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Some Quantum f-divergence Inequalities for Convex Functions of Self-adjoint Operators

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ABSTRACT: In this paper, some new inequalities for convex functions of self-adjoint operators are obtained. As applications, we present some inequalities for quantum f-divergence of trace class operators in Hilbert Spaces.

Key Words: Self-adjoint bounded linear operator, Trace inequality, Convex function, Quantum f-divergence.

Contents

1. Introduction

Let $(X, \mathcal{A})$ be a measurable space satisfying $|\mathcal{A}| > 2$ and μ be a σ-finite measure on $(X, \mathcal{A})$. Let $\mathcal{P}$ be the set of all probability measures on $(X, \mathcal{A})$ which are absolutely continuous with respect to μ. For P, $Q \in \mathcal{P}$, let $p = \frac{dP}{d\mu}$ and $q = \frac{dQ}{d\mu}$ denote the *Radon-Nikodym* derivatives of P and Q with respect to μ.

Two probability measures P, $Q \in \mathcal{P}$ are said to be *orthogonal* and we denote this by $Q \perp P$ if

$$P(\{q = 0\}) = Q(\{p = 0\}) = 1.$$

Let $f : [0, \infty) \to (-\infty, \infty]$ be a convex function that is continuous at 0, i.e., $f(0) = \lim_{u \downarrow 0} f(u)$.

In 1963, I. Csiszár [6] introduced the concept of f-divergence as follows.

Definition 1.1. *Let P, $Q \in \mathcal{P}$. Then*

$$I_f(Q, P) = \int_X p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x), \tag{1.1}$$

is called the f-divergence of the probability distributions Q and P.

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Remark 1.2. *Observe that, the integrand in the formula (1.1) is undefined when $p(x) = 0$. The way to overcome this problem is to postulate for f as above that*

$$0f\left[\frac{q(x)}{0}\right] = q(x)\lim_{u\downarrow 0}\left[uf\left(\frac{1}{u}\right)\right], \quad x \in X. \tag{1.2}$$

We now give some examples of f-divergences that are well-known and often used in the literature (see also [3]).

1.1. The Class of χ^α-Divergences

The f-divergences of this class, which is generated by the function χ^α, $\alpha \in [1,\infty)$, defined by

$$\chi^\alpha(u) = |u-1|^\alpha, \quad u \in [0,\infty)$$

have the form

$$I_f(Q,P) = \int_X p\left|\frac{q}{p}-1\right|^\alpha d\mu = \int_X p^{1-\alpha}|q-p|^\alpha d\mu. \tag{1.3}$$

From this class only the parameter $\alpha = 1$ provides a distance in the topological sense, namely the *total variation distance* $V(Q,P) = \int_X |q-p|\, d\mu$. The most prominent special case of this class is, however, *Karl Pearson's* χ^2*-divergence*

$$\chi^2(Q,P) = \int_X \frac{q^2}{p}d\mu - 1$$

that is obtained for $\alpha = 2$.

1.2. Dichotomy Class

From this class, generated by the function $f_\alpha : [0,\infty) \to \mathbb{R}$

$$f_\alpha(u) = \begin{cases} u - 1 - \ln u & \text{for } \alpha = 0; \\ \frac{1}{\alpha(1-\alpha)}\left[\alpha u + 1 - \alpha - u^\alpha\right] & \text{for } \alpha \in \mathbb{R}\setminus\{0,1\}; \\ 1 - u + u\ln u & \text{for } \alpha = 1; \end{cases}$$

only the parameter $\alpha = \frac{1}{2}$ $\left(f_{\frac{1}{2}}(u) = 2\left(\sqrt{u}-1\right)^2\right)$ provides a distance, namely, the *Hellinger distance*

$$H(Q,P) = \left[\int_X \left(\sqrt{q}-\sqrt{p}\right)^2 d\mu\right]^{\frac{1}{2}}.$$

Another important divergence is the *Kullback-Leibler divergence* obtained for $\alpha = 1$,

$$KL(Q,P) = \int_X q\ln\left(\frac{q}{p}\right) d\mu.$$

1.3. Divergences of Arimoto-type

This class is generated by the functions

$$\Psi_\alpha(u) := \begin{cases} \frac{\alpha}{\alpha-1}\left[(1+u^\alpha)^{\frac{1}{\alpha}} - 2^{\frac{1}{\alpha}-1}(1+u)\right] & \text{for } \alpha \in (0,\infty)\setminus\{1\}; \\ (1+u)\ln 2 + u\ln u - (1+u)\ln(1+u) & \text{for } \alpha = 1; \\ \frac{1}{2}|1-u| & \text{for } \alpha = \infty. \end{cases}$$

It has been shown in [19] that this class provides the distances $[I_{\Psi_\alpha}(Q,P)]^{\min\left(\alpha,\frac{1}{\alpha}\right)}$ for $\alpha \in (0,\infty)$ and $\frac{1}{2}V(Q,P)$ for $\alpha = \infty$.

For f continuous convex on $[0,\infty)$ we obtain the $*$-*conjugate* function of f by

$$f^*(u) = uf\left(\frac{1}{u}\right), \quad u \in (0,\infty)$$

and

$$f^*(0) = \lim_{u\downarrow 0} f^*(u).$$

It is also known that if f is continuous convex on $[0,\infty)$ then so is f^*.

The following two theorems contain the most basic properties of f-divergences. For their proofs we refer the reader to Chapter 1 of [17] (see also [3]).

Theorem 1.3 (Uniqueness and Symmetry Theorem). *Let f, f_1 be continuous convex on $[0,\infty)$. We have*

$$I_{f_1}(Q,P) = I_f(Q,P),$$

for all $P, Q \in \mathcal{P}$ if and only if there exists a constant $c \in \mathbb{R}$ such that

$$f_1(u) = f(u) + c(u-1),$$

for any $u \in [0,\infty)$.

Theorem 1.4 (Range of Values Theorem). *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous convex function on $[0,\infty)$.*

For any $P, Q \in \mathcal{P}$, we have the double inequality

$$f(1) \le I_f(Q,P) \le f(0) + f^*(0). \tag{1.4}$$

(i) If $P = Q$, then the equality holds in the first part of (1.4).

If f is strictly convex at 1, then the equality holds in the first part of (1.4) if and only if $P = Q$;

(ii) If $Q \perp P$, then the equality holds in the second part of (1.4).

If $f(0) + f^*(0) < \infty$, then equality holds in the second part of (1.4) if and only if $Q \perp P$.

The following result is a refinement of the second inequality in Theorem 1.4 (see [3, Theorem 3]).

Theorem 1.5. *Let f be a continuous convex function on $[0,\infty)$ with $f(1)=0$ (f is normalised) and $f(0)+f^*(0)<\infty$. Then*

$$0 \le I_f(Q,P) \le \frac{1}{2}\left[f(0)+f^*(0)\right]V(Q,P) \tag{1.5}$$

for any $Q,P\in\mathcal{P}$.

For other inequalities for f-divergence see [2], [7]- [12] and [13] Motivated by the above results, in this paper we obtain some new inequalities for quantum f-divergence of trace class operators in Hilbert spaces. It is shown that for normalised convex functions it is nonnegative. Some upper bounds for quantum f-divergence in terms of variational and χ^2-distance are provided. Applications for some classes of divergence measures such as Umegaki and Tsallis relative entropies are also given.

In what follows we recall some facts we need concerning the trace of operators and quantum f-divergence for trace class operators in infinite dimensional complex Hilbert spaces.

2. Some inequalities for convex functions of self-adjoint operators

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f: I\to\mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x,y\in\mathring{I}$ and $x<y$, then $f'_-(x)\le f'_+(x)\le f'_-(y)\le f'_+(y)$, which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

Let A be a self-adjoint operator on the complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$ with $\mathrm{Sp}(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any continuous function $f:[m,M]\to\mathbb{R}$, it is known that we have the following spectral representation in terms of the Riemann-Stieltjes integral (see for instance [14, p. 257]):

$$\langle f(A)x,y\rangle = \int_{m-0}^{M} f(\lambda)d(\langle E_\lambda x,y\rangle),$$

and

$$\|f(A)x\|^2 = \int_{m-0}^{M} |f(\lambda)|^2 d\|E_\lambda x\|^2,$$

for any $x,y\in H$.

Theorem 2.1. *Let A be a self-adjoint operator on the complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$ and $\mathrm{Sp}((A)\subseteq[m,M]\subset\mathring{I}$ where I is an interval. If the function $f:I\to\mathbb{R}$*

is convex on I, then

$$\begin{aligned}\langle f(A)x, x\rangle &- \frac{f(M)-f(m)}{M-m}\langle Ax, x\rangle \\ &\geq \left[\frac{2}{M-m}\int_m^M f(t)dt - \frac{Mf(M)-mf(m)}{M-m}\right]\|x\|^2.\end{aligned}$$

Proof. Without loosing the generality, we can assume that f is differentiable on $\mathring{I}$. By the convexity of f on I, we have

$$f(t) - f(s) \geq f^{'}(s)(t-s), \tag{2.1}$$

for all $t, s \in [m, M] \subset \mathring{I}$.
If $\{E_t\}_{t\in\mathbb{R}}$ is the spectral family of A then the mapping $[m, M] \ni t \mapsto \langle E_t x, x\rangle$ is monotonic real mapping for any $x \in H$.
If we integrate (2.1) over $t \in [m, M]$ with the integraler $\langle E_t x, x\rangle$ then we get

$$\begin{aligned}\int_{m-0}^M f(t)d\langle E_t x, x\rangle &- f(s)\int_{m-0}^M d\langle E_t x, x\rangle \\ &\geq f^{'}(s)\left[\int_{m-0}^M td\langle E_t x, x\rangle - s\int_{m-0}^M d\langle E_t x, x\rangle\right]\end{aligned}$$

for any $s \in [m, M]$ and $x \in H$.
Using the spectral representation theorem for self-adjoint operators we get

$$\langle f(A)x, x\rangle - f(s)\|x\|^2 \geq f^{'}(s)\left[\langle Ax, x\rangle - s\|x\|^2\right] \tag{2.2}$$

for any $s \in [m, M]$ and $x \in H$.
Now if we integrate (2.2) over s on $[m, M]$ and divide by $M-m$, we get

$$\begin{aligned}\langle f(A)x, x\rangle &- \|x\|^2\frac{1}{M-m}\int_m^M f(s)ds \\ &\geq \left(\langle Ax, x\rangle\frac{1}{M-m}\int_m^M sf^{'}(s)ds\right) =: K\end{aligned} \tag{2.3}$$

However, we know that

$$\frac{1}{M-m}\int_m^M f^{'}(s)ds = \frac{f(M)-f(m)}{M-m}$$

and

$$\frac{1}{M-m}\int_m^M sf^{'}(s)ds = \frac{1}{M-m}\left[Mf(M) - mf(m) - \int_m^M f(s)ds\right].$$

Therefore

$$\begin{aligned} K &= \langle Ax,x\rangle\frac{f(M)-f(m)}{M-m} - \frac{\|x\|^2\left[Mf(M)-mf(m)-\int_m^M f(s)ds\right]}{M-m} \\ &= \langle Ax,x\rangle\frac{f(M)-f(m)}{M-m} - \frac{\|x\|^2\left[Mf(M)-mf(m)-\int_m^M f(s)ds\right]}{M-m} \\ &\quad +\frac{\|x\|^2}{M-m}\int_m^M f(s)ds \end{aligned}$$

By (2.3) we then get

$$\begin{aligned} \langle f(A)x,x\rangle &- \frac{f(M)-f(m)}{M-m}\langle Ax,x\rangle \\ &\geq \frac{2\|x\|^2}{M-m}\int_m^M f(s)ds - \frac{\|x\|^2\left[Mf(M)-mf(m)\right]}{M-m} \\ &= \|x\|^2\left[\frac{2}{M-m}\int_m^M f(s)ds - \frac{Mf(M)-mf(m)}{M-m}\right]. \end{aligned}$$

□

Remark 2.2. *Since $f(t)=t^{-1}$ is convex function for $t>0$, by above theorem we have*

$$\langle mMA^{-1}x,x\rangle + \langle Ax,x\rangle \geq \frac{2mMln(\frac{M}{m})}{M-m}$$

for $\|x\|=1$.

Theorem 2.3. *Let A and B be two self-adjoint operators on the Hilbert space $(H,\langle\cdot,\cdot\rangle)$ with $\mathrm{Sp}(A),\mathrm{Sp}(B)\subseteq[m,M]\subset\mathring{I}$ and $f:I\to\mathbb{R}$ a continuously differentiable convex function on I. Then*

$$\begin{aligned} \langle f(A)x,x\rangle\|y\|^2 &- \langle f(B)y,y\rangle\|x\|^2 \\ &\geq \left\langle\left(\langle Ax,x\rangle f^{'}(B)-\|x\|^2f^{'}(B)B\right)y,y\right\rangle \end{aligned}$$

for any $x,y\in H$.

Proof. Since $f:I\to\mathbb{R}$ is convex on I we have

$$f(t)-f(s)\geq f^{'}(s)(t-s),\qquad \text{for any } t,s\in\mathring{I}. \tag{2.4}$$

Let $\{E_t\}_{t\in\mathbb{R}}$ be the spectral family of A and $\{F_t\}_{t\in\mathbb{R}}$ the spectral family of B. The mappings $[m,M]\ni t\mapsto\langle E_tx,x\rangle$ and $[m,M]\ni s\mapsto\langle F_sy,y\rangle$ are monotonic real mapping for any $x,y\in H$.

Integrating (2.4) over t on $[m, M]$ with integraler $\langle E_t x, x\rangle$ we get

$$\begin{aligned}\int_{m-0}^{M} f(t)d\langle E_t x, x\rangle \quad - \quad & f(s)\int_{m-0}^{M} d\langle E_t x, x\rangle \\ & \geq f^{'}(s)\left(\int_{m-0}^{M} td\langle E_t x, x\rangle - s\int_{m-0}^{M} d\langle E_t x, x\rangle\right)\end{aligned}$$

which is equivalent to

$$\langle f(A)x, x\rangle - \|x\|^2 f(s) \geq f^{'}(s)\langle Ax, x\rangle - sf^{'}(s)\|x\|^2,$$

for any $s \in [m, M]$ and $x \in H$.

Integrating above inequality over s on $[m, M]$ with integraler $\langle F_s x, x\rangle$ we then get

$$\begin{aligned}&\langle f(A)x, x\rangle \int_{m-0}^{M} d\langle F_s y, y\rangle - \|x\|^2 \int_{m-0}^{M} f(s)d\langle F_s y, y\rangle \\ &\geq \langle Ax, x\rangle \int_{m-0}^{M} f^{'}(s)d\langle F_s y, y\rangle - \|x\|^2 \int_{m-0}^{M} sf^{'}(s)d\langle F_s y, y\rangle\end{aligned}$$

which is equivalent to

$$\begin{aligned}\langle f(A)x, x\rangle\|y\|^2 \quad - \quad & \langle f(B)y, y\rangle\|x\|^2 \\ & \geq \langle Ax, x\rangle\langle f^{'}(B)y, y\rangle - \|x\|^2\langle f^{'}(B)By, y\rangle.\end{aligned}$$

We know that

$$\left\langle\left(\langle Ax, x\rangle f^{'}(B) - \|x\|^2 f^{'}(B)B\right) y, y\right\rangle = \left(\langle Ax, x\rangle I - \|x\|^2 B\right) f^{'}(B)\langle y, y\rangle$$

and

$$\langle f(A)x, x\rangle\|y\|^2 - \langle f(B)y, y\rangle\|x\|^2 = \langle\left(\langle f(A)x, x\rangle I - \langle x, x\rangle f(B)\right) y, y\rangle$$

On the other hand, we have

$$\left(\langle Ax, x\rangle I - \|x\|^2 B\right) f^{'}(B) \leq \langle f(A)x, x\rangle I - \|x\|^2 f(B)$$

and

$$\langle f(A)x, x\rangle I - \|x\|^2 f(B) \geq f^{'}(B)(\langle Ax, x\rangle I - \|x\|^2 B)$$

for all $x \in H$. We obtain the desired result. □

Remark 2.4. *Let $B = \sum_{j=1}^{n} \alpha_j X_j$ in above theorem and $\alpha_j \geq 0$, $j \in \{1, \ldots, n\}$ such that $\sum_{j=1}^{n} \alpha_j = 1$, then we have*

$$\begin{aligned}\langle f(A)x,x\rangle I \quad - \quad & \|x\|^2 f\left(\sum_{j=1}^{n}\alpha_j X_j\right) \\ & \geq f^{'}\left(\sum_{j=1}^{n}\alpha_j X_j\right)\left(\langle Ax,x\rangle I - \|x\|^2\sum_{j=1}^{n}\alpha_j X_j\right).\end{aligned}$$

Multiply above inequality by α_k and summing on $k \in \{1,\ldots,n\}$, we get

$$\begin{aligned}&\left\langle \sum_{i=1}^{n}\alpha_k f(X_k)x,x\right\rangle I - \|x\|^2 f\left(\sum_{j=1}^{n}\alpha_j X_j\right) \\ &\geq f^{'}\left(\sum_{j=1}^{n}\alpha_j X_j\right)\left(\left\langle\left(\sum_{k=1}^{n}\alpha_k X_k\right)x,x\right\rangle I - \|x\|^2\sum_{j=1}^{n}\alpha_j X_j\right).\end{aligned}$$

Equivalently,

$$\begin{aligned}&\left\langle \sum_{i=1}^{n}\alpha_k f(X_k)x,x\right\rangle I - f\left(\sum_{j=1}^{n}\alpha_j X_j\right) \\ &\geq f^{'}\left(\sum_{j=1}^{n}\alpha_j X_j\right)\left(\left\langle\left(\sum_{k=1}^{n}\alpha_k X_k\right)x,x\right\rangle I - \sum_{j=1}^{n}\alpha_j X_j\right)\end{aligned}$$

for $\|x\| = 1$ which is a Jensen type inequality.

3. Some quantum f-divergence inequalities for convex functions

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I}\|Ae_i\|^2 < \infty. \tag{3.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I}\|Ae_i\|^2 = \sum_{j\in I}\|Af_j\|^2 = \sum_{j\in I}\|A^* f_j\|^2 \tag{3.2}$$

showing that the definition (3.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I}\|Ae_i\|^2\right)^{1/2} \tag{3.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm*.

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (3.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{Tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{3.4}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (3.4) converges absolutely and it is independent from the choice of basis.

Utilising the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{Tr}(B^*A) = \operatorname{Tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{Tr}(A^*A) = \operatorname{Tr}\left(|A|^2\right)$$

for any A, $B \in \mathcal{B}_2(H)$.

If $A \geq 0$ and $P \in \mathcal{B}_1(H)$ with $P \geq 0$, then

$$0 \leq \operatorname{Tr}(PA) \leq \|A\| \operatorname{Tr}(P). \tag{3.5}$$

Indeed, since $A \geq 0$, then $\langle Ax, x\rangle \geq 0$ for any $x \in H$. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, then

$$0 \leq \left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle \leq \|A\| \left\|P^{1/2}e_i\right\|^2 = \|A\| \langle Pe_i, e_i\rangle$$

for any $i \in I$. Summing over $i \in I$ we get

$$0 \leq \sum_{i\in I} \left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle \leq \|A\| \sum_{i\in I} \langle Pe_i, e_i\rangle = \|A\| \operatorname{Tr}(P)$$

and since

$$\sum_{i\in I} \left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle = \sum_{i\in I} \left\langle P^{1/2}AP^{1/2}e_i, e_i\right\rangle = \operatorname{Tr}\left(P^{1/2}AP^{1/2}\right) = \operatorname{Tr}(PA)$$

we obtain the desired result (3.5).

This obviously imply the fact that, if A and B are self-adjoint operators with $A \leq B$ and $P \in \mathcal{B}_1(H)$ with $P \geq 0$, then

$$\operatorname{Tr}(PA) \leq \operatorname{Tr}(PB). \tag{3.6}$$

Now, if A is a self-adjoint operator, then we know that

$$|\langle Ax, x\rangle| \leq \langle |A|\, x, x\rangle \text{ for any } x \in H.$$

This inequality follows by Jensen's inequality for the convex function $f(t) = |t|$ defined on a closed interval containing the spectrum of A.

If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, then

$$\begin{aligned} |\operatorname{Tr}(PA)| &= \left|\sum_{i\in I}\left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle\right| \leq \sum_{i\in I}\left|\left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle\right| \\ &\leq \sum_{i\in I}\left\langle |A|\, P^{1/2}e_i, P^{1/2}e_i\right\rangle = \operatorname{Tr}(P\,|A|), \end{aligned} \tag{3.7}$$

for any A a self-adjoint operator and $P \in \mathcal{B}_1^+(H) := \{P \in \mathcal{B}_1(H) \text{ with } P \geq 0\}$.

For the theory of trace functionals and their applications the reader is referred to [22].

For some classical trace inequalities see [4], [5], [18] and [23], which are continuations of the work of Bellman [1].

On complex Hilbert space $(\mathcal{B}_2(H), \langle\cdot,\cdot\rangle_2)$, where the Hilbert-Schmidt inner product is defined by

$$\langle U, V\rangle_2 := \operatorname{Tr}(V^*U),\ U,\ V \in \mathcal{B}_2(H),$$

for $A,\ B \in \mathcal{B}^+(H)$ consider the operators $\mathfrak{L}_A : \mathcal{B}_2(H) \to \mathcal{B}_2(H)$ and $\mathfrak{R}_B : \mathcal{B}_2(H) \to \mathcal{B}_2(H)$ defined by

$$\mathfrak{L}_A T := AT \text{ and } \mathfrak{R}_B T := TB.$$

We observe that they are well defined and since

$$\langle \mathfrak{L}_A T, T\rangle_2 = \langle AT, T\rangle_2 = \operatorname{Tr}(T^*AT) = \operatorname{Tr}\left(|T^*|^2 A\right) \geq 0$$

and

$$\langle \mathfrak{R}_B T, T\rangle_2 = \langle TB, T\rangle_2 = \operatorname{Tr}(T^*TB) = \operatorname{Tr}\left(|T|^2 B\right) \geq 0$$

for any $T \in \mathcal{B}_2(H)$, they are also positive in the operator order of $\mathcal{B}(\mathcal{B}_2(H))$, the Banach algebra of all bounded operators on $\mathcal{B}_2(H)$ with the norm $\|\cdot\|_2$ where $\|T\|_2 = \operatorname{Tr}\left(|T|^2\right)$, $T \in \mathcal{B}_2(H)$.

Since $\operatorname{Tr}\left(|X^*|^2\right) = \operatorname{Tr}\left(|X|^2\right)$ for any $X \in \mathcal{B}_2(H)$, then also

$$\begin{aligned} \operatorname{Tr}(T^*AT) &= \operatorname{Tr}\left(T^*A^{1/2}A^{1/2}T\right) = \operatorname{Tr}\left(\left(A^{1/2}T\right)^* A^{1/2}T\right) \\ &= \operatorname{Tr}\left(\left|A^{1/2}T\right|^2\right) = \operatorname{Tr}\left(\left|\left(A^{1/2}T\right)^*\right|^2\right) = \operatorname{Tr}\left(\left|T^*A^{1/2}\right|^2\right) \end{aligned}$$

for $A \geq 0$ and $T \in \mathcal{B}_2(H)$.

We observe that $\mathfrak{L}_A$ and $\mathfrak{R}_B$ are commutative, therefore the product $\mathfrak{L}_A\mathfrak{R}_B$ is a self-adjoint positive operator in $\mathcal{B}(\mathcal{B}_2(H))$ for any positive operators $A, B \in \mathcal{B}(H)$.

For $A, B \in \mathcal{B}^+(H)$ with B invertible, we define the *Araki transform* $\mathfrak{A}_{A,B} : \mathcal{B}_2(H) \to \mathcal{B}_2(H)$ by $\mathfrak{A}_{A,B} := \mathfrak{L}_A\mathfrak{R}_{B^{-1}}$. We observe that for $T \in \mathcal{B}_2(H)$ we have $\mathfrak{A}_{A,B}T = ATB^{-1}$ and

$$\langle \mathfrak{A}_{A,B}T, T\rangle_2 = \left\langle ATB^{-1}, T\right\rangle_2 = \operatorname{Tr}\left(T^*ATB^{-1}\right).$$

Observe also, by the properties of trace, that

$$\begin{aligned}\operatorname{Tr}\left(T^*ATB^{-1}\right) &= \operatorname{Tr}\left(B^{-1/2}T^*A^{1/2}A^{1/2}TB^{-1/2}\right)\\ &= \operatorname{Tr}\left(\left(A^{1/2}TB^{-1/2}\right)^*\left(A^{1/2}TB^{-1/2}\right)\right) = \operatorname{Tr}\left(\left|A^{1/2}TB^{-1/2}\right|^2\right)\end{aligned}$$

giving that

$$\langle \mathfrak{A}_{A,B}T, T\rangle_2 = \operatorname{Tr}\left(\left|A^{1/2}TB^{-1/2}\right|^2\right) \geq 0 \tag{3.8}$$

for any $T \in \mathcal{B}_2(H)$.

We observe that, by the definition of operator order and by (3.8) we have $r1_{\mathcal{B}_2(H)} \leq \mathfrak{A}_{A,B} \leq R1_{\mathcal{B}_2(H)}$ for some $R \geq r \geq 0$ if and only if

$$r\operatorname{Tr}\left(|T|^2\right) \leq \operatorname{Tr}\left(\left|A^{1/2}TB^{-1/2}\right|^2\right) \leq R\operatorname{Tr}\left(|T|^2\right) \tag{3.9}$$

for any $T \in \mathcal{B}_2(H)$.

We also notice that a sufficient condition for (3.9) to hold is that the following inequality in the operator order of $\mathcal{B}(H)$ is satisfied

$$r|T|^2 \leq \left|A^{1/2}TB^{-1/2}\right|^2 \leq R|T|^2 \tag{3.10}$$

for any $T \in \mathcal{B}_2(H)$.

Let U be a self-adjoint linear operator on a complex Hilbert space $(K; \langle\cdot,\cdot\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(\operatorname{Sp}(U))$ of all *continuous functions* defined on the *spectrum* of U, denoted $\operatorname{Sp}(U)$, and the C^*-algebra $C^*(U)$ generated by U and the identity operator 1_K on K as follows:

For any $f, g \in C(\operatorname{Sp}(U))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t\in\operatorname{Sp}(U)}|f(t)|$;

(iv) $\Phi(f_0) = 1_K$ and $\Phi(f_1) = U$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \operatorname{Sp}(U)$.

With this notation we define

$$f(U) := \Phi(f) \quad \text{for all } f \in C(\operatorname{Sp}(U))$$

and we call it the *continuous functional calculus* for a self-adjoint operator U.

If U is a self-adjoint operator and f is a real valued continuous function on $\mathrm{Sp}(U)$, then $f(t) \geq 0$ for any $t \in \mathrm{Sp}(U)$ implies that $f(U) \geq 0$, i.e. $f(U)$ is a positive operator on K. Moreover, if both f and g are real valued functions on $\mathrm{Sp}(U)$ then the following important property holds:

$$f(t) \geq g(t) \quad \text{for any} \quad t \in \mathrm{Sp}(U) \quad \text{implies that} \quad f(U) \geq g(U) \tag{P}$$

in the operator order of $B(K)$.

Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. Utilising the continuous functional calculus for the Araki self-adjoint operator $\mathfrak{A}_{Q,P} \in \mathcal{B}(\mathcal{B}_2(H))$ we can define the *quantum f-divergence* for

$$Q, P \in D_1(H) := \{P \in \mathcal{B}_1(H), \quad P \geq 0 \text{ with } \mathrm{Tr}(P) = 1\}$$

and P invertible, by

$$D_f(Q, P) := \left\langle f(\mathfrak{A}_{Q,P}) P^{1/2}, P^{1/2} \right\rangle_2 = \mathrm{Tr}\left(P^{1/2} f(\mathfrak{A}_{Q,P}) P^{1/2}\right). \tag{3.11}$$

If we consider the continuous convex function $f : [0, \infty) \to \mathbb{R}$, with $f(0) := 0$ and $f(t) = t \ln t$ for $t > 0$ then for $Q, P \in D_1(H)$ and Q, P invertible we have

$$D_f(Q, P) = \mathrm{Tr}[Q(\ln Q - \ln P)] =: U(Q, P),$$

which is the *Umegaki relative entropy*.

If we take the continuous convex function $f : [0, \infty) \to \mathbb{R}$, $f(t) = |t - 1|$ for $t \geq 0$ then for $Q, P \in D_1(H)$ with P invertible we have

$$D_f(Q, P) = \mathrm{Tr}(|Q - P|) =: V(Q, P),$$

where $V(Q, P)$ is the *variational distance*.

If we take $f : [0, \infty) \to \mathbb{R}$, $f(t) = t^2 - 1$ for $t \geq 0$ then for $Q, P \in D_1(H)$ with P invertible we have

$$D_f(Q, P) = \mathrm{Tr}\left(Q^2 P^{-1}\right) - 1 =: \chi^2(Q, P),$$

which is called the *χ^2-distance*

Let $q \in (0, 1)$ and define the convex function $f_q : [0, \infty) \to \mathbb{R}$ by $f_q(t) = \frac{1-t^q}{1-q}$. Then

$$D_{f_q}(Q, P) = \frac{1 - \mathrm{Tr}\left(Q^q P^{1-q}\right)}{1 - q},$$

which is *Tsallis relative entropy*.

If we consider the convex function $f : [0, \infty) \to \mathbb{R}$ by $f(t) = \frac{1}{2}\left(\sqrt{t} - 1\right)^2$, then

$$D_f(Q, P) = 1 - \mathrm{Tr}\left(Q^{1/2} P^{1/2}\right) =: h^2(Q, P),$$

which is known as *Hellinger discrimination*.

If we take $f:(0,\infty)\to\mathbb{R}$, $f(t)=-\ln t$ then for $Q,P\in D_1(H)$ and Q,P invertible we have

$$D_f(Q,P)=\operatorname{Tr}\left[P(\ln P-\ln Q)\right]=U(P,Q).$$

The reader can obtain other particular quantum f-divergence measures by utilizing the normalized convex functions from Introduction, namely the convex functions defining the dichotomy class, Matsushita's divergences, Puri-Vincze divergences or divergences of Arimoto-type. We omit the details.

In the important case of finite dimensional space H and the generalized inverse P^{-1}, numerous properties of the quantum f-divergence, mostly in the case when f is *operator convex,* have been obtained in the recent papers [15], [16], [20], [21] and the references therein.

In the following theorem we apply the same proof of Theorem 2.3.

Theorem 3.1. *Let $\mathfrak{A}_{Q,P}$ be Araki self-adjoint operator on $\mathcal{B}(\mathcal{B}_2(H))$ and* $\operatorname{Sp}(\mathfrak{A}_{Q,P})\subseteq[m,M]\subset\mathring{I}$ *where I is an interval. If the function $f:I\to\mathbb{R}$ is convex on I, then*

$$\begin{aligned}\langle f(\mathfrak{A}_{Q,P})T,T\rangle_2 \quad &- \quad \frac{f(M)-f(m)}{M-m}\langle\mathfrak{A}_{Q,P}T,T\rangle_2\\ &\geq\left[\frac{2}{M-m}\int_m^M f(t)dt-\frac{Mf(M)-mf(m)}{M-m}\right]\|T\|_2^2.\end{aligned}$$

for $T\in B_2(H)$.

Remark 3.2. *Let $T=P^{\frac{1}{2}}$ in above theorem we get*

$$\begin{aligned}\langle f(\mathfrak{A}_{Q,P})P^{\frac{1}{2}},P^{\frac{1}{2}}\rangle_2 \quad &- \quad \frac{f(M)-f(m)}{M-m}\langle\mathfrak{A}_{Q,P}P^{\frac{1}{2}},P^{\frac{1}{2}}\rangle_2\\ &\geq\left[\frac{2}{M-m}\int_m^M f(t)dt-\frac{Mf(M)-mf(m)}{M-m}\right].\end{aligned}$$

for $\|P^{\frac{1}{2}}\|_2=1$.
On the other hand, by (3.11) we have

$$D_f(Q,P)\geq\frac{f(M)-f(m)}{M-m}+\left[\frac{2}{M-m}\int_m^M f(t)dt-\frac{Mf(M)-mf(m)}{M-m}\right] \tag{3.12}$$

since $Q\in D_1(H)$.

If we take the continuous convex function $f:[0,\infty)\to\mathbb{R}$, $f(t)=|t-1|$ for $t\geq 0$ then for $P,Q\in D_1(H)$ with P invertible we have the following for $m\geq 1$

$$\begin{aligned}D_f(Q,P) &= \operatorname{Tr}(|Q-P|)\\ &= V(Q,P)\geq 2(M+m-1).\end{aligned}$$

In the following theorem we apply the same proof of Theorem 2.1.

Theorem 3.3. *Let $\mathfrak{A}_{Q,P}$ and $\mathfrak{B}_{V,U}$ be two Araki self-adjoint operators on $\mathcal{B}(\mathcal{B}_2(H))$ with $\mathrm{Sp}(\mathfrak{A}_{Q,P}), \mathrm{Sp}(\mathfrak{B}_{V,U}) \subseteq [m, M] \subset \mathring{I}$ and $f : I \to \mathbb{R}$ a continuously differentiable convex function on I. Then*

$$\begin{aligned} \langle f(\mathfrak{A}_{Q,P})T, T\rangle_2 \|S\|^2 \quad &- \quad \langle f(\mathfrak{B}_{V,U})S, S\rangle_2 \|T\|^2 \\ &\geq \quad \left\langle \left(\langle \mathfrak{A}_{Q,P}T, T\rangle_2 f^{'}(\mathfrak{B}_{V,U}) - \|T\|^2 f^{'}(\mathfrak{B}_{V,U})\mathfrak{B}_{V,U} \right) S, S \right\rangle_2 \end{aligned}$$

for any $x, y \in H$.

Remark 3.4. *Let $T = P^{\frac{1}{2}}$ and $S = U^{\frac{1}{2}}$ in above theorem we get*

$$\begin{aligned} \langle f(\mathfrak{A}_{Q,P})P^{\frac{1}{2}}, P^{\frac{1}{2}}\rangle_2 \quad &- \quad \langle f(\mathfrak{B}_{V,U})U^{\frac{1}{2}}, U^{\frac{1}{2}}\rangle_2 \\ &\geq \quad \left\langle \left(\langle \mathfrak{A}_{Q,P}P^{\frac{1}{2}}, P^{\frac{1}{2}}\rangle_2 f^{'}(\mathfrak{B}_{V,U}) - f^{'}(\mathfrak{B}_{V,U})\mathfrak{B}_{V,U} \right) U^{\frac{1}{2}}, U^{\frac{1}{2}} \right\rangle_2 \end{aligned}$$

for $\|P^{\frac{1}{2}}\|_2 = \|U^{\frac{1}{2}}\|_2 = 1$.

From above inequality we have

$$D_f(Q, P) - D_f(V, U) \geq \mathrm{Tr}\left(f^{'}(\mathcal{B}_{V,U})U - f^{'}(\mathcal{B}_{V,U})V \right) \tag{3.13}$$

since $Q \in D_1(H)$.

Put $f(t) = t^2 - 1$ for $t \geq 0$ in above inequality (3.13) we get

$$\chi^2(Q, P) - \chi^2(V, U) \geq 2\,\mathrm{Tr}\left(\mathcal{B}_{V,U}U - \mathcal{B}_{V,U}V\right)$$

where $P, Q, U, V \in D_1(H)$ and P, U are invertible. Equivalently, we have

$$\chi^2(Q, P) - \chi^2(V, U) \geq 2\,\mathrm{Tr}\left(V - V^2U^{-1}\right).$$

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