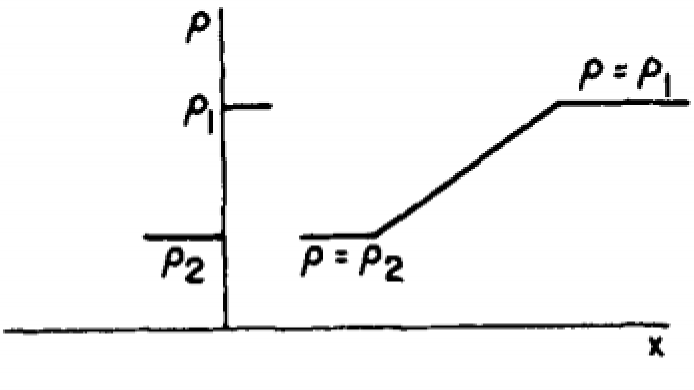


Figure 2.23: Centered expansion wave

In the fan, the solution (2.5), (2.6) then reads

c=F, x=Ft, for $c_2 < F < c_1$

and by elimination of F we have the simple explicit solution for c:

c=$\frac{x}{t}$, $c_2 < \frac{x}{t} < c_1$

The complete solution for c is

$$c=\begin{cases} c_1, \text{ if } c_1 < \frac{x}{t} \\ x_{\bar{t}}, \text{ if } c_2 < \frac{x}{t} < c_1 \\ c_2, \text{ if } \frac{x}{t} < c_2 \end{cases}$$

The relation c = c(ϕ) can be solved to determine ϕ. For the compressive step,

$c_2 > c_1$ the fan in the (x,t) plane is reversed to produce the overlap shown in Fig. 2.13.

We will discuss those problems in details in rarefraction Waves

Chapter 3

Conservation Laws

While the wave equation has many solutions which illustrate waves and their properties, wave behavior can be found in applications which are modeled by other partial differential equations. In the following chapters we will look at a class of mathematical models which are derived from conservation laws. Later it will be shown that many of these models possess solutions with wave behavior.

3.1 Derivation of a general scalar conservation law

A conservation law is an equation which accounts for all of the ways that the amount of a particular quantity can change. This accounting is one of the basic principles of mathematical modeling and can be applied to a variety of quantities such as mass, momentum, energy, and population. Suppose that a medium, essentially one-dimensional and positioned along the x—axis, contains some substance which can move or flow. This quantity could be, for example, cars moving along a section of road, particles of pollutant in a narrow stream of water, or heat energy flowing along a wire. For brevity, let Q represent this quantity (cars, particles, energy, etc.). In this section we will derive a general conservation law which describes the amount of Q in the medium at time t.

Let u(x, t) measure the density or concentration (amount per unit length) of Q at position x of the medium at time t (Figure 15.1). The value of u could indicate, for example, the density of traffic (cars per mile) or concentration of pollutant (grams per meter) at position x.

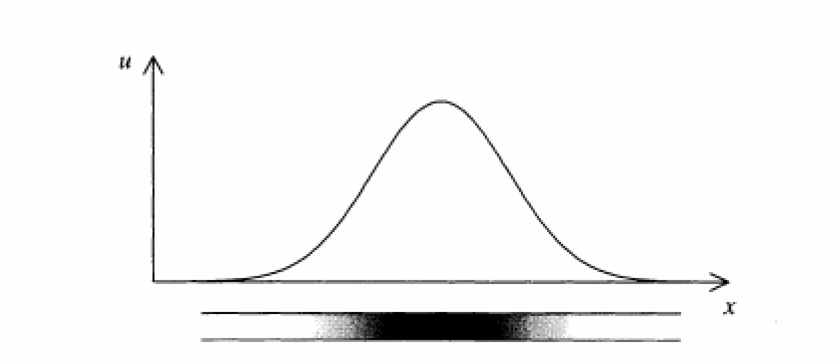


Figure 3.1: Centered expansion wave

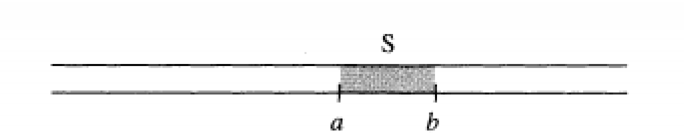


Figure 3.2: Centered expansion wave

Let u(x,t) measure the density or concentration (amount per unit length) of Q at position x of the medium at time t (Figure 3.1). The value of u could indicate, for example, the density of traffic (cars per mile) or concentration of pollutant (grams per meter) at position x. Now let S be any small segment of the medium with endpoints located at x = a and x = b with a < b (Figure 3.2). It will be assumed that changes in the amount of the quantity Q within this segment can occur in only two ways: either Q enters or leaves S through its ends at x =a and x = b, or Q is somehow being added (created) or removed (destroyed) from the medium within the segment. By accounting for all of the ways in which the total amount of Q can change within S,we are forming a general conservation law for Q:

★The net (time) rate of change of the total amount of Q in S = The rate at which Q enters or leaves S through the left end x = a + which Q enters or leaves S through the right end x = b + The rate at which Q is created or removed within S (3.1)

The next step will be to quantify the different parts of this conservation principle. Since u(x, t) is the amount of Q per unit length along the medium, the total amount of Q in the segment S at time t is computed by the integral $\int_a^b u(x,t)dx$. As the quantity Q flows through the medium, the amount of Q within S can change over time; the rate at which this amount changes with respect to time is given by the derivative

$$\frac{d}{dt}\int_a^b u(x,t)dx \tag{3.2}$$

The rate at which Q enters S through either of its ends will be described by a Flux Function. Let ϕ(x, t) denote the rate (amount per unit time) at which Q is flowing past position x at time t. A positive value $\phi(x,t) > 0$ indicates that the flow is in the direction of increasing x, while $\phi(x,t) < 0$ means the flow is in the opposite direction. Such a function is called the Flux. The rate at which Q enters S through the end x = a is then $\phi(a,t)$. If $\phi(a,t)$ is positive, then Q is flowing into S through the left end at x = a, while $\phi(a,t) < 0$ indicates Q is flowing out of S through the left end. Similarly, the rate at which Q enters S through the right end at x = b is —$\phi(b,t)$. The extra minus sign at x = b is needed since $\phi(b,t) > 0$ indicates Q is flowing to the right at x =b, which decreases (negative rate) the amount of Q in the segment S (see Figure 3.2). The net rate at which Q enters S through its ends is then given by

$$\phi(a,t) - \phi(b,t) \tag{3.3}$$

The addition or removal of Q within the segment S will be represented by a source function. Let f(x,t) be the rate (amount per unit time per unit length) at which Q is being added to or removed from the medium at position x and time t. Such a function f is called a Source Function. A positive value $f(x,t) > 0$ indicates that Q is being created or added to the medium at position x, while $f(x,t) < 0$ means Q is being destroyed or removed. The total rate (amount per unit time) at which Q is being created within the segment S at time t is

$$\int_a^b f(x,t)dx = SourceFunction \tag{3.4}$$

Substituting the measurements (3.2), (3.3), and (3.4) into the conservation principle (3.1) results in an equation called a conservation law in integral form:

$$\boxed{\frac{d}{dt}\int_a^b u(x,t)dx = \phi(a,t) - \phi(b,t) + \int_a^b f(x,t)dx} \tag{3.5}$$

An alternative form of the integral conservation law can be derived when u and ϕ are assumed to have continuous first derivatives. With this assumption (3.5) can be rewritten as

$\int_a^b u_t(x,t)dx =$ -$\int_a^b \phi_x(x,t)dx$ +$\int_a^b f(x,t)dx$

So that

$\int_a^b (u_t(x,t) + \phi_x(x,t) - f(x,t))dx$=0.

If u_t, ϕ_x, and f are all continuous, then the fact that this integral is zero for every $a < b$ along the medium implies that the integrand $u_t + \phi_x - f$ must be zero. This results in a conservation law in differential equation form:

$$\boxed{u_t + \phi_x = f} \tag{3.6}$$

3.2 Constitutive equations

The conservation law (3.6) is a very general equation which relates three functions: the density function u,the flux ϕ, and the source term f. It simply states that the rate of change of the amount of Q at position x depends on the rate at which Q flows past x (flux) and the rate at which Q is created at x (source). In order to determine u(x, t), more must be known about the flux ϕ and the source term f.

The source term f is usually determined or specified from the particular physical problem behind the conservation law. In many cases, it is zero.

Even when f=0, $u_t + \phi_x$=0 is still only one differential equation for two unknowns u and ϕ. A second equation relating u and ϕ is often given, based on an assumption about the physical process being modeled or on experimental evidence. Such an equation is called a constitutive equation. In general, our models will consist of two parts,

$u_t + \phi_x = f$ Conservation Law(Fundamental law of nature)

Relation between u and ϕ $\implies$ Constitutive Equation(approximation based on experience).

The flux ϕ often depends on u. For example, if the rate (amount per time) at which the quantity Q flows past a point depends on the concentration of Q, then the flux is a function of density and forms an explicit constitutive law $\phi = \phi(u)$. When this is the case, the chain rule gives $\phi_x = \phi'(u)u_x$, so that the conservation law (3.6) can be written as

$$u_t + \phi'(u)u_x = f \tag{3.7}$$

The inviscid Burgers equation

$u_t + uu_x = 0$ is an example of a conservation law in the form (3.7). In this equation the source term f(x,t) is zero and the flux ϕ is a function of u for which $\phi'(u) = u$.

One possibility for the flux term is the constitutive equation $\phi = \frac{1}{2}u^2$

The inviscid Burgers equation can then be written in conservation law form,

$u_t + (\frac{1}{2}u^2)_x$=0

3.3 Example of Conservation Laws: Traffic Flow

Since its first developments in the mid 1950's by M.J. Lighthill and G.B. Whitham [LW] and P.I. Richards [R], the deterministic modeling of traffic flow has yielded several examples of wave behavior. Here we will follow parts of the books by Haberman [Habl] and Whitham [Whi] to form conservation laws which model traffic flow, and later observe wave phenomena arising from these models. As a simplified example, consider automobile traffic moving along a section of single lane road with no exits or entrances. Let u(x, t) represent the density of cars (number of cars per mile) at position x along the road at time t. The function u(x, t) in principle should be a discrete valued function since cars are discrete objects; however, we will assume that u(x,t) is a continuous representation of the traffic density such as the one shown in Figure 3.3. As before, the basic conservation law for the traffic density u(x,t) is

$u_t + \phi_x = f$

In this conservation law, the source f(x,t) represents the rate (cars/hour per mile) at which cars are added or removed from the road at position x. With the assumption that there are no exits or entrances to the road and that cars do not appear or disappear from the road for any other reason, the source function f(x,t) is zero. The flux function ϕ(x,t) represents the rate (cars per hour) at which cars are passing position x along the road at time t. To an observer standing along the side of a road, the rate at which cars pass by depends not only on the traffic density u, but also on the traffic velocity v. If v is measured in miles per hour, then the flux ϕ is the product

ϕ= u(cars/mile)x v(miles/hour)= uv (cars/hour).

Traffic velocity v is generally not constant and is related to factors such as traffic density, weather, and time of day. As a simple model, we will assume

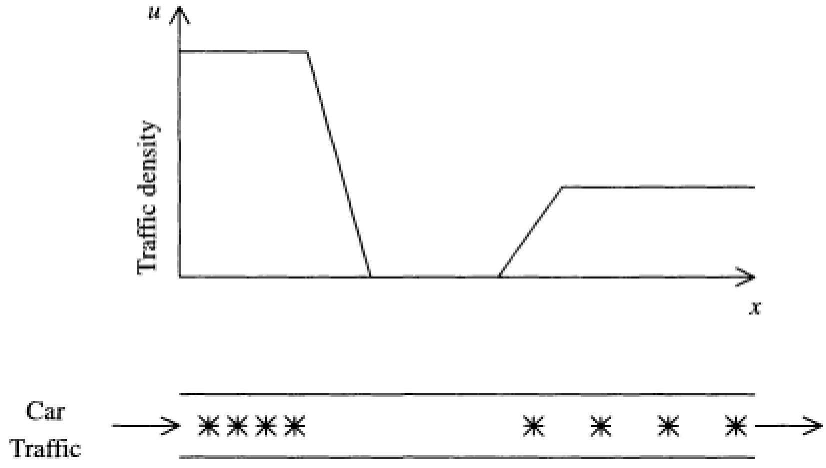


Figure 3.3: Continuous representation of traffic density along a single lane road

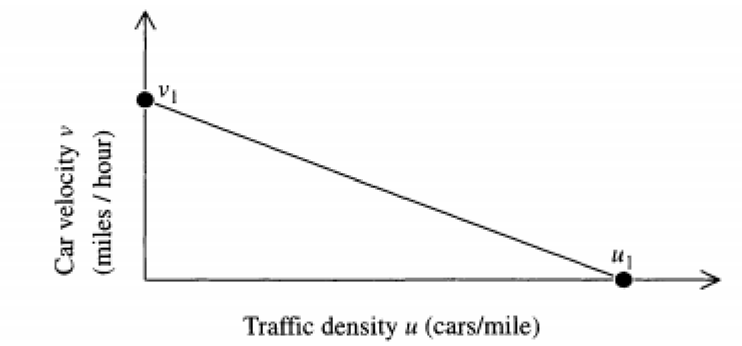


Figure 3.4: Higher traffic density generally results in lower traffic speed.

that the velocity v of the cars depends only on the traffic density, and in particular, denser traffic results in lower speeds. Suppose that drivers will travel at a maximum speed of v_1 miles per hour on a road which has little or no traffic (u = 0). We will also assume that traffic is at its maximum density u_1 cars per mile when the cars have come to a complete stop (v = 0). A linear model of this connection between traffic velocity and traffic density is shown in Figure 3.4 and is described by the equation

$v = v_1 - \frac{v_1}{u_1}u,\ 0 \leq u \leq u_1.$

The constitutive equation relating flux ϕ and traffic density u is then

$$\phi = uv = v_1(u - \frac{u^2}{u_1})(cars/hour) \tag{3.8}$$

With the car flux and source function f(x,t) = 0, the conservation law $u_t + \phi_x = f$ modeling traffic density along the road becomes

$$\boxed{u_t + v_1(1 - \frac{2u}{u_1})u_x = 0}$$

Chapter 4

Kinematic Waves

$$\boxed{\rho(x,t) = \text{Density per unit length}; q(x,t) = \text{Flux per unit length}; v(x,t) = \text{Flow velocity}} \tag{4.1}$$

In many problems of wave propagation there is a continuous distribution of either material or some state of the medium, and (for a one dimensional problem) we can define a density ρ(x, t) per unit length and a flux q(x,t) per unit time. We can then define a flow velocity v(x,t) by

$$v = \frac{q}{\rho} \tag{4.2}$$

Assuming that the material (or state) is conserved, we can stipulate that the rate of change of the total amount of it in any section $x_1 > x > x_2$ must be balanced by the net inflow across x_1 and x_2. That is,

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx + q(x_1,t) - q(x_2,t) = 0 \tag{4.3}$$

If p(x,t) has continuous derivatives, we may take the limit as xl-*x2 and obtain the conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{4.4}$$

The simplest wave problems arise when it is reasonable, on either theoretical or experimental grounds, to postulate (in a first approximation!) a functional relation between q and ρ. If this is written as

$$q = Q(\rho) \tag{4.5}$$

(4.4) and (4.5) form a complete system. On substitution we have

$$\rho_t + c(\rho)\rho_x = 0 \tag{4.6}$$

where

$$c(\rho) = Q'(\rho) \tag{4.7}$$

This leads to our (2.2) and a typical solution is given by (2.5) to (2.6). The breaking requires us to reconsider both the mathematical assumption that ρ and q have derivatives and the physical assumption that q=Q(ρ) is a good approximation. To fix ideas for the further development of the theory some specific examples are noted briefly here.

★ An amusing case (which is also important) concerns traffic flow. It is reasonable to suppose that some essential features of fairly heavy traffic flow may be obtained by treating a stream of traffic as a continuum with an observable density ρ(x,t), equal to the number of cars per unit length, and a flow q(x,t), equal to the number of cars crossing the position x per unit time. For a stretch of highway with no entries or exits, cars are conserved! So we stipulate (4.3). For traffic it also seems reasonable to argue that the traffic flow q is determined primarily by the local density ρ and to propose (4.5) as a first approximation. Such functional relations have been studied and documented to some extent by traffic engineers. We can then apply the theory. But it is clear in this case that when breaking occurs there is no lack of possible explanations for some breakdown in the formulation. Certainly the assumption $q = Q(\rho)$ is a very simplified view of a very complicated phenomenon. For example, if the density is changing rapidly (as it is near breaking), one expects the drivers to react to more than the local density and one also expects that there will be a time lag before they respond adequately to the changing conditions. One might also question the continuum assumption itself.

★ Another example is flood waves in long rivers. Here ρ is replaced by the cross-sectional area of the channel, A, and this varies with x and t as the level of the river rises. If q is the volume flux across the section, then (4.3) between A and q expresses the conservation of water. Although the fluid flow is extremely complicated, it seems reasonable to start with a functional relation $q = Q(A)$ as a first approximation to express the increase in flow as the level rises. Such relations have been plotted from empirical observations on various rivers. But it is again clear that this assumption is an oversimplification which may well have to be corrected if troubles arise in the theory.

★ A similar example, proposed and studied extensively by Nye (1960), is the example of glacier flow. The flow velocity is expected to increase with the thickness of the ice, and it seems reasonable to assume a functional dependence between the two.

★ In chromatography(It is a laboratory technique for the separation of mixture) and in similar exchange processes studied in problems of chemical engineering, the same theory arises. The formulation is a little more complicated. The situation is that a fluid carrying dissolved substances or particles or ions flows through a fixed bed and the material being carried is partially adsorbed on the fixed solid material in the bed. The fluid flow is idealized to have a constant velocity V. Then if ρ_f is the density of the material carried in the fluid, and ρ_s is the density deposited on the solid,

$\rho = \rho_f + \rho_s$, $q = V\rho_f$

Hence the conservation equation (4.4) reads

$\frac{\partial}{\partial t}(\rho_f + \rho_s) + \frac{\partial}{\partial x}(V\rho_f) = 0$

A second relation concerns the rate of deposition on the solid bed. The exchange equation

$\frac{\partial \rho_s}{\partial t}=k_1(A - \rho_s)\rho_f$-$k_2\rho_s(B - \rho_f)$

is apparently the simplest equation with the required properties.

The first term $k_1(A - \rho_s)\rho_f$ represents deposition from the fluid to the solid at a rate proportional to the amount in the fluid, but limited by the amount already on the solid up to a capacity A.

The second term $k_2\rho_s(B-\rho_f)$ is the reverse transfer from the solid to the fluid. (In some processes, the second term is just proportional to ρ_s; this is the limit B$\rightarrow \infty$,k$_2B$ finite.)

In equilibrium, $\frac{\partial \rho_s}{\partial t}$ is zero. i,e. the right hand side of the equation vanishes and ρ_s is a definite function of ρ_f. In slowly varying conditions, with relatively large reaction rates k_1, and k_2, we may take a first approximation in which the right hand side still vanishes (”quasi-equilibrium”) and we have

ρ_s=A$\frac{k_1\rho_f}{k_2B+(k_1-k_2)\rho_f}$

Thus ρ_s is a function of ρ_f; since q=Vρ_f, hence q is a function of ρ. When changes become rapid, just before breaking, the term $\frac{\partial \rho_s}{\partial t}$ in the rate equation can no longer be neglected. Since at breaking change of density deposited on the solid w.r.t to time is not small.

★ As a different type of example, the concept of group velocity can be fitted into this general scheme.

In linear dispersive waves, $\phi = a\cos\theta$, there are oscillatory solutions with a local wave number k(x,t) and a local frequency $\omega(x,t)$.

Thus k is the density of the waves— the number of wave crests per unit length—and w is the flux—number of wave crests crossing the position x per unit time.

Let n=The number of wave crests; x=Length; t=Time. Then k=$\frac{n}{x}$ and ω=$\frac{n}{t}$.

If we expect that wave crests will be conserved in the propagation, the conservation equation

$$\frac{\partial k}{\partial t}+\frac{\partial \omega}{\partial x}=0$$

In addition, k and ω are related by the dispersion relation

$$\omega = \omega(k)$$

Hence, $\frac{\partial k}{\partial t} + \omega'(k)\frac{\partial k}{\partial x} = 0$

We have a wave propagation for the variations of the local wave number of the "carrier" wavetrain, and the propagation velocity is $\frac{d\omega}{dk}$. This is the group velocity.

These ideas will be considered in full detail in the later discussion of dispersive waves.

The wave problems listed here depend primarily on the conservation equation (4.4), and for this reason they were given the name kinematic waves (Lighthill and Whitham, 1955).

Chapter 5

Shock Waves-I

The derivation of the differential equation form of a conservation law $u_t + \phi_x = 0$ assumes that the solution u has continuous first derivatives. The method of characteristics can construct such a solution, but only up until the time of a gradient catastrophe. In this chapter the solution u(x,t) will be extended beyond the breaking time by permitting u(x, t) to be a piecewise smooth function. In doing so, we will have to return to the original integral form of the conservation law at points (x,t) where u(x,t) is discontinuous. The resulting discontinuous solution of the conservation law is called a Shock Wave.

5.1 Piecewise smooth solutions of a conservation law

As we have seen, characteristic curves for the initial value problem

$$\begin{cases} \mathrm{u}_t + c(u)u_x = 0, -\infty < x < \infty, t > 0, \\ \mathrm{u(x,0)}=\mathrm{u}_0(x) \end{cases} \tag{5.1}$$

can be used to construct a solution u(x,t) starting at time t = 0, but ending at the breaking time t_b of a gradient catastrophe. In the following section we will modify the method of characteristics to allow the profile u(x,t) to literally break at time t=t_b

, forming a function which is only piecewise smooth for time t_b (Figure 5.1).

To describe piecewise smooth functions, suppose (x_s(t),t) is a curve in the xt—plane which divides the upper half of the plane into two parts (Figure 5.2). Let R^- represent the region to the left of the curve and R^+ the region

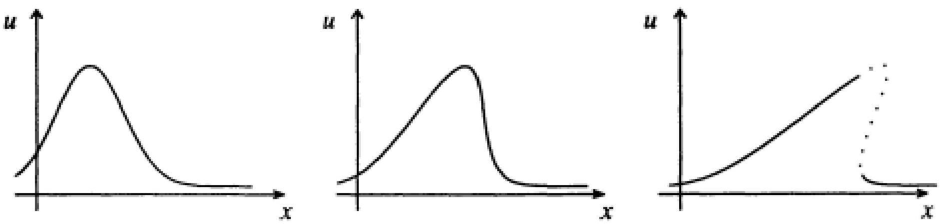


Figure 5.1: Profiles of a function u(x,t) which "breaks" after a gradient catastrophe.

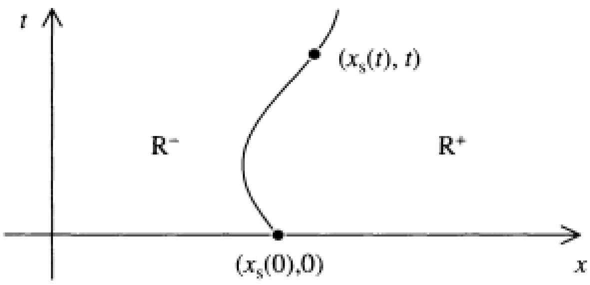


Figure 5.2: Curve $(x_s(t), t)$

to the right of the curve. A function u(x,t) is called a piecewise smooth solution of

$$\begin{cases} u_t + \phi_x = 0, -\infty < x, \infty, t > 0; \\ u(x,0) = u_0(x) \end{cases} \tag{5.2}$$

with jump discontinuity along x_s if u(x,t) has the following properties:

(1)u(x,t) has continuous first derivatives u_t and u_x in R^+ and R^-, and satisfies the initial value problem in region R^-

$$\begin{cases} u_t + \phi_x = 0, \text{for}(x,t)\text{in}R^-; \\ u(x,0) = u_0(x), \text{for} x < x_s(0), \end{cases} \tag{5.3}$$

and in region R^+

$$\begin{cases} u_t + \phi_x = 0, \text{for}(x,t)\text{in}R^+; \\ u(x,0) = u_0(x), \text{for} x > x_s(0), \end{cases} \tag{5.4}$$

(2) At each point (x_0, t_0) on the curve ($x_s(t)$,t), the limit of u(x,t) as (x,t) $\rightarrow$ (x_0, t_0) in R^- and the limit of u(x,t) as (x,t)$\rightarrow$ (x_0, t_0) in R^+ both exist but are not necessarily equal.

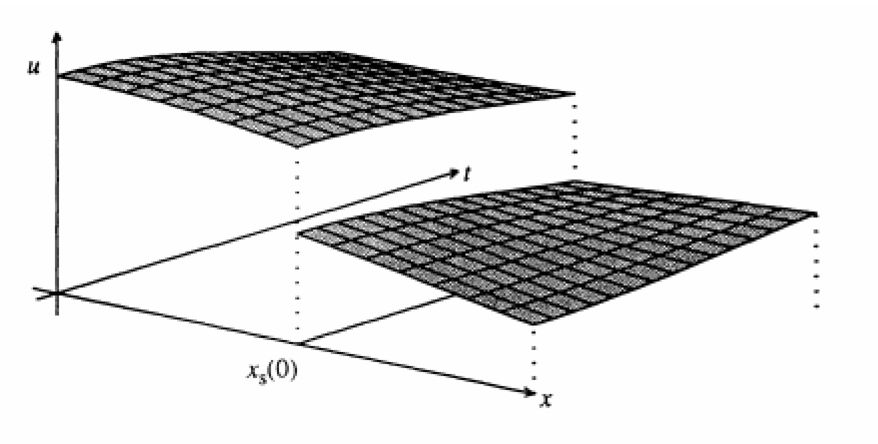


Figure 5.3: The graph of a piecewise smooth function u(x,t)

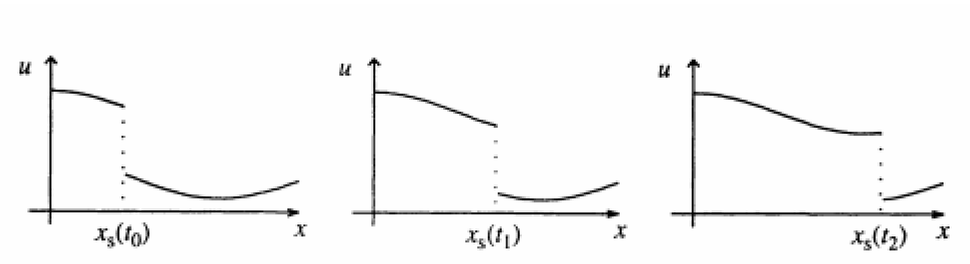


Figure 5.4: Profiles of a piecewise smooth function u(x,t) with discontinuity at x_s(t)

The graph of such a function appears as two sections of surface with a jump along the curve (x_s(t),t) in the xt—plane (Figure 5.3). The animation of a piecewise smooth function, formed by taking slices of the surface at a sequence of increasing times, has a profile with a moving jump discontinuity located at x_s(t) (Figure 5.4).

5.2 Shock wave solutions of a conservation law

The construction of a solution of $u_t + \phi_x = 0$ by the method of characteristics temporarily stops when a gradient catastrophe occurs. The physical process that the conservation law models, however, does not necessarily end. In this section we will describe how to extend the solution u(x,t) beyond the breaking time by permitting u(x, t) to be only piecewise smooth, but in a way which continues to obey the underlying conservation principle. The formation of a discontinuity after a gradient catastrophe is a dramatic change in the nature of u(x,t). Such a function will be called a shock wave solution of the conservation law.

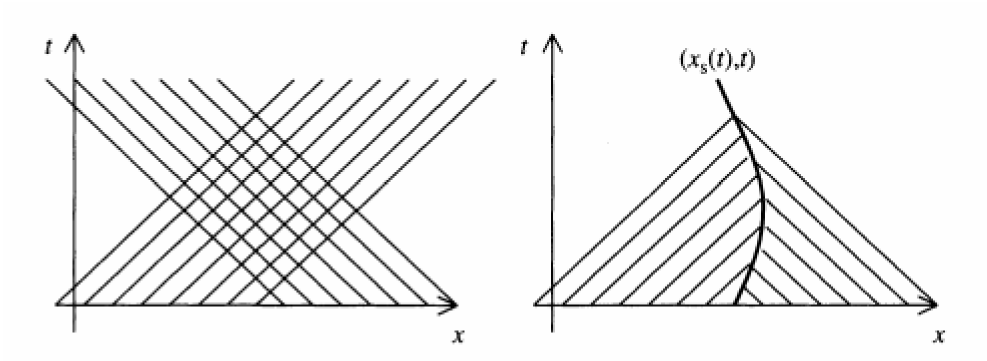


Figure 5.5: Using a curve to divide a region of crossing characteristics.

Suppose that characteristics of

$$\begin{cases} u_t + \phi_x = 0, -\infty < x, \infty, t > 0; \\ u(x,0) = u_0(x) \end{cases} \tag{5.5}$$

begin intersecting at time t_b, which we will assume is $t_b = 0$ as shown in Figure 5.5. In order to proceed with the method of characteristics, a curve (x_s(t),t) is drawn through the region of crossing characteristics to separate the characteristics approaching from the left and right (Figure 5.5). While many curves can be drawn to separate the crossing characteristics, it will now be shown that the underlying conservation law selects out one choice of x_s(t).

Suppose u(x, t) is a piecewise smooth solution of the initial value problem (5.5) with jump discontinuity along x_s(t). While u(x,t) satisfies $u_t + \phi_x$= 0 at each point (x, t) in R^- and R^+ , the derivatives of u(x, t) do not necessarily exist at points (x, t) on the curve. To see what happens at points $(x_s(t), t)$ on the curve, we have to return to the original integral form of the conservation law (5.5). With no source term, the integral form of the conservation law is

$$\frac{d}{dt}\int_a^b u(x,t)dx = \phi(a,t) - \phi(b,t) \tag{5.6}$$

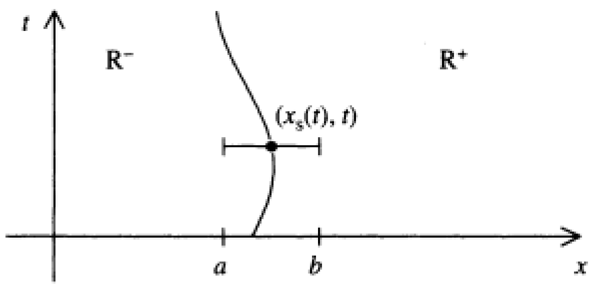


Figure 5.6

Fixing a point ($x_s(t)$, t) on the curve, pick a and b so that a ¡ $x_s(t)$ ¡ b as shown in Figure 5.6. The integral in the conservation law (5.6) can then be split into two parts as

$\int_a^b u(x,t)dx = \int_a^{x_s(t)^-} u(x,t)dx+\int_{x_s(t)^+}^b u(x,t)dx$

Substituting into the conservation law (5.6) and using the chain rule to compute the derivative of these integrals with respect to t results in

$\int_a^{x_s(t)^-} u_t(x,t)dx+u(x_s^-,t)\frac{dx_s}{dt}+\int_{x_s(t)^+}^b u_t(x,t)dx$-$u(x_s^+,t)\frac{dx_s}{dt}=\phi(a,t) - \phi(b,t)$

Letting $a \to x_s^-$ and $b \to x_s^+$ reduces this to the equation

$u(x_s^-,t)\frac{dx_s}{dt}$-$u(x_s^+,t)\frac{dx_s}{dt}=\phi(x_s^-,t) - \phi(x_s^+,t)$

from which we can solve for $\frac{dx_s}{dt}$ to obtain the ordinary differential equation

$$\boxed{\frac{dx_s}{dt} = \frac{\phi(x_s^+, t) - \phi(x_s^-, t)}{u(x_s^+, t) - u(x_s^-, t)}} \tag{5.7}$$

This derivation shows that in order for a piecewise smooth solution of the initial value problem (5.5) to satisfy the integral form of the conservation law (5.6), the curve along which u(x,t) has a jump discontinuity must be picked to satisfy (5.7). The differential equation (5.7) is called the Rankine-Hugoniot jump condition for u(x, t). The expressions $\phi(x_s^+, t) - \phi(x_s^-, t)$ and $u(x_s^+, t) - u(x_s^-, t)$ calculate the jump in the values of ϕ and u as (x, t) crosses the curve ($x_s(t)$,t) from left to right. Using the jump notation

$[\phi](x, t)$=$\phi(x_s^+, t) - \phi(x_s^-, t)$ and $[u](x, t)$=$u(x_s^+, t) - u(x_s^-, t)$

the Rankine-Hugoniot jump condition is written as

$$\boxed{\frac{dx_s}{dt} = \frac{[\phi]}{[u]}} \tag{5.8}$$

A piecewise smooth solution u(x,t) of $u_t + \phi_x = 0$ with a jump along a curve $x_s(t)$ satisfying the Rankine-Hugoniot condition is called a shock wave solution of the conservation law. The curve $x_s(t)$ is called a shock path.

★Example

Consider the following initial value problem for the inviscid Burgers equation

$$\begin{cases} u_t + uu_x = 0, -\infty < x < \infty, t > 0, \\ u(x, 0) = \begin{cases} 1, \text{if} x \leq 0 \\ 0, \text{if} x > 0 \end{cases} \end{cases} \tag{5.9}$$

The characteristics

$$x=\begin{cases} 0.t+x_0, \text{when} x_0 > 0 \\ 1.t + x_0, \text{when} x_0 < 0 \end{cases}$$

Based on the diagram of characteristics (Figure 5.7), it appears that the characteristics begin crossing at (0, 0) with a breaking time of $t_b = 0$. For this reason we will look for a shock wave solution with shock path starting at (0, 0).

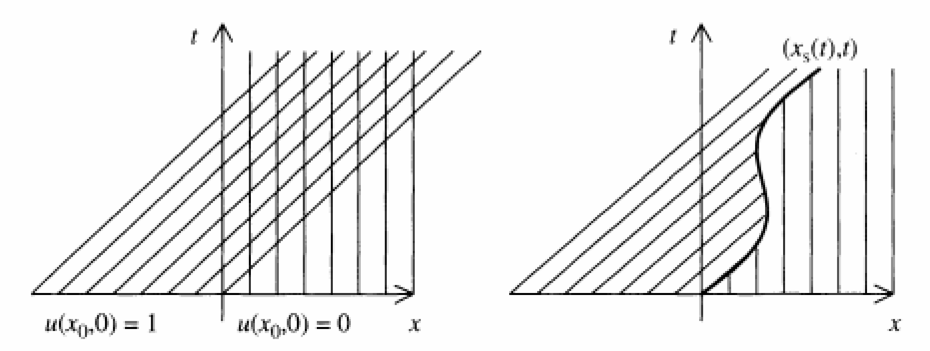


Figure 5.7: Characteristic diagram and a curve separating crossing characteristics for equation 5.9

Once (x_s(t),t) is found to separate the crossing characteristics, the method of characteristics can be used in the regions R^- to the left and R^+ to the right of the path (Figure 5.7). If (x,t) is a point in R^-, then there is one characteristic line extending back from (x, t) to a point (x_0, 0) on the negative x—axis. Since u is constant along this line and the value of u(x_0,0) $=$ 1 for x_0 ¡ 0 the value of u at (x,t) is u(x,t)=(x_0,0) $=$ 1. Similarly, if (x,t) is a point in R^+ , then the characteristic through it extends back to a point (x_0, 0) on the positive x—axis where u(x_0,0) $=$ 0. In this case, u(x,t) $=$ u(x_0,0) $=$ 0. Once the shock path is found to separate the regions R^- and R^+, the solution u will be given by

$$u(x,t) = \begin{cases} 1, \text{if}(x,t) \in R^- \\ 0, \text{if}(x,t) \in R^+ \end{cases}$$

The curve (x_s(t),t) separating the two regions will be found using the Rankine-Hugoniot jump condition; starting the shock path at (0, 0) forms the initial value problem

$\frac{dx_s}{dt} = \frac{[\phi]}{[u]}$, $x_s(0) = 0$

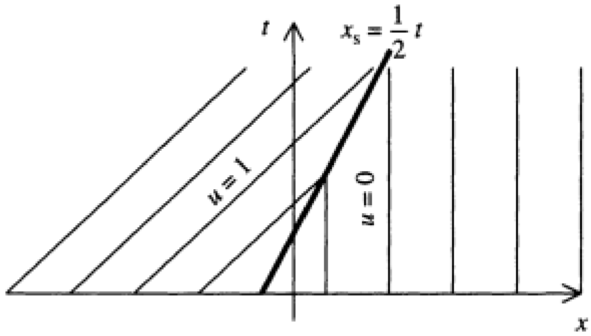


Figure 5.8: Shock path for Equation 5.9

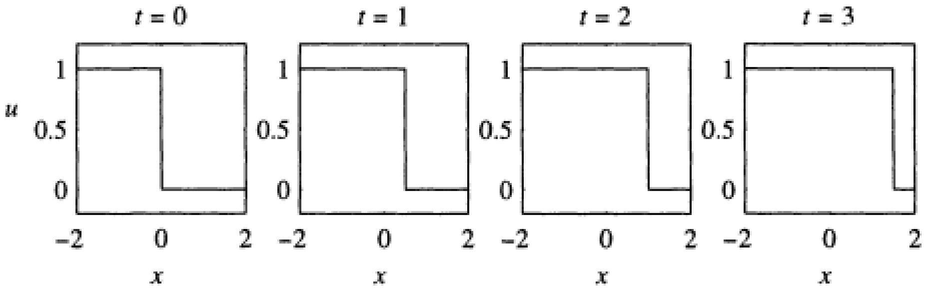


Figure 5.9: Animation of the shock wave solution of Example 5.9

The flux ϕ for the Burgers equation $u_t + uu_x = 0$ is $\phi = \frac{1}{2}u^2$, so

$\frac{dx_s}{dt} = \frac{[\frac{1}{2}u^2]}{[u]} = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{u^+ + u^-}{2}$

Since u=1 in R^- and u $= 0$ in R^+ , the value of was (x, t) approaches the curve from the left is $u^- = 1$, while the value from the right is $u^+ = 0$. The jump condition then simplifies to $\frac{dx_s}{dt} = \frac{1}{2}$, which together with the initial condition $x_s(0) = 0$ implies the shock path is the line $x_s = \frac{t}{2}$. An xt—diagram showing the shock path x $= \frac{t}{2}$ and characteristics (Figure 5.8) illustrates the resulting shock wave solution,

$$\text{u(x,t)}=\begin{cases} 1, \text{if} x < \frac{1}{2}t \\ 0, \text{if} x > \frac{1}{2}t \end{cases}$$

Four frames of animation of this function are shown in Figure 5.9. Note in particular the jump discontinuity moving to the right with speed $\frac{1}{2}$.

5.3 Shock Wave Example: Traffic at a Red Light

Shock wave solutions for conservation laws are piecewise smooth solutions which satisfy the Rankine-Hugoniot jump condition along curves of discontinuity. The resulting moving discontinuity models an abrupt change propagating through a medium. In this chapter a shock wave will be constructed to model traffic backing up at a red light.

5.3.1 An initial value problem

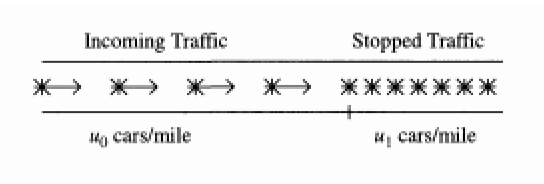


Figure 5.10: Incoming cars encountering stopped traffic

Suppose that car traffic, moving uniformly along a single lane road, encounters the end of a line of traffic which has stopped at a traffic light (Figure 5.10). The cars which have already stopped are lined up with maximum density u_1 cars per mile, while the cars approaching the end of the line have a uniform density u_0 cars per mile. Since u_1 is the maximum possible traffic density, the value of u_0 will satisfy $0 < u_0 < u_1$.

Returning to Section 3.3, let u(x, t) represent the density (cars per mile) of traffic at position x along the road at time t. The flux $\phi(x,t)$ represents the rate (cars per hour) at which traffic passes by position x and time t. Letting v_1 denote maximum traffic velocity, the linear model for traffic velocity $v = v_1(1 - \frac{u}{u_1})$ results in the constitutive equation (see Section 3.3).

$$\phi = uv = v_1(u - \frac{u^2}{u_1}) \tag{5.10}$$

Assuming that the road has no entrances or exits, the basic conservation law

$u_t + \phi_x = f$ with flux ϕ and sourse f=0 becomes

$u_t + v_1(1 - \frac{2u}{u_1})u_x$=0

Let x = 0 represent the location of the end of the stopped traffic at time t = 0. For now, it will be assumed that the stopped traffic extends indefinitely in one direction and the incoming traffic extends indefinitely in the other. In this case, the initial value problem

$$\begin{cases} u_t + v_1(1 - \frac{2u}{u_1})u_x, -\infty < x < \infty, t > 0 \\ u(x,0) = \begin{cases} u_0, \text{if} x < 0 \\ u_1, \text{if} x \geq 0 \end{cases} \end{cases} \tag{5.11}$$

models the profile of traffic density u(x,t) at later times t.

5.3.2 Shock wave solution

In this section we will use the method of characteristics to find a solution of the initial value problem (5.11). Since the conservation law in (5.11) is of the form $u_t + c(u)u_x$ = 0, a solution u of (5.11) will be constant along the characteristic lines

$x = c(u(x_0, 0))t + x_0$,

where $u(x_0, 0)$ is determined by the initial condition in (5.11), and c(u) is given by

$c(u) = v_1(1 - \frac{2u}{u_1})$

If $x_0 \geq 0$, then the characteristic starting at (x_0,0) is

$x = c(u_1)t + x_0 = -v_1 t + x_0$.

In an xt—diagram, this shows that characteristics starting at points(x_0,0) on the positive x—axis are parallel lines with negative slope $\frac{-1}{v_1}$.

On the other hand, if x_0 ¡ 0, then the characteristic starting at (x_0,0) is

$x = c(u_0)t + x_0 = v_1(1 - \frac{2u_0}{u_1})t + x_0$

Note that this line can have positive or negative slope $\frac{1}{c}$ depending on whether c =$v_1(1 - \frac{2u_0}{u_1})$ is positive or negative, i.e., if u_0 is smaller or larger than $\frac{u_1}{2}$ (see

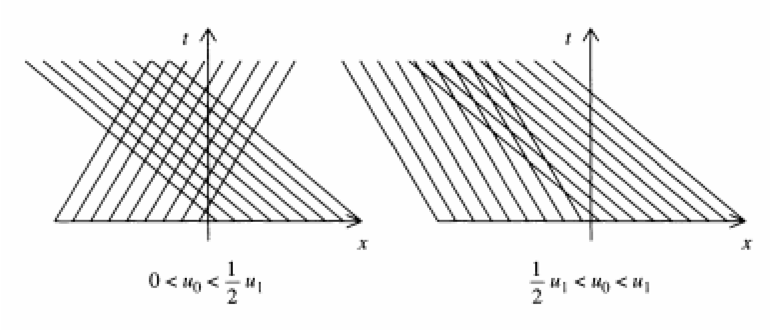


Figure 5.11: Breaking time for the solution of the initial value problem (5.11) is $t_b = 0$.

Figure 5.11). In either case, however, the slope $\frac{1}{c}$ will be between $\frac{-1}{v_1}$ and $\frac{1}{v_1}$ since the incoming traffic density u_0 satisfies $0 < u_0 < u_1$.

As shown in Figure 5.11, the characteristics will begin crossing at the origin. For this problem we will need to look for a shock wave solution whose shock path $x_s(t)$ starts at $x_s(0) = 0$ and extends upward to divide the region in which characteristics intersect (Figure 5.12).

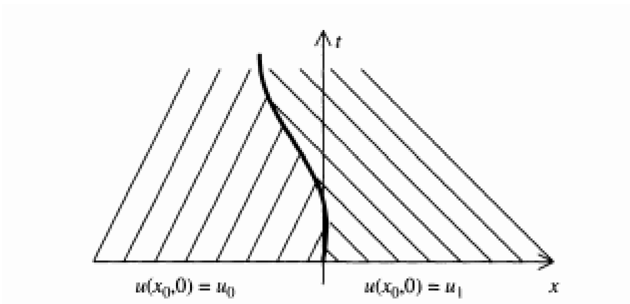


Figure 5.12: Setting up an xt—diagram for the initial value problem (5.9)

At a point (x, t) to the left of the shock path, the characteristic passing through the point extends back to the negative x—axis where u(x_0,0) $= u_0$. Since u is constant along characteristics, u(x,t) $=u_0$. Similarly, a point (x,t) to the right of the shock path lies on a characteristic which extends back to the positive x—axis where $u(x_0, 0)$ $=u_1$, so u(x,t) $= u_1$. The traffic density function u(x,t) will then have the form

$$\text{u(x,t)}=\begin{cases} u_0, if x < x_s(t) \\ u_1, \text{if} x > x_s(t) \end{cases}$$

The Rankine-Hugoniot jump condition $\frac{dx_s}{dt} = \frac{[\phi]}{[u]}$ will determine the shock path with the flux ϕ given by (5.10). At a point (x,t) on the shock path, we already determined that the values of u from the right and left are $u^+ = u_1$ and $u^- = u_0$. The jump condition

$\frac{dx_s}{dt} = \frac{[\phi]}{u} = \frac{\phi(u^+)-\phi(u^-)}{u^+-u^-} = \frac{\phi(u_1)-\phi(u_0)}{u_1-u_0}$

then simplifies to

$\frac{dx_s}{dt} = \frac{0-v_1(u_0-\frac{u_0^2}{u_1})}{u_1-u_0} = -v_1\frac{u_0}{u_1}$

Integrating this differential equation with respect to t and using the starting point $x_s(0) = 0$ gives the only allowed shock path, the line

$x_s = -v_1\frac{u_0}{u_1}t$

The resulting shock wave solution to (5.11) is then

$$u(x,t) = \begin{cases} u_0, if x < -v_1\frac{u_0}{u_1}t \\ u_1, if x \geq -v_1\frac{u_0}{u_1}t \end{cases} \quad (5.12)$$

with an xt—diagram shown in Figure 5.13. Note that this shock path indicates that the end of the line of stopped traffic will back up at the rate of $v_1\frac{u_0}{u_1}$ miles per hour.

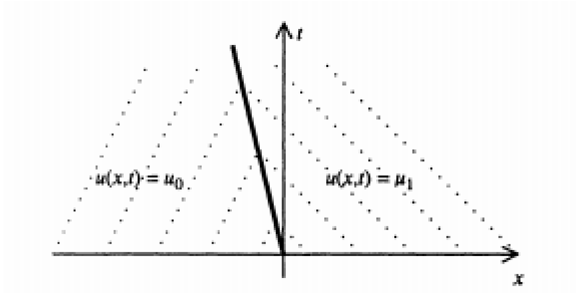


Figure 5.13: Shock path x_s=$v_1\frac{u_0}{u_1}t$

★ Example:

As a particular example, suppose that the stopped traffic is at a maximum density u_1 = 300 cars per mile, and the maximum velocity along this stretch of road is v_1 = 45 miles per hour. If the incoming traffic is traveling at 30 miles per hour, then the velocity model v = $v_1(1|\frac{}{uu_1)}$ predicts that the incoming traffic density u_0 satisfies

$30 = 45(1 — \frac{u_0}{300})$, so $u_0 = 100$ cars per mile. With these values, the solution (5.11) becomes

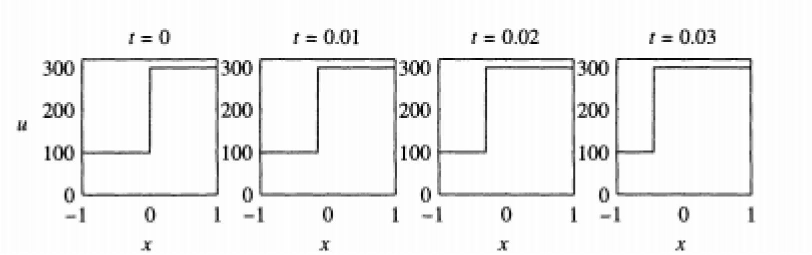


Figure 5.14: Traffic backing up at a rate of 15 miles per hour

$$u(x,t) = \begin{cases} 100, \text{if} x < -15t \\ 300, \text{if} x \geq -15t \end{cases}$$

The resulting shock path, representing the location of the end of the line of stopped traffic, is given by x = $—v\frac{u_0}{u_1}t$ = -15t, indicating that the end of the stopped traffic is backing up at 15 miles per hour. Four frames of animation of this traffic flow are shown in Figure 5.14.

Chapter 6

Rarefaction Waves

Earlier we saw how intersecting characteristics led to the construction of shock wave solutions of a conservation law. In this chapter we will examine a problem at the other extreme: in nonlinear conservation laws, it is possible to have regions in the xt—plane which contain no characteristics. For these regions, the method of characteristics will be modified to form rarefaction waves. Later in this chapter a rarefaction wave will be constructed which models traffic flow after a red light turns green.

6.1 An example of a rarefaction wave

The characteristics $x = c(u(x_0, 0))t + x_0$ for the initial value problem

$$\begin{cases} u_t + uu_x = 0, -\infty < x < \infty, t > 0, \\ u(x,0) = \begin{cases} 0, \text{if} x \leq 0, \\ 1, \text{if} x > 0 \end{cases} \end{cases} \tag{6.1}$$

constructed using the characteristic speed c(u)=u are

$$x = \begin{cases} 0.t + x_0, \text{if} x_0 \leq 0 \\ 1.t + x_0, \text{if} x_0 > 0 \end{cases}$$

When drawn in the xt—plane (Figure 6.1), note that the characteristics do not enter the wedge-shaped region $0 < x < t < \infty$. In this section we will look at rarefaction waves as one way of constructing a solution u(x, t) of the initial value problem (6.1) in this region.

Suppose the initial profile u(x,0) is modified to make a smooth transition from u = 0 to u = 1 within an interval of length Δx around x - 0. As shown in Figure 6.2, the resulting characteristics then make a smooth transition from lines with speed c = 0 (vertical) to lines with speed c = 1 (slope 1).

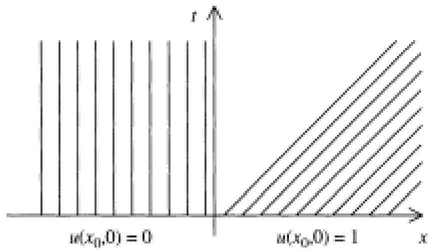


Figure 6.1: Characteristics which do not enter part of the xt—plane.

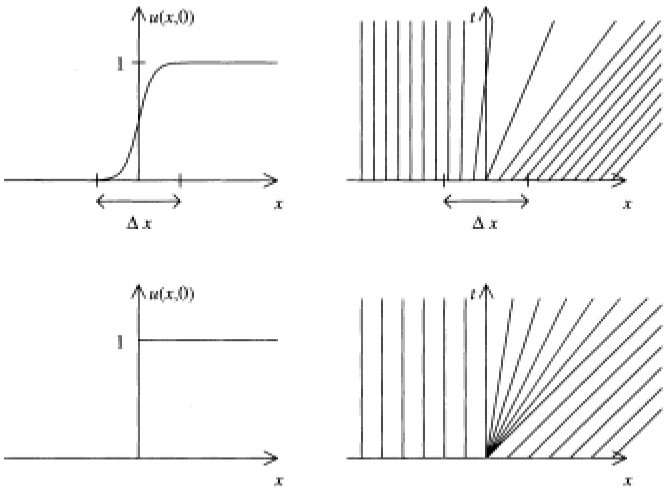


Figure 6.2: Smoothing the initial data u(x,0) to create a fan of characteristics, then letting $x \to 0$.

Letting the interval of transition Δx shrink to 0 (Figure 6.2) suggests that we might be able to find a solution of $u_t + uu_x = 0$ in the region 0 ¡ x ¡ t by filling it with a "fan of characteristics". This fan consists of lines x = ct, originating from the origin, whose speeds vary from c = 0 (vertical line) to c = 1. A function u(x,t) which is constant along each of these inserted "characteristics" would be of the form u(x, t)

$=g(\frac{x}{t})$, a function of the speed (or slope) of the lines x = ct.

To search for a solution of $u_t + uu_x = 0$ of the form u(x,t)=$g(\frac{x}{t})$ first note that by the chain rule, the derivatives u_t and u_x are

$u_t(x,t) = -\frac{x}{t^2}g'(\frac{x}{t})$, $u_x(x,t) = \frac{1}{t}g'(\frac{x}{t})$.

Substituting these derivatives into ut-uux = 0 produces the equation

$-\frac{x}{t^2}g'(\frac{x}{t}) + g(\frac{x}{t})\frac{1}{t}g'(\frac{x}{t}) = 0$

from which it follows by factoring that

$\frac{1}{t}g'(\frac{x}{t})(g(\frac{x}{t}) - \frac{x}{t}) = 0$

This shows that either $g' = 0$ (g is constant) or $g(\frac{x}{t}) = \frac{x}{t}$. The following exercise shows that we can discard the first possibility.

★ Example

Consider the initial value problem given in (6.1). Use the method of characteristics to show that u(x,t)= 0 in the region x≤ 0 and u(x,t)= 1 in the region $x > t$. Now suppose that $u(x,t) = g(\frac{x}{t}) = A$ in the wedge-shaped region $0 < x < t$, resulting in the function

$$u(x,t) = \begin{cases} 0, \text{if} x \leq 0; \\ A, \text{if} 0 < x \leq t; \\ 1, \text{if} t < x \end{cases}$$

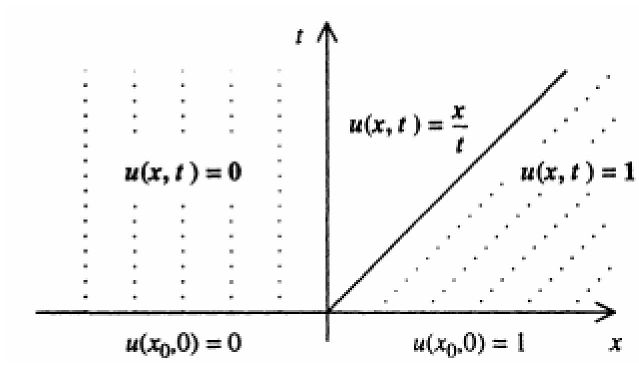


Figure 6.3: An xt—diagram using u(x,t) = $\frac{x}{t}$ to fill the center wedge-shaped region.

Use the Rankine-Hugoniot jump condition along the lines x=0 and x = t to show that u(x,t) cannot be a shock wave solution of (6.1). The other possibility for g

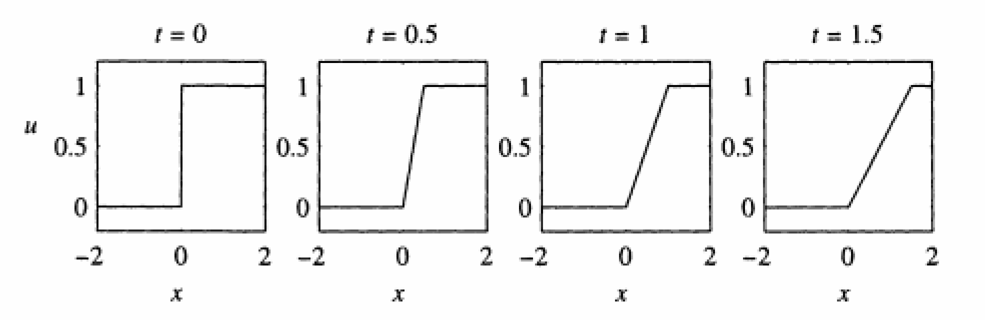


Figure 6.4: Animation of the function u(x,t) in (6.2)

is $g(\frac{x}{t}) = \frac{x}{t}$. Figure 6.3 shows the resulting xt—diagram that is formed by taking u(x,t)= $g(\frac{x}{t}) = \frac{x}{t}$ in the wedge-shaped region 0¡x¡t, and using the method of characteristics in the left ($x < 0$) and right ($x > 1$) regions. Thefunction u(x,t) is now piecewise defined by

$$\begin{cases} 0, & \text{if} x \leq 0; \\ \frac{x}{t}, & \text{if} 0 < x \leq \text{ t }; \\ 1, & \text{if} t < x \end{cases} \tag{6.2}$$

The four frames of animation displayed in Figure 6.4 show that the profile of the solution "thins out" or "rarefies" as time increases. Such a function is an example of a rarefaction wave. Note that although the function u(x, t) defined in (6.2) is continuous for $t > 0$, the derivatives u_t and u_x do not exist along the lines x = 0 and x = t and so u does not satisfy the differential equation $u_t + uu_x = 0$ at these points. This function, however, satisfies the conditions to be a weak solution of $u_t + uu_x = 0$, as we will describe later.

In general, a Rarefaction wave is a nonconstant function of the form u(x,t) = g($\frac{x-a}{t}$). The lines $\frac{x-a}{t}$ = c in the xt—plane are often called characteristics since u is constant along them; however, they are not constructed by the characteristic equation $\frac{dx}{dt}$ = c(u) derived from $u_t + c(u)u_x = 0$. These lines are distinguished by their fan shape originating from the point x = a on the x—axis (Figure 6.5).

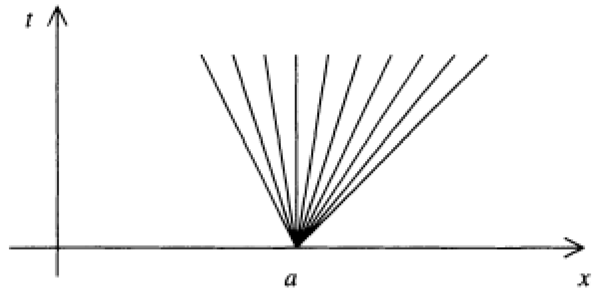


Figure 6.5: Characteristics for a rarefaction wave u(x, t) = g($\frac{x-a}{t}$)

6.2 Stopped traffic at a green light

Suppose traffic is backed up indefinitely in one direction behind a red light. The light, located at position x = 0, turns green at time t=0 and the traffic begins to move forward. As shown in Figure 6.6, it will be assumed that prior to the changing of the light, traffic behind the light is at its maximum density u_1 and no traffic exists ahead of the light.

Using the constitutive equation $\phi = v_1(u - \frac{u^2}{u_1})$ derived from the linear velocity model $v = v_1(1 - \frac{u}{u_1})$, an initial value problem which describes the traffic density u after the light turns green is

$$\begin{cases} u_t + v_1(1 - \frac{2u}{u1})u_x = 0, -\infty < x < \infty, t > 0, \\ u(x, 0) = \begin{cases} u_1, \text{if} x \leq 0; \\ 0, \text{if} x > 0 \end{cases} \end{cases} \tag{6.3}$$

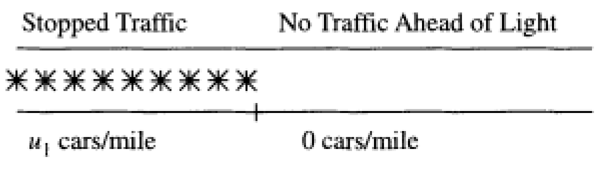


Figure 6.6: Traffic stopped at a red light.

The characteristic lines for this initial value problem are of the form $x = c(u(x_0, 0))t + x_0$ with c given by c(u) = $v_1(1 — \frac{2u}{u_1})$. Characteristics which start at points $(x_0, 0)$

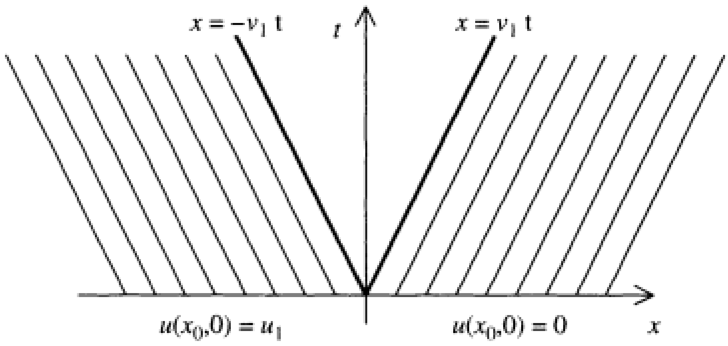


Figure 6.7: Characteristic lines of the initial value problem (6.3).

on the negative x—axis ($x_0 < 0$) have speed

$c(u(x_0, 0)) = c(u_1) = v_1(1 - \frac{2u_1}{u_1}) = -v_1$

while those starting at points on the positive x—axis have speed

$c(u(x_0, 0)) = c(0) = v_1(1 - 0) = -v_1$

The resulting characteristic lines are then

$$\text{x}=\begin{cases} \text{-v}_1 t + x_0, & \text{if} \text{x}_0 \leq 0 \\ \text{v}_1 t + x_0, & \text{if} \text{x}_0 > 0 \end{cases}$$

The characteristic diagram shown in Figure 6.7 separates into three parts: $x < -v_1 t$, $-v_1 t < x < v_1 t$, and $x > v_1 t$. No characteristics enter the middle region; however, as shown in the following exercise, a rarefaction wave can be constructed to fill this wedge-shaped area.

The resulting rarefaction wave solution is then

$$u(x,t) = \begin{cases} u_1, & \text{ifx} \leq -v_1 t, \\ \frac{1}{2}u_1(1 - \frac{1}{v_1}\frac{x}{t}), & \text{if—v}_1 t < x < v_1 t \\ 0, & \text{ifx} \geq v_1 t \end{cases} \tag{6.4}$$

Chapter 7

An Important Example with Rarefaction and Shock Waves

In general, nonlinear conservation laws may possess solutions which are constructed using a combination of shock and rarefaction waves.

In this chapter we will construct an example of such a solution.

Consider the initial value problem for Burgers' equation

$$\begin{cases} u_t + uu_x = 0, & -\infty < x < \infty, t > 0; \\ u(x,0) = \begin{cases} 0, & \text{if} x \leq 0, \\ 1, & \text{if} 0 < x < 1, \\ 0, & \text{if} x \geq 1 \end{cases} \end{cases} \tag{7.1}$$

With c(u) = u, the characteristics $x = c(u(x_0, 0))t + x_0 are$

$$x = \begin{cases} 0.t + x_0, & \text{if} x_0 \leq 0, \\ 1.t + x_0, & \text{if} 0 < x_0 < 1, \\ 0.t + x_0, & \text{if} x_0 \geq 1 \end{cases}$$

The characteristic diagram shown in Figure 7.1 has intersecting characteristics as well as a wedge-shaped region with no characteristics. Since u is constant along characteristics lines, the initial condition and the characteristic diagram show that u(x, t)= 0 for $x < 0$, u(x, t) = 1 for $0 < t < x < 1$, and u(x, t) = 0 for $0 < t < x - 1 < \infty$ (see Figure 7.2). A piecewise smooth solution to (7.1) will be completed using a combination of shock and rarefaction waves in the remaining regions of the xt—plane.

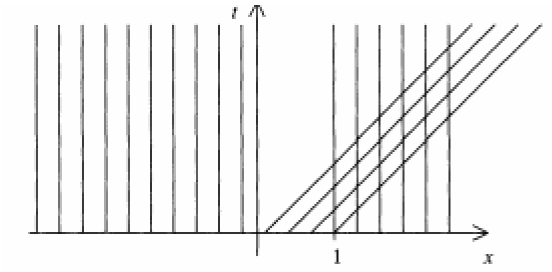


Figure 7.1: Characteristics of the initial value problem (7.1).

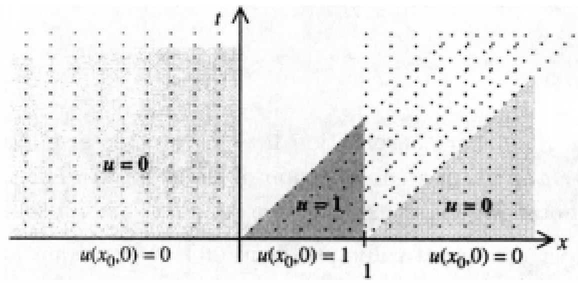


Figure 7.2: The value of u is constant along characteristics in regions of single characteristics.

★ Step 1: A rarefaction

We will begin by constructing a rarefaction wave to fill the wedge-shaped region in the xt—plane that does not contain any characteristic lines. As shown in Section 6.1, a rarefaction wave solution of $u_t + uu_x = 0$ with a fan of characteristic lines originating from (0,0) is

$u(x,t) = \frac{x}{t}$

Drawing a fan of characteristic lines for this rarefaction in the triangular wedge results in the characteristic diagram shown in Figure 7.3 and the updated xt—diagram in Figure 7.4.

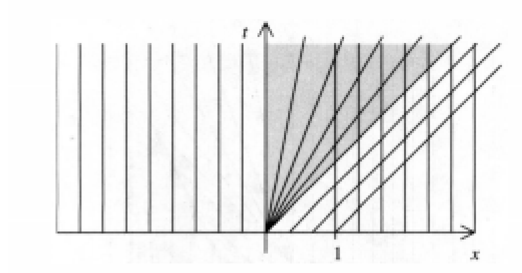


Figure 7.3: Characteristics of the rarefaction u(x,t) $= \frac{x}{t}$ fill the wedge-shaped region originating from the origin.

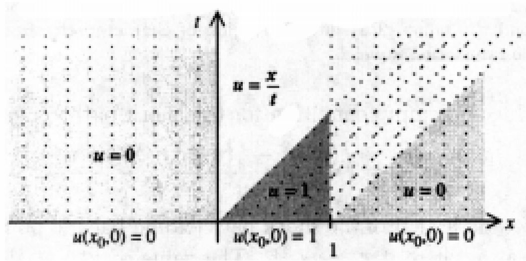


Figure 7.4: The xt—diagram of the solution including the rarefaction wave.

The characteristic diagram in Figure 7.3 shows a region of intersecting characteristics near the x—axis. We shall sketch a possible shock path in Figure 7.3 starting at the point (1,0).

★ Step 2: A Shock

The diagram in Figure 7.3 shows intersecting characteristics with a breaking time of $t_b = 0$. The next step will be to construct a shock path, starting at the point (x,t) = (1,0), which separates the characteristics $x = t + x_0$ from the vertical lines $x = x_0$. With the flux $\phi u = \frac{1}{2}u^2$ from Burgers' equation $u_t + uu_x = 0$, the

Rankine-Hugoniot jump condition for the shock path becomes

$\frac{dx_s}{dt} = \frac{[\phi]}{[u]} = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{u^+ + u^-}{2}$

The characteristics left of the shock path extend back to points $(x_0,\ 0)$on the x—axis where $0 < x_0 < 1$. The value of u(x,t) along these lines will be u(x,t) = $u(x_0, 0) = 1$, so the value of u(x,t) as (x,t) approaches the shock path from the left

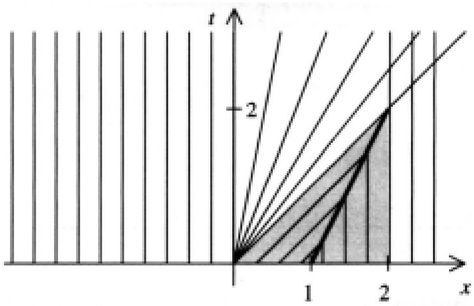


Figure 7.5: Shock path $x_s(t) = \frac{1}{2}t + 1$ for $0 \leq t \leq 2$ separates a region where u(x,t) = 1 from a region where u(x,t) = 0. The shock path will need to be extended beyond the point (2,2) into the region of characteristics from the rarefaction wave.

is $u^- = 1$. Similarly, the characteristics to the right of the shock path are vertical lines which extend back to points (x_0, 0) on the x—axis where $x_0 > 1$. The value of u(x, t) along these lines will be u(x, t) $= u(x_0, 0) = 0$, so the value of u(x, t) as (x, t) approaches the shock path from the right is $u^+ = 0$.

The jump condition for the path then becomes

$\frac{dx_s}{dt} = \frac{1+0}{2} = \frac{1}{2}$

which gives $x_s = \frac{1}{2}t + k$. The constant k is found using the condition that the shock starts at $(x_s, t) = (1, 0)$. In this case k = 1, and the resulting shock path is

$x_s = \frac{1}{2}t + 1, 0 \leq t \leq 2$

As shown in Figure 7.5, this part of the shock path ends at t = 2, where the vertical characteristics begin intersecting the characteristics inserted for the rarefaction wave.

★ Extension of the Shock

The shock path constructed in Step 2 separates the characteristics x = t + x_0 from the vertical lines x= x_0. As a final step in the construction of w(x, t), the shock will be extended from (x,t) = (2,2) into the region $t > 2$ where the vertical lines x = x_0 intersect the fan of characteristics from the rarefaction wave (Figure 7.5).

As in Step 2, the jump condition for the shock path is

$\frac{dx_s}{dt} = \frac{[\phi]}{[u]} = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{u^+ + u^-}{2}$

The characteristics to the right of the shock are vertical lines which extend back to points (x_0, 0) on the x—axis with $x_0 > 1$. The value of u(x, t) along these lines will be u(x, t) = u(x_0, 0) = 0, so the value of u(x,t) as (x,t) approaches the shock path from the right is $u^+ = 0$. To the left of the path, we have already determined that the value of u is u(x,t) = $\frac{x}{t}$ from the rarefaction wave, so the value of u(x,t) as (x,t) approaches the path from the left is $u^- = \frac{x}{t}$. The jump condition for points on the shock path is then

$$\frac{dx_s}{dt} = \frac{0+\frac{x_s}{t}}{2} = \frac{x_s}{2t}$$

This first order differential equation for xs is separable; rewriting the equation as

$$\frac{1}{x_s}\frac{dx_s}{dt} = \frac{1}{2t}$$

and integrating shows that $\ln x_s = \ln\sqrt{t} + k$, and so $x_s = k_1\sqrt{t}$ for some constant k_1. Since this part of the shock path starts at the point (x,t) = (2,2), the condition $x_s(2) = 2 determinesthat$ $\mathrm{k}_1 = \sqrt{2}$, and so the shock path here is

$$x_s = \sqrt{2t},\ t \geq 2$$

As shown in Figure 7.6, this curve separates the region of rarefaction characteristics from the vertical characteristics for time $t \geq 2$.

The characteristic diagram in Figure 7.6 completes the construction of a piecewise smooth solution to the initial value problem (7.1); the final xt—diagram of the solution is shown in Figure 7.7.

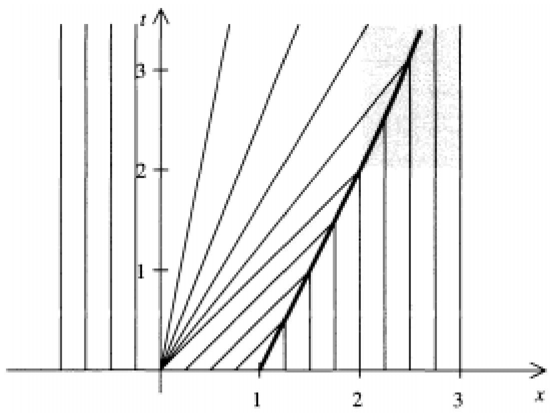


Figure 7.6: Extending the shock path by $x_s = \sqrt{2t}$, $t \geq 2$ to separate the region of rarefaction characteristics from the vertical characteristics.

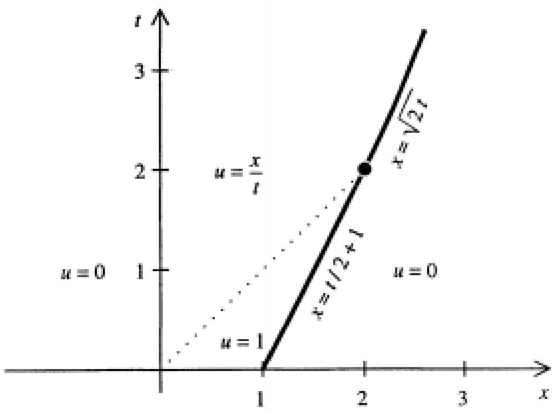


Figure 7.7: An xt—diagram for a function u(x, t) consisting of a shock and a rarefaction

During the first two units of time, the profile of u(x, t) is given by

$$u(x,t) = \begin{cases} 0, & \text{if} x < 0 \\ \frac{x}{t}, & \text{if} 0 < x < t \\ 1, & \text{if} t < x < \frac{1}{2}t + 1 \\ 0, & \text{if} \frac{1}{2}t + 1 < x \end{cases} \tag{7.2}$$

Once past time t = 2, the profile of u(x,t) is defined by

$$\begin{cases} 0, & \text{if} x < 0 \\ \frac{x}{t}, & \text{if} 0 < x < \sqrt{2t} \\ 0, & \text{if} \sqrt{2t} < x \end{cases} \tag{7.3}$$

Chapter 8

Nonunique Solutions and the Entropy Condition

Rarefaction and shock waves are special solutions of conservation laws that exhibit wave behavior. In the process of constructing them, however, we have relaxed the notion of "solution" from a function u(x, t) which satisfies $u_t + \phi_x = 0$ for all (x, t), to a piecewise smooth solution which satisfies the integral form of the conservation law where u is not continuous. In this chapter, we will see that this more general notion of solution makes it possible for an initial value problem to possess many different solutions. The entropy condition will then be introduced as an example of a condition which is used to select one solution over all others.

8.1 Nonuniqueness of piecewise smooth solutions

The rarefaction wave from Section 6.1 and equation 6.2

$$u(x,t) = \begin{cases} 0, & \text{if} x \leq 0; \\ \frac{x}{t}, & \text{if} 0 < x < t; \\ 1, & \text{if} x \geq t \end{cases} \tag{8.1}$$

was constructed as a piecewise smooth solution of the initial value problem

$$\begin{cases} u_t + uu_x = 0, -\infty < x < \infty, t > 0, \\ u(x,0) = \begin{cases} 0, & \text{if} x \leq 0, \\ 1, & \text{if} x > 0, \end{cases} \end{cases} \tag{8.2}$$

It is also possible, however, to find other solutions of this problem using shocks waves. In fact, if A is any number satisfying $0 < A < 1$, then the function

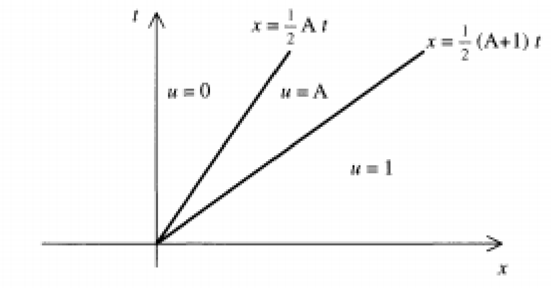


Figure 8.1: A shock wave solution of the initial value problem (8.2) with two shock paths.

$$u(x,t) = \begin{cases} 0, & \text{if} x \leq \frac{1}{2}At; \\ A, & \text{if} \frac{1}{2}At < x < \frac{1}{2}(A+1)t; \\ 1, & \text{if} x \geq \frac{1}{2}(A+1)t \end{cases} \tag{8.3}$$

represented by the xt—diagram in Figure 24.1 is a shock wave solution with two shock paths (see Exercise 24.1). Thus there are many solutions of the initial value problem (24.2)—a rarefaction wave solution and an infinite number of shock wave solutions.

★ Consider the function u(x,t) given by (8.3).

(a) Verify that u(x,t) satisfies ut + uux = 0 in each of the three regions $x < \frac{1}{2}At, \frac{1}{2}At < x < \frac{1}{2}(A+1)t, x > \frac{1}{2}(A+1)t$

(b) Verify that the paths of discontinuity $x_s = \frac{1}{2}At$ and $x_s = \frac{1}{2}(A+1)t$ satisfy the

Rankine-Hugoniot jump condition.

8.2 The Entropy Condition

When an initial value problem has more than one solution, additional information must be specified if one particular solution is to be selected.In gas dynamics, for example, the entropy condition is used to select a solution which is most physically realistic.

A function u(x,t) satisfies the entropy condition if it is possible to find a positive constant E so that

$\frac{u(x+h,t)-u(x,t)}{h} \leq \frac{E}{t}$

for all $t > 0, h > 0$, and x. Graphically, this is a condition on the slope of the profile of u(x,t) at each time t—the slope between any two points on the profile (secant slope) at time t is less than $\frac{E}{t}$:

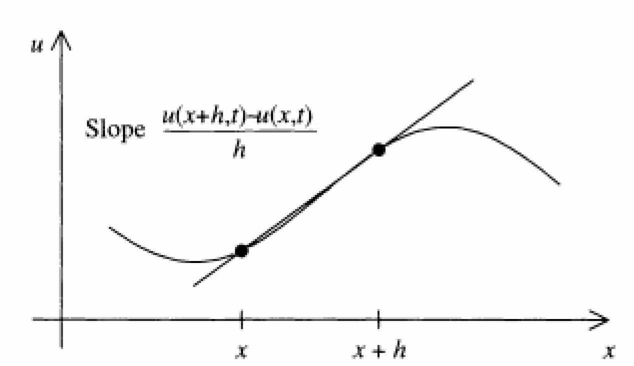


Figure 8.2

Note that this condition restricts how large the positive secant slope can be, and does not prohibit the curve from having steep negative slopes. Furthermore, the bound $\frac{E}{t}$ restricting the size of positive slopes decreases to zero as t increases.

For the initial value problem (8.2) in the previous section, there are an infinite number of shock wave solutions given by (8.3). Figure 8.2 shows the profile of these solutions, and indicates that large positive secant slopes are possible by picking x and x + h on opposite sides of the shock. The secant slope

$\frac{u(x+h,t)-u(x,t)}{h} = \frac{1-A}{h}$

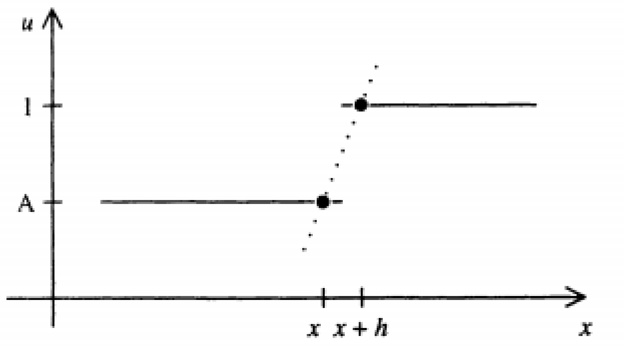


Figure 8.3: Large positive secant slopes occur in the profiles of the shock wave solutions (8.3).

grows arbitrarily large as x and x + h approach the location of the jump, so it is not possible to find a constant E such that this secant slope is less than E/t for all x and $h > 0$. The shock wave solutions (8.3) do not satisfy the entropy condition.

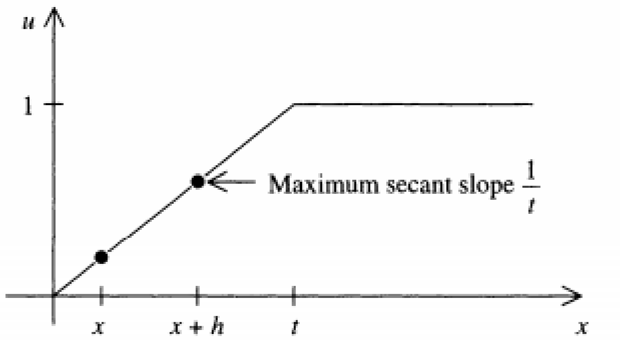


Figure 8.4: Maximum positive secant slope is $\frac{1}{t}$ in the profiles of the rarefaction wave (8.1).

The rarefaction wave (8.1), however, does satisfy the entropy condition. The profile of this function at time t shown in Figure 8.3 indicates that a maximum positive secant slope of $\frac{1}{t}$ occurs when x and x + h are between 0 and t. For this function the entropy condition is met by picking E=1 , since

$\frac{u(x+h,t)-u(x,t)}{h} = \frac{1}{t}$

The entropy condition would then select this rarefaction wave solution over the shock waves solutions in the initial value problem (8.2).

The entropy condition plays an important role in the design of numerical methods for constructing approximations to solutions of conservation laws. Since a conservation law may possess several solutions, care must be taken to ensure that the numerical method not only converges, but converges to the desired solution. For further reading on the entropy condition, its variations, and its role in numerical algorithms.

Chapter 9

Weak Solutions of Conservation Laws

9.1 Classical Solutions

Constructing solutions of conservation laws by piecing together shocks and rarefactions can become quite tedious if the initial condition is anything more than a very simple function. Furthermore, constructing a particular solution is sometimes not as important as determining more general properties of the conservation law. In this chapter the weak form of a conservation law is introduced as an alternative to the differential equation form $u_t + \phi_x = 0$. This view of the conservation law has several mathematical advantages over the differential equation form.

The solutions of differential equations that we have focused on are often called classical solutions in order to distinguish them from the weak solutions described in the next section. Consider the general initial value problem

$u_t + \phi_x = 0, -\infty < x < \infty, t > 0$

$u(x, 0) = u_0(x),$

where $\phi(x, t)$ has continuous first derivatives and u_0(x) is continuous. A function u(x, t) is called a classical solution of this initial value problem if

(a) u is continuous for all x and $t > 0$,

(b) u_x and u_t exist and are continuous for all x and $t > 0$,

(c) u satisfies $u_t + \phi_x = 0$ for all x and $t > 0$, and

(d) u(x,0) $= u_0$(x) for all x.

The notion of weak solution will allow us to proceed directly to functions u(x, t) which are not necessarily continuous or differentiable, but are solutions in a different sense.

9.2 The weak form of a conservation law

The weak form of $u_t + \phi_x = 0$ is an alternative integral form of the conservation law. The underlying idea is to use special functions of x and t, called test functions, to examine the solution of $u_t + \phi_x = 0$ in regions of the xt—plane. A real valued function T(x, t) is called a Test Function if

(a) T_t and T_x exist and are continuous for all (x,t), and

(b) there is some circle in the xt—plane such that T(x, t) $= 0$ for all (x,t) on or outside the circle.

An example of a test function is

$$T(x,t) = \begin{cases} e^{\frac{-1}{1-x^2-t^2}}, & \text{if} x^2 + t^2 < 1, \\ 0, & \text{if} x^2 + t^2 \geq 1, \end{cases}$$

whose graph is shown in Figure 9.1. The exponential decay of T(x, t) to zero as(x,t) approaches the boundary of the circle $x^2 + t^2 = 1$ from the inside leads to this function having continuous first derivatives T_t and T_x for all (x,t), even on the unit circle.

To derive the weak form of a conservation law, begin by assuming that u is a classical solution of

$$u_t + \phi_x = 0, -\infty < x < \infty, t > 0 \tag{9.1}$$

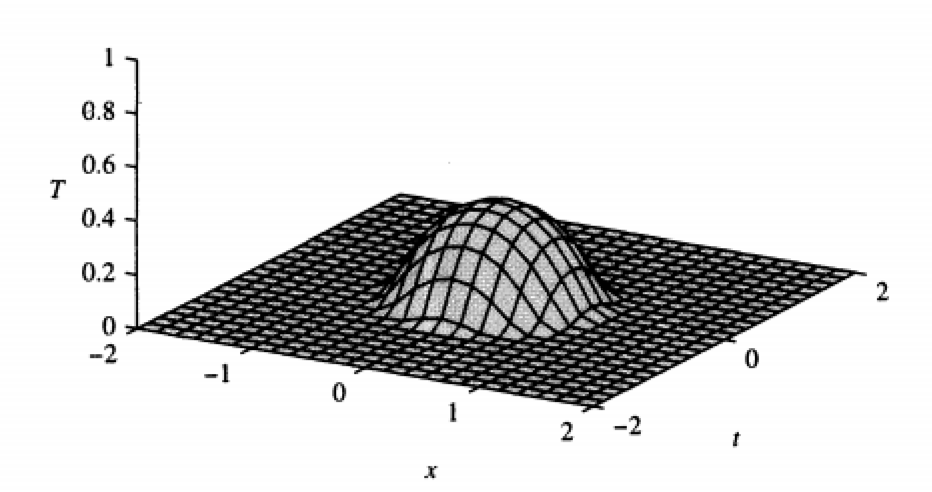


Figure 9.1: A test function.

$$u(x,0) = u_0(x) \tag{9.2}$$

Multiplying the differential equation (9.1) by T(x,t) and integrating over all possible x and all t $\geq 0 gives$

$$\int_0^\infty \int_{-\infty}^\infty [u_t(x,t)T(x,t) + \phi_x(x,t)T(x,t)]dxdt = 0 \tag{9.3}$$

The left side can be written as the sum of two integrals I_1 and I_2, where

$I_1 = \int_0^\infty \int_{-\infty}^\infty [u_t(x,t)T(x,t)dxdt$
$I_2 = \int_0^\infty \int_{-\infty}^\infty [\phi_t(x,t)T(x,t)dxdt$

Interchanging the order of integration in the double integral I_1 and applying integration by parts to the resulting inside integral rewrites I_2 as

$I_1 = \int_{-\infty}^\infty [\int_0^\infty [u_t(x,t)T(x,t)dt]dx$
$= \int_{-\infty}^\infty [u(x,t)T(x,t)|_{t=0}^{t\to\infty} - \int_0^\infty u(x,t)T_t(x,t)dt]dx$

The value of u(x, t)T(x, t) is zero as $t \to \infty$ since T(x, t) is zero for all (x, t) outside some circle in the xt—plane. The value of u(x, 0)T(x, 0) is $u_0(x)T(x,0)$ by the initial condition (9.2). The expression for I_1 is then

$$I_1 = -\int_{-\infty}^{\infty} [u_0(x)T(x,0)dx - \int_0^{\infty}\int_{-\infty}^{\infty} u(x,t)T_t(x,t)dxdt \tag{9.4}$$

A similar calculation can be carried out for I_2. Applying integration by parts to the inside integral of the double integral I_2 results in

$I_2 = \int_0^{\infty}[\int_{-\infty}^{\infty} \phi_x(x,t)T(x,t)dx]dt$

$= \int_0^{\infty}[\phi(x,t)T(x,t)|_{t\to-\infty}^{t\to\infty} - \int_{-\infty}^{\infty} \phi(x,t)T_x(x,t)dx]dt$

The value of u(x, t)T(x, t) is zero as $x \to \pm\infty$ since T(x, t) is zero for all (x, t) outside some circle in the xt—plane, so

$$I_2 = -\int_0^{\infty}\int_{-\infty}^{\infty} \phi(x,t)T_x(x,t)dxdt \tag{9.5}$$

Using the two calculations (9.4) and (9.5) for I_1 and I_2, the integral of the conservation law $u_t + \phi_x = 0$ in (9.3) can be rewritten as

$$\int_0^{\infty}\int_{-\infty}^{\infty}(u(x,t)T_t(x,t) + \phi(x,t)T_x(x,t))dxdt + \int_{-\infty}^{\infty}[u_0(x)T(x,0)dx = 0 \tag{9.6}$$

This is called the weak form of the initial value problem (9.1 and 9.2) for the conservation law $u_t + \phi_x = 0$.

Note that the weak form (9.6) does not involve any derivatives of u(x,t). A weak solution of the initial value problem (9.1 and 9.2) is a function u(x,t) which satisfies (9.6) for every test function T(x,t). For a weak solution there is no requirement that ut or u_x even exist. Furthermore, the partial differential equation and the initial condition in (9.1 and 9.2) are both accounted for in this single equation.

★*Example*

Consider the initial value problem

$u_t + u^2 u_x = 0, -\infty < x < \infty, t > 0$

$u(x,0) = \frac{1}{1+x^2}$

The flux for this conservation law is $\phi(u) = \frac{u^3}{3}$. Taking this flux and the initial function $u_0(x) = \frac{1}{1+x^2}$ in (9.6) gives the weak form of the initial value problem as

$\int_0^\infty \int_{-\infty}^\infty (u(x,t)T_t(x,t) + \frac{1}{3}u^3(x,t)T_x(x,t))dxdt + \int_{-\infty}^\infty \frac{T(x,0)}{1+x^2}dx = 0$

for all test functions T(x,t).

Chapter 10

Shock Wave-II

WE OBTAINED THE solution of the equation

$\rho_t + q_x = 0$

on the assumptions

(1) ρ and q are continuously differentiable.

(2) There exists a functional relation between q and ρ; that is q = Q(ρ).

In our discussions we found the phenomena of breaking in some cases. At the time of breaking we have to reconsider our assumptions.

We will approach this in two directions.

(i) We still assume a functional relation between q and ρ i.e. q = Q(ρ), but allow jump discontinuities for ρ and q.

(ii) We assume ρ and q are continuously differentiable and q is a function of ρ and $_x$. For simplicity we take this in the form

$q = Q(\rho) - v\rho_x$, where v¿0.

We already discussed (i) in Shock Wave-I part.

10.1 Equal Area Rule

The general question of fitting in a discontinuous shock to replace a multivalued region can be answered elegantly by the following argument. The integrated form of the conservation equation, i.e. ,

$$\frac{d}{dt}\int_{x_2}^{x_1} \rho dx + q_1 - q_2 = 0$$

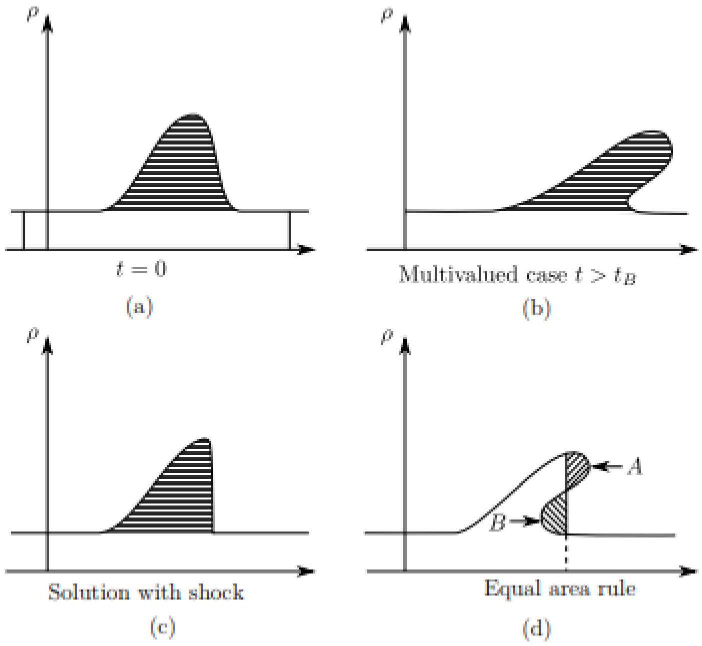


Figure 10.1

holds for both the multivalued solution and the discontinuous solution. If we take the case of a single hump disturbance as shown in the figure 10.1(a), with $\rho = \rho_0$ on both sides of the disturbance, and if we take x_1, x_2 far away from the disturbance with $q_1 = q_2 = Q(\rho_0), then$

$$\int_{x_2}^{x_1} \rho dx = \text{Constant in time}$$

This is so for both the multivalued solution in figure 10.1(b) and the discontinuous solution in figure 10.1(c). Hence the position of the shock must be chosen to give equal areas A = B for the two lobes as shown in figure 10.1(d).

10.2 Shock Fitting

10.2.1 Quadratic Q(ρ)

This determination, although quite general, is not in a convenient form for analytic work. The general case gets complicated and it is worthwhile to do a special case first. The special case is again a quadratic expression for Q(ρ). This includes the case of weak disturbances about a value $\rho = \rho_0$, since Q(ρ) can then be approximated by

$Q = Q(\rho_0) + Q'(\rho_0)(\rho - \rho_0) + \frac{1}{2}Q''(\rho_0)(\rho - \rho_0)^2$

and for this reason it has considerable generality.

We consider

$Q(\rho) = \alpha\rho^2 + \beta\rho + \gamma$

Then

$c(\rho) = Q'(\rho) = 2\alpha\rho + \beta$

and the shock velocity

$U = \frac{Q(\rho_2) - Q(\rho_1)}{\rho_2 - \rho_1}$ becomes

$U = \frac{1}{2}(c_1 + c_2)$

where $c_1 = c(\rho_1), c - 2 = c(\rho_2)$.

10.2.2 Asymptotic Behavior

We are interested in finding out what happens to the solution as t$t \to \infty$, and this can be obtained directly without going through the previous construction in detail. We first study a special Q(ρ) which simplifies the results.

The equation is,

$$\rho_t + q_x = 0 \tag{10.1}$$

with the shock condition

$$-U[\rho] + [q] = 0 \tag{10.2}$$

If $q = Q(\rho)$ and $c(\rho) = Q'(\rho)$ then, as noted already, (10.1) can be written as

$C_t + CC_x = 0$, or $C_t + (\frac{1}{2}C^2)_x = 0$

where C(x, t) = c(ρ(x, t)). From the second form of the equation for C, we may be tempted to write the shock condition (10.2) as

$$-U[C] + [\frac{1}{2}C^2] = 0 \tag{10.3}$$

But this is not always true,(It depends on the physical problem and integral terms) i.e. conservation of ρ does not imply the conservation of C. However, when Q is quadratic, say,

$Q(\rho) = \alpha\rho^2 + \beta\rho + \gamma$

then conservation of ρ implies the conservation of C, since C is linear in ρ.

This can be easily checked as follows: We have

$c(\rho) = Q'(\rho) = 2\alpha\rho + \beta$

by equation (10.2)

$$-U[\rho] + [\alpha\rho^2 + \beta\rho + \gamma] = 0 \tag{10.4}$$

Now from (10.3)

$-U[C] + [\frac{1}{2}C^2] = 0$

$\Rightarrow -U[2\alpha\rho + \beta] + [\frac{1}{2}(2\alpha\rho + \beta)^2] = 0$

$\Rightarrow -2\alpha U[\rho] + \frac{1}{2}[4\alpha^2\rho^2 + 4\alpha\beta\rho] = 0$

$\Rightarrow 2\alpha(-U[\rho] + [\alpha\rho + \beta\rho + \gamma]) = 0$

where, $[\beta] = [\frac{1}{2}\beta^2] = [\gamma] = 0$ since β and γ are constants.

In this case we can work with C alone and the shock condition is

$U = \frac{c_1+c_2}{2}$

The initial value problem is

$$\begin{cases} C_t + CC_x = 0, t > 0, -\infty < x < \infty \\ C = F(x), t = 0 : -\infty < x < \infty. \end{cases} \tag{10.5}$$

We will now consider the asymptotic behavior of a single hump, i.e

$$\text{F}(\xi)=\begin{cases} c_0, & \text{in} x \le a; \\ g(x), & \text{in}[a, L]; \\ c_0, & \text{in} x \ge L. \end{cases}$$

where g is continuous in [a, L] with g(a) = g(L) = c_0, as shown in figure 10.1(a).

In this case breaking will occur at the front and we fit a shock to remove multi-valuedness. As time increases, much of the initial detail is lost. As this process is continued, it is plausible to reason that the remaining disturbance becomes linear in x. In any event, there is such a simple solution with C $=\frac{x}{t}$. We propose, therefore, that the solution is

$$C = \begin{cases} c_0, & \text{x} \le c_0 a; \\ \frac{x}{t}, & c_0 t \le x \le s(t); \\ c_0, & s(t) < x. \end{cases} \tag{10.6}$$

where x=s(t) is the position of the shock still to be determined.

The shock condition is U $=\frac{c_1+c_2}{2}$; therefore, since $c_1 = c_0, c_2 = \frac{s(t)}{t}$, we have

$$\frac{ds}{dt} = \frac{1}{2}(c_0 + \frac{s}{t}) \tag{10.7}$$

The solution of this is easily found to be

$S = c_0 t + bt^{\frac{1}{2}}$

where b is a constant. So we have a triangular wave for C as shown in figure 10.2. The area of the triangle is $\frac{1}{2}b^2$ and this must remain equal to the area A under the initial hump. Hence $b = (2A)^{1/2}$. Only the area of the initial wave appears in this final asymptotic solution; all other details are lost. It shold be remarked that this behavior is completely different from linear theory.

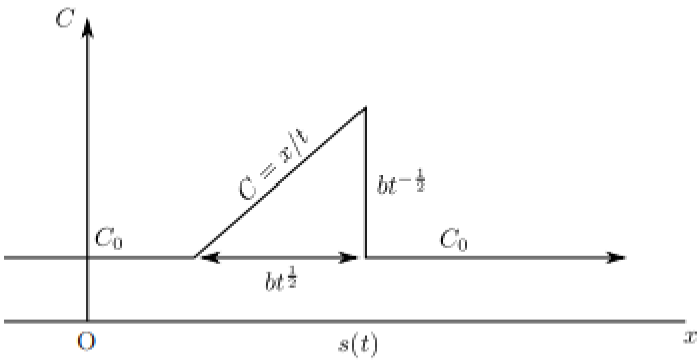


Figure 10.2

★$\underline{Triangular Wave}$:- A triangular wave is a non-sinusoidal wave form named for it's triangle shape. It is periodic, peicewise linear, continuous real function.

The wave fig of sine, square, Triangle and sawtooth waves are given below. .

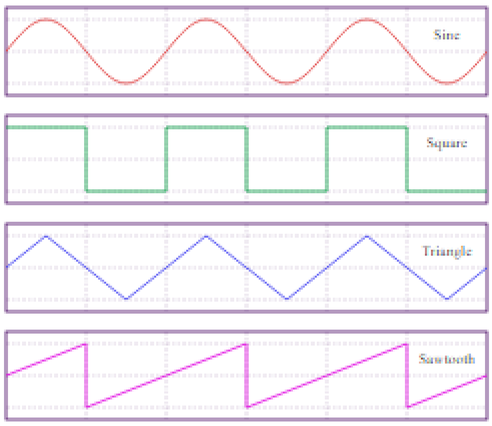


Figure 10.3

10.3 Shock structure

In the first approach to resolve breaking we have assumed a functional relation in and q with appropriate shock conditions. Now we consider the second approach, namely that q and ρ are continuously differentiable but that q is a function of ρ and ρ_x. For simplicity we take

$$q = Q(\rho) - v\rho_x \tag{10.8}$$

where > 0. (Here the sign of is important). When ρ_x is small, q = Q(ρ) is a good approximation; but near breaking where ρ_x is large, (10.8) gives a better approximation. A motivation for (10.8) can be seen from traffic flow. In traffic flow, the density ρ is the number of cars per unit length. When the density is increasing ahead,then the number of cars is increasing in a fixed length, so $\rho_x > 0$. For that $q_1 = Q(\rho)$-(+ve term), i.e, $q_1 < q$(where q=Q(ρ) is equilibrium condition). So a little below equilibrium.

Similarly, when density is decreasing then, $\rho_x < 0$. For that $q_2 = Q(\rho)$-(-ve term), i.e, $q_2 > q$. So a little above equilibrium. This is represented by the extra term ρ_x in (10.8). The other examples in chapter have similar correction terms in an improved description.

The conservation equation is

$\frac{d}{dt}\int_{x_2}^{x_1} \rho dx + q_1 - q_2 = 0$

and for differentiable ρ, q we have the differential equation

$$\rho_t + q_x = 0 \tag{10.9}$$

as before. Using (10.8), (10.9) becomes

$$\rho_t + c(\rho)\rho_x = v\rho_{xx} \tag{10.10}$$

where $c(\rho) = Q'(\rho)'$. Before considering the solution of (10.10) in detail, we note the general qualitative effects of the terms $c(\rho)\rho_x$ and ρ_{xx}. To see this we take the initial function to be a step function.

$$t = 0 : \rho = \begin{cases} \rho_2, & \text{if} x < 0; \\ \rho_1, & \text{if} x > 0 \end{cases} \tag{10.11}$$

with $\rho_2 > \rho_1$. Omitting the term $c(\rho).\rho_x$, the equation (10.10) becomes the heat equation,

$$\rho_t = v\rho_{xx} \tag{10.12}$$

The solution to (10.12) with the initial conditions (10.11) is

$$\rho = \rho_2 - \frac{\rho_2 - \rho_1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4vt}}} e^{-\xi^2} d\xi \tag{10.13}$$

This shows that the effect of the term ρ_{xx} is to smooth out the initial distribution like $(t)^{-\frac{1}{2}}$. Neglecting the term ρ_{xx} in (10.10) we have the immediate breaking discussed earlier. Thus our equation (10.10) will have both the effects, namely stretching and steepening, and it seems reasonable that there will be solutions having the balance between the two. We will now look for simple solutions to test the idea. Let us assume that

We now look within the framework of this more accurate theory . One obvious idea is to look for a steady profile solution in which

$$\rho = \rho(X), X = x - Ut \tag{10.14}$$

where U is constant.

is a solution of (10.10). We also assume that

$$\begin{cases} \rho \to \rho_1, & \text{as} x \to \infty \\ \rho \to \rho_2, & \text{as} x \to -\infty \\ \rho_x \to 0, & \text{as} x \to \pm\infty \end{cases} \tag{10.15}$$

Now (10.10) becomes

$$c(\rho)\rho_X - U_{\rho X} = v\rho_{XX} \tag{10.16}$$

Using $c(\rho) = Q(\rho)$ and integrating (10.16) with reference to X we obtain

$$Q(\rho) - U\rho + A = v\rho_x \tag{10.17}$$

where A is the constant of integration. Equations (10.17) and (10.15) imply

$U = \frac{Q(\rho_2)-Q(\rho_1)}{\rho_2-\rho_1}$

which is exactly the same as the shock velocity in the discontinuity theory. Equation (10.17) can be written as

$\frac{1}{v} = \frac{1}{Q(\rho)-U\rho+A}\frac{d\rho}{dX}$

Integrating this with reference to X we get

$$\frac{X}{v} = \int \frac{d\rho}{Q(\rho) - U\rho + A} \tag{10.18}$$

Since ρ_1, ρ_2 are zeroes of $Q(\rho)U\rho+A$ the integrals taken over the neighbourhoods of ρ_1, ρ_2 diverge; so $X \to \pm\infty$ as $\rho \to \rho_1$ or ρ_2. This is consistent with our assumptions (10.15).

The values ρ_1, ρ_2 are zeros of $Q(\rho) - U\rho + A$, and in general they are simple zeros. As $p \to \rho_1$ or ρ_2 in (10.18), the integral diverges and $X \to \pm\infty$ as required. If $Q(\rho) - U\rho + A < 0$ between the two zeros, and if v is positive, we have $\rho_X < 0$ and the solution is as shown in Fig. 10.4 with ρ increasing monotonically from ρ_1, at $+\infty$

to ρ_2 at $-\infty$. If $Q(\rho) - U\rho + A > 0$ and $v > 0$, the solution increases from ρ_2 at $-\infty$ to ρ_1 at $+\infty$.

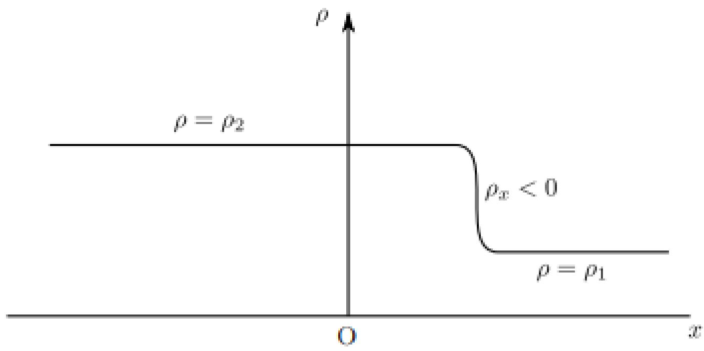


Figure 10.4: Shock Structure

It is clear from (3.22) that if ρ_1,ρ_2 are kept fixed (so that U,A are fixed), a change in v can be absorbed by a change in the X scale. As $v \to 0$, the profile in Fig. 10.4 is compressed in the X direction and tends in the limit to a step function increasing ρ from ρ_1, to ρ_2 and traveling with the velocity given by Rankine-Hugoniot condition. For small nonzero v the shock is a rapid but continuous increase taking place over a narrow region. The breaking due to the nonlinearity is balanced by the diffusion in this narrow region to give a steady profile.

One very important point is the sign of the change in rho. A continuous wave carrying an increase of ρ will break forward and require a shock with $\rho_2 > \rho_1$ if $c'(\rho) > 0$; it will break backward and require a shock with $\rho_2 < \rho_1$ if $c'(\rho) < 0$. The shock structure given by (3.22) must agree. As remarked above, v is always positive for stability, so the direction of increase of ρ depends on the sign of $Q(\rho) - U\rho + A$ between the two zeros ρ_1, and ρ_2.

But $c'(\rho)$=$Q''(\rho)$. Hence when $c'(\rho) > 0$, $Q(\rho) - U\rho + A < 0$ between zeros and the solution is as seen in Fig.10.4 with $\rho_2 > \rho_1$ as required. If $c'(\rho) < 0$, the step is reversed and $\rho_2 < \rho_1$. The breaking argument and the shock structure agree.

An explicit solution for (10.10) and (10.11) can be obtained when Q is the quadratic

$Q = \alpha\rho^2 + \beta\rho + \gamma$

Then

$Q(\rho) - U\rho + A = -\alpha(\rho - \rho_1)(\rho_2 - \rho)$

and by 10.18

$\frac{X}{v} = -\int \frac{d\rho}{\alpha(\rho-\rho_1)(\rho_2-\rho)} = \frac{1}{\alpha(\rho_2-\rho_1)} \log(\frac{\rho_2-\rho}{\rho-\rho_1})$

Hence we obtain a solution

$$\rho = \rho_1 + (\rho_2 - \rho_1)\frac{e^{-\frac{(\rho_2-\rho_1)\alpha X}{v}}}{1 + e^{-\frac{(\rho_2-\rho_1)\alpha X}{v}}} \tag{10.19}$$

when v is small, the transition region between ρ_1 to ρ_2 is very thin.

Chapter 11

Applications

Here we shall discuss some example in short which are the applicaion of our previous theories. We shall discuss those topic precisely in next.

Our problems which lead to the non-linear equation $\phi_t + c(\phi)\phi_x = 0$.

In most of the problems we relate two quantities:

ρ(x, t)= the density of something per unit length;

q(x, t)= the flow per unit time.

If the 'something' is conserved, then for a section $x_2 \leq x \leq x_1$ we have the conservation equation

$$\frac{d}{dt}\int_{x_2}^{x_1} \rho dx + q(x_1, t) - q(x_2, t) = 0 \tag{11.1}$$

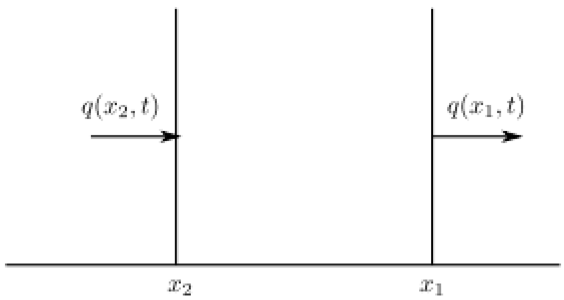


Figure 11.1

If ρ and q are continuously differentiable then in the limit $x_1 \to x_2$, equation (10.1) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{11.2}$$

If there exists also a functional relation $\rho = Q(\rho)$ (this is so to a first approximation in many cases) then (10.2) can be written as

$$\rho_t + c(\rho)\rho_x = 0, \tag{11.3}$$

where $c(\rho) = Q'(\rho)$.

We will now give some specific examples.

11.1 Traffic Flow

We consider the flow of cars on a long highway. Here,

- ρ= The number of cars per unit length.
- V=The average local velocity of the cars.
- q= The flow per unit time, is given by $q = \rho v$.

For a long section of highway with no exits or entrances the cars are conserved so that (10.1) holds. It also seems reasonable to assume that on the average v is a function of ρ to a first approximation. Hence ρ satisfies (10.3). The velocity v will be a decreasing function V(ρ), and Q(ρ) = ρ V(ρ). When the density is small the velocity will be some upper limiting value, and when the density is maximum the velocity will be zero. Therefore the graph of V will take a form as shown in the figure 10.2.

Since q = Q(ρ) = ρV, there will be no flow when the velocity is maximum (i.e. ρ = 0) and when ρ is maximum (i.e. V = 0). Hence the graph of Q(ρ) will look like the figure 10.3.

It was found in one set of observations on U.S. highways that the maximum density is approximately 225 vehicles per mile (per traffic lane), and the maximum flow is

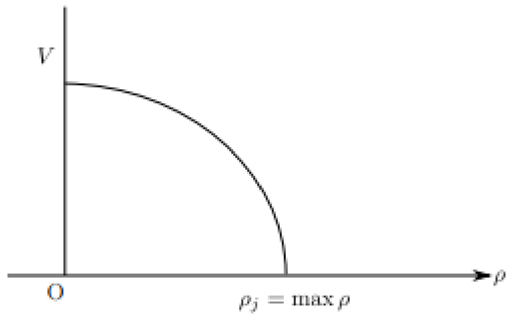


Figure 11.2

approximately 1500 vehicles per hour. When the flow q is maximum the density is found to be around 80 vehicles per mile.

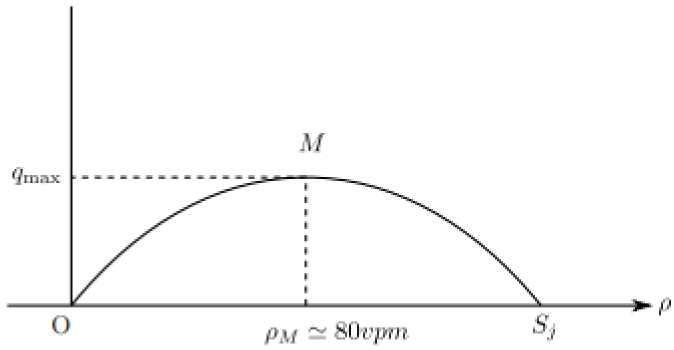


Figure 11.3

The propagation speed for the wave is c(ρ) = $Q'(\rho)$=V (ρ)+ $\frac{dV}{d\rho}\rho$.

Since V is a decreasing function of $\rho, \frac{dV}{d\rho} < 0$. Thus c($\rho$) $< V(\rho)$ i.e. the propagation velocity is less than the average velocity. Relative to individual cars the waves arrive from ahead.

Referring to the Q(ρ) diagram decreasing in $[\rho_M, \rho_j]$, Q attains a maximum at ρ_M. Therefore c(ρ) = Q(ρ) is positive in [0, ρ_M), zero at ρ_M and negative in $(\rho_M, \rho_j]$. That is waves move forward relative to the highway in [0, ρ_M), are stationary at ρ_M and move backward in $(\rho_M, \rho_j]$.

Greenberg in 1959, found a good fit with data for the Lincoln Tunnel in New York by taking

$$Q(\rho) = a\rho \log(\frac{\rho_j}{\rho}) \tag{11.4}$$

with a = 17.2 mph and ρ_j = 228 vpm. For this formula,

$$V(\rho) = \frac{Q(\rho)}{\rho} = a \log(\frac{\rho_j}{\rho}) \tag{11.5}$$

and c(ρ) = Q(ρ) = $a(\log(\frac{\rho_j}{\rho})1) = V(\rho)$ a. Hence the relative propagation velocity is equal to the constant 'a' at all densities and this relative speed is about 17 mph. The values of ρ_M and q_{max} are:

ρ_M=83 vpm and q$_{max}$ = $1430vph$

(Since, $\rho_M = \frac{\rho_j}{e}$ and $q_{max} = \frac{a\rho_j}{e}$)

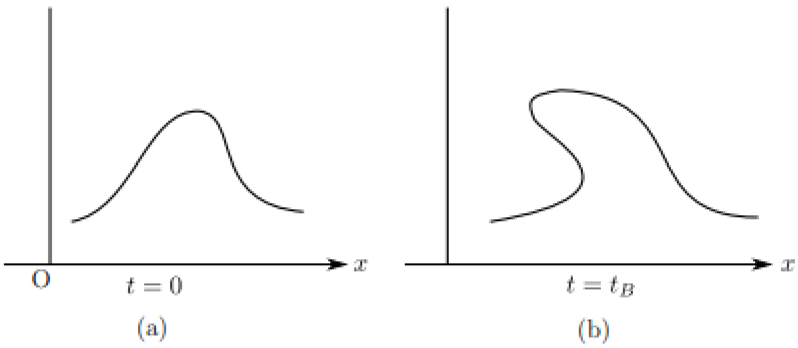


Figure 11.4

Let f be the initial distribution function as shown in the figure 11.4(a). Since $c'(\rho) = V'(\rho) < 0$, breaking occurs on the left. The solution of the problem is

$\rho = \rho(\xi)$

$x = tF(\xi) + \xi$, where $F(\xi) = c(f(\xi))$

Breaking occurs when $F'(\xi) < 0$.

But $F'(\xi) = c'(f(\xi)).f'(\xi) < 0$ iff $f'(\xi) > 0$.

i.e. when f is increasing.

In most other examples c(ρ) > 0, so that a wave of increasing density breaks at the front.

11.2 Flood waves in rivers

Another example comes from an approximate theory for flood waves in rivers. For simplicity we take a rectangular channel of constant breadth, and assume that the disturbance is roughly the same across the breadth. Then the height h(x, t) plays the role of 'density'. Let ρ be the flow per unit breadth, per unit time. Then from the conservation law we have

$$\frac{d}{dt}\int_{x_2}^{x_1} h dx + q_1 - q_2 = 0 \tag{11.6}$$

Taking the limit $x_2 \to x_1$, we obtain

$$h_t + q_x = 0 \tag{11.7}$$

A functional relation q = Q(h) is a good first approximation when the river is flooding. Therefore the governing equation is

$$h_t + c(h)h_x = 0 \tag{11.8}$$

where c(h)=$\frac{dQ}{dh}$

This formula for the wave speed was first proposed by Kleitz and Seddon. The function Q(h) is determined from a balance between gravitational acceleration down the sloping bed and frictional effects. When the function is given by the Chezy formula V $\propto h^{\frac{1}{2}}$. i.e. V $= kh^{\frac{1}{2}}$, where V is the velocity of the flow, we have

$$Q(h) = Vh = kh^{\frac{3}{2}}, c(h) = \frac{3}{2}kh^{\frac{1}{2}} = \frac{3}{2}V \tag{11.9}$$

According to this, flood waves move roughly half as fast again as the stream.

★ Friction is a force that resists motion of sliding or rolling of one object moving relative to another.

11.3 Chemical exchange processes

In chemical engineering various processes concern a flow of fluid carrying some substances or particles through a solid bed. In the process some part of the material in the fluid will be deposited on the solid bed. In a simple formulation we assume that the fluid has constant velocity V. We take density to be $\rho = \rho_f + \rho_s$, where ρ_f is the density of the substance concerned in the fluid and ρ_s is the density of the material deposited on solid bed. The total flow of material across any section is

$$q = \rho_f V \tag{11.10}$$

The conservation equation becomes

$$\frac{\partial}{\partial t}(\rho_f + \rho_s) + V\frac{\partial \rho_f}{\partial x} = 0 \tag{11.11}$$

To complete the system we require more equations. When the changes are slow we can assume to a first approximation that is a quasi - equilibrium between the amounts in the fluid and on the solid and that this balance leads to a functional relation $\rho_s = R(\rho_f)$. Then (11.11) becomes

$$\frac{\partial}{\partial t}\rho_f + c(\rho_f)\frac{\partial}{\partial x}\rho_f = 0 \tag{11.12}$$

where

$$\mathrm{c}(\rho_f) = \tfrac{V}{1+R'(\rho_f)}$$

The relation between ρ_f and ρ_s is discussed in more detail later.

11.4 Glaciers

Nye (1960, 1963) has pointed out that the ideas on flood waves apply equally to the study of waves on glaciers and has developed the particular aspects that are most important there. He refers to Finsterwalder (1907) for the first studies of wave motion on glaciers and to independent formulations by Weertman (1958). For order of magnitude purposes, one may take

$$Q(h) \propto h^N \tag{11.13}$$

with N roughly in the range 3 to 5.

The propagation speed is

$$c = \frac{dQ}{dh} = Nv, \tag{11.14}$$

where v is the average velocity $\frac{Q}{h}$. Thus the waves move about three to five times faster than the average flow velocity. Typical velocities are of the order of 10 to 100 metres per year.

An interesting question considered by Nye is the effect of periodic accumulation and evaporation of the ice; depending on the period, this may refer either to seasonal or climatic changes. To do this, a prescribed source term f(x, t) is added to the continuity equation; that is one takes

$$h_t + q_x = f(x,t), q = Q(h,x). \tag{11.15}$$

The consequences are determined from integration of the characteristic equations

$$\frac{dh}{dt} = f(x,t) - Q_x(h,x), \frac{dx}{dt} = Q_h(h,x). \tag{11.16}$$

The main results in that parts of the glacier may be very sensitive, and relatively rapid local changes can be triggered by the source term.

11.5 Erosion

Erosion in mountains was studied by Luke. Let h(x, t) be the height of the mountain from the ground level. It is reasonable to assume a functional relation between h_t and h_x as:

$$h_t = -Q(h_x) \tag{11.17}$$

(When the slope of the mountain is greater, it is more vulnerable to erosion).

Let

$$s = h_x \tag{11.18}$$

Then differentiating (11.18) with respect to x we obtain

$$h_{tx} = -Q'(h_x)h_{xx}. \tag{11.19}$$

i.e, $\frac{\partial s}{\partial t} + Q'(s)\frac{\partial s}{\partial x}$=0, which is our one dimensional non-linear wave equation with c(s) = Q(s). When breaking occurs we introduce discontinuities in s, which is h_x, and h remains continuous but with a sharp corner.

Chapter 12

Shallow Water Waves(Second Order System)

The extension of the ideas presented so far to higher order systems can be adequately explained on a typical example. We shall use the so called shallow water wave theory for this purpose, although the pioneering work was originally done in gas dynamics.

- Basic Concept

★**Question:-** What happens when waves enter shallow water?

Answer:- Shoaling and refraction of waves occur when the waves are in shallow water. If the water depth less than half the wave length, then the wave is considered to be in shallow water.

λ= Wave Length;

h= Water Depth;

Then $h < \frac{\lambda}{2}$ (More accurately it's $h < \frac{\lambda}{20}$)(But here we shall consider first inequality).

For deep water waves $h \geq \frac{\lambda}{2}$.

When the waves move into shallow water, they begin feel the bottom of the ocean.

★**Question:-** Do waves move faster in shallow or deep water?

Answer:- Briefly, the deeper water, the faster a surface wave will travel and the lower the height will be. As waves comes ashore on the ocean, they slow down and get taller, preserving the amount of energy in the wave. The wave speed and variation with depth also depends on the wavelength.

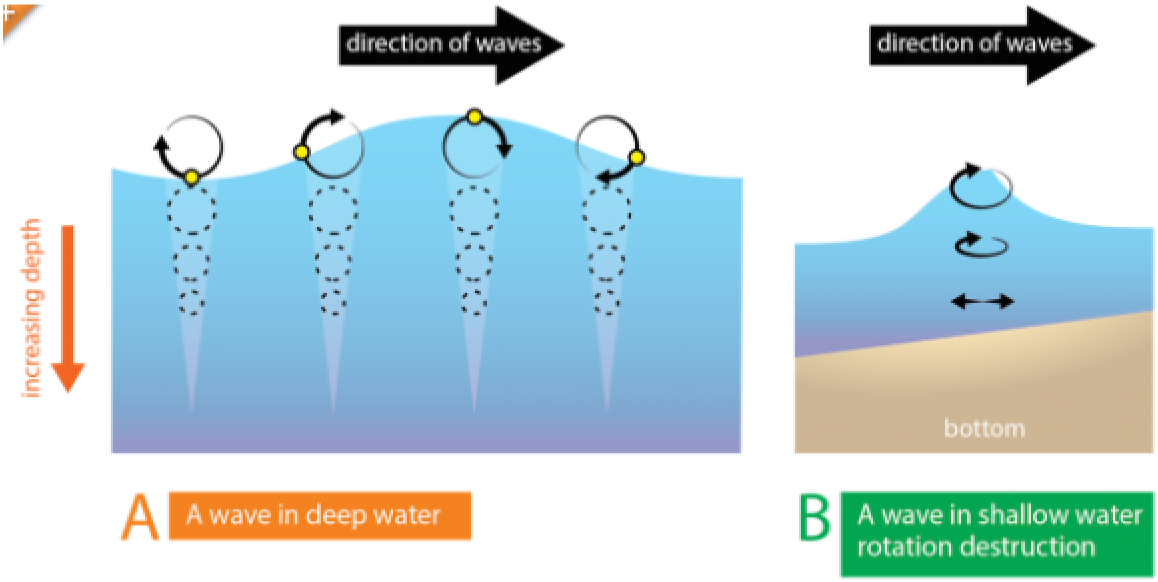


Figure 12.1: Shallow and Deep Water

★**Question:-** Why are waves slower in shallow water?

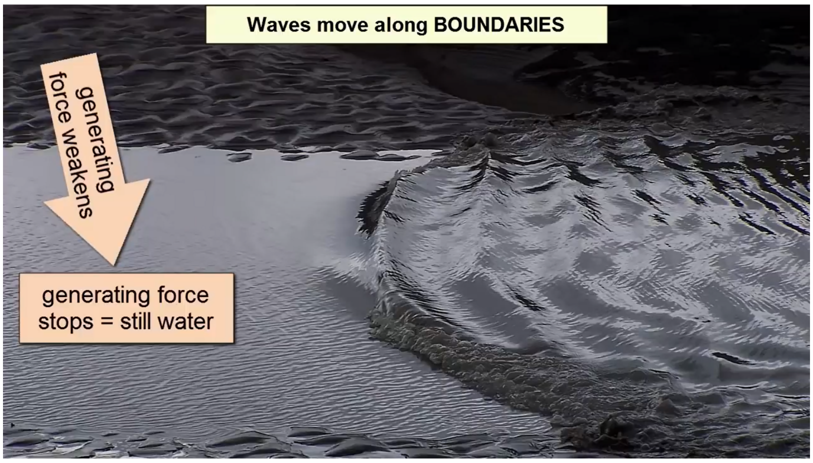
Answer:- In shallow water near the coast, waves slow down because of the force exerted on them by the sea-bed. If a wave is approaching the coast at an angle, the near shore part of the wave slows more than the off shore part of the wave(Because it's shallow water).

In shallow water wave height is more than normal. And waves are slower is shallow water wave.

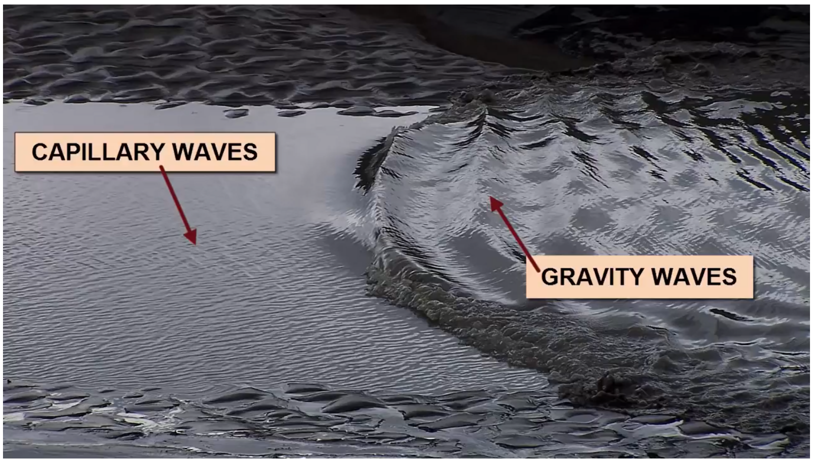
★**Question:-** Are Tsunami's shallow water wave?

Answer:- Tides and tsunamis shallow water waves, even in the deep ocean. The deep ocean is shallow with respect to wave with a wavelength longer them twice the ocean's depth.

Next we are providing some pictures which will make sense about the behaviour of Shallow water, Deep water and some different type of water waves.

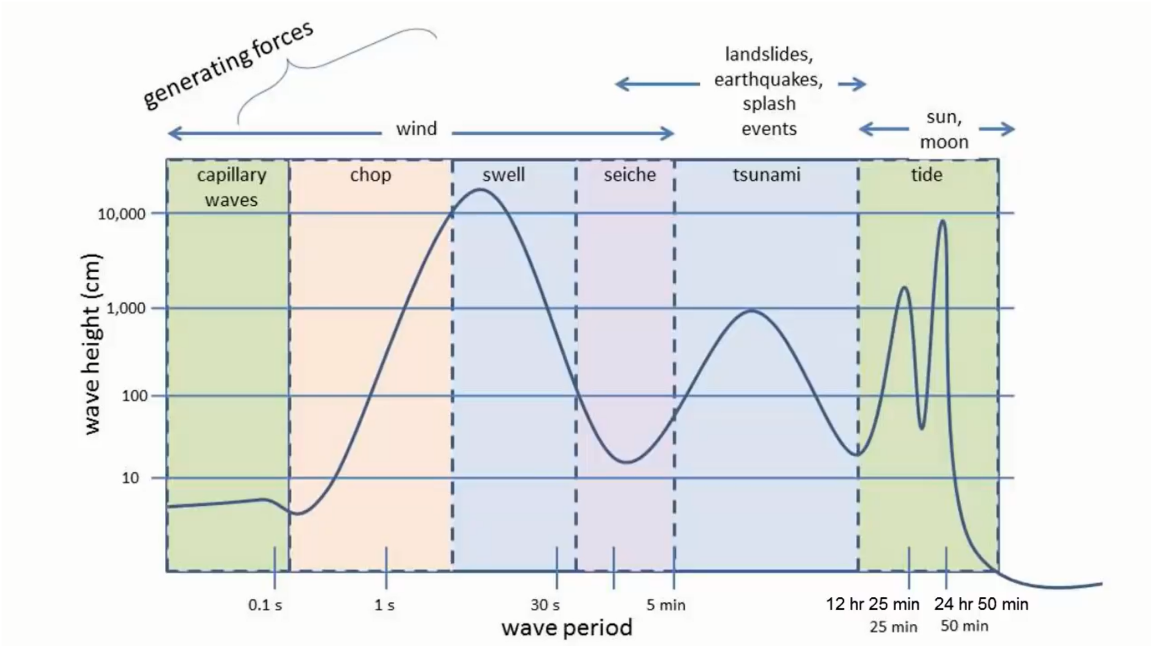


(a)



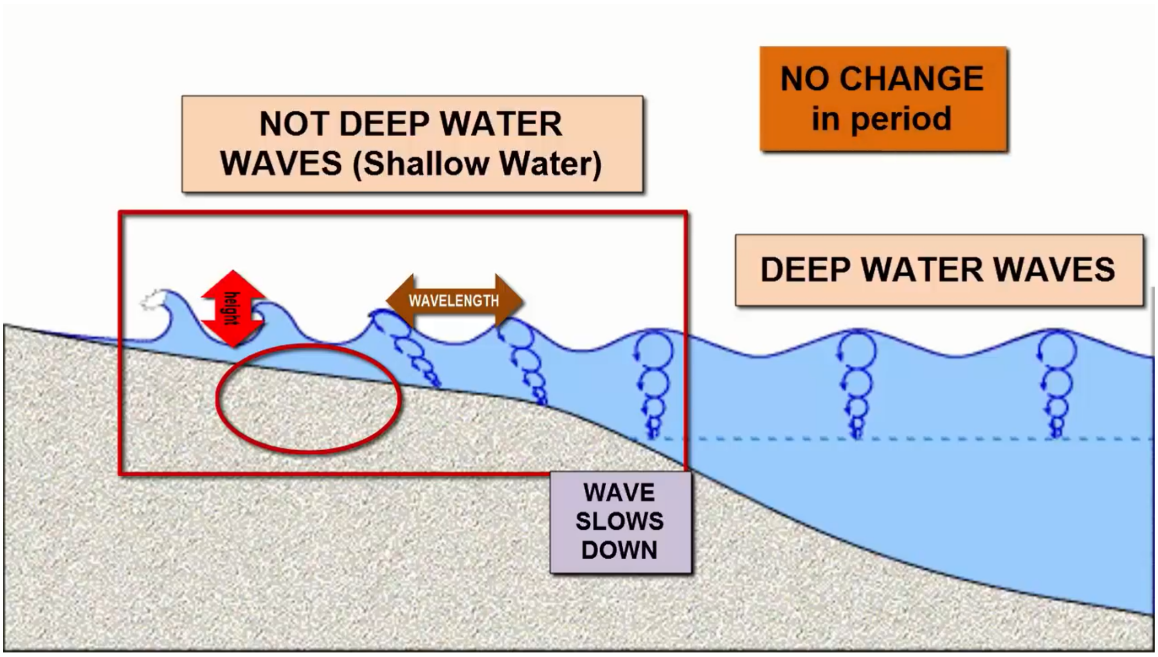
(b)

Figure 12.2: Capillary Waves and Gravity Waves



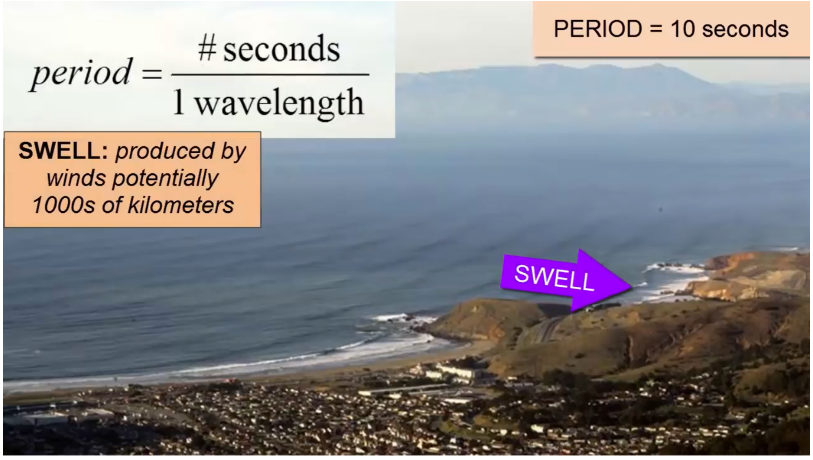
(a)

Figure 12.3: Wave period vs Wave height graph of different types of Waves

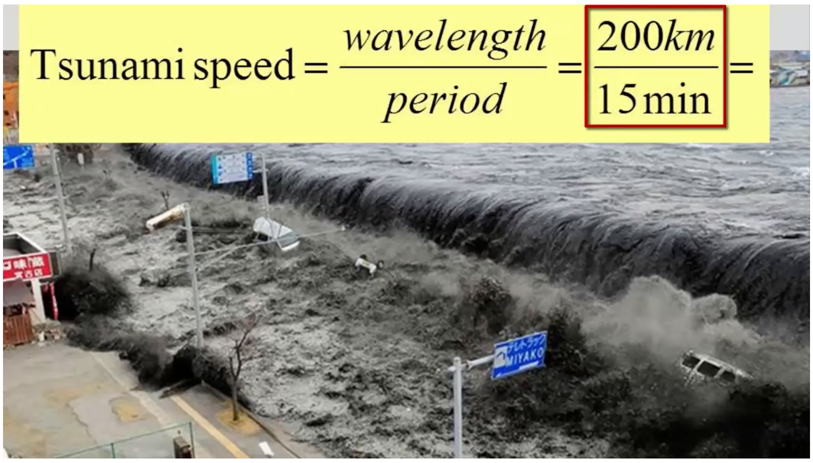


(a)

Figure 12.4: Shallow and Deep water waves



(a)



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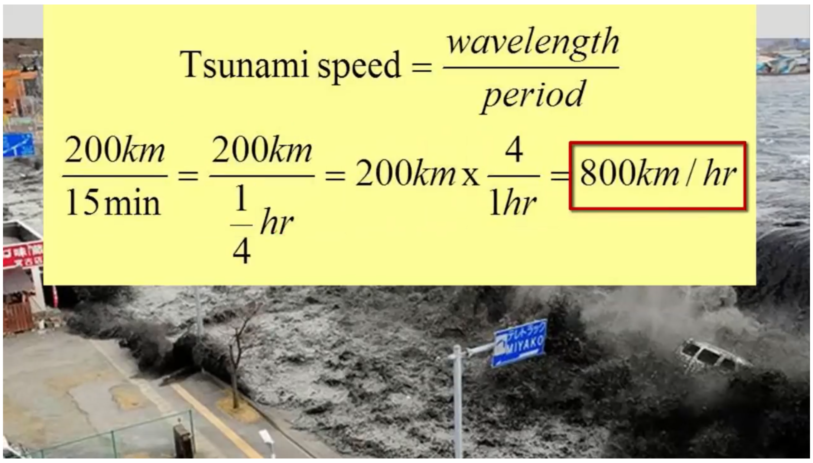
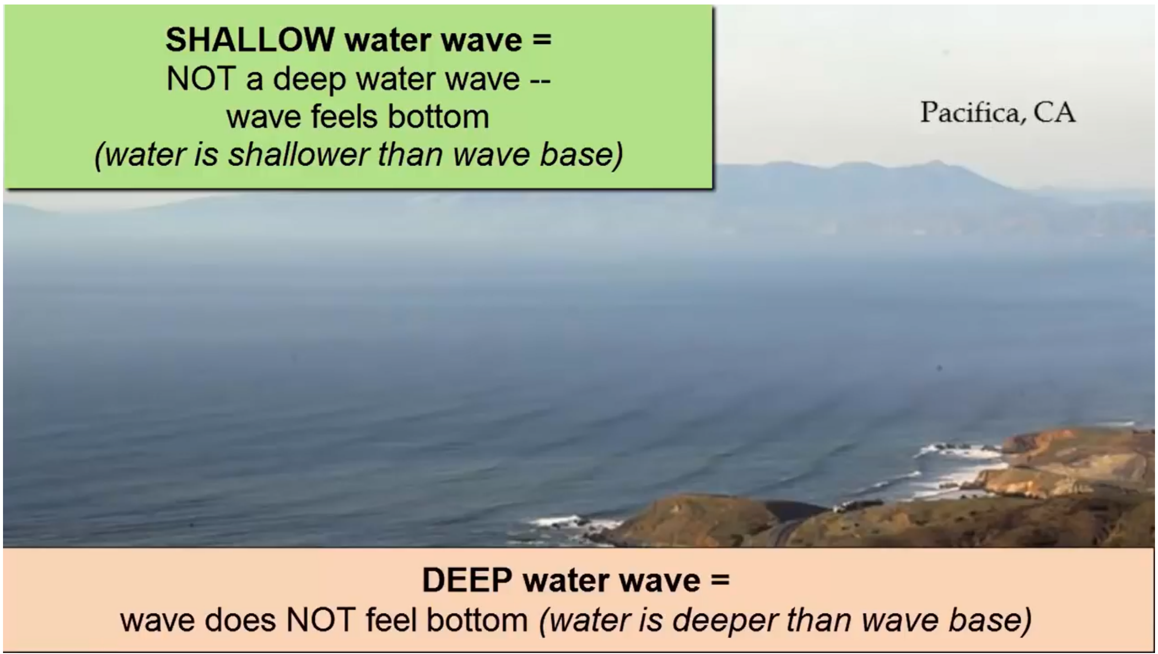


(c)

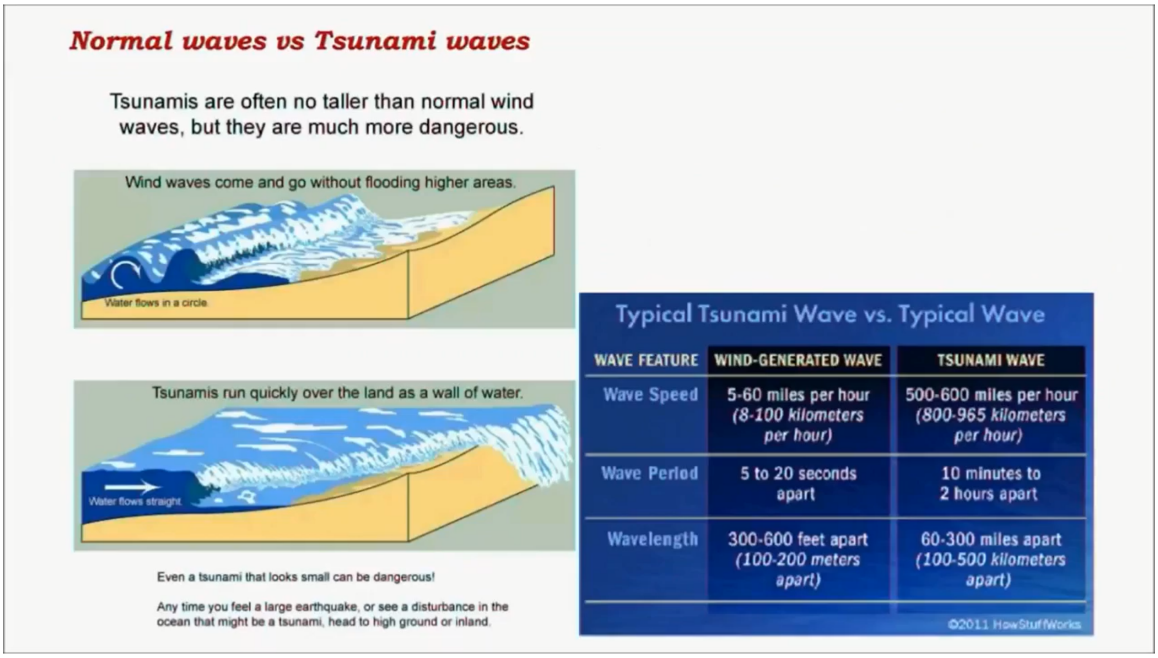
(d)

Figure 12.5: Period, Tsunami speed and Normal ocean water wave speed



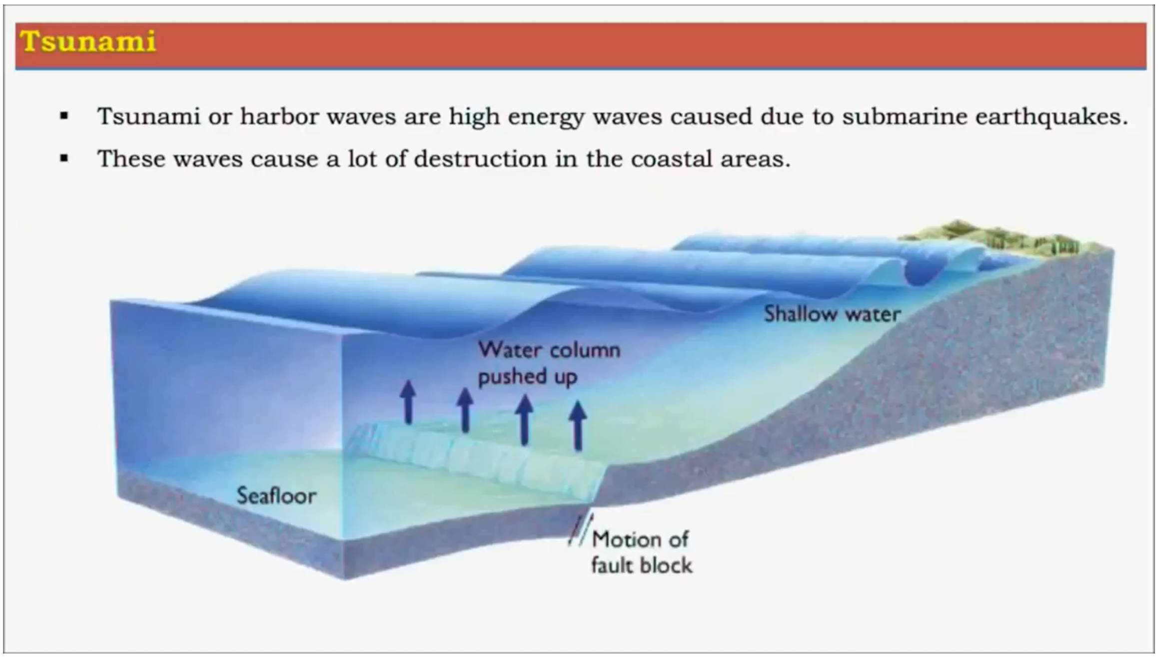
(a)

Figure 12.6: Shallow and Deep ocean waves



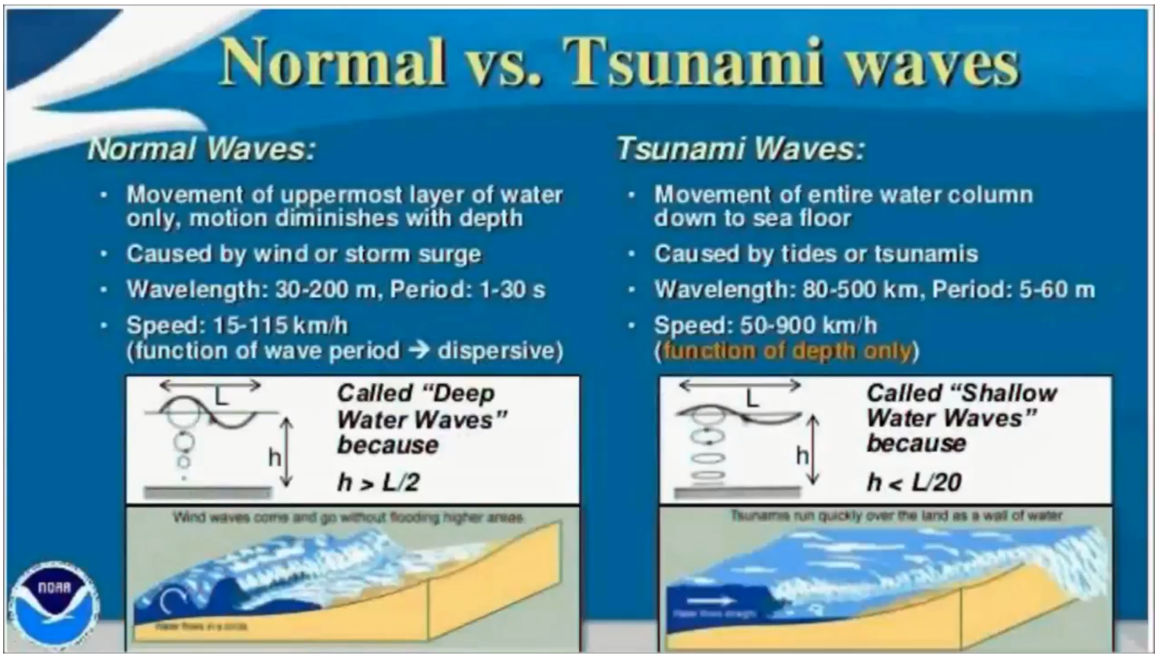
(a)

Figure 12.7: Data of Normal Wave vs. Tsunami Waves



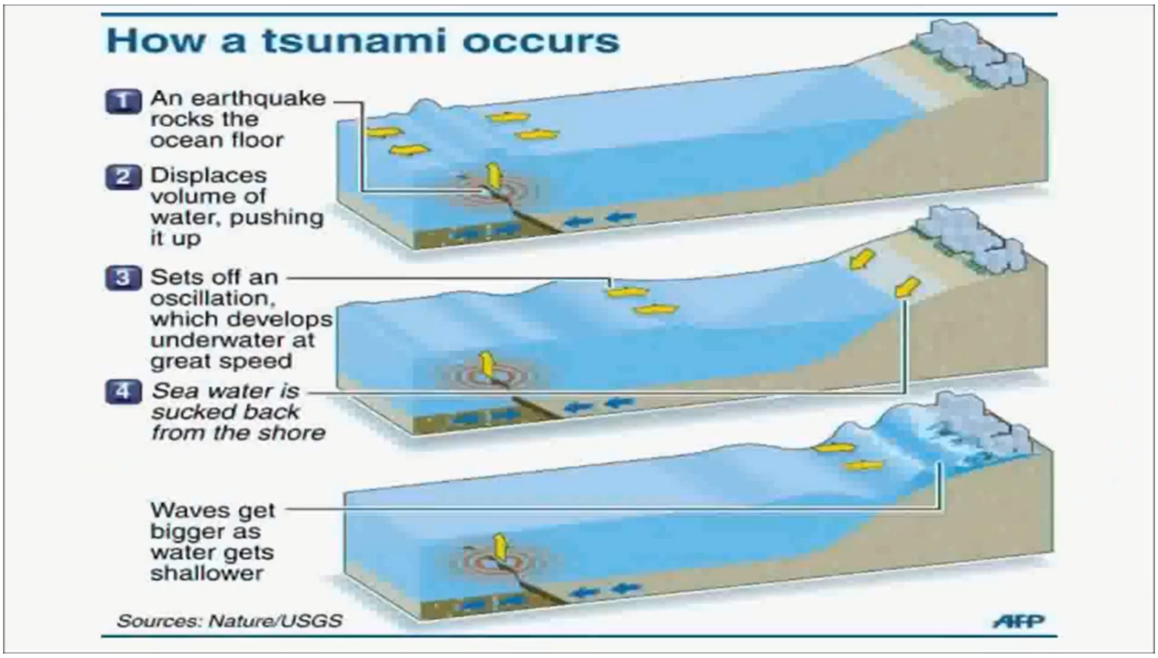
(a)

Figure 12.8: Tsunami Waves moving to the shore



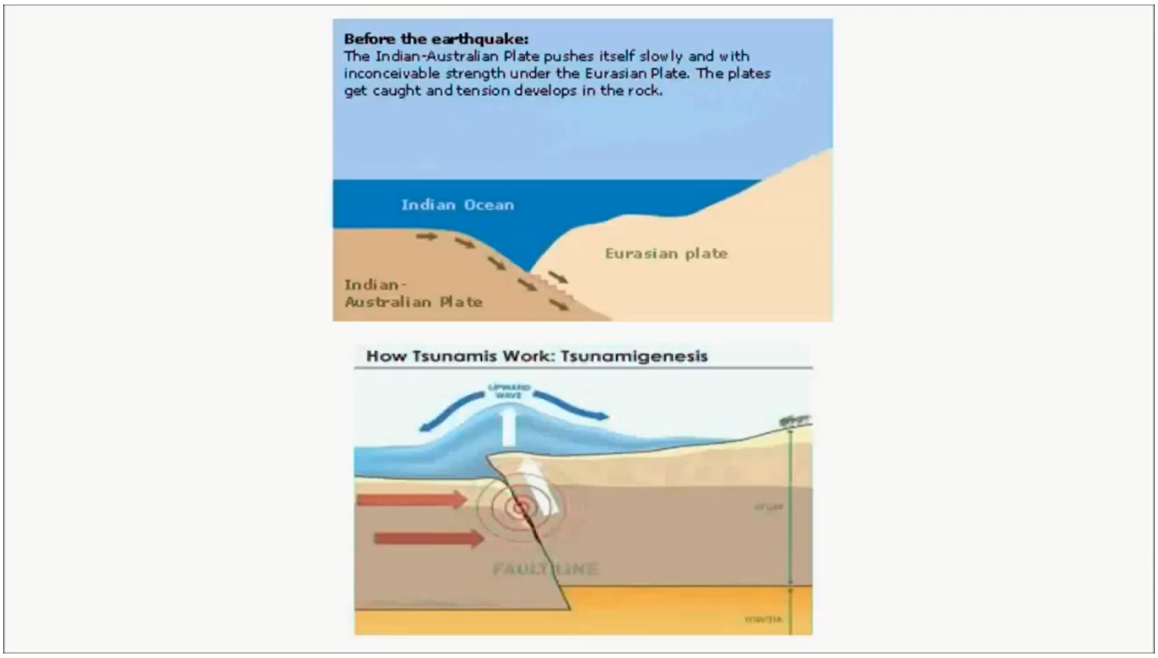
(a)

Figure 12.9: Normal Wave vs. Tsunami Waves



(a)

Figure 12.10: How a Tsunami Waves occures



(a)

Figure 12.11: Example of two plates which causes Tsunami in Indian Ocean

12.1 The equations of shallow water theory

Here let us take,

• The bottom to be horizontal and neglect friction.(★) • The density of the water be normalized to unity and let the width be one unit.(★)

Some notations,

• u(x,t)=Velocity,

• p_0= Atmospheric pressure,

• $p_0 + p'(x,t)$= Pressure in the fluid.

Next go to the page no 97.

In every section $x_2 \leq x \leq x_1$ the mass is conserved, i.e.

$\frac{d}{dt}\int_{x_2}^{x_1} h(x,t)dx + q_1 - q_2 = 0$, where q=uh

• q=Flux of flow,

• h= Height,

• u=Velocity.

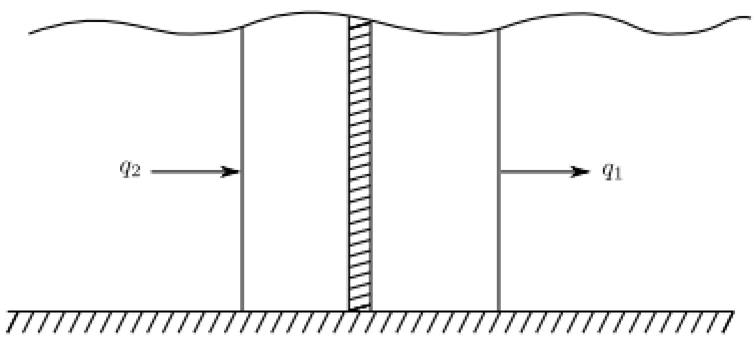


Figure 12.12

Taking the limit $x_2 \to x_1$, we obtain

$h_t + q_x = 0$

i.e,

$$h_t + (uh)_x = 0 \tag{12.1}$$

This time a second relation between u and h is obtained from the conservation of the momentum in the x-direction. If we consider a section $x_2 \leq x \leq x_1$, as shown in figure 12.12, a constant pressure p_0 acting all around the boundary, including free surface and bottom, is self-equilibrating. Therefore, only the excess pressure p contributes to the momentum balance. If P(x, t) denotes the total excess pressure,

$$P = \int_0^h p' dy \tag{12.2}$$

acting across a vertical section, the momentum equation is then

$$\frac{d}{dt}\int_{x_2}^{x_1} hudx = hu^2|_{x_2} - hu^2|_{x_1} + P_2 - P_1 \tag{12.3}$$

where $P_i = P(x_i, t), i = 1, 2$.

•$\underline{NOTE: -Newton's 2nd Law}$

F=m×a,

where • F= Force,

• m=Mass,

• a= Acceleration.

⇒ F=m×$\frac{dv}{dt}$

⇒ $F = \frac{d}{dt}(m \times v)$=Rate of change of momentum.

The term in the left hand side of (12.3) is the total rate of change of momentum in the section $x_2 \leq x \leq x_1$, and $hu^2|_{x=x_i}$, on the right, denotes the momentum transport across the surface through $x = x_i, (i = 1, 2)$.

The basic assumption is shallow water theory is that the pressure is hydrostatic, i.e.

$$\frac{\partial p}{\partial y} = -g \tag{12.4}$$

where, g=Acceleration due to gravity.

Integrating (12.4) and assuming the condition p = p0 at the top y = h,

$\int_{p_0}^{p} \partial p = -\int_{h}^{y} g\partial y$

$\Rightarrow p - p_0 = -g(y-h)$

$\Rightarrow p = p_0 + g(h-y)$

Hence from (12.2) the total excess pressure is

$$P = \int_0^h g(h-y)dy = gh^2 - \frac{gh^2}{2} = \frac{1}{2}gh^2 \tag{12.5}$$

Equations (12.3) and (12.5) yield

$$\frac{d}{dt}\int_{x_2}^{x_1} hudx + [hu^2 + \frac{1}{2}gh^2]_{x_2}^{x_1} = 0 \tag{12.6}$$

The conservation form should be noted.

In the case of river flow discussed earlier, there would also be further terms on the right hand side of (12.6) due to the slope effect and friction; the slope is now omitted and frictional effects are neglected.

In the limit $x_2 \to x_1$, (12.6) becomes

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x = 0 \tag{12.7}$$

Equations (12.1) and (12.7) provide the system for the determination of u and h.

If h and u have jump discontinuities, the shock conditions corresponding to (12.1) and (12.7) (but deduced basically from the original integrated form) are

$$-U[h] + [uh] = 0 \tag{12.8}$$

$$-U[uh] + [hu^2 + \frac{1}{2}gh^2] = 0 \tag{12.9}$$

respectively, where U is the shock velocity.

Using equations (12.1) in (12.7) we obtain

$h_t u + hu_t + (uh)_x u + (uh)u_x + ghh_x = 0$, (Spliting 12.7)

$\Rightarrow h_t u + hU_t - h_t u + (uh)u_x + ghh_x = 0$, $((uh)_x = -h_t)$

$$u_t + uu_x + gh_x = 0 \tag{12.10}$$

equation (12.1) can be written as

$$h_t + uh_x + hu_x = 0 \tag{12.11}$$

(12.10) and (12.11) together is a hyperbolic system of equation.

12.2 Simple Wave

Each of the conservation equations

$h_t + (uh)_x = 0$ and

$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x = 0$

which are earlier form $\rho_t + q_x = 0$

In those earlier cases, a relation $q = Q(\rho)$ was provided in the basic formulation. In the present case, we might ask in relation to (12.1) whether there are solutions in which q = uh is a function of h, where the appropriate functional relation is provided not from outside observations but from the second equation (12.7). We might equally well ask with respect to (12.7) whether there are solutions in which $hu^2 + \frac{1}{2}gh^2$ is a function of hu, where the functional relation is provided by (12.1). The two are equivalent and come down to the question of whether there are solutions in which, say, h is a function of u. We try

$$h = H(u) \tag{12.12}$$

and consider the consistancy of the two equations. We use the simplified equations (12.10) and (12.11) for the actual substitution. (This approach is equivalent to

Earnshaw's approach in gas dynamics). After the substitution h = H(u), we have

$$u_t + uu_x + gH'(u)u_x = 0 \tag{12.13}$$

And

$$H'(u)u_t + uH'(u)u_x + H(u)u_x = 0$$

$$\Rightarrow u_t + uu_x + \frac{H(u)}{H'(u)}u_x \tag{12.14}$$

For consistancy we require (12.14)

$$gH'(u) = \tfrac{H(u)}{H'(u)}$$

which implies

$$\sqrt{g}H'(u) = \pm\sqrt{H} \tag{12.15}$$

- $\underline{CASE-1:}\sqrt{g}H'(u) = +\sqrt{H}$

$$\Rightarrow \sqrt{g}\int_{k=0}^{k=u}\tfrac{d(H(k))}{\sqrt{H(k)}} = \int_{k=0}^{k=u}dk$$

$$2\sqrt{gH} - 2\sqrt{gH_0} = u \tag{12.16}$$

where $H_0 = H(0)$. Then (12.13) becomes

$$u_t + uu_x + \sqrt{gH}u_x = 0$$

$$\Rightarrow u_t + (u + \sqrt{gH})u_x = 0 \tag{12.17}$$

Thus $u + \sqrt{gH}$ is the velocity of propagation.

If we use (12.16) $2\sqrt{gH} - 2\sqrt{gH_0} = u$ and set $c_0 = \sqrt{gH_0}$, equation (12.17) can be written as

$$u_t + (c_0 + \frac{3u}{2})u_x = 0 \tag{12.18}$$

We now have exactly the form discussed in the earlier chapters and can take over the results from there. Equation (12.16) is the functional relation equivalent to $q = Q(\rho)$.

$$2\sqrt{gH} - 2\sqrt{gH_0} = u$$

$$\Rightarrow \sqrt{gH} = \tfrac{u}{2} + \sqrt{gH_0}$$

$$\Rightarrow H(u) = (\tfrac{u}{2\sqrt{g}} + \sqrt{H_0})^2$$

• $\underline{CASE - 2:}$ $\sqrt{g}H'(u) = -\sqrt{H(u)}$

Then similarly the equation becomes

$$2\sqrt{gH} - 2\sqrt{gH_0} = -u; where H_0 = H(0) \tag{12.19}$$

And similarly the hyperbolic equation becomes

$$u_t + (u - \sqrt{gH})u_x = 0 \tag{12.20}$$

Each of these equations represents a so called 'simple wave'. The choice of signs in (12.16) and (12.18) correspond to wave moving to the right, the other signs correspond to one moving to the left.

•$\underline{Example: -}$

We consider a piston 'wave maker' moving parallel to the x-axis in the negative direction with given velocity. Initially when the piston is at rest, the water is at rest.

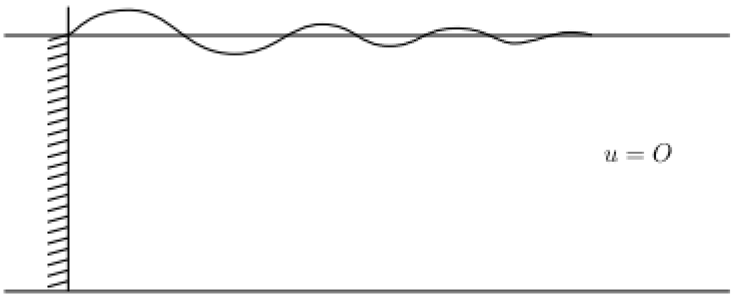


Figure 12.13

The movement of the piston is represented in the x, t plane by the curve

$$x = X(t), u = X'(t), \tag{12.21}$$

where X(t) is given function.

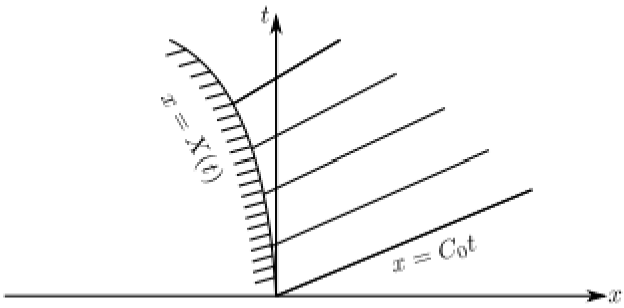


Figure 12.14

The flow of water is governed by the equation

$$u_t + (c_0 + \frac{3u}{2})u_x = 0 \tag{12.22}$$

since the wave moving to the right is produced.

Characteristic curve on which

$$\frac{dx}{dt} = c_0 + \frac{3u}{2}, u(x,0) = 0 \tag{12.23}$$

On this characteristic $\frac{dx}{dt} = 0$; therefore, u=constant=$X'(\tau)$, , if the characteristic is passing through $(X(\tau),\tau)$.

Therefore

$$\frac{dx}{dt} = c_0 + \frac{3}{2}X'(\tau)$$

$$\int_{x(\tau)}^{x(t)} dx = \int_{\tau}^{t}(c_0 + \frac{3}{2}X'(\tau))dt$$

$$x = x(\tau) + (c_0 + \frac{3}{2}X'(\tau))(t-\tau)$$

Hence the solution of the piston problem is

$$\begin{cases} x = x(\tau) + (c_0 + \frac{3}{2}X'(\tau))(t-\tau), \\ u = X'(\tau), \end{cases} \tag{12.24}$$

where τ is the characteristic parameter. As in the previous cases, expansion waves (in this case $X''(t) \leq 0$) do not break and the solution is valid for all t. On the other hand, moving the piston forward or providing a positive acceleration, will produce a breaking wave. The inclusion of discontinuities based on the jump conditions (noted after equation (12.7), i.e, $U[] + [] = 0$) is similar in spirit to the discussion of chapter 11, but is somewhat more complicated than before. The relation (12.16), $2\sqrt{gH} - 2\sqrt{gH_0} = u$ is not strictly valid across discontinuities (note it was deduced from the differential equations), Since at discontinuity $H'(u)$ does not exist, so our main equation (12.13) and (12.14) are useless.

And approximations have to be made if the simple wave solutions are still used. (See for details in the equivalent gas dynamics case).

12.3 Method of characteristics for a system

The above simple wave solutions provide an interesting approach and tie the discussion closely to the earlier material on a single equation. However, they are limited to waves moving in one direction only. We want to consider questions of waves moving in both directions and interacting with each other. We shall also find via Riemann's arguments a further understanding of the role of the simple waves.

Figure 12.15: Square Waves moving in two different direction

Since we already know that c $=\sqrt{gh}$ is a useful quantity here, we shall introduce it at the outset to simplify the expressions but it is in no way crucial. The equations (12.10) and (12.11) then become

$$u_t + uu_x + 2cc_x = 0 \tag{12.25}$$

$$c_t + uc_x + \frac{1}{2}cu_x = 0 \tag{12.26}$$

Since, $c = \sqrt{gh}, \Rightarrow gh = c^2, \Rightarrow gh_x = 2cc_x$ and $gh_t = 2cc_t$.

Now we note that each equation relates the directional derivatives of u and c for different directions. If the directions were the same we might make progress as in Chapter 2. But we can try linear combinations of (12.25) and (12.26) that have the desired property. Accordingly, we consider (12.25) +m $\times$ (12.26), where m is a quantity to be determined. We have

$$(u_t + uu_x + 2cc_x) + m(c_t + uc_x + \frac{1}{2}cu_x) = 0 \tag{12.27}$$

$$\Rightarrow u_t + (u + \tfrac{mc}{2})u_x + (mu + 2c)c_x + mc_t = 0$$

$$\Rightarrow (u_t + vu_x) + m(c_t + vc_x) = 0$$

$$\Rightarrow (u_x, u_t).(v, 1) + m(c_x, c_t).(v, 1) = 0$$

Hence they have the same direction (v,1), provided

$u + \frac{m}{2}c = u + \frac{2c}{m} = v$(For this condition the directional derivatives are identical).

$$\Rightarrow m^2 = 4$$

$$\Rightarrow m = \pm 2$$

•$\underline{Case - I}$:- Take m=2 in (12.27) we have $u_t + uu_x + 2cc_x + 2(c_t + uc_x + \frac{1}{2}cu_x) = 0$

$$\Rightarrow (u + 2c)_t + (u + c)(u + 2c)_x = 0 \qquad (12.28)$$

We choose the ς_+ characteristic to be

$$\varsigma_+ : \frac{dx}{dt} = u + c$$

On ς_+, (12.28) becomes, $\frac{d}{dt}(u + 2c) = 0$

$$\Rightarrow \text{u+2c= Constant on } \varsigma_+$$

•$\underline{Case - II}$:- Take m=-2 in (12.27) we have $u_t + uu_x + 2cc_x - 2(c_t + uc_x + \frac{1}{2}cu_x) = 0$

$$\Rightarrow (u - 2c)_t + (u - c)(u - 2c)_x = 0 \qquad (12.29)$$

We choose the ς_- characteristic to be

$$\varsigma_+ : \frac{dx}{dt} = u - c$$

On ς_-, (12.29) becomes, $\frac{d}{dt}(u - 2c) = 0$

$$\Rightarrow \text{u-2c= Constant on } \varsigma_-$$

Thus we obtain

$$\begin{cases} u + 2c = Constant, & \text{on} \frac{dx}{dt} = u + c; \\ u - 2c = Constant, & \text{on} \frac{dx}{dt} = u - c \end{cases} \qquad (12.30)$$

The constants may differ from characteristic to characteristic.

This is the method of characteristics for higher order systems. For an n^{th} order system of first order equations for $u_1, ..., u_n$, one looks for a linear combination of the equations so that the directional derivatives of each u_i is the same. If there are n real different combinations with the characteristic property the system is *hyperbolic*.

In the present case the characteristic equations will be useful in various ways. We first reconsider the simple wave solutions.

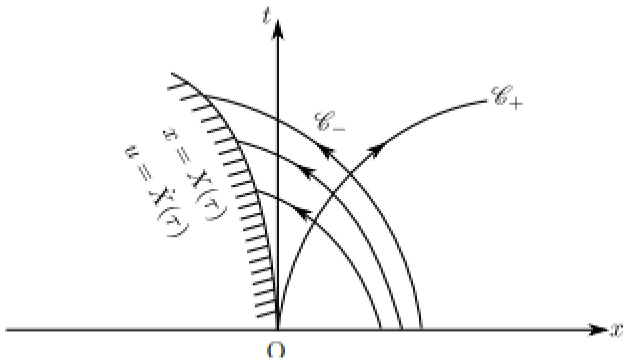


Figure 12.16

12.4 Riemann's argument for simple waves

We focus on the piston problem to show how the argument goes through and refer to figure 12.15.

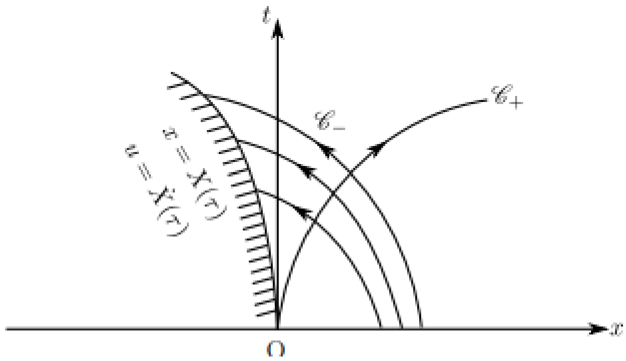


Figure 12.17

Using the fact that $u - c \leq u$ we can show that the ς_- characteristics cover the whole region $(x,t) : t \geq 0, x \geq X(t)$. On each ς_- we have u - 2c = Constant; from the initial condition

$$t = 0 : u = 0, c = c_0$$

we find that $u - 2c = -2c_0$. But this is true for each ς_- with the same constant.

Therefore

$$u - 2c = -2c_0 \tag{12.31}$$

everywhere. This is exactly the relation $(2\sqrt{gH} - 2\sqrt{gH_0} = u....(12.16))$: We could now refer to the previous discussion to complete the solution. To complete the solution in the present context, we use the ς_+ relation

$$\text{u+2c= Constant on } \frac{dx}{dt} = u + c$$

From (4.18) this becomes

$$\text{u= Constant on } \frac{dx}{dt} = c_0 + \frac{3}{2}u$$

exactly the information contained in $([u_t + (c_0 + \frac{3}{2}u)u_x = 0]......(12.18))$. We conclude that

$$\begin{cases} u = X'(\tau) \\ x = X(\tau) + (c_0 + \frac{3}{2}X'(\tau))(t - \tau) \end{cases} \tag{12.32}$$

as before.

<u>Problem:- Dam break</u>

In an idealization, the flow of water out of a dam is governed by the equations

$$u_t + uu_x + 2cc_x = 0$$

$$c_t + uc_x + \frac{1}{2}cu_x = 0$$

with the initial conditions

$$t = 0 : \begin{cases} u = 0, -\infty < x < \infty, \\ h = \begin{cases} h_1, \infty < x < \infty \\ 0, 0 < x < \infty \end{cases} \end{cases}$$

From the above system

$$u_t + uu_x + 2cc_x + m(c_t + uc_x + \frac{1}{2}cu_x) = 0 \tag{12.33}$$

$$\Rightarrow u_t + (u + \tfrac{mc}{2})u_x + (mu + 2c)c_x + mc_t = 0$$

$$\Rightarrow (u_t + vu_x) + m(c_t + vc_x) = 0$$

$$\Rightarrow (u_x, u_t).(v, 1) + m(c_x, c_t).(v, 1) = 0$$

Hence they have the same direction (v,1), provided
$u + \frac{m}{2}c = u + \frac{2c}{m} = v$ (For this condition the directional derivatives are identical).

$$\Rightarrow m^2 = 4$$

$$\Rightarrow m = \pm 2$$

•$\underline{Case - I}$:- Take m=2 in (12.33) we have $u_t + uu_x + 2cc_x + 2(c_t + uc_x + \frac{1}{2}cu_x) = 0$

$$\Rightarrow (u + 2c)_t + (u + c)(u + 2c)_x = 0 \tag{12.34}$$

We choose the ς_+ characteristic to be

$$\varsigma_+ : \frac{dx}{dt} = u + c$$

On ς_+, (12.34) becomes, $\frac{d}{dt}(u + 2c) = 0$

$$\Rightarrow \text{u+2c= Constant on } \varsigma_+$$

•$\underline{Case - II}$:- Take m=-2 in (12.33) we have $u_t + uu_x + 2cc_x - 2(c_t + uc_x + \frac{1}{2}cu_x) = 0$

$$\Rightarrow (u - 2c)_t + (u - c)(u - 2c)_x = 0 \tag{12.35}$$

We choose the ς_- characteristic to be

$$\varsigma_- : \frac{dx}{dt} = u - c$$

On ς_-, (12.35) becomes, $\frac{d}{dt}(u-2c)=0$

$$\Rightarrow \text{u-2c= Constant on } \varsigma_-$$

Thus we obtain

$$\begin{cases} u+2c=Constant, & \text{on}\frac{dx}{dt}=u+c; \\ u-2c=Constant, & \text{on}\frac{dx}{dt}=u-c \end{cases} \tag{12.36}$$

The constants may differ from characteristic to characteristic.

On each ς_- we have u-2c=Constant; from the initial conditions

At

$$\begin{cases} \text{t=0: u=0, -}\infty < x < \infty \\ h = \begin{cases} h_1, -\infty < x < 0 \\ 0, 0 < x < \infty \end{cases} \end{cases}$$

$$\text{c=}\begin{cases} \sqrt{gh_1}, -\infty < x < 0 \\ 0, 0 < x < \infty \end{cases}$$

•$\underline{Case - I}$:- For t=0,

$$\text{0-2}\begin{cases} \sqrt{gh_1} \\ 0 \end{cases} \text{=Constant}$$

i.e,

$$\text{Constant=}\begin{cases} \text{-2}\sqrt{gh_1}, \infty < x < 0 \\ 0, 0 < x < \infty \end{cases}$$

Then

$$\text{u-2c=}\begin{cases} \text{-2}\sqrt{gh_1}, -\infty < x < 0 \\ 0, 0 < x < \infty \end{cases}$$

This is exactly the relation

$$2\sqrt{gH} - 2\sqrt{gH_0} = u$$

where h=H(u)

•$\underline{Case - II}$:- Use the same concept for u+2c= Constant on $\frac{dx}{dt} = u + c$.

This the solution of the Dam Break.

12.5 Hodograph transformation

In the interaction of waves, where both families of characteristics carry nontrivial disturbances (i.e. (12.31) does not hold), solutions are much more difficult, and numerical methods are often used.

However, one alternative analytic method for studying the interaction of waves, or the two interacting families of waves produced by general initial conditions, is the 'hodograph' method. The equations are

$$\begin{cases} c_t + uc_x + \frac{1}{2}cu_x = 0 \\ u_t + uu_x + 2cc_x = 0 \end{cases} \tag{12.37}$$

and we note that the coefficients are functions of the dependent variables only. We try to make use of that fact by interchanging the role of dependent and independent variables.

....

Implicit Function Theorem:-

Let

(i) $f = (f_1, f_2, ..., f_n)$ be a vector valued function defined on an open set S in $\mathbb{R}^{n+k}$ with values in $\mathbb{R}^n$.

i.e, $f : S(\subset \mathbb{R}^{n+k}) \to \mathbb{R}^n$

(ii) $f \in C^1$ on S,

(iii) $(x_0; t_0)$ be a point in S for which $f(x_0; t_0) = 0$ for which $\det[D_j f_i(x_0; t_0)]_{n\times n} \neq 0$

Then,

$\exists$ a k- dimension open set T_0 containing t_0 and one and only one vector valued function g, defined on T_0 and having values in $\mathbb{R}^n$.

i.e, $g : T_0(\subset \mathbb{R}^k) \to \mathbb{R}^n, t_0 \in T_0$

such that

(i) $g \in C^1$ on T_0,

(ii) $g(t_0) = x_0$,

(iii)$f(g(t); t) = 0, \forall t \in T_0$,

(iv) $g'(b) = -(A_x)^{-1}A_y$.

• The function g is implicitly defined by f(g(t);t)=0.

....

We have u = u(x, t), c = c(x, t) and consider the inverse functions

$$x = x(u, c),\ t = t(u, c).$$

Then using the implicit function theorem we shall get

$$g' = -(A_x)^{-1}A_y$$

$$\begin{bmatrix} u_x & u_t \\ c_x & c_t \end{bmatrix} = -1\frac{}{g} \begin{bmatrix} t_c & -x_c \\ -t_u & x_u \end{bmatrix}$$

where g=det(A_x).

The term 'hodograph' is used when the velocities u and c are considered as independent variables. We have the relations

$$u_x = -\frac{t_c}{g}, u_t = \frac{x_c}{g}, c_x = \frac{t_u}{g}, c_t = -\frac{x_u}{g}$$

where $g = \frac{(c,t)}{(u,c)} = x_u t_c - x_c t_u$.

For the system (12.37), the highly non-linear factor g cancels through and we have

$$\begin{cases} x_u = ut_u - \frac{1}{2}ct_c \\ x_c = ut_c - 2ct_u \end{cases} \tag{12.38}$$

Notice g would not cancel if there were undifferentiated terms. Equations (12.38) are now linear and this offers considerable simplification.

Differentiating the first equation in (12.38) with respect to c, partially, and the second one with respect to u and subtracting, we find

$$4t_{uu} = t_{cc} + \frac{3}{c}t_c. \tag{12.39}$$

This is a linear equation for t(u, c) which can be solved by standard methods.

However, the difficulties in this method are:

(1) The transformed boundary conditions in the u-c plane will sometimes be awkward.

(2) When breaking occurs g = 0, corresponding to the multivaluedness, and fitting in shocks may sometimes be difficult in this plane.

For these reasons a numerical method is often preferred. However, in the case of waves on a sloping beach an analogous method has led to an extremely valuable solution; it will be described in chapter 5(Nonlinear waves on a sloping beach).

In that connection, a particularly elegant form of the transformation is useful and we note it here for the case of the horizontal bottom. We 56 use the characteristic form

p=u+2c=constant on $\frac{dx}{dt} = u + c$

q = u - 2c = constant on $\frac{dx}{dt} = u - c$

If p, q are used as variables, we can write

$$\frac{dx}{dt} = u + c$$

as $x_q = (u + c)t_q$

since p is a constant on that characteristic and q can be used as parameter. Similarly

$$x_p = (u - c)t_p$$

We then substitute for u and c in terms of p and q to obtain

$$x_q = \frac{3p+q}{4}t_q, x_p = \frac{p+3q}{4}t_p$$

These are the linear hodograph equations equivalent to (12.38) . Eliminating x, we have

$$2(p - q)t_{pq} - 3(t_q - t_p) = 0$$

which is equivalent to (12.39).

Chapter 13

Basic concept of Hyperbolic Partial Differential Equation

13.1 Hyperbolic PDE:- Hyperbolic System

13.1.1 Overview of Hyperbolic Partial Differential Equations

• The One-Way Wave Equation:-

The prototype for all hyperbolic partial differential equations is the one-way wave equation:

$$u_t + au_x = 0, \tag{13.1}$$

where,

- a is a constant,
- t represents time,
- x represents the spatial variable.

The subscript denotes differentiation, i.e., $u_t = \frac{\partial u}{\partial t}$.

We give u(t, x) at the initial time, which we always take to be 0. i.e., u(0, x) is required to be equal to a given function u_0(x) for all real numbers x —and we wish to determine the values of u(t, x) for positive values of t. This is called an initial value problem. By inspection we observe that the solution of (13.1) is

$$u(t,x) = u_0(x - at). \tag{13.2}$$

The formula (13.2) tells us several things. First, the solution at any time t_0 is a copy of the original function, but shifted to the right, if a is positive, or to the left, if a is negative, by an amount $|a|t_0$. Another way to say this is that the solution at (t, x) depends only on the value of $\xi = x - at$. The lines in the (t, x) plane on which x - at is constant are called characteristics. The parameter a has dimensions of distance divided by time and is called the speed of propagation along the characteristic. Thus the solution of the one-way wave equation (13.1) can be regarded as a wave that propagates with speed a without change of shape, as illustrated in Figure 13.1

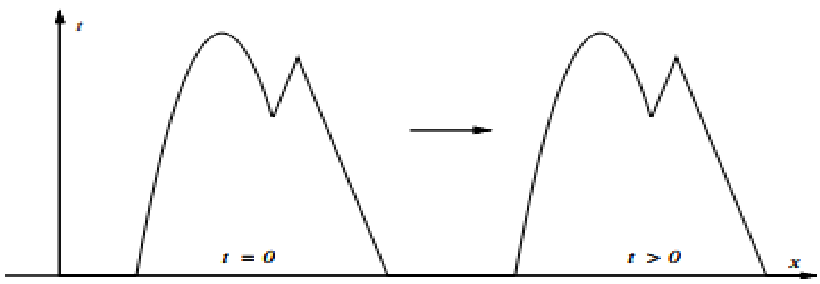


Figure 13.1: The solution of the one-way wave equation is a shift.

Second, whereas equation (13.1) appears to make sense only if u is differentiable, the solution formula (13.2) requires no differentiability of u_0. In general, we allow for discontinuous solutions for hyperbolic problems. An example of a discontinuous solution is a shock wave, which is a feature of solutions of nonlinear hyperbolic equations. To illustrate further the concept of characteristics, consider the more general hyperbolic equation

$$u_t + au_x + bu = f(t,x), u(0,x) = u_0(x), \tag{13.3}$$

where a and b are constants. Based on our preceding observations we change variables from (t, x) to (τ, ξ), where τ and ξ are defined by

$$\tau = t, \xi = x - at.$$

The inverse transformation is then

$$t = \tau, x = \xi + a\tau.$$

and we define $\overline{u}(\tau, \xi) = u(t, x)$, where (τ, ξ) and (t, x) are related by the preceding relations. (Both u and $\overline{u}$ represent the same function, but the tilde is needed to distinguish

between the two coordinate systems for the independent variables.) Equation (13.3) then becomes

$$\frac{\partial \overline{u}}{\partial \tau} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial \tau} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial \tau} = u_t + au_x = -bu + f(\tau, \xi + a\tau)$$

$$\frac{\partial \overline{u}}{\partial \tau} = -b\overline{u} + f(\tau, \xi + a\tau)$$

$\frac{\partial \overline{u}}{\partial \tau} + b\overline{u} = f(\tau, \xi + a\tau)$. This is an ordinary differential equation in τ

$$\frac{d}{dt}(\overline{u}e^{b\tau}) = e^{b\tau} = f(\tau, \xi + a\tau)$$

$$\int_0^\tau d(\overline{u}e^{b\tau}) = \int_0^\tau f(\sigma, \xi + a\sigma)e^{b\sigma}d\sigma$$

The solution is

$$\overline{u}(\tau, \xi) = u_0(\xi)e^{-b\tau} + \int_0^\tau f(\sigma, \xi + a\sigma)e^{-b(\tau-\sigma)}d\sigma$$

Returning to the original variables, we obtain the representation for the solution of equation (13.3) as

$$u(t, x) = u_0(x - at)e^{-bt} + \int_0^t f(s, x - a(t - s))e^{-b(t-s)}ds \qquad (13.4)$$

We see from (13.4) that u(t, x) depends only on values of (t', x') such that $x' - at'$ = x - at, i.e., only on the values of u and f on the characteristic through (t, x) for $0 \leq t' \leq t$

This method of solution of (13.3) is easily extended to nonlinear equations of the form

$$u_t + au_x = f(t, x, u) \tag{13.5}$$

13.1.2 Systems of Hyperbolic Equations

We now examine systems of hyperbolic equations with constant coefficients in one space dimension. The variable u is now a vector of dimension d.

Definition:- A system of the form

$$u_t + Au_x + Bu = F(t, x) \tag{13.6}$$

is hyperbolic if the matrix A is diagonalizable with real eigenvalues.

By saying that the matrix A is diagonalizable, we mean that there is a nonsingular matrix P such that PAP^{-1} is a diagonal matrix, that is,

$$PAP^{-1}=$$

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_d \end{bmatrix}$$

$$=\Lambda$$

The eigenvalues ai of A are the characteristic speeds of the system. Under the change of variables w = P u we have, in the case B = 0,

$$w_t + \Lambda w_x = PF(t, x) = \overline{F}(t, x)$$

or

$$w_t^i + a_i w_x^i = \overline{f}^i(t, x)$$

which is the form of equation (13.3). Thus, when matrix B is zero, the one-dimensional hyperbolic system (13.6) reduces to a set of independent scalar hyperbolic equations. If B is not zero, then in general the resulting system of equations is coupled together, but only in the undifferentiated terms. The effect of the lower order term, Bu, is to cause growth, decay, or oscillations in the solution, but it does not alter the primary feature of the propagation of the solution along the characteristics.

Example:-

As an example of a hyperbolic system, we consider the system

$$u_t + 2u_x + v_x = 0$$

$$v_t + u_x + 2v_x = 0$$

which can be written as

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix}_t + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} u_x \\ v_x \end{bmatrix}$$

As initial data we take

$$u(0,x) = u_0(x) = \begin{cases} 1 \text{ if —x—} \leq 1, \\ 0\text{if}|x| > 1 \end{cases} ; \text{ v(0, x) = 0.}$$

By adding and subtracting the two equations, the system can be rewritten as

$$(u+v)_t + 3(u+v)_x = 0,$$

$$(u-v)_t + (u-v)_x = 0$$

or

$$w^1_t + 3w^1_x = 0, w^1(0,x) = u_0(x),$$

$$w_t^2 + 3w_x^2 = 0, w^2(0,x) = u_0(x),$$

The matrix P is

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

for this transformation. The solution is, therefore,

$$w^1(t,x) = w_0^1(x - 3t),$$

$$w^2(t,x) = w2_0(x - t)$$

or

$$u(t,x) = \frac{1}{2}(w^1 + w^2) = \frac{1}{2}[u_0(x - 3t) + u_0(x - t)],$$

$$v(t,x) = \frac{1}{2}(w^1 -^2) = \frac{1}{2}[u_0(x - 3t) - u_0(x - t)]$$

These formulas show that the solution consists of two independent parts, one propagating with speed 3 and one with speed 1, which are the the eigen values of A.

13.1.3 Equations with Variable Coefficients

We now examine equations for which the characteristic speed is a function of t and x. Consider the equation

$$u_t + a(t,x)u_x = 0 \tag{13.7}$$

with initial condition u(0, x) = $u_0(x)$, which has the variable speed of propagation a(t, x). If, as we did after equation (13.3), we change variables to τ and ξ , where τ = t and ξ is as yet undetermined, we have

$$\frac{\partial \overline{u}}{\partial \tau} = \frac{\partial t}{\partial \tau}u_t + \frac{\partial x}{\partial \tau}u_x = u_t + \frac{\partial x}{\partial \tau}u_x$$

In analogy with the constant coefficient case, we set

$$\frac{dx}{d\tau} = a(t,x) = a(\tau, x).$$

This is an ordinary differential equation for x giving the speed along the characteristic through the point (τ, x) as a(τ, x). We set the initial value for the characteristic curve through (τ, x) to be ξ. Thus the equation (13.7) is equivalent to the system of ordinary differential equations

$$\begin{cases} \frac{d\overline{u}}{d\tau} = 0, \overline{u}(0,\xi) = u_0(\xi) \\ \frac{dx}{dt} = a(\tau, x), x(0) = \xi \end{cases} \tag{13.8}$$

As we see from the first equation in (13.8), u is constant along each characteristic curve, but the characteristic determined by the second equation need not be a straight line. We now present an example to illustrate these ideas.

<u>Example:-</u>Consider the equation

$$u_t + xu_x = 0,$$

$$\text{u(0,x)=}\begin{cases} 1, \text{if} 0 \le x \le 1, \\ 0, \text{otherwise.} \end{cases}$$

Corresponding to the system (13.8) we have the equations

$$\frac{d\overline{u}}{d\tau} = 0, \frac{dx}{d\tau} = x, x(0) = \xi$$

The general solution of the differential equation for x(τ) is x(τ) $= ce^{\tau}$. Because we specify that ξ is defined by x(0) =ξ , we have x(τ) $= \xi$ e^{τ} , or $\xi = x^e - t$. The equation for $\overline{u}$ shows that $\overline{u}$ is independent of τ, so by the condition at τ equal to zero we have that

$$\overline{u}(\tau, \xi) = u_0(\xi).$$

Thus

$$u(t, x) = \overline{u}(\tau, \xi) = u_0(\xi) = u_0(xe^{-t}).$$

So we have, for t ¿ 0

$$u(t, x) = \begin{cases} 1 if 0 \leq x \leq e^t, \\ 0, \text{otherwise} \end{cases}$$

As for equations with constant coefficients, these methods apply to nonlinear equations of the form

$$u_t + a(t, x)u_x = f(t, x, u), \tag{13.9}$$

as shown in Exercise 13.9. Equations for which the characteristic speeds depend on u, i.e., with characteristic speed a(t, x, u), require special care, since the characteristic curves may intersect.

13.1.4 Systems with Variable Coefficients

For systems of hyperbolic equations in one space variable with variable coefficients, we require uniform diagonalizability.

Definition:- The system

$$u_t + A(t, x)u_x + B(t, x)u = F(t, x) \tag{13.10}$$

with

$$u(0, x) = u_0(x)$$

is hyperbolic if there is a matrix function P (t, x) such that

$$P(t,x)A(t,x)P^{-1}(t, x) = \begin{bmatrix} a_1(t, x) & 0 & \cdots & 0 \\ 0 & a_2(t, x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_d(t, x) \end{bmatrix} = \Lambda(t, x)$$

is diagonal with real eigenvalues and the matrix norms of P (t, x) and $P^{-1}(t, x)$ are bounded in x and t for x R, $t \geq 0$.

The characteristic curves for system (13.10) are the solutions to the differential equations

$$\frac{dx^i}{dt} = a_i(t, x), x^i(0) = \xi^i$$

Setting v = P (t, x)u, we obtain the system for v:

$$v_t + \Lambda v_x = P(t, x)F(t, x) + G(t, x)v,$$

where

$$G = (P_t + \Lambda P_x P B)P^{-1}$$

In terms of directional derivatives this system is equivalent to

$$\frac{dv^i}{dt}|_{\text{along} x^i} = \overline{f}(t, x) + \sum_{j=1}^{d} g_j^i(t, x)v^j$$

This formula is not a practical method of solution for most problems because the ordinary differential equations are often quite difficult to solve, but the formula does show the importance of characteristics for these systems.

Chapter 14

Waves on a Sloping Beach; Shallow Water Theory

In the last chapter we considered flow over a horizontal level surface. In the case of a non-uniform bottom, we will get an additional term in the horizontal momentum equation due to the force acting on the bottom surface.

14.1 Shallow water equations

We choose a coordinate system x, y such that $y = -h_0(x)$ denotes the bottom and $y = \eta(x,t)$ the water surface. Hence the total depth h(x, t) is

$$h(x,t) = h_0(x) + \eta(x,t)$$

The equation of conservation of mass is

$$\frac{d}{dt}\int_{x_2}^{x_1} h(x,t)dx + [uh]_{x_2}^{x_1} = 0 \tag{14.1}$$

as before, and if u and h are continuously differentiable, then

$$h_t + (uh)_x = 0 \tag{14.2}$$

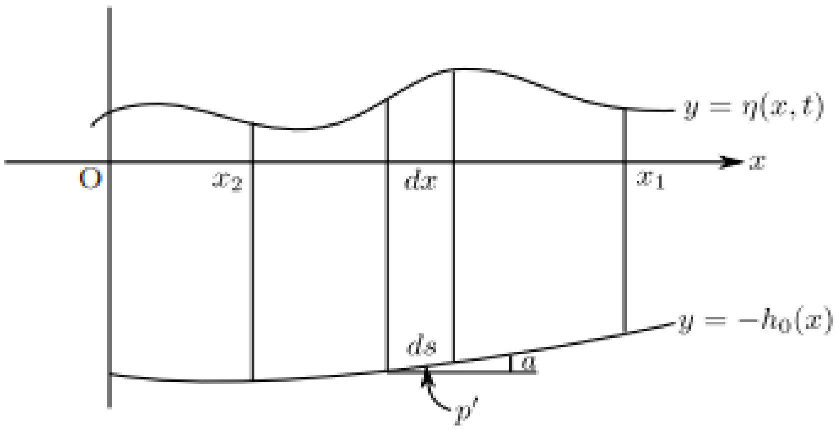


Figure 14.1

Let us now consider the momentum balance in the x-direction. Let p' be the excess pressure as before. When the bottom is not horizontal, the contribution of p' from the bottom surface will have a non-zero horizontal component. Let us consider a thin section of thickness dx and let ds be the line element along the bottom $y = -h_0(x)$. Let α be the inclination of ds to the x-axis. Then

$$ds = \tfrac{dx}{cos\alpha}$$

Hence the momentum balance in the horizontal direction is

$$\frac{d}{dt}\int_{x_2}^{x_1} hudx + [hu^2 + \frac{1}{2}gh^2]_{x_2}^{x_1} = -\int_{x_2}^{x_1}(p'\frac{dx}{cos\alpha})sn\alpha \tag{14.3}$$

In the shallow water theory we have $p' = g(\eta - y)$. At $y = -h_0$, $p' = g(\eta + h_0) = gh$. Therefore (14.3) becomes

$$\frac{d}{dt}\int_{x_2}^{x_1} hudx + [hu^2 + \frac{1}{2}gh^2]_{x_2}^{x_1} = \int_{x_2}^{x_1} gh\frac{dh_0}{dx}dx \tag{14.4}$$

since $\frac{dh_0}{dx} = tan\alpha$. If all the quantities are smooth, in the limit $x_2 \to x_1$, we obtain

$$(ht)_t + (hu^2 + \frac{1}{2}gh^2)_x = gh\frac{dh_0}{dx} \tag{14.5}$$

When there are discontinuities, the shock condition derived from (14.4) is

$$-U[hu] + [hu^2 + \frac{1}{2}gh^2] = 0$$

since the right hand side of (14.4) becomes zero in the limit $x_2 \to x_1$. Thus, the shock conditions are unaffected by the extra term $gh\frac{dh_0}{dx}$ due to the non-uniform bottom.

Using the mass conservation equation (14.2) the momentum equation (14.5) can be written as

$$(ht)_t + (hu^2 + \frac{1}{2}gh^2)_x = gh\frac{dh_0}{dx}$$

split the equation and separate like that

$$\Rightarrow u(h_t + u_x h + uh_x) + h(u_t + uu_x + g\eta_x) = 0 (Since, h(x,t) = h_o(x) + \eta(x,t)).$$

$$\Rightarrow u_t + uu_x + g\eta_x = 0 (By, 14.2)$$

Hence the system of equations for the flow of shallow water over a non-uniform bottom is

$$\begin{cases} h_t + u_x h + uh_x = 0 \\ u_t + uu_x + g\eta_x \\ h = h_0 + \eta. \end{cases} \tag{14.6}$$

14.2 Linearized equations

We assume that the disturbances are small of order $\epsilon << 1$ i.e. $\frac{\eta}{h_0} = O(\epsilon)$ and $\frac{u}{\sqrt{gh_0}} = o(\epsilon)$ where, u=Velocity of the fluid and $\sqrt{gh_0}$=Propagation Velocity. We also assume that the derivative are also of the same order.

Since $h = \eta(x,t) + h_0(x)$, equations (14.6) can be written down as

$$\eta_t + uh_0' + h_0 u_x + u\eta_x + \eta u_x = 0 \tag{14.7}$$

$$u_t + uu_x + g\eta_x = 0 \tag{14.8}$$

The first three terms of (14.7) are of order $O(\epsilon)$ whereas the last two terms are of order $O(\epsilon^2)$. In the equation (14.8) $uu_x = O(\epsilon^2)$ and the other terms are of order o. Hence to a first order approximation(i.e avoid 2nd order and higher order terms of ϵ. We have

$$\begin{cases} \eta_t + h_0 u_x + h_0' u = 0 \\ u_t + g\eta_x = 0 \end{cases} \tag{14.9}$$

Equations (14.9) are the linearized versions of equations (14.6). Differentiating the first equation of (14.9) partially w.r.t. t and using the second equation, we obtain

$$\eta_{tt} = gh_0\eta_{xx} + gh_0'\eta_x \tag{14.10}$$

This is the wave equation with an additional term. If h_0 were constant then

$$\eta_{tt} = gh_0\eta_{xx}$$

and the general solution of this is

$$\eta_1(x - \sqrt{gh_0}t) + f_2(x - \sqrt{gh_0}t)$$

The velocity of propagation is $\sqrt{gh_0}$.

14.3 Linear theory for waves on a sloping beach

We now consider a sloping beach with inclination β to the horizontal. We assume β to be small so that linearized shallow water theory can be applied. However there will be some questions about validity to be considered later. These are

(i) The question of using the shallow water theory as $x \to \infty$, when the water becomes deep.

(ii) The question of the assumption $\frac{\eta}{h_0} << 1$ near x = 0 where $h_0 \to 0$.

We have to solve equation (14.10) with $h_0 = xtan\beta$ and we take $h_0 \simeq \beta x$ since β is very small. Hence the equation can be written as

$$\eta_{tt} = g\beta x\eta_{xx} + g\beta\eta_x. \tag{14.11}$$

Let $\eta = N(x)e^{-i\omega t}$ be a solution of equation (14.11). Then we obtain an ordinary differential equation for N as follows:

$$N(-i\omega)^2e^{-i\omega t} = g\beta xN''(x)e^{-i\omega t} + g\beta N'(x)e^{-i\omega t}$$

$$N'' + \frac{1}{x}N' + \frac{\omega^2}{g\beta}\frac{1}{x}N = 0 \tag{14.12}$$

This is to be solved in $0 < x < \infty$.

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Regular Singular Point:-

Let $y'' + p_1(x)y' + p_2(x)y = 0$. If p_1 and p_2 are not analytic at $x = x_0$ then $x = x_0$ is called singular point.

If $(x-x_0)p_1(x)$ and $(x-x_0)^2p_2$ are both analytic at $x = x_0$ then x_0 is called regular singular point.

If any one of $(x-x_0)p_1(x)$ and $(x-x_0)^2p_2$ or both are not analytic at $x = x_0$ then x_0 is called irregular singular point.

....

The point x = 0 is a regular point of equation (14.12), and x = ∞ is an irregular point. This suggests a transformation to Bessel's equation or some other confluent hypergeometric equation. In fact the transformation

$$x = \frac{g\beta}{\omega^2}\frac{X^2}{4}$$

Then

$$N'(x) = \frac{dN}{dx} = \frac{dN}{dX}\frac{dX}{dx} = \frac{dN}{dX}\frac{2\omega^2}{g\beta X}$$

And

$$N''(x) = \frac{d^2N}{dx^2} = \frac{d}{dX}(\frac{dN}{dx})\frac{dX}{dx} = \frac{2\omega^2}{g\beta}\frac{d}{dX}(\frac{1}{X}\frac{dN}{dX})\frac{2\omega^2}{g\beta X} = \frac{4\omega^4}{(g\beta)^2}\frac{1}{X}\frac{d}{dX}(\frac{1}{X}\frac{dN}{dX})$$

$x = \frac{g\beta}{\omega^2}\frac{X^2}{4}$ converts (14.12) into the Bessel equation of order zero.

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Bessel Equation:- $x^2y'' + xy' + (x^2 - p^2)y = 0$ is the Bessel's equation of order p.

Bessel equation of order zero is $y'' + \frac{1}{x}y' + y = 0$ and it's two linearly independent solutions are

$$J_0(X) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}(\frac{X}{2})^{2n}$$

and

$$Y_0(X) = \frac{2}{\pi}[(\ln\frac{X}{2} + \gamma)J_0(X) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}X^{2n}}{2^{2n}(n!)^2}(1 + \frac{1}{2} + ... + \frac{1}{n})]$$

where

$$\gamma = \lim_{x\to\infty}((1 + \frac{1}{2} + ... + \frac{1}{n}) - \ln n) \cong 0.5772.$$

....

$$\frac{d^2N}{dX^2} + \frac{1}{X}\frac{dN}{dX} + N = 0 \tag{14.13}$$

The Bessel functions $J_0(X), Y_0(X)$ are two linearly independent solutions of the equation (14.13). Hence a general solution of (14.12) is

$$N = AJ_0(2\omega\sqrt{\frac{x}{g\beta}}) - iBY_0(2\omega\sqrt{\frac{x}{g\beta}})$$

where A and B are constants. Since the power series for $J_0(X)$ contains only even powers of X, $J_0(2\omega\sqrt{\frac{x}{g\beta}})$ is an integer power series in x and is regular at the beach x = 0. We note that Y_0 has a logarithmic singularity at x = 0.

The complete solution of (14.11) is

$$\eta(x,t) = [AJ_0(2\omega\sqrt{\frac{x}{g\beta}}) - iBY_0(2\omega\sqrt{\frac{x}{g\beta}})]e^{-i\omega t} \tag{14.14}$$

As $x \to \infty$ the asymptotic formula for η is

$$\eta \sim \frac{1}{\sqrt{\pi}}(\frac{g\beta}{\omega^2 x})^{\frac{1}{4}}\frac{A+B}{2}e^{-2i\omega\sqrt{\frac{x}{g\beta}}-i\omega t+\frac{\pi i}{4}} + \frac{A-B}{2}e^{2i\omega\sqrt{\frac{x}{g\beta}}-i\omega t-\frac{\pi i}{4}} \tag{14.15}$$

The first term in the bracket corresponds to an incoming wave and the second one to an outgoing wave. In a uniform medium an outgoing periodic wave is given by

$$ae^{ikx-i\omega t}$$

where

- k=Wave number
- ω=Frequency
- a=Amplitude.

The terms in (14.15) are generalizations to the form

$$a(x,t)e^{i\theta(x,t)}$$

....

Period:- The time taken (T) for any particle to complete one vibration is called Period.

Amplitude:- The maximum displacement of any particle from it's mean position is called amplitude(A) of the wave.

Frequency:- The number of vibration per second by particle is called frequency(N) of the waves, where $N = \frac{L}{T}$.

Wave Length:-The between two consecutive particles of the medium which are in the same phase or which differ in phase by twice of radian(2π) is called wave-length(Y) of the wave.

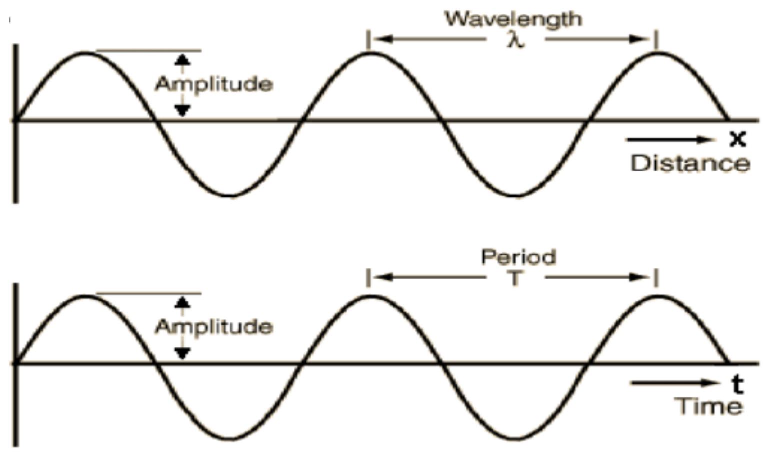

Figure 14.2

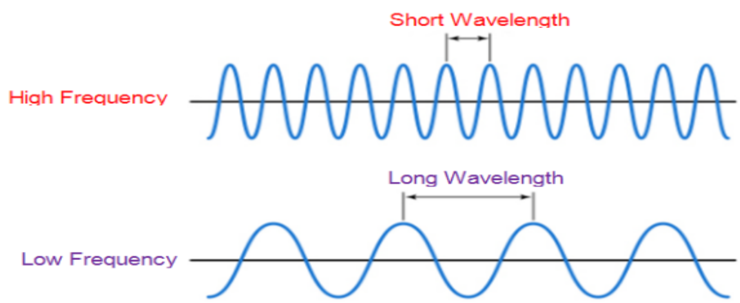

Figure 14.3: High and Low frequency radio wave

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A generalized wave number and frequency can be defined in terms of the phase function $\theta(x, t)$ by

$$k(x, t) = \theta_x, \nu(x, t) = -\theta_t; \tag{14.16}$$

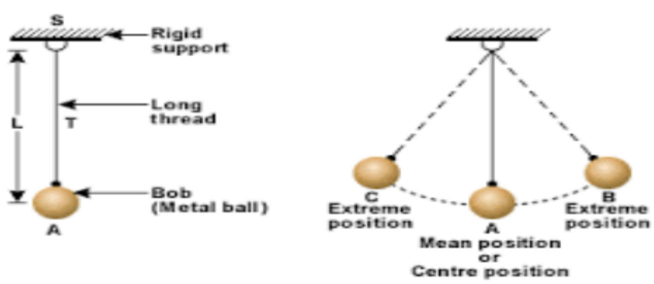

Figure 14.4: Extreme and Mean position

the generalized phase velocity is

$$c(x,t) = \frac{\nu}{k} = -\frac{\theta_t}{\theta_x} \qquad (14.17)$$

The function a(x, t) is the amplitude.

In our particular case the outgoing wave has

$$\theta(x,t) = 2\omega\sqrt{\tfrac{x}{g\beta}} - \omega t - \tfrac{\pi}{4}$$

Hence the wave number, frequency and phase velocity are

$$k(x,t) = \theta_x = \tfrac{\omega}{\sqrt{g\beta x}}$$

$$\nu(x,t) = -\theta_t = \omega$$

$$c(x,t) = \sqrt{g\beta x}$$

We note that the waves get shorter as $x \to 0$(since $k \to \infty$), and that $c = \sqrt{gh_0(x)}$ is the generalization of the result for constant depth. The incoming wave is similar with the opposite sign of propagation.

Behavior as x $\to \infty$:-

We note that the amplitude a varies proportional to $x^{-1/4}$. As $x \to \infty, a \to 0$. This means that, within shallow water, we cannot pose the natural problem of a prescribed incoming wave at infinity with a given nonzero amplitude. This is due to the failure of the shallow water assumptions at ∞, one of the questions noted at the beginning of this section. It it found from the full theory , (for the solution corresponding to J_0) that the ratio of amplitude at infinity a_∞ to amplitude at shoreline a_0 is in fact $(\frac{2\beta}{\pi})^{\frac{1}{2}}$. Therefore, $\frac{a_\infty}{a_0} \to 0$ as $\beta \to 0$, and the $x^{-\frac{1}{4}}$ behavior is the shallow water theory's somewhat inadequate attempt to cope with this. However, the full solution does show that the shallow water theory is a good approximation near the shore. And it is valuable there since, for example, the corresponding nonlinear solution can be found in the shallow water theory but not in the full theory.

Behavior as x $\to \infty$ and breaking:-

We see from (14.15) that the ratio of B to A, which controls the amount of J_0 and Y_0 in the solution, also determines the proportion of incoming wave that is reflected back to infinity.

For B = 0, we have perfect reflection with

$$\eta = AJ_0(2\omega\sqrt{\frac{x}{g\beta}})e^{-i\omega t} \tag{14.18}$$

and the solution is bounded and regular at the shoreline x = 0.

In the other extreme, A = B, there is no reflection, we have a purely incoming wave

$$\eta = A(J_0 - iY_0)e^{-i\omega t} \tag{14.19}$$

but it is now singular at the shoreline. The interpretation of the singularity is that it is the linear theory's crude attempt to represent the breaking of waves and the associated loss of energy. As B increases, more energy goes into the singularity (breaking) and less is reflected.

Although breaking is the most obvious phenomenon we observe at the seashore, a number of long wave phenomena (long swells, edge waves, tsunamis) are in the range where breaking does not occur so that the J_0 solution (B = 0) with perfect reflection is relevant. This is fortunate since practical use of the Y_0 solution would be limited, although the situation is mathematically interesting.

The singular solution is related also to the second question noted at the beginning of this section: The breakdown of the linearizing assumption $\frac{\eta}{h_0} << 1$ as $h_0 \to 0$ at the shoreline. On this we can say that the nonlinear solution corresponding to J_0 can be found without this assumption (next section), and it endorses the linear approximation. The Y_0 solution with its crucial ties to complicated nonlinear breaking is clearly a different matter.

Tidal estuary problem:-

In a channel where the breadth b(x) varies, as well as the depth $h_0(x)$, the shallow water equations are modified to

$$\frac{\partial}{\partial t}[(h_0+\eta)b] + \frac{\partial}{\partial x}[(h_0+\eta)ub] \text{ (By } h_t + (uh)_x = 0)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0$$

For the case $h_0(x) = \beta x$, $b(x) = \alpha x$ the linearized equation for η can again be solved in Bessel functions. G.I. Taylor used this solution to study the large tidal variations in the Bristol channel. In this application to extremely long waves, breaking is not an issue and only the J_n solution is accepted.

14.4 Nonlinear waves on a sloping beach

In Section 14.3 we considered the linear approximation of the equations for waves on a sloping beach. Carrier and Greenspan in 1958 gave an exact solution of the nonlinear equations using a modified type of hodograph transformation applied to characteristic variables. We recall that the governing equations are

$$h_t + uh_x + hu_x = 0$$

$$u_t + uu_x + gh_x - g\beta = 0 \text{ where } h = \beta x + \eta(x,t).$$

Introducing the variable $c = \sqrt{gh}$= velocity of propagation, which we know to be significant, the above equations become

$$\begin{cases} 2c_t + 2uc_x + cu_x = 0 \\ u_t + uu_x + 2cc_x - g\beta = 0 \end{cases} \tag{14.20}$$

Due to the presence of the term $g\beta$, the straight forward hodograph transformation (u, c) $\rightarrow$ (x, t) will not simplify the equations, since this time the Jacobian g would not cancel through. However, Carrier and Greenspan introduced new variables suggested by the characteristic forms and applied a hodograph transformation to these.

Use Hodograph Transformation as x=x(u,c) and t=t(u,c).

Then, $c_t = -\frac{x_u}{\varrho}, c_x = \frac{t_u}{\varrho}, u_t = \frac{x_c}{\varrho}, u_x = -\frac{t_c}{\varrho}$. Here ϱ is the Jacobian which we denoted by g in Hodograph Transformation(Chapter 12). For the presence of g= Gravity notation we are defining ϱ as Jacobian.

The characteristic forms of the equations (14.20) are

$$\begin{cases} c_t + uc_x + \frac{1}{2}cu_x = 0 \\ u_t + uu_x + 2cc_x - g\beta = 0 \end{cases}$$

Each equation relates the directional derivatives of u and c for difference direction.

Take linear combination of above two,

$$u_t + uu_x + 2cc_x - g\beta + m(c_t + uc_x + \tfrac{1}{2}cu_x) = 0$$

$$[u_t + (u + \tfrac{m}{2}c)u_x] + [c_t + (u + \tfrac{2c}{m})c_x] - g\beta = 0$$

Provided, $u + \frac{m}{2}c = u + \frac{2c}{m} = v(\text{say})$

$$m = \pm 2$$

For $m = \pm 2$ the characteristic forms of the equations (14.20) are

$$\begin{cases} (u + 2c)_t + (u + c)(u + 2c)_x - g\beta = 0, \\ (u - 2c)_t + (u - c)(u - 2c)_x - g\beta = 0. \end{cases} \tag{14.21}$$

These can be written as

$$\begin{cases} (u + 2c - g\beta t)_t + (u + c)(u + 2c - g\beta t)_x = 0, \\ (u - 2c - g\beta t)_t + (u - c)(u - 2c - g\beta t)_x = 0. \end{cases} \tag{14.22}$$

The ς_+ and ς_- characteristic curves are defined by

$$\begin{cases} \varsigma_+ : \frac{dx}{dt} = u + c, u + 2c - g\beta t = Constant \\ \varsigma_- : \frac{dx}{dt} = u - c, u - 2c - g\beta t = Constant \end{cases} \tag{14.23}$$

We define the characteristic variables p, q by

$$p = u + 2c - g\beta t, \tag{14.24}$$

$$q = u - 2c - g\beta t. \tag{14.25}$$

Then equations (14.23) can be written

$$x_q = (u + c)t_q,$$

$$x_p = (u - c)t_p,$$

which introduces the hodograph transformation (p, q) $\rightarrow$ (x, t). Solving (14.24), (14.25) for u, c and inserting them in the above equations we obtain

$$\begin{cases} x_q = (\frac{3p+q}{4} + g\beta t)t_q, \\ x_p = (\frac{p+3q}{4} + g\beta t)t_p, \end{cases} \tag{14.26}$$

Equations (14.26) are still nonlinear, but by good fortune the nonlinear terms are in the form $(\frac{1}{2}g\beta t^2)_q$, $(\frac{1}{2}g\beta t^2)_p$)p so that when we take cross derivatives and subtract to obtain an equation for t, these terms cancel each other. This was the remarkable fact observed by Carrier and Greenspan. Differentiating the first equation in (14.26) partially with respect to p and the second equation with respect to q and subtracting we obtain

$$2(p - q)t_{pq} + 3(t_q - t_p) = 0. \tag{14.27}$$

Equation (14.27) is a linear equation which can ve solved by standard methods.

This is the main step, but further transformations can be used to convert (14.27) into the cylindrical wave equation whose solutions are already well documented. First, by the transformation

$$\begin{cases} \sigma = p - q, \\ \lambda = -(p+q) \end{cases} \tag{14.28}$$

Then

$$t_p = \frac{\partial t}{\partial \sigma}\frac{\partial \sigma}{\partial p} + \frac{\partial t}{\partial \lambda}\frac{\partial \lambda}{\partial p} = \frac{\partial t}{\partial \sigma} - \frac{\partial t}{\partial \lambda}$$

Similarly determine t_{pq} and t_q and equation (14.27) becomes

$$t_{\lambda\lambda} = t_{\sigma\sigma} + \frac{3}{\sigma}t_\sigma \tag{14.29}$$

This can be further simplified by introducing the transformation

$$g\beta t = \frac{\lambda}{2} - \frac{\phi_\sigma}{\sigma}; \tag{14.30}$$

the term $-\frac{\phi_\sigma}{\sigma}$ is for transforming (14.29) into the cylindrical wave equation and the term $\frac{\lambda}{2}$ is included to give a simple final form for u. Thus we obtain cylindrical wave equation

$$\phi_{\lambda\lambda} = \phi_{\sigma\sigma} + \frac{1}{\sigma}\phi_\sigma \tag{14.31}$$

Equations (14.24), (14.25) give u, c in terms of p, q. From the transformation (14.28) we obtain p, q in terms of σ, λ. These together with equation (14.30) lead to

$$c = \frac{\sigma}{4} \tag{14.32}$$

$$u = -\frac{\phi_\sigma}{\sigma} \tag{14.33}$$

$$g\beta t = \frac{\lambda}{2} - \frac{\phi_\sigma}{\sigma} \tag{14.34}$$

It can be shown from (14.26), with a use of (14.31), that

$$(g\beta x)_\sigma = (-\tfrac{1}{4}\phi_\lambda + \tfrac{1}{2}\tfrac{\phi_\sigma{}^2}{\sigma^2} + \tfrac{\sigma^2}{16})_\sigma$$

$$(g\beta x)_\lambda = (-\tfrac{1}{4}\phi_\lambda + \tfrac{1}{2}\tfrac{\phi_\sigma{}^2}{\sigma^2})_\lambda$$

After some big calculation we shall get this above two equations.

From these we obtain

$$g\beta x = -\frac{1}{4}\phi_\lambda + \frac{1}{2}\frac{\phi_\sigma{}^2}{\sigma^2} + \frac{\sigma^2}{16} \tag{14.35}$$

The final set of transformations (14.32)-(14.35) is sufficiently involved that it seems inconceivable that anyone would discover them directly. One can note that

$$u - g\beta t = -\tfrac{\lambda}{2} \text{ and } \tfrac{1}{2}u^2 + c^2 - g\beta x = \tfrac{1}{4}\phi_\lambda$$

take simple forms and these combinations appear in two alternative ways of absorbing gβ in conservation forms for the second of (14.20), i.e.

$$(u - g\beta t)_t + (\tfrac{1}{2}u^2 + c^2)_x = 0$$

$$u_t + (\tfrac{1}{2}u^2 + c^2 - g\beta x)_x = 0$$

But this comment does not appear to lead any further.

Almost equally important as the linearity of (14.31) is the fact that the moving shoreline c = 0 is now fixed at $\sigma = 0$ in the new independent variables. We can now work in a fixed domain. The simplest separable solution of (14.31) is

$$\phi = N(\sigma)\cos\alpha\lambda \tag{14.36}$$

where α is an arbitrary separation constant. The equation for N(σ) is then the Bessel equation of order zero.

$$N'' + \frac{1}{\sigma}N' + \alpha^2 N = 0 \tag{14.37}$$

The solution bounded at the shoreline $\sigma = 0$ is

$$N = AJ_0(\alpha\sigma)$$

where A is a constant. Hence

$$\phi = AJ_0(\alpha\sigma)\cos\alpha\lambda \tag{14.38}$$

Equation (14.38) together with the above transformations and relations give an exact solution for the non linear equation (14.20).

Linear approximation :-

It will be useful to note how the linearized approximation is obtained from (14.38). In the linear theory u is small which implies ϕ is small. Hence to a first order approximation we obtain from (14.34), (14.35) that

$$\begin{cases} g\beta t \simeq \frac{\lambda}{2}, \\ g\beta x \simeq \frac{\sigma^2}{16} \end{cases} \tag{14.39}$$

Thus

$$\phi \simeq AJ_0(4\alpha\sqrt{g\beta x})\cos 2\alpha g\beta t$$

Taking $\alpha = \frac{\omega}{2g\beta}$ we obtain

$$\phi \simeq AJ_0(2\omega\sqrt{\tfrac{x}{g\beta}})\cos\omega t$$

which is in agreement with our result obtained in section 14.3. To relate ϕ to the particle velocity u and elevation η, we first note that

$u = -\frac{\phi_\sigma}{\sigma} = -\alpha^2 A \frac{J_0'(\alpha\sigma)}{\alpha\sigma} \cos\alpha\lambda \simeq \frac{2\omega a_0}{\beta} \frac{J_0'(2\omega\sqrt{\frac{x}{g\beta}})}{2\omega\sqrt{\frac{x}{g\beta}}} \cos\omega t$

where

$$a_0 = \frac{\beta}{2\omega}\alpha^2 A = \frac{\omega}{8g\beta^2} A \tag{14.40}$$

Then rather than trying to improve on the approximation for σ and hence c to find η, we rather note that the above linearized approximation for u goes along in linear theory with

$$\eta = -a_0 J_0(2\omega\sqrt{\frac{x}{g\beta}}) \sin\omega t \tag{14.41}$$

These approximations provide a rough way to interpret the variables in the non-linear form (14.38). In particular we see it as the nonlinear counterpart of the wave with perfect reflection at the beach.

Run-up:-

Perhaps the most important quantity among the results is the range of x at the shoreline $\sigma = 0$, since this provides the amplitude of the run-up.

If $x = F(\lambda, \sigma)$ then the range of x at $\sigma = 0$ is $[\min_\lambda F(\lambda, 0), \max_\lambda F(\lambda, 0)]$.

Using (14.35), (14.38) and the fact that

$$\lim_{z\to 0} \frac{J_0'(z)}{z} = -\frac{1}{2}$$

we obtain:

at the shore $\sigma = 0, g\beta x = \frac{1}{4}\alpha A \sin\alpha\lambda + \frac{1}{8}\alpha^4 A^2 \cos^2\alpha\lambda$

At maximum or minimum run-up, u = $-\frac{\phi_\sigma}{\sigma} = 0$. Therefore, from (14.38),

$$\cos\alpha\lambda = 0 \text{ and hence } \sin\alpha\lambda = \pm 1$$

Therefore at the maximum run up: $g\beta x = \frac{1}{4}\alpha A$, at the minimum run up: $g\beta x = -\frac{1}{4}\alpha A$. Hence the range of x is

$$-\frac{\alpha A}{4g\beta} \leq x \leq \frac{\alpha A}{4g\beta}$$

If a_0 is the vertical amplitude, we have

$$a_0 = \frac{\alpha A}{4g} \tag{14.42}$$

This agrees with (14.40) when the linearized relation $\alpha = \frac{\omega}{2g\beta}$ is used. The latter will not be quite accurate in the nonlinear theory for the relation of α to the frequency ω, but it is probably a good enough approximation; the exact relation could, of course, be calculated.

Breaking condition:-

In finding a solution for equations (14.20) we made use of many transformations and got a solution which is single valued, bounded and smooth in terms of the variables λ, σ. When the Jacobian of the transformation $(\lambda, \sigma) \rightarrow$ (x, t) becomes zero the solution in the xt-plane will be multivalued i.e. breaking will occur. We will find the condition for breaking to occur. By (14.26) and (14.28),

$$\varrho = x_\lambda t_\sigma - x_\sigma t_\lambda = (ut_\lambda + ct_\sigma)t_\sigma - (ct_\lambda + ut_\sigma)t_\lambda = c(t_\sigma^2 - t_\lambda^2)$$[Use equations (14.32),(14.33),(14.34) and (14.35)]

Differentiating (14.34) partially with respect to σ and λ and using (14.38), we obtain

$$g\beta t_\sigma = \tfrac{A\alpha^3}{z}(J_0 + \tfrac{2}{z}J_0')\cos\alpha\lambda$$

$$g\beta t_\lambda = \tfrac{1}{2} + \tfrac{A\alpha^3}{z}J_0'\sin\alpha\lambda$$

where $z = \alpha\sigma$. Using the relations

$$J_0' = -J_1; J_0 + \tfrac{2}{z}J_0' = -J_2,$$

We obtain

$$g\beta(t_\lambda - t_\sigma) = \frac{1}{2} - A\alpha^3(\frac{J_1\sin\alpha\lambda - J_2\cos\alpha\lambda}{z}) \tag{14.43}$$

$$g\beta(t_\lambda + t_\sigma) = \frac{1}{2} - A\alpha^3(\frac{J_1 \sin\alpha\lambda + J_2\cos\alpha\lambda}{z}) \tag{14.44}$$

Now

$$\frac{J_1 \sin\alpha\lambda \pm J_2 \cos\alpha\lambda}{z} = \frac{\sqrt{J_1^2+J_2^2}}{z}\sin(\alpha\lambda \pm \eta)$$

where $\eta = \tan^{-1}(\frac{J_2}{J_1})$. Hence, the maximum values of these expressions are

$$\frac{\sqrt{J_1^2+J_2^2}}{z}$$

It can be shown that

$$\frac{d}{dz}(\frac{J_1^2+J_2^2}{z^2})$$

Hence for positive z, $\frac{J_1^2+J_2^2}{z^2}$ is a decreasing function and its maximum value is attained at z = 0, where it is equal to $\frac{1}{4}$.Therefore the factors in (14.43), (14.44) first vanish when $A\alpha^3 = 1$, and breaking first occurs at the shoreline.

If we again use the approximate relation $\alpha = \frac{\omega}{2g\beta}$, together with (14.42) a necessary and sufficient condition for breaking to occur is

$$\frac{\omega^2 a_0}{g\beta^2} \geq 1 \tag{14.45}$$

This is a very fruitful result obtained from the nonlinear theory. Breaking is obviously a complicated phenomenon with wide variations in type and conditions. But (14.45) gives a valuable result on the significant combination of parameters. From observations also it is found that the quantity $P = \frac{\omega^2 a_0}{g\beta^2}$ plays an important role. Galvin's experiments and observations group breaking phenomena into different ranges of P. He distinguishes the ranges (although with some overlap).

P	Type
≤ 0.045	Surging; No breaking
0.045-0.81	Collapsing; Fig. 14.5
0.28 19	Plunging; Fig. 14.6
14 64	Spilling; Fig. 14.7

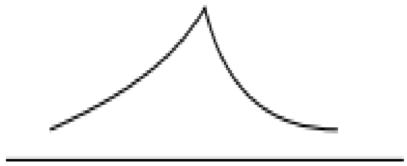

Figure 14.5: Collapsing

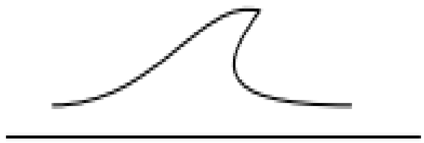

Figure 14.6: Plunging

Carrier and Greenspan give other solutions and include the analysis for solving the general initial value problem.

14.5 Bore on beach

When breaking occurs, a discontinuous ”bore”, corresponding to the shocks discussed earlier would be fitted in. The appropriate jump conditions were noted in Section 12.1. This has not been carried through in the Carrier-Greenspan solutions. However the simpler problem of what happens when a bore initially moving with constant speed and strength in an offshore region of constant depth impinges on a sloping beach has been studied by approximate and numerical methods.

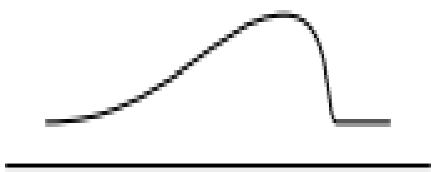

Figure 14.7: Spilling

14.6 Edge Waves

In the previous section we have considered only normal incidence with dependence only on distance x normal to the shore. We now turn to phenomena that include longshore dependence. If x_1 is normal to and x_2 is along the shore, the linearized equation for the surface elevation $\eta(x_1, x_2, t)$ is modified from (14.11) to

$$\eta_{tt} = g\beta x_1(\eta_{x_1x_1}) + g\beta\eta_{x_1} \tag{14.46}$$

The modification is slight and we do not give the derivation. We use separation of variables and let

$$\eta = N(x_1)e^{\pm ikx_2 \pm i\omega t} \tag{14.47}$$

Then $N(x_1)$ satisfies

$$N'' + \frac{1}{x_1}N' + (\frac{\omega^2}{g\beta x_1} - k^2)N = 0 \tag{14.48}$$

The interval of interest is $0 < x_1 < \infty$. The origin $x_1 = 0$ is a regular singular point; one solution is analytic and the other has a logarithmic singularity. At ∞, the equation is roughly

$$N'' - k^2 N \simeq 0 \tag{14.49}$$

with solutions

$$N \simeq e^{-kx_1}, e^{kx_1} \tag{14.50}$$

In this case only the solutions bounded at both $x_1 = 0$ and appear to be of interest. We shall see that the solutions represent waves running along the beach, and no-one seems to have interpreted the logarithmic solution in any sense such as breaking. So we choose the analytic solution near $x_1 = 0$. Then, in general, this solution will be a linear combination of both e^{-kx_1} and e^{+kx_1} at ∞. For an acceptable physical solution the term in e^{kx_1} should be absent. This is possible only for special values of $\frac{\omega^2}{g\beta}$. We have a singular eigenvalue problem. If we set

$$N = e^{-kx_1}F(X), X = 2kx_1, k > 0 \tag{14.51}$$

it becomes a standard one. We have

$$XF_{XX} + (1 - X)F_X + \frac{1}{2}(\frac{\omega^2}{g\beta k} - 1)F = 0 \tag{14.52}$$

and the required solutions are Laguerre polynomials

$$L_n(X) = \frac{e^X}{n!}\frac{d^n}{dx^n}(X^n e^{-X}) \tag{14.53}$$

with

$$\omega^2 = gk(2n + 1)\beta, n = \text{Positive integer} \tag{14.54}$$

The solution for $N(x_1)$ is

$$N(x_1) = e^{-kx_1}L_n(2kx_1) \tag{14.55}$$

The final solutions for η are

$$\eta = e^{-|k|x_1}L_n(2|k|x_1)e^{\pm ikx_2 \pm i\omega t} \tag{14.56}$$

where $|k|$ is appropriate if negative values of k are used.

These solutions all decay away from the shoreline and have crests perpendicular to the shoreline. For this reason they are known as ‘edge waves’. The lowest mode n = 0 has

$$N(x_1) = e^{-kx_1}, \omega^2 = gk\beta, k > 0$$

and one might take for example

$$\eta = e^{-kx_1} \cos{(KX_2 - \omega t)}. \tag{14.57}$$

This corresponds to a solution first found by Stokes. It is interesting to note how the different terms in (14.57) are balanced by this solution. One might note the propagation speed is $\sqrt{g\beta x_1}$ and expect the waves to swing round to the beach due to the increase of speed with x_1. The final result avoids this and we see from (14.57) that the balance is

$$\eta_{x_1 x_1} + \eta_{x_2 x_2} = 0, \eta = g\beta\eta_{x_1} \tag{14.58}$$

The propagation speed argument applies directly when η_{tt} balances the second derivatives in (x_1, x_2); the balance in (14.58) avoids this.

The equation is hyperbolic but these particular solutions avoid the hyperbolic character and appear as 'dispersive waves' with dispersion relations given in (14.54). (See Chapter 2 for a discussion of the distinctions, and Chapter 11 for the main properties of dispersive waves). We also note there is no possibility of an oblique wave at ∞. This would require

$$\eta \sim e^{\pm i\iota x_1 \pm ikx_2 \pm i\omega t}$$

with real ι and k. We have only the wave of normal incidence found in Section 14.5,

$$\eta = J_0(2\omega\sqrt{\frac{x}{g\beta}})e^{\pm i\omega t} \tag{14.59}$$

or the edge waves travelling along the beach. As noted earlier, (14.59) does not have a finite nonzero amplitude at ∞, but it does at least represent a normal wave. For the oblique case there is not even a corresponding solution. This again is a breakdown of the shallow water assumption in deep water. We can interpret the result roughly by remarking that oblique deep water waves would in reality swing around towards the shore when they feel the depth decrease. They do this completely, and achieve normal incidence as in (14.59), by the time the shallow water theory applies. In the linear theory, edge waves are not stimulated directly by incoming waves at infinity. We check these explanations from the full linear theory.

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