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# Hermite–Hadamard’s trapezoid and mid-point type inequalities on a disk

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available at the end of the article**Abstract**

Some trapezoid and mid-point type inequalities related to the Hermite–Hadamard inequality on the disk of center  $C = (a, b)$  and radius  $R$ ,  $D(C, R) \subseteq \mathbb{R}^2$ , are investigated. It is shown that the estimated value obtained in the trapezoid and mid-point type inequalities has a relation with the integral of the partial derivative of the considered function on  $\partial(C, R)$ , the boundary of  $D(C, R)$ .

**MSC:** 26D15; 26A51; 26D07**Keywords:** Hermite–Hadamard inequality; Convex functions of double variable; Trapezoid and mid-point type inequalities**1 Introduction**

Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex mapping defined on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ . The following double inequality

$$f\left(\frac{a+b}{2}\right)(b-a) \leq \int_a^b f(x) dx \leq (b-a) \frac{f(a)+f(b)}{2}, \quad (1)$$

is known in the literature as Hermite–Hadamard inequality for convex mappings. For more results and generalization about (1), see [1, 5–11] and the references therein.

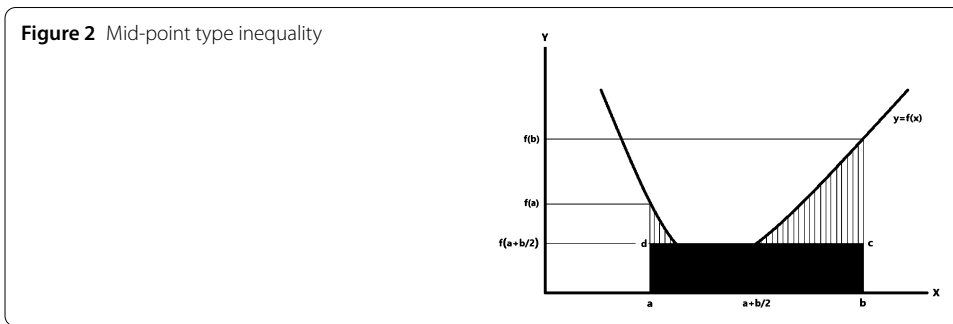
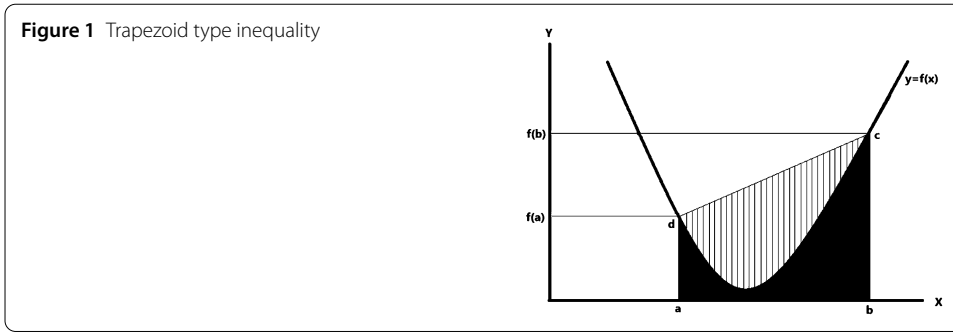
An interesting problem in (1) is estimating the difference between the right term and the integral of  $f$  on  $[a, b]$  and also estimating the difference between the left term and the integral of  $f$  on  $[a, b]$ .

In [3], the authors have obtained an estimation for the difference between the right term of (1) and the integral of  $f$  as follows.

**Theorem 1.1** *Let  $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$ , then the following inequality holds:*

$$\left| \int_a^b f(x) dx - (b-a) \frac{f(a)+f(b)}{2} \right| \leq \frac{1}{8} (b-a)^2 (|f'(a)| + |f'(b)|). \quad (2)$$

As we can see in Theorem 1.1, the estimation value is in connection with the absolute value of the derivative of the considered function on the boundary points of the corresponding interval  $[a, b]$ . In fact the striped area shown in Fig. 1, which is equivalent to the



difference between the area of trapezoid  $abcd$  and the area under the graph of  $f$ , is estimated in (2) as well. Due to this geometric property, we call inequality (2) trapezoid type inequalities related to the Hermite–Hadamard inequality.

Also in [4], the author obtained an estimation for the difference between the left term of (1) and the integral of  $f$ :

**Theorem 1.2** ([4]) *Let  $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$ , then we have*

$$\left| \int_a^b f(x) dx - (b-a)f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{8}(b-a)^2(|f'(a)| + |f'(b)|). \tag{3}$$

According to (3), the striped area shown in Fig. 2, which is in fact equivalent to the difference between the area under the graph of  $f$  and the area of rectangle  $abcd$ , is estimated. Due to this geometric property, we call inequality (3) mid-point type inequalities related to the Hermite–Hadamard inequality.

Now let us consider a point  $C = (a, b) \in \mathbb{R}^2$  and the disk  $D(C, R)$  centered at the point  $C$  and having the radius  $R > 0$ . The following inequality has been obtained in [2], which is a Hermite–Hadamard inequality related to convex functions defined on the disk  $D(C, R)$  in  $\mathbb{R}^2$ .

**Theorem 1.3** *If the mapping  $f : D(C, R) \rightarrow \mathbb{R}$  is convex on  $D(C, R)$ , then one has the inequality*

$$f(C) \leq \frac{1}{\pi R^2} \iint_{D(C, R)} f(x, y) dx dy \leq \frac{1}{2\pi R} \int_{\partial(C, R)} f(\gamma) dl(\gamma), \tag{4}$$

where  $\partial(C, R)$  is the circle centered at the point  $C = (a, b)$  with radius  $R$ . The above inequalities are sharp.

Motivated by the above-mentioned works, we investigate the trapezoid and mid-point type inequalities related to (4). We show that on a disk  $D(C, R)$ , these kinds of estimations have a relation with the integral of  $|\frac{\partial f}{\partial r}|$  (in polar coordinates) on  $\partial(C, R)$ , the boundary of the disk  $D(C, R)$ , provided that  $|\frac{\partial f}{\partial r}|$  is convex with respect to the variable  $r \in [0, R]$ .

## 2 Main results

The first result of this section is the trapezoid type inequality related to (4).

**Theorem 2.1** Consider a set  $I \subset \mathbb{R}^2$  with  $D(C, R) \subset I^\circ$ . Suppose that the mapping  $f : D(C, R) \rightarrow \mathbb{R}$  has continuous partial derivatives in the disk  $D(C, R)$  with respect to the variables  $r$  and  $\theta$  in polar coordinates. If, for any constant  $\theta \in [0, 2\pi]$ , the function  $|\frac{\partial f}{\partial r}|$  is convex with respect to the variable  $r$  on  $[0, R]$ , then

$$\left| \frac{1}{2\pi R} \int_{\partial(C, R)} f(\gamma) dl(\gamma) - \frac{1}{\pi R^2} \iint_{D(C, R)} f(x, y) dx dy \right| \leq \frac{1}{6\pi} \int_{\partial(C, R)} \left| \frac{\partial f}{\partial r} \right|(\gamma) dl(\gamma). \tag{5}$$

*Proof* For a constant  $\theta \in [0, 2\pi]$ , if we consider

$$x(r) = a + r \cos \theta$$

and

$$y(r) = b + r \sin \theta,$$

then we have  $([\dot{x}(r)]^2 + [\dot{y}(r)]^2)^{\frac{1}{2}} = (\sin^2(\theta) + \cos^2(\theta))^{\frac{1}{2}} = 1$ , where  $\dot{x}, \dot{y}$  are the derivatives of  $x, y$ , respectively, with respect to the variable  $r$  on  $[0, R]$ . So, by the use of integration by parts, we have the following equalities:

$$\begin{aligned} & \int_0^R \frac{\partial f}{\partial r}(a + r \cos \theta, b + r \sin \theta) r^2 dr = r^2 f(a + r \cos \theta, b + r \sin \theta) \Big|_0^R \\ & - 2 \int_0^R f(a + r \cos \theta, b + r \sin \theta) r dr = R^2 f(a + R \cos \theta, b + R \sin \theta) \\ & - 2 \int_0^R f(a + r \cos \theta, b + r \sin \theta) r dr. \end{aligned} \tag{6}$$

The integration of (6) with respect to  $\theta$  on  $[0, 2\pi]$  implies that

$$\begin{aligned} & R^2 \int_0^{2\pi} f(a + R \cos \theta, b + R \sin \theta) d\theta - 2 \int_0^{2\pi} \int_0^R f(a + r \cos \theta, b + r \sin \theta) r dr d\theta \\ & = \int_0^{2\pi} \int_0^R \frac{\partial f}{\partial r}(a + r \cos \theta, b + r \sin \theta) r^2 dr d\theta. \end{aligned}$$

Since  $|\frac{\partial f}{\partial r}|$  is convex with respect to the variable  $r$  on  $[0, R]$  for any  $\theta \in [0, 2\pi]$ , then

$$\begin{aligned} & \left| R^2 \int_0^{2\pi} f(a + R \cos \theta, b + R \sin \theta) d\theta - 2 \int_0^{2\pi} \int_0^R f(a + r \cos \theta, b + r \sin \theta) r dr d\theta \right| \\ & \leq \int_0^{2\pi} \int_0^R \left| \frac{\partial f}{\partial r} \right| (a + r \cos \theta, b + r \sin \theta) r^2 dr d\theta \\ & = \int_0^{2\pi} \int_0^R \left| \frac{\partial f}{\partial r} \right| \left( \frac{r}{R} (a + R \cos \theta, b + R \sin \theta) + \left( 1 - \frac{r}{R} \right) (a, b) \right) r^2 dr d\theta \\ & \leq \int_0^{2\pi} \int_0^R \frac{r^3}{R} \left| \frac{\partial f}{\partial r} \right| (a + R \cos \theta, b + R \sin \theta) dr d\theta \\ & \quad + \int_0^{2\pi} \int_0^R r^2 \left( 1 - \frac{r}{R} \right) \left| \frac{\partial f}{\partial r} \right| (C) dr d\theta \\ & = \frac{R^3}{4} \int_0^{2\pi} \left| \frac{\partial f}{\partial r} \right| (a + R \cos \theta, b + R \sin \theta) d\theta + \frac{\pi R^3}{6} \left| \frac{\partial f}{\partial r} \right| (C). \end{aligned} \tag{7}$$

Now, consider the curve  $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$  given by

$$\gamma : \begin{cases} x(\theta) = a + R \cos \theta, \\ y(\theta) = b + R \sin \theta, \end{cases} \quad \theta \in [0, 2\pi].$$

Then  $\gamma([0, 2\pi]) = \partial(C, R)$ , and we write (integrating with respect to arc length)

$$\begin{aligned} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right| (\gamma) dl(\gamma) &= \int_0^{2\pi} \left| \frac{\partial f}{\partial r} \right| (x(\theta), y(\theta)) \left( [\dot{x}(\theta)]^2 + [\dot{y}(\theta)]^2 \right)^{\frac{1}{2}} d\theta \\ &= R \int_0^{2\pi} \left| \frac{\partial f}{\partial r} \right| (a + R \cos \theta, b + R \sin \theta) d\theta. \end{aligned} \tag{8}$$

From (7) and (8) we obtain

$$\begin{aligned} & \left| R^2 \int_0^{2\pi} f(a + R \cos \theta, b + R \sin \theta) d\theta - 2 \int_0^{2\pi} \int_0^R f(a + r \cos \theta, b + r \sin \theta) r dr d\theta \right| \\ & \leq \frac{R^2}{4} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right| (\gamma) dl(\gamma) + \frac{\pi R^3}{6} \left| \frac{\partial f}{\partial r} \right| (C). \end{aligned} \tag{9}$$

Also using the convexity of  $|\frac{\partial f}{\partial r}|$  in (4) we have

$$\begin{aligned} \left| \frac{\partial f}{\partial r} \right| (C) &\leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R \left| \frac{\partial f}{\partial r} \right| (a + r \cos \theta, b + r \sin \theta) dr d\theta \\ &\leq \frac{1}{2\pi R} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right| (\gamma) dl(\gamma). \end{aligned} \tag{10}$$

So by replacing (10) in (9) we obtain

$$\begin{aligned} & \left| R \int_{\partial(C,R)} f(\gamma) dl(\gamma) - 2 \int_0^{2\pi} \int_0^R f(a + r \cos \theta, b + r \sin \theta) r dr d\theta \right| \\ & \leq \frac{R^2}{3} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right| (\gamma) dl(\gamma). \end{aligned} \tag{11}$$

Finally dividing (11) with  $2\pi R^2$  we get

$$\left| \frac{1}{2\pi R} \int_{\partial(C,R)} f(\gamma) dl(\gamma) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y) dx dy \right| \leq \frac{1}{6\pi} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right|(\gamma) dl(\gamma). \quad \square$$

*Example 2.2* Consider the bifunction  $f(x,y) = R - \sqrt{(x-a)^2 + (y-b)^2}$  defined on the disk  $D(C,R)$ . In polar coordinates we have that

$$f(a + r \cos \theta, b + r \sin \theta) = R - r$$

for  $0 \leq r \leq R, \theta \in [0, 2\pi]$  and specially  $f(a + R \cos \theta, b + R \sin \theta) = 0$  for all  $\theta \in [0, 2\pi]$ . So

$$\begin{aligned} & \left| \frac{1}{2\pi R} \int_{\partial(C,R)} f(\gamma) dl(\gamma) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y) dx dy \right| \\ &= \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y) dx dy = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R (R-r)r dr d\theta = \frac{R}{3}. \end{aligned} \tag{12}$$

On the other hand, it is not hard to see that  $\left| \frac{\partial f}{\partial r} \right|(a + R \cos \theta, b + R \sin \theta) = 1$  for all  $\theta \in [0, 2\pi]$ , and so

$$\frac{1}{6\pi} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right|(\gamma) dl(\gamma) = \frac{R}{3}. \tag{13}$$

Then identities (12) and (13) show that inequality (5) is sharp.

The following result is the mid-point type inequality related to (4).

**Theorem 2.3** Consider a set  $I \subset \mathbb{R}^2$  with  $D(C,R) \subset I^\circ$ . Suppose that the mapping  $f : D(C,R) \rightarrow \mathbb{R}$  has continuous partial derivatives in the disk  $D(C,R)$  with respect to the variables  $r$  and  $\theta$  in polar coordinates. If, for any constant  $\theta \in [0, 2\pi]$ , the function  $\left| \frac{\partial f}{\partial r} \right|$  is convex with respect to the variable  $r$  on  $[0, R]$ , then

$$\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y) dx dy - f(C) \right| \leq \frac{2}{3\pi} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right|(\gamma) dl(\gamma). \tag{14}$$

*Proof* As we have seen in the proof of Theorem 2.1, for a constant  $\theta \in [0, 2\pi]$ , if we consider  $x(r) = a + r \cos \theta$  and  $y(r) = b + r \sin \theta$ , then we have  $([\dot{x}(r)]^2 + [\dot{y}(r)]^2)^{\frac{1}{2}} = 1$ . So from fundamental theorem of calculus we have

$$\int_0^R \frac{\partial f}{\partial r}(a + r \cos \theta, b + r \sin \theta) dr = f(a + R \cos \theta, b + R \sin \theta) - f(C).$$

Hence

$$\begin{aligned} & \int_0^{2\pi} \int_0^R \frac{\partial f}{\partial r}(a + r \cos \theta, b + r \sin \theta) dr d\theta \\ &= \int_0^{2\pi} f(a + R \cos \theta, b + R \sin \theta) d\theta - 2\pi f(C), \end{aligned}$$

which implies that

$$\int_0^{2\pi} \int_0^R \frac{\partial f}{\partial r}(a + r \cos \theta, b + r \sin \theta) dr d\theta = \frac{1}{R} \int_{\partial(C,R)} f(\gamma) dl(\gamma) - 2\pi f(C). \tag{15}$$

Now from (15) we obtain

$$\begin{aligned} & \left| \frac{1}{2\pi R} \int_{\partial(C,R)} f(\gamma) dl(\gamma) - f(C) \right| \\ & \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^R \left| \frac{\partial f}{\partial r} \right|(a + r \cos \theta, b + r \sin \theta) dr d\theta. \end{aligned}$$

Since  $\left| \frac{\partial f}{\partial r} \right|$  is convex, then it follows that

$$\begin{aligned} & \left| \frac{1}{2\pi R} \int_{\partial(C,R)} f(\gamma) dl(\gamma) - f(a, b) \right| \\ & \leq \frac{1}{2\pi} \left[ \int_0^{2\pi} \int_0^R \left| \frac{\partial f}{\partial r} \right| \left( \frac{r}{R}(a + R \cos \theta, b + R \sin \theta) + \left(1 - \frac{r}{R}\right)(a, b) \right) dr d\theta \right] \\ & \leq \frac{1}{2\pi} \left[ \int_0^{2\pi} \int_0^R \frac{r}{R} \left| \frac{\partial f}{\partial r} \right|(a + R \cos \theta, b + R \sin \theta) dr d\theta \right. \\ & \quad \left. + \int_0^{2\pi} \int_0^R \left(1 - \frac{r}{R}\right) \left| \frac{\partial f}{\partial r} \right|(C) dr d\theta \right] \\ & = \frac{1}{4\pi} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right|(\gamma) dl(\gamma) + \frac{R}{2} \left| \frac{\partial f}{\partial r} \right|(C). \end{aligned} \tag{16}$$

From the triangle inequality and (16) we get

$$\begin{aligned} & \left| \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R f(a + r \cos \theta, b + r \sin \theta) dr d\theta - f(C) \right| \\ & \leq \frac{1}{4\pi} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right|(\gamma) dl(\gamma) + \frac{R}{2} \left| \frac{\partial f}{\partial r} \right|(C) \\ & \quad + \left| \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R f(a + r \cos \theta, b + r \sin \theta) r dr d\theta - \frac{1}{2\pi R} \int_{\partial(C,R)} f(\gamma) dl(\gamma) \right|. \end{aligned} \tag{17}$$

Since  $\left| \frac{\partial f}{\partial r} \right|$  satisfies the Hermite–Hadamard inequality (4), then

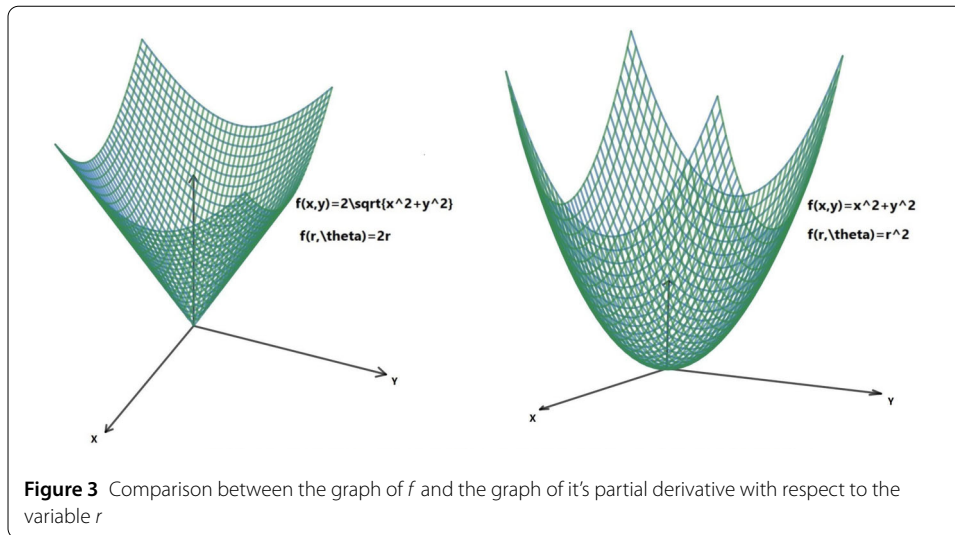
$$\left| \frac{\partial f}{\partial r} \right|(C) \leq \frac{1}{2\pi R} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right|(\gamma) dl(\gamma).$$

So, by replacing (5) and the inequality in (17) above, we obtain

$$\left| \frac{1}{\pi R^2} \iint_{D(C,R)} f(x, y) dx dy - f(C) \right| \leq \frac{2}{3\pi} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right|(\gamma) dl(\gamma). \quad \square$$

**Remark 2.4** If the functions  $f$  and  $\left| \frac{\partial f}{\partial r} \right|$  are convex on  $D(C, R)$ , then by the use of inequalities (5), (14), and (4) we have

$$0 \leq \frac{1}{\pi R^2} \iint_{D(C,R)} f(x, y) dx dy - f(C) \leq \frac{2}{3\pi} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right|(\gamma) dl(\gamma)$$



and

$$0 \leq \frac{1}{2\pi R} \int_{\partial(C,R)} f(\gamma) dl(\gamma) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y) dx dy \leq \frac{1}{6\pi} \int_{\partial(C,R)} \left| \frac{\partial f}{\partial r} \right|(\gamma) dl(\gamma).$$

**Example 2.5** There exists a function satisfying all the conditions of Remark 2.4 as well. Consider the function  $f(x,y) = x^2 + y^2$  with  $(x,y) \in \mathbb{R}^2$  defined on a disk  $D((0,0),R)$ . It is clear that  $f(r,\theta) = r^2$  and  $|\frac{\partial f}{\partial r}| = 2r$ , which is equivalent to  $f(x,y) = 2\sqrt{x^2 + y^2}$  with  $(x,y) \in \mathbb{R}^2$  defined on a disk  $D((0,0),R)$ . As we can see in Fig. 3, the functions  $f$  and  $|\frac{\partial f}{\partial r}|$  are convex.

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**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors read and approved the final manuscript.

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